

# Differential Geometry of Surfaces

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# 1 Introduction

In this section, we will introduce some of the basic definition of curves and surfaces.

## 1.1 Curves

Intuitively, A curves can be thought as the trace of a moving particle in the space. Mathematical, a curves is defined to be the image of a function,  $\gamma : U \rightarrow \mathbb{R}^n$ , where  $U \subset \mathbb{R}$ .

**Definition 1** (Parametrised curve). A **parametrised curve** in  $\mathbb{R}^n$  is a smooth function  $\gamma : U \rightarrow \mathbb{R}^n$ , where  $U \subset \mathbb{R}$ .

Throughout, this report we will assume that smoothness mean  $C^\infty$ , i.e. the function is differentiable infinitely many times.

**Definition 2** (Regular curve). Let  $\gamma : U \rightarrow \mathbb{R}^n$  be a curve. It is called regular if its derivative is non-vanishing, i.e.  $\|\gamma'(t)\| \neq 0, \forall t \in U$ .

There are many different ways to parametrise a curve, e.g.  $\gamma(t) = (t, t^2)$  and  $\tilde{\gamma}(t) = (t^2, t^4)$ . However, only one of these curve is regular, which is  $\gamma(t)$ . Moreover, there are many different ways to parametrise a curves such that all the parametrisations are regular.

**Definition 3** (Unit speed curve). Let  $\gamma : U \rightarrow \mathbb{R}^n$  be a curve. It is called unit-speed, if  $\|\gamma'(t)\| = 1, \forall t \in U$ .

We will see later on that a lot of the formulas and results relating to curves take on a much simpler form when the curve is unit-speed, e.g. curvature of a unit-speed curve, see definition 4, is just the norm of it's second derivative.

**Proposition 1.** *A parametrised curve is unit-speed if and only if it is regular.*

proof ??

Explain in more detail why curvature is defined in the following manner:

**Definition 4** (Curvature of a curve). Let  $\gamma : U \rightarrow \mathbb{R}^n$  be a unit-speed curve. The curvature at point  $\gamma(t)$  is defined as

$$\kappa(t) = \|\gamma''\|$$

These are all the definitions and results about curves that we need to know to understand this report.

## 1.2 Surfaces

Intuitively, a surface is a subset of  $\mathbb{R}^3$  that looks like a  $\mathbb{R}^2$  in the neighbourhood of any given point, e.g. the surface of the Earth is spherical; however, it appear to be a flat plane( $\mathbb{R}^2$ ) to an observer on the surface.

**Definition 5** (Diffeomorphism). if  $f : U \rightarrow W$  is continuous, bijective, and smooth, and if its inverse maps  $f^{-1} : W \rightarrow U$  is also continuous and smooth, then f is called a diffeomorphism and  $U$  and  $W$  are called diffeomorphic.

**Definition 6** (Regular Surface). A subset of  $\mathbb{R}^3$  is a regular surface, if every point  $P \in S$ , there exists a open set  $U$  in  $\mathbb{R}^2$  and an open set  $W$  in  $\mathbb{R}^3$  containing  $P$  such that  $S \cap W$  is diffeomorphic to  $U$ .

Therefore, a surface is collection of diffeomorphisms,  $\sigma : U \rightarrow S \cap W$ , which we call regular surface patches.

**Definition 7** (Reparametrisation of surface patches). Let  $\sigma : U \rightarrow S$  and  $\tilde{\sigma} : \tilde{U} \rightarrow S$  be surface patches for a surface  $S$ , then  $\tilde{\sigma}$  is called a reparametrisation of  $\sigma$  if there exists a map,  $\Phi : \tilde{U} \rightarrow U$ , which is smooth and bijective with smooth inverse,  $\Phi^{-1} : U \rightarrow \tilde{U}$ .

