III Условия Коши – Римана. Гармонические функции

§5: №1 2) $f(z) = |\bar{z}|^2 = |x - iy|^2 = x^2 + y^2$

~~1) способ: $\Delta f = f(z_0 + \Delta z) - f(z_0) =$~~

~~$= |\bar{z}_0 + \Delta\bar{z}|^2 - |\bar{z}_0|^2 = |x_0 + \Delta x - i(y_0 + \Delta y)|^2 - |x_0 - iy_0|^2 =$~~

~~$= (x_0 + \Delta x)^2 + (y_0 + \Delta y)^2 - x_0^2 - y_0^2 = 2x_0 \Delta x + 2y_0 \Delta y + \Delta x^2 + \Delta y^2$~~

$u(x,y) = x^2 + y^2 \quad v(x,y) = 0$

условие К-Р $\begin{cases} \dfrac{\partial u}{\partial x} = 2x \neq \dfrac{\partial v}{\partial y} = 0 \quad \forall x,y \\ \dfrac{\partial u}{\partial y} = 2y \neq -\dfrac{\partial v}{\partial x} = 0 \quad \forall x,y \end{cases}$ но верно в точке (0,0)

$\Rightarrow f(z)$ дифференцируема в точке ~~$z_0 = (0,0)$~~ $z_0 = 0$

4) $f(z) = x^2 - y^2 - 2xy \cdot i = (x - iy)^2 = (\bar{z})^2$

Уточнить у Семера

$u = x^2 - y^2 \quad v = -2xy$

условие К-Р $\begin{cases} \dfrac{\partial u}{\partial x} = 2x \neq \dfrac{\partial v}{\partial y} = -2x \\ \dfrac{\partial u}{\partial y} = -2y \neq -\dfrac{\partial v}{\partial x} = 2y \end{cases}$ но верно в точке (0,0)

$f'(z) = 2x + i \cdot (-2y) = 2 \cdot \bar{z}$

По идее диф-ма в ~~(0,0)~~ $z_0 = 0$

6) $f(z) = z \operatorname{Re} z = (x + iy)x =$

$= x^2 + xy\, i$

$\boxed{\text{ЗАДАЧА НА ТРОЙКУ}}$

Условие Коши-Римана $\begin{cases} \dfrac{\partial u}{\partial x} = 2x = \dfrac{\partial v}{\partial y} = x \\ \dfrac{\partial u}{\partial y} = 0 = -\dfrac{\partial v}{\partial x} = -y \end{cases}$

имеет единственное решение $z = 0$

Функция диф-ма только в одной точке несмотря на то, что $u, v \in C^\infty$

№6 2) $\cos(2e^{z}) =$

$$= \frac{e^{2ie^{z}}+e^{-2ie^{z}}}{2} = \frac{1}{2}\left(e^{2ie^{x}\cdot e^{iy}}+e^{-2ie^{x}e^{iy}}\right) =$$

$$= \frac{1}{2}\left(e^{2e^{x}(-\sin y+i\cos y)}+e^{-2e^{x}(-\sin y+i\cos y)}\right) =$$

$$= \frac{1}{2}\left(e^{-2e^{x}\sin y}\cdot e^{i\cdot 2e^{x}\cos y}+e^{2e^{x}\sin y}\cdot e^{-2\cos y\, e^{x}\cdot i}\right) =$$

$$= \frac{1}{2}\left[e^{-2e^{x}\sin y}\left(\cos(2e^{x}\cos y)+i\sin(2e^{x}\cos y)\right)+\right.$$
$$\left.+e^{2e^{x}\sin y}\left(\cos(2e^{x}\cos y)-i\sin(2e^{x}\cos y)\right)\right] =$$

$$= ch(2e^{x}\sin y)\cdot\cos(2e^{x}\cos y)-i\sin(2e^{x}\cos y)\cdot sh(2e^{x}\sin y)$$

$$u(x,y) = ch(2e^{x}\sin y)\cdot\cos(2e^{x}\cos y)$$

$$v(x,y) = -sh(2e^{x}\sin y)\sin(2e^{x}\cos y)$$

$$\frac{\partial u}{\partial x} = sh(2e^{x}\sin y)\cdot 2\sin y\cdot e^{x}\cdot\cos(2e^{x}\cos y)+$$
$$+ch(2e^{x}\sin y)\cdot(-\sin(2e^{x}\cos y))\cdot 2e^{x}\cos y$$

$$\frac{\partial v}{\partial y} = -ch(2e^{x}\sin y)\cdot 2e^{x}\cdot\cos y\cdot\sin(2e^{x}\cos y)+$$
$$+\ sh(2e^{x}\sin y)\cdot\cos(2e^{x}\cos y)\cdot(-2e^{x}\sin y)$$

