UNIT 4
ELECTROMAGNETIC THEORY

1. Charge Density:

If a charge is distributed continuously in a medium, it can be expressed in terms of a physical quantity known as **charge density.** There are three types of charge densities, namely linear charge density, surface charge density and volume charge density.

1. 1 Linear Charge Density (λ)

If a charge is distributed continuously on a linear conductor, the charge on its unit length is called the linear charge density. It is generally represented by λ and is measured in the units of C/m. If *l* be the length of a conductor and λ the linear charge density, then the total charge on the conductor will be,

$$q = \int_0^l \lambda dl$$

For a uniform charge distribution, λ is constant, and is given as $\lambda = q/l$. Here q is the total charge, given by $q = \lambda l$.

1.2 Surface Charge Density (σ)

If the charge is distributed over a surface, the charge on the unit area of the conductor is called the *surface charge density* (σ) and its unit is C/m^2.

If the surface charge density at a point of the conductor is σ, then the charge contained in a small element of area *dS* will be σdS. Therefore, the total charge on the surface of the conductor,

$$q = \int_s \sigma dS$$

If the charge distribution is uniform, then the value of σ will be constant, and is given by σ =q/S where S is total area of the surface and q is the total charge, given by q=σS.

1.3 Volume Charge Density(ρ)

If the charge is distributed in the volume of a conductor, the charge contained in a unit volume of the conductor is called the *volume charge density* (ρ) and its unit is C/m^3**.**

If the volume charge density at a point of the volume of a conductor is ρ, then the charge contained in a small element of the volume dV will be ρdV. Therefore, the total charge contained in the conductor,

$$q = \int_v \rho dV$$

For the uniformdistribution of the charge, ρ will be constant, and is given by

$$\rho = q / V$$

$$\text{or } q = \rho V$$

where q is the total charge and V is the total volume, which is equal to $\frac{4}{3}\pi r^3$ for a spherical shaped conductor of radius r.

❖ Del Operator

The del operator is the differential operator, which is represented by $\vec{\nabla}$ and is given by

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

It is not a vector in itself, but when it operates on a scalar function it provides the resultant as a vector. For ex., when $\vec{\nabla}$ is operated on a scalar function F(x,y,z), we get

$$\vec{\nabla} F = \hat{i}\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z}$$

$\vec{\nabla} F$ does not mean multiplication of $\vec{\nabla}$ with F rather it is an instruction to differentiate. Here we should say that $\vec{\nabla}$ is a vector operator that acts upon F.

The del operator $\vec{\nabla}$ can act in different ways. For example, when it acts on a scalar function F, the resultant $\vec{\nabla}$ F is called the gradient of a scalar function F. When it acts on a vector function $\vec{A}$ via the dot product, the resultant is $\vec{\nabla} \bullet \vec{A}$, which is called the divergence of a vector $\vec{A}$. When it acts on a vector function $\vec{A}$ via the cross product, the resultant is $\vec{\nabla} \times \vec{A}$, which is called the curl of a vector $\vec{A}$. Finally, the Laplacian of a scalar function F is written as $\nabla^2 F$ together with

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

2.1 Gradient

If we think of the derivative of a function of one variable we notice that it simply tells us how fast the function varies if we move a small distance. It means the gradient is the rate of change of a quantity wit distance. For ex., temperature gradient in a metal bar is the rate of change of temperature along the bar. However, for a function of three variables the situation is more complicated, as it will depend on what direction we choose to move. For a function F *(x,y,z)* of three variables, we obtain from a theorem on partial derivative

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

Here dF is a measure of changes in F that occurs when we alter all three variables by small amounts dx, dy and dz. The above expression for dF can be written in terms of a dot product of vectors as,

$$dF = \vec{\nabla} F \bullet \vec{dl}$$

where $\vec{\nabla}F = \hat{i}\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z}$ is nothing but gradient of F. Clearly the gradient is a vector quantity, i.e., it has both magnitude and direction. The meaning of gradient becomes clearer when we write,

$$dF = \vec{\nabla}F \bullet \vec{dl}$$

$$\text{Or } dF = \left|\vec{\nabla}F\right|\left|\vec{dl}\right|\cos\alpha$$

where α is the angle between $\vec{\nabla}F$ and $\vec{dl}$.

2.2 Divergence

As mentioned earlier, the divergence of a vector field $\vec{A}$ is represented by $\vec{\nabla} \bullet \vec{A}$. So it is given by,

$$\vec{\nabla} \bullet \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Clearly the divergence of a vector field is a scalar. Also, the divergence of a scalar cannot be obtained, as the dot product of $\vec{\nabla}$ with a scalar is not possible.

