Fast Calculation of the Riemann Zeta Function (s) for Integer Values of the Argument s

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FAST CALCULATION OF THE RIEMANN ZETA FUNCTION $\zeta(s)$ FOR INTEGER VALUES OF THE ARGUMENT s

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We suggest an algorithm for fast calculation of the Riemann zeta function for integer values of the argument, which is based on the method for fast calculation of Siegel's E-functions. The computational complexity is near to optimal.

1. Introduction

In [1–5], a method called the FEC (Fast E-function Calculation) was suggested for fast calculation of the values of functions that have the type of Siegel's E-function. It was also proved there that, using the FEC, one can calculate quickly any elementary transcendental function, classical constants e, π, and the Euler constant γ, as well as higher transcendental functions such as the gamma function, Bessel functions, and other special functions for algebraic values of the argument and the parameters.

Below we shall assume that numbers are written in the binary notation.

By the complexity of multiplication of two n-digit numbers we shall mean the number of elementary (binary) operations $M(n)$ sufficient for calculation of the product of these two n-digit numbers.

From here on, elementary (binary) operations, for the sake of brevity, will simply be called operations.

Suppose that the function $y = f(z)$ is defined on some bounded domain $\mathbb{D} \subset \mathbb{C}$, does not have singularities in $\mathbb{D}$, and is bounded together with its derivative. Then to calculate $y = f(z)$ at the point $z = z_0 \in \mathbb{D}$ with accuracy 2^{-n} (with accuracy up to n digits) means to find a number y_n that satisfies the inequality

$$|y_n - f(z_0)| \le c2^{-n},$$

where the constant c does not depend on n.

The number of operations that is sufficient for calculation of $y = f(z)$ with accuracy 2^{-n} at any point of the domain of the function is denoted by $s_f(n)$ and called the complexity of calculation of the function $f(z)$.

In [1–5], it is proved that the complexity of calculation by FEC of the aforementioned elementary and higher transcendental functions and constants is estimated by

$$s_f(n) = O\big(M(n)\log^2 n\big).$$

The representation of the Riemann zeta function $\zeta(s)$ in the form of the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

does not permit fast calculating of $\zeta(s)$, since this series converges very slowly. The well-known formulas for the zeta function [6, p. 61] make it possible to calculate $\zeta(s)$ fast, using the FEC, only for $s = 2m$ and $s = -2m + 1$; m is a natural number. In this case, the complexity of the calculation, as proved in [4], is estimated by

$$s_\zeta(n) = O\big(M(n)\log^2 n\big).$$

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In [4], it was proved that, by the FEC, the value $\zeta(3)$ can be calculated equally fast. This proof used a formula that represented $\zeta(3)$ in the form of a rapidly converging series. This formula is contained in the paper by van der Poorten [7] devoted to Apery's proof of irrationality of $\zeta(3)$.

In the present paper, we prove that, using the FEC, one can calculate quickly the function $\zeta(s)$ for any natural values of the argument s ($s = k,\ k \geq 2$) with the complexity of the calculation estimated by

$$s_\zeta(n) = O\big(M(n)\log^2 n\big). \tag{1}$$

The proof is based on new formulas for $\zeta(s)$ (Lemma 1) and on application of the FEC to calculation of special integrals (Lemma 2). These lemmas and their proofs are placed in Sec. 2 of the present paper. In Sec. 3, the theorem of the fast calculation of $\zeta(k)$, $k \geq 2$, is proved.

In conclusion, it is worth noting that the estimate (1) can be rewritten, expanding the expression for the multiplication complexity $M(n)$.

The first algorithm of fast multiplication was found by A. A. Karatsuba [8] (see also [9]) and has computational complexity estimated by

$$M(n) \leq c n^{\log_2 3},$$

where c is a constant.

Improving the method of Karatsuba, other algorithms were constructed, and, particularly, the algorithm of Schönhage–Strassen [10], with an estimate that is the best at the present time:

$$M(n) \leq cn \log n \log\log n,$$

where c is a constant.

