12–16 PARSEVAL'S IDENTITY

Using Parseval's identity, prove that the series have the indicated sums. Compute the first few partial sums to see that the convergence is rapid.

12. $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96} = 1.01467\,8032$

(Use Prob. 15 in Sec. 11.1.)

13. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} = 1.23370\,0550$

(Use Prob. 13 in Sec. 11.1.)

14. $\frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \cdots$

$= \frac{\pi^2}{16} - \frac{1}{2} = 0.11685\,0275$

(Use Prob. 5, this set.)

15. $1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} = 1.08232\,3234$

(Use Prob. 21 in Sec. 11.1.)

16. $1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots = \frac{\pi^6}{960} = 1.00144\,7078$

(Use Prob. 9, this set.)

11.7 Fourier Integral

Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only. Sections 11.3 and 11.5 first illustrated this, and various further applications follow in Chap. 12. Since, of course, many problems involve functions that are **nonperiodic** *and are of interest on the whole x-axis,* we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to "Fourier integrals."

In Example 1 we start from a special function f_L of period $2L$ and see what happens to its Fourier series if we let $L \to \infty$. Then we do the same for an *arbitrary* function f_L of period $2L$. This will motivate and suggest the main result of this section, which is an integral representation given in Theorem 1 (below).

EXAMPLE 1 Rectangular Wave

Consider the periodic rectangular wave $f_L(x)$ of period $2L > 2$ given by

$$f_L(x) = \begin{cases} 0 & \text{if} \quad -L < x < -1 \\ 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < L. \end{cases}$$

The left part of Fig. 277 shows this function for $2L = 4, 8, 16$ as well as the nonperiodic function $f(x)$, which we obtain from f_L if we let $L \to \infty$,

$$f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We now explore what happens to the Fourier coefficients of f_L as L increases. Since f_L is even, $b_n = 0$ for all n. For a_n the Euler formulas (6), Sec. 11.2, give

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}, \qquad a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\,\frac{\sin(n\pi/L)}{n\pi/L}.$$

This sequence of Fourier coefficients is called the **amplitude spectrum** of f_L because $|a_n|$ is the maximum amplitude of the wave $a_n \cos(n\pi x/L)$. Figure 277 shows this spectrum for the periods $2L = 4, 8, 16$. We see that for increasing L these amplitudes become more and more dense on the positive w_n-axis, where $w_n = n\pi/L$. Indeed, for $2L = 4, 8, 16$ we have 1, 3, 7 amplitudes per "half-wave" of the function $(2 \sin w_n)/(Lw_n)$ (dashed in the figure). Hence for $2L = 2^k$ we have $2^{k-1} - 1$ amplitudes per half-wave, so that these amplitudes will eventually be everywhere dense on the positive w_n-axis (and will decrease to zero).

The outcome of this example gives an intuitive impression of what about to expect if we turn from our special function to an arbitrary one, as we shall do next. ∎

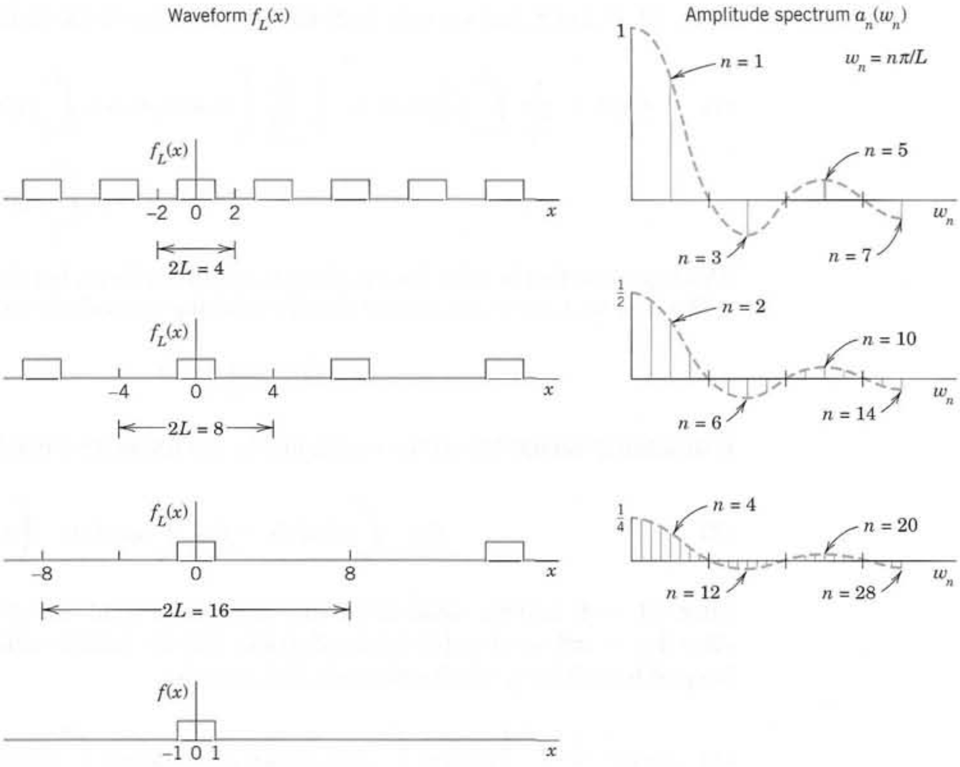

Fig. 277. Waveforms and amplitude spectra in Example 1

From Fourier Series to Fourier Integral

We now consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \qquad w_n = \frac{n\pi}{L}$$

and find out what happens if we let $L \to \infty$. Together with Example 1 the present calculation will suggest that we should expect an integral (instead of a series) involving $\cos wx$ and $\sin wx$ with w no longer restricted to integer multiples $w = w_n = n\pi/L$ of π/L but taking *all* values. We shall also see what form such an integral might have.

If we insert a_n and b_n from the Euler formulas (6), Sec. 11.2, and denote the variable of integration by v, the Fourier series of $f_L(x)$ becomes

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\, dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v) \cos w_n v\, dv + \sin w_n x \int_{-L}^{L} f_L(v) \sin w_n v\, dv\right].$$

We now set

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

Then $1/L = \Delta w/\pi$, and we may write the Fourier series in the form

$$(1) \qquad f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Bigg[(\cos w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\,dv + (\sin w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\,dv\Bigg].$$

This representation is valid for any fixed L, arbitrarily large, but finite.

We now let $L \to \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L\to\infty} f_L(x)$$

is **absolutely integrable** on the x-axis; that is, the following (finite!) limits exist:

$$(2) \qquad \lim_{a\to-\infty}\int_a^0 |f(x)|\,dx + \lim_{b\to\infty}\int_0^b |f(x)|\,dx \quad \left(\text{written } \int_{-\infty}^{\infty} |f(x)|\,dx\right).$$

Then $1/L \to 0$, and the value of the first term on the right side of (1) approaches zero. Also $\Delta w = \pi/L \to 0$ and it seems ***plausible*** that the infinite series in (1) becomes an integral from 0 to ∞, which represents $f(x)$, namely,

$$(3) \quad f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\,dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\,dv\right] dw.$$

If we introduce the notations

$$\textbf{(4)} \qquad A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

we can write this in the form

$$\textbf{(5)} \qquad f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\,dw.$$

This is called a representation of $f(x)$ by a **Fourier integral.**

It is clear that our naive approach merely ***suggests*** the representation (5), but by no means establishes it; in fact, the limit of the series in (1) as Δw approaches zero is not the definition of the integral (3). Sufficient conditions for the validity of (5) are as follows.

THEOREM 1

Fourier Integral

If $f(x)$ is piecewise continuous (see Sec. 6.1) *in every finite interval and has a right-hand derivative and a left-hand derivative at every point* (see Sec 11.1) *and if the integral* (2) *exists, then $f(x)$ can be represented by a Fourier integral* (5) *with A and B given by* (4). *At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point* (see Sec. 11.1). (Proof in Ref. [C12]; see App. 1.)

Applications of Fourier Integrals

The main application of Fourier integrals is in solving ODEs and PDEs, as we shall see for PDEs in Sec. 12.6. However, we can also use Fourier integrals in integration and in discussing functions defined by integrals, as the next examples (2 and 3) illustrate.

EXAMPLE 2 Single Pulse, Sine Integral

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases} \qquad \text{(Fig. 278).}$$

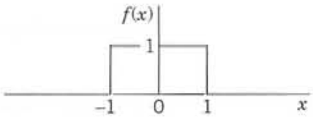

Fig. 278. Example 2

Solution. From (4) we obtain

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv = \frac{1}{\pi}\int_{-1}^{1}\cos wv\, dv = \left.\frac{\sin wv}{\pi w}\right|_{-1}^{1} = \frac{2\sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{1}\sin wv\, dv = 0$$

and (5) gives the answer

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\cos wx \sin w}{w}\, dw. \tag{6}$$

The average of the left- and right-hand limits of $f(x)$ at $x = 1$ is equal to $(1 + 0)/2$, that is, $1/2$.