In order to make clear the meaning of the divergence, we consider the net flux $\oint \vec{A} \bullet d\vec{S}$ of a vector field $\vec{A}$ from a closed surface S.

2.3 Curl:

As mentioned earlier, the curl of vector field $\vec{A}$ is represented by $\vec{\nabla} \times \vec{A}$. So it is given by

$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Clearly the curl is a vector quantity, i.e. it has both direction and magnitude. Also, it is evident that we cannot have the curl of a scalar as the cross product of $\vec{\nabla}$ with a scalar is meaningless.

In order to make clear the meaning of curl, we consider the circulation of a vector field $\vec{A}$ around a closed path, i.e. $\oint \vec{A} \bullet \vec{dl}$. It is evident that the curl of $\vec{A}$ is a rotational vector. Its magnitude would be the maximum circulation of $\vec{A}$ per unit area.

- ❖ **Gauss's or Green's Theorem**

  According to this theorem,

$$\int_v \left(\vec{\nabla} \bullet \vec{F}\right) dV = \oint_s \vec{F} \bullet d\vec{S}$$

Here dV is the volume element and $d\vec{S}$ is the surface element. In the same manner as we stated fundamental theorem for gradient, the Green's theorem states that the integral of a derivative (here the divergence) over a region (here the volume) is equal to the value of the function at the boundary (here the surface). Since the boundary of a volume is always a closed surface, the R.H.S. is the integral over closed surface.

Evidently this theorem converts the volume integral into the surface integral. Therefore, this theorem is very useful in the situations where it is difficult to calculate the volume integral.

- **Stokes' Theorem**

Stokes' theorem states that the integral of the curl of a vector function over a patch of surface is equal to the value of the function at the perimeter of the patch. So here the derivative is the curl, region is the surface and the boundary is the perimeter of the patch of the surface. Therefore,

$$\int_s \left(\vec{\nabla}\times\vec{\boldsymbol{F}}\right)\cdot\overrightarrow{\boldsymbol{dS}} = \oint_c \vec{\boldsymbol{F}}\cdot\overrightarrow{\boldsymbol{dl}}$$

Clearly, this theorem converts the surface integral into the line integral. Here the L.H.S. is the surface integral whereas the R.H.S. is the closed line integral. So a point of confusion is that which way we should go around, i.e. clockwise or anticlockwise when we integrate the line integral. Moreover, we should know about the direction of the surface element $\overrightarrow{\boldsymbol{dS}}$. For example, for a closed surface $\overrightarrow{\boldsymbol{dS}}$ points outwards normal but for fingers point in the direction of line integral, then the thumb gives the direction of $\overrightarrow{\boldsymbol{dS}}$.

Based on the statement of Stokes' theorem, we can make some more observations. For ex., $\int_s \left(\vec{\nabla}\times\vec{\boldsymbol{F}}\right)\cdot\overrightarrow{\boldsymbol{dS}}$ does not depend on the shape of the surface rather it depends on the boundary line. Also for a closed surface, $\oint \left(\vec{\nabla}\times\vec{\boldsymbol{F}}\right)\cdot\overrightarrow{\boldsymbol{dS}} = 0$ as the boundary shrinks down to a point..

Electric Field and Electric Potential:

Electric field is the region around a charge in which another charge experiences a force. The electrostatic field $\vec{\mathbf{E}}$ is a special kind of field whose curl is always zero, i.e. $\vec{\nabla}\times\vec{\mathbf{E}}=\mathbf{0}$. Since any vector whose curl is zero is equal to the gradient of some scalar quantity, we make use of this property to introduce the scalar quantity as the electric potential V. In order to find this, we use the Stokes' theorem $\int\left(\vec{\nabla}\times\vec{\mathbf{E}}\right)\cdot\mathbf{d}\vec{\mathbf{S}} = \oint\vec{\mathbf{E}}\cdot\mathbf{d}\vec{\mathbf{l}}$. This gives

$$\oint\vec{\mathrm{E}}\cdot\mathrm{d}\vec{\mathrm{l}} = 0 \quad \text{----------}\rightarrow \quad \text{(i)}$$

