3 неделя **Вариационное исчисление**

(13) $J(y) = \int_1^2 \underbrace{\left[\frac{3y^2}{x^2} + \frac{(y')^2}{x} + 8y \right]}_{F(x,y,y')} dx$, $y(1)=0$; $y(2)=8\ln 2$

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) =$

$= \frac{6y}{x^2} + 8 - \frac{2y''}{x} + \frac{2y'}{x^2} = 0$

$2y''\cdot x^2 - 2y'\cdot x - 6y = 8x^3$

$y_1 = \frac{1}{x}$ решения одн. ур-ния $\Rightarrow y = C_1 x^3 + C_2 \cdot \frac{1}{x}$

$y_2 = -x^3$

$\begin{vmatrix} \frac{1}{x} & y \\ -\frac{1}{x^2} & y' \end{vmatrix} = e^{+\int \frac{2x}{2x^2}dx}$

$\Rightarrow \left(\frac{y}{y_1}\right)' = \frac{1}{y_1^2} e^{\int \frac{1}{x}dx} = \frac{1}{y_1^2} \cdot C \cdot x = C \cdot x^3$

$\Rightarrow \frac{y}{y_1} = \frac{C_1}{4}x^4 + C_2$

$y = \frac{C_1}{4}x^3 + \frac{C_2}{x} = C_1 \cdot x^3 + \frac{C_2}{x}$

$\begin{cases} C_1' x^3 + \frac{C_2'}{x} = 0 \\ C_1' \cdot 3x^2 - \frac{C_2'}{x^2} = 4x \end{cases}$ $\Rightarrow -\frac{C_2'}{x^2} = C_1' x^2$

$4x^2 \cdot C_1' = 4x \qquad C_2' = -C_1' \cdot x^4 = -x^3$

$C_1 = \frac{-1}{x^2} + C_1 \qquad C_2 = -\frac{1}{4}x^4 + C_2$

$\Rightarrow y = \cancel{x^5 + C_1 x^3 - \frac{1}{5}x^5 - \frac{1}{x}C_2 = \frac{3}{5}x^5 - \frac{1}{x}C_2 + C_1 x^3}$

~~$y(1) = \frac{2}{3} - C_2 + C_1 = 0$~~

~~$y(2) = \frac{64}{3} - \frac{1}{2}C_2 + 8C_1 = 8\ln 2$~~

$$y = C_1 x^3 + \frac{C_2}{x} + x^3 \ln|x| - \frac{x^3}{2}$$

Найдем экстремаль

$$\begin{cases} y(1) = 0 + C_1 - \frac{1}{2} + C_2 = 0 \Rightarrow C_1 = \frac{1}{2} - C_2 \\ y(2) = \cancel{8\ln 2} + 8C_1 - 4 + \frac{C_2}{2} = \cancel{8\ln 2} \Rightarrow \cancel{4} - 8C_2 - \cancel{4} + \frac{C_2}{2} = 0 \Rightarrow C_2 = 0 \end{cases} \qquad C_1 = \frac{1}{2}$$

$$\Rightarrow \hat{y} = x^3 \ln x$$

Исследуем ее на экстремум

$$\Delta J[\hat{y}] = \int_1^2 \left[\frac{3}{x^3}(\hat{y}+h)^2 + \frac{1}{x}(\hat{y}'+h')^2 + 8(\hat{y}+h) - \frac{3}{x^3}\hat{y}^2 - \frac{1}{x}(\hat{y}')^2 - 8\hat{y} \right] dx =$$

$$= \int_1^2 \left[\frac{6}{x^3}\hat{y}h + \frac{3}{x^3}h^2 + \frac{1}{x}2\hat{y}'h' + \frac{1}{x}(h')^2 + 8h \right] dx \ \text{⊜}$$

$$\int_1^2 \frac{2}{x}\hat{y}'h'\,dx = \frac{2}{x}\hat{y}'h\Big|_1^2 - \int_1^2 h\left[\frac{2\hat{y}''}{x} - \frac{2\hat{y}'}{x^2}\right]dx$$

ур-ние Эйлера ⇒ занул.

$$\text{⊜} \int_1^2 \left[\frac{3}{x^3}h^2 + \frac{1}{x}(h')^2 \right] dx > 0 \Rightarrow \text{абс. минимум}$$

Ответ: $\hat{y} = x^3 \ln x$; абс. минимум

38 $J[y] = \int_0^1 \left[(y')^2 \sqrt{4-x^2} - 2y \right] dx \qquad y(0) = 2 \quad (0;2) \qquad y(1) = \sqrt{3} \quad (1;\sqrt{3})$

