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Volume and Surface Area for Polyhedra and Polytopes

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When the curious calculus student first realizes the relationship between the area and circumference of the circle, $dA/dr = C$, and the similar relationship between the volume and surface area of the sphere, $dV/dr = A$, the question arises: "Is this always the case?" We show that, indeed, such a relation exists in all dimensions for (convex) regular polytopes when the derivative is taken with respect to the *inner radius*, i.e., the (minimal) distance from the center to the boundary. With some modification, a similar relation holds for all polytopes. This approach leads to a related, turn-of-the-century result due to Minkowski.

1. Introductory examples The circle and sphere are generalized by the $(n-1)$-dimensional hypersphere, defined as the set of points in n-space that are a constant distance r from the center point. The *content* (n-dimensional volume) bounded by a hypersphere of radius r is known to be (see [**4**], for example)

$$V = \frac{2r^n\pi^{n/2}}{n\Gamma(n/2)},$$

where Γ is the gamma function. It is a standard exercise (see [**3**, p. 125]) to verify that the derivative dV/dr is the content of the bounding hypersphere.

Using standard trigonometry, a regular k-gon with inner radius r can be shown to have circumference $C = 2rn\tan(\pi/n)$ and area $A = r^2 n\tan(\pi/n)$. Note that $dA/dr = C$.

A hypercube with inner radius r has side length $2r$, so the content of an n-dimensional hypercube is $V = 2^n r^n$. The surface of an n-dimensional hypercube has $2n$ faces, each an $(n-1)$-dimensional hypercube. Therefore, an n-dimensional hypercube has surface content $A = 2n \cdot 2^{n-1} r^{n-1} = n2^n r^{n-1}$. Again, as predicted, $dV/dr = A$.

It is well known that there are exactly five convex regular polyhedra in dimension 3, the Platonic solids. In dimension 4, there are exactly six convex regular polytopes. In dimensions $n \geq 5$, there are exactly three regular polytopes: the hypercube, regular simplex, and cross polytope (see, e.g., [**3**, p. 136]). After introducing and exploring the regular n-dimensional simplex using standard content formulae expressed *in terms of edge length*, we give a separate argument that verifies the "volume-surface area" relationship for *all* regular n-polytopes.

2. The regular *n*-simplex The regular n-dimensional simplex Δ_n, determined by $n+1$ points arranged equidistantly in n-space, enjoys the expected relationship between content and surface content. We have already observed this relationship in dimension 2, for the equilateral triangle. The proof for all dimensions will use the following lemma.

LEMMA. *For a regular n-dimensional simplex Δ_n $(n \geq 2)$, the inner radius r, altitude a, and edge length e satisfy*

$$a = (n+1)r \quad \text{and} \quad e = \sqrt{2n(n+1)}\, r.$$

Proof. The result (easily verified when $n = 2$) is shown by induction on the dimension n. Assuming the result for dimension n, construct a triangle in Δ_{n+1} determined by a vertex, the center of the opposite n-dimensional face, and the center of a common $(n-1)$-dimensional face, as shown in FIGURE 1. This right triangle has an altitude of Δ_{n+1} and altitude and radius of n-dimensional faces as edges; label their lengths as a_{n+1}, a_n, and r_n, respectively. The altitudes pass through the center O of Δ_{n+1}; mark this point and drop a perpendicular to the opposite triangle edge, piercing the center of the n-dimensional face as shown in FIGURE 2. The length of the constructed radius of Δ_{n+1} is r_{n+1}, as marked.