It means the line integral of $\vec{\mathbf{E}}$ from point a_1 to point a_2 will be the same for all the paths between these points. Hence, the line integral of $\vec{\mathbf{E}}$ is independent of path. Since changing the path would not alter the value of integral, we can define a function, say V, such that

$$V = -\int_a^r \vec{\mathbf{E}} \bullet d\vec{l} \quad \text{--------}\rightarrow \text{(ii)}$$

The differential form of the above equation is written as

$$\vec{\mathbf{E}} = -\vec{\nabla} V \quad \text{-----------}\rightarrow \text{(iii)}$$

Here V is called the electric potential. Actually all the potentials are relative and there is no absolute zero potential. However, convention is that the potential is zero at infinite distance from the charge. In view of this, the lower limit a in eq.(ii) is called as a standard reference point where V is zero. The upper limit is nothing but the point where V is to be calculated. So V depends only on the point $\vec{r}$.

➢ Superposition Principle:

According to the original principle of superposition of electrodynamics, the total force $\vec{F}$ on a charge q (test charge) is equal to the vector sum of the forces due to all the source charges (considering them individually). It means

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots\ldots \quad \text{------------}\rightarrow \text{(iv)}$$

Since $\vec{F} = q\vec{E}$, from Eq. (iv) we find the following for the electric field $\vec{E}$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots\ldots \quad \text{----------}\rightarrow \text{(v)}$$

If we write a for the common reference point, the above equation can be written as

$$-\int_a^r \vec{E}.\vec{dl} = -\int_a^r \vec{E}_1.\vec{dl} - \int_a^r \vec{E}_2.\vec{dl} - \int_a^r \vec{E}_3.\vec{dl} - \ldots\ldots \text{----------------}\rightarrow \text{(vi)}$$

$$\text{Or } \vec{\nabla} V = \vec{\nabla} V_1 + \vec{\nabla} V_2 + \vec{\nabla} V_3 + \ldots \text{----------------------}\rightarrow \text{(vii)}$$

Now it is clear from Eq. (vii) that

$$V = V_1 + V_2 + V_3 + \ldots \quad \text{-------------}\rightarrow \text{(viii)}$$

The above equation reveals that the potential V at a given point $\vec{r}$ is the sum of the potentials due to all the charges. It means the electric potential also satisfies the principle of superposition and the sum is simply an ordinary sum. However, from Eq.(v) it is clear that in case of the electric field $\vec{E}$ this sum is the vector sum.

Poisson's and Laplace's Equations:

For deriving these equations, we start with the following Gauss's law for a linear medium

$$\vec{\nabla} \times \vec{D} = \rho \text{ or}$$

$$\vec{\nabla} \bullet \varepsilon \vec{E} = \rho \quad \text{----------}\rightarrow \text{(i)}$$

Since $\vec{\mathbf{E}} = -\vec{\nabla} V$, the above equation for a homogeneous medium (where ε is constant) can be written as

$$\nabla^2 V = -\rho / \varepsilon \quad \text{-----------}\rightarrow \text{(ii)}$$

This equation is called as Poissons's equation. For a charge free region, i.e. where $\rho = 0$, the Poisson's equation takes the form

$$\nabla^2 V = 0 \quad \text{-------}\rightarrow \text{(iii)}$$

This equation is known as Laplace's equation. This equation is much useful in solving electrostatic problems where a set of conductors are maintained at different potentials; for ex., capacitors and vacuum tube diodes.

Using the expressions for Laplacian operator ∇^2 in Cartesian system, we can write Laplace's eq. (iii) in this coordinate system as,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- **Gauss's Theorem**

The net outward electric flux through any closed surface drawn in an electric field is equal to $\frac{1}{\varepsilon}$ times the total charge enclosed within the surface. It is expressed as

$$\phi = \int_s \vec{E} \bullet \vec{ds} = \frac{1}{\varepsilon_0} \sum q = \frac{1}{\varepsilon_0}$$

where Q is the sum of all the charges present within the surface. The charge outside of the surface is not counted and therefore the flux ϕ due to the charge q sitting outside the surface

$$\phi = \int_s \vec{E} \bullet \vec{ds} = 0$$

- **Differential form of Gauss's Theorem**

When a charge is distributed over a volume such that ρ is the volume charge density, then the charge enclosed by the surface enclosing the volume is given by

$$q = \int_v \rho dV \quad \text{--}\rightarrow \text{(i)}$$

Substituting this expression of q in $\int_s \vec{E} \bullet \vec{dS} = \frac{q}{\varepsilon_0}$, we get

$$\varepsilon_0 \oint_s \vec{E} \bullet \vec{dS} = \int_v \rho dV \quad \text{---}\rightarrow \text{(ii)}$$

