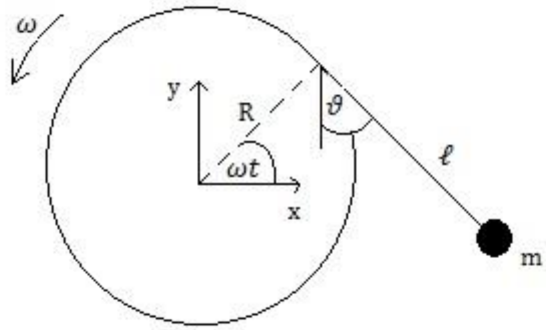
Solving Lagrangian Mechanics Problems

Classical Mechanics – PHY 3221

a) Write down generalized x, y, z coordinates of the masses (usually two-dimensional).

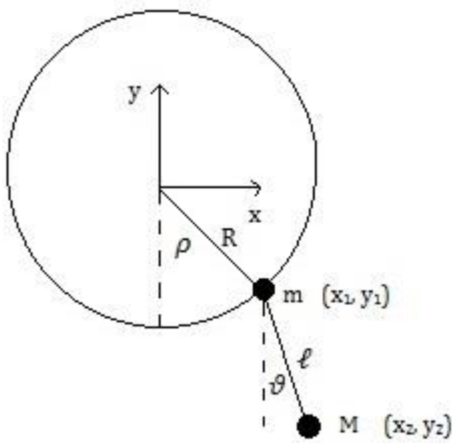
If there is one mass, there is only one set of coordinates. Use trigonometry to write coordinates.
Example:



$$x = R\cos\omega t + \ell\sin\vartheta$$

$$y = R\sin\omega t - \ell\cos\vartheta$$

If two masses are involved, there are two sets of coordinates; (x_1, y_1) and (x_2, y_2). *m_2*'s coordinates are usually the same as *m_1*'s coordinates but with an additional term dependent on a different angle.



$$x_1 = R\sin\rho$$
$$y_1 = -R\cos\rho$$

$$x_2 = R\sin\rho + \ell\sin\vartheta$$
$$y_2 = -R\cos\rho - \ell\cos\vartheta$$

b) Calculate kinetic (T) and potential energy (U).

First, the x- and y- velocities must be calculated by taking the derivatives of x & y and using the Pythagorean Theorem to solve for v. Therefore, kinetic energy is:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad \textit{if two masses} \rightarrow \quad T = \frac{1}{2}m\left({\dot{x}_1}^2 + {\dot{y}_1}^2\right) + \frac{1}{2}M\left({\dot{x}_2}^2 + {\dot{y}_2}^2\right)$$

And the potential energy is:

$$U = mgy \qquad \textit{if two masses} \rightarrow \quad U = mgy_1 + Mgy_2$$

If there is a spring involved in the system, then U will also consist of the spring's stored energy.

$$U_{spring} = \frac{1}{2}kx^2 \;\rightarrow\; \frac{1}{2}k(r - \Delta\ell)^2 \quad (\textit{distance changes as the spring stretches/compresses})$$

c) Write down Lagrangian.

The Lagrangian is a function of the position and the velocities of a mechanical system $L(x, v)$. It is usually the difference between kinetic (T) and potential energy (U). Make sure to correctly determine T and U before attempting to write down L.

$$L = T - U$$

d) Determine the equations of motion.

The EQ's of motion can be determined using the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \qquad \textit{or written as} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

The time-derivative of the partial derivative of L with respect to $\dot{x}$ is equal to the partial derivative of L with respect to x. If there are n coordinates (which is most likely to be the case), there will be n EQ's of motion. In general, these equations are coupled, nonlinear second-order differential equations.