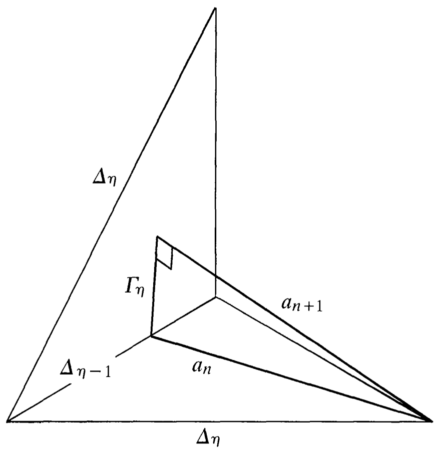

FIGURE 1
The altitudes of Δ_{n+1} and Δ_n.

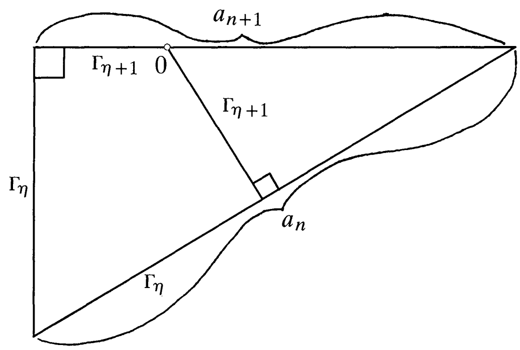

FIGURE 2
The center O and inner radius r_{n+1} of Δ_{n+1}.

By similarity of triangles,

$$\frac{r_{n+1}}{a_n - r_n} = \frac{r_n}{a_{n+1}}.$$

By the induction hypothesis, $r_{n+1}a_{n+1} = r_n(nr_n)$, so

$$r_n^2 = \frac{r_{n+1}a_{n+1}}{n}. \tag{1}$$

Since $a_{n+1}^2 = a_n^2 - r_n^2$, we use the induction hypothesis $a_n = (n+1)r_n$ to conclude that $a_{n+1}^2 = (n+1)^2 r_n^2 - r_n^2$. By (1), it follows that $a_{n+1}^2 = (n+2)r_{n+1}a_{n+1}$. Therefore,

$$a_{n+1} = (n+2)\,r_{n+1}, \tag{2}$$

and the first part of the lemma is shown.

To verify the second equality, substitute (2) and the induction hypothesis into (1), and solve for r_{n+1}. ■

An n-simplex is the n-dimensional cone over an $(n-1)$-simplex. By induction, the content V of Δ_n with edge length e is therefore

$$V = \frac{1}{n}a_n V' = \frac{\sqrt{n+1}}{n!\sqrt{2^n}}e^n,$$

where V' is the content of Δ_{n-1}. By the chain rule and the preceding lemma, the derivative of V with respect to the inner radius is

$$\begin{aligned}\frac{dV}{dr_n} &= \frac{\sqrt{n+1}}{n!\sqrt{2^n}}ne^{n-1}\frac{de}{dr_n}\\ &= \frac{\sqrt{n+1}}{n!\sqrt{2^n}}ne^{n-1}\sqrt{2n(n+1)}\\ &= (n+1)\frac{\sqrt{n}}{(n-1)!\sqrt{2^{n-1}}}e^{n-1},\end{aligned}$$

which, as the content of $n+1$ simplices of edge length e and dimension $n-1$, is exactly the surface content of Δ_n.

The following formula-free argument yields the same result. One may view Δ_n as being constructed from $n+1$ smaller (non-regular) simplices created by coning from the centroid p of Δ_n to its *principal* ($(n-1)$-dimensional) faces. Dilation of Δ_n from its centroid corresponds to dilation of each smaller simplex from vertex p to its opposite face; the inner radius r_n of Δ_n is the height of each smaller simplex. Hence, the derivative of the content of a cone over a principal face with respect to its height is the content of that principal face. This relationship—which is as natural as the volume-surface relationship for the sphere which motivated this inquiry—holds for cones over any principal face in any dimension. It is the key to the argument for all regular polyhedra and polytopes.

