Assignment No.5

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Download latex-tikz codes from

https://github.com/Panisha707/ASSIGNMENT05/blob/main/main.tex

Download python codes from

https://github.com/Panisha707/ASSIGNMENT05/blob/main/untitled31.py

Question taken from

quad_form, exercises 2.24

1 Question No 1

Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$

2 Solution

Given equation of the parabola is:

$$y^2 = 8x \tag{2.0.1}$$

$$y^2 - 8x = 0 \tag{2.0.2}$$

$$y^2 + 2(-4) = 0 \tag{2.0.3}$$

comparing it with standard equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.4}$$

$\therefore$ a = b = e = 0, d = -4, c = 1, f = 0

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{2.0.6}$$

Lemma 2.1. *The equation of a parabola is:*

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.7}$$

Then its vertex can be calculated as :

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -\mathbf{f} \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.8}$$

$$\textit{where, } \eta = \mathbf{u^T}\mathbf{p_1} \tag{2.0.9}$$

Equation of the parabola can be written as

$$\implies \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -4 & 0 \end{pmatrix}\mathbf{x} + 0 = 0 \tag{2.0.10}$$

We can find the eigen values corresponding to the $\mathbf{V}$,

$$|\mathbf{V} - \lambda\mathbf{I}| = \left|\begin{pmatrix} 0-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}\right| \tag{2.0.11}$$

$$(-\lambda)(1-\lambda) = 0 \tag{2.0.12}$$

$\therefore$ Eigen values are $\lambda_1 = 0, \lambda_2 = 1$
Calculating the eigen vectors corresponding to $\lambda_1 = 0, \lambda_2 = 1$ respectively

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \tag{2.0.13}$$

$$\implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.14}$$

$$\implies \mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.15}$$

The vertex of the parabola can be given as

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -\mathbf{f} \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.16}$$

$$\text{where, } \eta = \mathbf{u^T}\mathbf{p_1} = \begin{pmatrix} -4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.17}$$

$$\begin{pmatrix} -8 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.18}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.19}$$

Since $\lambda_2 > \lambda_1$
Hence, the axis using $\mathbf{p_2}$ is given by

$$\mathbf{p_2}^T\left(\mathbf{x} - \mathbf{c}\right) = 0 \tag{2.0.20}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{2.0.21}$$

$$\implies y = 0 \tag{2.0.22}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.23}$$

Theorem 2.1. *The eccentricity, directrices and foci of parabola are given by*
Eccentricity,

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{2.0.24}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1 \tag{2.0.25}$$

$$c = \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2e^2\mathbf{u}^\top\mathbf{n}} \tag{2.0.26}$$

Focus,

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{2.0.27}$$

Directrix,

$$\mathbf{n}^T\mathbf{x} = c \tag{2.0.28}$$

From Equation 2.0.24, eccentricity can be calculated as

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{2.0.29}$$

$$e = \sqrt{1 - \frac{0}{1}} = \sqrt{1} \tag{2.0.30}$$

$$\implies e = 1 \tag{2.0.31}$$

Substituting λ_2 and $\mathbf{p_1}$ values in Equation 2.0.25

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1 \tag{2.0.32}$$

$$\mathbf{n} = \sqrt{1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.33}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.34}$$

Equation 2.0.26 can be calculated as

$$c = \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2e^2\mathbf{u}^\top\mathbf{n}} \tag{2.0.35}$$

$$c = \frac{16 - 1 \times 0}{2 \times 1^2 \times \begin{pmatrix} -4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \tag{2.0.36}$$

$$= \frac{16}{-8} \tag{2.0.37}$$

$$\implies c = -2 \tag{2.0.38}$$

Focus of the parabolic equation can be calculated from the equation 2.0.27

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{2.0.39}$$

$$\mathbf{F} = \frac{-2 \times 1 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -4 \\ 0 \end{pmatrix}}{1} \tag{2.0.40}$$

$$\implies \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{2.0.41}$$

Directrix of the parabolic equation can be calculated from equation 2.0.28

$$\mathbf{n}^T\mathbf{x} = c \tag{2.0.42}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2 \tag{2.0.43}$$

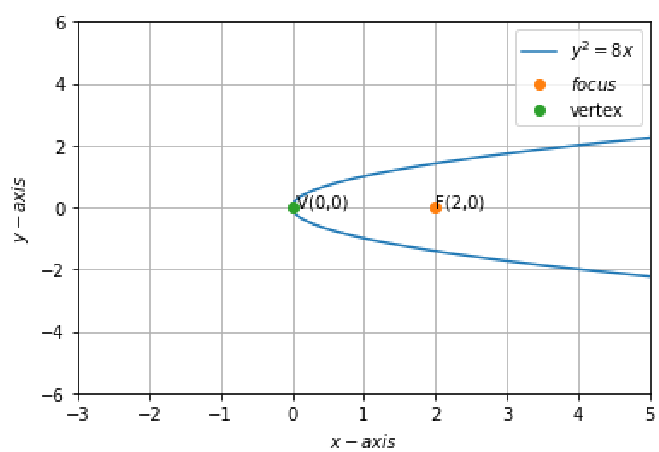

Fig. 2.1: Parabola