Запишем ур-ние Эйлера:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = -2 - \left(2y'\cdot\sqrt{4-x^2}\right)' = -2 - 2y''\cdot\sqrt{4-x^2} + 2\frac{y'\cdot x}{\sqrt{4-x^2}} = 0$$

$y_1 = C_1$ $\quad (2y'\sqrt{4-x^2})' = -2$

$y_2 =$ $\quad \Rightarrow 2y'\sqrt{4-x^2} = -2x + C_1$

$$y' = \frac{-x}{\sqrt{4-x^2}} + \frac{C_1}{\sqrt{4-x^2}}$$

$$y = \int \frac{-x}{\sqrt{4-x^2}}dx + \int \frac{C_1}{\sqrt{4-x^2}} = +\sqrt{4-x^2} + C_1 \arcsin\left(\frac{x}{2}\right) + C_2$$

$y(1) = \sqrt{3} + C_1 \cdot \frac{\pi}{6} + C_2 = \sqrt{3}$
$y(0) = 2 + 0 + C_2 = 2$ $\quad \Big| \Rightarrow C_1 = 0,\ C_2 = 0 \Rightarrow y_0 = \sqrt{4-x^2}$ - доп. экстремаль

Исследуем ее на экстремум

$$\Delta J[y_0] = \int_0^1 \left[(y_0'+h')^2\sqrt{4-x^2} - 2(\cancel{y_0}+h) - (y_0')^2\sqrt{4-x^2} + \cancel{2y_0}\right]dx =$$

$$= \int_0^1 \left[\underline{2y_0' \cdot h' \cdot \sqrt{4-x^2}} + (h')^2 \cdot \sqrt{4-x^2} - 2h\right]dx \ = \quad \text{ур-ние Эйлера}$$

$$\int_0^1 2y_0' \cdot \sqrt{4-x^2} \cdot h' dx = \cancel{\left(2y_0'\sqrt{4-x^2} \cdot h\right)\Big|_0^1} - \int_0^1 \left[2y_0''\sqrt{4-x^2} + 2y_0' \cdot \frac{-x}{\sqrt{4-x^2}}\right] h\, dx$$

$$= \int_0^1 (h')^2 \cdot \sqrt{4-x^2}\, dx > 0 \Rightarrow \text{абс. мин.}$$

Ответ: $y_0 = \sqrt{4-x^2}$; абс. минимум

(57) $J[y] = \int_1^2 \left[24x^3y - yy' - x^2(y')^2\right]dx \qquad (1;1);\ (2;-7)$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = (24x^3 - y') - (-y - 2x^2y')' = 24x^3 - \cancel{y'} + \cancel{y'} + 4xy' +$$

$$+ 2x^2y'' = 0 \qquad x^2y'' + 2xy' + 12x^3 = 0$$

$y_1 = -x^3$ ~~реш~~ ур общего ур-ния
$y_2 = C_1$ - решение ОДУ
$y_2 = \frac{C_2}{x}$ - 2с решение ОДУ

$\Rightarrow y = C_1 + \frac{C_2}{x} - x^3$

$y(1) = C_1 + C_2 - 1 = 1 \qquad C_1 + C_2 = 2$

$y(2) = C_1 + \frac{C_2}{2} - 8 = -7 \qquad 2C_1 + C_2 = 2$

$\Rightarrow C_2 = 2, \quad C_1 = 0$

$\Rightarrow y_0 = \frac{2}{x} - x^3$ - доп. экстремаль.

Исследуем ~~[illegible]~~ функционал на ней

$$J[y_0] = \int_1^2 \left[24x^3(y_0+h) - (y_0+h)(y_0'+h') - x^2(y_0'+h')^2 - \cancel{24x^3 y_0} + y_0 y_0' + x^2 (y_0')^2 \right] dx =$$

$$= \int_1^2 \left[\boxed{24x^3 h} - \underline{y_0 h'} - \cancel{y_0' h} - \cancel{h h'} - x^2 \left(\underline{2y_0' h'} + (h')^2 \right) \right] dx \; \textcircled{=}$$

ур-ние Эйлера

$$\int_1^2 h h' dx = \frac{1}{2} \int_1^2 d(h^2) = \frac{h^2}{2} \Big|_1^2 = 0$$

$$\int_1^2 (-y_0 - 2y_0' x^2) h' dx = \cancel{(\ldots) h \Big|_1^2} + \int_1^2 \left[\cancel{y_0'} + 2y_0'' x^2 + 4x y_0' \right] h \, dx$$

$$\textcircled{=} \int_1^2 -x^2 (h')^2 dx < 0 \quad \Rightarrow \text{абс. максимум}$$

Ответ: $\frac{2}{x} - x^3$; абс. макс.