According to Gauss's divergence theorem, we can convert the surface integral into the volume integral as

$$\varepsilon_0 \oint_s \vec{E} \bullet \vec{dS} = \int_v div \vec{E} dV \quad \text{-----------}\rightarrow \text{(iii)}$$

Therefore, from Eqs. (ii) and (iii), we have

$$\varepsilon_0 \int_v div\vec{E} dV = \int_v \rho dV$$

Since the above equality is true for every volume, so the integrands of left and right sides should be equal, i.e.

$$\varepsilon_0 div\vec{E} = \rho$$

$$\vec{\nabla} \cdot \left(\varepsilon_0 \vec{E}\right) = \rho \qquad \text{----}\rightarrow \text{(iv)}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \text{-------------}\rightarrow \text{(v)}$$

Where, $\vec{D}$ is the electric flux density, given by $\vec{D} = \varepsilon_0 \vec{E}$. Eq.(v) is the differential form of Gauss's theorem.

Ampere's Circuital Law:

Ampere's circuital law in magetostatics is analogous to the Gauss's law in electrostatics. This law says that the line integral of magnetic field $\vec{B}$ around any closed loop is equal to μ_0 times the net current I flowing through the area enclosed by the loop i.e.

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I$$

Here, μ_0 is the permeability of the free space.

Proof: Consider a long straight conductor carrying a current I. By Biot-Savart law, the magnitude of the magnetic field at a point O, at a distance r from the conductor, is given by

$$B = \frac{\mu_0}{4\pi} \frac{2I}{r} \qquad \text{-------------}\rightarrow \text{(i)}$$

Let us draw a circle with a radius r taking C as centre around the current carrying conductor as shown in the fig.. $\vec{B}$ will be the same in magnitude at all points on this circle. Again we consider a circle element of length dl at the point O. From the figure it is clear that $\vec{dl}$ and $\vec{B}$ are in the same direction.

$$\therefore \oint \vec{B} \cdot \vec{dl} = \oint B dl \cos\theta$$

$$= B \oint dl \left[\because \theta = 0^0\right]$$

$$= \frac{\mu_0}{4\pi} \frac{2I}{r} 2\pi r = \mu_0 I$$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I$$

This is the required Ampere's circuital law.

Scalar and Vector Potentials:

As mentioned earlier, the zero curl of electrostatic field $\vec{E}$, i.e. $\vec{\nabla}\times\vec{E}=0$, introduces a scalar potential V such that $\vec{E}=-\vec{\nabla}V$. When we analyze $\vec{\nabla}\bullet\vec{B}=0$, we find that the field $\vec{B}$ can be written as a curl of another vector (say $\vec{A}$), i.e.

$\vec{B}=\vec{\nabla}\times\vec{A}$---------→(i)

Since a field can be completely determined if we know its divergence as well as its curl, the divergence of $\vec{A}$ remains to be explored. In this context, with the use of eq. (i), Ampere's law reads

$\vec{\nabla}\times\vec{B}=\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\bullet\vec{A}\right)-\nabla^2\vec{A}=\mu_0\vec{J}$ --------→ (ii)

It is clear that eq.(ii) will resemble Poisson's equation, if $\vec{\nabla}\bullet\vec{A}=0$. This condition is known as Coulomb gauge. With the application of this condition, the Ampere's law simply yields

$\nabla^2\vec{A}=-\mu_0\vec{J}$ -----→ (iii)

The solution of the above equation can be obtained if the current density $\vec{J}$ vanishes at infinity. Then the solution comes out to be

$\vec{A}\left(\vec{r}\right)=\frac{\pi_0}{4\pi}\int\frac{\vec{J}}{r}dX$ ---------→(iv)

Here dX is the volume element and the vector $\vec{A}$ is called magnetic vector potential. Like the electric scalar potential V, the magnetic vector potential $\vec{A}$ cannot be uniquely defined as we can add to it another vector whose curl is zero. This addition does not change the field $\vec{B}$. On the other side, it is a point of observation that we cannot introduce a magnetic scalar potential U such that $\vec{B}=-\vec{\nabla}U$. The reason is that it is incompatible with Ampere's law, since the curl of a gradient is always zero.

- **Continuity Equation**

The continuity equation says that the total current flowing out of some volume must be equal to the rate of decrease of the charge within that volume, if the charge is neither being created nor lost. Since the charge is flowing, we consider that the charge density ρ is a function of time. The transportation of the charge constitutes the current, i.e.

