Darinka Dentcheva darinka.dentcheva@stevens.edu

Homework 5
Solutions

Problem 1. Consider the chess game from Assignment 2. Assume that if you play timid your probability of making a draw is p = 0.8 and the probability to win is the same as the probability to lose.

(a) Solve the problem again with this new probability setting.

(b) Investigate how your strategy changes when the probability of wining changes in the case of a timid play. The probability of wining can change from 0 to 0.15.

In this problem, we have to re-calculate the value of the tie-position: $v_6^*(0)$. In order to do this, we can set up a new infinite-horizon dynamic programming problem to solve the question of optimal strategy in the "suddent death" phase. Any state different than 0 is absorbing. For this problem, the state space is $\{-1, 0, 1\}$ with the states designating loss, tie, and win, respectively. Any state different than 0 is absorbing. The control space is $\{0, 1\}$ with $u = 0$ for timid and $u = 1$ for bold. We have the following dynamic programming equation

$$\begin{aligned}
\tilde{v}(-1) &= 0, \\
\tilde{v}(1) &= 1, \\
\tilde{v}(0) &= \max\{0.1\tilde{v}(1) + 0.8\tilde{v}(0) + 0.1\tilde{v}(-1), 0.45\tilde{v}(1) + 0.55\tilde{v}(-1)\} \\
&= \{0.1 + 0.8\tilde{v}(0), 0.45\}
\end{aligned}$$

This is a positive model. If we play timid in state $x = 0$, then

$$0.1 + 0.8\tilde{v}(0) = \tilde{v}(0) \quad \Rightarrow \quad \tilde{v}(0) = 0.5.$$

On the other hand, if we play bold in state $x = 0$, then $\tilde{v}(0) = 0.45 < 0.5$. Therefore, playing timid is optimal; we can verify this also numerically. Denoting the probability to win when you play timid by p_w, $p_w \in (0, 0.2)$, the dynamic programming equation is as follows:

$$\tilde{v}(0) = \max\{p_w + 0.8\tilde{v}(0), 0.45\}$$

It is good to play timid if

$$p_w + 0.8\tilde{v}(0) \geq 0.45 \quad \Leftrightarrow \quad \tilde{v}(0) \geq \frac{0.45 - p_w}{0.8}$$

In that case also

$$\tilde{v}(0) = p_w + 0.8\tilde{v}(0) \quad \Leftrightarrow \quad \tilde{v}(0) = \frac{p_w}{0.2}.$$

Putting the two together, we obtain that it is good to play timid in state $x = 0$, when

$$\frac{0.45 - p_w}{0.8} \le \frac{p_w}{0.2} \quad \Leftrightarrow \quad p_w \ge 0.09.$$

Problem 2. We have a tree farm. At any time, the size s of a tree is $0, 1, 2, 3, 4$, where 0 means that the tree has died, and 4 is the size of a mature tree. We need to decide when to harvest a given tree. Each year it costs about \$ 10+s to maintain a tree, and \$ 30+5s to harvest a tree. The sales price of a tree of each size is as follows:

Tree size	1	2	3	4
Sales Price	\$ 150	\$ 180	\$ 210	\$ 260

The transition probability matrix for the size of the tree is as follows:

sizes	0	1	2	3	4
0	1	0	0	0	0
1	0.05	0.15	0.7	0.1	0
2	0.05	0	0.2	0.7	0.05
3	0.05	0	0	0.5	0.45
4	0.05	0	0	0	0.95

(a) Describe a dynamic programming problem to determine an optimal harvesting policy.

- State: the size of the tree $\mathcal{X} = \{0, 1, 2, 3, 4\}$.
- Control set: $\mathcal{U} = \{0, 1\}$ maintain $u = 0$ or harvest $u = 1$.
- Feasible control mapping: $U(0) = \{0\}$, for $i = 1, 2, 3, 4$, we have $U(i) = \{0, 1\}$.
- The transition probabilities under $u = 0$, $p(j|i, 0)$, are given by the matrix in the problem formuation, while under $u = 1$, we have for any $i > 0$:

$$p(j|i,1) = \begin{cases} 1 & \text{for } j = 0; \\ 0 & \text{for } j > 0. \end{cases}$$

- Dynamic Programming Equation:

$$v^*(i) = \max \left\{ -(10 + i) + \sum_{j=0}^{4} p(j|i,0)v^*(j); \;\; h_i - (30 + 5i) \right\}, \quad i = 1, 2, 3, 4.$$

$$v^*(0) = 0.$$

Here h_i is the selling price of a tree of size i.

(b) Solve the problem numerically. What numerical methods are applicable to this problem and why.

This is an optimal stopping problem with finitely many states. We can use value iteration and linear programming to solve the problem. The optimal value function is

$$v^*(0) = 0, \quad v^*(1) = 123.824, \quad v^*(2) = 142.5, \quad v^*(3) = 165, \quad v^*(4) = 210.$$

The optimal policy is to grow the tree in sizes 1 and 2 and to harvest in sizes 3 and 4.