The algorithms of multiplication ("naive," or "schoolish," of Karatsuba, of Schönhage-Strassen, and their modifications) are described in detail in [11]. Also there, for each of these algorithms, the domain of its greater efficiency compared to the others is established.

Consequently, the estimate (1) of the complexity of calculating $\zeta(k)$, $k \geq 2$, can be written in the form

$$s_\zeta(n) = O(n \log^3 n \log\log n).$$

2. Auxiliary lemmas

Let us prove two auxiliary statements.

Lemma 1. *Suppose that k is a natural number, $k \geq 2$; $n_1, n_2, \ldots, n_k$ are nonnegative integers, and*

$$G_i = \sum_{\substack{n_1+n_2+\ldots+n_k=i \\ n_1+2n_2+\ldots+kn_k=k}} \frac{k!}{n_1!\,n_2!\ldots n_k!} \prod_{j=1}^{k} \left(\frac{J_j}{j!}\right)^{n_j}, \tag{2}$$

where

$$J_j = \int_0^\infty e^{-t} \log^j t \, dt. \tag{3}$$

Then the following identity holds:

$$\zeta(k) = \frac{(-1)^k}{(k-1)!} \sum_{i=1}^{k} (-1)^{i-1}(i-1)!\, G_i. \tag{4}$$

PROOF. First we show that for any natural k, $k \geq 2$, we have the identity

$$\frac{d^k}{ds^k}\big(\log \Gamma(s+1)\big)\Big|_{s=0} = (-1)^k (k-1)!\, \zeta(k). \tag{5}$$

By definition of the Euler gamma function $\Gamma(s)$ (see [6, p. 51]),

$$\Gamma(s) = \frac{1}{s} e^{-\gamma s} \left(\prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{\frac{s}{n}} \right). \tag{6}$$

where γ is the Euler constant. Taking the logarithm of both sides of (6), we obtain

$$\log \Gamma(s) = -\log s - \gamma s + \sum_{n=1}^{\infty} \left(\frac{s}{n} - \log \left(1 + \frac{s}{n}\right)\right). \tag{7}$$

For $|s| \le 1 - \delta$, $0 < \delta < 1$, consider the Taylor-series expansion of the function $\log\left(1 + \frac{s}{n}\right)$ in powers of $\frac{s}{n}$:

$$\log\left(1 + \frac{s}{n}\right) = \frac{s}{n} + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} \frac{s^k}{n^k}.$$

Hence,

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{s}{n} - \log\left(1 + \frac{s}{n}\right)\right) &= \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \frac{s^k}{n^k} \\ &= \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} s^k \sum_{n=1}^{\infty} \frac{1}{n^k} = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} s^k \zeta(k). \end{aligned} \tag{8}$$

By (7) and (8), taking into account $\Gamma(s+1) = s\Gamma(s)$, we have

$$\log \Gamma(s+1) + \gamma s = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} s^k \zeta(k). \tag{9}$$

Differentiating (9) k times in s and then putting $s = 0$, we obtain the identity (5).

Now let us expand the left-hand side of (5). Toward this end, let us differentiate k times the composite function $y = \log \Gamma(s+1)$ by applying the formula of composite differentiation (see, e.g., [12]). We obtain

$$\begin{aligned} \frac{d^k}{ds^k}(\log \Gamma(s+1)) &= \sum_{i=1}^{k} (-1)^{i-1} \frac{(i-1)!}{(\Gamma(s+1))^i} \\ &\times \sum_{\substack{n_1+n_2+\ldots+n_k=i \\ n_1+2n_2+\ldots+kn_k=k \\ n_1,n_2,\ldots,n_k \ge 0 \text{ are integers}}} \frac{k!}{n_1!\, n_2! \ldots n_k!} \prod_{j=1}^{k} \left(\frac{1}{j!} \frac{d^j}{ds^j} \Gamma(s+1) \right)^{n_j}. \end{aligned} \tag{10}$$

The gamma function $\Gamma(s+1)$ can be defined (see [6, p. 53]) by the integral

$$\Gamma(s+1) = \int_0^{\infty} e^{-t} t^s \, dt, \quad \operatorname{Re} s > -1. \tag{11}$$