$I=\frac{dq}{dt}=\frac{d}{dt}\int_v\rho dV$ ----→ (i)

Here, it is assumed that the current is extended in space of volume V closed by a surface S. The net amount of charge which crosses a unit area normal to the directed surface in unit time is defined as the current density $\vec{J}$. This current density $\vec{J}$ is related to the total current I flowing through the surface S as

$I=\oint_s\vec{J}\bullet\vec{dS}$ -----→ (ii)

Here the integral is over closed surface, as the surface bounding the volume is closed surface. Form eqs. (i) and (ii), we have

$$\oint_s \vec{J} \bullet \vec{dS} = -\frac{dq}{dt} = -\frac{d}{dt}\oint_v \rho dV \quad \text{---}\rightarrow \text{(iii)}$$

The minus sign above is needed in view of decreasing charge ρ in the volume V. so

$$\oint_s \vec{J} \bullet \vec{dS} = -\oint_v \frac{d\rho}{dt} dV \qquad \text{-----}\rightarrow \text{(iv)}$$

From Gauss's divergence theorem, we have

$$\oint_s \vec{J} \bullet \vec{dS} = \int_v \left(div\vec{J}\right) dV$$

Or $\int_v \left(div\vec{J}\right) dV = -\int_v \frac{\partial\rho}{\partial t} dV$ -------→(v)

Since the Eq.(v) holds good for any arbitrary volume, we can put the integrands to be equal. Then

$$div\vec{J} + \frac{\partial\rho}{\partial t} = 0 \qquad \text{--------}\rightarrow \text{(vi)}$$

This is the continuity equation.

In case of stationary currents, i.e. when the charge density at any point within the region remains constant.

$$\frac{\partial\rho}{\partial t} = 0 \qquad \text{-----}\rightarrow \text{(vii)}$$

So that $div\vec{J} = 0$ or $\vec{\nabla} \bullet \vec{J} = 0$

which expresses the fact that there is no net outward flux of current density $\vec{J}$.

Maxwell's Equations: Differential Form

When the charges are in motion, the electric and magnetic fields are associated with this motion which will have variations in both the space and time. These electric and magnetic fields are inter related. This phenomenon is called electromagnetism which is summarized by the set of equations, known as Maxwell's equations. The Maxwell's equations are nothing but are the representation of the basic laws of electromagnetism.

In differential form, four Maxwell's equations are given below (S.I. units)

$$\vec{\nabla} \bullet \vec{D} = \rho \text{ or } \vec{\nabla} \bullet \vec{E} = \rho / \varepsilon_0 \qquad \text{------}\rightarrow \text{(i)}$$

$$\vec{\nabla} \bullet \vec{B} = 0 \quad \text{---------}\rightarrow \text{(ii)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t} \quad \text{-------}\rightarrow \text{(iii)}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \text{------}\rightarrow \text{(iv)}$$

Derivation of Maxwell's First Equation

Let us consider a surface S bounding a volume V in a dielectric medium, which is kept in the $\vec{E}$ field. The application of external field $\vec{E}$ polarizes the dielectric medium and charges are induced, called bound charges or charges due to polarization. The total charge density at a point in a small volume element dV would then be $(\rho + \rho_p)$, where ρ_p is the polarization charge density, given by $\rho_p = -div\vec{P}$ and ρ is the free charge density at that point in the small volume element dV.

Thus, the total charge density at that point will be $\rho - (div\vec{P})$. Then Gauss's theorem can be expressed as

$$\oint_s \vec{E} \bullet d\vec{S} = \oint_v (div\vec{E}) dV = \frac{1}{\varepsilon_0} \int_v (\rho - div\vec{P}) dV$$

$$\varepsilon_0 \int_v (div\vec{E}) dV = \int_v (\rho - div\vec{P}) dV$$

$$\int_v div(\varepsilon_0 \vec{E} + \vec{P}) dV = \int_v \rho dV$$

The quantity $(\varepsilon_0 \vec{E} + \vec{P})$ is denoted by a quantity $\vec{D}$ called the electric displacement. Therefore,

$$\int_v (div\vec{D}) dV = \int_v \rho dV$$

Since this equation is true for all the arbitrary volumes, the integrands in this equation must be equal, i.e.

$div\vec{D} = \rho$ or

$$\vec{\nabla} \bullet \vec{D} = \rho$$

This is the Maxwell's first equation.

