Линейные однородные уравнения в частных производных первого порядка

1 неделя

§17 4) $(x^2+z^2)\frac{\partial u}{\partial x} + 2(xy - xz^3)\frac{\partial u}{\partial y} + 2xz\frac{\partial u}{\partial z} = 0$

• $u = \frac{y}{z} - \frac{z^2}{2}$
• $x^2 - z^2 = z$

$$\begin{cases} \dot{x} = x^2+z^2 \\ \dot{y} = 2(xy - xz^3) \\ \dot{z} = 2xz \end{cases}$$

$$\frac{dx}{dz} = \frac{x^2+z^2}{2xz} = \frac{x}{2z} + \frac{z}{2x} \Rightarrow 2xx' - \frac{1}{z}x^2 = z \text{ Бернулли}$$

$t = x^2 \Rightarrow t' = 2xx'$

$t' - \frac{1}{z}t = z$ ← лин. ур-ние

$t' = \frac{1}{z}t$

$\frac{dt}{t} = \frac{dz}{z} \Rightarrow t = C\cdot z,\ C\neq 0$

$C'(z)z + \cancel{C} - \cancel{C} = z$

$C'(z) = 1 \Rightarrow C(z) = z + C$

$\Rightarrow t = z^2 + Cz = x^2 \Rightarrow u = \frac{x^2 - z^2}{z}$ — I интеграл

$$\frac{dy}{dz} = \frac{2x(y - z^3)}{2xz} = \frac{y}{z} - z^2$$

$y' - \frac{y}{z} = -z^2$ лин. уравнение

$y' = \frac{y}{z}$

$\frac{dy}{y} = \frac{dz}{z} \Rightarrow y = C\cdot z,\ C\neq 0.\quad y = C(z)z$

$C'(z)z + \cancel{C(z)} - \cancel{C(z)} = -z^2 \Rightarrow C(z) = -\frac{z^2}{2} + C$

2й I интеграл

$\Rightarrow \frac{y + \frac{z^3}{2}}{z} = u_2$

Для решения ЗК запишем систему

$$\begin{cases} \frac{x^2-y^2}{z}=2 \\ u_1=\frac{x^2-z^2}{z} \\ u_2=\frac{y}{z}+\frac{z^2}{2} \end{cases} \Rightarrow u_1=\frac{2}{z}$$

$$u=\frac{y}{z}-\frac{z^2}{2}=u_2-\cancel{4}\frac{z^2}{\cancel{4}}=u_2-\frac{2\cdot 4}{u_1^2}=\frac{y}{z}+\frac{z^2}{2}-\frac{4z^2}{(x^2-z^2)^2}$$

$$u=F\left(\frac{x^2-z^2}{z};\ \frac{y}{z}+\frac{z^2}{2}\right)$$

(14) $(z+2x-2y)\frac{\partial u}{\partial x}+(z-2x+2y)\frac{\partial u}{\partial y}-2z\frac{\partial u}{\partial z}=0.$

- $u=xz^4$
- $x+y=0$

$$\begin{cases} \dot{x}=(z+2x-2y) \\ \dot{y}=(z-2x+2y) \\ \dot{z}=-2z \end{cases}$$

1) $x+y+z=u_1$ 1й I интеграл

2) $\begin{cases} (\dot{x-y})=4(x-y) \\ \dot{z}=-2z \end{cases} \Rightarrow \frac{d(x-y)}{dz}=-2\frac{x-y}{z}$

$\ln|x-y|=-2\ln|z|+C$

$(x-y)z^2=u_2$ 2й I интеграл

$\Rightarrow$ $u=F\left(x+y+z,\ (x-y)z^2\right)$

Решение ЗК:

$$\begin{cases} x+y+z=u_1 \\ (x-y)z^2=u_2 \\ x+y=0 \end{cases} \Rightarrow \begin{matrix} u_1=z \\ 2x z^2=u_2 \end{matrix} \Rightarrow \begin{matrix} x=\frac{u_2}{2u_1^2} \\ y=-\frac{u_2}{2u_1^2} \end{matrix}$$

$$u=xz^4=\frac{u_2}{2u_1^2}\cdot u_1^4=\frac{1}{2}u_2\cdot u_1^2=\frac{1}{2}(x-y)z^2\cdot(x+y+z)^2$$

14) $(x^2y+2x)\frac{\partial u}{\partial x}+(2xy^2+y)\frac{\partial u}{\partial y}-(xyz+2z)\frac{\partial u}{\partial z}=0$ $\quad \cdot\ u=yz+y+\frac{1}{y}$

$\cdot\ x=y$

$$\begin{cases} \dot{x}=2x+x^2y \\ \dot{y}=2xy^2+y \\ \dot{z}=-2z-xyz \end{cases}$$

