$$\vec{\nabla}\cdot\vec{D}=\rho \qquad \text{-------------}\rightarrow(v)$$

Where, $\vec{D}$ is the electric flux density, given by $\vec{D}=\varepsilon_0\vec{E}$. Eq.(v) is the differential form of Gauss's theorem.

Ampere's Circuital Law:

Ampere's circuital law in magetostatics is analogous to the Gauss's law in electrostatics. This law says that the line integral of magnetic field $\vec{B}$ around any closed loop is equal to μ_0 times the net current I flowing through the area enclosed by the loop i.e.

$$\oint\vec{B}\cdot\vec{dl}=\mu_0 I$$

Here, μ_0 is the permeability of the free space.

Proof: Consider a long straight conductor carrying a current I. By Biot-Savart law, the magnitude of the magnetic field at a point O, at a distance r from the conductor, is given by

$$B=\frac{\mu_0}{4\pi}\frac{2I}{r} \qquad \text{-------------}\rightarrow \text{(i)}$$

Let us draw a circle with a radius r taking C as centre around the current carrying conductor as shown in the fig.. $\vec{B}$ will be the same in magnitude at all points on this circle. Again we consider a circle element of length dl at the point O. From the figure it is clear that $\vec{dl}$ and $\vec{B}$ are in the same direction.

$$\therefore\oint\vec{B}\cdot\vec{dl}=\oint Bdl\cos\theta$$

$$=B\oint dl \qquad [\because\theta=0^o]$$

$$=\frac{\mu_0}{4\pi}\frac{2I}{r}2\pi r=\mu_0 I$$

$$\oint\vec{B}\cdot\vec{dl}=\mu_0 I$$

This is the required Ampere's circuital law.

Scalar and Vector Potentials:

As mentioned earlier, the zero curl of electrostatic field $\vec{E}$, i.e. $\vec{\nabla}\times\vec{E}=0$, introduces a scalar potential V such that $\vec{E}=-\vec{\nabla}V$. When we analyze $\vec{\nabla}\cdot\vec{B}=0$, we find that the field $\vec{B}$ can be written as a curl of another vector (say $\vec{A}$), i.e.

$\vec{B}=\vec{\nabla}\times\vec{A}$ ---------→(i)

Since a field can be completely determined if we know its divergence as well as its curl, the divergence of $\vec{A}$ remains to be explored. In this context, with the use of eq. (i), Ampere's law reads

$$\vec{\nabla}\times\vec{B}=\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)-\nabla^2\vec{A}=\mu_0\vec{J} \qquad \text{--------}\rightarrow \text{(ii)}$$

It is clear that eq.(ii) will resemble Poisson's equation, if $\vec{\nabla}\cdot\vec{A}=0$. This condition is known as Coulomb gauge. With the application of this condition, the Ampere's law simply yields

$$\nabla^2\vec{A}=-\mu_0\vec{J} \quad \text{-----}\rightarrow \text{(iii)}$$

The solution of the above equation can be obtained if the current density $\vec{J}$ vanishes at infinity. Then the solution comes out to be

$$\vec{A}(\vec{r})=\frac{\pi_0}{4\pi}\int\frac{\vec{J}}{r}dX \quad \text{---------}\rightarrow \text{(iv)}$$

Here dX is the volume element and the vector $\vec{A}$ is called magnetic vector potential. Like the electric scalar potential V, the magnetic vector potential $\vec{A}$ cannot be uniquely defined as we can add to it another vector whose curl is zero. This addition does not change the field $\vec{B}$. On the other side, it is a point of observation that we cannot introduce a magnetic scalar potential U such that $\vec{B}=-\vec{\nabla}U$. The reason is that it is incompatible with Ampere's law, since the curl of a gradient is always zero.