К-Р:

$$\begin{cases}\dfrac{\partial u}{\partial x} = +\dfrac{\partial v}{\partial y} \ \checkmark \\ \dfrac{\partial u}{\partial y} = sh(2e^{x}\sin y)\cdot 2e^{x}\cdot\cos y\cdot\cos(2e^{x}\cos y)+ch(2e^{x}\sin y)(-\sin(2e^{x}\cos y))\cdot 2e^{x}(-\sin y)\end{cases}$$

$$-\frac{\partial v}{\partial x} = ch(2e^{x}\sin y)\cdot 2e^{x}\sin y\cdot\sin(2e^{x}\cos y)+sh(2e^{x}\sin y)\cdot\cos(2e^{x}\cos y)\cdot 2e^{x}\cos y$$

$\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ ✓ $\Rightarrow$ $f(z)$ диф-ма на $\mathbb{C}$ и ее $f'(z) =$

или

$$2e^{z} = 2e^{x+iy} =$$
$$= 2e^{x}(\cos y+i\sin y)$$
и
$$\cos z = ch\,y\cos x - i\,sh\,y\sin x$$

$$f'(z) = \frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} =$$

$$= sh(2e^x \sin y) \cdot 2e^x \sin y \cdot \cos(2e^x \cos y) + \quad e^z$$

$$+ ch(2e^x \sin y) \cdot (-\sin(2e^x \cos y)) \cdot 2e^x \cos y + e^{iy}$$

$$+ i\left[ch(2e^x \sin y) \cdot 2e^x \sin y \cdot (-\sin(2e^x \cos y)) + (+sh(2e^x \sin y)) \cos(2e^x \cos y) \cdot 2e^x \cos y\right] =$$

$$= 2e^z \cdot ch(2e^x \sin y) \cdot (-\sin(2e^x \cos y)) - i \cdot 2e^z \cdot sh(2e^x \sin y) \cos(2e^x \cos y) =$$

$$= -2e^z \left[ch(2e^x \sin y) \cdot \sin(2e^x \cos y) + i\, sh(2e^x \sin y) \cos(2e^x \cos y)\right]$$

похоже на $\sin(2e^z)$, проверим.

$$\sin(2e^z) = \sin(2e^x \cos y + 2e^x \cdot \sin y \cdot i) =$$

$$\sin(z) = ch y \sin x + sh y \cos x \cdot i$$

$$= ch(2e^x \sin y) \cdot \sin(2e^x \cos y) + sh(2e^x \sin y) \cos(2e^x \cos y)$$ верно

$$\Rightarrow f'(z) = -2e^z \cdot \sin(2e^z)$$

5) $f(z) = \frac{z}{e^z} = z \cdot e^{-z} = (x + iy)\, e^{-x} \cdot (\cos y + i \sin y) =$

$$= e^{-x}(x \cos y - y \sin y + (x \sin y + y \cos y) i)$$

$$u(x,y) = e^{-x}(x \cos y - y \sin y)$$

$$v(x,y) = e^{-x}(x \sin y + y \cos y)$$

$$f'(z) = (z \cdot e^{-z})' = e^{-z} + (-z) e^{-z} = (1 - z) e^{-z}$$

№7 1) $\operatorname{tg} z = \frac{\sin z}{\cos z} = \frac{f_1}{f_2}$

$f_1 = \sin z = \operatorname{ch} y \sin x + i \operatorname{sh} y \cos x = u_1 + i \cdot v_1$

$f_2 = \cos z = \operatorname{ch} y \cos x - i \operatorname{sh} y \sin x = u_2 + i v_2$

$\begin{cases} \frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial y} \\ \frac{\partial u_1}{\partial y} = -\frac{\partial v_1}{\partial x} \end{cases}$ верно, u_1 и v_1 диф-мы $\Rightarrow$ f_1 диф-ма

$\begin{cases} \frac{\partial u_2}{\partial x} = \frac{\partial v_2}{\partial y} \\ \frac{\partial u_2}{\partial y} = -\frac{\partial v_2}{\partial x} \end{cases}$ верно, u_2 и v_2 диф-мы $\Rightarrow$ f_2 диф-ма