(102) $J[y] = \int_0^{\pi} \left[(y')^2 - \frac{16}{9} y^2 + 2y \sin x \right] dx \qquad (0;0) \quad (\pi; -\frac{\sqrt{3}}{2})$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = -\frac{32}{9} y + 2\sin x - 2y'' = 0$$

$$8y'' + \frac{16}{9} y = + \sin x$$

$$y = C_1 \sin\left(\frac{4}{3}x\right) + C_2 \cos\left(\frac{4}{3}x\right)$$

$$\begin{cases} C_1' \sin(\frac{4}{3}x) + C_2' \cos(\frac{4}{3}x) = 0 \qquad C_2' = -C_1' \operatorname{tg}(\frac{4}{3}x) \\ C_1' \cdot \frac{4}{3}\cos(\frac{4}{3} x) - C_2' \frac{4}{3} \sin(\frac{4}{3} x) = +\sin x \end{cases}$$

$$\frac{4}{3} C_1' \frac{\cos^2(\frac{4}{3} x)+\sin^2(\frac{4}{3} x)}{\cos(\frac{4}{3} x)} = +\sin x$$

$$\begin{cases} C_1' = +\frac{3}{4} \sin x \cdot \cos(\frac{4}{3} x) = +\frac{3}{8}\left[\sin(\frac{7x}{3}) - \sin(\frac{x}{3})\right] \\ C_2' = -\frac{3}{4} \sin x \cdot \sin\frac{4}{3} x = -\frac{3}{8}\left[\cos(\frac{x}{3}) - \cos(\frac{7x}{3})\right] \end{cases}$$

$$C_1 = \int +\frac{3}{8}\left[\sin(\frac{7x}{3}) - \sin(\frac{x}{3})\right] = -\frac{9}{56}\cos(\frac{7x}{3}) + \frac{9}{8}\cos(\frac{x}{3}) + C_1$$

$$C_2 = \int -\frac{3}{8}\left[\cos(\frac{x}{3}) - \cos(\frac{7x}{3})\right] = -\frac{9}{8}\sin(\frac{x}{3}) + \frac{9}{56}\sin(\frac{7x}{3}) + C_2$$

$\cancel{y = \sin(\frac{4}{3}x) \cdot [\frac{9}{28}}$

$y = \frac{9}{7} \sin x$ — ЧР НДУ

с) $y = C_1 \sin(\frac{4}{3} x) + C_2\cos(\frac{4}{3} x) + \frac{9}{7} \sin x$

$$y(0) = C_2 = 0$$
$$y(\pi) = -\frac{\sqrt{3}}{2}C_1 - \frac{1}{2} C_2 + 0 = -\frac{\sqrt{3}}{2}$$
$\Rightarrow C_1 = 1$, $C_2 = 0$

$y_0 = \sin(\frac{4}{3} x) + \frac{9}{7}\sin x$ — допустимая экстремаль.

Исследуем на экстремум

$$\Delta J[y_0] = \int_0^\pi \left[2y_0'h' + (h')^2 - \frac{16}{9}(2y_0 h + h^2) + 2h\sin x\right] dx =$$

ур-ние Эйлера

$$\int_1^2 2y_0'h' = 2y_0'h\Big|_1^2 - \int 2y_0'' h\, dx$$

$$= \int_0^\pi \left[(h')^2 - \frac{16}{9}h^2\right] dx$$

$\exists\ h = \sin kx \Rightarrow h' = k\cdot\cos kx$

$\Rightarrow \int_0^{\pi}\left[k^2\cos^2(kx) - \frac{16}{9}\sin^2(kx)\right]dx = \int_0^{\pi}\left[\left(\frac{k^2}{2} - \frac{8}{9}\right) + \underbrace{\left(\frac{k^2}{2} + \frac{8}{9}\right)\cos(2kx)}_{\text{зануляется при инт-ии}}\right]dx =$

$= \left(\frac{k^2}{2} - \frac{8}{9}\right)\pi$

$\Rightarrow$ при $|k| > \frac{4}{3}$ $\Delta J[y_0] > 0$

при $|k| < \frac{4}{3}$ $\Delta J[y_0] < 0$ $\Bigg| \Rightarrow$ допустимая экстремаль $y_0 =$

$= \sin\left(\frac{4}{3}x\right) + \frac{9}{4}\sin x$

не дает экстремума функционала.

