UNIT-1: SIMPLE STRESSES AND STRAIN

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Simple Stress and Strain:

1.1. INTRODUCTION 2

1.1.1 Types of Loads 2

1.1.2 Classification of Materials 3

1.2. STRESS 4

1.2.1 Definition of Stress 4

1.2.2 Units of Stress 4

1.2.3 Types of Stresses 5

1.3. STRAIN 6

1.4. LINEAR ELASTICITY AND ELASTIC LIMIT 8

1.5 Hook’s law 8

1.6 poison’s ratio 9

1.6. Stress – strain relationships 10

1.7. Extension / Shortening of a bar 15

1.8 bars with cross sections varying in steps 16

1.9. bars with continuously varying cross sections 16

1.9.1 Bars with varying Circular cross section 16

1.9.2. Bars with varying rectangular cross section 18

1.10. ELONGATION OF BAR DUE TO SELF WEIGHT 19

1.11. Principle of Superposition. 20

Worked Examples 21

UNIT 1:

SIMPLE STRESS AND STRAIN:

Syllabus

Introduction, Stress, strain, mechanical properties of materials, Linear elasticity, Hooke's Law and Poisson's ratio, Stress-Strain relation - behaviour in tension for Mild steel, cast iron and non ferrous metals. Extension / Shortening of a bar, bars with cross sections varying in steps, bars with continuously varying cross sections (circular and rectangular), Elongation due to self weight, Principle of super position.

1.1. INTRODUCTION

When an external force acts on a body, the body tends to undergo some deformation. Due to cohesion between the molecules, the body resists deformation. This resistance by which material of the body opposes the deformation is known as strength of material. Within a certain limit (i.e., in the elastic stage) the resistance offered by the material is proportional to the deformation brought out on the material by the external force. Also within this limit the resistance is equal to the external force (or applied load). But beyond the elastic stage, the resistance offered by the material is less than the applied load. In such a case, the deformation continues, until failure takes place. Within elastic stage, the resisting force equals applied load. This resisting force per unit area is called stress or intensity of stress.

1.1.1 Types of Loads

In the mechanics of the deformable bodies, the following types of loads are commonly considered:

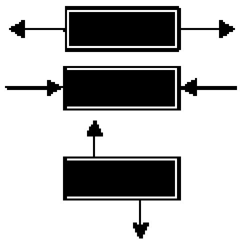
- Dead loads—static in nature, such as the self-weight of the roof.
- Live loads—fluctuating in nature, do not remain constant- such as a weight of a vehicle moving on a bridge.
- Tensile loads.
- Compressive loads.
- Shearing loads.

Depending on the nature of the forces mentioned, the stress can be called the tensile stress or the compressive stress. The tensile stress is induced when the applied force has pulling effect on the body as shown in Table 1.1. Generally, the tensile stress is considered positive.

The compressive stress is induced when the applied load has pushing effect towards a point. Generally, the compressive stress is considered negative. On the other hand, the shearing stress is induced when the applied load is parallel or tangent to the surface.

Table 1.1 Description of load and corresponding stress

Load	*Stress*	*Strain*
Tensile	Tensile	Tensile
Compressive	Compressive	Compressive
Shearing	Shearing	Shearing



1.1.2 Classification of Materials

From an engineering point of view, properties concerned with metals are:

1. Elasticity
2. Plasticity
3. Brittleness
4. Malleability
5. Ductility

Many of these properties are contrasting in nature so that a given metal cannot exhibit simultaneously all these properties. For example, mild steel exhibits the property of elasticity, copper possesses the property of ductility, wrought iron is malleable, lead is plastic and cast iron is brittle.

Elastic Material

II undergoes a deformation when subjected to an external loading such that the deformation disappears on the removal of the loading (rubber).

Plastic Material

It undergoes a continuous deformation during the period of loading and the deformation is permanent. It does not regain its original dimensions on the removal o\ the loading (aluminium).

Rigid Material

It does not undergo any deformation when subjected to an external loading (glass and cast iron).

Malleability

Materials ability to be hammered out into thin sheets, such as lead, is called malleability.

Brittle Materials

They exhibit relatively small extensions to fracture such as glass and cast iron. There is little or no necking at fracture for brittle materials.

1.2. STRESS

1.2.1 Definition of Stress

Stress is an internal resistance offered by a unit area of the material, from which a member is made, to an externally applied load. Alternatively, the force per unit area or intensity of the forces distributed over a given section is called the stress on that section. The resistance of material or the internal force acting on a unit area may act in any direction.

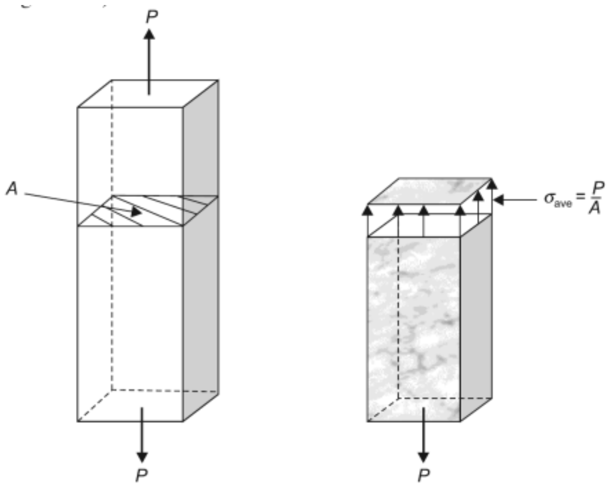


Fig. 1.1: Stress

Direct or normal stress G is calculated by using the following formula:

$$\sigma = \frac{\text{Applied load}}{\text{Original cross-sectional area resisting the force}}$$

$$\sigma = \frac{P}{A} \tag{1.1}$$

1.2.2 Units of Stress

The unit of stress depends upon the unit of load (or force) and unit of area. In M.K.S. units, the force is expressed in kgf and area in meter square (i.e., m^2). Hence unit of stress becomes as kgf/m^2. If area is expressed in centimeter square *(i.e.,* cm^2), the stress is expressed as kgf/cm^2.

In the S.I. units, the force is expressed in newtons (written as N) and area is expressed as m^2. Hence unit of stress becomes as N/m^2. The area is also expressed in millimeter square then unit of force becomes as N/mm^2

l N/m^2 = l $N/(100cm)^2$ = 1 $N/(10^4\ cm^2)$ = $10^{-4}\ N/cm^2$ or $10^{-6}\ N/mm^2$ or 1 MPa= 1 N/mm^2

1.2.3 Types of Stresses

The stress may be normal stress or a shear stress. Normal stress is the stress which acts in a direction perpendicular to the area. It is represented by σ (sigma). The normal stress is further divided into tensile stress and compressive stress.

Tensile Stress:

The stress induced in a body, when subjected to two equal and opposite pulls as shown in Fig. 1.2 as a result of which there is an increase in length, is known as tensile stress. The ratio of increase in length to the original length is known as *tensile strain.* The tensile stress acts normal to the area and it pulls on the area.

