II Элементарные функции. Функциональные ряды

§3

№8 1) e^{z} $z=2\pi i$

$e^{i2\pi}=1$

2) $e^{i\pi}=-1$

4) $e^{-\pi i/2}=-i$

№9 3) $\sin\left(z+\frac{\pi}{2}\right)=\cos z$ Д-ть

$$\sin\left(z+\frac{\pi}{2}\right)=\underbrace{\sin z\cdot\underbrace{\cos\frac{\pi}{2}}_{0}}_{0}+\cos z\,\overbrace{\sin\frac{\pi}{2}}^{1}=\cos z$$

4) $\sin(\pi+z)=-\sin z$

$$\underbrace{\underbrace{\sin\pi}_{0}\cdot\cos z}_{0}+\overbrace{\cos\pi}^{-1}\cdot\sin z=-\sin z$$

Другой способ

3) $$\sin\left(z+\frac{\pi}{2}\right)=\frac{e^{i\left(\frac{\pi}{2}+z\right)}-e^{-i\left(z+\frac{\pi}{2}\right)}}{2i}=$$

$$=\frac{\cancel{i}\left(e^{iz}+e^{-iz}\right)}{2\cancel{i}}=\cos z$$

4) $$\sin(\pi+z)=\frac{e^{i(\pi+z)}-e^{-i(\pi+z)}}{2i}=\frac{e^{-iz}-e^{iz}}{2i}=-\sin z$$

8) $$\cos(z_1+z_2)=\frac{e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}}{2}=$$

$$=\frac{e^{i(z_1+z_2)}+e^{i(z_1-z_2)}-e^{i(z_1-z_2)}+e^{-i(z_1+z_2)}}{4}+$$

$= \frac{e^{iz_1}}{2}\cos z_2 - \frac{e^{-iz_2}}{2}\cdot\left(e^{iz_1} - e^{-iz_1}\right)$

$\cos z_1 \cos z_2 - \sin z_1 \sin z_2 = \frac{e^{iz_1}+e^{-iz_1}}{2}\cdot\frac{e^{iz_2}+e^{-iz_2}}{2} -$

$+ \frac{e^{iz_1}-e^{-iz_1}}{2\cancel{i}}\cdot\frac{e^{iz_2}-e^{-iz_2}}{2\cancel{i}} = \frac{1}{4}\left(e^{i(z_1+z_2)} + \cancel{e^{i(z_1-z_2)}} + \cancel{e^{i(z_2-z_1)}} + e^{-i(z_1+z_2)}\right)$

$+ \frac{1}{4}\left(e^{i(z_1+z_2)} + e^{-i(z_1+z_2)} - \cancel{e^{i(z_1-z_2)}} - \cancel{e^{i(z_2-z_1)}}\right) =$

$= \cos(z_1+z_2)$

N11 1) $\operatorname{sh} z = -i\sin(iz)$

$\operatorname{sh} z = \frac{e^{z}-e^{-z}}{2} = -i\,\frac{e^{-z}-e^{z}}{2i} = -i\sin(+iz)$

2) $\operatorname{sh}(iz) = \frac{e^{iz}-e^{-iz}}{2} = i\cdot\frac{e^{iz}-e^{-iz}}{2i} = i\sin z$

3) $\cos(iz) = \cancel{\frac{e^{-z+iz}+e^{1+iz}}{2}}\ \frac{e^{-z}+e^{+z}}{2} = \operatorname{ch} z$

4) $\operatorname{ch}(iz) = \frac{e^{iz}+e^{-iz}}{2} = \cos z$

N12 1) $\operatorname{Re}\sin z = \sin x \operatorname{ch} y$ D-Tb.

$\operatorname{Re}\frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{ix-y}-e^{-ix+y}}{2i} = \frac{e^{-y}}{2i}(\cos x + i\sin x) - \frac{e^{y}}{2i}(\cos x - i\sin$

$\cancel{\sin x \operatorname{ch} y = \frac{e^{ix}-e^{-ix}}{2i}\cdot\frac{e^{y}+e^{-y}}{2} = \frac{e^{ix+y}+e^{ix-y}-e^{-(ix-y)}-e^{-(ix+y)}}{4i}}$

$\cancel{\operatorname{Re}}$

$$= \frac{e^{-y}}{2i}\cos x + \frac{e^{-y}}{2}\sin x - \frac{e^{y}}{2i}\cos x + \frac{e^{y}}{2x}\sin x =$$

$$= ch\, y \cdot \sin x + \cos x \cdot \frac{e^{y} - e^{-y}}{2} \cdot i = \underline{ch\, y \cdot \sin x + \cos x \cdot sh\, y \cdot i}$$