Furthermore, from (6) and Theorem 1 we obtain (multiply by $\pi/2$)

$$\int_0^{\infty}\frac{\cos wx \sin w}{w}\, dw = \begin{cases} \pi/2 & \text{if} \quad 0 \leqq x < 1, \\ \pi/4 & \text{if} \quad x = 1, \\ 0 & \text{if} \quad x > 1. \end{cases} \tag{7}$$

We mention that this integral is called **Dirichlet's discontinous factor.** (For P. L. Dirichlet see Sec. 10.8.)

The case $x = 0$ is of particular interest. If $x = 0$, then (7) gives

$$\int_0^{\infty}\frac{\sin w}{w}\, dw = \frac{\pi}{2}\,. \tag{8*}$$

We see that this integral is the limit of the so-called **sine integral**

$$\mathrm{Si}(u) = \int_0^{u}\frac{\sin w}{w}\, dw \tag{8}$$

as $u \to \infty$. The graphs of $\mathrm{Si}(u)$ and of the integrand are shown in Fig. 279.

In the case of a Fourier series the graphs of the partial sums are approximation curves of the curve of the periodic function represented by the series. Similarly, in the case of the Fourier integral (5), approximations are obtained by replacing ∞ by numbers a. Hence the integral

$$\frac{2}{\pi}\int_0^{a}\frac{\cos wx \sin w}{w}\, dw \tag{9}$$

approximates the right side in (6) and therefore $f(x)$.

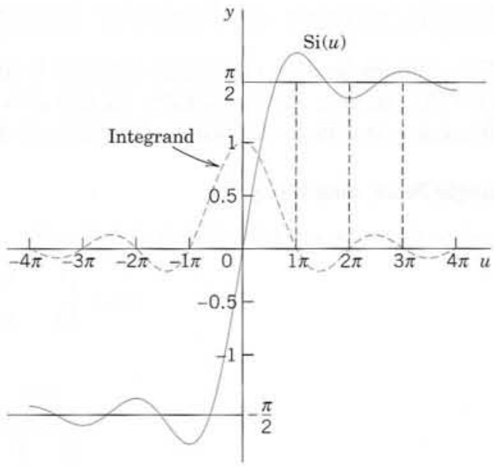

Fig. 279. Sine integral Si(u) and integrand

Figure 280 shows oscillations near the points of discontinuity of $f(x)$. We might expect that these oscillations disappear as a approaches infinity. But this is not true; with increasing a, they are shifted closer to the points $x = \pm 1$. This unexpected behavior, which also occurs in connection with Fourier series, is known as the **Gibbs phenomenon.** (See also Problem Set 11.2.) We can explain it by representing (9) in terms of sine integrals as follows. Using (11) in App. A3.1, we have

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^a \frac{\sin (w + wx)}{w}\,dw + \frac{1}{\pi}\int_0^a \frac{\sin (w - wx)}{w}\,dw.$$

In the first integral on the right we set $w + wx = t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x + 1)a$. In the last integral we set $w - wx = -t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x - 1)a$. Since $\sin(-t) = -\sin t$, we thus obtain

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^{(x+1)a} \frac{\sin t}{t}\,dt - \frac{1}{\pi}\int_0^{(x-1)a} \frac{\sin t}{t}\,dt.$$

From this and (8) we see that our integral (9) equals

$$\frac{1}{\pi}\,\mathrm{Si}(a[x + 1]) - \frac{1}{\pi}\,\mathrm{Si}(a[x - 1])$$

and the oscillations in Fig. 280 result from those in Fig. 279. The increase of a amounts to a transformation of the scale on the axis and causes the shift of the oscillations (the waves) toward the points of discontinuity -1 and 1. ■

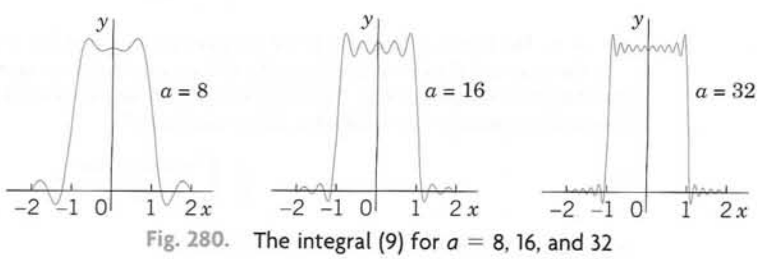

Fig. 280. The integral (9) for $a = 8, 16$, and 32

Fourier Cosine Integral and Fourier Sine Integral

For an even or odd function the Fourier integral becomes simpler. Just as in the case of Fourier series (Sec. 11.3), this is of practical interest in saving work and avoiding errors. The simplifications follow immediately from the formulas just obtained.

Indeed, if $f(x)$ is an ***even*** function, then $B(w) = 0$ in (4) and

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv. \tag{10}$$

The Fourier integral (5) then reduces to the **Fourier cosine integral**

$$f(x) = \int_0^\infty A(w)\cos wx\,dw \qquad (f \text{ even}). \tag{11}$$

Similarly, if $f(x)$ is ***odd,*** then in (4) we have $A(w) = 0$ and

$$B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv\,dv. \tag{12}$$

The Fourier integral (5) then reduces to the **Fourier sine integral**

$$f(x) = \int_0^\infty B(w)\sin wx\,dw \qquad (f \text{ odd}). \tag{13}$$

Evaluation of Integrals

Earlier in this section we pointed out that the main application of the Fourier integral is in differential equations but that Fourier integral representations also help in evaluating certain integrals. To see this, we show the method for an important case, the Laplace integrals.

EXAMPLE 3 Laplace Integrals

We shall derive the Fourier cosine and Fourier sine integrals of $f(x) = e^{-kx}$, where $x > 0$ and $k > 0$ (Fig. 281). The result will be used to evaluate the so-called Laplace integrals.

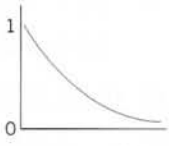

Fig. 281. $f(x)$ in Example 3

Solution. **(a)** From (10) we have $A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\,dv$. Now, by integration by parts,

$$\int e^{-kv}\cos wv\,dv = -\frac{k}{k^2+w^2}\,e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right).$$

If $v = 0$, the expression on the right equals $-k/(k^2 + w^2)$. If v approaches infinity, that expression approaches zero because of the exponential factor. Thus

$$A(w) = \frac{2k/\pi}{k^2+w^2}. \tag{14}$$

By substituting this into (11) we thus obtain the Fourier cosine integral representation

$$f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}\,dw \qquad (x > 0,\quad k > 0).$$

From this representation we see that

$$\textbf{(15)}\qquad \int_0^\infty \frac{\cos wx}{k^2 + w^2}\, dw = \frac{\pi}{2k}\, e^{-kx} \qquad (x > 0,\quad k > 0).$$

(b) Similarly, from (12) we have $B(w) = \dfrac{2}{\pi}\displaystyle\int_0^\infty e^{-kv} \sin wv\, dv$. By integration by parts,

$$\int e^{-kv} \sin wv\, dv = -\frac{w}{k^2 + w^2}\, e^{-kv} \left(\frac{k}{w} \sin wv + \cos wv\right).$$

This equals $-w/(k^2 + w^2)$ if $v = 0$, and approaches 0 as $v \to \infty$. Thus

$$(16)\qquad B(w) = \frac{2w/\pi}{k^2 + w^2}.$$

From (13) we thus obtain the Fourier sine integral representation

$$f(x) = e^{-kx} = \frac{2}{\pi} \int_0^\infty \frac{w \sin wx}{k^2 + w^2}\, dw.$$

From this we see that

$$\textbf{(17)}\qquad \int_0^\infty \frac{w \sin wx}{k^2 + w^2}\, dw = \frac{\pi}{2}\, e^{-kx} \qquad (x > 0,\quad k > 0).$$

The integrals (15) and (17) are called the **Laplace integrals.** ■

PROBLEM SET 11.7

1–6 EVALUATION OF INTEGRALS

Show that the given integral represents the indicated function. *Hint.* Use (5), (11), or (13); the integral tells you which one, and its value tells you what function to consider. (Show the details of your work.)

1. $\displaystyle\int_0^\infty \frac{\cos xw + w \sin xw}{1 + w^2}\, dw = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$

2. $\displaystyle\int_0^\infty \frac{\sin w - w \cos w}{w^2} \sin xw\, dw = \begin{cases} \pi x/2 & \text{if } 0 < x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$

3. $\displaystyle\int_0^\infty \frac{\cos xw}{1 + w^2}\, dw = \frac{\pi}{2} e^{-x}$ if $x > 0$

4. $\displaystyle\int_0^\infty \frac{\sin w}{w} \cos xw\, dw = \begin{cases} \pi/2 & \text{if } 0 \leqq x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$

5. $\displaystyle\int_0^\infty \frac{\cos(\pi w/2)}{1 - w^2} \cos xw\, dw = \begin{cases} \frac{\pi}{2} \cos x & \text{if } 0 < |x| < \pi/2 \\ 0 & \text{if } |x| \geqq \pi/2 \end{cases}$

6. $\displaystyle\int_0^\infty \frac{\sin \pi w \sin xw}{1 - w^2}\, dw = \begin{cases} \frac{\pi}{2} \sin x & \text{if } 0 \leqq x \leqq \pi \\ 0 & \text{if } x > \pi \end{cases}$

7–12 FOURIER COSINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (11).