3. The regular case Suppose Γ_n is a cone with height h over a principal face Γ_{n-1}. If Γ_n is dilated from the cone point, then the content of Γ_{n-1} is a function of h given by γh^{n-1} for some constant γ. The content of Γ_n is given, however, by the familiar formula $\frac{1}{n}hF_{n-1}$, where F_{n-1} is the content of Γ_{n-1}. Like its special cases for the area of a triangle and the volume of a 3-dimensional cone, this formula may be found by integrating the cross sectional content γh^{n-1} of Γ_n along its altitude. Hence, the content of Γ_n is $\frac{1}{n}\gamma h^n$. The next lemma follows immediately:

LEMMA. *Let Γ_n be a cone of height h over principal face Γ_{n-1}. The derivative of the content of cone Γ_n with respect to its height h is the content of the principal face Γ_{n-1}.*

THEOREM 1. *The derivative of the content of any convex regular n-dimensional polytope with respect to its inner radius is the content of the boundary of the polytope.*

Proof. Any convex regular polytope P is the union of κ identical cones Γ_n with principal faces Γ_{n-1} opposite the cone point and height r, which is the inner radius of the regular polytope, as illustrated in FIGURE 3. The content of P is $V = \kappa\frac{1}{n}\gamma r^n$, so that $dV/dr = \kappa\gamma r^{n-1}$, the content of the boundary of P. ∎

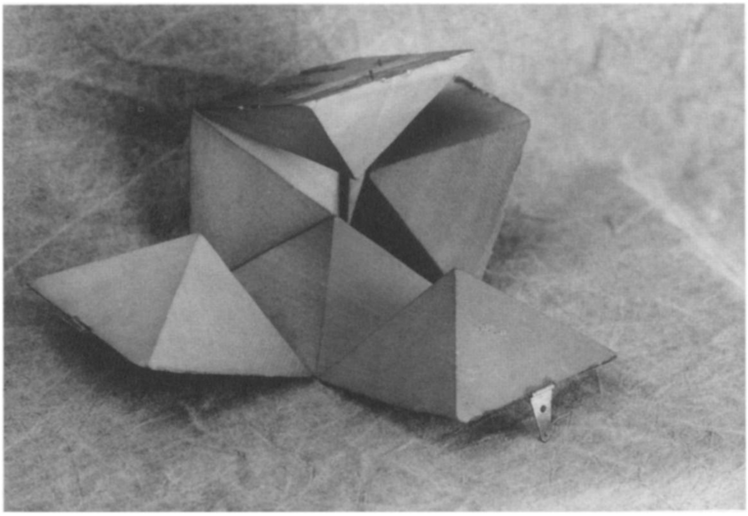

FIGURE 3
The cube as a union of six simplices.

For some convex sets, $dV/dr = \xi A$, where ξ is a constant not equal to 1. Consider, for example, the ellipse formed by stretching a circle in one direction by a factor of $a > 1$. Area increases by the factor a while arc length increases by a factor b ($b \approx \sqrt{\frac{1+a^2}{2}}$ by approximating an elliptic integral) which is strictly less than a. Hence the derivative of area with respect to inner radius (length of the semi-minor axis) is $\frac{a}{b}$ $\left(\frac{a}{b} \approx \sqrt{\frac{2a^2}{1+a^2}}\right)$ times the perimeter. One can also show that the derivative of the area of a rectangle with length l and width w $(l \geq w)$ with respect to the distance from the center to the near side is $\frac{2l}{w+l}$ times the perimeter. When the ellipse is a circle or the rectangle is a square, the constant $\xi = 1$. In general, when is ξ equal to 1?