- **Continuity Equation**

The continuity equation says that the total current flowing out of some volume must be equal to the rate of decrease of the charge within that volume, if the charge is neither being created nor lost. Since the charge is flowing, we consider that the charge density ρ is a function of time. The transportation of the charge constitutes the current, i.e.

$$I=\frac{dq}{dt}=\frac{d}{dt}\int_v \rho dV \quad \text{----}\rightarrow \text{(i)}$$

Here, it is assumed that the current is extended in space of volume V closed by a surface S. The net amount of charge which crosses a unit area normal to the directed surface in unit time is defined as the current density $\vec{J}$. This current density $\vec{J}$ is related to the total current I flowing through the surface S as

$$I=\oint_s \vec{J}\cdot\vec{dS} \quad \text{-----}\rightarrow \text{(ii)}$$

Here the integral is over closed surface, as the surface bounding the volume is closed surface. Form eqs. (i) and (ii), we have

$$\oint_s \vec{J}\cdot\vec{dS}=-\frac{dq}{dt}=-\frac{d}{dt}\oint_v \rho dV \quad \text{---}\rightarrow \text{(iii)}$$

The minus sign above is needed in view of decreasing charge ρ in the volume V. so

$$\oint_s \vec{J}\cdot\vec{dS}=-\oint_v \frac{d\rho}{dt}dV \quad \text{-----}\rightarrow \text{(iv)}$$

From Gauss's divergence theorem, we have

$$\oint_s \vec{J}\cdot\vec{dS}=\int_v\left(div\vec{J}\right)dV$$

Or $\int_v \left(div\vec{J}\right)dV = -\int_v \frac{\partial \rho}{\partial t}dV$ -------→(v)

Since the Eq.(v) holds good for any arbitrary volume, we can put the integrands to be equal. Then

$div\vec{J} + \frac{\partial \rho}{\partial t} = 0$ --------→ (vi)

This is the continuity equation.

In case of stationary currents, i.e. when the charge density at any point within the region remains constant.

$\frac{\partial \rho}{\partial t} = 0$ -----→ (vii)

So that $div\vec{J} = 0$ or $\vec{\nabla}\cdot\vec{J} = 0$

Which expresses the fact that there is no net outward flux of current density $\vec{J}$.

Maxwell's Equations: Differential Form

When the charges are in motion, the electric and magnetic fields are associated with this motion which will have variations in both the space and time. These electric and magnetic fields are inter related. This phenomenon is called electromagnetism which is summarized by the set of equations, known as Maxwell's equations. The Maxwell's equations are nothing but are the representation of the basic laws of electromagnetism.

In differential form, four Maxwell's equations are given below (S.I. units)

$\vec{\nabla}\cdot\vec{D} = \rho$ or $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ ------→ (i)

$\vec{\nabla}\cdot\vec{B} = 0$ ---------→ (ii)

$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$ -------→ (iii)

$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ ------→(iv)

<u>Derivation of Maxell's First Equation</u>

Let us consider a surface S bounding a volume V in a dielectric medium, which is kept in the $\vec{E}$ field. The application of external field $\vec{E}$ polarises the dielectric medium and charges are induced, called bound charges or charges due to polarisation. The total charge density at a point in a small volume element dV would then be $(\rho + \rho_p)$, where ρ_p is the polarization charge density, given by $\rho_p = -div\vec{P}$ and ρ is the free charge density at that point in the small volume element dV.

Thus, the total charge density at that point will be $\rho - \left(div\vec{P}\right)$. Then Gauss's theorem can be expressed as

$$\oint_s \vec{E}\cdot d\vec{S} = \oint_v \left(div\vec{E}\right)dV = \frac{1}{\varepsilon_0}\int_v \left(\rho - div\vec{P}\right)dV$$

$$\varepsilon_0 \int_v \left(div\vec{E}\right)dV = \int_v \left(\rho - div\vec{P}\right)dV$$

$$\int_v div\left(\varepsilon_0 \vec{E} + \vec{P}\right)dV = \int_v \rho dV$$

The quantity $\left(\varepsilon_0 \vec{E} + \vec{P}\right)$ is denoted by a quantity $\vec{D}$ called the electric displacement. Therefore,

$$\int_v \left(div\vec{D}\right)dV = \int_v \rho dV$$

Since this equation is true for all the arbitrary volumes, the integrands in this equation must be equal, i.e.

$div\vec{D} = \rho$ or

$\vec{\nabla}\cdot\vec{D} = \rho$

This is the Maxell's first equation.

When the medium is isotropic, the three vectors $\vec{D}, \vec{E}$ and $\vec{P}$ are in the same direction and for small field $\vec{E}$, $\vec{D}$ is proportional to $\vec{E}$, i.e.

$\vec{D} = \varepsilon \vec{E}$

Where ε is called the permittivity of the dielectric medium. The ratio $\varepsilon / \varepsilon_0$ is called the dielectric constant of the medium.