Differentiating (11) j times in the parameter s and putting then $s = 0$, we find

$$\frac{d^j}{ds^j} \Gamma(s+1) \bigg|_{s=0} = \int_0^{\infty} e^{-t} \log^j t \, dt. \tag{12}$$

Taking into account that $\Gamma(s+1)|_{s=0} = 1$, by (5), (10), and (12) we obtain the statement of the lemma. $\triangle$

Let us now estimate the complexity of computation by the FEC of the integrals J_j, $j = 1, 2, \ldots, k$, defined by (3). Denote by $s_J(n)$ the complexity of computation of J_j for some natural j, $1 \le j \le k$. Then the following lemma is valid.

Lemma 2. *For any fixed natural number k and any natural parameter j, $1 \le j \le k$, the following estimate holds*:

$$s_J(n) = O(n \log^3 n \log\log n).$$

PROOF. From here on, we assume that

$$n \ge 2k \log 2k, \quad k \ge 2. \tag{13}$$

Let us represent the integral $J_j = \int\limits_0^\infty e^{-t} \log^j t\, dt$ in the form of the sum of two integrals:

$$J_j = A_j + B_j, \tag{14}$$

where

$$A_j = \int\limits_0^p e^{-t} \log^j t\, dt \tag{15}$$

and

$$B_j = \int\limits_p^\infty e^{-t} \log^j t\, dt; \tag{16}$$

here we assume that

$$p = n. \tag{17}$$

Let us bound the integral B_j from above. Integrating (16) by parts and proceeding to bounds, we successively obtain

$$\begin{aligned} B_j &= -e^{-t} \log^j t \Big|_p^\infty + j \int\limits_p^\infty e^{-t} t^{-1} \log^{j-1} t\, dt \le e^{-p} \log^j p + \frac{j}{p} \int\limits_p^\infty e^{-t} \log^{j-1} t\, dt \\ &\le e^{-p} \log^j p \left(1 + \frac{j}{p \log p} + \ldots + \frac{j!}{(p \log p)^j}\right) \le e^{-p} \log^j p \, \frac{1 - \left(\frac{j}{p \log p}\right)^j}{1 - \frac{j}{p \log p}}. \end{aligned} \tag{18}$$

From (13) and (17), it follow that $\dfrac{j}{p \log p} \le \dfrac{1}{2 \log 4}$ $(1 \le j \le k)$. Then, by (18), we obtain the following estimate for B_j:

$$B_j \le \frac{5}{3} e^{-p} \log^j p \le \frac{5}{3} e^{-p} \log^k p. \tag{19}$$

Before we apply the FEC to calculation of the integral A_j, let us transform A_j into a convenient form, as was done in [3] for computing the gamma function. For $0 \le t \le p$, we have

$$e^{-t} = \sum_{i=0}^{r} (-1)^i \frac{t^i}{i!} + R(t), \quad R(t) = \sum_{i=r+1}^{\infty} (-1)^i \frac{t^i}{i!}. \tag{20}$$

Let us represent A_j, defined by (15), in the form of the sum of two integrals:

$$A_j = S_j + R_j, \tag{21}$$

where, taking into account (20), S_j and R_j can be written in the form

$$\begin{aligned} S_j &= \sum_{i=0}^{r}(-1)^i \frac{1}{i!} \int_0^p t^i \log^j t\, dt, \\ R_j &= \sum_{i=r+1}^{\infty}(-1)^i \frac{1}{i!} \int_0^p t^i \log^j t\, dt. \end{aligned} \tag{22}$$

In order to bound the sum R_j from above, we represent it in the form of a sum of two terms. We have

$$|R_j| \le |U_j| + |W_j|, \tag{23}$$

where

$$U_j = \sum_{i=r+1}^{\infty}(-1)^i \frac{1}{i!} \int_0^1 t^i \log^j t\, dt, \tag{24}$$

$$W_j = \sum_{i=r+1}^{\infty}(-1)^i \frac{1}{i!} \int_1^p t^i \log^j t\, dt. \tag{25}$$