7. $f(x) = \begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$

9. $f(x) = \begin{cases} x & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$

10. $f(x) = \begin{cases} x/2 & \text{if} \quad 0 < x < 1 \\ 1 - x/2 & \text{if} \quad 1 < x < 2 \\ 0 & \text{if} \quad x > 2 \end{cases}$

11. $f(x) = \begin{cases} \sin x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

12. $f(x) = \begin{cases} e^{-x} & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$

13. **CAS EXPERIMENT. Approximate Fourier Cosine Integrals.** Graph the integrals in Prob. 7, 9, and 11 as functions of x. Graph approximations obtained by replacing ∞ with finite upper limits of your choice. Compare the quality of the approximations. Write a short report on your empirical results and observations.

14–19 FOURIER SINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (13).

14. $f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$

15. $f(x) = \begin{cases} \sin x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

16. $f(x) = \begin{cases} 1 - x^2 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$

17. $f(x) = \begin{cases} \pi - x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

18. $f(x) = \begin{cases} \cos x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

19. $f(x) = \begin{cases} a - x & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$

20. **PROJECT. Properties of Fourier Integrals**
(a) Fourier cosine integral. Show that (11) implies

(a1) $$f(ax) = \frac{1}{a} \int_0^\infty A\left(\frac{w}{a}\right) \cos xw \, dw$$

$(a > 0)$ *(Scale change)*

(a2) $$xf(x) = \int_0^\infty B^*(w) \sin xw \, dw,$$

$$B^* = -\frac{dA}{dw}, \qquad A \text{ as in (10)}$$

(a3) $$x^2 f(x) = \int_0^\infty A^*(w) \cos xw \, dw,$$

$$A^* = -\frac{d^2A}{dw^2}.$$

(b) Solve Prob. 8 by applying (a3) to the result of Prob. 7.

(c) Verify (a2) for $f(x) = 1$ if $0 < x < a$ and $f(x) = 0$ if $x > a$.

(d) Fourier sine integral. Find formulas for the Fourier sine integral similar to those in (a).

11.8 Fourier Cosine and Sine Transforms

An **integral transform** is a transformation in the form of an integral that produces from given functions new functions depending on a different variable. These transformations are of interest mainly as tools for solving ODEs, PDEs, and integral equations, and they often also help in handling and applying special functions. The **Laplace transform** (Chap. 6) is of this kind and is by far the most important integral transform in engineering.

The next in order of importance are Fourier transforms. We shall see that these transforms can be obtained from the Fourier integral in Sec. 11.7 in a rather simple fashion. In this section we consider two of them, which are real, and in the next section a third one that is complex.

Fourier Cosine Transform

For an ***even*** function $f(x)$, the Fourier integral is the Fourier cosine integral

$$(1) \quad \text{(a)} \quad f(x) = \int_0^\infty A(w) \cos wx \, dw, \qquad \text{where} \qquad \text{(b)} \quad A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv$$

[see (10), (11), Sec. 11.7]. We now set $A(w) = \sqrt{2/\pi}\, \hat{f}_c(w)$, where c suggests "cosine." Then from (1b), writing $v = x$, we have

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx \tag{2}$$

and from (1a),

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw. \tag{3}$$

ATTENTION! In (2) we integrate with respect to x and in (3) with respect to w. Formula (2) gives from $f(x)$ a new function $\hat{f}_c(w)$, called the **Fourier cosine transform** of $f(x)$. Formula (3) gives us back $f(x)$ from $\hat{f}_c(w)$, and we therefore call $f(x)$ the **inverse Fourier cosine transform** of $\hat{f}_c(w)$.

The process of obtaining the transform $\hat{f}_c$ from a given f is also called the **Fourier cosine transform** or the *Fourier cosine transform method.*

Fourier Sine Transform

Similarly, for an ***odd*** function $f(x)$, the Fourier integral is the Fourier sine integral [see (12), (13), Sec. 11.7]

$$(4) \quad \text{(a)} \quad f(x) = \int_0^\infty B(w) \sin wx \, dw, \qquad \text{where} \qquad \text{(b)} \quad B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv.$$

We now set $B(w) = \sqrt{2/\pi}\, \hat{f}_s(w)$, where s suggests "sine." Then from (4b), writing $v = x$, we have

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx. \tag{5}$$

This is called the **Fourier sine transform** of $f(x)$. Similarly, from (4a) we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw. \tag{6}$$

This is called the **inverse Fourier sine transform** of $\hat{f}_s(w)$. The process of obtaining $\hat{f}_s(w)$ from $f(x)$ is also called the **Fourier sine transform** or the *Fourier sine transform method.*

Other notations are

$$\mathscr{F}_c(f) = \hat{f}_c, \qquad \mathscr{F}_s(f) = \hat{f}_s$$

and $\mathscr{F}_c^{-1}$ and $\mathscr{F}_s^{-1}$ for the inverses of $\mathscr{F}_c$ and $\mathscr{F}_s$, respectively.

EXAMPLE 1 **Fourier Cosine and Fourier Sine Transforms**

Find the Fourier cosine and Fourier sine transforms of the function

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases} \qquad \text{(Fig. 282).}$$

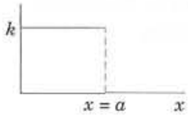

Fig. 282. $f(x)$ in Example 1

Solution. From the definitions (2) and (5) we obtain by integration

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \cos wx \, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{\sin aw}{w}\right)$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \sin wx \, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{1 - \cos aw}{w}\right).$$

This agrees with formulas 1 in the first two tables in Sec. 11.10 (where $k = 1$).

Note that for $f(x) = k = const$ $(0 < x < \infty)$, these transforms do not exist. (Why?) ■

EXAMPLE 2 **Fourier Cosine Transform of the Exponential Function**

Find $\mathcal{F}_c(e^{-x})$.

Solution. By integration by parts and recursion,

$$\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos wx \, dx = \sqrt{\frac{2}{\pi}}\, \frac{e^{-x}}{1 + w^2}\,(-\cos wx + w \sin wx)\Bigg|_0^\infty = \frac{\sqrt{2/\pi}}{1 + w^2}.$$

This agrees with formula 3 in Table I, Sec. 11.10, with $a = 1$. See also the next example. ■

What did we do to introduce the two integral transforms under consideration? Actually not much: We changed the notations A and B to get a "symmetric" distribution of the constant $2/\pi$ in the original formulas (10)–(13), Sec. 11.7. This redistribution is a standard convenience, but it is not essential. One could do without it.

What have we gained? We show next that these transforms have operational properties that permit them to convert differentiations into algebraic operations (just as the Laplace transform does). This is the key to their application in solving differential equations.

Linearity, Transforms of Derivatives

If $f(x)$ is absolutely integrable (see Sec. 11.7) on the positive x-axis and piecewise continuous (see Sec. 6.1) on every finite interval, then the Fourier cosine and sine transforms of f exist.

Furthermore, if f and g have Fourier cosine and sine transforms, so does $af + bg$ for any constants a and b, and by (2),

$$\begin{aligned}\mathcal{F}_c(af + bg) &= \sqrt{\frac{2}{\pi}} \int_0^\infty [af(x) + bg(x)] \cos wx \, dx \\ &= a\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx + b\sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos wx \, dx.\end{aligned}$$

The right side is $a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$. Similarly for $\mathcal{F}_s$, by (5). This shows that the Fourier cosine and sine transforms are **linear operations,**

$$\text{(7)} \qquad \begin{aligned} &\text{(a)} \quad \mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g), \\ &\text{(b)} \quad \mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g). \end{aligned}$$

THEOREM 1

Cosine and Sine Transforms of Derivatives

Let $f(x)$ be continuous and absolutely integrable on the x-axis, let $f'(x)$ be piecewise continuous on every finite interval, and let let $f(x) \to 0$ *as* $x \to \infty$. *Then*

$$\text{(8)}\qquad \begin{aligned} &\text{(a)} \qquad \mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f(0), \\ &\text{(b)} \qquad \mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}. \end{aligned}$$

PROOF This follows from the definitions by integration by parts, namely,

$$\begin{aligned} \mathcal{F}_c\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[f(x) \cos wx \Big|_0^\infty + w \int_0^\infty f(x) \sin wx \, dx \right] \\ &= -\sqrt{\frac{2}{\pi}}\, f(0) + w\mathcal{F}_s\{f(x)\}; \end{aligned}$$

and similarly,

$$\begin{aligned} \mathcal{F}_s\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin wx \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[f(x) \sin wx \Big|_0^\infty - w \int_0^\infty f(x) \cos wx \, dx \right] \\ &= 0 - w\mathcal{F}_c\{f(x)\}. \end{aligned}$$

Formula (8a) with f' instead of f gives (when f', f'' satisfy the respective assumptions for f, f' in Theorem 1)

$$\mathcal{F}_c\{f''(x)\} = w\mathcal{F}_s\{f'(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0);$$

hence by (8b)

$$\text{(9a)}\qquad \mathcal{F}_c\{f''(x)\} = -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0).$$

Similarly,

$$\text{(9b)}\qquad \mathcal{F}_s\{f''(x)\} = -w^2\mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\, wf(0).$$

A basic application of (9) to PDEs will be given in Sec. 12.6. For the time being we show how (9) can be used for deriving transforms.

