Интеграл Фурье. Преобразование Фурье

§ R

253) $V.P. \int_{-\infty}^{\infty} \frac{dx}{x} = \lim_{R\to\infty} \int_{-R}^{+R} \frac{dx}{x} = \lim_{R\to\infty} \ln|x| \Big|_{-R}^{R} = \lim_{R\to+\infty} (\ln|R| - \ln|-R|) = 0$

255) $\cancel{V.P. \int_{-\infty}^{\infty} \frac{dx}{x^2-3x+2} = \lim_{R\to+\infty} \int_{-R}^{R} \frac{dx}{x^2-3x+2} = \lim_{R\to+\infty} \left(\int_{-R}^{R} \frac{dx}{x-2} - \int_{-R}^{R} \frac{dx}{x-1} \right) =}$

$\cancel{\frac{1}{x^2-3x+2} = \frac{1}{(x-1)(x-2)} = \frac{1}{x-2} - \frac{1}{x-1}}$

$\cancel{= \lim_{R\to+\infty} \left(\ln|x-2| \Big|_{-R}^{R} - \ln|x-1| \Big|_{-R}^{R} \right) = \lim_{R\to+\infty} \left(\ln\left|\frac{R-2}{R+2}\right| - \ln\left|\frac{R-1}{R+1}\right| \right) = 0}$

$V.P. \int_0^{+\infty} \frac{dx}{x^2-3x+2} = V.P. \int_0^{+\infty} \left(\frac{1}{x-2} - \frac{1}{x-1} \right) dx = \cancel{\lim_{\varepsilon_1\to 0}}$

$= \lim_{\varepsilon_2\to+\infty} \lim_{\varepsilon_1\to 0} \left(\int_0^{1-\varepsilon_1} f(x)\,dx + \int_{1+\varepsilon_1}^{2-\varepsilon_1} f(x)\,dx + \int_{2+\varepsilon_1}^{\varepsilon_2} f(x)\,dx \right)$

$\int_0^{1-\varepsilon_1} \left(\frac{1}{x-2} - \frac{1}{x-1} \right) dx = (\ln|x-2| - \ln|x-1|) \Big|_0^{1-\varepsilon_1} = \ln|1+\varepsilon_1| - \ln 2 - \ln|\varepsilon_1| + 0$

$\int_{1+\varepsilon_1}^{2-\varepsilon_1} \left(\frac{1}{x-2} - \frac{1}{x-1} \right) dx = (\ln|x-2| - \ln|x-1|) \Big|_{1+\varepsilon_1}^{2-\varepsilon_1} = \cancel{\ln|\varepsilon_1|} - \ln|\varepsilon_1+1| - \ln|\varepsilon_1-1| +$

$+ \cancel{\ln|\varepsilon_1|}$

$\int_{2+\varepsilon_1}^{\varepsilon_2} \left(\frac{1}{x-2} - \frac{1}{x-1} \right) dx = (\ln|x-2| - \ln|x-1|) \Big|_{2+\varepsilon_1}^{\varepsilon_2} =$

$= \ln|\varepsilon_2-2| - \ln|\varepsilon_1| - \ln|\varepsilon_2-1| + \ln|\varepsilon_1+1|$

$\cancel{+2\ln|\varepsilon_1|}$

2) $V.P. \int_0^{+\infty} \frac{dx}{x^2+3x+2} = \lim_{\varepsilon_2\to\infty} \lim_{\varepsilon_1\to 0} \left(\cancel{\ln|\varepsilon_1+1|} - \ln 2 - \cancel{\ln|\varepsilon_1|} - \cancel{\ln|\varepsilon_1+1|} - \ln|\varepsilon_1-1| + \right.$

$\left. + \ln|\varepsilon_2-2| - \cancel{\ln|\varepsilon_1|} - \ln|\varepsilon_2-1| + \ln|\varepsilon_1+1| \right) =$

$$= \lim_{\xi_2\to\infty} \lim_{\xi_1\to 0} \left(-\ln 2 + \ln\left|\frac{\xi_1+1}{\xi_1-1}\right| + \ln\left|\frac{1-\frac{2}{\xi_2}}{1-\frac{1}{\xi_2}}\right|\right) =$$

$$= -\ln 2$$

5.2

$\boxed{\S 17}$

(1.3) $f(x) = \text{sign}(x-a) - \text{sign}(x-b)$, $b > a$

$$f(x) \sim \begin{cases} 2, & \cancel{a<x<b}\; x \in (a;b) \\ 1; & x=a \cup x=b \\ 0; & x \in (-\infty; a) \cup (b; +\infty) \end{cases}$$

c) $f(x) = \frac{1}{\pi}\int_0^{+\infty} dy \int_a^b 2\cdot\cos y(x-t)\, dt =$

$= \frac{2}{\pi}\int_0^{+\infty} \frac{\sin(y(x-b)) - \sin(y(x-a))}{-y}\, dy =$

$= \frac{2}{\pi}\int_0^{+\infty} \frac{\sin(y(x-a)) - \sin(y(x-b))}{y}\, dy$

(3.1) $f(x) = \underbrace{\underbrace{e^{-\alpha|x|}}_{\text{чет}}\ \underbrace{\sin\beta x}_{\text{нечет}}}_{\text{нечет}}$,