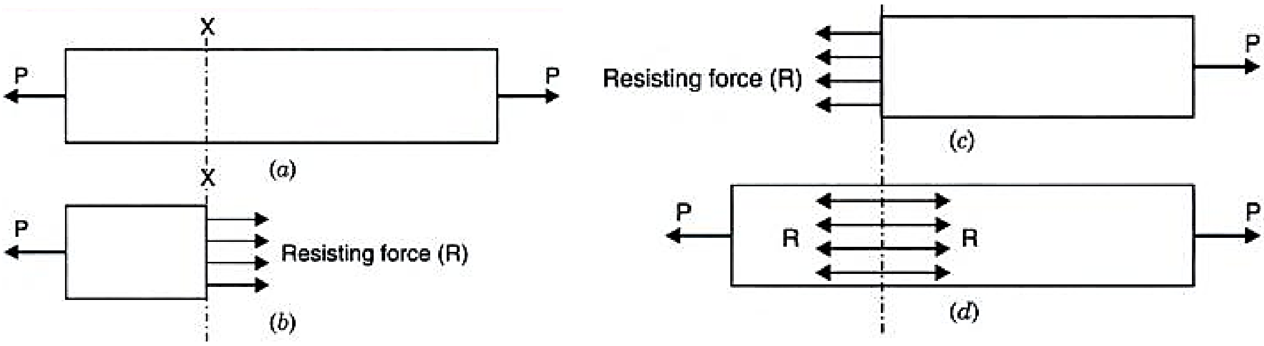


Fig. 1.2: Tensile stress

Fig. 1.2 (a) shows a bar subjected to a tensile force *P* at its ends. Consider a section *x-x,* which divides the bar into two parts. The part left to the section *x-x,* will be in equilibrium if *P* = Resisting force *(R).* This is shown in Fig. 1.2 (b). Similarly the part right to the section *x-x,* will be in equilibrium if *P* = Resisting force as shown in Fig. 1.2 (c). This resisting force per unit area is known as stress or intensity of stress.

$$\therefore \quad \text{Tensile stress} = \sigma = \frac{\text{Resisting force } (R)}{\text{Cross-sectional area}} = \frac{\text{Tensile load } (P)}{\text{A}}$$

$$\sigma = \frac{P}{A}$$

And tensile strain is given by,

$$e = \frac{\text{Increase in length}}{\text{Original length}} = \frac{dL}{L}.$$

Compressive Stress:

The stress induced in a body, when subjected to two equal and opposite pushes as shown in Fig. 1.3 (a) as a result of which there is a decrease in length of the body, is known as compressive stress. And the ratio of decrease in length to the original length is known as *compressive strain.* The compressive stress acts normal to the area and it pushes on the area.

Let an axial push *P* is acting on a body in cross-sectional area *A.* Due to external push *P*, let the original length *L* of the body decreases by *dL.*

Then compressive stress is given by,

$$\sigma = \frac{\text{Resisting Force } (R)}{\text{Area } (A)} = \frac{\text{Push } (P)}{\text{Area } (A)} = \frac{P}{A}$$

And compressive strain is given by,

$$e = \frac{\text{Decrease in length}}{\text{Original length}} = \frac{dL}{L}$$

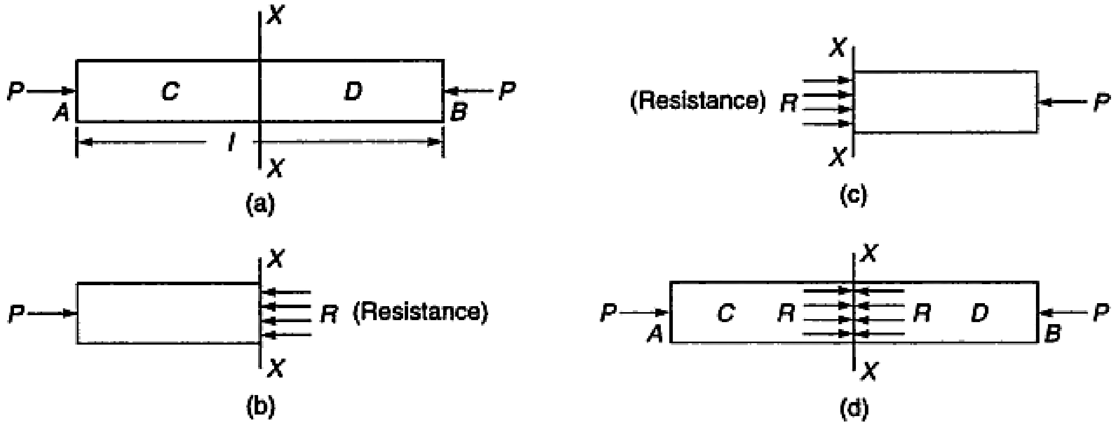


Fig. 1.3: Compressive Stress

Shear Stress:

The stress induced in a body, when subjected to two equal and opposite forces which are acting tangentially across the resisting section as shown in Fig. 1.4 as a result of which the body tends to shear off across the section, is known as shear stress. The corresponding strain is known as *shear strain.* The shear stress is the stress which acts tangential to the area. It is represented by **τ**.

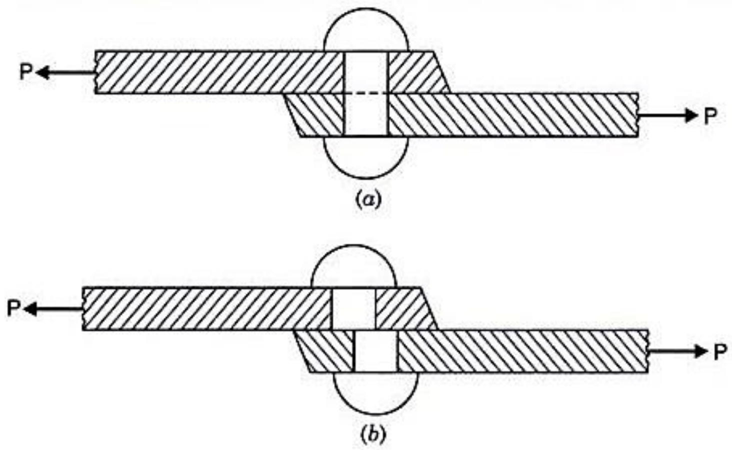


Fig. 1.4: Shear Stress

1.3. STRAIN

When a body is subjected to some external force, there is some change of dimension of the body. The ratio of change of dimension of the body to the original dimension is known as strain. Strain is dimensionless. Strain may be :

- Tensile strain

- Compressive strain,
- Volumetric strain
- Shear strain.

If there is some increase in length of a body due to external force, then the ratio of increase of length to the original length of the body is known as *tensile strain.* But if there is some decrease in length of the body, then the ratio of decrease of the length of the body to the original length is known as *compressive strain.* The ratio of change of volume of the body to the original volume is known as *volumetric strain.* The strain produced by shear stress is known as shear strain.

Linear Strain

It is defined as

$$\varepsilon = \frac{\text{Change in length}}{\text{Small original length}} = \frac{\delta L}{L_o}$$

Linear strain may be either *tensile* or *compressive.* If there is some increase in the length of a body due to external force, then the strain is known as tensile strain. On the other hand, if there is some decrease in the length of the body due to external force, then the strain is known as compressive strain. Please note that both are linear strain only.