1) $(2xy^2+y)\dot{x}=(2x+x^2y)\dot{y}$

$2xy^2\dot{x}+y\dot{x}=2x\dot{y}+x^2y\dot{y}$

$xy(2y\dot{x}-x\dot{y})=(2x\dot{y}-y\dot{x})$

$\frac{2x\dot{x}y-x^2\dot{y}}{y^2}=\frac{2x\dot{y}}{y^3}-\frac{\dot{x}}{y^2}$

$\dot{\left(\frac{x^2}{y}\right)}=\dot{\left(\frac{-x}{y^2}\right)} \Rightarrow \frac{x^2}{y}+\frac{x}{y^2}=u_1$ I интеграл

2) $\frac{dx}{dz}=\frac{x(2+xy)}{-z(2+xy}=-\frac{x}{z}$

$\frac{dx}{x}=-\frac{dz}{z} \Rightarrow xz=u_2$ — 2й интеграл

$$u=F\left(xz;\ \frac{x^2}{y}+\frac{x}{y^2}\right)$$

Решим ЗК:

$$\begin{cases} u_1=xz \\ u_2=\frac{x^3}{y}+\frac{x}{y^2} \\ x=y \end{cases} \Rightarrow u_2=y+\frac{1}{y} \Rightarrow u=u_1+u_2=xz+\frac{x^2}{y}+\frac{x}{y^2}$$

(42) $z\cos x \frac{\partial u}{\partial x} + z(1-y\sin x)\frac{\partial u}{\partial y} + (1-z)\sin x \frac{\partial u}{\partial z} = 0$

$u = e^{z}(z-1)$

$y = 1+\sin x$

$0 < x < \frac{\pi}{2}$

(91)

$$\begin{cases} \dot{x} = z\cos x \\ \dot{y} = z(1-y\sin x) \\ \dot{z} = (1-z)\sin x \end{cases}$$

1) $\frac{dy}{dx} = \frac{1-y\sin x}{\cos x}$

$y' + \operatorname{tg}x \cdot y = \frac{1}{\cos x}$ лин. уравнение

$y' = -\operatorname{tg}x \cdot y$

$\frac{dy}{y} = -\operatorname{tg}x\, dx$

$\ln|y| = \int \frac{d(\cos x)}{\cos x} = \ln|\cos x| + C$

$\Rightarrow y = C \cdot \cos x$

$C'(x)\cdot\cos x - \cancel{C(x)\cdot\sin x + \sin x\cdot C(x)} = \frac{1}{\cos x}$

$C'(x) = \frac{1}{\cos^2 x} \Rightarrow C(x) = \operatorname{tg}x + C$

$\Rightarrow y = \sin x + C\cdot\cos x \Rightarrow \frac{y-\sin x}{\cos x} = u_1$ I интеграл

2) $\frac{dz}{dx} = \frac{(1-z)}{z}\cdot\operatorname{tg}x$

$\frac{z}{1-z}\,dz = \operatorname{tg}x\,dx$

$\frac{1}{1-z} - 1$

$C + \ln|1-z| - z = -\ln|\cos x| \Rightarrow u_2 = \frac{z-1}{\cos x}e^{z}$

$\Rightarrow u = F\left(\frac{y-\sin x}{\cos x}; \frac{z-1}{\cos x}e^{z}\right)$

Решим ЗК:

$$\begin{cases} \dfrac{y-\sin x}{\cos x} = u_1 \\ \dfrac{z-1}{\cos x} e^z = u_2 \\ y = 1+\sin x \end{cases} \Rightarrow u_1 = \frac{1}{\cos x}$$

$$e^z \cdot (z-1) = \frac{u_2}{u_1} = u$$

$$u = \frac{e^z \cdot (z-1)}{y-\sin x}$$

(91) $x^4 \frac{\partial u}{\partial x} + (x^4 z + x^3 y - 2x)\frac{\partial u}{\partial y} + (2 - x^3 z)\frac{\partial u}{\partial z} = 0$ $\quad \cdot\ u = (y+1)^2$ $\quad \cdot\ xz = 1$

$$\begin{cases} \dot{x} = x^4 \\ \dot{y} = x^4 z + x^3 y - 2x \\ \dot{z} = 2 - x^3 z \end{cases}$$

1) $\frac{dz}{dx} = \frac{2}{x^4} - \frac{z}{x}$

$z' + \frac{1}{x} z = \frac{2}{x^4}$

$\frac{dz}{z} = -\frac{dx}{x} \Rightarrow \ln|z| = -\ln|x| + C \Rightarrow z = \frac{C}{x}$

$\frac{C'(x)}{x} - \cancel{\frac{C(x)}{x^2}} + \cancel{\frac{C(x)}{x^2}} = \frac{2}{x^4}$