EXAMPLE 3 An Application of the Operational Formula (9)

Find the Fourier cosine transform $\mathcal{F}_c(e^{-ax})$ of $f(x) = e^{-ax}$, where $a > 0$.

Solution. By differentiation, $(e^{-ax})'' = a^2e^{-ax}$; thus

$$a^2 f(x) = f''(x).$$

From this, (9a), and the linearity (7a),

$$\begin{aligned} a^2\mathcal{F}_c(f) &= \mathcal{F}_c(f'') \\ &= -w^2\mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}}\, f'(0) \\ &= -w^2\mathcal{F}_c(f) + a\sqrt{\frac{2}{\pi}}\,. \end{aligned}$$

Hence

$$(a^2 + w^2)\mathcal{F}_c(f) = a\sqrt{2/\pi}.$$

The *answer* is (see Table I, Sec. 11.10)

$$\mathcal{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2 + w^2}\right) \qquad (a > 0).$$ ■

Tables of Fourier cosine and sine transforms are included in Sec. 11.10.

PROBLEM SET 11.8

1–10 FOURIER COSINE TRANSFORM

1. Let $f(x) = -1$ if $0 < x < 1$, $f(x) = 1$ if $1 < x < 2$, $f(x) = 0$ if $x > 2$. Find $\hat{f}_c(w)$.
2. Let $f(x) = x$ if $0 < x < k$, $f(x) = 0$ if $x > k$. Find $\hat{f}_c(w)$.
3. Derive formula 3 in Table 1 of Sec. 11.10 by integration.
4. Find the inverse Fourier cosine transform $f(x)$ from the answer to Prob. 1. *Hint.* Use Prob. 4 in Sec. 11.7.
5. Obtain $\mathcal{F}_c^{-1}(1/(1 + w^2))$ from Prob. 3 in Sec. 11.7.
6. Obtain $\mathcal{F}_c^{-1}(e^{-w})$ by integration.
7. Find $\mathcal{F}_c((1 - x^2)^{-1}\cos(\pi x/2))$. *Hint.* Use Prob. 5 in Sec. 11.7.
8. Let $f(x) = x^2$ if $0 < x < 1$ and 0 if $x > 1$. Find $\mathcal{F}_c(f)$.
9. Does the Fourier cosine transform of $x^{-1}\sin x$ exist? Of $x^{-1}\cos x$? Give reasons.
10. $f(x) = 1$ $(0 < x < \infty)$ has no Fourier cosine or sine transform. Give reasons.

11–20 FOURIER SINE TRANSFORM

11. Find $\mathcal{F}_s(e^{-\pi x})$ by integration.
12. Find the answer to Prob. 11 from (9b).
13. Obtain formula 8 in Table II of Sec. 11.11 from (8b) and a suitable formula in Table I.
14. Let $f(x) = \sin x$ if $0 < x < \pi$ and 0 if $x > \pi$. Find $\mathcal{F}_s(f)$. Compare with Prob. 6 in Sec. 11.7. Comment.
15. In Table II of Sec. 11.10 obtain formula 2 from formula 4, using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [(30) in App. 3.1].
16. Show that $\mathcal{F}_s(x^{-1/2}) = w^{-1/2}$ by setting $wx = t^2$ and using $S(\infty) = \sqrt{\pi/8}$ in (38) of App. 3.1.
17. Obtain $\mathcal{F}_s(e^{-ax})$ from (8a) and formula 3 in Table I of Sec. 11.10.
18. Show that $\mathcal{F}_s(x^{-3/2}) = 2w^{1/2}$. *Hint.* Set $wx = t^2$, integrate by parts, and use $C(\infty) = \sqrt{\pi/8}$ in (38) of App. 3.1.
19. **(Scale change)** Using the notation of (5), show that $f(ax)$ has the Fourier sine transform $(1/a)\hat{f}_s(w/a)$.
20. **WRITING PROJECT. Obtaining Fourier Cosine and Sine Transforms.** Write a short report on ways of obtaining these transforms, giving illustrations with examples of your own.

11.9 Fourier Transform. Discrete and Fast Fourier Transforms

The two transforms in the last section are real. We now consider a third one, called the **Fourier transform,** which is complex. We shall obtain this transform from the complex Fourier integral, which we explain first.

Complex Form of the Fourier Integral

The (real) Fourier integral is [see (4), (5), Sec. 11.7]

$$f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx]\, dw$$

where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv\, dv, \qquad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv\, dv.$$

Substituting A and B into the integral for f, we have

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(v)\, [\cos wv \cos wx + \sin wv \sin wx]\, dv\, dw.$$

By the addition formula for the cosine [(6) in App. A3.1] the expression in the brackets $[\cdots]$ equals $\cos (wv - wx)$ or, since the cosine is even, $\cos (wx - wv)$. We thus obtain

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^{\infty} f(v) \cos (wx - wv)\, dv \right] dw. \tag{1*}$$

The integral in brackets is an *even* function of w, call it $F(w)$, because $\cos (wx - wv)$ is an even function of w, the function f does not depend on w, and we integrate with respect to v (not w). Hence the integral of $F(w)$ from $w = 0$ to ∞ is 1/2 times the integral of $F(w)$ from $-\infty$ to ∞. Thus (note the change of the integration limit!)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^{\infty} f(v) \cos (wx - wv)\, dv \right] dw. \tag{1}$$

We claim that the integral of the form (1) with sin instead of cos is zero:

$$\frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^{\infty} f(v) \sin (wx - wv)\, dv \right] dw = 0. \tag{2}$$

This is true since $\sin (wx - wv)$ is an odd function of w, which makes the integral in brackets an odd function of w, call it $G(w)$. Hence the integral of $G(w)$ from $-\infty$ to ∞ is zero, as claimed.

We now take the integrand of (1) plus $i\ (= \sqrt{-1})$ times the integrand of (2) and use the **Euler formula** [(11) in Sec. 2.2]

$$e^{ix} = \cos x + i \sin x. \tag{3}$$

Taking $wx - wv$ instead of x in (3) and multiplying by $f(v)$ gives

$$f(v)\cos(wx - wv) + if(v)\sin(wx - wv) = f(v)e^{i(wx-wv)}.$$

Hence the result of adding (1) plus i times (2), called the **complex Fourier integral,** is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)e^{iw(x-v)}\,dv\,dw \qquad (i = \sqrt{-1}). \tag{4}$$

It is now only a very short step to our present goal, the Fourier transform.

Fourier Transform and Its Inverse

Writing the exponential function in (4) as a product of exponential functions, we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-iwv}\,dv\right]e^{iwx}\,dw. \tag{5}$$

The expression in brackets is a function of w, is denoted by $\hat{f}(w)$, and is called the **Fourier transform** of f; writing $v = x$, we have

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx. \tag{6}$$

With this, (5) becomes

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}\,dw \tag{7}$$

and is called the **inverse Fourier transform** of $\hat{f}(w)$.

Another notation for the Fourier transform is

$$\hat{f} = \mathcal{F}(f),$$

so that

$$f = \mathcal{F}^{-1}(\hat{f}).$$

The process of obtaining the Fourier transform $\mathcal{F}(f) = \hat{f}$ from a given f is also called the **Fourier transform** or the *Fourier transform method.*

Conditions sufficient for the existence of the Fourier transform (involving concepts defined in Secs. 6.1 and 11.7) are as follows, as we state without proof.

THEOREM 1

Existence of the Fourier Transform

If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform $\hat{f}(w)$ of $f(x)$ given by (6) exists.

EXAMPLE 1 Fourier Transform

Find the Fourier transform of $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise.

Solution. Using (6) and integrating, we obtain

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-iwx}}{-iw} \bigg|_{-1}^{1} = \frac{1}{-iw\sqrt{2\pi}} (e^{-iw} - e^{iw}).$$

As in (3) we have $e^{iw} = \cos w + i \sin w$, $e^{-iw} = \cos w - i \sin w$, and by subtraction

$$e^{iw} - e^{-iw} = 2i \sin w.$$

Substituting this in the previous formula on the right, we see that i drops out and we obtain the answer

$$\hat{f}(w) = \sqrt{\frac{\pi}{2}}\, \frac{\sin w}{w}.$$

■

EXAMPLE 2 Fourier Transform

Find the Fourier transform $\mathcal{F}(e^{-ax})$ of $f(x) = e^{-ax}$ if $x > 0$ and $f(x) = 0$ if $x < 0$; here $a > 0$.