5.20.1

(3) $J[y] = \int_1^2 [x^2(y')^2 + 6y^2 + 2x^3 y]dx \quad y(1) = \frac{1}{6}$

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 12y + 2x^3 - 4xy' - 2x^2\cdot y'' = 0$

$y = \frac{1}{6}x^3$ — ч.р. неодн

$y_1 = x^2$ — реш. ОДУ

$y_2 = \frac{1}{x^3}$ — реш. ОДУ

$\Rightarrow y = C_1x^2 + C_2\frac{1}{x^3} + \frac{1}{6}x^3$

$\begin{cases} y(1) = C_1 + C_2 + \frac{1}{6} = \frac{1}{6} \Rightarrow C_1 + C_2 = 0 \\ \left.\frac{\partial f}{\partial y'}\right|_{x=2} = 0 \end{cases}$

$\frac{\partial f}{\partial y'} = 2x^2 y' = 8y' \Rightarrow y'(2) = 0$

$y'(2) = C_1\cdot 2x - \frac{3C_2}{x^4} + \frac{1}{2}x^2 = 4C_1 - \frac{3}{16}C_2 + 2 = 0$

$\frac{61}{16}C_1 = -2 \Rightarrow C_1 = -\frac{32}{61}$

$C_2 = +\frac{32}{61}$

$\Rightarrow y_0 = -\frac{32}{61}x^2 + \frac{32}{61}\frac{1}{x^3} + \frac{1}{6}x^3$ ← допустимая экстремаль

$\Delta J[y_0] = \int_1^2 \left[x^2 \cdot \left(\underline{2y_0' h'} + (h')^2\right) + 6\left(2y_0 h + h^2\right) + \underline{2x^3 h}\right] dx \;=$ ур-ние Эйлера

$\int_1^2 2y_0' x^2 \cdot h' dx = \cancel{2y_0' x^2 h \Big|_1^2} - \int_1^2 \left[2y_0'' x^2 + 4y_0' \cdot x\right] h\, dx$

0

$= \int_1^2 \left[x^2 (h')^2 + 6h^2\right] dx > 0 \Rightarrow$ абс. минимум

⑨ $J[y] = \int_1^3 \left[8yy' \ln x - x(y')^2 + 6xy'\right] dx, \; y(3) = 15$

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 8y' \ln x - \left(8y \ln x - 2xy' + 6x\right)' = \cancel{8y' \ln x} - \cancel{8y' \ln x} -$

$- 8y \cdot \frac{1}{x} + 2y' + 2xy'' - 6 = 0$

$y'' + \frac{1}{x} y' - \frac{4}{x^2} y = \frac{3}{x}$

$y_1 = x^2$ — решение ОДУ

$y_2 = \frac{1}{x^2}$ — решение ОДУ $\Rightarrow y = C_1 x^2 + C_2 \cdot \frac{1}{x^2} - x$

$y_3 = -x$ — решение НДУ

$\frac{\partial F}{\partial y'}\Big|_{x=1} = 0 = \left(8y \ln x - 2xy' + 6x\right)\Big|_{x=1} = \cancel{8y \ln x} - 2y' + 6 \Rightarrow y'(1) = 3$

$y(3) = 9C_1 + \frac{1}{9} C_2 - 3 = 15$

$y'(1) = 2C_1 - 2C_2 - 1 = 3$ $\Rightarrow C_1 = 2, C_2 = 0 \Rightarrow y_0 = 2x^2 - x$ — допустимая экстремаль

$\Delta J[y] = \int_1^3 \left[8 \ln x \left(\underline{y_0 \cdot h'} + \cancel{y_0' h + hh'}\right) - x\left(\underline{2y_0' h'} + (h')^2\right) + \underline{6xh'}\right] dx$

$\int_1^3 8 \ln x \cdot y_0 h' + \cancel{\ldots} - 2xy_0' h' dx = - \int_1^3 \left[\frac{8y_0}{x} + \cancel{8 \ln x\, y_0'} + \cancel{\ldots} -\right.$

$\left. -2y_0' - 2xy_0'' \cancel{\ldots} - 6\right] h\, dx$ ур-ние Эйлера

$\int_1^3 8 \ln x\, hh' dx = 4 \ln x\, h^2 \Big|_1^3 - \int 4h^2 \cdot \frac{1}{x} dx$

