CS795 Large Scale Optimization in Machine Learning

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1. We attached the code in our submission and below is our observation.
   a) The AGD diverse and reach to infinity with 100 iteration. on the other hand extra-gradient fast convergence to optima.
   b) After bounding the variable x and y from 0 to 1. The solution converge fast.
   c) We implemented the both SGD and extra SGD algorithm but generator does not perform very well. Also as we try to increase number of epochs the loss for generator also increased. Please find the attached code pdfs for the same.

2. Consider function $f(x, y) = \frac{y}{x+y}$ with $\mathcal{X} = [1, +\infty)$ and $\mathcal{Y} = [1, +\infty)$

   - Show that the function f is convex-concave on $\mathcal{X}$ x $\mathcal{Y}$.

     Proof. First, we show f is convex in $\mathcal{X}$. For any $\lambda \in [0, 1]$, for any $x_1 \neq x_2 \in \mathcal{X}$, and for any $y \in \mathcal{Y}$,

     $$f(\lambda x_1 + (1-\lambda)x_2, y) = \frac{y}{\lambda x_1 + (1-\lambda)x_2 + y}$$

     $$= \frac{y}{\lambda(x_1 + y) + (1-\lambda)(x_2 + y)} \leq \frac{\lambda y}{x_1 + y} + \frac{(1-\lambda)y}{x_2 + y}$$

     $$= \lambda f(x_1, y) + (1-\lambda) f(x_2, y)$$

     Second, we show $g = -f$ is convex in $\mathcal{Y}$, i.e. f is concave in $\mathcal{Y}$. For any $\lambda \in [0, 1]$, for any $y_1 \neq y_2 \in \mathcal{Y}$, and for any $x \in \mathcal{X}$,

     $$g(x, \lambda y_1 + (1-\lambda)y_2) = \frac{-(\lambda y_1 + (1-\lambda)y_2)}{x + \lambda y_1 + (1-\lambda)y_2}$$

     $$= \frac{-(\lambda y_1 + (1-\lambda)y_2)}{\lambda(x + y_1) + (1-\lambda)(x + y_2)} = \frac{-\lambda y_1}{\lambda(x + y_1) + (1-\lambda)(x + y_2)} + \frac{-(1-\lambda)y_2}{\lambda(x + y_1) + (1-\lambda)(x + y_2)}$$

     $$\leq \frac{-\lambda y_1}{\lambda(x + y_1)} + \frac{-(1-\lambda)y_2}{x + y_2}$$

     $$= \lambda g(x, y_1) + (1-\lambda) g(x, y_2)$$

     Hence, f is convex-concave on $\mathcal{X}$ x $\mathcal{Y}$.

     □

   - Show that Nash equilibrium does not exist.

     Proof. Let $y_* = \arg\max_{y \in Y} \min_{x \in X} f(x, y)$ and $x_* = \arg\min_{x \in X} \max_{y \in Y} f(x, y)$. So,

     $$\max_{y \in Y} \min_{x \in X} f(x, y) = \min_{x \in X} f(x, y_*) = \min_{x \in X} \frac{y_*}{x + y_*} = 0$$

     On the other hand,

     $$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} f(x_*, y) = \max_{y \in Y} \frac{y}{x_* + y} = \max_{y \in Y} \frac{1}{\frac{x_*}{y} + 1} = 1$$

     By the above, $\max_{y \in Y} \min_{x \in X} f(x, y) \neq \min_{x \in X} \max_{y \in Y} f(x, y)$, and thus, Nash equilibrium does not exist. □

3. Existence or Nonexistence of Nash Equilibrium for Noncompact Convex-concave Min-max Problems.

Proof. For non-compact cases, though question 2 is a perfect example showing that the Nash equilibrium does not exist even for strictly-convex-strictly-concave functions. We here show another counterexample. Consider below function f on domain $[-\infty, 0]$ x $[0, \infty]$

$$f(x, y) = -x^3y^2$$

It's very obvious that f is strictly convex in $\mathcal{X} = [-\infty, 0]$ and is strictly concave in $\mathcal{Y} = [0, \infty]$. This can be evaluated by simply looking at the hessian of f with respect to different variables in each domain.

Let $y_* = \arg\max_{y\in Y} \min_{x\in X} f(x, y)$ and $x_* = \arg\min_{x\in X} \max_{y\in Y} f(x, y)$. So,

$$\max_{y\in Y} \min_{x\in X} f(x, y) = \min_{x\in X} f(x, y_*) = \min_{x\in X} -x^3y_*^2 \longrightarrow \infty$$

On the other hand,

$$\min_{x\in X} \max_{y\in Y} f(x, y) = \max_{y\in Y} f(x_*, y) = \max_{y\in Y} -x_*^3y^2 = 0$$

Hence, Nash equilibrium does not exist.

□

4. We attached question 4 in the next page.