Let us bound from above the sums U_j and W_j. Considering that the members of the series U_j and W_j with alternating signs are monotonically decreasing in their magnitude and tend to zero, we have for U_j and W_j, respectively,

$$|U_j| \le \frac{1}{(r+1)!}\left|\int_0^1 t^{r+1} \log^j t\, dt\right| \le \frac{j!}{(r+j+2)!}, \tag{26}$$

$$|W_j| \le \frac{1}{(r+1)!}\int_1^p t^{r+1} \log^j t\, dt \le \frac{p^{r+2}}{(r+2)!}\log^j p. \tag{27}$$

From (23)–(27), we obtain the estimate for R_j:

$$|R_j| \le \frac{j!}{(r+j+2)!} + \frac{p^{r+2}}{(r+2)!}\log^j p. \tag{28}$$

Under the conditions (13) ($1 \le j \le k$), from the expressions (14) and (21) for the integrals J_j and A_j and from (19) and (28), it follows that the integral J_j can be represented in the form

$$J_j = S_j + \Theta, \tag{29}$$

where

$$|\Theta| \le \frac{5}{3}e^{-p}\log^k p + \frac{1}{(r+3)!} + \frac{p^{r+2}}{(r+2)!}\log^k p.$$

Since

$$\frac{1}{(r+3)!} \le \left(\frac{e}{r+3}\right)^{r+3}, \qquad \frac{p^{r+2}}{(r+2)!} \le \left(\frac{ep}{r+2}\right)^{r+2},$$

and because of (17), we choose $r \ge 4n$ and obtain the following estimate for Θ:

$$|\Theta| \le 2e^{-n}\log^k n.$$

From this and from (13), we have

$$|\Theta| \le 2^{-n}. \tag{30}$$

Replacing the integrals in the sum S_j by their values, we obtain from (22), (29), and (30) that in order to compute the integral $J_j = \int\limits_0^\infty e^{-t} \log^j t\, dt$ with accuracy 2^{-n}, it suffices to calculate with the same accuracy the sum

$$S_j = \sum_{i=0}^{r} (-1)^i \frac{p^{i+1}}{(i+1)!} \sum_{m=0}^{j} \frac{(-1)^m}{(i+1)^m} \frac{j!}{(j-m)!} \log^{j-m} p, \tag{31}$$

where

$$p = n, \quad r \geq 4n, \quad n \geq 2k \log 2k, \quad k \geq 2, \quad 1 \leq j \leq k. \tag{32}$$

Let us write the expression (31) in the following form:

$$S_j = \sum_{m=0}^{j} (-1)^m \frac{j!}{(j-m)!} (\log^{j-m} p) \sigma_m, \tag{33}$$

where

$$\sigma_m = \sum_{i=0}^{r} \frac{(-1)^i}{(i+1)^m} \frac{p^{i+1}}{(i+1)!}. \tag{34}$$

We first compute the sum σ_m. Take $r + 1 = 2^q$ ($q \geq 1$, $2^{q-1} < 4n \leq 2^q$) terms of the series (34). Let the numbers $a_{r+1-\nu}(0)$, $\nu = 0, 1, \ldots, r$, be defined by the equalities

$$a_{r+1-\nu}(0) = (-1)^{r-\nu} \frac{p^{r+1-\nu}}{(r+1-\nu)!\,(r+1-\nu)^m}.$$

By the definition of σ_m, we have

$$\sigma_m = a_1(0) + a_2(0) + \ldots + a_{r+1}(0). \tag{35}$$

The computation of σ_m will be performed in q steps of the FEC process, described in detail in [3].