Solution. From the definition (6) we obtain by integration

$$\begin{aligned}\mathcal{F}(e^{-ax}) &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-iwx}\, dx \\ &= \frac{1}{\sqrt{2\pi}}\, \frac{e^{-(a+iw)x}}{-(a+iw)} \bigg|_{x=0}^{\infty} = \frac{1}{\sqrt{2\pi}\,(a+iw)}.\end{aligned}$$

This proves formula 5 of Table III in Sec. 11.10. ■

Physical Interpretation: Spectrum

The nature of the representation (7) of $f(x)$ becomes clear if we think of it as a superposition of sinusoidal oscillations of all possible frequencies, called a **spectral representation.** This name is suggested by optics, where light is such a superposition of colors (frequencies). In (7), the **"spectral density"** $\hat{f}(w)$ measures the intensity of $f(x)$ in the frequency interval between w and $w + \Delta w$ (Δw small, fixed). We claim that in connection with vibrations, the integral

$$\int_{-\infty}^{\infty} |\hat{f}(w)|^2\, dw$$

can be interpreted as the **total energy** of the physical system. Hence an integral of $|\hat{f}(w)|^2$ from a to b gives the contribution of the frequencies w between a and b to the total energy.

To make this plausible, we begin with a mechanical system giving a single frequency, namely, the harmonic oscillator (mass on a spring, Sec. 2.4)

$$my'' + ky = 0.$$

Here we denote time t by x. Multiplication by y' gives $my'y'' + ky'y = 0$. By integration,

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}ky^2 = E_0 = const$$

where $v = y'$ is the velocity. The first term is the kinetic energy, the second the potential energy, and E_0 the total energy of the system. Now a general solution is (use (3) in Sec. 11.4 with $t = x$)

$$y = a_1 \cos w_0 x + b_1 \sin w_0 x = c_1 e^{iw_0 x} + c_{-1} e^{-iw_0 x}, \qquad {w_0}^2 = k/m$$

where $c_1 = (a_1 - ib_1)/2$, $c_{-1} = \bar{c}_1 = (a_1 + ib_1)/2$. We write simply $A = c_1 e^{iw_0 x}$, $B = c_{-1} e^{-iw_0 x}$. Then $y = A + B$. By differentiation, $v = y' = A' + B' = iw_0(A - B)$. Substitution of v and y on the left side of the equation for E_0 gives

$$E_0 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}ky^2 = \tfrac{1}{2}m(iw_0)^2(A - B)^2 + \tfrac{1}{2}k(A + B)^2.$$

Here ${w_0}^2 = k/m$, as just stated; hence $m{w_0}^2 = k$. Also $i^2 = -1$, so that

$$E_0 = \tfrac{1}{2}k[-(A - B)^2 + (A + B)^2] = 2kAB = 2kc_1 e^{iw_0 x} c_{-1} e^{-iw_0 x} = 2kc_1 c_{-1} = 2k|c_1|^2.$$

Hence *the energy is proportional to the square of the amplitude* $|c_1|$.

As the next step, if a more complicated system leads to a periodic solution $y = f(x)$ that can be represented by a Fourier series, then instead of the single energy term $|c_1|^2$ we get a series of squares $|c_n|^2$ of Fourier coefficients c_n given by (6), Sec. 11.4. In this case we have a **"discrete spectrum"** (or **"point spectrum"**) consisting of countably many isolated frequencies (infinitely many, in general), the corresponding $|c_n|^2$ being the contributions to the total energy.

Finally, a system whose solution can be represented by an integral (7) leads to the above integral for the energy, as is plausible from the cases just discussed.

Linearity. Fourier Transform of Derivatives

New transforms can be obtained from given ones by

THEOREM 2

Linearity of the Fourier Transform

The Fourier transform is a ***linear operation;*** *that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and b, the Fourier transform of $af + bg$ exists, and*

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g). \tag{8}$$

PROOF This is true because integration is a linear operation, so that (6) gives

$$\begin{aligned}\mathcal{F}\{af(x) + bg(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{-iwx}\, dx \\ &= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-iwx}\, dx \\ &= a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}. \qquad \blacksquare\end{aligned}$$

In applying the Fourier transform to differential equations, the key property is that differentiation of functions corresponds to multiplication of transforms by iw:

THEOREM 3

Fourier Transform of the Derivative of $f(x)$

Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}. \tag{9}$$

PROOF From the definition of the Fourier transform we have

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-iwx}\, dx.$$

Integrating by parts, we obtain

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \left[f(x)e^{-iwx} \Big|_{-\infty}^{\infty} - (-iw) \int_{-\infty}^{\infty} f(x)e^{-iwx}\, dx \right].$$

Since $f(x) \to 0$ as $|x| \to \infty$, the desired result follows, namely,

$$\mathcal{F}\{f'(x)\} = 0 + iw\mathcal{F}\{f(x)\}.$$

Two successive applications of (9) give

$$\mathcal{F}(f'') = iw\mathcal{F}(f') = (iw)^2\mathcal{F}(f).$$

Since $(iw)^2 = -w^2$, we have for the transform of the second derivative of f

$$\mathcal{F}\{f''(x)\} = -w^2\mathcal{F}\{f(x)\}. \tag{10}$$

Similarly for higher derivatives.

An application of (10) to differential equations will be given in Sec. 12.6. For the time being we show how (9) can be used to derive transforms.

EXAMPLE 3

Application of the Operational Formula (9)

Find the Fourier transform of xe^{-x^2} from Table III, Sec 11.10.

Solution. We use (9). By formula 9 in Table III.

$$\begin{aligned}
\mathcal{F}(xe^{-x^2}) &= \mathcal{F}\left\{-\frac{1}{2}\left(e^{-x^2}\right)'\right\} \\
&= -\frac{1}{2}\,\mathcal{F}\left\{\left(e^{-x^2}\right)'\right\} \\
&= -\frac{1}{2}\,iw\mathcal{F}(e^{-x^2}) \\
&= -\frac{1}{2}\,iw\,\frac{1}{\sqrt{2}}\,e^{-w^2/4} \\
&= -\frac{iw}{2\sqrt{2}}\,e^{-w^2/4}.
\end{aligned}$$

Convolution

The **convolution** $f * g$ of functions f and g is defined by

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)\,dp = \int_{-\infty}^{\infty} f(x-p)g(p)\,dp. \tag{11}$$

The purpose is the same as in the case of Laplace transforms (Sec. 6.5): taking the convolution of two functions and then taking the transform of the convolution is the same as multiplying the transforms of these functions (and multiplying them by $\sqrt{2\pi}$):

THEOREM 4

Convolution Theorem

Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then

$$\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g). \tag{12}$$

PROOF By the definition,

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(x-p)\,dp\; e^{-iwx}\,dx.$$

An interchange of the order of integration gives

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(x-p)e^{-iwx}\,dx\,dp.$$

Instead of x we now take $x - p = q$ as a new variable of integration. Then $x = p + q$ and

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(q)e^{-iw(p+q)}\,dq\,dp.$$

This double integral can be written as a product of two integrals and gives the desired result

$$\begin{aligned}\mathcal{F}(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)e^{-iwp}\,dp \int_{-\infty}^{\infty} g(q)e^{-iwq}\,dq \\ &= \frac{1}{\sqrt{2\pi}} [\sqrt{2\pi}\,\mathcal{F}(f)][\sqrt{2\pi}\,\mathcal{F}(g)] = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g).\end{aligned}$$ ■

By taking the inverse Fourier transform on both sides of (12), writing $\hat{f} = \mathcal{F}(f)$ and $\hat{g} = \mathcal{F}(g)$ as before, and noting that $\sqrt{2\pi}$ and $1/\sqrt{2\pi}$ in (12) and (7) cancel each other, we obtain

$$(f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(w)\hat{g}(w)e^{iwx}\,dw, \tag{13}$$

a formula that will help us in solving partial differential equations (Sec. 12.6).

Discrete Fourier Transform (DFT), Fast Fourier Transform (FFT)

In using Fourier series, Fourier transforms, and trigonometric approximations (Sec. 11.6) we have to assume that a function $f(x)$, to be developed or transformed, is given on some interval, over which we integrate in the Euler formulas, etc. Now very often a function $f(x)$ is given only in terms of values at finitely many points, and one is interested in extending Fourier analysis to this case. The main application of such a "discrete Fourier analysis" concerns large amounts of equally spaced data, as they occur in telecommunication, time series analysis, and various simulation problems. In these situations, dealing with sampled values rather than with functions, we can replace the Fourier transform by the so-called **discrete Fourier transform (DFT)** as follows.