4. The circumscribing polytope case Let V be the content of a polytope P having faces with content A_i. We cone from an arbitrary point p of the polytope P and let a provisional inner radius r be the distance to a nearest face. This inner radius coincides with the height of the cone over this face, which has content A_0. Then there are constants a_i and k_i such that $A_i = a_i A_0$ and $h_i = k_i r$, where h_i is the height of the ith cone. These relations hold, of course, for any *dilation* of the polytope P. Since h_i is the distance from p to the hyperplane determined by a principal face, $k_i \geq 1$. Note that the total surface content A of the polytope is then $(\Sigma a_i)A_0$. Furthermore, if V_i is the content of the ith cone, then

$$\begin{aligned}\frac{dV}{dr} &= \sum \frac{dV_i}{dr} = \sum \frac{dV_i}{dh_i} = \sum k_i A_i \\ &= \sum k_i a_i A_0 = \frac{\Sigma a_i k_i}{\Sigma a_i}\left(\sum a_i\right) A_0 \\ &= \frac{\Sigma k_i a_i}{\Sigma a_i} A.\end{aligned}$$

Some choice of coning point minimizes the constant $\xi = \frac{\Sigma k_i a_i}{\Sigma a_i}$ at a value greater than or equal to 1. We have $\xi = 1$ if and only if $k_i = 1$ for each i, which is equivalent to the statement that P circumscribes an $(n-1)$-sphere centered at the coning point p. We have shown the following result.

THEOREM 2. *Suppose P is a polytope with content V and surface content A. Suppose also that P undergoes a dilation centered at an interior point p. Let r be the distance from p to the boundary of P. Then $dV/dr = A$ if and only if the sphere centered at p with radius r is circumscribed by P.*

The polytopes that can circumscribe a sphere include the regular polytopes, the semiregular polyhedra and their duals, and many other non-regular polytopes, such as in FIGURE 4. In general it is not possible to pick an "inner radius" r so that $dV/dr = A$. But is there a different variable that can replace r to achieve this "volume-area" relationship?

5. The general case Adopting a new point of view toward the circumscribing polytope case enables us to generalize the "volume-surface area" relationship to all polytopes. When each principal face is the same distance r from a central point p, multiplying r by a fixed proportion has the same effect as adding a fixed amount ϵ to r. Theorem 2 tells us that this coincidence occurs only for circumscribing polytopes. We adopt this latter point of view for the general situation.

Given a polytope P, choose any point p in the interior of P and a positive ϵ. Suppose P expands by pushing out each principal face Γ a distance ϵ to a parallel face $\tilde{\Gamma}$. See FIGURE 5. In general this expansion is not a dilation since the resulting polytope may not be similar to the original. (By this process each non-principal face will typically be carried to different locations by each of the principal faces containing it. As these faces will not align, as in FIGURE 6, the resulting shears must be patched.) Denote the resulting "ϵ-collared polytope" of P as $P_{\epsilon, p}$.

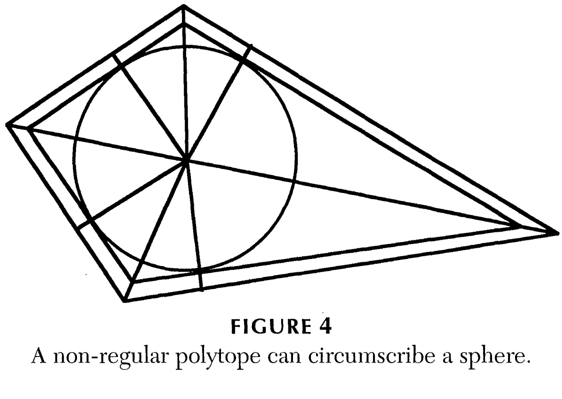

FIGURE 4
A non-regular polytope can circumscribe a sphere.

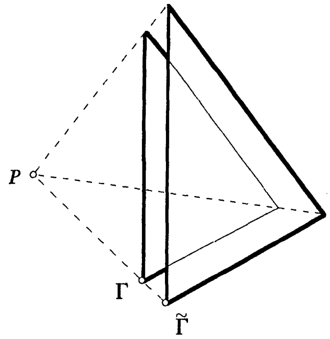

FIGURE 5
Pushing out each principal face.

