# Discussions on Simulation of Max-ID Processes


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# Max-ID Processes

For the convenience of notations, we adopt the same notations as in Dombry and Eyi-Minko (2013).

General Representation for Max-ID Processes:

A max-id process $\eta(t)$, $t \in T$ can be represented by

$$\eta(t) = \bigvee_{i=1}^{N} \phi_i(t), t \in T; \quad \Phi = \sum_{i=1}^{N} \delta_{\phi_i}$$

where $\phi_i(t)$ are continuous functions (random variables) on T, and $\Phi$ is the Poisson point process (or Poisson random measure) with mean measure $\mu$ on the space of continuous functions on $T$, denoted by $\mathcal{C}$. $\phi_i$ are called the atoms of $\Phi$.
$N = \Phi(\mathcal{C}) \sim \mathit{Poisson}(\mu(\mathcal{C}))$ if $\mu(\mathcal{C})$ is finite, otherwise $N = +\infty$ almost surely.

# More Notations

- $[\Phi] = \{\phi_i\}_{i=1}^N$, is the set of atom functions of $\Phi$.
- $\max(\Phi)(t) = \max\{f(t); f \in [\Phi]\}$, which equals to $\eta(t)$ in distribution.
- W.L.O.G, assume $\mathcal{C}$ is the space of non-negative and non-null continuous functions on $T$.
- $M(\mathcal{C})$ be the space of point processes $M = \sum_{i \in I} \delta_{f_i}$ on $\mathcal{C}$ such that $\{f_i \in \mathcal{C} : \|f_i\| = sup_t(f_i(t)) > \epsilon)\}$ is finite $\forall \epsilon > 0$.
- Let $K \subset T$ be a closed subset of $T$, we have the following notations,

$$\begin{cases} f_1 =_K f_2 & \Leftrightarrow \quad \forall\, t \in K,\ f_1(t) = f_2(t) \\ f_1 <_K f_2 & \Leftrightarrow \quad \forall\, t \in K,\ f_1(t) < f_2(t) \\ f_1 \not< f_2 & \Leftrightarrow \quad \exists\, t \in K,\ f_1(t) \geq f_2(t). \end{cases}$$

- Let $\boldsymbol{s} = (s_1, \dots, s_\ell) \in T^\ell$ and $f(\boldsymbol{s}) = (f(s_1), \dots, f(s_\ell))$ we denote $\mu_s(A) = \mu(\{f \in \mathcal{C} : f(s) \in A\})$, $A \subset [0, +\infty)^\ell \backslash \{0\}$ and $\bar{\mu}_s(x) = \mu(\{f \in \mathcal{C}; f(s) \not< x\})$, $x \in [0, +\infty)^\ell \backslash \{0\}$.

# K-extremal functions and K-sub-extremal functions

Definition (Extremal Functions)

An atom function $f \in [\Phi]$ is called $K$-sub-extremal if and only if $f <_k \max(\Phi)$. Otherwise, we call it $K$-extremal function. In other words, a sub-extremal function has no contribution to the maximum $\max(\Phi)$ on $K$.

Definition (Decomposition of PPP)

We can decompose the PPP $\Phi$ to $K$-extremal PPP and $K$-sub-extremal PPP as following

$$\Phi_K^+ = \sum_{i=1}^{N} 1_{\{\phi_i \not<_K \eta\}} \delta_{\phi_i}; \quad \Phi_K^- = \sum_{i=1}^{N} 1_{\{\phi_i <_K \eta\}} \delta_{\phi_i}$$

# Hitting Scenarios

Definition (Hitting Scenarios)

A hitting scenarios on $K = \{t_1, t_2, \ldots, t_k\}$ is a ordered partition set $\Theta = (\theta_1, \ldots, \theta_{\ell(\Theta)}) \in \mathcal{P}_K$ of $K$.

Let $\{P_s(x, df);\, x \in [0, +\infty)^I \backslash \{0\}\}$ be the conditional measure $\mu(df)$ given $f(s) = x$.

# Regular Conditional Distributions for Max-ID Processes

- Let

$$\pi_t(y,\cdot) = Pr[\Theta \in \cdot | \eta(t) = y];$$
$$Q_t(y,\tau,\cdot) = Pr[(\phi_j^+) \in \cdot | \eta(t) = y, \Theta = \tau]$$
$$R_t(y,\tau,(f_j),\cdot) = Pr[\Phi_K^{-1} \in \cdot | \eta(t) = y, \Theta = \tau, (\phi_j^+) = (f_j)]$$

Theorem (3.2 in Dombry and Eyi-Minko (2013))

*Suppose $C_t^-(y) = \{M \in M(\mathcal{C}); \forall f \in [M], f(t) < y\}$ and assumption A holds, we have*

$$\pi_t(y,\tau) = \frac{d\nu_t^\tau}{d\nu_t}(y) \tag{1}$$

$$Q_t(y,\tau,df_1\cdots df_l) = \bigotimes_{j=1}^{\ell} \left\{ \frac{1_{\{f_j(t_{\tau_j^c}) < y_{\tau_j^c}\}} P_{t_{\tau_j}}(y_{\tau_j}, df_j)}{P_{t_{\tau_j}}(y_{\tau_j}, \{f(t_{\tau_j^c}) < y_{\tau_j^c}\})} \right\} \tag{2}$$

$$R_t(y,\tau,(f_j),B) = R_t(y,B) = \frac{Pr[\Phi \in B \cap C_t^-(y)]}{Pr[\Phi \in C_t^-(y)]}. \tag{3}$$

# Conditional Independence

From (3), we have conditional independence between $\Phi_K^-$ and $\Phi_K^+$ given $\eta(t) = y$.

