Differential Geometry of Surfaces

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1 Introduction

In this section, we will introduce some of the basic definition of curves and surfaces.

1.1 Curves

Intuitively, A curves can be thought as the trace of a moving particle in the space. Mathematical, a curves is defined to be the image of a function, $\gamma : U \to \mathbb{R}^n$, where U $\subset \mathbb{R}$..

Definition 1 (Parametrised curve)**.** A **parametrised curve** in $\mathbb{R}^n$ is a smooth function $\gamma : U \to \mathbb{R}^n$, where U $\subset \mathbb{R}$.

Throughout, this report we will assume that smoothness mean C$^\infty$, i.e. the function is differentiable infinitely many times.

Definition 2 (Regular curve)**.** Let $\gamma : U \to \mathbb{R}^n$ be a curve. It is called regular if its derivative is non-vanishing, i.e. $\|\gamma'(t)\| \neq 0$, $\forall \in U$.

There are many different ways to parametrise a curve, e.g. $\gamma(t) = (t, t^2)$ and $\tilde{\gamma}(t) = (t^2, t^4)$. However, only one of these curve is regular, which is $\gamma(t)$. Moreover, there are many different ways to parametrise a curves such that all the parametrisations are regular.

Definition 3 (Unit speed curve)**.** Let $\gamma : U \to \mathbb{R}^n$ be a curve. It is called unit-speed, if $\|\gamma'(t)\| = 1$, $\forall \in U$.

We will see later on that a lot of the formulas and results relating to curves take on a much simpler form when the curve is unit-speed, e.g. curvature of a unit-speed curve, see definition 4, is just the norm of it's second derivative.

Proposition 1. *A parametrised curve is unit-speed if and only if it is regular.*

proof ??
Explain in more detail why curvature is defined in the following manner:

Definition 4 (Curvature of a curve)**.** Let $\gamma : U \to \mathbb{R}^n$ be a unit-speed curve. The curvature at point $\gamma(t)$ is defined as

$$\kappa(t) = \|\gamma''\|$$

These are all the definitions and results about curves that we need to know to understand this report.

1.2 Surfaces

Intuitively, a surface is a subset of $\mathbb{R}^3$ that looks like a $\mathbb{R}^2$ in the neighbourhood of any given point, e.g. the surface of the Earth is spherical; however, it appear to be a flat plane($\mathbb{R}^2$) to an observer on the surface.

Definition 5 (Diffeomorphism)**.** if $f : U \to W$ is continuous, bijective, and smooth, and if its inverse maps $f^{-1} : W \to U$ is also continuous and smooth, then f is called a diffeomorphism and U and W are called diffeomorphic.

Definition 6 (Regular Surface)**.** A subset of $\mathbb{R}^3$ is a regular surface, if every point P $\in S$, there exists a open set U in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing P such that $S \cap W$ is diffeomorphic to U.

Therefore, a surface is collection of diffeomorphisms, $\sigma : U \to S \cap W$, which we call regular surface patches.

Definition 7 (Reparametrisation of surface patches)**.** Let $\sigma : U \to S$ and $\tilde{\sigma} : \tilde{U} \to S$ be surface patches for a surface S, then $\tilde{\sigma}$ is called a reparametrisation of σ if there exists a map, $\Phi : \tilde{U} \to U$, which is smooth and bijective with smooth inverse, $\Phi^{-1} : U \to \tilde{U}$.

Definition 8 (Tangent space)**.** Let S be a regular surface. The **tangent plane** to S at the point $p \in S$ is the set of all initial velocity vectors of regular curves in S with initial position p, i.e

$$T_pS = \{\gamma'(0) | \gamma \text{ is a regular curve in S with } \gamma(0) = p\}$$

These are all the definitions about surfaces that we need to understand this report.

2 First Fundamental Form

In this section, we will define one of the most important object that lets us compute lengths, angles and areas on surface. It is called the **first fundamental form**.

Definition 9 (The fundamental form)**.** The **first fundamental form** is the restriction of the inner product of the ambient space($\mathbb{R}^3$) to the tangent space(T_pS) at point $p \in S$.