Derivation of maxwell's Second Equation

Since the magnetic lines of force are either closed or go off to infinity, the number of magnetic lines of force entering any arbitrary surface is exactly the same as leaving it. It means the flux of magnetic induction $\vec{B}$ across any closed surface is always zero, i.e.

$$\oint_s \vec{B}\cdot d\vec{S} = 0$$

Transforming the surface integral to volume integral using Gauss's divergence theorem, we have

$$\oint_s \vec{B}\cdot d\vec{S} = \int_v \left(div\vec{B}\right)dV = 0$$

The integrand in the above equation should vanish for the surface boundary as the volume is arbitrary. Therefore

$div\vec{B} = 0$ or $\vec{\nabla}\cdot\vec{B} = 0$

This is the maxwell's second equation.

<u>Derivation of Maxwell's Third Equation</u>

According to Faraday's law, the emf induced in a closed loop is given by

$$E_{emf} = -\frac{\partial\phi}{\partial t} = -\int_s \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} = -\frac{\partial}{\partial t}\oint_s \vec{B}\cdot d\vec{S}$$

Here the flux $\phi = \oint \vec{B} \cdot \overrightarrow{dS}$ where S is any closed surface having the loop as boundary. The emf (E_{emf}) can also be found by calculating the work done in carrying a unit charge completely around the loop. Thus,

$$E_{emf} = \oint_c \vec{E} \cdot \overrightarrow{dl}$$

Here $\vec{E}$ is the intensity of the field associated with the induced emf on equating the above two equations, we get

$$\oint_c \vec{E} \cdot \overrightarrow{dl} = -\oint_s \frac{\partial \vec{B}}{\partial t} \cdot \overrightarrow{dS}$$

According to the Stokes's theorem, the line integral can be transformed in to surface integral with the help of

$$\oint_c \vec{E} \cdot \overrightarrow{dl} = \int_s \left(\vec{\nabla} \times \vec{E}\right) \cdot \overrightarrow{dS}$$ therefore,

$$\int_s \left(\vec{\nabla} \times \vec{E}\right) \cdot \overrightarrow{dS} = -\oint_s \frac{\partial \vec{B}}{\partial t} \cdot \overrightarrow{dS}$$

This equation must be true for any surface whether small or large in the field. So the two vectors in the integrands must be equal at every point, i.e.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This is the Maxwell's third equation.

<u>Derivation of Maxwell's Fourth Equation</u>

According to the Ampere's law, the work done in carrying a unit magnetic pole once around a closed arbitrary path linked with the current is expressed by

$$\oint_c \vec{H} \cdot \overrightarrow{dl} = I$$

Or $\oint_c \vec{H} \cdot \overrightarrow{dl} = \oint_c \vec{J} \cdot \overrightarrow{dS}$

As per Stokes' theorem,

$$\oint_c \vec{H} \cdot \overrightarrow{dl} = \oint_S \left(\vec{\nabla} \times \vec{H}\right) \cdot \overrightarrow{dS}$$

Therefore,

$$\oint_S \left(\vec{\nabla} \times \vec{H}\right) \cdot \overrightarrow{dS} = \int_s \vec{J} \cdot \overrightarrow{dS}$$

This gives $\vec{\nabla} \times \vec{H} = \vec{J}$

The above relation is derived on the basis of Ampere's law, which holds good only for the steady current however, for the changing electric fields, the current density should be modified. The difficulty with the above equation is that, if we take divergence of this equation, then

$$div\left(\vec{\nabla} \times \vec{H}\right) = div\vec{J}$$ [Since divergence of a curl=0]

$$\Rightarrow 0 = div\vec{J}$$

$$\Rightarrow div\vec{J} = 0$$

$$div\vec{J} = -\frac{\partial \rho}{\partial t}$$

Therefore, Maxwell realized that the definition of the total current density is incomplete and suggested to add another density $\vec{J}'$. Therefore

$$curl\vec{H} = \vec{J} + \vec{J}'$$

Now, taking divergence of the above equation, we get

$$div\left(curl\vec{H}\right) = div\vec{J} + div\vec{J}'$$

Or $0 = div\vec{J} + div\vec{J}'$

$$div\vec{J}' = -div\vec{J} = \frac{\partial \rho}{\partial t}$$

Since,

$$\rho = \vec{\nabla}\cdot\vec{D}$$

$$div\vec{J}' = \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{D}\right)$$

$$\vec{\nabla}\cdot\vec{J}' = \vec{\nabla}\cdot\frac{\partial \vec{D}}{\partial t}$$

Hence

$$\vec{J}' = \frac{\partial \vec{D}}{\partial t}$$

Therefore, the Maxwell's fourth equation can be written as

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

The last term of R.H.S. of this equation is called Maxwell's correction and is known a *displacement current density.* The above equation is called modified Ampere's law for unstead or changing current which is responsible for electromagnetic fields.