$C'(x) = \frac{2}{x^3} \Rightarrow C(x) = -\frac{1}{x^2} + C$

$\Rightarrow z = -\frac{1}{x^3} + \frac{1}{x} C \Rightarrow \left(xz + \frac{1}{x^2}\right) = u_1$ I интеграл

2) $\cancel{z}\ \frac{\dot{y}}{x} + \dot{z} - \frac{y\dot{x}}{x^2} = 0$

$\Rightarrow u_2 = z + \frac{y}{x}$ 2й I интеграл

$$\Rightarrow u = F\left(xz + \frac{1}{x^2};\ z + \frac{y}{x}\right)$$

Решим ЗК: $\begin{cases} xz + \frac{1}{x^2} = u_1 & z^2+1 = u_1 \\ z + \frac{y}{x} = u_2 & z(y+1) = u_2 \\ xz = 1 \end{cases} \Bigg| \Rightarrow u = (y+1)^2 = \left(\frac{u_2}{z}\right)^2 = \frac{u_2^2}{u_1 - 1} = \frac{(zx+y)^2}{x^2\left(\frac{1}{x^2} + xz - 1\right)}$

$$= \frac{(y+xz)^2}{x^3 z - x^2 + 1}$$

(T4) $2\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial y} = 0$

$$\begin{cases} \dot{x} = 2 \\ \dot{y} = 3 \end{cases} \Rightarrow \frac{dy}{dx} = \frac{3}{2}$$

$$y = \frac{3}{2}x + C$$

$$3x - 2y = u_1$$

$$\Rightarrow \underline{u = F(3x - 2y)}$$

а) $u = 10$ при $3x - 2y = 5$

$$\begin{cases} 3x - 2y = 5 \quad u = 10 \\ u_1 = 3x - 2y \end{cases}$$

$\dot{g}(x, y) = 0 \Rightarrow$ Кривая содержит хар-кие точки и теорема о $\exists u!$ невыполнена $\Rightarrow$ решений много ($\dot{g} = 0$, $\varphi = const$)

$u = F(5) = 10$

$u = 2 \cdot (3x - 2y)$

или

$u = \frac{2}{5}(3x - 2y)^2$

или

$u = 10 \cdot \frac{\sin(3x - 2y)}{\sin 5}$

б) $u = e^{y}$ при $3x - 2y = 5$

$$\begin{cases} 3x - 2y = 5 \\ u_1 = 3x - 2y \end{cases}$$

$\dot{g}(x, y) = 0 \Rightarrow$ нарушается условие постановки ЗК

$\varphi \neq const \Rightarrow$ решений нет

в) $u = \sin y$ при $x = 0$

$$\begin{cases} x = 0 \\ u_1 = 3x - 2y = -2y \end{cases}$$

$\dot{g}(x, y) = 2 \neq 0 \xRightarrow{\text{усл-ия}}$ Т-ма о $\exists u!$ выполнена

$\Rightarrow u = \sin\left(-\frac{u_1}{2}\right) = \sin\left(y - \frac{3}{2}x\right)$

(T5)

1)

2)

5) $\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0$

$$\begin{cases} \dot{t} = 1 \\ \dot{x} = u \\ \dot{y} = v \end{cases} \Rightarrow \begin{matrix} u_1 = x - ut \\ u_2 = y - vt \end{matrix} \Rightarrow u = F(x-ut;\, y-vt)$$

1) $f = \frac{1}{x^2+y^2} e^{-b(u^2+v^2)}$ при $t=0$

$$\begin{cases} t=0 \\ u_1 = x - 0 \\ u_2 = y - 0 \end{cases} \Rightarrow f = \frac{1}{u_1^2+u_2^2} \cdot e^{-b(u^2+v^2)} = \frac{1}{(x-ut)^2+(y-vt)^2} e^{-b(u^2+v^2)}$$

2) $f = \frac{1+\frac{1}{2}\sin\Omega t}{x^2} e^{-b(u^2+v^2)}$

$$\begin{cases} y=0 \\ u_1 = x - ut \\ u_2 = y - vt \end{cases} \Rightarrow t = -\frac{u_2}{v} \Rightarrow x = u_1 - \frac{u}{v}u_2$$

$$\Rightarrow f = \frac{1+\frac{1}{2}\sin\Omega\left(-\frac{u_2}{v}\right)}{\left(u_1 - \frac{u}{v}u_2\right)^2} e^{-b(u^2+v^2)} =$$

$$= \frac{1+\frac{1}{2}\cdot\sin\Omega\left(t-\frac{y}{v}\right)}{\left(x-\frac{u}{v}y\right)^2} \cdot e^{-b(u^2+v^2)}$$