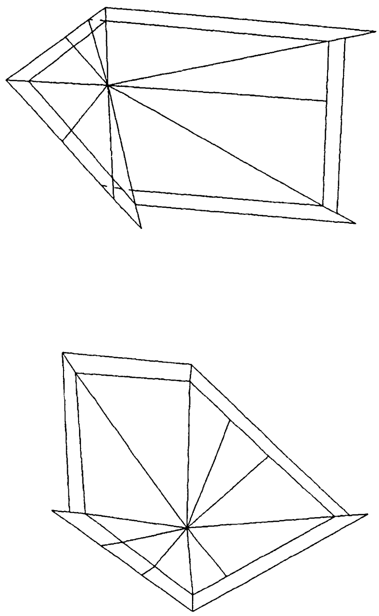

FIGURE 6
Pushing can create sheared principal faces.

THEOREM 3. *Suppose P is a polytope with content V and surface content A. Let p be any point in P. Suppose P expands by pushing each principal face out ϵ from p to form an ϵ-collared polytope $P_{\epsilon, p}$. Then $dV/d\epsilon = A$.*

Proof. Let h_i be the distance from p to the i-th principal face, as before, and note that $\frac{d\epsilon}{dh_i} = 1$. Then

$$\frac{dV}{d\epsilon} = \sum \frac{dV_i}{d\epsilon} = \sum \frac{dV_i}{dh_i} = \sum A_i = A. \qquad \blacksquare$$

The (non-polytope) ϵ-neighborhood of P, $P_\epsilon = \{x | d(x, P) \leq \epsilon\}$, can also be formed. Compare the example in FIGURE 7 to the polytope collarings of FIGURE 6. For a fixed polytope P and constant ϵ, P_ϵ and the various $P_{\epsilon, p}$ are all distinct, but the differences in their contents are of the order ϵ^n.

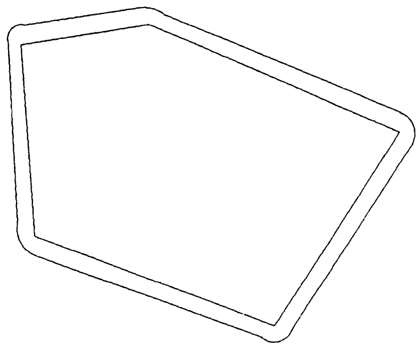

FIGURE 7
An ϵ-neighborhood of a polytope.

If $\mathscr{V}(P)$ represents the content of P, Theorem 3 implies that

$$\lim_{e \to 0} \frac{\mathscr{V}(P_{\epsilon, p}) - \mathscr{V}(P)}{\epsilon} = A.$$

In a 1901 article in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Minkowski used ϵ-neighborhoods to show that

$$\lim_{e \to 0^+} \frac{\mathscr{V}(P_\epsilon) - \mathscr{V}(P)}{\epsilon} = A.$$

See [**1**, II, p. 550] or [**2**, II, p. 18] for Minkowski's alternate approach to this result.

With modifications, Theorem 3 can be extended to other regions, such as *starlike* polygons and their n-dimensional analogues. Recall that a region R is starlike if there is some point p in the interior of R (a *star point*) that can see, through R, every point of R. Define $R_{\epsilon, p}$, where p is a star point, to be an ϵ-collaring of R, as before. In this setting the argument for Theorem 3 extends to starlike polygons and their n-dimensional analogues. Minkowski's approach can also extend to such regions if certain overlapped regions are considered "signed volume."

The results of Theorem 3 and Minkowski can also be extended to general (curved) convex sets. A convex set C can be approximated arbitrarily closely by polytopes $\{P_i\}$ so that the content and surface content of the P_i tend in the limit to the content and surface content of C. Similarly, the ϵ-neighborhood C_ϵ is approximated by ϵ-collarings or ϵ-neighborhoods of the P_i. The polytope approximations show that Theorem 3 extends to convex sets.

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