$$u(0, y) = f(y),\ u(a, y) = g(y),\ 0 < y < b,$$

$$u(x, 0) = 0,\ u(x, b) = 0,\ 0 < x < a.$$

Find the temperature distribution.

50. The boundaries of a thin rectangular plate are $x = 0$, $x = a$, $y = 0$ and $y = b$. The steady state temperature distribution in the plate is to be determined when the vertical edges are insulated. A boundary value problem modelling the temperature distribution is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b,$$

$$\frac{\partial u}{\partial x}(0, y) = 0,\ \frac{\partial u}{\partial x}(a, y) = 0,\ 0 < y < b,$$

$$u(x, 0) = 0,\ u(x, b) = f(x),\ 0 < x < a.$$

Find the temperature distribution, if $f(x)$ is given by

$$f(x) = \begin{cases} x, & 0 < x < a/2, \\ a - x, & a/2 < x < a. \end{cases}$$

Solve the Laplace equation $(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = 0$ for a rectangular or a square plate, subject to the following conditions.

51. $u(0, y) = 0,\ u(a, y) = 0;\ u(x, 0) = f(x),\ u(x, b) = 0.$

52. $u(0, y) = 0,\ u(a, y) = 0;\ u(x, 0) = 0,\ u(x, b) = g(x).$

53. $u(0, y) = 0,\ u(a, y) = 0;\ \dfrac{\partial u}{\partial y}(x, 0) = 0,\ u(x, b) = g(x).$

54. $u(0, y) = 0,\ u(a, y) = a - y;\ \dfrac{\partial u}{\partial y}(x, 0) = 0,\ \dfrac{\partial u}{\partial y}(x, a) = 0.$

55. $u(0, y) = y,\ \dfrac{\partial u}{\partial x}(1, y) = -5;\ u(x, 0) = 0,\ u(x, 1) = 0.$

9.6 Fourier Transforms

In this section, we define the *Fourier transform* which is an integral transform similar to Laplace transform. Fourier transforms are used in many areas of science, engineering, medicine etc.

Let $f(t)$ be piecewise continuous on $(-\infty, \infty)$. Assume that $f(t)$ is absolutely convergent, that is $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges. Then, the Fourier transform of $f(t)$ denoted by $\mathcal{F}[f(t)]$ is defined as

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = F(\omega). \tag{9.117}$$

In Laplace transforms, we have defined $\mathcal{L}[f(t)] = F(s)$. Similar notation is also used in Fourier transforms.

Assume now that $\int_{-\infty}^{\infty} |F(\omega)|\, d\omega$ converges. Then, we define the *inverse Fourier transform* of $F(\omega)$ as

$$\mathscr{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega = f(t). \tag{9.118}$$

The Fourier transform and its inverse are called a *transform pair*. Therefore, the Fourier transform defined in Eq. (9.117) and its inverse defined in Eq. (9.118) form a transform pair.

Example 9.21 Find the Fourier transform of the following functions defined on $(-\infty, \infty)$.

(i) $f(t) = \begin{cases} a, & -l < t < 0, \\ 0, & \text{otherwise}, a > 0. \end{cases}$ (ii) $f(t) = \begin{cases} a, & -l < t < l, \\ 0, & \text{otherwise}, a > 0. \end{cases}$

(iii) $f(t) = \begin{cases} a, & -l < t < 0, \\ b, & 0 < t < l, \\ 0, & \text{otherwise}, a > 0, b > 0. \end{cases}$

Solution We use the definition to find the Fourier transform.