$\Rightarrow \operatorname{tg} z$ диф-м

$\cos z = 0 \Leftrightarrow \begin{cases} \operatorname{ch} y \cos x = 0 \\ \operatorname{sh} y \sin x = 0 \end{cases}$ $\cos x = 0 \Rightarrow x = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$; $\operatorname{sh} y = 0 \Rightarrow y = 0$

$\Rightarrow \operatorname{tg} z$ диф-м на $\mathbb{C} / \{z = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}\}$

$(\operatorname{tg} z)' = \left(\frac{f_1}{f_2}\right)' = \frac{f_1'}{f_2} - \frac{f_1 f_2'}{f_2^2} = \frac{f_1' f_2 - f_1 f_2'}{f_2^2} = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z}$

3) $\frac{e^z + 2}{e^z - 2} = \frac{f_1}{f_2}$

$f_1 = e^z + 2 = e^x \cdot \cos y + i e^x \cdot \sin y + 2$

$f_2 = e^x \cos y - 2 + i e^x \sin y$

$u_1 = e^x \cos y + 2$ $\quad v_1 = e^x \sin y$ условия К-Р выполнены $\Rightarrow$ f_1 и f_2 диф-мы

$u_2 = e^x \cos y - 2$ $\quad v_2 = e^x \sin y$

$f_2 = 0 \Leftrightarrow \begin{cases} e^x \cos y = 2 \\ e^x \sin y = 0 \end{cases}$ $\quad \sin y = 0 \quad y = \pi k \Rightarrow \cos = \pm 1$ но $e^x > 0$ $\Rightarrow$

$\Rightarrow \cos y = 1 \Rightarrow y = 2\pi k, k \in \mathbb{Z}$

$e^x = 2 \Rightarrow x = \ln 2$

$\Rightarrow z = \ln 2 + i \cdot 2\pi k, k \in \mathbb{Z}$

$\Rightarrow$ при $z \neq \ln 2 + i \cdot 2\pi k$ $\quad \frac{e^z + 2}{e^z - 2}$ диф-~~ма~~ ма

$$\frac{f_1}{f_2} = \frac{f_1' f_2 - f_1 f_2'}{f_2^2} = \frac{e^z \cdot (e^z - 2) - (e^z + 2) e^z}{(e^z + 2)^2} = -\frac{4e^z}{(e^z + 2)^2}$$

№13 1) $u = xy$

$$\frac{\partial u}{\partial x} = y = \frac{\partial v}{\partial y} \Rightarrow v = \frac{y^2}{2} + C(x) \Rightarrow \frac{\partial v}{\partial x} = C'(x)$$

$$\frac{\partial u}{\partial y} = x = -\frac{\partial v}{\partial x} \Rightarrow C'(x) = -x \Rightarrow C(x) = -\frac{x^2}{2} + C$$

$$\Rightarrow v = \frac{y^2 - x^2}{2} + C$$

3) $v = y \sin x \operatorname{ch} y + x \operatorname{sh} y \cos x$

$$\frac{\partial v}{\partial y} = \sin x \operatorname{ch} y + y \sin x \operatorname{sh} y + x \operatorname{ch} y \cos x = \frac{\partial u}{\partial x}$$

$$\Rightarrow u = -\cos x \cdot \operatorname{ch} y - y \cos x \cdot \operatorname{sh} y + \operatorname{ch} y \int x \cos x \, dx \;=$$

$$\int x \cos x \, dx = \int x \, d(\sin x) = x \sin x - \int \sin x \, dx = x \sin x + \cos x$$

$$= -\cos x(\cancel{\operatorname{ch} y} + y \operatorname{sh} y) + \operatorname{ch} y(x \sin x + \cancel{\cos x}) =$$

$$= -\cos x \cdot y \operatorname{sh} y + x \sin x \operatorname{ch} y + C(y)$$

$$\frac{\partial u}{\partial y} = \operatorname{sh} y \, \cancel{x \cdot \sin x} - \cancel{\cos x \cdot \operatorname{sh} y} - \cancel{\cos x \cdot y \cdot \operatorname{ch} y} + C'(y) =$$

$$= -\frac{\partial v}{\partial x} = -\cancel{y \cos x \operatorname{ch} y} - \operatorname{sh} y \cancel{\cos x} + \cancel{x \operatorname{sh} y \sin x} \Rightarrow C'(y) = 0$$

