# From Theory to Practice:B-Spline in Computer Graphics Applications

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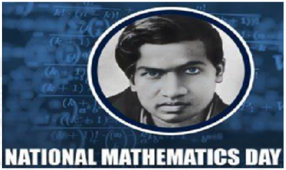


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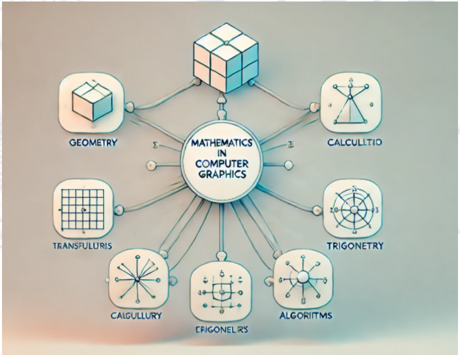
# Outline

- Applications of Mathematics in Computer Graphics
- History of Spline Curves
- Types of Spline
- Continuity
- Curve Representation
- Mathematical Form

# Applications of Mathematics in Computer Graphics

Mathematics forms the core of algorithms and techniques that enable computer graphics to function, from creating basic 2D shapes to rendering realistic 3D worlds.

| Math Concept | Application in Computer Graphics |
|---|---|
| Geometry | Modeling shapes, transformations, and animations. |
| Linear Algebra | 3D rendering, scene transformations, and projections. |
| Calculus | Surface modeling and simulating motion. |
| Trigonometry | Calculating angles, lighting, and shadows. |



# History of Spline Curves

Spline curves have a rich history, dating back to the early days of mechanical drafting. These curves were originally created by bending flexible strips of wood or metal, called splines.

- **Early Beginnings** The earliest splines were physical tools used by draftsmen for drawing smooth curves.
- **Mathematical Representation** In the 1960s, mathematicians **Pierre Bézier** developed mathematical formulas to represent spline curves, making them suitable for computer processing.
- **Computer Aided Design** The arrival of computers changed how splines were used, making it easier and faster to create and work with curves in many different applications.

# Types of Spline

A **spline** is a piecewise polynomial function that maintains smoothness at the points where segments join.

## Linear Spline

Linear splines connect two control points with a straight line. They do not curve smoothly and are not flexible enough to represent complex shapes.

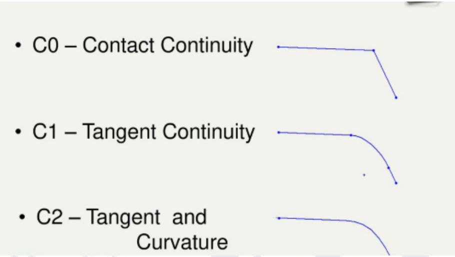
## Quadratic Spline

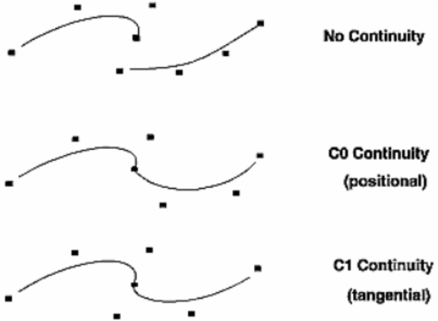
Quadratic splines connect three control points with parabolic segments, providing smoother curves than linear splines. However, they can create sharp corners, which may not be desirable for some applications.

## Cubic Spline

Cubic splines connect four control points using cubic segments, providing flexibility and smoothness for creating complex curves. They offer enhanced smoothness and better control over the curve's shape.