Let $f(x)$ be periodic, for simplicity of period 2π. We assume that N measurements of $f(x)$ are taken over the interval $0 \leqq x \leqq 2\pi$ at regularly spaced points

$$x_k = \frac{2\pi k}{N}, \qquad k = 0, 1, \cdots, N-1. \tag{14}$$

We also say that $f(x)$ is being **sampled** at these points. We now want to determine a **complex trigonometric polynomial**

$$q(x) = \sum_{n=0}^{N-1} c_n e^{inx_k} \tag{15}$$

that **interpolates** $f(x)$ at the nodes (14), that is, $q(x_k) = f(x_k)$, written out, with f_k denoting $f(x_k)$,

$$f_k = f(x_k) = q(x_k) = \sum_{n=0}^{N-1} c_n e^{inx_k}, \qquad k = 0, 1, \cdots, N-1. \tag{16}$$

Hence we must determine the coefficients $c_0, \cdots, c_{N-1}$ such that (16) holds. We do this by an idea similar to that in Sec. 11.1 for deriving the Fourier coefficients by using the orthogonality of the trigonometric system. Instead of integrals we now take sums. Namely, we multiply (16) by e^{-imx_k} (note the minus!) and sum over k from 0 to $N-1$. Then we interchange the order of the two summations and insert x_k from (14). This gives

$$\sum_{k=0}^{N-1} f_k e^{-imx_k} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} c_n e^{i(n-m)x_k} = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)2\pi k/N}. \tag{17}$$

Now

$$e^{i(n-m)2\pi k/N} = \left[e^{i(n-m)2\pi/N}\right]^k.$$

We donote $[\cdots]$ by r. For $n = m$ we have $r = e^0 = 1$. The sum of *these* terms over k equals N, the number of these terms. For $n \neq m$ we have $r \neq 1$ and by the formula for a geometric sum [(6) in Sec. 15.1 with $q = r$ and $n = N-1$]

$$\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} = 0$$

because $r^N = 1$; indeed, since k, m, and n are integers,

$$r^N = e^{i(n-m)2\pi k} = \cos 2\pi k(n-m) + i \sin 2\pi k(n-m) = 1 + 0 = 1.$$

This shows that the right side of (17) equals $c_m N$. Writing n for m and dividing by N, we thus obtain the desired coefficient formula

$$\textbf{(18*)} \qquad c_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-inx_k} \qquad f_k = f(x_k), \quad n = 0, 1, \cdots, N-1.$$

Since computation of the c_n (by the fast Fourier transform, below) involves successive halfing of the problem size N, it is practical to drop the factor $1/N$ from c_n and define the **discrete Fourier transform** of the given signal $\mathbf{f} = [f_0 \quad \cdots \quad f_{N-1}]^{\mathsf{T}}$ to be the vector $\hat{\mathbf{f}} = [\hat{f}_0 \quad \cdots \quad \hat{f}_{N-1}]$ with components

$$\textbf{(18)} \qquad \hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}, \qquad f_k = f(x_k), \quad n = 0, \cdots, N-1.$$

This is the frequency spectrum of the signal.

In vector notation, $\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$, where the $N \times N$ **Fourier matrix** $\mathbf{F}_N = [e_{nk}]$ has the entries [given in (18)]

$$(19) \qquad e_{nk} = e^{-inx_k} = e^{-2\pi ink/N} = w^{nk}, \qquad w = w_N = e^{-2\pi i/N},$$

where $n, k = 0, \cdots, N-1$.

EXAMPLE 4 Discrete Fourier Transform (DFT). Sample of $N = 4$ Values

Let $N = 4$ measurements (sample values) be given. Then $w = e^{-2\pi i/N} = e^{-\pi i/2} = -i$ and thus $w^{nk} = (-i)^{nk}$. Let the sample values be, say $\mathbf{f} = [0 \quad 1 \quad 4 \quad 9]^{\mathsf{T}}$. Then by (18) and (19),

$$(20) \qquad \hat{\mathbf{f}} = \mathbf{F}_4 \mathbf{f} = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}.$$

From the first matrix in (20) it is easy to infer what $\mathbf{F}_N$ looks like for arbitrary N, which in practice may be 1000 or more, for reasons given below. ■

From the DFT (the frequency spectrum) $\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$ we can recreate the given signal $\mathbf{f} = \mathbf{F}_N^{-1}\mathbf{f}$, as we shall now prove. Here $\mathbf{F}_N$ and its complex conjugate $\overline{\mathbf{F}}_N = \frac{1}{N}[\overline{w}^{nk}]$ satisfy

$$(21a) \qquad \overline{\mathbf{F}}_N \mathbf{F}_N = \mathbf{F}_N \overline{\mathbf{F}}_N = N\mathbf{I}$$

where $\mathbf{I}$ is the $N \times N$ unit matrix; hence $\mathbf{F}_N$ has the inverse

$$(21b) \qquad \mathbf{F}_N^{-1} = \frac{1}{N} \overline{\mathbf{F}}_N.$$

PROOF We prove (21). By the multiplication rule (row times column) the product matrix $\mathbf{G}_N = \overline{\mathbf{F}}_N\mathbf{F}_N = [g_{jk}]$ in (21a) has the entries g_{jk} = *Row* j *of* $\overline{\mathbf{F}}_N$ *times Column* k *of* $\mathbf{F}_N$. That is, writing $W = \overline{w}^j w^k$, we prove that

$$g_{jk} = (\overline{w}^j w^k)^0 + (\overline{w}^j w^k)^1 + \cdots + (\overline{w}^j w^k)^{N-1}$$

$$= W^0 + W^1 + \cdots + W^{N-1} = \begin{cases} 0 & \text{if} \quad j \neq k \\ N & \text{if} \quad j = k. \end{cases}$$

Indeed, when $j = k$, then $\overline{w}^k w^k = (\overline{w}w)^k = (e^{2\pi i/N}e^{-2\pi i/N})^k = 1^k = 1$, so that the sum of *these* N terms equals N; these are the diagonal entries of $\mathbf{G}_N$. Also, when $j \neq k$, then $W \neq 1$ and we have a geometric sum (whose value is given by (6) in Sec. 15.1 with $q = W$ and $n = N - 1$)

$$W^0 + W^1 + \cdots + W^{N-1} = \frac{1 - W^N}{1 - W} = 0$$

because $W^N = (\overline{w}^j w^k)^N = (e^{2\pi i})^j(e^{-2\pi i})^k = 1^j \cdot 1^k = 1$. ■

We have seen that $\hat{\mathbf{f}}$ is the frequency spectrum of the signal $f(x)$. Thus the components $\hat{f}_n$ of $\hat{\mathbf{f}}$ give a resolution of the 2π-periodic function $f(x)$ into simple (complex) harmonics. Here one should use only n's that are much smaller than $N/2$, to avoid **aliasing.** By this we mean the effect caused by sampling at too few (equally spaced) points, so that, for instance, in a motion picture, rotating wheels appear as rotating too slowly or even in the wrong sense. Hence in applications, N is usually large. But this poses a problem. Eq. (18) requires $O(N)$ operations for any particular n, hence $O(N^2)$ operations for, say, all $n < N/2$. Thus, already for 1000 sample points the straightforward calculation would involve millions of operations. However, this difficulty can be overcome by the so called **fast Fourier transform (FFT),** for which codes are readily available (e.g. in Maple). The FFT is a computational method for the DFT that needs only $O(N)\log_2 N$ operations instead of $O(N^2)$. It makes the DFT a practical tool for large N. Here one chooses $N = 2^p$ (p integer) and uses the special form of the Fourier matrix to break down the given problem into smaller problems. For instance, when $N = 1000$, those operations are reduced by a factor $1000/\log_2 1000 \approx 100$.

The breakdown produces two problems of size $M = N/2$. This breakdown is possible because for $N = 2M$ we have in (19)

$$w_N{}^2 = w_{2M}{}^2 = (e^{-2\pi i/N})^2 = e^{-4\pi i/(2M)} = e^{-2\pi i/M} = w_M.$$

The given vector $\mathbf{f} = [f_0 \quad \cdots \quad f_{N-1}]^\mathsf{T}$ is split into two vectors with M components each, namely, $\mathbf{f}_{\text{ev}} = [f_0 \quad f_2 \quad \cdots \quad f_{N-2}]^\mathsf{T}$ containing the even components of $\mathbf{f}$, and $\mathbf{f}_{\text{od}} = [f_1 \quad f_3 \quad \cdots \quad f_{N-1}]^\mathsf{T}$ containing the odd components of $\mathbf{f}$. For $\mathbf{f}_{\text{ev}}$ and $\mathbf{f}_{\text{od}}$ we determine the DFTs

$$\hat{\mathbf{f}}_{\text{ev}} = \left[\hat{f}_{\text{ev},0} \quad \hat{f}_{\text{ev},2} \quad \cdots \quad \hat{f}_{\text{ev},N-2}\right]^\mathsf{T} = \mathbf{F}_M\mathbf{f}_{\text{ev}}$$

and

$$\hat{\mathbf{f}}_{\text{od}} = \left[\hat{f}_{\text{od},1} \quad \hat{f}_{\text{od},3} \quad \cdots \quad \hat{f}_{\text{od},N-1}\right]^\mathsf{T} = \mathbf{F}_M\mathbf{f}_{\text{od}}$$

involving the same $M \times M$ matrix $\mathbf{F}_M$. From these vectors we obtain the components of the DFT of the given vector $\mathbf{f}$ by the formulas

$$\text{(22)} \qquad \begin{aligned} &\text{(a)} \quad \hat{f}_n = \hat{f}_{\text{ev},n} + w_N{}^n\hat{f}_{\text{od},n} && n = 0, \cdots, M-1 \\ &\text{(b)} \quad \hat{f}_{n+M} = \hat{f}_{\text{ev},n} - w_N{}^n\hat{f}_{\text{od},n} && n = 0, \cdots, M-1. \end{aligned}$$

For $N = 2^p$ this breakdown can be repeated $p - 1$ times in order to finally arrive at $N/2$ problems of size 2 each, so that the number of multiplications is reduced as indicated above.