**Definition 8** (Tangent space). Let  $S$  be a regular surface. The **tangent plane** to  $S$  at the point  $p \in S$  is the set of all initial velocity vectors of regular curves in  $S$  with initial position  $p$ , i.e

$$T_p S = \{\gamma'(0) | \gamma \text{ is a regular curve in } S \text{ with } \gamma(0) = p\}$$

These are all the definitions about surfaces that we need to understand this report.

## 2 First Fundamental Form

In this section, we will define one of the most important object that lets us compute lengths, angles and areas on surface. It is called the **first fundamental form**.

[Change this definition](#)

**Definition 9** (The fundamental form). The **first fundamental form** is the restriction of the inner product of the ambient space ( $\mathbb{R}^3$ ) to the tangent space ( $T_p S$ ) at point  $p \in S$ .

### 2.1 The first fundamental form in local coordinates

We'll now discuss the classical notation for expressing the first fundamental form in local coordinates. Suppose  $\sigma : U \rightarrow S$  is a surface patch and  $\gamma(t) : X \rightarrow S$  is a curve on the surface patch of  $S$ , defined as  $\gamma(t) = \sigma(u(t), v(t))$ . Let  $\gamma(t_0) = p \in S$ , then using the chain rule we know that  $\gamma' = \sigma_u u' + \sigma_v v'$ . We may then compute:

$$\begin{aligned} \langle \gamma', \gamma' \rangle &= \langle \sigma_u u' + \sigma_v v', \sigma_u u' + \sigma_v v' \rangle \\ &= \langle \sigma_u u', \sigma_u u' \rangle (u')^2 + \langle \sigma_u u', \sigma_v v' \rangle (u' v') + \langle \sigma_v v', \sigma_u u' \rangle v' u' + \langle \sigma_v v', \sigma_v v' \rangle (v')^2 \\ &= \langle \sigma_u u', \sigma_u u' \rangle (u')^2 + 2 \langle \sigma_u u', \sigma_v v' \rangle (u' v') + \langle \sigma_v v', \sigma_v v' \rangle (v')^2 \\ &= E(u')^2 + 2F u' v' + G(v')^2 \end{aligned} \tag{1}$$

where

$$E = \|\sigma_u\|^2, F = \|\sigma_u \sigma_v\|, G = \|\sigma_v\|^2 \tag{2}$$

**Definition 10** (the first fundamental form). The first fundamental form in local coordinates  $(u, v)$  is the expression  $\mathcal{F}_1 = Edu^2 + 2Fdudv + Gdv^2$ .

Now, a good question to ask would be, how reparametrisation of a surface affects the first fundamental form of the surface in terms of the local coordinates?

**Example 1.** Let  $S$  be a surface. Suppose  $\tilde{\sigma} : \tilde{U} \rightarrow S$  is a reparametrisation of  $\sigma : U \rightarrow S$ , where  $\sigma$  is a surface patch of  $S$  and  $U, \tilde{U} \subset \mathbb{R}^2$ . Assume that  $\Phi : \tilde{U} \rightarrow U$  is a smooth map and the following are the first fundamental form of  $\tilde{\sigma}$  and  $\sigma$ , respectively:

$$\tilde{E}d\tilde{u}^2 + 2\tilde{F}d\tilde{u}d\tilde{v} + Gd\tilde{v}^2 \text{ and } Edu^2 + 2Fdudv + Gdv^2$$

Then,

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J(\Phi)^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} J(\Phi) \tag{3}$$

, where  $J(\Phi)$  is the Jacobian of  $\Phi$ , i.e.