1ST STEP. Successively combining by pairs the summands of σ_m in (35) and taking the "obvious" common multiplier out of the parentheses, we obtain

$$\sigma_m = a_1(1) + a_2(1) + \ldots + a_{r_1}(1),$$

where $r_1 = 2^{-1}(r+1)$ and the numbers $a_{r_1-\nu}(1)$, $\nu = 0, 1, \ldots, r_1 - 1$, are defined by the equalities

$$\begin{aligned}
a_{r_1-\nu}(1) &= a_{r+1-2\nu}(0) + a_{r+1-(2\nu+1)}(0) \\
&= (-1)^{r-2\nu-1} \frac{p^{r-2\nu}}{(r+1-2\nu)!\,(r-2\nu)^m (r+1-2\nu)^m} \beta_{r_1-\nu}(1), \\
\beta_{r_1-\nu}(1) &= -p(r-2\nu)^m + (r+1-2\nu)^{m+1}.
\end{aligned} \tag{36}$$

At the 1st step, the numbers $\beta_{r_1-\nu}(1)$, $\nu = 0, 1, \ldots, r_1 - 1$, are calculated by Eq. (36).

At the ℓth step ($\ell \leq q$), we have

$$\sigma_m = a_1(\ell) + a_2(\ell) + \ldots + a_{r_\ell}(\ell), \tag{37}$$

where $r_\ell = 2^{-1} r_{\ell-1} = 2^{-\ell}(r+1)$ and the numbers $a_{r_\ell-\nu}(\ell)$, $\nu = 0, 1, \ldots, r_\ell - 1$, are defined by the equalities

$$\begin{aligned}
a_{r_\ell-\nu}(\ell) &= a_{r_{\ell-1}-2\nu}(\ell-1) + a_{r_{\ell-1}-(2\nu+1)}(\ell-1) \\
&= \frac{p^{r+2-2^\ell\nu-2^\ell}}{(r+1-2^\ell\nu)!\,(r+1-2^\ell\nu)^m (r-2^\ell\nu)^m \ldots (r+2-2^\ell\nu-2^\ell)^m} \beta_{r_\ell-\nu}(\ell), \\
\beta_{r_\ell-\nu}(\ell) &= \left(\frac{(r+1-2^\ell\nu-2^{\ell-1})!}{(r+1-2^\ell\nu-2^\ell)!}\right)^m p^{2^{\ell-1}} \beta_{r_{\ell-1}-2\nu}(\ell-1) \\
&+ \frac{(r-2^\ell\nu)!}{(r-2^\ell\nu-2^{\ell-1})!} \left(\frac{(r+1-2^\ell\nu)!}{(r+1-2^\ell\nu-2^{\ell-1})!}\right)^m \beta_{r_{\ell-1}-(2\nu+1)}(\ell-1).
\end{aligned} \tag{38}$$

At the ℓth step, the numbers $\beta_{r_\ell-\nu}(\ell)$, $\nu = 0, 1, \ldots, r_\ell - 1$, are calculated by Eq. (38).

$(\ell+1)$TH STEP. (We assume $\ell + 1 \le q$.) Successively combining by pairs the summands of σ_m in (37) and taking the "obvious" common multiplier out of the parentheses, we obtain

$$\sigma_m = a_1(\ell+1) + a_2(\ell+1) + \ldots + a_{r_{\ell+1}}(\ell+1),$$

where $r_{\ell+1} = 2^{-1}r_\ell = 2^{-(\ell+1)}(r+1)$ and the numbers $a_{r_{\ell+1}-\nu}(\ell+1)$, $\nu = 0, 1, \ldots, r_{\ell+1} - 1$, are defined by the equalities

$$a_{r_{\ell+1}-\nu}(\ell+1) = a_{r_\ell-2\nu}(\ell) + a_{r_\ell-(2\nu+1)}(\ell)$$
$$= \frac{p^{r+2-2^{\ell+1}\nu-2^{\ell+1}}}{(r+1-2^{\ell+1}\nu)!\,(r+1-2^{\ell+1}\nu)^m(r-2^{\ell+1}\nu)^m\ldots(r+2-2^{\ell+1}\nu-2^{\ell+1})^m}\beta_{r_{\ell+1}-\nu}(\ell+1).$$

$$\beta_{r_{\ell+1}-\nu}(\ell+1) = \left(\frac{(r+1-2^{\ell+1}\nu-2^\ell)!}{(r+1-2^{\ell+1}\nu-2^{\ell+1})!}\right)^m p^{2^\ell}\beta_{r_\ell-2\nu}(\ell)$$
$$+\frac{(r-2^{\ell+1}\nu)!}{(r-2^{\ell+1}\nu-2^\ell)!}\left(\frac{(r+1-2^{\ell+1}\nu)!}{(r+1-2^{\ell+1}\nu-2^\ell)!}\right)^m\beta_{r_\ell-(2\nu+1)}(\ell). \tag{39}$$