2) (14) $J[y]=\int_0^1 [(y')^2+y^2-4e^x y-8xe^x y']\,dx$

Первунецкин

Б01-001

2е задание

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=2y-4e^x-(2y'-8xe^x)'=$$

$$=2y-4e^x-2y''+8(x+1)e^x=$$

$$=2y''-2y-8xe^x-4e^x=0$$

$$y''-y-4xe^x-2e^x=0$$

$$y=C_1e^x+C_2e^{-x}$$

$$y=x^2e^x-\text{ЧР неодн} \Rightarrow y=C_1e^x+C_2e^{-x}+x^2e^x$$

$$\delta J[y,h]=\int_0^1\left[\frac{\partial f}{\partial y}\cdot h+\frac{\partial f}{\partial y'}\cdot h'\right]dx=0$$

$$\int_0^1\frac{\partial f}{\partial y'}h'\,dx=\frac{\partial f}{\partial y'}\cdot h\Big|_{x=0}^{x=1}-\int\left(\frac{\partial f}{\partial y'}\right)'h\,dx$$

$$\Downarrow$$

1) $\frac{\partial f}{\partial y'}\Big|_{x=0}=0 \Rightarrow y'(0)=0$

2) $\frac{\partial f}{\partial y'}\Big|_{x=1}=0 \Rightarrow y'(1)=4e$

$$\begin{cases} y'(0)=C_1-C_2=0 \Rightarrow C_1=C_2 \\ y'(1)=C_1\cdot e-\frac{C_2}{e}+3e=4e \Rightarrow C_1=\frac{e}{2\,\mathrm{sh}1}\end{cases}$$

~~[illegible]~~

$$\Rightarrow y_0=2\,\frac{e^2+1}{e^2}\cdot\mathrm{ch}\,x+x^2e^x=\frac{e}{2\,\mathrm{sh}1}\cdot\mathrm{ch}\,x+x^2e^x \text{ — допустимая экстремаль}$$

$$\delta J[y_0]=\int_0^1\left[(2y_0'\cdot h'+h'^2)+(2y_0h+h^2)-4e^x\cdot h-8x\cdot e^x h'\right]dx \;\textcircled{=}$$

y_0 — р-ние Эйлера

$$\int_0^1(2y_0'-8xe^x)h'\,dx=(2y_0'-8xe^x)\cdot h\Big|_0^1-\int_0^1\left[(2y_0'')-8(x+1)e^x\right]h\,dx$$

0

$\Delta J[y_0] = \int^2 \left[x^2\left(2y_0'h' + (h')^2\right) + 6\left(2y_0 h + h^2\right) + 2x^3 h\right] dx$ $\textcircled{=}$

ур-ние Эйлера

2) $\Delta J[y_0] = \int_1^3 \left[-\frac{4}{x}h^2 - x(h')^2\right] dx < 0$

$\Rightarrow$ абс. максимум

$\textcircled{=} \int_0^1 [(h')^2 + (h^2)] dx > 0 \Rightarrow$ абс. минимум

$\boxed{\S 20.2}$

(4) $J[y_1, y_2] = \int_0^1 \left[12y_1^2 + y_2^2 + x^2(y_1')^2 + (y_2')^2\right] dx$ $\quad y_1: (1;1); (2;8)$ $\quad y_2: (1;e); (2;e^2)$

$$\begin{cases} \frac{\partial f}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_1'}\right) = 24y_1 - 2x^2 y_1'' - 4x y_1' = 0 \quad (1) \\ \frac{\partial f}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_2'}\right) = 2y_2 - 2y_2'' = 0 \Rightarrow y_2 = C_1 e^{x} + C_2 e^{-x} \end{cases}$$

(1) $x^2 y_1'' + 2x y_1' - 12 y_1 = 0$

$y_{10} = x^3$ — ЧР

$y_{11} = \frac{1}{x^4}$ — ЧР $\quad \Big| \Rightarrow \tilde{y}_1 = C_1 \cdot x^3 + C_2 \cdot \frac{1}{x^4}$

$\tilde{y}_1(1) = C_1 + C_2 = 1$

$\tilde{y}_1(2) = 8C_1 + \frac{1}{16}C_2 = 8$ $\quad \Big| \Rightarrow C_1 = 1, C_2 = 0 \Rightarrow \tilde{y}_1 = x^3$

$\tilde{y}_2(1) = C_1 e + C_2 \frac{1}{e} = e$

$\tilde{y}_2(2) = C_1 e^2 + C_2 \cdot \frac{1}{e^2} = e^2$ $\quad \Big| \Rightarrow C_1 = 1, C_2 = 0 \Rightarrow \tilde{y}_2 = e^x$

$\Rightarrow \begin{cases} \tilde{y}_1 = x^3 \\ \tilde{y}_2 = e^x \end{cases}$ — допустимая экстремаль