**It's important to know which variables to differentiate and which to treat as constants when calculating the EQ's of motion. This also demonstrates the importance of the product and chain rule – don't make the mistake of differentiating incorrectly as it will lead to incorrect EQ's of motion, etc. Just as a reminder, the chain rule is: $h(x) = f\big(g(x)\big) \quad \rightarrow \quad h'(x) = f'\big(g(x)\big)g'(x)$

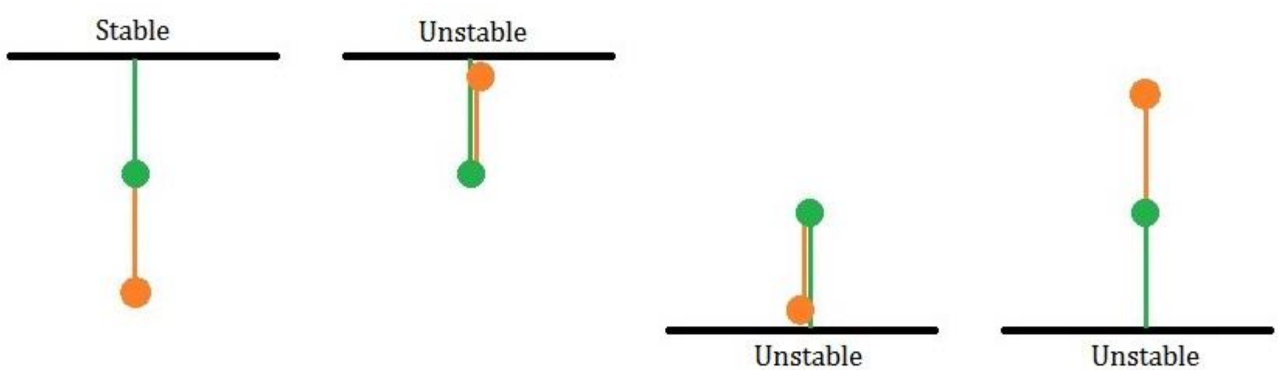
The ending result for the EQ's of motion can be left as either a second-order differential equation equal to zero, or solved for $\ddot{x}$ in terms of the other variables.

e) Determine the equilibrium position(s). Which are stable, unstable?

The equilibrium positions (aka Lagrange points) are determined by setting all time derivatives in the EQ's of motion equal to zero:

$$\ddot{x} = \dot{x} = \ddot{y} = \dot{y} = 0$$

Using common sense, you should be able to determine which points are stable or unstable by drawing a graph or through visualization of each equilibrium point. For example, a double pendulum has four equilibrium points, one of which is stable while the other three are unstable.



f) Approximate the equation(s) of motion for small angles, i.e. $\cos\vartheta \approx 1,\ \sin\vartheta \approx \vartheta$.

This is as simple as it seems; replace every $\cos\vartheta$ with 1, and every $\sin\vartheta$ with ϑ.

Example using one of the EQ's of motion for the double pendulum in part a:

EQ of motion: $\ell\ddot{\vartheta} + \ddot{x}\cos\vartheta + g\sin\vartheta = 0$

Small angle approximation: $\ell\ddot{\vartheta} + \ddot{x} + g\vartheta = 0$

g) Linearize the equations of motion by neglecting all nonlinear terms.

Most often, Euler-Lagrange equations produce non-linear, second-order ODE's of motion, which can be difficult to interpret without fancy graphing software. Therefore, we use a linearized form of the equation to visualize the system's motion. Linearization is only useful when assuming the angles are small, i.e. for minor deviations of the system from equilibrium[1]. To linearize, we assume:

- $\cos\vartheta \approx 1,\ \sin\vartheta \approx \vartheta$ (small angle approximations – part g),
- terms of second-order and higher are negligible, and
- non-linear terms are neglected $\left(\ddot{\vartheta}^2 = 0\right)$.

Example of linearizing EQ's of motion:

$$Nonlinearized:\quad \ell\ddot{\vartheta} + R\cos(\rho - \vartheta)\ddot{\rho} - R\dot{\rho}^2\sin(\rho - \vartheta) + g\sin\vartheta = 0$$

$$Linearized:\quad \ell\ddot{\vartheta} + R\ddot{\rho} + g\vartheta = 0$$

h) What are the normal frequencies and what type of movement do they represent?