In the case of rod having uniform cross-section *A*. the normal stress σ could be assumed to have a constant value *P/A.* Thus, it is appropriate to define ε as the ratio of the total deformation δL over the total length *L* of the rod.

Whereas in the case of a member of variable cross-section, however, the normal stress σ = *P/A* varies along the member, and it is necessary to define the strain at a given point as

$$\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta(\delta L)}{\Delta x} = \frac{d(\delta L)}{dx}$$

Shear Strain

It is a measure of the angle through which a body is deformed by the applied force, denoted by γ. The shear strain is represented by the angle through which the other two faces have rotated as shown in Fig.

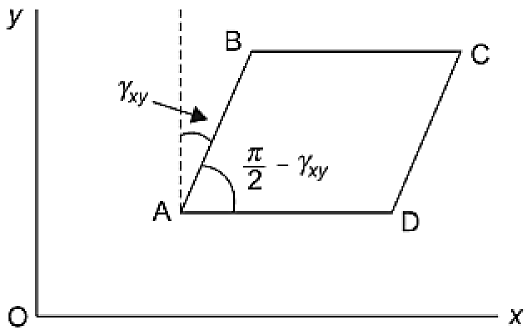


Fig. 1.5: Shear strain

Volumetric Strain

The ratio of change in the volume of the body to the original volume is known as volumetric strain.

1.4. LINEAR ELASTICITY AND ELASTIC LIMIT

When an external force acts on a body, the body tends to undergo some deformation. If the external force is removed and the body comes back to its original shape and size (which means the deformation disappears completely), the body is known as *elastic body*. This property, by virtue of which certain materials return back to their original position after the removal of the external force, is called *elasticity*. The body will regain its previous shape and size only when the deformation caused by the external force, is within a certain limit. Thus there is a limiting value of force up to and within which, the deformation completely disappears on the removal of the force. The value of stress corresponding to this limiting force is known as the *elastic limit* of the material. If the external force is so large that the stress exceeds the elastic limit, the material loses to some extent its property of elasticity. If now the force is removed, the material will not return to its original shape and size and there will be a residual deformation in the material.

1.5 HOOK’S LAW

For elastic bodies, the ratio of stress to strain is constant and is known as *Young's modulus* or the *modulus of elasticity* and is denoted by *E,* i.e.,

$$\sigma \propto \varepsilon$$

$$\sigma = E\varepsilon$$

$$E = \frac{\text{Tensile stress}}{\text{Tensile strain}} \quad \text{or} \quad \frac{\text{Compressive stress}}{\text{Compressive strain}}$$

$$E = \frac{\sigma}{e}$$

Strain has no units as it is a ratio. Thus, *E* has the same units as stress.

The materials that maintain this ratio are said to obey *Hooke s law* which states that within elastic limits, strain is proportional to the stress producing it. The elastic limit of a material is determined by plotting a tensile test diagram. Young's modulus is the stress required to cause a unit strain.

Similarly, for elastic materials, the shear strain is found to be proportional to the applied shear stress within the elastic limit. *Modulus of rigidity* or *shear modulus* denoted by *G* is the ratio of shear stress to shear strain, i.e.,

$$\tau = G\gamma$$

The ratio between the volumetric (Identical) stress and the volumetric strain is called Bulk modulus of elasticity and is denoted by K**.**

1.6 POISON'S RATIO

The ratio of lateral strain to the longitudinal strain is a constant for a given material, when the material is stressed within the elastic limit. This ratio is called **Poisson's ratio** and it is generally denoted by μ or ν or 1/m. Hence mathematically,

$$\text{Poisson's ratio, } \mu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

Longitudinal strain:

When a body is subjected to an axial tensile load, there is an increase in the length of the body. But at the same time there is a decrease in other dimensions of the body at right angles to the line of action of the applied load. Thus the body is having axial deformation and also deformation at right angles to the line of action of the applied load (i.e., lateral deformation).

The ratio of axial deformation to the original length of the body is known as longitudinal (or linear) strain. The longitudinal strain is also defined as the deformation of the body per unit length in the direction of the applied load.

Let *L* = Length of the body,

P = Tensile force acting on the body,

δL = Increase in the length of the body in the direction of *P*

Then, $$\text{Longitudinal strain} = \frac{\delta L}{L}$$

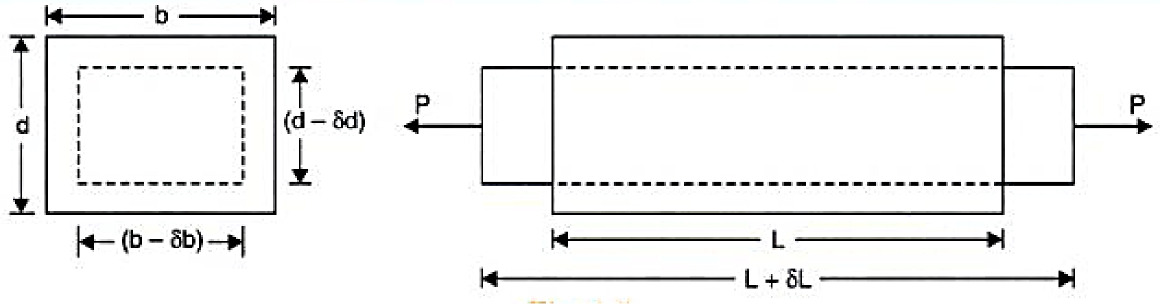


Fig. 1.6: longitudinal and lateral strain

Lateral strain:

The strain at right angles to the direction of applied load is known as lateral strain. Let a rectangular bar of length L, breadth b and depth d is subjected to an axial tensile load P as shown in Fig. 1.6. The length of the bar will increase while the breadth and depth will decrease.

Let δL = Increase in length,

δb= Decrease in breadth, and

δd = Decrease in depth.

$$\text{Lateral strain} = \frac{\delta b}{b} \text{ or } \frac{\delta d}{d}$$

Note:

1) If longitudinal strain is tensile, the lateral strains will be compressive.

2) If longitudinal strain is compressive then lateral strains will be tensile.

3) Hence every longitudinal strain in the direction of load is accompanied by lateral strains of opposite kind in all directions perpendicular to the load.

1.6. STRESS – STRAIN RELATIONSHIPS

For Structural Steel

Certain important properties of materials used for engineering applications can be determined by conducting laboratory tests on small specimens of the material. One such common test is tension test. Tension test involves application of gradually increasing axial tensile load on a standard specimen (the test is performed using Universal Testing Machine aptly called UTM). After performing tension or compression test and determining stress and strain at various magnitudes of load, we can obtain a diagram by plotting stress along Y-axis and strain along X-axis. The stress-strain diagram is of immense help in conveying information about mechanical properties and behaviour of the material. We shall restrict ourselves to behaviour of structural steel only. Our interest on structural steel stems out from the fact that, it is one of the most widely used metals, being used in buildings, bridges, towers, antennas and many more structures. Structural steel is also called low carbon steel or mild steel.