# Continuity


• C0 – Contact Continuity
• C1 – Tangent Continuity
• C2 – Tangent and Curvature


No Continuity
C0 Continuity (positional)
C1 Continuity (tangential)

# Curve Representation

A cubic B-Spline curve is defined by a linear combination of basis functions, called **B-spline basis functions**, multiplied by the control points. Each basis function is a piecewise polynomial of degree three

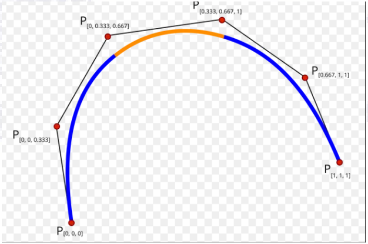
$C(t) = \sum_{i=0}^{n} B_{i,k}(t) P_i$

where

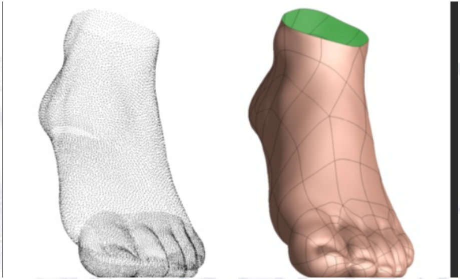
$C(t)$is the curve at parameter

$B_{i,k}(t)$ are the B-spline basis functions of order k

$P_i$ are the control points.



# How We Draw 3D Image



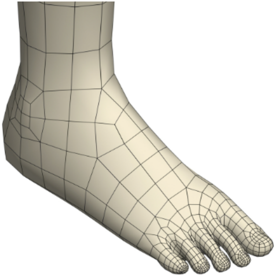
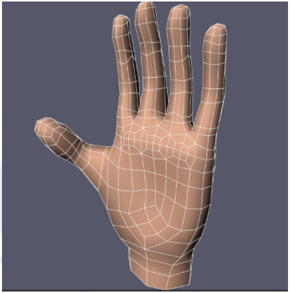



Figure: human feet and hand in computer Graphics

Click on the link to watch the video: Watch YouTube Video

# Cubic Splines: Mathematical Form

Given a set of control points(n+1), $(x_0, y_0), (x_1, y_1), (x_2, y_2)....., (x_n, y_n)$ a cubic spline is defined as:

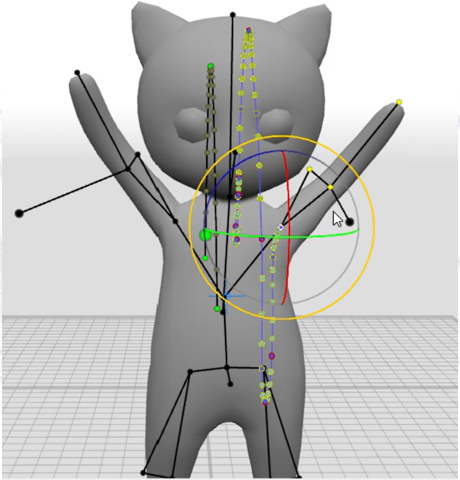
*

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad \text{for } x \in [x_i, x_{i+1}]$$

where $S_i(x)$ is the polynomial on the interval $[x_i, x_{i+1}]$, and the coefficients $a_i, b_i, c_i, d_i$ are chosen such that the following conditions hold:

1. **Continuity:** The spline is continuous at each control point.
2. **Smoothness:** The first and second derivatives of the spline are continuous at the control points.

# Robotic Arm



# Mathematical Form

Imagine a smooth animation with 3 control points $P_0 = (0, 0), P_1 = (1, 2), P_2 = (2, 0)$.


Figure: Data point

For two segments $S_0(x)$ (between $x_0 = 0$ and $x_1 = 1$) and $S_1(x)$ (between $x_1 = 1$ and $x_2 = 2$),

**cubic splines are expressed as:**

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3, \quad x \in [0, 1],$$

$$S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3, \quad x \in [1, 2].$$

To determine the coefficients $a_i, b_i, c_i, d_i$, we impose the following conditions:

# Interpolation Conditions

- $S_0(x_0) = 0$: $a_0 = 0.$
- $S_0(x_1) = 2$: Substituting $x = 1$ into $S_0(x)$:

$$a_0 + b_0(1-0) + c_0(1-0)^2 + d_0(1-0)^3 = 2.$$

**Simplifying:**

$$b_0 + c_0 + d_0 = 2 \quad \text{(Equation 1).}$$

- $S_1(x_1) = 2$: $a_1 = 2.$
- $S_1(x_2) = 0$: Substituting $x = 2$ into $S_1(x)$:

$$a_1 + b_1(2-1) + c_1(2-1)^2 + d_1(2-1)^3 = 0.$$

**Simplifying:**

$$b_1 + c_1 + d_1 = -2 \quad \text{(Equation 2).}$$

# Smoothness Conditions (Continuity of Derivatives)

- First Derivative Continuity at $x_1 = 1$:

$$S_0'(1) = S_1'(1).$$

Differentiating $S_0(x)$ and $S_1(x)$:

$$S_0'(x) = b_0 + 2c_0(x-0) + 3d_0(x-0)^2, \quad S_1'(x) = b_1 + 2c_1(x-1) + 3d_1(x-1)$$