2.1 The first fundamental form in local coordinates

We'll now discuss the classical notation for expressing the first fundamental form in local coordinates. Suppose $\sigma : U \to S$ is a surface patch and $\gamma(t) : X \to S$ is a curve on the surface patch of S, defined as $\gamma(t) = \sigma(u(t), v(t))$. Let $\gamma(t_0) = p \in S$, then using the chain rule we know that $\gamma' = \sigma_u u' + \sigma_v v'$. We may then compute:

$$\begin{aligned}
\langle \gamma', \gamma' \rangle &= \langle \sigma_u u' + \sigma_v v', \sigma_u u', \sigma_v v' \rangle \\
&= \langle \sigma_u u', \sigma_u u' \rangle (u')^2 + \langle \sigma_u u', \sigma_v v' \rangle (u'v') + \langle \sigma_v v', \sigma_u u' \rangle v'u' + \langle \sigma_v v', \sigma_v v' \rangle (v')^2 \\
&= \langle \sigma_u u', \sigma_u u' \rangle (u')^2 + 2 \langle \sigma_u u', \sigma_v v' \rangle (u'v') + \langle \sigma_v v', \sigma_v v' \rangle (v')^2 \\
&= E(u')^2 + 2Fu'v' + G(v')^2 \qquad (1)
\end{aligned}$$

where

$$E = \|\sigma_u\|^2, F = \|\sigma_u \sigma_v\|, G = \|\sigma_v\|^2 \qquad (2)$$

Definition 10 (the first fundamental form)**.** The first fundamental form in local coordinates(u, v) is the expression $\mathcal{F}_1 = Edu^2 + 2Fdudv + Gdv^2$.

Now, a good question to ask would be, how reparametrisation of a surface affects the first fundamental form of the surface in terms of the local coordinates?

Example 1. Let S be a surface. Suppose $\tilde{\sigma} : \tilde{U} \to S$ is a reparametrisation of $\sigma : U \to S$, where σ is a surface patch of S and $U, \tilde{U} \subset \mathbb{R}^2$. Assume that $\Phi : \tilde{U} \to U$ is a smooth map and the following are the first fundamental form of $\tilde{\sigma}$ and σ, respectively:

$$\tilde{E}d\tilde{u}^2 + 2\tilde{F}d\tilde{u}d\tilde{v} + Gd\tilde{v}^2 \text{ and } Edu^2 + 2Fdudv + Gdv^2$$

Then,

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J(\Phi)^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} J(\Phi) \qquad (3)$$

, where $J(\Phi)$ is the Jacobian of Φ, i.e.

$$J(\Phi) = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}$$

Proof. We know that $\tilde{\sigma}(\tilde{u},\tilde{v}) = \sigma(\Phi(\tilde{u},\tilde{v})) = \sigma(u,v)$ and $\tilde{E} = \|\tilde{\sigma}_{\tilde{u}}\|^2$, and $E = \|\sigma_u\|^2$. Then,

$$\begin{aligned}
\tilde{E} &= \|\tilde{\sigma}_{\tilde{u}}\|^2 \\
&= \|(\sigma \circ \Phi)_{\tilde{u}}\|^2 \\
&= \left(\sigma_u \frac{\partial u}{\partial \tilde{u}} + \sigma_v \frac{\partial v}{\partial \tilde{u}}\right)^2 \\
&= \sigma_u^2 \frac{\partial u}{\partial \tilde{u}}^2 + 2\sigma_u\sigma_v \frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{u}} + \sigma_v^2 \frac{\partial v}{\partial \tilde{u}}^2 \\
&= E\frac{\partial u}{\partial \tilde{u}}^2 + 2F\frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{u}} + G\frac{\partial v}{\partial \tilde{u}}^2
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tilde{G} &= \sigma_u^2 \frac{\partial u}{\partial \tilde{v}}^2 + 2\sigma_u\sigma_v \frac{\partial u}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{v}} + \sigma_v^2 \frac{\partial v}{\partial \tilde{v}}^2 \\
&= E\frac{\partial u}{\partial \tilde{v}}^2 + 2F\frac{\partial u}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{v}} + G\frac{\partial v}{\partial \tilde{v}}^2 \\
\tilde{F} &= \sigma_u^2 \frac{\partial u}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{v}} + \sigma_u\sigma_v \frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{v}} + \sigma_v\sigma_u \frac{\partial v}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{v}} + \sigma_v^2 \frac{\partial v}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{v}} \\
&= E\frac{\partial u}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{v}} + F\frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{v}} + F\frac{\partial v}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{v}} + G\frac{\partial v}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{v}}
\end{aligned}$$

Therefore, if we write this in a matrix form we would get equation 3.

$\square$