A typical stress strain diagram for mild steel is as shown in Figure. The initial behaviour is portrayed by straight line OA. In this region the stress is proportional to strain and thus the behaviour is linear. Beyond point A. the linear relationship no longer exists, correspondingly, the stress at A is called proportionality limit. However, the material remains elastic even beyond the limit of proportionality. The stress up to which the material behaves elastic is called elastic limit.

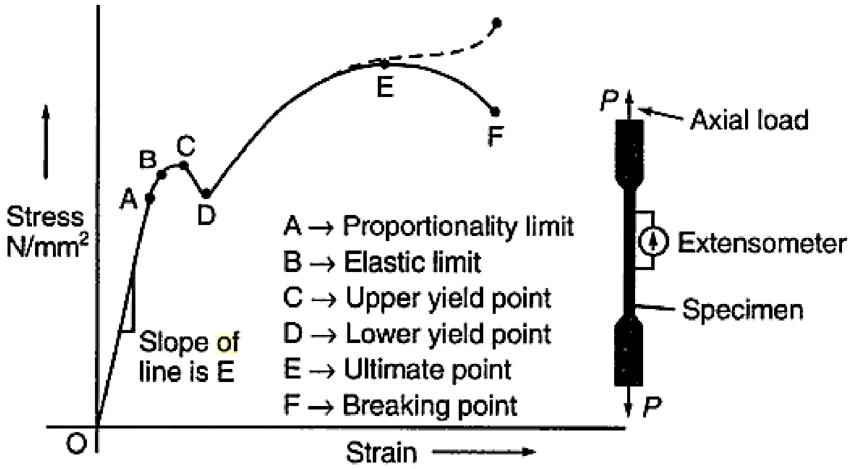
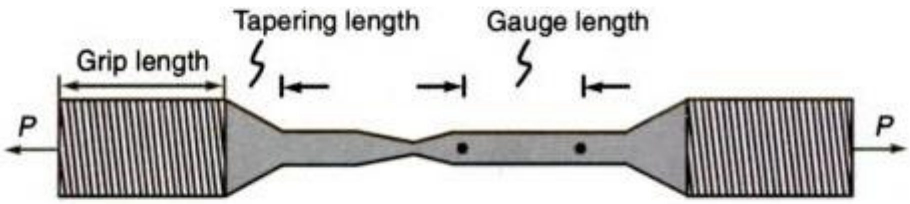


Figure: Stress-strain curve for structural steel

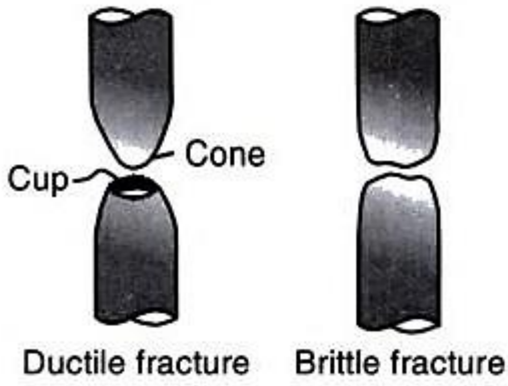
which is shown by point B on the curve. If the load is further increased, the material reaches a point where sudden and appreciable increase in strain occurs without appreciable increase in stress. This behaviour is shown by point C on the curve. The stress corresponding to point C (upper yield point)is called upper yield stress. An accurate testing of the specimen would reveal that the curve drops at point D (lower yield point) and the corresponding stress is called lower yield stress. In the region of upper and lower yield points, the material becomes perfectly plastic, which indicates that it can deform without an increase in applied load.

After undergoing the large strains in the region of upper and lower yield points, the steel begins to strain harden. Strain hardening is a process, where material undergoes changes in its atomic and crystalline structure. This process brings in new lease of life for the material and it picks up increased resistance to further loading (hence resistance to deformation). Thus, additional elongation requires an increase in tensile load, and stress-strain diagram mounts up with a positive slope from D to E. Point E signifies the maximum stress the material can bear and this point is called ultimate point and the corresponding stress is ultimate stress. Further, stretching of the bar is actually accompanied by drastic reduction in area and in load, and fracture finally occurs as shown by point F on the diagram.

Being a ductile material, steel specimen sustains uniform strain over the entire length up to the ultimate strength point. Figure shows that the stress decreases beyond the ultimate strength of the material and rupture does not occur until a strain considerably in excess of the strain corresponding to the ultimate stress has been reached. The strain that occurs during this phase tends to be localised over a very short length of the test specimen, leading to necking phenomenon (also called waist formation) depicted in Figure (b). This necking is typical of a metal which behave in a ductile manner. Figure (c) shows type of failures for ductile and brittle materials.



(b)



(c)

After conducting tension test on steel we can determine the following items

- Elastic modulus
- Proportional limit
- Yield stress
- Ultimate stress

- **Percentage increase in length is a measure of ductility of the metal. It is given by**

$$\text{percentage elongation} = \frac{l_f - l_o}{l_o}$$

where, l_f = length of test specimen at fracture, l_o = original length.

- **Percentage reduction in cross sectional area: Ductility can also be measured by percentage decrease in cross sectional area as given by**

$$\text{percentage reduction in area} = \frac{A_o - A_f}{A_o}$$

where, A_o is original area of cross section and A_f is area of cross section at fracture.

True Stress-Strain Diagram

In plotting stress-strain diagram, we make use of original area of cross section while computing all stress values and original length while calculating corresponding strains. In this context it is pertinent to define the following:

Nominal or Conventional or Engineering Stress

The ratio of load over original area of cross section of a component is nominal stress.

True Stress

The ratio of load over instantaneous area of cross section is true stress. Thus, under tensile load, instantaneous area is less than original area and under compressive load, instantaneous area is more than original area.

Nominal or Engineering Strain

Strain values are calculated at various intervals of gradually increasing load considering original gauge length of the specimen, such a strain is nominal or engineering strain. Nominal strain is change in dimension to corresponding original dimension.

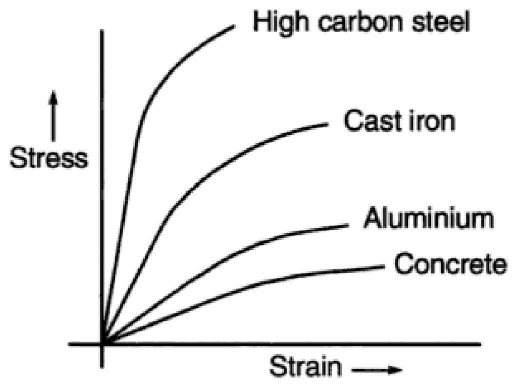
True Strain

As the load keeps on increasing, the gauge length will also keep on varying (e.g., gauge length increases under tensile loading). If actual length is used in calculating the strain, the strain obtained is true strain. Crisply, change in dimension to instantaneous dimension is true strain. In most of the engineering designs, the stresses considered will be well within proportional limit and as the strain involved up to this limit is very small, the change in area is not at all appreciable. Therefore, original area of cross section is considered while defining the stress for all practical purposes.

