MATH 11247: Optimization under Uncertainty
Assignment #1

Due Date: February 7, 4pm

Instructions

- This is an individual assignment.
- You **must** cite **any** references (texts, papers, websites, etc.) you have used to help you solve these problems.
- Submit your written/typed solutions via Gradescope **(pdf file only)**. For Question 4, please add your code's screenshot (or typed version) to your answer (and *do not submit the code as a separate file*).
- For any clarifications on the questions, feel free to post in Piazza.

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1 [15pts] A manufacturing factory produces two types of items. The cost of making these items is c_1 & c_2 and the selling price is $s_1 > c_1$ & $s_2 > c_2$, respectively. They dispatch these items at the end of the day based on the order they receive. Any unsold item is sold to a vendor at a cheaper rate $r_1 < s_1$ and $r_2 < s_2$, respectively. The demands for the two items are represented by random variables D_1 and D_2 that follow a joint distribution $H_{D_1,D_2}(t_1,t_2) = \mathbb{P}(D_1 \le t_1, D_2 \le t_2)$, where the **joint density** is:

$$h_{D_1,D_2}(t_1,t_2) = \begin{cases} K(\alpha\, t_1 + \beta\, t_2^2), & \text{if } 0 \le t_1 \le u_1 \text{ and } 0 \le t_2 \le u_2 \\ 0, & \text{otherwise} \end{cases}$$

where α, β, K are given positive parameters. Then, the probability of $D_1 \in [\ell_1, u_1]$ and $D_2 \in [\ell_2, u_2]$ is

$$\mathbb{P}(\ell_1 \le D_1 \le u_1, \ell_2 \le D_2 \le u_2) = \int_{t_1=\ell_1}^{u_1} \int_{t_2=\ell_2}^{u_2} h_{D_1,D_2}(t_1,t_2)\,dt_1 dt_2.$$

Determine the optimal quantity of each type of item to produce in order to maximize the expected profit.

Hint: Let $x_1^\star$ and $x_2^\star$ be the optimal number of items of each type that should be produced, in order to maximize the profit. First show that $x_1^\star$ and $x_2^\star$ have the same expression as in the formula shown for the single item case (e.g., on slide 25 of Chapter 1), then substitute the cdf (obtained with the help of the given density function).

2 [20pts] Scotland is renowned for its commitment to renewable energy and its vast wind energy resources. A leading energy company is planning the development of a new wind farm to meet electricity demand over a time horizon consisting of a set of time periods T. The wind farm can utilize a set of turbine types K, where each type $k \in K$ has distinct characteristics. The company must decide how many turbines of each type to install now to operate in the future $|T|$-period horizon. The electricity generated by these turbines depends on uncertain wind

speeds, which vary across time periods. Any shortfall in electricity generation during a time period must be covered by purchasing electricity from external suppliers at a premium price.

Let g_{tk} denote the random electricity generation (in MWh) of one turbine of type $k \in K$ during time period $t \in T$, while d_t denotes the random electricity demand in the same period. The installation cost for one turbine of type k is given by f_k, the maintenance cost for operating a turbine of type k during one time period is given by v_k (while zero operational/maintenance cost is incurred for a generator if it has not been put into operational mode during a time period), and the cost of outsourcing one MWh of electricity from external suppliers is given by o. The company's objective is to minimize the total expected cost, which includes installation costs, operational/maintenance costs, and outsourced electricity costs.
(P.S.There is no energy storage available in the system.)

Write a two-stage stochastic programming model for this problem. Please explain your model in detail.

3 When the Cannich wildfire devastated the Scottish Highlands in 2023, the water sources around the impacted areas became swiftly polluted and the access of numerous residents to safe drinking water got interrupted. Even to date, those areas are still struggling with the long-lasting impacts of the wildfires on its forested watersheds. The government has decided to invest in preventive measures that can immediately take action should another such wildfire occur. This includes pre-positioning of emergency water treatment plants containing (costly) chemicals for rapidly treating the water in the event of wildfires. Because such investments are quite expensive, the government needs to make the best decisions regarding opening the plants and their inventory levels. Assuming that up to $\Gamma \in \mathbb{N}$ plants might get hit by a disaster, the inventory levels of the opened plants have to be enough to accommodate for the absence of the damaged ones, so that the demand for drinking water can always be met for all residents.

There are different types of chemicals, $p \in P$, that need to be stored in the water treatment plants, each with unit storage cost c_p and deterministic demand d_p. Plants themselves can be of different types, $k \in K$, each with its own characteristics. The installation cost of a water treatment plant of type k is G_k, and it is required to have at least L_{kp}, but no more than U_{kp}, inventory level of chemical of type p. For these plants, n potential locations have been identified, indexed by $i \in I$. At most one plant can be built at each location. Scenarios $\xi \in \Xi$ are characterized by parameters $a_i(\xi)$ which is equal to 1 if the plant $i \in I$ remains functional under scenario $\xi \in \Xi$, and 0 otherwise.

The goal is to determine the emergency water treatment plants to be opened, their type and inventory levels of chemicals, such that the demand is met even when up to Γ plants become inaccessible after a wildfire while minimizing total cost.

(a) [20pts] What modeling paradigm is more suitable for this problem and why? Accordingly, write down a mathematical program for the natural disaster preparedness problem; please explain it in detail.

(b) [5pts] At most, how many scenarios are possible for this problem? That is, what is the largest value of $|\Xi|$?

4. [10pts] Consider the following function:

$$F(x, \ell_1, \ell_2, \ell_3) = e^{-\sin(|\ell_1 x|)} + \frac{\lfloor x - \ell_2 \rfloor}{\sqrt{(x + \ell_3)}}$$

where ℓ_1, ℓ_2 and ℓ_3 are independent random variables with the following distributions:

- ℓ_1: Cauchy distribution with location $x_0 = -2$ and scale $\gamma = 2$
- ℓ_2: Poisson distribution with mean $\lambda = 5$
- ℓ_3: Uniform distribution with minimum $a = 3$ and maximum $b = 10$

For a candidate solution $\hat{x} = 16$, estimate $\mathbb{E}[F(\hat{x}, \ell_1, \ell_2, \ell_3)]$ using 15000 scenarios with equal probabilities.

5. **(Quick research)** For each of the below tasks, find a published paper or a preprint available online (e.g., in Optimization Online, arXiv, SSRN, etc.) and provide *brief* answers.

(a) [10pts] For an application of *two-stage stochastic programming*, describe the problem and the evolution of information therein (i.e., clearly identify the decision stages, random variables, decisions to be made at each stage), mention the form of the assumed probability distribution of the random variables, and provide a justification why stochastic programming might be a suitable paradigm for the studied problem.

(b) [10pts] For an application of *(static or two-stage adaptive) robust optimization*, describe the problem and the evolution of information therein (i.e., clearly identify the decision stages, random variables, decisions to be made at each stage), mention the form of the used uncertainty set, and provide a justification why robust optimization might be a suitable paradigm for the studied problem.

(c) [10pts] For an application of *(static or two-stage adaptive) chance-constrained programming*, describe the problem and the evolution of information therein (i.e., clearly identify the decision stages, random variables, decisions to be made at each stage), mention the form of the used uncertainty model, and provide a justification why chance-constrained programming might be a suitable paradigm for the studied problem.