№14

$\Rightarrow C(y) = C$

$\Rightarrow u = x \cdot \sin x \operatorname{ch} y - y \operatorname{sh} y \cdot \cos x$

2) $u = x^2 - y^2 + 2xy$

$\frac{\partial u}{\partial x} = 2x + 2y = \frac{\partial v}{\partial y} \Rightarrow v = 2xy + y^2 + C(x)$

$-\frac{\partial v}{\partial x} = -\cancel{2y} - C'(x) = -\cancel{2y} + 2x \qquad C'(x) = -2x \Rightarrow C(x) = -x^2 + C$

$\Rightarrow v = -x^2 + y^2 + 2xy + C$

(117) 3) $\operatorname{Im} f(z) = y \operatorname{ch} x \cos y + x \sin y \operatorname{sh} x \quad f(0) = 1$

v

$\frac{\partial v}{\partial y} = \operatorname{ch} x \cos y - y \operatorname{ch} x \sin y + x \cos y \operatorname{sh} x = \frac{\partial u}{\partial x}$

$\Rightarrow u = \cancel{\operatorname{sh} x \cos y} - y \operatorname{sh} x \sin y + \cos y \int x \operatorname{sh} x \, dx \;(=)$

$\int x \operatorname{sh} x \, dx = x \operatorname{ch} x - \int \operatorname{ch} x \, dx = x \operatorname{ch} x - \operatorname{sh} x$

$(=)\; x \operatorname{ch} x \cos y - y \operatorname{sh} x \sin y + C(y)$

$\cancel{-\frac{\partial v}{\partial x} = \operatorname{ch} x \cos y + x \operatorname{sh} x \cos y - y \operatorname{ch} x}$

$\frac{\partial u}{\partial y} = -x \cancel{\operatorname{ch} x \sin y} - \operatorname{sh} \cancel{x \sin y} - y \cancel{\operatorname{sh} x \cos y} + C'(y) =$

$= -\frac{\partial v}{\partial x} = -y \cancel{\operatorname{sh} x \cos y} - \sin y \operatorname{sh} x - x \sin y \cancel{\operatorname{ch} x} \Rightarrow C'(y) = 0$

$\Rightarrow u = x \operatorname{ch} x \cos y - y \operatorname{sh} x \sin y + C$

$f(z=0) = \underset{C}{u} + i \underset{=0}{v} = C = 1 \Rightarrow u = x \operatorname{ch} x \cos y - y \operatorname{sh} x \sin y + 1$

$f = z \cdot (\operatorname{ch} x \cos y + i \sin y \operatorname{sh} x) + 1 = z \cdot \operatorname{ch} z + 1$

$f(0)=i$

б) $u = \operatorname{Re} f(z) = x e^x \cos y - (y+1) e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y + x e^x \cos y - (y+1) e^x \sin y = \frac{\partial v}{\partial y}$$

$$v = \cancel{e^x}\,\cancel{e^x}\,(x+1) e^x \sin y + e^x \cos y - e^x \int y \sin y \, dy =$$

$$\int y \sin y \, dy = -y\cos y + \int \cos y \, dy = -y \cos y + \sin y$$

$$= x e^x \sin y + (y+1) e^x \cos y + C(x)$$

$$-\frac{\partial v}{\partial x} = -\cancel{e^x \sin y} - \cancel{x e^x \sin y} - \cancel{(y+1) e^x \cos y} - C'(x) =$$

$$= \frac{\partial u}{\partial y} = -\cancel{x e^x \sin y} - \cancel{e^x \sin y} - \cancel{(y+1) e^x \cos y} \Rightarrow C'(x) = 0$$

$$\Rightarrow C(x) = c$$

$$\Rightarrow v = x e^x \sin y + (y+1) e^x \cos y + C$$

$$f(0) = u + iv = 0 + i(1+c) = i$$

$$\Rightarrow C = 0$$

$$\Rightarrow v = x e^x \sin y + (y+1) e^x \cos y =$$

$$= e^x \left(x \sin y + (y+1) \cos y \right)$$

$$u = e^x \left(x \cos y - (y+1) \sin y \right)$$

$$f = u + iv = e^x \cdot (x + i(y+1)) (\cos y + i \sin y) = \cancel{e^x} e^z (z + i)$$