Coming back to true stress-strain diagram, as mentioned above, the lateral contraction of the metal occurs when it is stretched under tensile load, this results in decreased cross sectional area. However, this decrease is too small to show a noticeable effect on calculated value of stress upto point D, but beyond point D, the reduction begins to alter the shape of the diagram. If the actual area is used to calculate stress, the true stress-strain curve will follow dashed line that is superposed on the diagram.

Stress-Strain Diagram for Other Materials

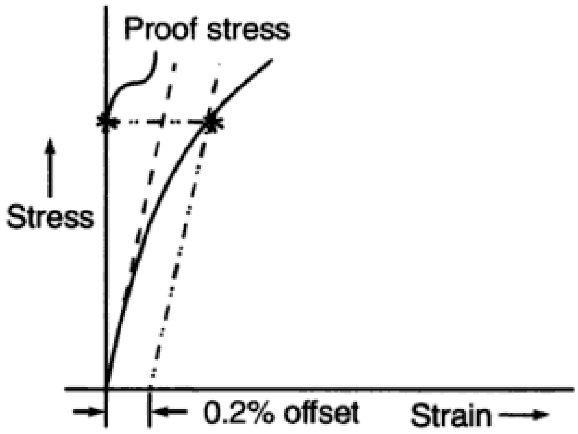
Every material has its own strength characteristics. Unlike steel, other materials do not show clear points of yield stress. But initial linear behaviour is shown by almost all materials. Figure presents the stress-strain behaviour of some important materials. Table presents elastic properties of certain metals.



13 Stress-strain diagram for non-ferrous metals.

Proof Stress

Most of the metals except steel, do not show well-defined yield point and yet undergoes large strains after the proportional limit is exceeded. An arbitrary yield stress called proof stress for these metals can be found out by offset method. On the stress-strain diagram of the metal under consideration, a line is drawn parallel to initial linear part of the curve (Figure 2.14) this line is drawn at a standard offset of strain value, such as 0.002 (0.2%). The intersection of the offset line and the stress-strain curve (point A in the figure) defines the yield point for the metal and hence yield stress. Proof stress is not an inherent property of the metal. Proof stress is also called offset yield stress.



1.7. EXTENSION / SHORTENING OF A BAR

Consider a prismatic bar of length L and cross-sectional area A subjected to axial force P. We have the relation

$$\sigma = E\varepsilon$$

upon substitution of ε and σ in that equation, we get

$$E = \frac{P/A}{\delta L/L} = \frac{PL}{A(\delta L)}$$

where

E = Young's Modulus, N/mm^2

L = original length , mm

δL = change in length , mm

A = original cross-sectional area, mm^2 and

P = axial force , N

The above Eq. can also be written as,

$$\delta L = \frac{PL}{AE}$$

Table 1.2 gives the values of Young's modulus of some commonly used materials.

Table 1.2: Young's modulus of some materials

Sl. no.	Material	Young's modulus (kN/mm^2)
1	Mild steel	200
2	Aluminium	70
3	Copper	100
4	Cast iron	90
5	Bronze	120
6	Wood	10

1.8 BARS WITH CROSS SECTIONS VARYING IN STEPS

Consider a bar of varying three sections of lengths L_1 L_2 and L_3 having respective areas of cross-sections A_1, A_2 and A_3 subjected to an axial pull P. Let δL_1, δL_2, δL_3 be the changes in length of the respective three sections of the bar, then we have

$$\delta L_1 = \frac{PL_1}{A_1E}, \quad \delta L_2 = \frac{PL_2}{A_2E}, \quad \delta L_3 = \frac{PL_3}{A_3E}$$

Now the total elongation of the bar,

$$\delta L = \delta L_1 + \delta L_2 + \delta L_3 = \frac{PL_1}{A_1E} + \frac{PL_2}{A_2E} + \frac{PL_3}{A_3E} = \frac{P}{E}\left(\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3}\right)$$

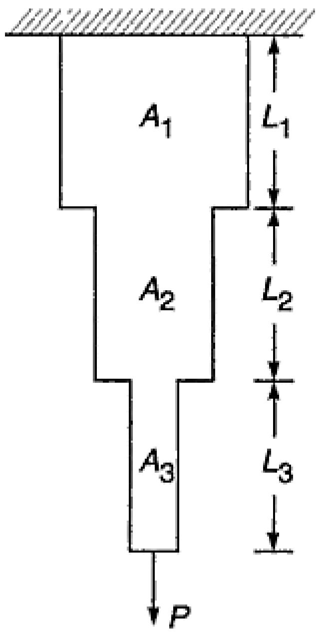


Fig. 1.7: Stepped bar

1.9. BARS WITH CONTINUOUSLY VARYING CROSS SECTIONS

1.9.1 Bars with varying Circular cross section

A bar uniformly tapering from a diameter D_1 at one end to a diameter D_2 at the other end is shown in Fig. 18.

Let P = Axial tensile load on the bar

L = Total length of the bar

E = Young's modulus.

Consider a small element of length dx of the bar at a distance x from the left end. Let the diameter of the bar be D at a distance x from the left end.

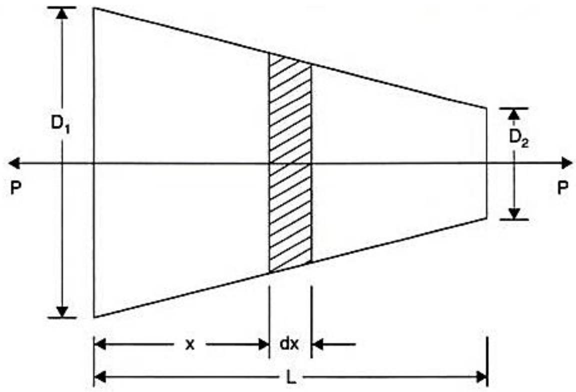


Fig. 1.8: Taper rod of circular cross section

$$D_x = D_1 - \left(\frac{D_1 - D_2}{L}\right)x$$
$$= D_1 - kx \qquad \text{where } k = \frac{D_1 - D_2}{L}$$

Area of cross-section of the bar at a distance x from the left end,

$$A_x = \frac{\pi}{4} D_x^2 = \frac{\pi}{4} (D_1 - k.x)^2.$$

Now the stress at a distance x from the left end is given by,

$$\sigma_x = \frac{\text{Load}}{A_x}$$
$$= \frac{P}{\frac{\pi}{4}(D_1 - k.x)^2} = \frac{4P}{\pi (D_1 - k.x)^2}$$

The strain e_x in the small element of length dx is obtained by using equation

$$e_x = \frac{\text{Stress}}{E} = \frac{\sigma_x}{E}$$
$$= \frac{4P}{\pi (D_1 - k.x)^2} \times \frac{1}{E} = \frac{4P}{\pi E(D_1 - k.x)^2}$$

Extension of the small elemental length dx

$$= \text{Strain}.\ dx = e_x \,.\, dx$$
$$= \frac{4P}{\pi E(D_1 - k\,.\,x)^2}\ .\ dx$$

Total extension of the bar is obtained by integrating the above equation between the limits 0 and L.