**At $x = 1$:**

$$b_0 + 2c_0 + 3d_0 = b_1 \quad \text{(Equation 3)}.$$

- Second Derivative Continuity at $x_1 = 1$:

$$S_0''(1) = S_1''(1).$$

# II

Differentiating again:

$$S_0''(x) = 2c_0 + 6d_0(x-0), \quad S_1''(x) = 2c_1 + 6d_1(x-1).$$

**At $x = 1$:**

$$2c_0 + 6d_0 = 2c_1 \quad \text{(Equation 4)}.$$

# Boundary Conditions (Natural Spline)

- Second Derivative at

**$x_0 = 0$:**

$$S_0''(0) = 0 \implies 2c_0 = 0 \implies c_0 = 0 \quad \text{(Equation 5).}$$

- Second Derivative at

**$x_2 = 2$:**

$$S_1''(2) = 0 \implies 2c_1 + 6d_1 = 0 \implies c_1 + 3d_1 = 0 \quad \text{(Equation 6).}$$

# System of Equations and Solutions

From the above conditions, the system of equations is:

$$
\begin{aligned}
b_0 + c_0 + d_0 &= 2 \quad \text{(Equation 1)}, \\
b_1 + c_1 + d_1 &= -2 \quad \text{(Equation 2)}, \\
b_0 + 2c_0 + 3d_0 &= b_1 \quad \text{(Equation 3)}, \\
2c_0 + 6d_0 &= 2c_1 \quad \text{(Equation 4)}, \\
c_0 &= 0 \quad \text{(Equation 5)}, \\
c_1 + 3d_1 &= 0 \quad \text{(Equation 6)}.
\end{aligned}
$$

Solving step-by-step, we get:

$$
d_0 = 1, \quad b_0 = 2, \quad a_0 = 0,
$$

$$
d_1 = -\frac{1}{3}, \quad c_1 = 1, \quad b_1 = 0, \quad a_1 = 2.
$$

# Final Cubic Splines

The two cubic spline segments are:

*

$$S_0(x) = 2x + x^3, \quad x \in [0, 1],$$

$$S_1(x) = 2 + (x-1)^2 - \frac{1}{3}(x-1)^3, \quad x \in [1, 2].$$

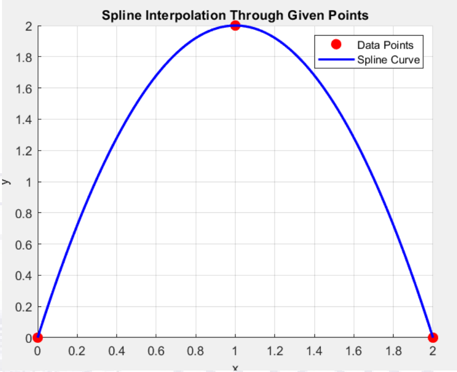



Figure: Caption

# Data Point

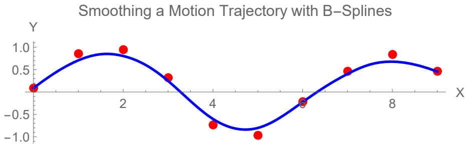



Figure: 10 data points where the red dot represents the data point and the blue line represents cubic spline

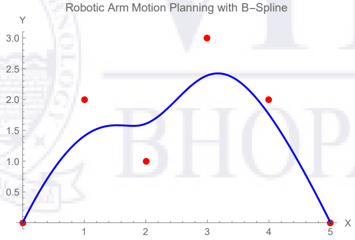



Figure: Robatic motion

thank you!