Normal frequencies are denoted by ω and are calculated using the linearized EQ's of motion determined in the previous step. The three terms below are synonymous:

$$\begin{matrix} Normal \\ frequency \end{matrix} = \begin{matrix} Natural \\ frequency \end{matrix} = \begin{matrix} Resonance \\ frequency \end{matrix}$$

The matrix method is the best way to solve for normal frequencies because it is applicable to ALL coupled linear differential equations[2]. Start by writing a matrix using the coefficients of the terms in each EQ of motion. **Example** using a double pendulum:

$EQ\ \#1:\quad \ell\ddot{\vartheta} + R\ddot{\rho} + g\vartheta = 0$ $\qquad EQ\ \#2:\quad \ell\ddot{\vartheta} + R(1+\varsigma)\ddot{\rho} + (1+\varsigma)g\rho = 0$

$$\begin{pmatrix} \ell & R \\ \ell & R(1+\varsigma) \end{pmatrix}\begin{pmatrix} \ddot{\vartheta} \\ \ddot{\rho} \end{pmatrix} + \begin{pmatrix} g & 0 \\ 0 & (1+\varsigma)g \end{pmatrix}\begin{pmatrix} \vartheta \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

[1]The word small is ambiguous but there are ways to define what can be considered 'small' per problem. Keeping all non-linear terms in the equation, it's possible to determine at what minimum angle the system behaves *chaotically*.

[2]The matrix method is not the only way to solve for normal frequencies; other methods may be more direct, depending on the problem.

We guess solutions to be of the form:

$$\begin{pmatrix}\vartheta\\ \rho\end{pmatrix} = \begin{pmatrix}V_\vartheta\\ V_\rho\end{pmatrix} e^{i\omega t} \qquad \textit{taking the second derivative yields} \rightarrow \qquad \begin{pmatrix}\ddot{\vartheta}\\ \ddot{\rho}\end{pmatrix} = -\omega^2 e^{i\omega t}\begin{pmatrix}V_\vartheta\\ V_\rho\end{pmatrix}$$

Substituting these matrices in terms of V into the original matrix:

$$\begin{pmatrix}-\omega^2\ell + g & -\omega^2 R\\ -\omega^2\ell & -\omega^2 R(1+\varsigma) + (1+\varsigma)g\end{pmatrix}\begin{pmatrix}V_\vartheta\\ V_\rho\end{pmatrix} e^{i\omega t} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Since $e^{i\omega t} > 0$, we can divide it out and then we're left with the determinant:

$$\begin{vmatrix}-\omega^2\ell + g & -\omega^2 R\\ -\omega^2\ell & -\omega^2 R(1+\varsigma) + (1+\varsigma)g\end{vmatrix} = 0$$

Solving the determinant:

$$\omega^4 R\ell(1+\varsigma) - \omega^2\ell g(1+\varsigma) - \omega^2 Rg(1+\varsigma) + g^2(1+\varsigma) - \omega^4 R\ell = 0$$

$$= \omega^4 R\ell\varsigma - \omega^2[\,\ell g(1+\varsigma) + gR(1+\varsigma) + g^2\,] + g^2(1+\varsigma) = 0$$

At this point, this is a solvable polynomial but it's helpful to be given values for the constants:

$$R = g = \varsigma = 1, \textit{and } \ell = 2 \qquad \rightarrow \qquad 2\omega^4 - 6\omega^2 + 2 = 0 \qquad \rightarrow \qquad \omega_1 \approx \pm 0.62, \qquad \omega_2 \approx \pm 1.62$$

i) Calculate the normal modes and write down the general solution of the linearized system.

We look for normal mode solutions where all elements oscillate at the same frequency (resonance). For each ω, we get a different normal mode. We determined there are four normal frequencies, but $\pm\omega$ is irrelevant in this case so we use only the $+\omega$'s. Therefore, we can expect two normal modes (makes sense because there are two masses).

