Differential Geometry of Surfaces

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1 Introduction

In this section, we will introduce some of the basic definition of curves and surfaces.

1.1 Curves

Intuitively, A curves can be thought as the trace of a moving particle in the space. Mathematical, a curves is defined to be the image of a function, $\gamma : U \to \mathbb{R}^n$, where U $\subset \mathbb{R}$..

Definition 1 (Parametrised curve)**.** A **parametrised curve** in $\mathbb{R}^n$ is a smooth function $\gamma : U \to \mathbb{R}^n$, where U $\subset \mathbb{R}$.

Throughout, this report we will assume that smoothness mean C^∞, i.e. the function is differentiable infinitely many times.

Definition 2 (Regular curve)**.** Let $\gamma : U \to \mathbb{R}^n$ be a curve. It is called regular if its derivative is non-vanishing, i.e. $\|\gamma'(t)\| \neq 0$, $\forall \in U$.

There are many different ways to parametrise a curve, e.g. $\gamma(t) = (t, t^2)$ and $\tilde{\gamma}(t) = (t^2, t^4)$. However, only one of these curve is regular, which is $\gamma(t)$. Moreover, there are many different ways to parametrise a curves such that all the parametrisations are regular.

Definition 3 (Unit speed curve)**.** Let $\gamma : U \to \mathbb{R}^n$ be a curve. It is called unit-speed, if $\|\gamma'(t)\| = 1$, $\forall \in U$.

We will see later on that a lot of the formulas and results relating to curves take on a much simpler form when the curve is unit-speed, e.g. curvature of a unit-speed curve, see definition 4, is just the norm of it's second derivative.

Proposition 1. *A parametrised curve is unit-speed if and only if it is regular.*

proof ??
Explain in more detail why curvature is defined in the following manner:

Definition 4 (Curvature of a curve)**.** Let $\gamma : U \to \mathbb{R}^n$ be a unit-speed curve. The curvature at point $\gamma(t)$ is defined as

$$\kappa(t) = \|\gamma''\|$$

These are all the definitions and results about curves that we need to know to understand this report.

1.2 Surfaces

Intuitively, a surface is a subset of $\mathbb{R}^3$ that looks like a $\mathbb{R}^2$ in the neighbourhood of any given point, e.g. the surface of the Earth is spherical; however, it appear to be a flat plane($\mathbb{R}^2$) to an observer on the surface.

Definition 5 (Diffeomorphism)**.** if $f : U \to W$ is continuous, bijective, and smooth, and if its inverse maps $f^-1 : W \to U$ is also continuous and smooth, then f is called a diffeomorphism and U and W are called diffeomorphic.

Definition 6 (Regular Surface)**.** A subset of $\mathbb{R}^3$ is a regular surface, if every point P $\in S$, there exists a open set U in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing P such that $S \cap W$ is diffeomorphic to U.

Therefore, a surface is collection of diffeomorphisms, $\sigma : U \to S \cap W$, which we call regular surface patches.

Definition 7 (Reparametrisation of surface patches)**.** Let $\sigma : U \to S$ and $\tilde{\sigma} : \tilde{U} \to S$ be surface patches for a surface S, then $\tilde{\sigma}$ is called a reparametrisation of σ if there exists a map, $\Phi : \tilde{U} \to U$, which is smooth and bijective with smooth inverse, $\Phi^{-1} : U \to \tilde{U}$.

Definition 8 (Tangent space)**.** Let S be a regular surface. The **tangent plane** to S at the point $p \in S$ is the set of all initial velocity vectors of regular curves in S with initial position p, i.e

$$T_pS = \{\gamma'(0) | \gamma \text{ is a regular curve in S with } \gamma(0) = p\}$$

These are all the definitions about surfaces that we need to understand this report.

2 First Fundamental Form

In this section, we will define the one of the most important object that lets us compute lengths, angles and areas on surface. It is called the **first fundamental form**.

Definition 9 (The fundamental form)**.** The **first fundamental form** is the restriction of the inner product of the ambient space($\mathbb{R}^3$) to the tangent space(T_pS) at point $p \in S$.

2.1 The first fundamental form in local coordinates