Maxwell's Equations: Integral Form

There are situations where the integral form of Maxwell's equations is useful. Therefore, now w derive these equations in integral form.

Maxwell's First Equation

Differential form of the Maxwell's first equation is

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \text{------------}\rightarrow \text{(i)}$$

On integrating Eq.(i) over a volume V, we have

$$\int_v \left(\vec{\nabla}\cdot\vec{D}\right)dV = \int_v \rho dV$$

Using Gauss's divergence theorem, the above equation reads

$$\oiint_S \vec{D}\cdot\vec{dS} = \int_v \rho dV = q$$

$$\oint_s \vec{D} \cdot \vec{dS} = q$$

Here q is the net charge contained in the volume V and S is the surface bounding the volume V. This integral form of the Maxwell's first equation says that the total electric displacement through the surface S enclosing a volume V is equal to the total charge contained within this volume.

This statement can also be put in the following form: The total outward flux corresponding to the displacement vector $\vec{D}$ through a closed surface $\vec{S}$ is equal to the total charge q within the volume V enclosed by the surface $\vec{S}$.

Maxwell's Second Equation:

Differential form of the Maxwell's second equation is

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{---}\rightarrow\text{(ii)}$$

Exactly in a manner adopted above, we can show that

$$\oint_s \vec{B} \cdot \vec{dS} = 0$$

Which signifies that the total outward flux of magnetic induction $\vec{B}$ through any closed surface $\vec{S}$ is equal to zero.

Maxwell's Third Equation:

Differential form of the Maxwell's third equation is

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{--}\rightarrow\text{(iii)}$$

On integrating Eq.(iii) over a surface $\vec{S}$ bounded by a closed path, we have

$$\oint_L \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t}\int_s \vec{B} \cdot \vec{dS}$$

Which signifies that the electromotive force around a closed path is equal to the time derivative of the magnetic displacement through any closed surface bounded by that path.

Maxwell's Fourth Equation:

Differential form of the Maxwell's fourth equation is

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{----------------}\rightarrow \text{(iv)}$$

Exactly, in a manner adopted above, we can have this equation in the following form

$$\oint_L \vec{H} \cdot \vec{dl} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot \vec{dS}$$

The above equation signifies that the magnetomotive force around a closed path is equal to the conduction current plus the time derivative of the electric displacement through any surface bounded by that path.

Significance of Maxwell's Equations:

Maxwell's equations represent concisely the fundamentals of electricity and magnetism. From them one can develop most of the working relationships in the field.

Maxwell's first equation represents the Gauss's law for electricity which says that the electric flux out of any closed surface is proportional to the total charge enclosed within the surface. The integral form of this equation finds applications in calculating electric fields around charge objects. It is consistent with Coulomb's law when applied to the electric field of a point charge. The area integral of the electric field gives a measure of the net charge enclosed. However, the divergence of the electric field gives a measure of the density of sources.

As mentioned, the area integral of a vector field determines the net source of the field(function). The integral form $\oiint_S \vec{B}\cdot\overrightarrow{dS}=0$ of the Maxwell's second equation says that the net magnetic flux out of any closed surface is zero. This is because the magnetic flux directed inward toward the south pole, of a magnetic dipole kept in any closed surface, will be equal to the flux outward the north pole. Therefore, the net flux is zero for dipole sources. If we imagine a magnetic monopole source, the area integral $\oiint_S \vec{B}\cdot\overrightarrow{dS}=0$ would have some finite value.

Because of this and since the divergence of a vector field is proportional to the density of point source, the form of the Gauss's law for magnetic field simply says that there are no magnetic monopoles.

The Maxwell's third equation when written in the integral form states that the line integral of the electric field around a closed loop is equal tot the negative of the rate of change of the magnetic flux through the area enclosed by the loop. The line integral basically is the generated voltage or emf in the loop. Therefore, the physical interpretation of Maxwell's third equation is that the changing magnetic field induces electric field.

For static electric field $\vec{E}$, the second term of the R.H.S. of the Maxwell's fourth equation vanishes and then the integral form of this equation says that the line integral of the magnetic field around a closed loop is proportional to the electric current flowing through the loop. This form of the Maxwell's equation is useful for calculating the magnetic field for simple geometries. However, this equation more specifically reveals that the changing electric field induces magnetic field. This seems complimentary to the meaning of the Maxwell's third equation. Therefore, they together yield the formulation of electromagnetic fields or electromagnetic waves, where both electric and magnetic field propagate together and the change in one field induces the other field.

Maxwell's Equations in Isotropic Dielectric Medium:

In an isotropic dielectric medium, the current density $\vec{J}$ and volume charge density ρ are zero. Further, the displacement vector $\vec{D}$ and the magnetic field $\vec{B}$ are defined as $\vec{D}=\varepsilon\vec{E}$ and $\vec{B}=\mu\vec{H}$. In fact $\vec{D}=\varepsilon_0\vec{E}+\vec{P}\equiv\varepsilon\vec{E}$ and $\vec{B}=\mu_0\vec{H}+\mu_0\vec{M}$ for the isotropic linear dielectric (polarizable and magnetic) medium. Here, the vectors $\vec{P}$ and $\vec{M}$ give respectively the measure of polarization and magnetization of the medium. However, for the dielectric medium, it would be sufficient to remember that ε_0 and μ_0 of free space have been simply replaced with ε and μ. Hence, for dielectric medium

$\vec{J}=0$ (or $\sigma=0$), $\rho=0$, $\vec{D}=\varepsilon\vec{E}$ and $\vec{B}=\mu\vec{H}$

Where ε and μ are respectively the absolute permittivity and permeability of the medium.

Under this situation, we can express the Maxwell's equation as

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \text{------------}\rightarrow \text{(i)}$$

$$\vec{\nabla}\cdot\vec{H}=0 \qquad \text{------------}\rightarrow \text{(ii)}$$

$$\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t} \qquad \text{------------}\rightarrow \text{(iii)}$$

$$\vec{\nabla}\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t} \qquad \text{------------}\rightarrow \text{(iv)}$$

Taking curl of eq. (iii), we get

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\times\left[-\mu\frac{\partial\vec{H}}{\partial t}\right]$$

$$\text{Or } \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right)-\nabla^2\vec{E}=-\mu\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{H}\right)$$

$$\text{Or } 0-\nabla^2 E=-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t} \qquad \text{[using eqs. (i) and (iv)]}$$

$$\nabla^2 E=\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{----------}\rightarrow \text{(v)}$$

Similarly, taking curl of Eq. (iv) and using Eqs. (ii) and (iii), we get

$$\nabla^2\vec{H}=\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{----------}\rightarrow \text{(vi)}$$

As discussed earlier, $\frac{1}{\sqrt{\mu\varepsilon}}$ gives the phase velocity of the wave in the medium. If we represent this as v, we obtain from Eqs. (v) and (vi)

$$\nabla^2 E-\frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2}=0 \quad \text{and}$$

$$\nabla^2\vec{H}-\frac{1}{v^2}\frac{\partial^2\vec{H}}{\partial t^2}=0$$

Eqs. (v) and (vi) are the wave equations in an isotropic linear dielectric medium.

$$\text{Now, } v=\frac{1}{\sqrt{\mu\varepsilon}}=\frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}$$

$$\text{Or } v=\frac{c}{\sqrt{\mu_r\varepsilon_r}}\left[\because c=\frac{1}{\sqrt{\mu_0\varepsilon_0}}\right] \qquad \text{-------}\rightarrow \text{(vii)}$$

Eq. (vii) shows that the propagation velocity of an electromagnetic wave in a dielectric medium is less than that in free space.

Also, refractive index = $\frac{c}{v} = \sqrt{\mu_r \varepsilon_r}$

For non-magnetic dielectric medium $\mu_r \approx 1$. Hence, refractive index = $\sqrt{\varepsilon_r}$ or Refractive index = √Relative Permittivity.