Example using part h:

$$\begin{pmatrix}-2\omega^2 + 1 & -\omega^2\\ -2\omega^2 & -2\omega^2 + 2\end{pmatrix}\begin{pmatrix}V_\vartheta\\ V_\rho\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Solving for V_ϑ in terms of V_ρ:

$$(-2\omega^2 + 1)V_\vartheta = \omega^2 V_\rho \qquad \textit{and} \qquad -2\omega^2 V_\vartheta = (2\omega^2 - 2)V_\rho$$

$$V_\vartheta = \left(\frac{1}{\omega^2} - 1\right)V_\rho \qquad \omega = 0.62 \rightarrow V_\vartheta = 1.6\,V_\rho \qquad \omega = 1.62 \rightarrow V_\vartheta = -0.62\,V_\rho$$

The normal modes are then:

$$\vec{V}_1 = \begin{pmatrix}1.6\\ 1\end{pmatrix} \quad \textit{and} \quad \vec{V}_2 = \begin{pmatrix}-0.62\\ 1\end{pmatrix}$$

Since the the EQ's of motion are linear ODE's, the superposition principle can be applied and the general solution can be written as a linear combination of the normal modes.

There are several ways to solve for the general solution. For example, if the EQ of motion is relatively uncomplicated, then you can use the method of integrating twice to solve for $\vartheta(t)$ or $\rho(t)$.

If it's more complicated, use the characteristic equation to solve for the general solution. The following method is used when the discriminant is less than zero (complex roots).

$$ar^2 + br + c = 0 \qquad and \qquad b^2 - 4ac < 0$$

$$then\ the\ solution\ is \qquad r = \lambda \pm i\mu \qquad \rightarrow \qquad y(t) = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t)$$

j) Determine whether energy is conserved.

The total energy of the system is equal to the kinetic (T) plus potential energy (U):

$$E = T + U$$

To determine whether energy is conserved (assuming frictionless systems), take the derivative of the total energy with respect to time. If it equals zero, then energy is conserved[3].

$$Energy\ conserved\ if{:}\ \ \frac{dE}{dt} = -\frac{\partial L}{\partial t} = 0$$

Then substitute the EQ's of motion in terms $\ddot{r}, \ddot{\vartheta}$, etc. into the $\frac{dE}{dt}$ equation.

Example using a mass attached to a hanging spring (*spring constant = k*):

$$E = T + U = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) - mgr\cos\vartheta + \tfrac{1}{2}k(r-\ell)^2$$

$$\frac{dE}{dt} = m\dot{r}\left(\ddot{r} + r\dot{\vartheta}^2 - g\cos\vartheta + k(r-\ell)\right) + mr\dot{\vartheta}\left(r\ddot{\vartheta} + g\sin\vartheta\right)$$

EQ's of motion: $\quad \ddot{r} = r\dot{\vartheta}^2 + g\cos\vartheta - \frac{k}{m}(r-\ell) \qquad and \qquad \ddot{\vartheta} = -\frac{1}{r}2\dot{r}\dot{\vartheta} - \frac{g}{r}\sin\vartheta$

Substituting $\ddot{r}$ and $\ddot{\vartheta}$ into $\frac{dE}{dt}$:

$$\frac{dE}{dt} = m\dot{r}\left(r\dot{\vartheta}^2 + g\cos\vartheta - \frac{k}{m}(r-\ell) + r\dot{\vartheta}^2 - g\cos\vartheta + \frac{k}{m}(r-\ell)\right) + mr\dot{\vartheta}\left(-2\dot{r}\dot{\vartheta} - g\sin\vartheta + g\sin\vartheta\right)$$

$$= 2mr\dot{r}\dot{\vartheta}^2 - 2mr\dot{r}\dot{\vartheta}^2 = 0 \qquad \rightarrow \qquad energy\ is\ conserved$$

[3]Hint: Energy is conserved when the Lagrangian is *independent* of time. If there is a time dependent term in the Lagrangian (e.g. the example in part a with $x = R\cos\omega t + \ell\sin\vartheta$) then energy is **not** conserved. A giveaway for non-conservative energy is when angular frequency, ω, is involved in the system (rotating component).

Misc. Formulas:

$$Simple\ Harmonic\ Motion\ (SHM): \quad \ddot{x} + \omega^2 x = 0 \qquad (\omega = \sqrt{k/m})$$

$$x_m = A_0 \cos(\omega t + \phi)$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \ddot{\vartheta} \\ \ddot{\rho} \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} \ddot{\vartheta} \\ \ddot{\rho} \end{pmatrix}$$