$b(y) = \frac{2}{\pi}\int_0^{+\infty} e^{-\alpha t}\sin\beta t\,\sin yt\, dt =$

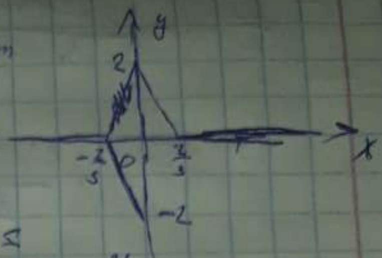
$= \frac{1}{\pi}\int_0^{+\infty} e^{-\alpha t}\cos((\beta-y)t)\, dt - \frac{1}{\pi}\int_0^{+\infty} e^{-\alpha t}\cos((\beta+y)t)\, dt =$

$= \frac{\alpha}{\pi(\alpha^2+(\beta-y)^2)} - \frac{\alpha}{\pi(\alpha^2+(\beta+y)^2)} =$

$= \frac{\alpha}{\pi}\cdot\frac{\cancel{\alpha^2}+(\beta+y)^2-\cancel{\alpha^2}-(\beta-y)^2}{(\alpha^2+(\beta-y)^2)(\alpha^2+(\beta+y)^2)} = \frac{4\alpha\beta y}{\pi(\alpha^2+(\beta-y)^2)(\alpha^2+(\beta+y)^2)}$

$f(x) = \frac{4\alpha\beta}{\pi}\int_0^{+\infty} \frac{y\cdot\sin(xy)\, dy}{(\alpha^2+(\beta-y)^2)(\alpha^2+(\beta+y)^2)}$

5.2 $f(x) = \begin{cases} 2-3x & 0 \le x \le \frac{2}{3} \\ 0 & x > \frac{2}{3} \end{cases} \Longleftrightarrow$ нечет



$f(x)$ нечет: $b(y) = \frac{2}{\pi} \int_0^{+\infty} (2-3t) \sin yt \, dt =$

$= \frac{2}{\pi} \int_0^{2/3} (2-3t) \sin yt \, dt = \frac{4}{\pi} \int_0^{2/3} \sin yt \, dt - \frac{6}{\pi} \int_0^{2/3} t \sin yt \, dt =$

$= -\frac{4}{\pi} \frac{\cos yt}{y} \Big|_0^{2/3} + \frac{6}{\pi y} (t \cdot \cos yt) \Big|_0^{2/3} - \frac{6}{\pi y} \int_0^{2/3} \cos yt \, dt =$

$= +\frac{4}{\pi y}\left(1 - \cancel{\cos\left(\frac{2}{3}y\right)}\right) + \cancel{\frac{4}{\pi y} \cdot \cos\left(\frac{2}{3}y\right)} - \frac{6}{\pi y^2} \sin \frac{2}{3}y =$

$= \frac{2}{\pi y^2}\left(2y - 3\sin\left(\frac{2}{3}y\right)\right)$

$\Rightarrow f(x) = \frac{2}{\pi} \int_0^{+\infty} \frac{2y - 3\sin\left(\frac{2}{3}y\right)}{y^2} \sin xy \, dy$

6.1 $f(x) = e^{-\alpha x}, \ x \ge 0, \ \alpha > 0$

продолжаем четно $\Rightarrow a(y) = \frac{2}{\pi} \int_0^{+\infty} e^{-\alpha t} \cos yt \, dt =$

$= \frac{2}{\pi} \cdot \frac{\alpha}{\alpha^2 + y^2}$

$\Rightarrow f(x) = \frac{2\alpha}{\pi} \int_0^{+\infty} \frac{\cos(xy) \, dy}{\alpha^2 + y^2}$

7.3 $f(x) = \begin{cases} \cos x, & |x| \le \pi \\ 0 & |x| > \pi \end{cases}$

$f(x)$ четная $\Rightarrow$ Фурье-преобразование совпадает с косинус-преобразованием.