When the medium is isotropic, the three vectors $\vec{D}, \vec{E}$ and $\vec{P}$ are in the same direction and for small field $\vec{E}$, $\vec{D}$ is proportional to $\vec{E}$, i.e.

$$\vec{D} = \varepsilon \vec{E}$$

where ε is called the permittivity of the dielectric medium. The ratio $\varepsilon / \varepsilon_0$ is called the dielectric constant of the medium.

Derivation of Maxwell's Second Equation

Since the magnetic lines of force are either closed or go off to infinity, the number of magnetic lines of force entering any arbitrary surface is exactly the same as leaving it. It means the flux of magnetic induction $\vec{B}$ across any closed surface is always zero, i.e.

$$\oint_s \vec{B} \bullet \vec{dS} = 0$$

Transforming the surface integral to volume integral using Gauss's divergence theorem, we have

$$\oint_s \vec{B} \bullet \vec{dS} = \int_v (div\vec{B}) dV = 0$$

The integrand in the above equation should vanish for the surface boundary as the volume is arbitrary. Therefore

$div\vec{B}=0$ or $\vec{\nabla}\bullet\vec{B}=0$

This is the maxwell's second equation.

Derivation of Maxwell's Third Equation

According to Faraday's law, the emf induced in a closed loop is given by

$$E_{emf}=-\frac{\partial\phi}{\partial t}=-\int_s\frac{\partial\vec{B}}{\partial t}\bullet\overrightarrow{dS}=-\frac{\partial}{\partial t}\oint_s\vec{B}\bullet\overrightarrow{dS}$$

Here the flux $\phi=\oint_c\vec{B}\bullet\overrightarrow{dS}$ where S is any closed surface having the loop as boundary. The emf (E_{emf}) can also be found by calculating the work done in carrying a unit charge completely around the loop. Thus,

$$E_{emf}=\oint_c\vec{E}\bullet\overrightarrow{dl}$$

Here $\vec{E}$ is the intensity of the field associated with the induced emf on equating the above two equations, we get

$$\oint_c\vec{E}\bullet\overrightarrow{dl}=-\oint_s\frac{\partial\vec{B}}{\partial t}\bullet\overrightarrow{dS}$$

According to the Stokes's theorem, the line integral can be transformed in to surface integral with the help of

$\oint_c\vec{E}\bullet\overrightarrow{dl}=\int_s\left(\vec{\nabla}\times\vec{E}\right)\bullet\overrightarrow{dS}$ therefore,

$$\int_s\left(\vec{\nabla}\times\vec{E}\right)\bullet\overrightarrow{dS}=-\oint_s\frac{\partial\vec{B}}{\partial t}\bullet\overrightarrow{dS}$$

This equation must be true for any surface whether small or large in the field. So the two vectors in the integrands must be equal at every point, i.e.

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

This is the Maxwell's third equation.

Derivation of Maxwell's Fourth Equation

According to the Ampere's law, the work done in carrying a unit magnetic pole once around a closed arbitrary path linked with the current is expressed by,

$$\oint_c\vec{H}\bullet\overrightarrow{dl}=I$$

Or $\oint_c\vec{H}\bullet\overrightarrow{dl}=\oint_C\vec{J}\bullet\overrightarrow{dS}$

As per Stokes' theorem,

$$\oint_c\vec{H}\bullet\overrightarrow{dl}=\oint_S\left(\vec{\nabla}\times\vec{H}\right)\bullet\overrightarrow{dS}$$

Therefore,

$$\oint_S \left(\vec{\nabla}\times\vec{H}\right)\bullet\overrightarrow{dS} = \int_s \vec{J}\bullet\overrightarrow{dS}$$

This gives $\vec{\nabla}\times\vec{H} = \vec{J}$

The above relation is derived on the basis of Ampere's law, which holds good only for the steady current however, for the changing electric fields, the current density should be modified. The difficulty with the above equation is that, if we take divergence of this equation, then

$$div\left(\vec{\nabla}\times\vec{H}\right) = div\vec{J} \qquad \text{[Since divergence of a curl=0]}$$

$$\Rightarrow 0 = div\vec{J}$$

$$\Rightarrow div\vec{J} = 0$$

$$div\vec{J} = -\frac{\partial\rho}{\partial t}$$

Therefore, Maxwell realized that the definition of the total current density is incomplete and suggested to add another density $\vec{J}\,'$. Therefore

$$curl\vec{H} = \vec{J} + \vec{J}\,'$$

Now, taking divergence of the above equation, we get

$$div\left(curl\vec{H}\right) = div\vec{J} + div\vec{J}\,'$$

Or $0 = div\vec{J} + div\vec{J}\,'$

$$div\vec{J}\,' = -div\vec{J} = \frac{\partial\rho}{\partial t}$$

Since,

$$\rho = \vec{\nabla}\bullet\vec{D}$$

$$div\vec{J}\,' = \frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{D}\right)$$

$$\vec{\nabla}\bullet\vec{J}\,' = \vec{\nabla}\bullet\frac{\partial\vec{D}}{\partial t}$$