2) $Re \cos z = \cos x \cdot ch\, y$

$$\underline{\cos z} = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2} = \frac{e^{-y}}{2}(\cos x + i\sin x) +$$

$$+ \frac{e^{y}}{2}(\cos x - i\sin x) = \underline{\cos x \cdot ch\, y} - \sin x \cdot sh\, y \cdot i$$

№13 3) Д-ть, что $|sh\, z| = \sqrt{ch^2 x - \cos^2 y}$

$$|sh\, z|^2 = sh\, z \cdot \overline{sh\, z} = ch^2 x - \cos^2 y$$

$$\overline{sh\, z} = \overline{\left(\frac{e^{z} - e^{-z}}{2}\right)} = \frac{e^{\bar z} - e^{-\bar z}}{2} = sh\, \bar z$$

$$sh\, z \cdot sh\, \bar z = \frac{1}{4}\left(e^{z+\bar z} - \left(e^{z-\bar z} + e^{\bar z - z}\right) + e^{-z-\bar z}\right) =$$

$$= \frac{1}{4}\left(e^{2x} + e^{-2x} - \left(e^{+2yi} + e^{-2yi}\right)\right) =$$

$$= \frac{1}{2} ch\, 2x - \frac{1}{2}\cos 2y = ch^2 x - \cos^2 y$$

№14 1) $\cos z$ принимает действительные значения, т.е. $Im(\cos z) = 0$

$$Im(\cos z) = Im\left(\frac{e^{iz} + e^{-iz}}{2}\right) = Im\left(\frac{e^{-y}}{2}(\cos x + i\sin x) + \frac{e^{y}}{2}(\cos x - i\sin x)\right) =$$

$$= 0 \quad \left(\frac{e^{y}}{2} - \frac{e^{-y}}{2}\right) \cdot \sin x = 0 \quad \left[\begin{array}{l} \sin x = 0 \quad x = \pi k, k \in \mathbb{Z} \\ y = 0 \end{array}\right. \quad y = 0$$

$\Rightarrow$ вся действительная ось

и вертикальные линии в точках $x = \pi k, k \in \mathbb{Z}$

$x = Re\, z$

№13 1) $|\sin z| = \sqrt{ch^2 y - \cos^2 x}$

$\sin z \cdot \overline{\sin z} = ch^2 y - \cos^2 x$

$$\overline{\sin z} = \overline{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)} = \frac{\overline{(e^{iz} - e^{-iz})}}{-2i} = \frac{e^{\overline{iz}} - e^{\overline{-iz}}}{-2i} = \frac{e^{-i\bar z} - e^{i\bar z}}{-2i}$$

$$\Rightarrow \text{лев. ч.}^2 = \frac{1}{4}\left(e^{iz} - e^{-iz}\right)\left(e^{-i\bar z} - e^{i\bar z}\right) = \frac{1}{4}\Big(\underbrace{e^{i(z-\bar z)}}_{e^{2i(iy)}} - \underbrace{e^{i(z+\bar z)}}_{e^{2ix}} -$$

$$- \underbrace{e^{-i(z+\bar z)}}_{e^{-2ix}} + \underbrace{e^{i(\bar z - z)}}_{e^{2i(-iy)}}\Big)$$

$2ch2y + -2\cos 2x$

$$= \frac{2ch2y - 2\cos 2x}{4} = \frac{4ch^2y - \cancel{2} - 4\cos^2 x + \cancel{2}}{\cancel{4}} = ch^2 y - \cos^2 x$$

№19 1) $e^z \to \infty$

$x \to +\infty$

и равн-о по y

2) $e^z \to 0$ при $x \to -\infty$ и р-но по y

3) $\sin z$ и $\cos z \to \infty$ при $y \to \pm\infty$ и это р-но по x

$\to \infty$

$\Leftrightarrow$

$\forall M > 0\ \exists N > 0\ \forall y : |y| > N\ \forall x \in \mathbb{R}$

$|\sin z| > M$

Нужно: $\sqrt{ch^2 y - \cos^2 x} > M$

Достаточно доказать $\sqrt{ch^2 y - 1} > M$

$\operatorname{ch} y > \frac{e^y}{2}$

Дост. если $y > N \Rightarrow \left(\frac{e^N}{2}\right)^2 - 1 > M^2$

N14 4) $\operatorname{tg} z$

$Im(\operatorname{tg} z) = 0$

$Im(\operatorname{tg} z) = Im\left(\frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right) = 0$

$= Im\left(\frac{e^{ix-y} - e^{-ix+y}}{i(e^{ix-y} + e^{-ix+y})}\right) = Im\left(-i \cdot \frac{e^{-y} \cdot (\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}\right)$