$F[f](x) = \mathcal{F}_c f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \cos t \cdot \cos tx \, dt = \frac{1}{\sqrt{2\pi}} \int_0^{\pi} \left(\cos(t+tx) + \cos(tx-t)\right) dt$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\sin(t(x+1))}{x+1} \Big|_0^{\pi} + \frac{\sin(t(x-1))}{x-1} \Big|_0^{\pi} \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{\sin(\pi(x+1))}{x+1} + \frac{\sin(\pi(x-1))}{x-1} \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\sin \pi x}{x-1} + \frac{\sin \pi x}{x+1} \right) = \frac{2 \sin(\pi x)}{} \sqrt{\frac{2}{\pi}} \frac{x \sin(\pi x)}{x^2 - 1}$$

$$\Rightarrow F[f](x) = \sqrt{\frac{2}{\pi}} \frac{x \sin(\pi x)}{x^2-1}$$

(8.2) $f(x) = e^{-x^2/2}$ – четная

$$F[f](x) = F_c^f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} e^{-t^2/2} \cdot \cos(tx)\, dt =$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^{+\infty} [\underbrace{e^{itx - t^2/2}}_{I} + \underbrace{e^{-itx - t^2/2}}_{II}] dt$$

$$I = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-t^2/2 + itx - \frac{i^2x^2}{2} + \frac{i^2x^2}{2}} dt =$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\left(\frac{t}{\sqrt{2}} - \frac{ix}{\sqrt{2}}\right)^2} \cdot e^{-x^2/2} dt = \frac{1}{\sqrt{\pi}} \cdot e^{-x^2/2} \int_0^{+\infty} e^{-p^2} dp =$$

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$p = \left(\frac{t}{\sqrt{2}} - \frac{ix}{\sqrt{2}}\right)$

$$= \frac{1}{2} e^{-x^2/2}$$

$$II = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-t^2/2 - itx - \frac{i^2x^2}{2} + \frac{i^2x^2}{2}} dt =$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\left(\frac{t}{\sqrt{2}} + \frac{ix}{\sqrt{2}}\right)^2} \cdot e^{-x^2/2} dt = \frac{1}{2} e^{-x^2/2}$$

$$\Rightarrow F[f](x) = e^{-x^2/2}$$

(8.6) $f(x) = \frac{d^2}{dx^2}\left(x e^{-|x|}\right) = y'', \ y$ – нечет

$F[f](x) = F[g''](x) = (ix)^2 F[g](x) = -x^2 F[g](x)$

$F[g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t e^{-|t|} e^{-ixt} dt = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} t e^{t(1-ix)} dt + \right.$
$\left. + \int_{0}^{+\infty} t \cdot e^{-t(1+ix)} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{te^{t(1-ix)}}{1-ix} \Big|_{-\infty}^{0} - \int_{-\infty}^{0} \frac{e^{t(1-ix)}}{1-ix} dx - \frac{t \cdot e^{-t(1+ix)}}{1+ix} \Big|_{0}^{+\infty} + \right.$
$\left. + \int_0^{+\infty} \frac{e^{-t(1+ix)}}{1+ix} dt \right] = \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{(1-ix)^2} + \frac{1}{(1+ix)^2} \right] =$

$= -\frac{1}{\sqrt{2\pi}} \cdot \frac{(1+ix)^2 - (1-ix)^2}{(1+x^2)^2} = 2\sqrt{\frac{2}{\pi}} \frac{i x^3}{(1+x^2)^2}$

(14.1) $\hat{f}(y)$ – преобразование Фурье $f(x) = \frac{1}{1+|x|^5}$

Функции f, xf, $x^2 f$, $x^3 f$ абс. инт.:

$\left(\left| \int_{-\infty}^{\infty} \frac{x^3}{1+|x|^5} dx = 2 \int_0^{+\infty} \frac{x^3}{1+x^5} dx = \underbrace{2 \int_0^1 \frac{x^3}{1+x^5} dx}_{\text{собственный, нет особенностей}} + 2 \int_1^{+\infty} \frac{x^3}{1+x^5} dx < \int_1^{\infty} \frac{dx}{x^2} \right. \right.$ по призн. ср. сх-ся

$\Rightarrow x^3 f$ абс. инт. $)$

Тогда $F^{'''}[f](y) = -i F[x^3 f](y)$

т.к. $x^3 f$ абс. инт, то $F[x^3 f](y)$ непр. $\Rightarrow F^{'''}[f](y)$ непр

(14.3) $\left(\frac{1}{1+|x|^5}\right)^{(x)}$ – непрерывна и абс. инт-ма

$F\left[\left(\frac{1}{1+|x|^5}\right)^{IV}\right] = i y^5 F\left[\frac{1}{1+|x|^5}\right] \Rightarrow y^5 F\left[\frac{1}{1+|x|^5}\right] = -i F\left[\frac{1}{1+|x|^5}\right] \xrightarrow[y\to\infty]{} 0$

по лемме Римана об осцилляции

$\Rightarrow \hat{f}(y) = o\left(\frac{1}{y^5}\right)$ при $y \to \infty$

T2 $f(x)$ - абс. инт. ; нечет

$$V.P. \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx =$$

$$= \lim_{R\to\infty} \left(\int_{-R}^{0} f(x)\,dx + \int_{0}^{R} f(x)\,dx \right) = \lim_{R\to\infty} \left(-\int_{0}^{R} f(t)\,dt + \int_{0}^{R} f(x)\,dx \right) =$$

$$= 0$$