Hence

$$\vec{J}\,' = \frac{\partial\vec{D}}{\partial t}$$

Therefore, the Maxwell's fourth equation can be written as

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

The last term of R.H.S. of this equation is called Maxwell's correction and is known as *displacement current density*. The above equation is called modified Ampere's law for unsteady or changing current which is responsible for electromagnetic fields.

Maxwell's Equations: Integral Form

There are situations where the integral form of Maxwell's equations is useful. Therefore, now we derive these equations in integral form.

Maxwell's First Equation

Differential form of the Maxwell's first equation is

$\vec{\nabla}\bullet\vec{D}=\rho$ ------------→ (i)

On integrating Eq.(i) over a volume V, we have

$$\int_v\left(\vec{\nabla}\bullet\vec{D}\right)dV=\int_v\rho dV$$

Using Gauss's divergence theorem, the above equation reads

$$\oint_s\vec{D}\bullet\vec{dS}=\int_v\rho dV=q$$

$$\oint_s\vec{D}\bullet\vec{dS}=q$$

Here q is the net charge contained in the volume V and S is the surface bounding the volume V. This integral form of the Maxwell's first equation says that the total electric displacement through the surface S enclosing a volume V is equal to the total charge contained within this volume.

This statement can also be put in the following form: The total outward flux corresponding to the displacement vector $\vec{D}$ through a closed surface $\vec{S}$ is equal to the total charge q within the volume V enclosed by the surface $\vec{S}$.

Maxwell's Second Equation:

Differential form of the Maxwell's second equation is

$\vec{\nabla}\bullet\vec{B}=0$ ---→(ii)

Exactly in a manner adopted above, we can show that,

$$\oint_s\vec{B}\bullet\vec{dS}=0$$

which signifies that the total outward flux of magnetic induction $\vec{B}$ through any closed surface $\vec{S}$ is equal to zero.

Maxwell's Third Equation:

Differential form of the Maxwell's third equation is

$\vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$ --→(iii)

On integrating Eq.(iii) over a surface $\vec{S}$ bounded by a closed path, we have

$$\oint_c\vec{E}\bullet\vec{dl}=-\frac{\partial}{\partial t}\int_s\vec{B}\bullet\vec{dS}$$

which signifies that the electromotive force around a closed path is equal to the time derivative of the magnetic displacement through any closed surface bounded by that path.

Maxwell's Fourth Equation:

Differential form of the Maxwell's fourth equation is

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \qquad \text{----------------}\rightarrow\text{(iv)}$$

Exactly, in a manner adopted above, we can have this equation in the following form as,

$$\oint_c \vec{H}\bullet\vec{dl}=\int_s\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)\bullet\vec{dS}$$

The above equation signifies that the magnetomotive force around a closed path is equal to the conduction current plus the time derivative of the electric displacement through any surface bounded by that path.

Significance of Maxwell's Equations:

Maxwell's equations represent concisely the fundamentals of electricity and magnetism. From them one can develop most of the working relationships in the field.Maxwell's first equation represents the Gauss's law for electricity which says that the electric flux out of any closed surface is proportional to the total charge enclosed within the surface. The integral form of this equation finds applications in calculating electric fields around charge objects. It is consistent with Coulomb's law when applied to the electric field of a point charge. The area integral of the electric field gives a measure of the net charge enclosed. However, the divergence of the electric field gives a measure of the density of sources.

As mentioned, the area integral of a vector field determines the net source of the field (function). The integral form $\oint_s \vec{B}\bullet\vec{dS}=0$ of the Maxwell's second equation says that the net magnetic flux out of any closed surface is zero. This is because the magnetic flux directed inward toward the south pole, of a magnetic dipole kept in any closed surface, will be equal to the flux outward the north pole. Therefore, the net flux is zero for dipole sources. If we imagine a magnetic monopole source, the area integral $\oint_s \vec{B}\bullet\vec{dS}=0$ would have some finite value.