$= Im\left(-i \cdot \frac{-\operatorname{sh} y \cdot \cos x + \operatorname{ch} y \cdot \sin x \cdot i}{\operatorname{ch} y \cdot \cos x - \operatorname{sh} y \cdot \sin x \cdot i}\right) =$

$= Im\left(i \cdot \frac{(\operatorname{sh} y \cos x - \operatorname{ch} y \sin x \cdot i)(\operatorname{ch} y \cos x + \operatorname{sh} y \sin x \cdot i)}{(\operatorname{ch} y \cos x)^2 + (\operatorname{sh} y \cdot \sin x)^2}\right) =$

$= \frac{\operatorname{sh} y \cdot \operatorname{ch} y \cos^2 x + \operatorname{sh} y \operatorname{ch} y \sin^2 x}{(\operatorname{ch} y \cos x)^2 + (\operatorname{sh} y \sin x)^2} = \frac{\operatorname{sh} y \cdot \operatorname{ch} y}{(\operatorname{ch} y \cos x)^2 + (\operatorname{sh} y \sin x)^2} = 0$

$\operatorname{ch} y \neq 0$

$\Rightarrow \operatorname{sh} y = 0 \Rightarrow y = 0 \quad Im\, z = 0$

№17 3) $\cos z = \frac{3}{4} i$

$\cos z = \operatorname{ch} y \cdot \cos x - \operatorname{sh} y \cdot \sin x \cdot i = \frac{3}{4} i$

$\begin{cases} \operatorname{ch} y \cdot \cos x = 0 \\ \operatorname{sh} y \cdot \sin x = -\frac{3}{4} \end{cases}$ $\operatorname{ch} y \neq 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

$\Downarrow$

$\sin x = \pm 1$

$\cancel{\Rightarrow y = \operatorname{arsh}\left(\pm\frac{3}{4}\right)}$

I $x = \frac{\pi}{2} + 2\pi k, k \in \mathbb{Z}$

$\sin x = 1$

$\operatorname{sh} y = -\frac{3}{4}$

$y = \operatorname{arsh}\left(\frac{3}{4}\right) =$

$= \ln\left(\frac{3}{4} + \frac{5}{4}\right) = -\ln 2$

II $x = -\frac{\pi}{2} + 2\pi k \quad k \in \mathbb{Z}$

$\sin x = -1$

$\operatorname{sh} y = +\frac{3}{4}$

$y = \ln\left(-\frac{3}{4} + \frac{5}{4}\right) = +\ln 2$

$\Rightarrow z = \pm\frac{\pi}{2} + 2\pi k \pm \ln 2 \cdot i = \pm\left(\frac{\pi}{2} - i \cdot \ln 2\right) + 2\pi k, k \in \mathbb{Z}$

4) $\cos z = \frac{3}{4} + \frac{i}{4}$

$\begin{cases} \operatorname{ch} y \cos x = \frac{3}{4} \\ \operatorname{sh} y \sin x = -\frac{1}{4} \end{cases}$ $\sin^2 x + \cos^2 x = 1$

$\operatorname{ch}^2 y - \operatorname{sh}^2 y = 1$

$\begin{cases} \operatorname{ch}^2 y \cos^2 x = \frac{9}{16} \\ \operatorname{sh} y \sin x = -\frac{1}{4} \end{cases}$ $(1 + \operatorname{sh}^2 y)(1 - \sin^2 x) = \frac{9}{16}$

$1 + \operatorname{sh}^2 y - \sin^2 x = \frac{5}{8}$

$\sin^2 x - \operatorname{sh}^2 y = \frac{3}{8}$ $\quad \sin^2 x = \operatorname{sh}^2 y + \frac{3}{8}$

$\sin^2 x \operatorname{sh}^2 y = \frac{1}{16}$ $\quad \operatorname{sh}^4 y + \frac{3}{8}\operatorname{sh}^2 y - \frac{1}{16} = 0$ $\quad$ но $\operatorname{sh}^2 y > 0$

$D = \frac{9}{64} - 4 \cdot \frac{-1}{16} = \frac{25}{64}$ $\quad \Rightarrow \operatorname{sh}^2 y = \frac{1}{8}$

$\frac{-\frac{3}{8} \pm \frac{5}{8}}{2}$ $\quad -\frac{1}{2};\ \frac{1}{8}$

$\operatorname{sh} y =$

№17 8) $ch z = \frac{1}{2}$

$$ch z = \frac{e^z + e^{-z}}{2} = \frac{1}{2}\left(e^{x+iy} + e^{-x-iy}\right) =$$

$$= \frac{1}{2}e^x(\cos y + i\sin y) + \frac{e^{-x}}{2}(\cos y - i\sin y) =$$

$$= ch x \cos y + sh x \sin y \cdot i = \frac{1}{2}$$

$$\begin{cases} ch x \cdot \cos y = \frac{1}{2} \\ sh x \cdot \sin y = 0 \end{cases}$$