$\therefore$ **Total extension,**

$$dL = \int_0^L \frac{4P\,.\,dx}{\pi E(D_1 - k.x)^2} = \frac{4P}{\pi E}\int_0^L (D_1 - k.x)^{-2}\,.\,dx$$
$$= \frac{4P}{\pi E}\int_0^L \frac{(D_1 - k\,.\,x)^{-2} \times (-k)}{(-k)}\,.\,dx \quad \text{[Multiplying and dividing by } (-k)]$$
$$= \frac{4P}{\pi E}\left[\frac{(D_1 - k\,.\,x)^{-1}}{(-1)\times(-k)}\right]_0^L = \frac{4P}{\pi Ek}\left[\frac{1}{(D_1 - k\,.\,x)}\right]_0^L$$
$$= \frac{4P}{\pi Ek}\left[\frac{1}{D_1 - k\,.\,L} - \frac{1}{D_1 - k \times 0}\right]$$
$$= \frac{4P}{\pi Ek}\left[\frac{1}{D_1 - k\,.\,L} - \frac{1}{D_1}\right]$$

Substituting the value of k in the above equation, we get,

$$k = \frac{D_1 - D_2}{L}$$

Total extension,

$$dL = \frac{4P}{\pi E \cdot \left(\frac{D_1 - D_2}{L}\right)}\left[\frac{1}{D_1 - \left(\frac{D_1 - D_2}{L}\right) \cdot L} - \frac{1}{D_1}\right]$$

$$= \frac{4PL}{\pi E \cdot (D_1 - D_2)}\left[\frac{1}{D_1 - D_1 + D_2} - \frac{1}{D_1}\right]$$

$$= \frac{4PL}{\pi E \cdot (D_1 - D_2)}\left[\frac{1}{D_2} - \frac{1}{D_1}\right]$$

$$= \frac{4PL}{\pi E \cdot (D_1 - D_2)} \times \frac{(D_1 - D_2)}{D_1 D_2} = \frac{4PL}{\pi E D_1 D_2}$$

If the rod is of uniform diameter, then D_1 - D_2 = D

$$\text{Total extension, } dL = \frac{4PL}{\pi E \cdot D^2}$$

1.9.2. Bars with varying rectangular cross section

A bar of constant thickness and uniformly tapering in width from one end to the other end is shown in Fig. 1.9.

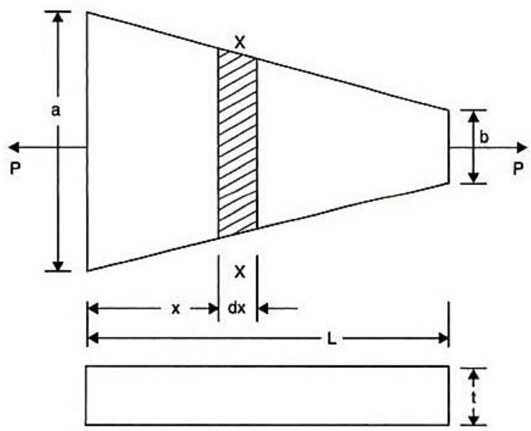


Fig. 1.9: Bars with rectangular cross section

Let P = Axial load on the bar
L = Length of bar
a = Width at bigger end
b = Width at smaller end
E = Young's modulus
t = Thickness of bar Consider any section X-X at a distance x from the bigger end.

Width of the bar at the section X-X

$$= a - \frac{(a-b)x}{L}$$

$$= a - kx \qquad \text{where } k = \frac{a-b}{L}$$

Thickness of bar at section X-X = t

∴ Area of the section X-X

$$= \text{Width} \times \text{thickness}$$
$$= (a - kx)t$$

∴ Stress on the section X-X

$$= \frac{\text{Load}}{\text{Area}} = \frac{P}{(a - kx)t}$$

Extension of the small elemental length dx

$$= \text{Strain} \times \text{Length } dx$$
$$= \frac{\text{Stress}}{E} \times dx \qquad \left(\because \ \text{Strain} = \frac{\text{Stress}}{E}\right)$$
$$= \frac{\left(\dfrac{P}{(a - kx)t}\right)}{E} \times dx \qquad \left(\because \ \text{Stress} = \frac{P}{(a - kx)t}\right)$$
$$= \frac{P}{E(a - kx)t} \, ax$$

Total extension of the bar is obtained by integrating the above equation between the limits 0 and L.

Total extension,

$$dL = \int_0^L \frac{P}{E(a - kx)t} \, dx = \frac{P}{Et} \int_0^L \frac{dx}{(a - kx)}$$
$$= \frac{P}{Et} \cdot \log_e \Big[(a - kx)\Big]_0^L \times \left(-\frac{1}{k}\right) = -\frac{P}{Etk} \left[\log_e (a - kL) - \log_e a\right]$$
$$= \frac{P}{Etk} \left[\log_e a - \log_e (a - kL)\right] = \frac{P}{Etk} \left[\log_e \left(\frac{a}{a - kL}\right)\right]$$
$$= \frac{P}{Et\left(\dfrac{a - b}{L}\right)} \left[\log_e \left(\frac{a}{a - \left(\dfrac{a - b}{L}\right) L}\right)\right] \qquad \left(\because \ k = \frac{a - b}{L}\right)$$
$$= \frac{PL}{Et(a - b)} \log_e \frac{a}{b}$$

1.10. ELONGATION OF BAR DUE TO SELF WEIGHT

Consider a prismatic or circular bar of cross-sectional area *A* and length *L* hanging freely under its own weight as shown in Fig. 1.10. This circular bar experiences zero load at the free end and maximum load at the top. Weight of a body is given by the product of density and volume. Let γ be the density of the material. Consider a small section of thickness *dx* at a distance *x* from the free end.

The deformation of the element is given by

$$\delta dx = \frac{W_x dx}{AE}$$

Where W_x = weight of the portion below the section = $\gamma A x$

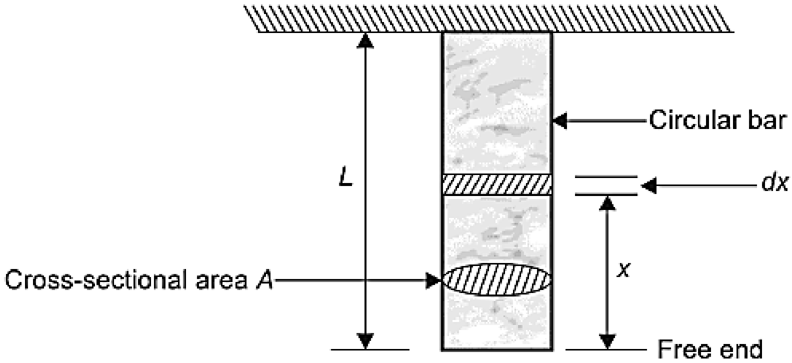


Fig.1.10: Elongation due to self weight

The extension of the entire bar can be obtained by integrating above Eq.

$$\int_0^L \delta dx = \delta L \int_0^L \frac{W_x dx}{AE}$$

$$\delta L = \int_0^L \frac{\gamma A x}{AE} dx$$

$$\delta L = \frac{\gamma}{E} \int_0^L x \times dx$$

$$\delta L = \frac{\gamma}{E} \left[\frac{x^2}{2} \right]_0^L$$

$$\delta L = \frac{\gamma L^2}{2E}$$

If W is the total weight of the bar, then

$$\gamma = \frac{W}{AL}$$

$$\delta L = \frac{WL}{2AE}$$

Note:

The deformation of the bar under its own weight is equal to the half of the deformation, if the body is subjected to the direct load equal to the weight of the body.