At the $(\ell+1)$th step, the numbers $\beta_{r_{\ell+1}-\nu}(\ell+1)$, $\nu = 0, 1, \ldots, r_{\ell+1} - 1$, are calculated by Eq. (39).

After q steps of such a process, we obtain

$$\sigma_m = a_{r_q}(q) = a_1(q) = \frac{p}{((r+1)!)^{m+1}}\beta_1(q), \tag{40}$$

where

$$\beta_1(q) = p^{2^{q-1}}((r+1-2^{q-1})!)^m\beta_{r_{q-1}}(q-1)$$
$$+ \frac{r!}{(r-2^{q-1})!}\left(\frac{(r+1)!}{(r+1-2^{q-1})!}\right)^m\beta_{r_{q-1}-1}(q-1). \tag{41}$$

i.e., the value of the sum σ_m will be calculated.

Let us count the number of operations that is sufficient for the computation of the numbers $\beta_{r_{\ell+1}-\nu}(\ell+1)$, $\nu = 0, 1, \ldots, r_{\ell+1} - 1$, at the $(\ell+1)$th step, $\ell + 1 \le q$, similarly to how it was done in [3–5]. We assume that the numbers $\beta_\mu(\ell)$ are already calculated. Let $\beta(\ell) = \max\limits_\mu \beta_\mu(\ell)$. The order of the numbers participating in the calculations at the $(\ell+1)$th step can be estimated from (39) and (36) as follows:

$$\beta(\ell+1) \le \beta(\ell)\left(p^{2^\ell}r^{m2^\ell} + r^{2^{\ell+1}}\right) \le 2\beta(\ell)p^{2^\ell}r^{m2^{\ell+1}}$$
$$\le 2^\ell\beta(1)p^{2^{\ell+1}}r^{m2^{\ell+2}} \le (2pr)^{m2^{\ell+3}}. \tag{42}$$

The complexity of the calculation of the products

$$\frac{(r-2^{\ell+1}\nu)!}{(r-2^{\ell+1}\nu-2^\ell)!},\qquad \left(\frac{(r+1-2^{\ell+1}\nu)!}{(r+1-2^{\ell+1}\nu-2^\ell)!}\right)^m,\qquad \left(\frac{(r+1-2^{\ell+1}\nu-2^\ell)!}{(r+1-2^{\ell+1}\nu-2^{\ell+1})!}\right)^m$$

is (see [3])

$$O\left(\sum_{\tau=1}^{\ell} M(2^\tau \log r) + M(m2^\ell \log r)\right) \tag{43}$$

operations. By (42) and (43), we obtain that $O(B(\ell+1))$ operations are sufficient for calculating $\beta_{r_{\ell+1}-\nu}(\ell+1)$ by Eq. (39), where

$$B(\ell+1) = \sum_{\tau=1}^{\ell} M(2^\tau \log r) + M(m2^\ell \log r) + M(m2^{\ell+3} \log pr).$$

In order to calculate all $\beta_{r_{\ell+1}-\nu}(\ell+1)$, the number of which equals $r_{\ell+1} = 2^{-(\ell+1)}(r+1)$, it suffices to perform $O(B(\ell+1)r_{\ell+1})$ operations. Consequently, we can calculate $\beta_1(q)$ by Eq. (41) in

$$O\left(\sum_{\ell=1}^{q-1} r_{\ell+1}B(\ell+1)\right) = O\left(\sum_{\ell=1}^{q-1} r2^{-(\ell+1)}\sum_{\tau=1}^{\ell} 2^{\tau}\log r(\tau+\log\log r)\log(\tau+\log\log r)\right.$$
$$\left.+\sum_{\ell=1}^{q-1} r2^{-(\ell+1)}2^{\ell+3}\log pr(\ell+\log\log pr)\log(\ell+\log\log pr)\right)$$
$$= O\left(\sum_{\ell=1}^{q-1} r\log pr(\ell+\log\log pr)\log(\ell+\log\log pr)\right) = O(r\log^2 r\log pr\log\log pr) \qquad (44)$$

operations.