$\Delta J[\tilde{y}_1, \tilde{y}_2] = \int^2 \left[12\left(2\tilde{y}_1 h_1 + h_1^2\right) + \left(2\tilde{y}_2 h_2 + h_2^2\right) + \underline{x^2\left(2\tilde{y}_1' h_1' + (h_1')^2\right)} + \right.$

$\left. + \left(\underbrace{2\tilde{y}_2' h_2'}_{-2\tilde{y}_2'' \cdot h_2} + (h_2')^2\right)\right] dx$ $\textcircled{=}$ $\quad = \left(-4x\tilde{y}_1' - 2x^2\tilde{y}_1''\right)h$

ур-ние Эйлера 2

ур-ние Эйлера 1

$\textcircled{=} \int_1^2 \left[12h_1^2 + h_2^2 + x^2(h_1')^2 + (h_2')^2\right] dx > 0 \Rightarrow$ абс. минимум

$\boxed{\S 20.3}$

Первушкин

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2е задание

(4) $J[y] = \int_0^1 \left[4(y')^2 + (y'')^2\right] dx$ $\quad (0;0;0); \left(1; \frac{e^2-3}{4}; \frac{e^2-1}{2}\right)$

Запишем ур-ние Эйлера-Пуассона

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = -8y'' + 2y^{IV} = 0$

$y^{IV} = 4y''$

$\lambda^4 - 4\lambda^2 = 0$

$\lambda^2(\lambda - 2)(\lambda + 2) = 0$

$\Rightarrow y = C_1 + C_2 x + C_3 e^{2x} + C_4 e^{-2x}$

$C_1 = -(C_3 + C_4) \quad C_2 = 2(C_4 - C_3)$

$$\begin{cases} y(0) = C_1 + C_3 + C_4 = 0 \\ y'(0) = C_2 + 2C_3 - 2C_4 = 0 \\ y(1) = C_1 + C_2 + C_3 e^2 + C_4 e^{-2} = \frac{e^2-3}{4} \quad \Big| \cdot 2 \\ y'(1) = C_2 + 2C_3 e^2 - 2C_4 e^{-2} = \frac{e^2-1}{2} \end{cases}$$

$$\begin{cases} 4C_4 - 8C_3 + 4C_3 e^2 = e^2 - 2 \\ 4C_3 - 4C_4 e^{-2} = 1 \end{cases}$$

$4C_4 = (4C_3 - 1)e^2$

$4C_3 e^2 - e^2 - 8C_3 + 4C_3 e^2 = e^2 - 2$

$C_3(8e^2 - 8) = 2(e^2 - 1)$

$C_1 = -\frac{1}{4}; C_2 = -\frac{1}{2} \Leftarrow C_3 = \frac{1}{4} \Rightarrow C_4 = 0$

$\Delta J[y_0] = \int^2 [x^2\cdot(2y_0'h' + (h')^2) + 6(2y_0 h + h^2) + 2x^3 h]\, dx$ ⊜ ур-ние Эйлера

⇒ абс. максимум

2) $y_0 = -\frac{1}{4} - \frac{1}{2}x + \frac{1}{4}e^{2x}$ – допустимая экстремаль

$\Delta J[y_0] = \int_0^1 [4(\underline{2y_0'h'} + (h')^2) + (2y_0''\cdot h'' + (h'')^2)]\, dx$ ⊜

$\int_0^1 2y_0'' h''\, dx = \cancel{2y_0''\cdot h'\big|_0^1} - \int_0^1 2y_0'''\cdot h'\, dx =$

$= 2y_0''\cdot h'\big|_0^1 - \cancel{2y_0''' h\big|_0^1} + \int_0^1 2y_0^{IV}\cdot h\, dx$ – ур-ние Эйлера-Пуассона

$\int_0^1 8y_0' h'\, dx = \cancel{8y_0' h\big|_0^1} - \int_0^1 8y_0'' h\, dx$

⊜ $\int_0^1 [4(h')^2 + (h'')^2]\, dx > 0$ ⇒ абс. минимум

$\boxed{\S 21}$

8) $J[y] = \int_1^2 x(y')^2\, dx$ $\quad (1;0)$, $(2;12)$ $\quad \int_1^2 \lambda y\, dx = 9$

$L = x(y')^2 + \lambda x y$

$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \lambda x - 2y' - 2xy'' = 0$