We show the reduction from $N = 4$ to $M = N/2 = 2$ and then prove (22).

EXAMPLE 5 Fast Fourier Transform (FFT). Sample of $N = 4$ Values

When $N = 4$, then $w = w_N = -i$ as in Example 4 and $M = N/2 = 2$, hence $w = w_M = e^{-2\pi i/2} = e^{-\pi i} = -1$. Consequently,

$$\hat{\mathbf{f}}_{\text{ev}} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_2 \end{bmatrix} = \mathbf{F}_2\mathbf{f}_{\text{ev}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_0 + f_2 \\ f_0 - f_2 \end{bmatrix}$$

$$\hat{\mathbf{f}}_{\text{od}} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_3 \end{bmatrix} = \mathbf{F}_2\mathbf{f}_{\text{od}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 + f_3 \\ f_1 - f_3 \end{bmatrix}.$$

From this and (22a) we obtain

$$\hat{f}_0 = \hat{f}_{\text{ev},0} + w_N{}^0 \hat{f}_{\text{od},0} = (f_0 + f_2) + (f_1 + f_3) = f_0 + f_1 + f_2 + f_3$$
$$\hat{f}_1 = \hat{f}_{\text{ev},1} + w_N{}^1 \hat{f}_{\text{od},1} = (f_0 - f_2) - i(f_1 + f_3) = f_0 - if_1 - f_2 + if_3.$$

Similarly, by (22b),

$$\hat{f}_2 = \hat{f}_{\text{ev},0} - w_N{}^0 \hat{f}_{\text{od},0} = (f_0 + f_2) - (f_1 + f_3) \qquad = f_0 - f_1 + f_2 - f_3$$
$$\hat{f}_3 = \hat{f}_{\text{ev},1} - w_N{}^1 \hat{f}_{\text{od},1} = (f_0 - f_2) - (-i)(f_1 - f_3) = f_0 + if_1 - f_2 - if_3.$$

This agrees with Example 4, as can be seen by replacing 0, 1, 4, 9 with f_0, f_1, f_2, f_3. ■

We prove (22). From (18) and (19) we have for the components of the DFT

$$\hat{f}_n = \sum_{k=0}^{N-1} w_N^{kn} f_k.$$

Splitting into two sums of $M = N/2$ terms each gives

$$\hat{f}_n = \sum_{k=0}^{M-1} w_N^{2kn} f_{2k} + \sum_{k=0}^{M-1} w_N^{(2k+1)n} f_{2k+1}.$$

We now use $w_N{}^2 = w_M$ and pull out $w_N{}^n$ from under the second sum, obtaining

$$\hat{f}_n = \sum_{k=0}^{M-1} w_M^{kn} f_{\text{ev},k} + w_N{}^n \sum_{k=0}^{M-1} w_M^{kn} f_{\text{od},k}. \tag{23}$$

The two sums are $f_{\text{ev},n}$ and $f_{\text{od},n}$, the components of the "half-size" transforms $\mathbf{F}\,\mathbf{f}_{\text{ev}}$ and $\mathbf{F}\,\mathbf{f}_{\text{od}}$.

Formula (22a) is the same as (23). In (22b) we have $n + M$ instead of n. This causes a sign change in (23), namely $-w_N{}^n$ before the second sum because

$$w_N{}^M = e^{-2\pi iM/N} = e^{-2\pi i/2} = e^{-\pi i} = -1.$$

This gives the minus in (22b) and completes the proof. ■

PROBLEM SET 11.9

1. (Review) Show that $1/i = -i$, $e^{ix} + e^{-ix} = 2 \cos x$, $e^{ix} - e^{-ix} = 2i \sin x$.

2–9 FOURIER TRANSFORMS BY INTEGRATION

Find the Fourier transform of $f(x)$ (without using Table III in Sec. 11.10). Show the details.

2. $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$

3. $f(x) = \begin{cases} k & \text{if } 0 < x < b \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

5. $f(x) = \begin{cases} k & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

7. $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

8. $f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$

9. $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } \quad 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

OTHER METHODS

10. Find the Fourier transform of $f(x) = xe^{-x}$ if $x > 0$ and 0 if $x < 0$ from formula 5 in Table III and (9) in the text. *Hint:* Consider xe^{-x} and e^{-x}.

11. Obtain $\mathcal{F}(e^{-x^2/2})$ from formula 9 in Table III.

12. Obtain formula 7 in Table III from formula 8.

13. Obtain formula 1 in Table III from formula 2.

14. TEAM PROJECT. Shifting. (a) Show that if $f(x)$ has a Fourier transform, so does $f(x - a)$, and $\mathcal{F}\{f(x - a)\} = e^{-iwa}\mathcal{F}\{f(x)\}$.

(b) Using (a), obtain formula 1 in Table III, Sec. 11.10, from formula 2.

(c) Shifting on the *w*-Axis. Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w - a)$ is the Fourier transform of $e^{iax}f(x)$.

(d) Using (c), obtain formula 7 in Table III from 1 and formula 8 from 2.

11.10 Tables of Transforms

Table I. Fourier Cosine Transforms

See (2) in Sec. 11.8.

	$f(x)$	$\hat{f}_c(w) = \mathcal{F}_c(f)$
1	$\begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\, \frac{\sin aw}{w}$
2	$x^{a-1} \quad (0 < a < 1)$	$\sqrt{\frac{2}{\pi}}\, \frac{\Gamma(a)}{w^a} \cos \frac{a\pi}{2}$ $\quad$ ($\Gamma(a)$ see App. A3.1.)
3	$e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + w^2} \right)$
4	$e^{-x^2/2}$	$e^{-w^2/2}$
5	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}}\, e^{-w^2/(4a)}$
6	$x^n e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}}\, \frac{n!}{(a^2 + w^2)^{n+1}} \operatorname{Re}\,(a + iw)^{n+1}$ $\quad$ Re = Real part
7	$\begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1 - w)}{1 - w} + \frac{\sin a(1 + w)}{1 + w} \right]$
8	$\cos (ax^2) \quad (a > 0)$	$\frac{1}{\sqrt{2a}} \cos \left(\frac{w^2}{4a} - \frac{\pi}{4} \right)$
9	$\sin (ax^2) \quad (a > 0)$	$\frac{1}{\sqrt{2a}} \cos \left(\frac{w^2}{4a} + \frac{\pi}{4} \right)$
10	$\frac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}\, (1 - u(w - a))$ $\quad$ (See Sec. 6.3.)
11	$\frac{e^{-x} \sin x}{x}$	$\frac{1}{\sqrt{2\pi}} \arctan \frac{2}{w^2}$
12	$J_0(ax) \quad (a > 0)$	$\sqrt{\frac{2}{\pi}}\, \frac{1}{\sqrt{a^2 - w^2}}\, (1 - u(w - a))$ $\quad$ (See Secs. 5.5, 6.3.)

Table II. Fourier Sine Transforms

See (5) in Sec. 11.8.

	$f(x)$	$\hat{f}_s(w) = \mathcal{F}_s(f)$
1	$\begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos aw}{w} \right]$
2	$1/\sqrt{x}$	$1/\sqrt{w}$
3	$1/x^{3/2}$	$2\sqrt{w}$
4	$x^{a-1} \quad (0 < a < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(a)}{w^a} \sin \frac{a\pi}{2}$ ($\Gamma(a)$ see App. A3.1.)
5	$e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \left(\frac{w}{a^2 + w^2} \right)$
6	$\frac{e^{-ax}}{x} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \arctan \frac{w}{a}$
7	$x^n e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{n!}{(a^2 + w^2)^{n+1}} \operatorname{Im} (a + iw)^{n+1}$ Im = Imaginary part
8	$xe^{-x^2/2}$	$we^{-w^2/2}$
9	$xe^{-ax^2} \quad (a > 0)$	$\frac{w}{(2a)^{3/2}} e^{-w^2/4a}$
10	$\begin{cases} \sin x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1 - w)}{1 - w} - \frac{\sin a(1 + w)}{1 + w} \right]$
11	$\frac{\cos ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}\, u(w - a)$ (See Sec. 6.3.)
12	$\arctan \frac{2a}{x} \quad (a > 0)$	$\sqrt{2\pi} \frac{\sinh aw}{w} e^{-aw}$

Table III. Fourier Transforms

See (6) in Sec. 11.9.

