MTH 308 - Mini Project (MP2)

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Introduction:

The aim of the project is to design and implement (in MATLAB) a numerical scheme that approximates a function from given data. The detailed description is as follows :

- Implement a function that takes as inputs the size of the grid to be used for approximation and the function to be approximated and returns your choice of grid of the given size to be used for approximation of the given function and the function data i.e the function values on the chosen grid.
- Implement an approximation procedure to approximately evaluate the function from the given data.

Approximation Theory and Optimal Polynomials:

Once we have decided the domain and degree of the polynomial to be approximated, the approximating polynomial is choosen in such a way so as to minimize the worst case error. For well-behaved functions, there exists an N^{th}-degree polynomial that will lead to an error curve that oscillates back and forth between ε and $-\varepsilon$ a total of $N+2$ times, giving a worst-case error of ε. It is seen that there exists an N^{th}-degree polynomial that can interpolate $N+1$ points in a curve. Such a polynomial is always optimal.

Chebyshev Approximation :

Polynomials very close to the optimal can be obtained by expanding the given function in terms of **Chebyshev polynomials** and then cutting off the expansion at the desired degree. This is similar to the Fourier analysis of the function, using the Chebyshev polynomials instead of the usual trigonometric functions.

The Chebyshev approximation of a function looks like :

$$f(x) \sim \sum_{i=0}^{\infty} c_i T_i(x)$$

where T_N is the N^{th} degree Chebyshev Polynomial of first kind.

Cutting off the series after T_N term gives the N^{th} degree polynomial approximating the given function $f(x)$.

Proof of Correctness :

For function with rapidly converging power series, if the series is cutoff after some term, the total error in this approximation is close to the first term after cutoff. In other words, the first term after cutoff dominates all the later terms. Same thing holds if we expand a function in terms of Chebyshev polynomials.

Cutting the Chebyshev expansion after T_N ensures the error will take the form of T_{N+1} or its multiple. Chebyshev polynomials have the property that they are level i.e they oscillate between $+1$ and 1 in the interval $[1, 1]$. T_{N+1} has $N + 2$ level extrema. This implies that the error between f(x) and its Chebyshev expansion out to T_N is close to a level function with $N + 2$ extrema,and hence is close to the optimal N^{th}-degree polynomial.

Implementation :

By **Runge Phenomenon** we know that arbitrary set of points or uniformly spaced points don't result in good approximation.To avoid this, we use the roots of **Chebyshev Polynomial** of the first kind scaled in the interval [a,b], as interpolation points. The expression for the same is:

$$x_k = \frac{b-a}{2} cos(\pi \frac{k}{N}) + \frac{b+a}{2}, \text{ for } k = 0, 1, ..., N$$

Next we apply Chebyshev Interpolation. We calculate the chebyshev series upto T_N term as :

$$T_k(x) = cos(k * cos^{-1}(x)) \text{ when } x \in [-1, 1]$$

The function $f(x)$ for $x \in [a, b]$ can be approximated as :

$$f \approx \sum_{j=0}^{N} a_j T_j(\frac{2x-(b+a)}{b-a})$$

where the coefficients a_j are:

$$a_j = \sum_{k=0}^{N} I_{jk} f(x_k) \text{ for } j = 0, 1, ..., N$$

I is the $(N + 1)$*$(N + 1)$ interpolation matrix

References

- Approximation Theory
- Finding the Zeros of a Univariate Equation: Proxy Rootfinders, Chebyshev Interpolation, and the Companion Matrix by **John P Boyd**