1.11. PRINCIPLE OF SUPERPOSITION.

When a number of loads are acting on a body, the resulting strain, according to principle of superposition, will be the algebraic sum of strains caused by individual loads.

While using this principle for an elastic body which is subjected to a number of direct forces (tensile or compressive) at different sections along the length of the body, first the free body diagram of individual section is drawn. Then the deformation of the each section is obtained. The total deformation of the body will be then equal to the algebraic sum of deformations of the individual sections.

WORKED EXAMPLES

1) The following observations were made during a tensile test on a mild steel specimen of 40 mm diameter and 200 mm long: Elongation with 40,000 N load (within the limit of proportionality) = 0.0304 mm, Yield load = 165,000 N, Maximum load = 245,000 N, Length of the specimen at fracture = 252 mm, Determine the yield stress, the modulus of elasticity, the ultimate stress and the percentage elongation.

Solution

Given:

Diameter of the specimen = 40 mm
Length of the specimen = 200 mm
Load = 40,000 N
Elongation within the limit of proportionality = 0.0304 mm
Yield load = 165,000 N
Maximum load = 245,000 N
Final length of the specimen = 252 mm

To find the yield stress:

Using the relation for yield stress, we have

$$\text{Yield stress} = \frac{\text{Yield load}}{\text{Area}} = \frac{165{,}000}{(\pi/4)\times 40^2} = 131.3 \text{ N/mm}^2$$

To find the modulus of elasticity:

Consider the load within the proportionality Limit. Then, stress is given by

$$\sigma = \frac{\text{Load (within the proportionality limit)}}{\text{Area}}$$

$$= \frac{40{,}000}{(\pi/4)\times 40^2} = \frac{40{,}000}{1256.64}$$

$$= 31.83 \text{ N/mm}^2$$

Strain is given by

$$\varepsilon = \frac{0.0304}{200} = 0.000152$$

Modulus of elasticity is given by

$$E = \frac{\sigma}{\varepsilon} = \frac{31.83}{0.000152} = 2.09 \times 10^5 \text{ N/mm}^2$$

To find the ultimate stress:

Using the relation for ultimate stress, we have

$$\text{Ultimate stress} = \frac{\text{Maximum load}}{\text{Area}} = \frac{245{,}000}{1256.64} = 194.96 \text{ N/mm}^2$$

To find the percentage elongation:

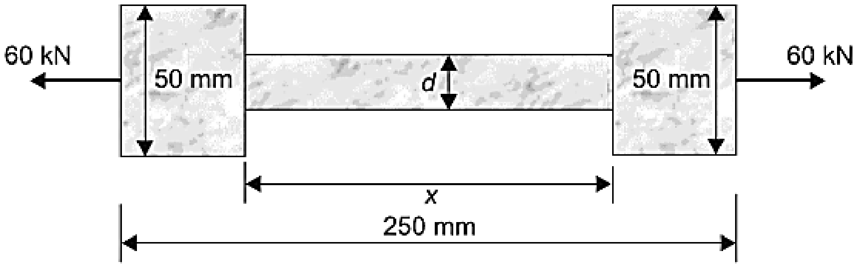
Using the relation, we have

Percentage elongation

$$= \frac{\text{Length of the specimen at failure} - \text{Original length}}{\text{Original length}} \times 100$$

$$= \frac{252 - 200}{200} \times 100 = 26\%$$

2) The bar shown in Fig. is subjected to a tensile load of 60 kN. Find the diameter of the middle portion of the bar if the stress is limited to 120 N/mm^2. Also find the length of the middle portion if the total elongation of the bar is 0.12 mm. Take E = 2 x 10^5 N/mm^2.



Solution

To find the diameter at the middle portion of the bar:

Stress in the middle portion of the bar is given by

$$\sigma = 120 = \frac{60{,}000}{(\pi/4)d^2}$$

$$d = 25.23 \text{ mm}$$

To find the length of the middle portion of the bar:

Let the length of the middle portion of the bar be x

Stress in the end portion is given by

$$\sigma' = \frac{60{,}000}{(\pi/4) \times 50^2} = 30.56 \text{ N/mm}^2$$

Also, total elongation = elongation of the end portion + elongation of the middle portion = 0.12 mm

$$\frac{30.56(250-x)}{2\times10^5}+\frac{120x}{2\times10^5}=0.12$$

$$30.56 \times 250 - 30.56x + 120x = 0.12 \times 2 \times 10^5$$

$$7640 + 89.44x = 24{,}000$$

$$x = 182.92 \text{ mm}$$

3) A flat steel plate is of trapezoidal form of uniform thickness of 8 mm and tapers uniformly from a width of 60 mm to 90 mm in a length of 300 mm. Determine the elongation of the plate under the axial force of 40 kN at each end. Assume E = 205 kN/mm^2.

Solution:

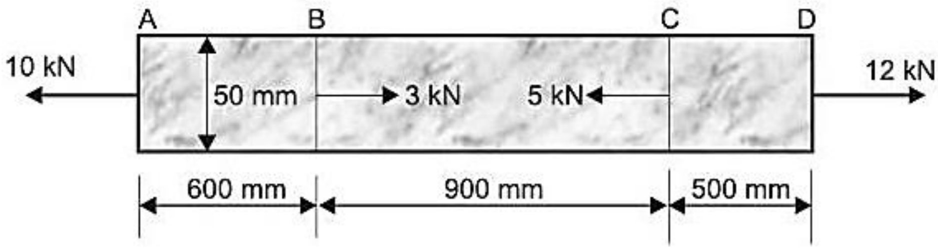
Thickness of the plate t = 8 mm
Width at one end b = 60 mm
Width at other end a = 90 mm
Length of the plate L = 300 mm
Axial force P = 40 kN
Modulus of elasticity E = 205 kN/mm^2