	$f(x)$	$\hat{f}(w) = \mathcal{F}(f)$
1	$\begin{cases} 1 & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin bw}{w}$
2	$\begin{cases} 1 & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$
3	$\dfrac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-a\lvert w\rvert}}{a}$
4	$\begin{cases} x & \text{if } 0 < x < b \\ 2x - b & \text{if } b < x < 2b \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{-1 + 2e^{ibw} - e^{-2ibw}}{\sqrt{2\pi}\, w^2}$
5	$\begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (a > 0)$	$\dfrac{1}{\sqrt{2\pi}(a + iw)}$
6	$\begin{cases} e^{ax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{(a-iw)c} - e^{(a-iw)b}}{\sqrt{2\pi}(a - iw)}$
7	$\begin{cases} e^{iax} & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin b(w - a)}{w - a}$
8	$\begin{cases} e^{iax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{i}{\sqrt{2\pi}}\,\dfrac{e^{ib(a-w)} - e^{ic(a-w)}}{a - w}$
9	$e^{-ax^2} \quad (a > 0)$	$\dfrac{1}{\sqrt{2a}}\, e^{-w^2/4a}$
10	$\dfrac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\dfrac{\pi}{2}}$ if $\lvert w\rvert < a$; 0 if $\lvert w\rvert > a$

CHAPTER 11 REVIEW QUESTIONS AND PROBLEMS

1. What is a Fourier series? A Fourier sine series? A half-range expansion?
2. Can a discontinuous function have a Fourier series? A Taylor series? Explain.
3. Why did we start with period 2π? How did we proceed to functions of any period p?
4. What is the trigonometric system? Its main property by which we obtained the Euler formulas?
5. What do you know about the convergence of a Fourier series?
6. What is the Gibbs phenomenon?
7. What is approximation by trigonometric polynomials? The minimum square error?
8. What is remarkable about the response of a vibrating system to an *arbitrary* periodic force?
9. What do you know about the Fourier integral? Its applications?
10. What is the Fourier sine transform? Give examples.

11–20 FOURIER SERIES

Find the Fourier series of $f(x)$ as given over one period. Sketch $f(x)$. (Show the details of your work.)

11. $f(x) = \begin{cases} -k & \text{if } -1 < x < 0 \\ k & \text{if } \ \ 0 < x < 1 \end{cases}$

12. $f(x) = \begin{cases} 0 & \text{if } -\pi/2 < x < \pi/2 \\ 1 & \text{if } \ \ \pi/2 < x < 3\pi/2 \end{cases}$

13. $f(x) = x \qquad (-2\pi < x < 2\pi)$

14. $f(x) = |x| \qquad (-2 < x < 2)$

15. $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 2 - x & \text{if } \ \ 1 < x < 3 \end{cases}$

16. $f(x) = \begin{cases} -1 - x & \text{if } -1 < x < 0 \\ 1 - x & \text{if } \ \ 0 < x < 1 \end{cases}$

17. $f(x) = |\sin 8\pi x| \ (-1/8 < x < 1/8)$

18. $f(x) = e^x \ (-\pi < x < \pi)$

19. $f(x) = x^2 \ (-\pi/2 < x < \pi/2)$

20. $f(x) = x \ (0 < x < 2\pi)$

21–23 Using the answers to suitable odd-numbered problems, find the sum of

21. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$

22. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots$

23. $1 + \frac{1}{9} + \frac{1}{25} + \cdots$

24. **(Parseval's identity)** Obtain the result of Prob. 23 by applying Parseval's identity to Prob. 12.

25. What are the sum of the cosine terms and the sum of the sine terms in a Fourier series whose sum is $f(x)$? Give two examples.

26. **(Half-range expansion)** Find the half-range sine series of $f(x) = 0$ if $0 < x < \pi/2$, $f(x) = 1$ if $\pi/2 < x < \pi$. Compare with Prob. 12.

27. **(Half-range cosine series)** Find the half-range cosine series of $f(x) = x \ (0 < x < 2\pi)$. Compare with Prob. 20.

28–29 MINIMUM SQUARE ERROR

Compute the minimum square errors for the trigonometric polynomials of degree $N = 1, \cdots, 8$:

28. For $f(x)$ in Prob. 12.

29. For $f(x) = x \ (-\pi < x < \pi)$.

30–31 GENERAL SOLUTION

Solve $y'' + \omega^2 y = r(t)$, where $|\omega| \neq 0, 1, 2, \cdots, r(t)$ is 2π-periodic and:

30. $r(t) = t(\pi^2 - t^2) \qquad (-\pi < t < \pi)$

31. $r(t) = t^2 \qquad (-\pi < t < \pi)$

32–37 FOURIER INTEGRALS AND TRANSFORMS

Sketch the given function and represent it as indicated. If you have a CAS, graph approximate curves obtained by replacing ∞ with finite limits; also look for Gibbs phenomena.

32. $f(x) = 1$ if $1 < x < 2$ and 0 otherwise, by a Fourier integral

33. $f(x) = x$ if $0 < x < 1$ and 0 otherwise, by a Fourier integral

34. $f(x) = 1 + x/2$ if $-2 < x < 0$, $f(x) = 1 - x/2$ if $0 < x < 2$, $f(x) = 0$ otherwise, by a Fourier cosine integral

35. $f(x) = -1 - x/2$ if $-2 < x < 0$, $f(x) = 1 - x/2$ if $0 < x < 2$, $f(x) = 0$ otherwise, by a Fourier sine integral

36. $f(x) = -4 + x^2$ if $-2 < x < 0$, $f(x) = 4 - x^2$ if $0 < x < 2$, $f(x) = 0$ otherwise, by a Fourier sine integral

37. $f(x) = 4 - x^2$ if $-2 < x < 2$, $f(x) = 0$ otherwise, by a Fourier cosine integral

38. Find the Fourier transform of $f(x) = k$ if $a < x < b$, $f(x) = 0$ otherwise.

39. Find the Fourier cosine transform of $f(x) = e^{-2x}$ if $x > 0$, $f(x) = 0$ if $x < 0$.

40. Find $\mathcal{F}_c(e^{-2x})$ and $\mathcal{F}_s(e^{-2x})$ by formulas involving second derivatives.

SUMMARY OF CHAPTER 11
Fourier Series, Integrals, Transforms

Fourier series concern **periodic functions** $f(x)$ of period $p = 2L$, that is, by definition $f(x + p) = f(x)$ for all x and some fixed $p > 0$; thus, $f(x + np) = f(x)$ for any integer n. These series are of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x\right) \qquad \text{(Sec. 11.2)} \tag{1}$$

with coefficients, called the **Fourier coefficients** of $f(x)$, given by the Euler formulas (Sec. 11.2)

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{aligned} \tag{2}$$

where $n = 1, 2, \cdots$. For period 2π we simply have (Sec. 11.1)

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1*}$$

with the *Fourier coefficients* of $f(x)$ (Sec. 11.1)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Fourier series are fundamental in connection with periodic phenomena, particularly in models involving differential equations (Sec. 11.5, Chap. 12). If $f(x)$ is even $[f(-x) = f(x)]$ or odd $[f(-x) = -f(x)]$, they reduce to **Fourier cosine** or **Fourier sine series,** respectively (Sec. 11.3). If $f(x)$ is given for $0 \leqq x \leqq L$ only, it has two **half-range expansions** of period $2L$, namely, a cosine and a sine series (Sec. 11.3).

The set of cosine and sine functions in (1) is called the **trigonometric system.** Its most basic property is its **orthogonality** on an interval of length $2L$; that is, for all integers m and $n \neq m$ we have

$$\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx = 0, \qquad \int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx = 0$$

and for all integers m and n,

$$\int_{-L}^{L} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx = 0.$$

This orthogonality was crucial in deriving the Euler formulas (2).

Partial sums of Fourier series minimize the **square error** (Sec. 11.6).

Ideas and techniques of Fourier series extend to nonperiodic functions $f(x)$ defined on the entire real line; this leads to the **Fourier integral**

$$f(x) = \int_{0}^{\infty} [A(w) \cos wx + B(w) \sin wx]\, dw \qquad \text{(Sec. 11.7)} \tag{3}$$

where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv\, dv, \qquad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv\, dv \tag{4}$$

or, in complex form (Sec. 11.9),

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\, dw \qquad (i = \sqrt{-1}) \tag{5}$$

where

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx. \tag{6}$$

Formula (6) transforms $f(x)$ into its **Fourier transform** $\hat{f}(w)$, and (5) is the inverse transform.

Related to this are the **Fourier cosine transform** (Sec. 11.8)

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos wx\, dx \tag{7}$$

and the **Fourier sine transform** (Sec. 11.8)

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin wx\, dx. \tag{8}$$

The **discrete Fourier transform (DFT)** and a practical method of computing it, called the **fast Fourier transform (FFT),** are discussed in Sec. 11.9.