$2xy'' + 2y' = \lambda x$

$y = ax^2$
$4ax + 2ax = \lambda x$
$a = \frac{\lambda}{8}$

$y_1 = C_1$, $y_2 = \ln x$ ⇒ $y = C_1 + C_2 \ln x$ – общее реш. ОДУ

$y = \frac{\lambda}{8}x^2$ – решение НДУ

⇒ $y = C_1 + C_2 \ln x + \frac{\lambda}{8}x^2$ – общее решение

$y(1) = C_1 + \frac{\lambda}{8} = 0 \quad C_1 = -\frac{\lambda}{8}$

$y(2) = C_1 + C_2 \ln 2 + \frac{\lambda}{2} = 12 \quad \frac{3}{8}\lambda + C_2 \ln 2 = 12$

$\int_1^2 C_1 x + C_2 x\ln x + \frac{\lambda}{8}x^3\, dx$

$\int_1^2 x\ln x\, dx = \frac{1}{2}\ln x\cdot x^2\big|_1^2 - \frac{1}{2}\int_1^2 x^2\cdot\frac{1}{x}\, dx =$

$= \frac{1}{2}\ln 2\cdot 4 - \frac{1}{2}\cdot\frac{4-1}{2} = 2\ln 2 - \frac{3}{4}$

$\int_1^2 [C_1 x + \frac{\lambda}{8}x^3]\, dx = C_1\frac{x^2}{2}\big|_1^2 + \frac{\lambda}{32}x^4\big|_1^2 = C_1\frac{3}{2} + \frac{\lambda}{32}\cdot(15)$

⇒ $\underbrace{\frac{3}{2}C_1}_{-\frac{6}{32}\lambda} + (2\ln 2 - \frac{3}{4})C_2 + \frac{15}{32}\lambda = 9$

$\begin{cases} (2\ln 2 - \frac{3}{4})C_2 + \frac{9}{32}\lambda = 9 \\ C_2\cdot \ln 2 + \frac{12}{32}\lambda = 12 \end{cases}$ ⇒ $C_2 = 0$, $\lambda = 32$, $C_1 = -4$

⇒ $y_0 = 4(x^2 - 1)$ – допустимая экстремаль

$\Delta J[y_0] = \int_1^2 x(\underline{2y_0'h'} + (h')^2)\, dx = \cancel{2y_0'\cdot x\cdot h\big|_1^2} + \int_1^2 [(h')^2 + (-2y_0' - 2xy_0'')h]\, dx =$

$= \int_1^2 [(h')^2 - \underbrace{\lambda\cdot x\cdot h}_{0}]\, dx$ ⊜

0 в силу условия связи

⊜ $\int_1^2 (h')^2\, dx \geq 0$ ⇒ y_0 дает абс. минимум

76) $J[y] = \int_0^{\pi} [y^2 - (y')^2]\, dx$ $\quad (0;1)$; $(\pi;-1)$

$\int_0^{\pi} y\cos x\, dx = \frac{\pi}{2}$

$\int_0^{\pi} y\sin x\, dx = \pi$

$L = y^2 - (y')^2 + \lambda_1 y\cos x + \lambda_2 y\sin x$

$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 2y + \lambda_1\cos x + \lambda_2 \sin x + 2y'' = 0$

$\Rightarrow \Delta J[y_0] = \int^{3} \left[-\frac{4}{x} h^2 - x (h')^4 \right] dx < 0$

$$y'' + y = -\frac{\lambda_1}{2} \cos x - \frac{\lambda_2}{2} \sin(nx)$$

$y = C_1 \cos x + C_2 \sin x$ – решение ОДУ

Неодн. ур-ие при $n = 1$

$$y_0 = x (A \cos x + B \sin x)$$

$$y_0' = A \cos x + B \sin x + x (B \cos x - A \sin x)$$

$$y_0'' = 2B \cos x - 2A \sin x + x (-A \cos x - B \sin x)$$

$$2B \cos x - 2A \sin x = -\frac{\lambda_1}{2} \cos x - \frac{\lambda_2}{2} \sin x$$

$$2B = -\frac{\lambda_1}{2} \Rightarrow B = -\frac{\lambda_1}{4}$$

$$-2A = -\frac{\lambda_2}{2} \qquad A = +\frac{\lambda_2}{4}$$

$$\Rightarrow y = C_1 \cos x + C_2 \sin x + \frac{\lambda_2}{4} x \cos x - \frac{\lambda_1}{4} x \sin x$$

$$\begin{cases} y(0) = C_1 = 1 \Rightarrow C_1 = 1 \\ y(\pi) = -C_1 - \frac{\lambda_2}{4}\pi = -1 \Rightarrow \lambda_2 = 0 \end{cases}$$

$$\Rightarrow y = \cos x + C_2 \sin x - \frac{\lambda_1}{4} x \sin x. \quad \exists\, \lambda_1 = \lambda$$