The complexity of calculating $((r+1)!)^{m+1}$ is

$$O(r\log rM(m\log r)) \qquad (45)$$

operations.

By (44), (45), and (32), the complexity of calculating σ_m by Eq. (40) with accuracy 2^{-n} is

$$O(n\log^3 n\log\log n)$$

operations.

Let us calculate the value of S_j by Eq. (33) with accuracy 2^{-3n}, assuming that each value of σ_m, $m=0,1,\dots,j$, is already calculated with the very same accuracy.

In order to calculate the value of $\log^{j-m} p$ with accuracy 2^{-3n}, it suffices, because of (32), to perform

$$O(\log^2 nM(n))$$

operations.

Consider

$$\widetilde{S}_j = \sum_{m=0}^{j}(-1)^m\frac{j!}{(j-m)!}(\log^{j-m} p+\Theta_1 2^{-3n})(\sigma_m+\Theta_2 2^{-3n}), \quad |\Theta_1|\le 1, \quad |\Theta_2|\le 1.$$

Then

$$|\widetilde{S}_j - S_j| = \sum_{m=0}^{j}(-1)^m\frac{j!}{(j-m)!}((\log^{j-m} p)\Theta_2 2^{-3n}+\sigma_m\Theta_1 2^{-3n}+\Theta_1\Theta_2 2^{-6n}), \quad |\Theta_1|\le 1, \quad |\Theta_2|\le 1.$$

Using (34) for estimating the value of σ_m, we obtain

$$|\widetilde{S}_j - S_j| \le j!(\log^j p)\Theta_2 2^{-3n} + j!e^p\Theta_1 2^{-3n} + j!\Theta_1\Theta_2 2^{-6n}, \quad |\Theta_1|\le 1, \quad |\Theta_2|\le 1.$$

Hence we have

$$|\widetilde{S}_j - S_j| \le k^{k+1}e^n 2^{-3n} \le 2^{-n},$$

because, according to (32), $1\le j\le k$, $p=n$, and $k\ge 2$.

From the previous estimates, we find that the complexity of the calculation of the sum S_j with accuracy 2^{-n} is

$$O(n\log^3 n\log\log n)$$

operations. From this and from (29) and (30), it follows that in order to calculate the integral

$$J_j = \int_0^{\infty} e^{-t}\log^j t\,dt$$

with accuracy 2^{-n}, it suffices to perform

$$O(n \log^3 n \log\log n)$$

operations. $\triangle$

3. Main result

Now we obtain an estimate for the complexity of calculation with accuracy 2^{-n} of the value $\zeta = \zeta(k)$, k is a natural number, $k \geq 2$, where $\zeta(s)$ is the Riemann zeta function.

Theorem. *The complexity of calculation of $\zeta = \zeta(k)$ for any natural k, $k \geq 2$, is estimated by*

$$s_\zeta(n) = O(n \log^3 n \log\log n).$$

PROOF. Let us use Eq. (4) for calculation of $\zeta = \zeta(s)$ for $s = k$, where k is some natural number, $k \geq 2$. By P_i we denote the expression

$$P_i = \sum_{\substack{n_1+n_2+\ldots+n_k=i \\ n_1+2n_2+\ldots+kn_k=k}} \frac{k!}{n_1!\,n_2!\ldots n_k!} \prod_{j=1}^{k} \left(\frac{S_j}{j!}\right)^{n_j}, \tag{46}$$

where S_j is the sum defined by Eq. (31).