Because of this and since the divergence of a vector field is proportional to the density of point source, the form of the Gauss's law for magnetic field simply says that there are no magnetic monopoles.

The Maxwell's third equation when written in the integral form states that the line integral of the electric field around a closed loop is equal to the negative of the rate of change of the magnetic flux through the area enclosed by the loop. The line integral basically is the generated voltage or emf in the loop. Therefore, the physical interpretation of Maxwell's third equation is that the changing magnetic field induces electric field.

For static electric field $\vec{E}$, the second term of the R.H.S. of the Maxwell's fourth equation vanishes and then the integral form of this equation says that the line integral of the magnetic field around a closed loop is proportional to the electric current flowing through the loop. This form of the Maxwell's equation is useful for calculating the magnetic field for simple geometries. However, this equation more specifically reveals that the changing electric field induces magnetic field. This seems complimentary to the meaning of the Maxwell's third equation. Therefore, they together yield the formulation of electromagnetic fields or electromagnetic waves, where both electric and magnetic field propagate together and the change in one field induces the other field.

Maxwell's Equations in Isotropic Dielectric Medium:

In an isotropic dielectric medium, the current density $\vec{J}$ and volume charge density ρ are zero. Further, the displacement vector $\vec{D}$ and the magnetic field $\vec{B}$ are defined as $\vec{D}=\varepsilon\vec{E}$ and $\vec{B}=\mu\vec{H}$. In fact $\vec{D}=\varepsilon_0\vec{E}+\vec{P}\equiv\varepsilon\vec{E}$ and $\vec{B}=\mu_0\vec{H}+\mu_0\vec{M}$ for the isotropic linear dielectric (polarizable and magnetic) medium. Here, the vectors $\vec{P}$ and $\vec{M}$ give respectively the measure of polarization and magnetization of the medium. However, for the dielectric medium, it would be sufficient to remember that ε_0 and μ_0 of free space have been simply replaced with ε and μ. Hence, for dielectric medium

$\vec{J}=0$ (or $\sigma=0$), $\rho=0$, $\vec{D}=\varepsilon\vec{E}$ and $\vec{B}=\mu\vec{H}$

Where ε and μ are respectively the absolute permittivity and permeability of the medium. Under this situation, we can express the Maxwell's equation as

$$\vec{\nabla}\bullet\vec{E}=0 \quad \text{--------------}\rightarrow\text{(i)}$$

$$\vec{\nabla}\bullet\vec{H}=0 \quad \text{--------------}\rightarrow\text{(ii)}$$

$$\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t} \quad \text{--------------}\rightarrow\text{(iii)}$$

$$\vec{\nabla}\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t} \quad \text{--------------}\rightarrow\text{(iv)}$$

Taking curl of eq. (iii), we get

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\times\left[-\mu\frac{\partial\vec{H}}{\partial t}\right]$$

Or $\vec{\nabla}\left(\vec{\nabla}\bullet\vec{E}\right)-\nabla^2\vec{E}=-\mu\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{H}\right)$

Or $0-\nabla^2 E=-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t}$ [using eqs. (i) and (iv)]

$$\nabla^2 E=\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{-----------}\rightarrow\text{(v)}$$

Similarly, taking curl of Eq. (iv) and using Eqs. (ii) and (iii), we get

$$\nabla^2 \vec{H} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{-----------}\rightarrow \text{(vi)}$$

As discussed earlier, $\frac{1}{\sqrt{\mu\varepsilon}}$ gives the phase velocity of the wave in the medium. If we represent this as ν, we obtain from Eqs. (v) and (vi)

$\nabla^2 E - \frac{1}{\nu^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$ and

$$\nabla^2 \vec{H} - \frac{1}{\nu^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Eqs. (v) and (vi) are the wave equations in an isotropic linear dielectric medium.

Now, $\nu = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}$

Or $\nu = \frac{c}{\sqrt{\mu_r\varepsilon_r}}\left[\because c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\right]$ --------→ (vii)

Eq. (vii) shows that the propagation velocity of an electromagnetic wave in a dielectric medium is less than that in free space.

Also, refractive index = $\frac{c}{\nu} = \sqrt{\mu_r\varepsilon_r}$

For non-magnetic dielectric medium $\mu_r \approx 1$. Hence, refractive index = $\sqrt{\varepsilon_r}$ or

Refractive index = √Relative Permittivity.

****