Theorem (4.1 in Dombry and Eyi-Minko (2013))

*Let $t \in T$ and suppose that the distribution for $\eta(t)$ has no atom. Then conditionally on $\eta(t) = y, \phi_t^+$ and $\Phi_{\{t\}}^-$ are independent. The conditional distribution of $\phi_t^+$ is equal to $P_t(y, \cdot)$. The conditional distribution of $\Phi_{\{t\}}^-$ is equal to the distribution of a Poisson point measure with intensity $1_{\{f(t)<y\}}\mu(df)$. Furthermore, for $\ell \geq 1$, $\boldsymbol{s} \in T^\ell$ and $\boldsymbol{z} \in [0, \infty)^\ell$, we have*

$$P[\eta(\boldsymbol{s}) < \boldsymbol{z} | \eta(t) = y] = P_t(y, \{f(s) < z\}) \exp[-\mu(\{f(\boldsymbol{s}) \not< \boldsymbol{z}, f(t) < y\})]$$

# Simulation via Extremal Functions

Theorem (Theorem based on 2 in Dombry *et al.* (2016))

*The distribution of* $(\phi^+_{x_n})_{1\leq n\leq N}$ *is given as follows:*

- *1. Initial distribution: the extremal function* $\phi^+_{x_1}$ *is distributed as* $P_{x_1}(Y,\cdot)$ *given* $Y$*, where* $Y$ *is distributed the same as the marginal distribution of* $\eta$*.*
- *2. Conditional distribution: for* $1 \leq n \leq N-1$*, the conditional distribution of* $\phi^+_{x_{n+1}}$ *with respect to* $(\phi^+_{x_i})_{1\leq i\leq n}$ *is equal to the distribution of*

$$\tilde{\phi}^+_{x_{n+1}} = \begin{cases} \arg\max_{\phi\in[\tilde{\Phi}_{\{x_1,\ldots,x_{n+1}\}}]} \phi(x_{n+1}), & [\tilde{\Phi}_{\{x_1,\ldots,x_{n+1}\}}] \neq \emptyset \\ \arg\max_{\phi\in[\Phi^+_{\{x_1,\ldots,x_n\}}]} \phi(x_{n+1}), & [\tilde{\Phi}_{\{x_1,\ldots,x_{n+1}\}}] = \emptyset \end{cases}$$

*where* $[\tilde{\Phi}_{x_1,\ldots,x_{n+1}}]$ *is a PPP with intensity*
$1_{\{f(x_i)<Z_n(x_i), i\leq i\leq n\}} 1_{\{f(x_{n+1})>Z_n(x_{n+1})\}} \mu(df)$ ,
$Z_n(x) = \max_{\phi\in\Phi^+_{\{x_1,\ldots,x_n\}}} \phi(x) = \max_{1\leq i\leq n} \phi^+_{x_i}(x)$.

# Algorithms

**Algorithm:** Simulation of a max-id process $Z$, exact at $\boldsymbol{x} = (x_1, \ldots, x_N)$.

Simulate $-\log(G(z)) \sim \mathrm{Exp}(1)$ and $\boldsymbol{Y} \sim P_{x_1}(z, \cdot)$, where $G$ is the marginal distribution.

Set $Z(\boldsymbol{x}) = \boldsymbol{Y}(\boldsymbol{x})$.

For $n = 2, \ldots, N$:

    Simulate $-\log(G(z)) \sim \mathrm{Exp}(1)$.

    while $(z > Z(x_n))$ $\{$

        Simulate $\boldsymbol{Y} \sim P_{x_n}(z, \cdot)$.

        If $Y(x_i) < Z(x_i)$ for all $i = 1, \ldots, n-1$,

            update $Z(\boldsymbol{x})$ by the componentwise $\max(Z(\boldsymbol{x}), Y())$

        Simulate $-\log(G(e)) \sim \mathrm{Exp}(1)$ and update $-\log(G(z))$ by $-\log(G(z)) - \log(G(e))$

    $\}$

Return $Z(\boldsymbol{x})$

# Algorithms for $P_x(z, \cdot)$

- What I don't understand well:
- $P_x(z, \cdot)$ is the conditional distribution of $\phi(x)$ given $\phi(x) = z$.
- From Theorem 4.1 in Dombry and Eyi-Minko (2013), we know that the conditional distribution of $\phi(x)^+$ is equal to $P_x(z, \cdot)$ given $\eta(x) = z$.
- Why are they equal?

Apart from this, if we have the spectral representation $\phi_i(\boldsymbol{x}) = R_i W_i(\boldsymbol{x})$, where $R_i$ is a Poisson point process on $[0, \infty)$ and $W_i(x)$ is a Gaussian random field. There are many ways to sample $R_i W_i(\boldsymbol{x})$ given its value at $x_0$.

- MCMC
- Adaptive Rejection Sampling

# Simulation Study: Comparison based on Extremal Coefficients I

Result file

# The End

# References I

Dombry, C., Engelke, S. and Oesting, M. (2016) Exact simulation of max-stable processes. *Biometrika* **103**, 303–317.

Dombry, C. and Eyi-Minko, F. (2013) Regular conditional distributions of continuous max-infinitely divisible random fields. *Electronic Journal of Probability* **18**, 1–21.