$$J(\Phi) = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}$$

*Proof.* We know that  $\tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma(\Phi(\tilde{u}, \tilde{v})) = \sigma(u, v)$  and  $\tilde{E} = \|\tilde{\sigma}_{\tilde{u}}\|^2$ , and  $E = \|\sigma_u\|^2$ . Then,

$$\begin{aligned} \tilde{E} &= \|\tilde{\sigma}_{\tilde{u}}\|^2 \\ &= \|(\sigma \circ \Phi)_{\tilde{u}}\|^2 \\ &= \left( \sigma_u \frac{\partial u}{\partial \tilde{u}} + \sigma_v \frac{\partial v}{\partial \tilde{u}} \right)^2 \\ &= \sigma_u^2 \frac{\partial u^2}{\partial \tilde{u}} + 2\sigma_u \sigma_v \frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{u}} + \sigma_v^2 \frac{\partial v^2}{\partial \tilde{u}} \\ &= E \frac{\partial u^2}{\partial \tilde{u}} + 2F \frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{u}} + G \frac{\partial v^2}{\partial \tilde{u}} \end{aligned}$$

Similarly,

$$\begin{aligned} \tilde{G} &= \sigma_u^2 \frac{\partial u^2}{\partial \tilde{v}} + 2\sigma_u \sigma_v \frac{\partial u}{\partial \tilde{v}} \frac{\partial v}{\partial \tilde{v}} + \sigma_v^2 \frac{\partial v^2}{\partial \tilde{v}} \\ &= E \frac{\partial u^2}{\partial \tilde{v}} + 2F \frac{\partial u}{\partial \tilde{v}} \frac{\partial v}{\partial \tilde{v}} + G \frac{\partial v^2}{\partial \tilde{v}} \\ \tilde{F} &= \sigma_u^2 \frac{\partial u}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{v}} + \sigma_u \sigma_v \frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} + \sigma_v \sigma_u \frac{\partial v}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{v}} + \sigma_v^2 \frac{\partial v}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} \\ &= E \frac{\partial u}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{v}} + F \frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} + F \frac{\partial v}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{v}} + G \frac{\partial v}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} \end{aligned}$$

Therefore, if we write this in a matrix form we would get equation 3. □

## 2.2 Length, Angle, and Area

Now, you may be wondering how do we calculate the lengths, angles and areas on the surface as we claimed before. To calculate the length, let  $\gamma : X \rightarrow S$  be a curve on the surface, where  $X \subset \mathbb{R}$ . Then,

$$l = \int_{t_0}^{t_1} \|\gamma(t)\| dt$$

$l$  is the length of the  $\gamma$  from  $t_0$  to  $t_1$ , where  $t_0, t_1 \in X$ . To calculate the angle between two tangent vectors, first we need to represent the inner product on tangent space in terms of the norm of vectors. Let  $x, y \in T_p S$  then,

$$\langle x, y \rangle = \frac{1}{2}(\|x\|^2 + \|y\|^2 + \|x - y\|^2)$$

Now, we can calculate the angle between two vectors in the tangent plane using the following formula:

$$\begin{aligned} \langle x, y \rangle &= \|x\| \|y\| \cos(\theta) \\ \therefore \theta &= \cos^{-1} \left( \frac{\langle x, y \rangle}{\|x\| \|y\|} \right) \\ &= \cos^{-1} \left( \frac{\frac{1}{2}(\|x\|^2 + \|y\|^2 + \|x - y\|^2)}{\|x\| \|y\|} \right) \end{aligned}$$

To calculate the area, we first need to represent the norm of the cross product of  $\sigma_u$  and  $\sigma_v$  in terms of the first fundamental form, where  $u$  and  $v$  are local coordinates.

$$\begin{aligned}\|\sigma_u \times \sigma_v\| &= \sqrt{|\sigma_u|^2 |\sigma_v|^2 - \langle \sigma_u, \sigma_v \rangle^2} \\ &= \sqrt{EG - F^2}\end{aligned}$$

Therefore, the area of the region  $\sigma(U)$ , where  $U \subset \mathbb{R}^2$ , is :

$$\begin{aligned}Area(U) &= \iint_U \|\sigma_u \times \sigma_v\| dA \\ &= \iint_U \sqrt{EG - F^2} dA\end{aligned}$$

### 3 Second Fundamental Form