Solution:

(a) State space: $X = \{1, 2\}$ – 1- Good, 2-Bad

Control space: $U = \{0, 1\}$ – 0- inspect, 1- do not inspect

Observation space: $O = \{1, 2, \dots, T\}$

Transition probabilities:

$$p(x_{t+1} = 1, o_{t+1} = o + 1 | x_t = 1, u_t = 0, o_t = o) = (1 - p)^o$$
$$p(x_{t+1} = 2, o_{t+1} = 1 | x_t = 1, u_t = 0, o_t = o) = (1 - p)^{o-1} p$$
$$p(x_{t+1} = 1, o_{t+1} = o + 1 | x_t = 1, u_t = 1, o_t = o) = (1 - p)^o$$
$$p(x_{t+1} = 2, o_{t+1} = 1 | x_t = 1, u_t = 1, o_t = o) = (1 - p)^{o-1} p$$
$$p(x_{t+1} = 1, o_{t+1} = 1 | x_t = 2, u_t = 0, o_t = o) = 1$$
$$p(x_{t+1} = 2, o_{t+1} = o + 1 | x_t = 2, u_t = 1, o_t = o) = 1$$

The probabilities of other conditions are 0.

Cost function:

$$c(x, u, o) = \begin{cases} C & , x = 2, u = 1, o = o \\ I + R & , x = 2, u = 0, o = o \\ 0 & , x = 1, u = 1, o = o \\ I & , x = 1, u = 0, o = o \end{cases}$$

Dynamic programming equations:

$$v_t{}^*(x, o) = \inf_{u \in U_t(x)} [c_t(x, u, o) + E\{v_{t+1}{}^*(x_{t+1}, o_{t+1}) | x_t = x, u_t = u,\ o_t = o\}],$$

$x \in X, o \in O, t = 1, \dots, T - 1$

$v_T{}^*(x, o) = c_T(x, o)\,, \quad x \in X$

So,

$$v_t{}^*(x, o) = min \Bigg\{ c_t\,(x, 0, o) + \sum_{x_{t+1} \in X, o_{t+1} \in O} p(x_{t+1}, o_{t+1} | x_t, 0, o_t) v^*(x_{t+1}, o_{t+1}),\ c_t\,(x, 1, o) + \sum_{x_{t+1} \in X, o_{t+1} \in O} p(x_{t+1}, o_{t+1} | x_t, 1, o_t) v^*(x_{t+1}, o_{t+1}), \Bigg\}$$

$$v_T{}^*(x) = c_T(x)$$

(b) If the initial state of the machine is good, and $p = 0.2,\ I = 1,\ R = 3,\ C = 2,\ T = 18$.

Transition probabilities:

$$p(x_{t+1} = 1, o_{t+1} = o + 1 | x_t = 1, u_t = 0, o_t = o) = (0.8)^o$$
$$p(x_{t+1} = 2, o_{t+1} = 1 | x_t = 1, u_t = 0, o_t = o) = (0.8)^{o-1} \times 0.2$$
$$p(x_{t+1} = 1, o_{t+1} = o + 1 | x_t = 1, u_t = 1, o_t = o) = (0.8)^o$$
$$p(x_{t+1} = 2, o_{t+1} = 1 | x_t = 1, u_t = 1, o_t = o) = (0.8)^{o-1} \times 0.2$$
$$p(x_{t+1} = 1, o_{t+1} = 1 | x_t = 2, u_t = 0, o_t = o) = 1$$
$$p(x_{t+1} = 2, o_{t+1} = o + 1 | x_t = 2, u_t = 1, o_t = o) = 1$$

Reward function:

$$c(x,u,o)=\begin{cases}2 & ,x=2,u=1,o=o\\ 4 & ,x=2,u=0,o=o\\ 0 & ,x=1,u=1,o=o\\ 1 & ,x=1,u=0,o=o\end{cases}$$

$$v_t{}^*(1,o)=min\{1+(0.8)^o v_{t+1}(1,o+1)+(0.8)^{o-1}\times 0.2v_{t+1}(2,1),0+(0.8)^o v_{t+1}(1,o+1)$$
$$+(0.8)^{o-1}\times 0.2v_{t+1}(2,1)\}$$

$$v_t{}^*(2,o)=min\{4+v_{t+1}(1,1),2+v_{t+1}(2,o+1)\}$$

$$t=1,\dots,17$$

$$v_{18}{}^*(1,o)=\min\{0,1\}=0$$

$$v_{18}{}^*(2,o)=\min\{2,4\}=2$$

In [37]:

```
import pandas as pd
import numpy as np
import random
```

In [63]:

```
def v(t, x, o):
    if t==18 and x==0:
        vt=0
    elif t==18 and x==1:
        vt=2
    elif t<18 and x==0:
        vt=0.8**o*v(t+1,x,o+1)+0.8**(o-1)*0.2*v(t+1,x+1,1)
    elif t<18 and x==1:
        vt=min(4+v(t+1,x-1,1), 2+v(t+1,x,o+1))
    return vt

i=17
j=1
while i>0:
    while j<=i:
        print('v', i, '(1,', j, ')=', v(i,0,j))
        print('v', i, '(2,', j, ')=', v(i,1,j))
        j+=1
    j=1
    i-=1
```

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v 17 (1, 1 )= 0.4
v 17 (2, 1 )= 4
v 17 (1, 2 )= 0.32000000000000006
v 17 (2, 2 )= 4
v 17 (1, 3 )= 0.25600000000000006
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v 17 (1, 4 )= 0.20480000000000007
v 17 (2, 4 )= 4
v 17 (1, 5 )= 0.16384000000000004
v 17 (2, 5 )= 4
v 17 (1, 6 )= 0.13107200000000005
v 17 (2, 6 )= 4
v 17 (1, 7 )= 0.10485760000000005
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v 1 (2, 1 )= 6.999048961434633
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In [ ]: