Unit 3

ELASTICITY

Elasticity: It is the property by virtue of which a body offers resistance to any deforming force and regains its original condition when the deforming force is removed.

Stress: A body under the action of external forces undergoes deformation. The displaced molecules inside the body develop a tendency to come back to their original positions because of existing Intermolecular binding forces. The average of this restoration tendency shown by all molecules together manifests as a balancing force – which we refer as **restoring force**.

The restoring force per unit area set up inside the body is called stress.

Since, restoring force is exactly equal and opposite to that of applied force, stress may also be defined as the ratio of applied force to the area of its application.

∴ Stress = $\frac{applied\ force}{area\ of\ application} = \frac{F}{m}$ unit – $\frac{N}{m^2}$: Dimension – $[ML^{-1}T^{-2}]$

Strain: The change produced in the body due to change in dimension of a body under a system of forces in equilibrium is called strain. Thus

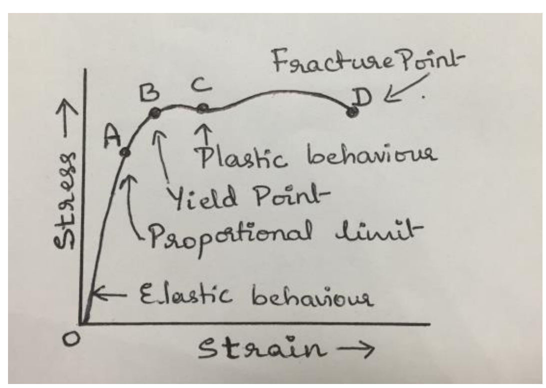
Strain = $\frac{\text{Change in dimension}}{\text{original dimension}}$: has no unit

Change in one dimension ⟹ linear strain = $\frac{\text{change in length}}{\text{original length}}$

Change in two dimension ⟹ Shear strain = angle of deformation

Change in three dimension ⟹ Bulk/volumetric strain = $\frac{\text{change in volume}}{\text{original volume}}$

Hooke's Law: Hooke found that stress applied is always proportional to the strain experienced by the body, but only when the deformation is small. For small deformation the molecular displacement against their binding forces is very small so that the molecules can come back to their previous position immediately after the removal of external force.



The plot of stress versus strain is shown in the figure.

Along OA: As stress increases from zero value, the strain experienced by the body will also increase linearly with respect to stress obeying Hook's law . A is a point up to which the linear proportionality between stress and strain is valid and is referred as **proportionality limit.**

Along AB: As the stress increases further, the strain also increases up to the point B but not linearly and the body still remains elastic, though the Hooke's law is not valid. B is a point at which the body losses its elastic nature and is referred as **Yield point.**

Along BC: along BC, the body cannot regain its original shape after the removal of applied force and further the body simply gets elongated even when applied force remain unchanged. – which is referred as **plastic deformation.**

At D: the deformation is large enough to break the intermolecular binding forces so that the body cannot remain as a single piece. Rather it undergoes fracture or become brittle. Hence the point D is referred as **Fracture Point.**

Hooke's law states that within elastic limit, the stress is directly proportional to strain

i.e., $\frac{\text{stress}}{strain} = const$ ⟶ called as Modulus of elasticity.

Corresponding to the three types of strain, there are three kinds of modulus of elasticity.

i) Young's modulus (y):

When a deformation force is applied to the body is one direction, the change per unit length produced in that direction is called longitudinal /linear /elongated strain. The corresponding force per unit area of cross section is called longitudinal/linear stress. Within the elastic limit, the ratio of linear stress to linear strain is a constant and is called the young's modulus.

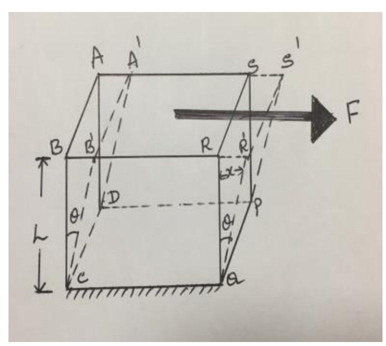
$$Y = \frac{linear\ stress}{linear\ strain} = \frac{F/a}{l/L} = \frac{\text{FL}}{al} \quad \text{-----------} \left(\frac{N}{m^2}\right)$$

Where F – force applied normal to the area of cross section a

l – change in original length L

ii) Rigidity modulus (*n*)

Consider a cube ABCDPQRS with its base DCQP fixed. A force is applied tangentially to the top surface ABRS. Due to this application of force on top surface, an equal and opposite force comes in to play on the lower surface PQCD. These two forces form a couple and cause the layers parallel to the two faces to move one over the other. Thus the top surface ABRS shifts to A′B′R′S′ through an angle $\boldsymbol{\theta}$ w.r.t the bottom fixed surface. In such cases, rigidity modulus is given by,



$$n = \frac{shearing\ stress}{shearing\ strain} = \frac{F/a}{\theta} = \frac{\text{F}}{a\,\theta} \quad \text{------------} \left(\frac{N}{m^2}\right)$$

where a =surface area of the face subjected to the force F.

iii) Bulk modulus (*K*)

A uniform force applied along normal direction on all the surfaces of the cube produces change in volume without changing its shape. Hence Bulk modulus given by

$$K = \frac{Bulk\ stress}{Bulk\ strain} = \frac{F/a}{v/V} = \frac{\mathrm{P\ V}}{v} \quad \text{----------------} \left(\frac{N}{m^2}\right)$$

Where F – Normal force acting on a face

a – surface area of a face

v – Change in volume

V – Original Volume of a Body

P = F/a = Pressure = Bulk stress

iv) Poisson's ratio (*σ*)

When a wire is stretched by applying force along its length, it gets elongated. Simultaneously there will be contraction in normal direction i.e. wire gets thinner, or diameter decreases. Such contraction in a direction $\perp^r$ to direction of applied force is known as lateral strain.

"Within the elastic limit, the ratio of the lateral strain to the longitudinal strain is called Poisson's ratio (*σ*) "

$$\boldsymbol{\sigma} = \frac{lateral\ strain}{longitudinal\ strain} = \frac{(decreases\ in\ diameter)/(original\ diameter)}{(increases\ \ length)\ /(original\ length)}$$

$$\boldsymbol{\sigma} = \frac{\Delta \mathrm{D}/\mathrm{D}}{\Delta \mathrm{L}/\mathrm{L}}$$

Let, α – longitudinal strain per unit stress = $\Delta \mathrm{L}/\mathrm{L}$

β– lateral strain per unit stress = $\Delta \mathrm{D}/\mathrm{D}$, then

$$\left[\!\left[\ \boldsymbol{\sigma} = \frac{\boldsymbol{\beta}}{\boldsymbol{\alpha}}\ \right]\!\right] \quad \text{--------------------}\ \textit{(no unit)}$$

Note: The modulus of elasticity is the basic property associated with the nature of the material of the body and is completely independent of its shape and dimensions.

The parameters which can affect modulus of elasticity are

1. Heavy stress
2. Temperature
3. Impurities
4. Heat treatment
5. Metal processing
6. Crystalline nature of body
7. When material is reduced to nano size, its elasticity varies

RELATION BETWEEN THE THREE MODULI AND POISSON'S RATIO

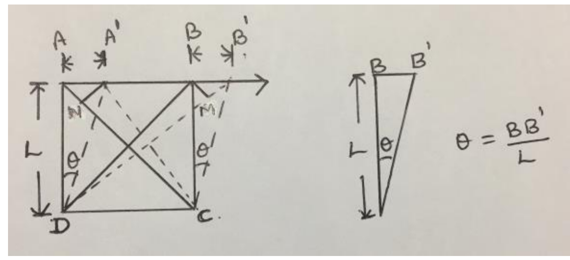
The relation between the three moduli and Poisson's ratio is established in three stages.

Stage I : To derive relation between Y, n and σ

Consider a cube ABCDPQRS of side L with its bottom surface PQCD fixed. F is the tangential force applied on the upper surface ABRS. Due to this, the face ABRS IS displaced to A'B'R'S' (but for clarity R'S' not shown) through an

angle θ . Meanwhile the diagonal DB increases to DB' and the diagonal AC reduces to A'C. M B' gives a measure

Of extension along DB & similarly NA' gives a measure of Contraction along AC.



Let α & β be the longitudinal and lateral strain per unit stress and per unit length,. Then

$y = 1/\alpha$; $\boldsymbol{\sigma} = \beta/\alpha$ --------------------------(1)

$n = \frac{shearing\ stress}{shearing\ strain} = \frac{F/a}{\theta} = \frac{F/L^2}{\theta} = \frac{T}{\theta}$ -------------------------- (2)

where T = F/L^2 = tensile stress.

We know that a shearing stress along AB is equivalent to an equal tensile stress along DB and an equal compression stress along AC at right angle. Hence

Extension along DB due to tensile stress along DB = DB. T .α

Extension along DB due to compression stress along AC = DB. T .β

∴ Total extension along DB = DB . T ($\alpha + \beta$)

i.e., $MB' = \sqrt{2}\, LT(\alpha+\beta)$ -------------------------- (3)

Since $\boldsymbol{\theta}$ is very small, angle AB'C $\cong 90^0 \Longrightarrow$ angle BB'M $\cong 45^0$

∴ In $\boldsymbol{\Delta}^{le,}$ BB'M, Cos $(BB'M)$ = $\frac{\mathrm{MB'}}{\mathrm{BB'}} = \frac{\mathrm{MB'}}{l} = \frac{1}{\sqrt{2}}$

$\Longrightarrow MB' = \frac{l}{\sqrt{2}}$ substituting this in eq (3), we get

$$\frac{l}{\sqrt{2}} = \sqrt{2}\, LT\,(\alpha+\beta)$$

$\therefore \quad \frac{TL}{l} = \frac{1}{2(\alpha+\beta)} \Longrightarrow \frac{T}{l/L} = \frac{1}{2(\alpha+\beta)}$ but, $l/L = \boldsymbol{\theta}$ $\because \boldsymbol{\theta}$ is very small

$$\therefore \quad \frac{T}{\theta} = \frac{(1/\alpha)}{2(1+\beta/\alpha)}$$

In case of unit length and unit tension, $T/\boldsymbol{\theta} = n$; $(1/\alpha) = Y$ and $(\beta/\alpha) = \boldsymbol{\sigma}$, we get

$$n = \frac{Y}{2(1+\sigma)} \quad \text{(or)} \quad \mathbf{Y = 2\,n\,(1+\boldsymbol{\sigma})} \quad \text{--------------------- (I)}$$

Stage II: To derive the relation between K, Y and $\boldsymbol{\sigma}$

Consider a unit cube ABCDPQRS. Let ($T_x : T_y : T_z$) be the stress acting along (X : Y : Z) directions as shown in fig. By using same terminology as before, the elongation produced along the edges (AB : BR : BC) will be ($\alpha T_x : \alpha T_y : \alpha T_z$) respectively

Similarly contraction produced in the direction perpendicular to the above three edges respectively will be ($\beta T_x : \beta T_y : \beta T_z$)

Hence final resultant length or length after deformation of these three edges are

$A'B' = 1+ \alpha T_x - \beta T_y - \beta T_z$

$B'R' = 1+ \alpha T_y - \beta T_x - \beta T_z$

$B'C' = 1+ \alpha T_z - \beta T_x - \beta T_y$ -------------------------- (5)

$\therefore$ Volume of deformed cube = V' = A'B' $\times$ B'R $\times$ B'C'

$$= (1+\alpha T_x - \beta T_y - \beta T_z)(1+\alpha T_y - \beta T_x - \beta T_z)(1+\alpha T_z - \beta T_x - \beta T_y)$$

$$= 1 + \alpha (T_x + T_y + T_z) - 2\beta (T_x + T_y + T_z)$$

$$= 1 + (\alpha - 2\beta)(T_x + T_y + T_z) \quad \text{-------------------------- (6)}$$

If $T_x = T_y = T_z = T$ (i.e cube subjected homogeneous force about all directions) , we get

Volume of deformed cube = V' = $1 + 3(\alpha - 2\beta) T$

Volume of the original cube = V = 1

$\therefore$ Change in volume = $\Delta V = (V' - V) = [(1 + 3(\alpha - 2\beta)T) - (1)] = 3(\alpha - 2\beta) T$

Volumetric strain $= (\Delta V/V) = 3(\alpha - 2\beta) T$ Since area of unit cube = 1, T = P = pressure:

$$\therefore (\Delta V/V) = 3(\alpha - 2\beta) P \quad \text{-------------------------- (7)}$$

$\therefore$ Bulk modulus = K = $\dfrac{T}{(\Delta V/V)} = \dfrac{P}{(\Delta V/V)}$

$$= \frac{P}{3(\alpha-2\beta)P} = \frac{1}{3(\alpha-2\beta)} = \frac{1}{3\,\alpha(1-2\beta/\alpha)} = \frac{(1/\alpha)}{3\left[1-2\left(\beta/\alpha\right)\right]} = \frac{y}{3(1-2\sigma)} \quad \because Y = 1/\alpha,\ \sigma = \beta/\alpha$$

$$\therefore \qquad Y = 3K(1-2\sigma) \qquad \text{------------------ (II)}$$

Stage III : combination of eq (I) & eq (II) (Relation between K, n and Y)

From equation (I) & (II), we have

$Y = 2n\,(1+\sigma) = 3K\,(1-2\sigma) \implies 3K\,(1\text{-}2\sigma) = 2n\,(1+\sigma)$ -------------------------- (8)

Eq(I) can be written as $\dfrac{Y}{n} = 2 + 2\,\sigma$

Eq(II) can be written as $\dfrac{Y}{3k} = 1\text{-}\,2\,\sigma$

$\therefore \quad \dfrac{y}{n} + \dfrac{y}{3k} = 3 \qquad \implies \dfrac{3}{Y} = \dfrac{1}{n} + \dfrac{1}{3k} \qquad \implies \dfrac{9}{Y} = \dfrac{3}{n} + \dfrac{1}{K}$

$\implies \quad \dfrac{\mathbf{3^{3-1}}}{\boldsymbol{Y}} = \dfrac{\mathbf{3^{3-2}}}{\boldsymbol{n}} + \dfrac{\mathbf{3^{3-3}}}{\boldsymbol{k}}$ is the relation among elastic moduli

$$Y = \left[\frac{9nK}{3K + n}\right]$$

Poisson's ratio and its limitation: ((Relation between K, n and σ)

The equation (8) can be written as,

$3K\,(1\text{-}2\sigma) = 2n\,(1+\sigma) \implies \quad 3K - 6K = 2n + 2n\sigma \quad \implies \sigma\,(-6\,K - 2n) = 2n - 3K$

$$\therefore \qquad \sigma = \left[\frac{3K - 2n}{6K + 2n}\right] \qquad \text{------------------(9)}$$

The elastic moduli are the physical quantities and hence cannot carry negative values. They always possess positive values varying from lowest of zero to highest of infinity.

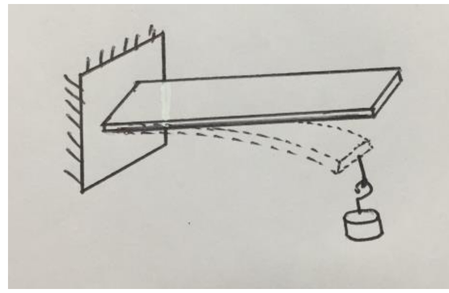
When $n \to 0 \implies \sigma \to \dfrac{3k}{6k} = \dfrac{1}{2}$: When $n \to \infty \implies \sigma \to \dfrac{3(k/n)-2}{6(k/n)+2} = -\dfrac{2}{2} = -1$

∴ Theoretical limits are $-1 < \sigma < ½$ since, σ being a physical quantity, is always ≥ 0

∴ Practical limits are $[\ 0 < \sigma < ½\ ]$

BENDING OF BEAMS

Beam: A homogeneous body of uniform cross section (either circular or rectangular) whose length is quite large compared to its other dimensions is called beam



Single-Cantilever: A set up wherein one end Of the uniform beam is fixed to a rigid support and the other end is subjected to a load is called single-cantilever.

Neutral-Surface: The beam can be thought of as made up of a number of parallel layers ,and each layer in turn can be thought of as made up of a number of infinitesimally thin straight parallel longitudinal filaments arranged one next to the other in the plane of the layer. When the free end of the beam is loaded, the beam bends. The successive layers undergo strain. All the filaments belonging to a layer will also undergo an identical strain.

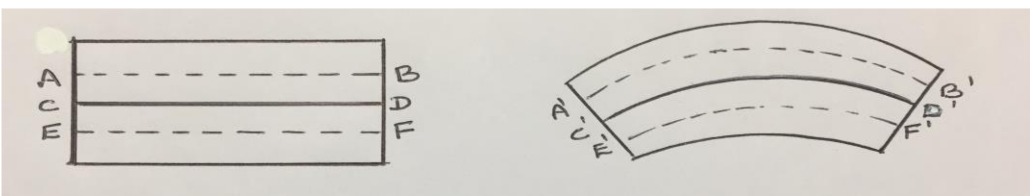


Figure before deformation Figure after deformation

A layer like AB in the upper area will be elongated to A′B′ and the layer like EF in the lower area will be contracted to E’F’ as shown in the above figures. During such a deformation, there exist a particular layer CD whose filaments do not changes its dimensions. Such layer of a uniform beam which does not undergo any change in its dimensions, when the beam is subjected to bending within its elastic limit is called **Neutral-Surface.**

Neutral axis: the intersection of plane of bending with neutral surface is called **Neutral axis.**

BENDING MOMENT OF A BEAM:

Consider a beam of cross sectional area A. The beam is viewed as made up of a number of parallel layers like A’B’, C’D’ , E’F’….etc. When one end of the beam is loaded by F , the successive layers are strained. The layer A’B’ above the neutral surface undergoes elongation. The layer E’F’ below the neutral surface undergoes contraction. C’D’ being the neutral surface do not undergoes any change in its dimensions.

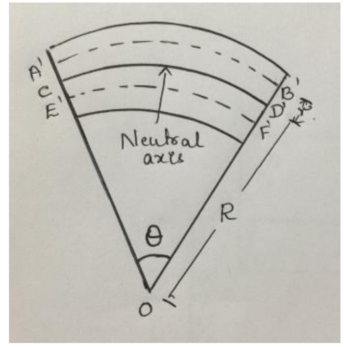
The radius of curvature of bending in a beam is taken as that radius of the circle obtained by completing the curvature of the neutral axis.

Hence, from the figure, we can write

Radius of curvature of bending = R = OC' =OD'

θ = common angle subtended by all the Layers like A'B', C'D', E'F'….etc.

$\therefore$ C'D' = R θ



The layers above neutral surface get elongated and experience a stretching force F. Similarly layers below the neutral surface get compressed and experience a compressive force F. These two forces are of equal magnitude and opposite in direction.

The moment of this force is known as bending moment of the beam.

To derive the expression for it, consider a filament say A'B' at a distance Z from the neutral filament CD . Select a small piece of filament $(a'b')$ on this filament. Let ($\Delta\theta$) be the angle subtended by this small piece of filament $(a'b')$ at the center of curvature O. let (ab) be the corresponding intercept on neutral axis CD. Hence , we can write,

$(O\ a')$ = (O a) + (aa') = R +Z

$(a'b')$ = (R + Z) ($\Delta\theta$) & (ab) = R($\Delta\theta$)

$\therefore$ Increase in length of small element $(a'b')$ = $(a'b')-(ab)$ = $[(\mathrm{R}+\mathrm{Z})(\Delta\theta)-\mathrm{R}(\Delta\theta)]$ = Z ($\Delta\theta$)

$\therefore$ Linear strain along $(a'b')$ $= \dfrac{increase\ in\ length}{original\ length} = \dfrac{\mathrm{Z}\ (\Delta\theta)}{\mathrm{R}(\Delta\theta)} = \dfrac{\mathrm{Z}}{R}$

Young's modulus = y $= \dfrac{linear\ stress}{linear\ strain} = \dfrac{\text{stress}}{\mathrm{Z/R}} = \dfrac{\mathrm{F}/\Delta\mathrm{A}}{\mathrm{Z/R}}$

Where F is the force acting on cross sectional area ΔA of small element$a'b'$. on simplification, we get,

$\therefore$ F = $\left(\frac{\mathrm{yZ}}{\mathrm{R}}\right)$. ΔA

Moment of this force about the neutral filament CD = F.Z = $\left(\frac{yZ^2}{R}\right)\Delta$A

So total moment of the force acting above and below the neutral surface in the entire beam is given by

$$\left(\frac{\mathbf{y}}{\mathbf{R}}\right) \Sigma\, \Delta\mathbf{A}.\ \mathbf{Z}^2 = \left(\frac{\mathbf{y}}{\mathbf{R}}\right) I_g$$

Bending moment

Here $\Sigma \Delta A . Z^2 = I_g$ – called geometrical moment of inertia of the cross sectional area of the beam
$= AK^2$

where A = cross sectional area of beam
K = radius of gyration about the neutral axis.

i) Bending moment of a beam of rectangular cross-section:

For a beam of rectangular cross—section, geometrical moment of inertia I_g is given by

$I_g = \frac{bd^3}{12}$; where b & d are the breadth & thickness of the rectangular cross-section

∴ Bending moment of a beam of rectangular cross-section = $\left(\frac{\mathrm{Y}}{\mathrm{R}}\right) I_g = \frac{\mathrm{Y}}{\mathrm{R}} \frac{bd^3}{12}$

Oscillation and Waves

Terminologies:

Amplitude (‘a’): The maximum distance covered by the body on either side of its mean or equilibrium position during the oscillation is called its amplitude.

Displacement (‘y’**):** The distance of the body from its mean position, at any given instant, measured along its path gives its displacement at that instant of time.

Frequency (‘ν’**):** Number of oscillations executed by an oscillating body in unit time is called its frequency. The SI unit of frequency is hertz, abbreviated as Hz. One Hz is one oscillation per second.

Angular frequency or angular velocity (‘ω’**):** It is the angle covered in unit time by a representative point moving on a circle whose motion is correlated to the motion of the vibrating body. The SI unit of angular frequency is radian per second.

Period (‘T’): It is the time taken by the body to complete one oscillation.

Equilibrium Position: It is the position a body assumes when at rest, and also the position about which it is displaced symmetrically while executing a simple harmonic motion.

Simple Harmonic Motion: If the motion of vibrating body is symmetric on either side of the equilibrium position, then it is said to be a simple harmonic motion.

Relation between ν and T:

Let the body execute n oscillations in t seconds.

∴ No. of oscillations/second $=\nu =\frac{n}{t}$ ………….. (1)

Also, since time taken for n oscillations = t seconds, time taken for one oscillation is,

$$\mathrm{T} = \frac{t}{n}$$

or, $\frac{1}{T} = \frac{n}{t}$ …………..(2)

from (1) & (2),

we have $\nu = \frac{1}{T}$

Relation between ω and T:

Let the body take T seconds to complete one oscillation (i.e., one complete rotation of the representative point on the circle). The angle covered in such a case is 2π radians.

∴ by the definition of angular frequency ω,

$\omega = \frac{angle covered}{time taken} = \frac{2\pi}{T} = 2\pi\nu.$

Equation of a Simple Harmonic Motion:

Suppose a particle P moves along the circumference of a circle of radius a in the anticlockwise direction with uniform angular velocity ω. Let the particle start from X and move over to P in a time t. Let θ be the angle covered by OP in the same time t. From P draw the perpendicular PN to the diameter YOY' (Fig. 1(A)).

As P moves further along the circumference of the circle YX'Y'XY, N moves to and fro along YOY'. Now, it could be observed that the acceleration of N is always towards O. It will have a higher magnitude when the distance ON is more. Therefore the motion of N is said to be simple harmonic. ON is the instantaneous displacement y, pertaining to the position of N.

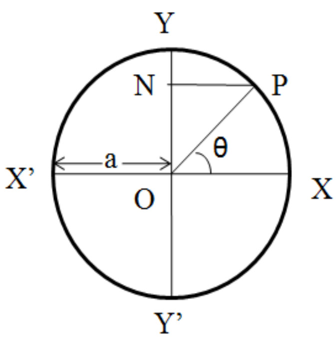


Fig. 1(A)

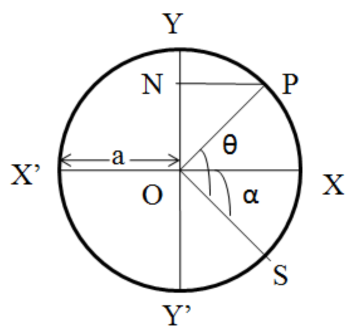


Fig. 1(B)

Fig.1. SIMPLE HARMONIC MOTION

Therefore the displacement of the particle at a given instant is

y= ON

y = OP sin θ, (since ∠XOP=θ)

$y = a sin\omega t$. (Since OP= a and $= \frac{\theta}{t}$)

The above equation represents a simple harmonic motion, in which y is the displacement at any instant t and a is the maximum displacement called the amplitude. But ω is the angular frequency= $\frac{2\pi}{T}$, since T the time period is the time taken by the particle to complete one rotation, which is also equal to the time taken by the point N to complete one oscillation.

Phase: If the particle P starts at X, then N, the foot of the perpendicular drawn from P, coincides with O so that the starting value for $y = 0$. Hence the initial displacement is zero. Instead, if the particle P starts from S (Fig. 1(B)) and the time is counted from that instant, then the particle executing SHM is said to have an initial displacement.

Let ∠SOX= α

$\angle$ XOP = (θ - α) = (ωt - α), instead of ωt.

ON = OP sin (ωt - α),

y= asin (ωt - α).

α is called the initial phase or epoch. If the starting position of the particle happens to lie above X then the equation becomes, y= asin ($\omega t + $ α).

Restoring Force and the Force Constant:

When a body is oscillating (or vibrating), its velocity variation has the following features. The velocity of the body,

(a) decreases when moving away from the equilibrium position

(b) increases while approaching the equilibrium position

(c) becomes maximum when crossing the equilibrium position and

(d) becomes zero at the maximum displacement position where the body will be reversing its direction of motion.

Such an effect on the body is attributed to the action of a force whose magnitude is proportional to, but the direction is opposite to the displacement of the body with respect to the equilibrium position. This force is called the restoring force, and is basically responsible for the oscillation of the body.

If F is the restoring force, and y is the displacement, then,

$F \propto -y$,

F= -ky,

where k is the proportional constant, called the force constant. The nature of this restoring force is identical to the restoring force that we encounter in elasticity.

FREE VIBRATIONS

Description of Free Vibrations & natural Frequency of vibration:

If a body is initiated to vibratory motion after being displaced from its equilibrium position and left free with no external force intervening in its motion, then the body executes free vibrations. It represents natural state of vibrations of the body. The amplitude remains undiminished throughout as shown in the Fig.2.

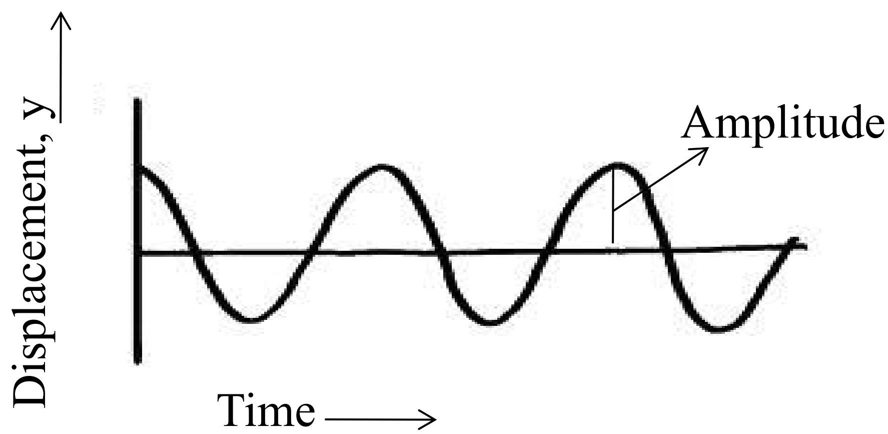


Fig.2. Free Vibrations

For such oscillations, the only force acting on the body will be the restoring force F.

We know that, for a vibrating body, F= $-$ k y.

If m is the mass of the body, then, as per Newton's second law of motion,

F= $m\frac{d^2y}{dt^2}$.

From the above two equations, we have,

$$m\frac{d^2y}{dt^2} = -\mathrm{k\,y} \qquad \Rightarrow \qquad \frac{d^2y}{dt^2} = -\left(\frac{k}{m}\right)\mathrm{y}$$

$$\frac{d^2y}{dt^2} + \left(\frac{k}{m}\right)\mathrm{y} = 0 \qquad \text{----------------------------------- (1)}$$

Where m is the mass of the vibrating body, $\frac{d^2y}{dt^2}$ is its acceleration, k is the force constant, y is the displacement from the mean position.

The above equation represents the equation of motion for a body executing free vibrations.

We have the equation for displacement y for a body executing simple harmonic motion (SHM) as

$$\mathrm{y} = a\,sin(\omega t) \qquad \text{-----------------------------------(2)}$$

Where a is the amplitude, ω is the angular frequency, t is the time of observation.

Differentiating eq (2), we get

$\frac{dy}{dt} = a\omega cos(\omega t) \Rightarrow \frac{d^2y}{dt^2} = -a\omega^2 \sin(\omega t) = -\omega^2 y$

$$\Rightarrow \frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \text{---------------------------------(3)}$$

Comparing eq (1) & (3), we get

$$\omega = \sqrt{\frac{k}{m}}$$

For a body executing free or undamped vibrations, its angular frequency ω is given by the above equation. ω is called as the natural frequency of the body.

Natural frequency of vibrations:

"The one's own characteristic frequency of vibration that a body always assumes spontaneously to vibrate after it is displaced from its equilibrium position and left free, is called the natural frequency of vibration of the body ".

Free vibrations:

"If a body is vibrating purely under the action of a restoring force, then it vibrates with undiminished amplitude at its own natural frequency, so long as no other external force, either varying or constant in nature, intervenes in its motion. Such vibrations are called free vibrations".

ANALYTICAL TREATMENT OF FREE VIBRATIONS

Consider a particle of mass *m* executing free vibrations with natural frequency ω. In free vibrations it is assumed that the particle is experiencing only the restoring force. Neither there is any resistance offered by the medium, nor there are any other external applied force acting on it. In such a case as discussed earlier, the vibrations are in accordance with S.H.M. Hence the differential equation for free vibrations is given by

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \text{-------------------------(1)}$$

The general solution for the above differential equation is of the form

$$y = Ae^{\alpha t} \quad \text{-------------------------(2)}$$

where A and α are the unknowns to be evaluated.

Differentiating eq (2) twice, we have

$$\Rightarrow \frac{dy}{dt} = \alpha A\, e^{\alpha t}$$

$$\Rightarrow \frac{d^2y}{dt^2} = \alpha^2 A e^{\alpha t} \quad \text{-------------------------(3)}$$

Substituting eq.(2) & (3) in eq(1),

$$\alpha^2 A e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$$

Since $Ae^{\alpha t} \neq 0 \Rightarrow \alpha^2 + \omega^2 = 0$

or $\alpha^2 = -\omega^2$.

$\Rightarrow \alpha = \pm(\sqrt{-1})\omega$

$\Rightarrow \alpha = \pm i\omega$.

Using the above equation in $y = Ae^{\alpha t}$, the general solution can be written as

$$y = Ce^{i\omega t} + De^{-i\omega t} \quad \text{-------------------------- (4)}$$

$$\frac{dy}{dt} = C\,(i\omega)e^{i\omega t} - D(i\omega)e^{-i\omega t} \quad \text{------------------------- (5)}$$

where C and D are arbitrary constants which can be evaluated by imposing the following boundary conditions.

A body, after being acted upon by an impulse, starts its vibratory motion from the maximum displacement, at which time its velocity is zero. If the time counted from that instant, then, we have,

Condition 1: if the counting time is started from maximum displacement, at t=0, y= a where a is the amplitude of vibration (which is equal to the maximum displacement).

Substituting *Condition 1* in eq (4), we have

$$a = C + D \quad \text{------------------------- (6)}$$

Also, since the velocity is zero, just when the body starts its vibratory motion, we have condition 2

Condition 2: at the starting time velocity is zero hence $\frac{dy}{dt} = 0$ at $t = 0$.

Applying *condition 2* in eq (5) gives

$$0 = C - D$$

$$\Rightarrow C = D \quad \text{-------------------------(7)}$$

Eq (4) now becomes

$$y = C\,(e^{i\omega t} + e^{-i\omega t}).$$

Euler's theorem is written as $e^{\pm i\omega t} = cos(\omega t) \pm isin(\omega t)$

$\therefore y = C\,\{(cos\omega t + isin\omega t) + (cos\omega t - isin\omega t)\} = 2C\,cos(\omega t)$.

From eq (6) & (7), the above equation becomes

$$y = a\,cos(\omega t) \quad \text{------------------------- (8)}$$

The above equation is the solution when the time is counted from the instant the body begins to move from maximum displacement position.

Generally, the phase of the body is taken to be $0°$ when it is in the equilibrium position and $90°$ at maximum displacement position.

In view of this, eq (8) can be written as

$y = a\,sin(\omega t \pm 90)$ is the equation when time is counted from the instant the body begins to move from maximum displacement position.

$y = a\,sin(\omega t \pm \varphi)$ is the equation if the counting of time is started when the body has already been displaced through certain distance at which time the corresponding phase is φ.

$y = a\, sin(\omega t)$ is the most commonly used solution where the counting of time begins when the body is just crossing the equilibrium position at which $\varphi = 0$.

The above equation is same as the equation for simple harmonic motion and hence it is clear that the motion of the body executing free vibrations is simple harmonic.

SOLVED EXAMPLES

Example 1:

A free particle is executing simple harmonic motion in a straight line. The maximum velocity it attains during any oscillation is 62.8 m/s. Find the frequency of oscillation, if its amplitude is 0.5m.

Data:

Maximum velocity during oscillation, v_{max}= 62.8 m/s

Amplitude, a=0.5 m.

To find:

Frequency of oscillation, ν =?

Solution:

We have the equation for free vibration,

$y = a \sin \omega t$

Velocity is given by, $v = \frac{dy}{dt} = a\omega \cos\omega t$

$= a\omega\sqrt{1 - sin^2\omega t}$

$= \omega\sqrt{a^2 - a^2 sin^2\omega t}$

$= \omega\sqrt{a^2 - y^2}$

The particle attains maximum velocity while passing through its equilibrium position, at which time, the displacement is zero (i.e., y=0).

$\therefore v_{max} = \omega\sqrt{a^2 - 0} = \omega a$

$\therefore$ Angular frequency, $\omega = \frac{Vmax}{a} = \frac{62.8}{0.5} = 125.6$ rad/s.

$\therefore$ Frequency of oscillations $= \nu = \frac{\omega}{2\pi} = \frac{125.6}{2\pi} = 20$ Hz.

$\therefore$ The frequency of oscillation of the particle is 20 Hz.

Example 2:

Calculate the displacement at the end of 10 seconds, and also the amplitude of oscillation for a free particle which is executing a simple harmonic motion in a straight line with a period of 25 seconds. 5 seconds after it has crossed the equilibrium point, the velocity is found to be 0.7m/s.

Data:

Period of oscillation, T= 25s

Velocity, v_1= 0.7m/s at time t_1= 5s, after crossing equilibrium position.

To find:

Amplitude of oscillation

$\therefore$ Velocity, $v = \frac{dy}{dt} = a\omega \cos \omega t$

At time t_1= 5s,v=0.7 m/s

$\therefore 0.7 = a \times \frac{2\pi}{25} \times \cos(\frac{2\pi}{25} \times 5)$

$\therefore 0.7 = 0.25 \times a \times 0.309$

$\therefore$ a=9.06m.

$\therefore$ Displacement at the end of 10 seconds is obtained by

$y = a \sin \omega t$

$=9.06 \times \sin (\frac{2\pi}{25} \times 10)$

$=9.06 \times 0.588 = 5.3$m.

∴ The displacement at the end of 10 seconds is 5.3 m and the amplitude of oscillation is 9.06 m.

Example 3:

A sonometer wire under tension is plucked and left free for vibrations. Find its frequency of vibrations, if the midpoint on the string attains a maximum velocity of 1.57 m/s, when its amplitude of oscillation is 5 mm. Treat the vibration as simple harmonic (Neglect the damping effect).

Data:

Max. Velocity, v_{max}= 1.57 m/s, when the amplitude, a= 5×10^{-3} m.

To find:

Frequency of vibration, ν=?

Solution:

When the damping effect is ignored, equation of free vibrations holds good. We have for free vibrations

$y = a \sin \omega t$, from which we get

$v=\frac{dy}{dt}$

$=\omega\sqrt{a^2 - y^2}$

$v = v_{max}$ when y=0.

∴ $v_{max} = \omega a = 2\pi \nu a$

∴ $\nu=\frac{v\ max}{2\pi a} = \frac{1.57}{2\ X \pi X a\ X\ 5 \times 10-3}$ = 50 Hz.

∴ Frequency of vibration of the wire is 50 Hz.

Example 4:

Calculate the frequency of oscillation for a spring if it is set for vertical oscillations with a load of 200 gm, attached to its bottom. The spring undergoes an extension of 5 cm for a load of 50 gm. Ignore the mass of the spring.

Data:

Extension of the spring, x= 5 x 10^{-2} m

Load acting during extension, M= 50 x 10^{-3} Kg.

Load acting during oscillation, m=200 x 10^{-3} Kg.

To find:

Frequency of oscillation of the spring, ν=?

Solution:

Force constant, k= $\frac{F}{x} = \frac{mg}{x} = \frac{50 \text{ X } 10-3 \text{ X } 9.8}{5 \text{ X } 10-2}$ = 9.8 N/m.

The wire executes oscillations with a frequency equal to its natural frequency ω equal to its natural frequency ν in the absence of any external force.

∴ Angular frequency of the spring,

ω= $\sqrt{\frac{k}{m}} = \sqrt{\frac{9.8}{200 \text{ X } 10-3}}$ = 7 rad/s.

∴ Frequency of oscillation, ν=$\frac{\omega}{2\pi}=\frac{7}{2\pi}$= 1.11 Hz.

∴ Frequency of oscillation of the spring is 1.11 Hz.

Example 5: Calculate the resonance frequency for a simple pendulum of length 1m.

Data:

Length of simple pendulum, L=1m.

To find:

Resonance frequency of oscillation, ν=?

Solution:

A simple pendulum set for oscillation, oscillates with a period,

$T=2\pi \frac{L}{g}=2\pi \frac{1}{9.8}$= 2 sec.

∴ It's frequency of oscillation is

$\nu=\frac{1}{T}=\frac{1}{2}$= 0.5 Hz.

This is its natural frequency of oscillation. Any external periodic force with this frequency causes resonant oscillations of the simple pendulum.

∴ Resonance frequency of the simple pendulum is 0.5 Hz.

Example 6:

Evaluate the resonance frequency of a spring of force constant 1974 N/m, carrying a mass if 2 Kg.

Data:

Force constant, k = 1974 N/m.

Mass suspend, m = 2 Kg.

To find:

Resonance frequency, ν=?

Solution:

We know that resonance frequency of any system is equal to its natural frequency of oscillation.

But the natural frequency,

ω= $\sqrt{\frac{k}{m}}$ and also ω=2π ν

∴ ν= $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$= $\frac{1}{2\pi}\sqrt{\frac{1974}{2}}$

ν =5 Hz

∴ Resonance frequency of the spring, ν =5 Hz

Questions

1. Explain the terms angular frequency, period and simple harmonic motion. (5M)
2. Write the relation between frequency (ν) & time period (T) and also relation between angular frequency (ω) & time period (T). (5M)
3. Construct a simple harmonic wave equation and explain the terms restoring force and force constant. (10 M)
4. Explain free vibrations and discuss about the analytical treatment of free vibrations. (10 M)
5. Calculate the displacement at the end of 10 seconds, and also the amplitude of oscillation for a free particle which is executing a simple harmonic motion in a straight line with a period of 25 seconds. 5 seconds after it has crossed the equilibrium point, the velocity is found to be 0.7m/s. (5M)
6. Calculate the frequency of oscillation for a spring if it is set for vertical oscillations with a load of 200 gm, attached to its bottom. The spring undergoes an extension of 5 cm for a load of 50 gm. Ignore the mass of the spring. (5M)