Put $R_j = G_i - P_i$, where G_i is defined by (2). Let us estimate $|R_j|$, taking into account that, by (29) and (30),

$$J_j = S_j + c2^{-n}, \quad |c| \leq 1. \tag{47}$$

For R_j, we have

$$R_j = \sum_{\substack{n_1+n_2+\ldots+n_k=i \\ n_1+2n_2+\ldots+kn_k=k}} \frac{k!}{n_1!\,n_2!\ldots n_k!} \left(\prod_{j=1}^{k} \left(\frac{J_j}{j!}\right)^{n_j} - \prod_{j=1}^{k} \left(\frac{S_j}{j!}\right)^{n_j}\right). \tag{48}$$

Let

$$\Delta = \prod_{j=1}^{k} \left(\frac{J_j}{j!}\right)^{n_j} - \prod_{j=1}^{k} \left(\frac{S_j}{j!}\right)^{n_j}. \tag{49}$$

After identical transformations, (49) can be written in the form

$$\Delta = \sum_{j=1}^{k} \left(\left(\frac{J_j}{j!}\right)^{n_j} - \left(\frac{S_j}{j!}\right)^{n_j}\right) \left(\frac{S_1}{1!}\right)^{n_1} \cdots \left(\frac{S_{j-1}}{(j-1)!}\right)^{n_{j-1}} \left(\frac{J_{j+1}}{(j+1)!}\right)^{n_{j+1}} \cdots \left(\frac{J_k}{k!}\right)^{n_k}. \tag{50}$$

Let us bound $|J_j|$ and $|S_j|$ from above.

$$|J_j| = \left|\int_0^\infty e^{-t} \log^j t\, dt\right| \leq \int_0^1 e^{-t} |\log^j t|\, dt + \int_1^\infty e^{-t} \log^j t\, dt. \tag{51}$$

Since

$$\int_0^1 |\log^j t| dt = \left|\int_0^\infty u^j e^{-u} du\right| = j!,$$

$$\int_1^\infty e^{-t} \log^j t\, dt = \int_0^\infty e^{-e^u+u} u^j\, du \leq \frac{1}{2}(j!),$$

by (47) and (51), we obtain the estimates

$$|J_j| \leq \frac{3}{2}(j!), \qquad |S_j| \leq 2(j!). \tag{52}$$

Using (52), we find an estimate for $|\Delta|$ from (50):

$$\begin{aligned} |\Delta| &\leq \sum_{j=1}^{k} \left| \left(\frac{J_j}{j!}\right)^{n_j} - \left(\frac{S_j}{j!}\right)^{n_j} \right| 2^i \left(\frac{3}{2}\right)^i \\ &\leq 3^i \sum_{j=1}^{k} \left(\frac{1}{j!}\right)^{n_j} n_j |J_j - S_j| \, |\max(J_j, S_j)|^{n_j - 1} \\ &\leq c3^i 2^{-n} \sum_{j=1}^{k} n_j \frac{2^{n_j - 1}}{j!}. \end{aligned}$$

From this and from (48), for $|R_j|$, $i \leq k$, we have

$$\begin{aligned} |R_j| &\leq c3^i 2^{-n} \sum_{\substack{n_1+n_2+\ldots+n_k=i \\ n_1+2n_2+\ldots+kn_k=k}} \sum_{j=1}^{k} \frac{k!}{n_1!\,n_2!\ldots n_k!} n_j \frac{2^{n_j-1}}{j!} \\ &\leq c3^i i 2^{-n} \sum_{j=1}^{k} \frac{k!}{j!} \leq c3^i i \frac{k^{k+1}}{k-1} 2^{-n} \leq 2c3^k k^{k+1} 2^{-n}. \end{aligned} \tag{53}$$

Let us calculate the value of $\zeta(k)$ by Eq. (4), substituting there the expression for P_i, defined by (46), for G_i. From (4), it follows that if we calculate the value G_i with some accuracy, then we calculate $\zeta(k)$ with the same accuracy. According to (53), in order to calculate G_i with accuracy 2^{-n}, it suffices to calculate P_i by Eq. (46) with the same accuracy.

Then we obtain that

$$O(n \log^3 n \log\log n)$$

operations are sufficient for calculating the value of $\zeta(k)$ with accuracy 2^{-n}. $\triangle$

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