This assignment consists of two parts.

Please use Python for implementation using only standard python libraries (e.g., numpy). **Do not use any third party libraries/implementations for algorithms**. For plotting, please use plotly library as it can generate interactive html plots. Here is an example showing how to plot 3D plots in plotly. Export all your plots in html format and submit them along with your code.

Please prepare a report to accompany your implementation. The report should contain detailed responses to questions. Briefly describe the key findings/insights for the graphs. Ensure the reproduction of graphs (modulo probabilistic execution) for your submission.

Additional readings: Artificial Intelligence: A Modern Approach (Ch. 15) and Probabilistic Robotics (Ch. 2 and Ch. 3).

Please refer to the submission instructions on the course webpage.

1. State Estimation using Kalman Filters (60 points)

Consider an airplane flying in a 3D world. A noisy sensor (e.g., GPS positions) provides measurements $z_t = [x'_t, y'_t, z'_t]$. The plane is controlled by providing velocity increments $u_t = [\delta\dot{x}_t, \delta\dot{y}_t, \delta\dot{y}_t]$, which get added to the velocity components $\dot{x}_t$, $\dot{y}_t$ and $\dot{y}_t$ at each time step. The uncertainty in the motions is characterized by a presence of Gaussian noise $\epsilon_t \sim \mathcal{N}(0, R)$. Similarly, the measurement uncertainty is characterized by presence of Gaussian noise $\delta_t \sim \mathcal{N}(0, Q)$. Our goal is to estimate its positions $[x_t, y_t, z_t]$ and velocities $[\dot{x}_t, \dot{y}_t, \dot{z}_t]$ at time instant t from noisy observations $[x'_t, y'_t, z'_t]$.

Environment Setup and Filtering (10 points)

(a) Implement the motion model for this problem. Initially, assume that the control inputs are all zero. Assume the noise parameters as: σ_{rx}, σ_{ry}, $\sigma_{rz} = 1.0$ and $\sigma_{r\dot{x}}$, $\sigma_{r\dot{y}}$, $\sigma_{r\dot{z}} = 0.008$ forming the covariance matrix R as $diag(\sigma^2_{rx}, \sigma^2_{ry}, \sigma^2_{rz}\sigma^2_{r\dot{x}}, \sigma^2_{r\dot{y}}, \sigma^2_{r\dot{z}})$ where $diag$ denotes the diagonal elements of R with off-diagonals as zero. Next, implement the observation model for this problem. Assume that the observation noise is distributed as an isotropic Gaussian with a standard deviation of $\sigma_s = 8$. Simulate the motion and the sensor models for $T = 300$ time steps. Plot the actual trajectory and the observed trajectory of the vehicle.

(b) Implement a Kalman filter for the problem to estimate the vehicle state given the assumptions above. Please formally write down the model for estimation. Let $\hat{x}_t$ denote the estimated state at time t. Implement a control policy where the velocity $\delta\dot{x}_t$ varies as a cosine wave and the velocities $\delta\dot{y}_t$ and $\delta\dot{z}_t$ vary as sine waves. Assume that the vehicle is initially at rest and at starting position $(0, 0, 0)$. Assume a prior belief over the vehicle's initial state has a standard deviation of 0.008 for each state variable.

Experiments (25 points)

(c) Plot the actual trajectory $[x_t, y_t, z_t]$, the noisy observations $[x'_t, y'_t, z'_t]$ and the trajectory estimated by the filter $[\hat{x}_t, \hat{y}_t, \hat{z}_t]$. Additionally, plot the uncertainty ellipses for the projection of the estimated trajectory on the XY plane. An uncertainty ellipse denotes the locus of points that are one standard deviation away from the mean.

(d) We have described 3 types of noise parameters : noise in position update (σ^2_{rx}, σ^2_{ry}, σ^2_{rz}), noise in velocity update ($\sigma_{r\dot{x}}$, $\sigma_{r\dot{y}}$, $\sigma_{r\dot{z}}$) and noise in the sensor measurements(σ_s). Qualitatively describe how the trajectories (actual, observed and estimated) and uncertainty ellipses change on varying the each of the 3 noise parameters. Explain your observations.

(e) Assume that the sensor observations drop out at time instants $t = 50$ and $t = 200$ for a period of 30 time steps and are re-acquired after that period. Simulate and show the evolution of uncertainty in the plane's position projected on XY plane $[x_t, y_t]$ by plotting the uncertainty ellipses. Explain your findings.

(f) Assume now that the X-sensor stops working at t = 100 but Y and Z sensors continue to work as usual. Plot the graphs of estimated and actual trajectory of the airplane in the XY plane. Also show uncertainty ellipses and explain your findings.

(g) Plot the estimated velocities $[\hat{\dot{x}}_t, \hat{\dot{y}}_t, \hat{\dot{z}}_t]$ and the true velocities of the vehicle $[\dot{x}_t, \dot{y}_t, \dot{z}_t]$. Briefly explain if the estimator can or cannot track the true values.

Data association (25 points)

(h) Simulate a second vehicle in the environments with linear-Gaussian motion and sensor models (as above) with different initial state estimates and noise characteristics. Let a and b index the two vehicles. The sensor receives two sets of measurements, $s_t^1 = [x_t'^1, y_t'^1, z_t'^1]$ and $s_t^2 = [x_t'^2, y_t'^2, z_t'^2]$. Estimate the latent states x_t^a and x_t^b at time t for the two vehicles. The estimator requires a strategy for associating observations s_t^1 and s_t^2 with the latent states x_t^a and x_t^b, known as the *data association* problem. Implement a data association strategy and study its behavior in your simulation.

(i) Scale your data association strategy to more number of agents. Explore how adding more agents impact your association strategy. Evaluate your approach on (4-5) agents.

2. Landmark Localization (40 points)

In this part, we extend the problem setup of the previous question (in the single agent setting) by incorporating an additional observation. For this part assume the airplane moves only in the XY plane.

Assume that there are certain landmarks (e.g., air traffic control towers) at the following known locations in the environment: $(150, 0)$, $(-150, 0)$, $(0, 150)$, $(0, -150)$ and $(25, 0)$. The aircraft can measure the Euclidean distance (range) to a landmark when its true position within a certain range of the landmark and the measurement is corrupted with Gaussian noise $\eta_t \sim \mathcal{N}(0, S)$. At any instant, the agent receives a single measurement from the nearest landmark and the measurement can be uniquely associated with the corresponding landmark. Note that the landmark-distance observation is in addition to the GPS-positions that the agent is receiving. The remaining problem setup is considered same as the previous question.

Extension for Landmark Observations (25 points)

(a) Formally describe how the additional *landmark*-distance observation can be incorporated in your estimator developed in the last question.

(b) Extend your simulation to account for the landmark observations as the agent moves through the environment. Assume an range of 50m for observing a landmark. Assume isotropic Gaussian noise with standard deviation of 0.01, 10 and 1 in the motion model, GPS-position observations and landmark-distance observations respectively.

Experiments (15 points)

(c) Simulate at least 1000 steps with the filter updates at 1Hz with the agent starting at $(-200, -50)$ with a velocity of 4 units with a 0.35 radian heading direction. Plot the true trajectory of the vehicle and the trajectory estimated by your filter. Overlay the uncertainty ellipses for the estimated trajectory and observe how the ellipse changes as the agent comes close to a landmark.

(d) Next, increase and decrease the uncertainty in the landmark measurements in relation to the uncertainty in the position measurements (as in the previous question). Vary the standard deviation in the landmark observations as 1 and 20 and explain your observations.

(e) Add an additional landmark in the environment and observe the impact on localization performance.