$$\frac{\pi}{2} = \int_0^{\pi} y \cos x \, dx = \int_0^{\pi} \left(\cos^2 x + C_2 \sin x \cos x - \frac{\lambda}{4} x \sin x \cos x\right) dx =$$

$$= \int_0^{\pi} \left(\frac{1}{2} + \underbrace{\frac{1}{2} \cos 2x}_{0} + \underbrace{\frac{C_2}{2} \sin 2x}_{0} - \frac{\lambda x}{8} \sin 2x\right) dx =$$

$$= \frac{\pi}{2} - \frac{\lambda}{8} \int_0^{\pi} x \sin 2x \, dx = \frac{\pi}{2} \Rightarrow \lambda = \lambda_1 = 0$$

$$y = \cos x + C_2 \sin x$$

$$\int_0^{\pi} [\cos x \sin x + C_2 \sin x \cdot \sin(nx)] \, dx = \int_0^{\pi} \left[\underbrace{\frac{1}{2} \sin 2x}_{0} + \frac{C_2}{2} (1 - \underbrace{\cos 2x}_{0})\right] dx$$

$= \frac{C_2}{2} \pi = \pi \Rightarrow C_2 = 2 \Rightarrow \tilde{y} = \cos x + 2 \sin x$ – допустимая экстремаль

Неодн. ур-ние при $n = -1$

$$y_0 = x (A \cos x + B \sin x)$$

$$2B \cos x - 2A \sin x = -\frac{\lambda_1}{2} \cos x + \frac{\lambda_2}{2} \sin x$$

$$B = -\frac{\lambda_1}{4}; \quad A = -\frac{\lambda_2}{4}$$

$$\Rightarrow y = C_1 \cos x + C_2 \sin x - \frac{\lambda_2}{4} x \cos x - \frac{\lambda_1}{4} x \sin x$$

$$y(0) = C_1 = 1$$

$$y(\pi) = -C_1 + \frac{\lambda_2 \pi}{4} = -1 \Rightarrow \lambda_2 = 0$$

$$y = \cos x + C_2 \sin x - \frac{\lambda_1}{4} x \sin x \quad \exists\, \lambda_1 = \lambda$$

$$\int_0^{\pi} \left[\cos^2 x + \underbrace{\frac{C_2}{2} \sin 2x}_{0} - \frac{\lambda}{8} x \sin 2x\right] dx =$$

$$= \frac{\pi}{2} - \frac{\lambda}{8} \int_0^{\pi} x \sin 2x \, dx = \frac{\pi}{2} \Rightarrow \frac{\lambda}{8} = 0 = \lambda_1$$

$$\Rightarrow y = \cos x + C_2 \sin x$$

$$-\int_0^{\pi} (\cos x \cdot \sin x + C_2 \sin^2 x) \, dx = -\int_0^{\pi} \left[\underbrace{\frac{1}{2} \sin 2x}_{0} + \frac{C_2}{2} (1 - \underbrace{\cos 2x}_{0})\right] dx$$

$$= -\frac{C_2}{2} \pi = \pi \Rightarrow C_2 = -2$$

$\tilde{y} = \cos x - 2 \sin x$ – допустимая экстремаль

При $n \neq \pm 1$

$$y_0 = A \sin nx + B x \sin x \Rightarrow y_0' = An \cdot \cos nx + B \sin x + B x \cos x$$

$$y_0'' = -An^2 \sin nx + 2B \cos x - B x \sin x$$

$$A(1 - n^2) \sin nx + 2B \cos x = -\frac{\lambda_1}{2} \cos x - \frac{\lambda_2}{2} \sin(nx)$$

$$\Rightarrow B = -\frac{\lambda_1}{4}; \quad A = -\frac{\lambda_2}{2(1 - n^2)}$$

$$y = C_1 \cos x + C_2 \sin x - \frac{\lambda_2}{2(1-n^2)} \sin(nx) - \frac{\lambda_1}{4} x \sin x$$

$$\left.\begin{aligned} y(0) &= C_1 = 1 \\ y(\pi) &= -C_1 = -1 \end{aligned}\right| \Rightarrow$$ нет единств. решения (т.к. есть вырожденные уравнения)

Ответ: $n = 1$: $\tilde{y} = \cos x + 2 \sin nx$

$n = -1$: $\tilde{y} = \cos x - 2 \sin x$

$n \neq \pm 1$: нет отдельной экстремали

2) $\Delta J[y_0] = \int_1^3 \left[-\frac{4}{x} h^2 - x (h')^2 \right] dx < 0$

$\Rightarrow$ абс. максимум