Using the relation, we have

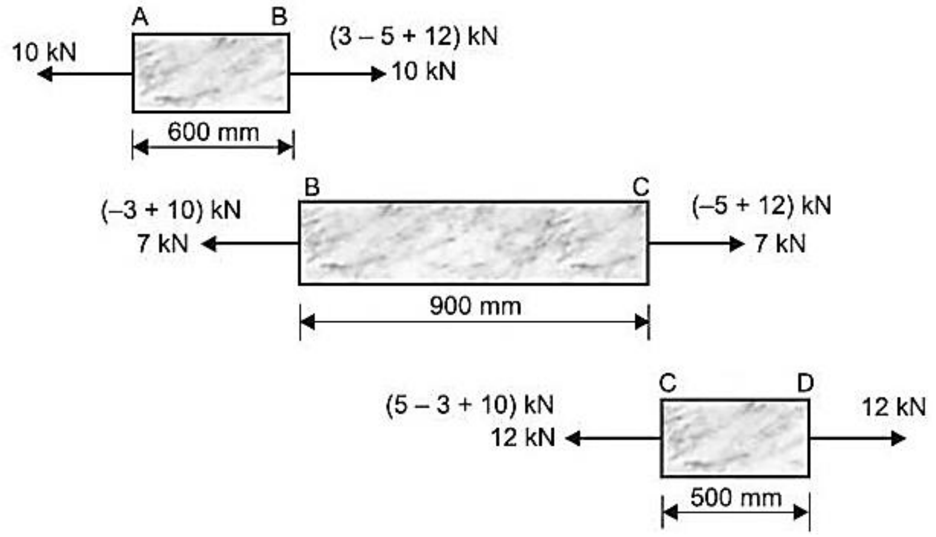
$$\delta l = \frac{2.302PL}{Et(a-b)}(\log a - \log b)$$

$$= \frac{2.302\times40\times300}{205\times8(90-60)}(\log 90 - \log 60) = 0.099 \text{ mm}$$

4) Figure shows the bar AB of uniform cross-sectional area is acted upon by several forces. Find the deformation of the bar, assuming E = 2 x 10^5 N/mm^2.



Solution: The free body diagram (F.B.D.) of individual sections is shown in Figure.



Total elongation is given by

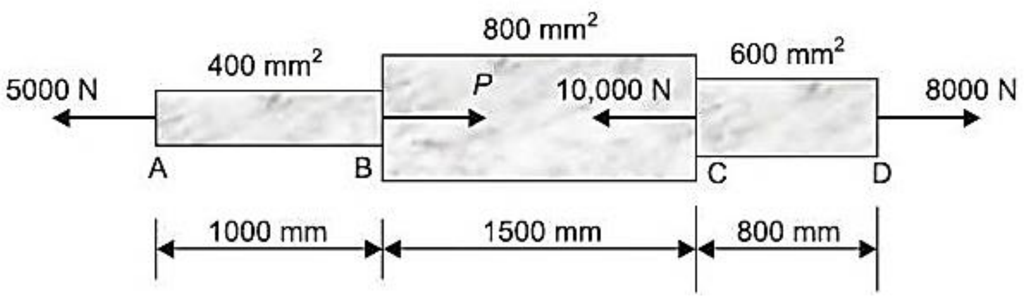
$$\delta L = \delta L_1 + \delta L_2 + \delta L_3$$

$$= \frac{P_1 L_1}{AE} + \frac{P_2 L_2}{AE} + \frac{P_3 L_3}{AE}$$

$$= \frac{1}{1963.5 \times 2 \times 10^5}(10{,}000 \times 600 + 7000 \times 900 + 12{,}000 \times 500$$

$$= 0.047 \text{ mm}$$

5) A steel bar ABCD of varying cross-section is subjected to the axial forces as shown in Fig. Find the value of P for equilibrium. If the modulus of elasticity E = 2.1 x 10^5 N/mm^2, determine the elongation of the bar.



Solution:

From the equilibrium condition:

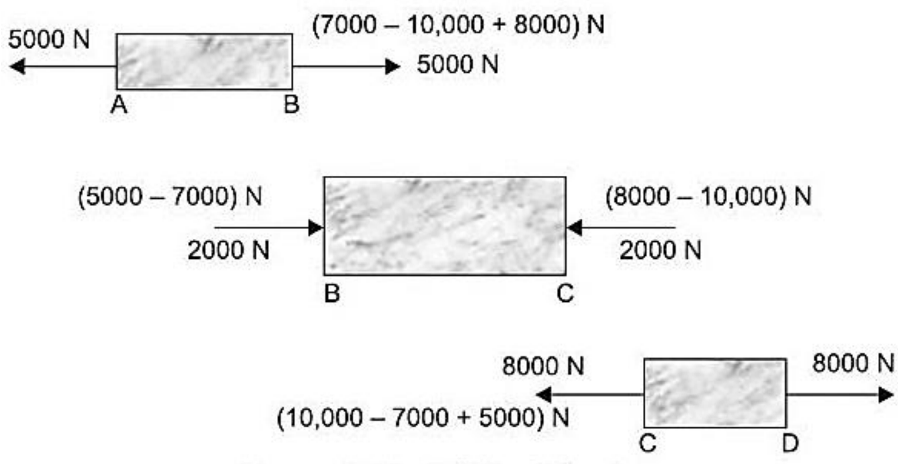
$$\Sigma F_x = 0$$

$$+8000 - 10{,}000 + P - 5000 = 0$$

$$P = 15{,}000 - 8000 = 7000 \text{ N}$$

To find the elongation of the bar:

Consider the free body diagram (F.B.D.) of the bar,



Total elongation is given by

$$\delta L = \delta L_1 + \delta L_2 + \delta L_3$$

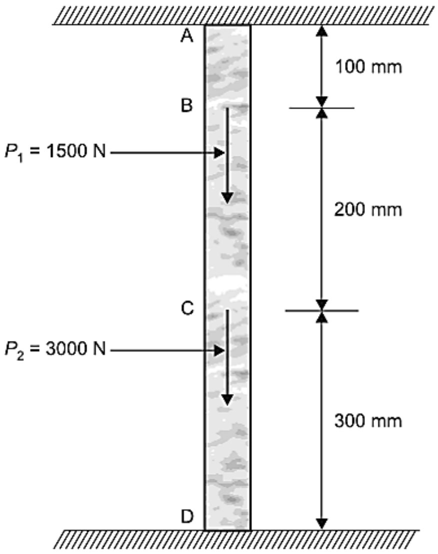
$$= \frac{1}{E}\left\{\frac{P_1L_1}{A_1} + \frac{P_2L_2}{A_2} + \frac{P_3L_3}{A_3}\right\}$$

$$= \frac{1}{2.1\times10^5}\left\{\frac{5000\times1000}{400} - \frac{2000\times1500}{800} + \frac{8000\times800}{600}\right\}$$

$$= \frac{1}{2.1\times10^5}(12{,}500 - 3750 + 10{,}666.67) = 0.092 \text{ mm}$$

6) A vertical prismatic bar is fastened at its upper end and supported at the lower end by an unyielding floor as shown in Fig. Determine the reaction *R* exerted by the floor of the bar if external loads P_1 = 1500 N and P_2 = 3000 N are applied at the intermediate points shown.

Solution



Let A be the cross-sectional area of the bar, and *E* be the modulus of elasticity.

Elongation of AD = elongation of AB + elongation of BC + elongation of the bar CD

$$\delta L = \frac{(1500+3000)100}{AE} + \frac{3000\times200}{AE} + 0$$

Since there is a rigid support at D, there is a reaction R at D which causes contraction of AD, i.e.

$$\delta L = \frac{R(100+200+300)}{AE}$$

As there is no change in the length of the bar AD, we have

$$\frac{R(100+200+300)}{AE} = \frac{(1500+3000)100}{AE} + \frac{3000\times 200}{AE} + 0$$

$$R = \frac{4500\times 100 + 3000\times 200}{(100+200+300)} = 1750 \text{ N}$$

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