I $sh x = 0$

$x = 0$

$ch x = 1$

$\cos y = \frac{1}{2}$

$y = \pm\frac{\pi}{3} + 2\pi k, k \in \mathbb{Z}$

II $\sin y = 0$ $y = \pi k, k \in \mathbb{Z}$

1) $\cos y = \pm 1$

$ch x \geq 1$

2) при $\cos y = -1$ решений нет

$y = 2\pi k, k \in \mathbb{Z}$

~~$ch x = 1$~~ Нет решений

Ответ: $z = \pm\frac{\pi i}{3} + 2\pi k \cdot i$ $k \in \mathbb{Z}$

№17 4) продолжение

$sh^2 y = \frac{1}{8} \Rightarrow \sin^2 x = \frac{1}{2}$

$\Rightarrow$ I: $\begin{cases} sh y = \frac{1}{2\sqrt{2}} \\ \sin x = -\frac{1}{\sqrt{2}} \end{cases}$

$ch y = \frac{3}{2\sqrt{2}}$

$\cos x = \frac{1}{\sqrt{2}}$

$x = -\frac{\pi}{4} + 2\pi k$ $k \in \mathbb{Z}$

II: $\begin{cases} sh y = -\frac{1}{2\sqrt{2}} \\ \sin x = \frac{1}{\sqrt{2}} \end{cases}$

$ch y = \frac{3}{2\sqrt{2}}$

$\cos x = \frac{1}{\sqrt{2}}$

$x = \frac{\pi}{4} + 2\pi k$

$\operatorname{arsh}(\operatorname{sh} y) = \ln(\operatorname{sh} y + \operatorname{ch} y)$ $\quad y = \ln\left(\frac{1+3}{2\sqrt{2}}\right) = \ln(\sqrt{2}) = \frac{1}{2}\ln 2 \quad y = \ln\left(\frac{-1+3}{2\sqrt{2}}\right) = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln(2)$

$\cancel{z = \sqrt{2}i - \frac{\pi}{4} + 2\pi k, k \in \mathbb{Z}} \quad \cancel{z = \frac{1}{\sqrt{2}} i + \frac{\pi}{4} + 2\pi k, k \in \mathbb{Z}}$

$$z = \frac{1}{2}\ln 2 \cdot i - \frac{\pi}{4} + 2\pi k, k \in \mathbb{Z} \qquad z = -\frac{1}{2}\ln 2 \cdot i + \frac{\pi}{4} + 2\pi k, k \in \mathbb{Z}$$

или $z = \pm\left(\frac{i}{2}\ln 2 - \frac{\pi}{4}\right) + 2\pi k, k \in \mathbb{Z}$ ← Ответ

§4 №6 4) Д-ть равномерную сходимость ряда $\sum_{n=1}^{\infty} 2^{-n} \cos nz$ на множестве $E = \{z : |\operatorname{Im} z| \le \delta < \ln 2\}$

$$\cancel{\sum_{n=1}^{\infty} |2|^{n}} \cos nz = \operatorname{ch} ny \cdot \cos(nx) - i \cdot \operatorname{sh} ny \cdot \sin(nx)$$

$$\sum_{n=1}^{\infty} 2^{-n} \cdot \cos(nz) = \sum_{n=1}^{\infty} \operatorname{ch}(ny)\cos(nx) \cdot 2^{-n} - \sum_{n=1}^{\infty} 2^{-n} \cdot \operatorname{sh}(ny)\sin(nx)$$

$$|2^{-n} \cdot \operatorname{ch}(ny)\cos(nx)| \le |2^{-n} \cdot \operatorname{ch} ny| \overset{|\operatorname{Im} z| \le \delta}{\le} |2^{-n} \operatorname{ch} n\delta| =$$

$$= 2^{-n} \cdot \frac{1}{2}\left(e^{n\delta} + e^{-n\delta}\right) \le 2^{-n} \cdot e^{n\delta} = \left(\frac{e^{\delta}}{2}\right)^n = q^n$$

$$\sum_{n=1}^{\infty} q^n < \infty, \text{ т.к. } q = \frac{e^{\delta}}{2} < \frac{e^{\ln 2}}{2} = 1$$

$\Rightarrow$ ряд $\sum_{n=1}^{\infty} 2^{-n} \cdot \operatorname{ch}(ny) \cdot \cos(nx)$ сх-ся по признаку Вейерштрасса

Аналогично сходится $\sum_{n=1}^{\infty} \left(-2^{-n} \operatorname{sh}(ny)\sin(nx)\right)$.

$\Rightarrow$ сходится ряд $\sum_{n=1}^{\infty} 2^{-n} \cos nz$, причем равномерно