(i) $$\mathscr{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = a\int_{-l}^{0} e^{-i\omega t}\, dt = -\frac{a}{i\omega}[e^{-i\omega t}]_{-l}^{0}$$

$$= -\frac{a}{i\omega}[1 - e^{i\omega l}] = \frac{ia}{\omega}[1 - e^{i\omega l}].$$

(ii) $$\mathscr{F}[f(t)] = a\int_{-l}^{l} e^{-i\omega t}\, dt = -\frac{a}{i\omega}[e^{-i\omega t}]_{-l}^{l} = -\frac{a}{i\omega}[e^{-i\omega l} - e^{i\omega l}]$$

$$= \frac{a}{i\omega}[e^{i\omega l} - e^{-i\omega l}] = \frac{2a}{\omega}\sin(\omega l).$$

(iii) $$\mathscr{F}[f(t)] = a\int_{-l}^{0} e^{-i\omega t}\, dt + b\int_{0}^{l} e^{-i\omega t}\, dt$$

$$= -\frac{a}{i\omega}[1 - e^{i\omega l}] - \frac{b}{i\omega}[e^{-i\omega l} - 1]$$

$$= \frac{1}{i\omega}[(b - a) + a e^{i\omega l} - b e^{-i\omega l}].$$

Example 9.22 Find the Fourier transform of the function

$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-\alpha t}, & t \geq 0, \end{cases} \qquad \alpha > 0.$$

Solution The function $f(t)$ has a jump discontinuity at $t = 0$ and is of magnitude 1. Also

$$\int_{-\infty}^{\infty} |f(t)|\, dt = \int_{0}^{\infty} e^{-\alpha t}\, dt = \frac{1}{\alpha}.$$

Therefore, Fourier transform of $f(t)$ exists. We have

$$\mathscr{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \lim_{s\to\infty} \int_0^s e^{-\alpha t}\, e^{-i\omega t}\, dt$$

$$= \lim_{s\to\infty} \left[-\frac{e^{-(\alpha+i\omega)t}}{\alpha + i\omega} \right]_0^s = \frac{1}{\alpha + i\omega}$$

since $\lim_{s\to\infty} [e^{-(\alpha+i\omega)s}] = \lim_{s\to\infty} e^{-\alpha s} [\cos(\omega s) - i \sin(\omega s)] = 0.$

Hence, $1/(\alpha + i\omega)$ and $f(t) = e^{-\alpha t}\, u_0(t)$ where $u_0(t)$ is the unit step function form a transform pair.

Therefore, $$\mathscr{F}[e^{-\alpha t} u_0(t)] = \frac{1}{\alpha + i\omega}. \qquad (9.119)$$

Example 9.23 Find the Fourier transform of the function $f(t) = e^{-a|t|}$, $-\infty < t < \infty$, $a > 0$. Write the inverse transform.

Solution We have $$f(t) = \begin{cases} e^{at}, & t < 0, \\ e^{-at}, & t > 0. \end{cases}$$

Therefore, $$\mathscr{F}[f(t)] = \int_{-\infty}^{0} e^{at}\, e^{-i\omega t}\, dt + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-at}\, e^{-i\omega t}\, dt$$

$$= \left[\frac{e^{(a-i\omega)t}}{a - i\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-(a+i\omega)t}}{-(a+i\omega)} \right]_0^{\infty} = \frac{1}{a - i\omega} + \frac{1}{a + i\omega} = \frac{2a}{a^2 + \omega^2}.$$

The inverse transform is given by

$$\mathscr{F}^{-1}\left[\frac{2a}{a^2 + \omega^2} \right] = e^{-a|t|}.$$

Example 9.24 Find the Fourier transform of e^{-at^2}, $a > 0$.

Solution From the definition, we obtain

$$\mathscr{F}[e^{-at^2}] = \int_{-\infty}^{\infty} e^{-at^2}\, e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} e^{-a[t^2 + (i\omega t/a)]}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-a[\{t + i\omega/(2a)\}^2 + \omega^2/(4a^2)]}\, dt = e^{-\omega^2/(4a)} \int_{-\infty}^{\infty} e^{-a[t + i\omega/(2a)]^2}\, dt$$

$$= e^{-\omega^2/(4a)} \int_{-\infty}^{\infty} e^{-\tau^2} \frac{d\tau}{\sqrt{a}} \qquad \left[\text{setting } \sqrt{a}\left(t + \frac{i\omega}{2a}\right) = \tau\right]$$

$$= \frac{\sqrt{\pi}}{\sqrt{a}}\, e^{-\omega^2/(4a)}$$

since $\int_{-\infty}^{\infty} e^{-\tau^2}\, d\tau = 2 \int_0^{\infty} e^{-\tau^2}\, d\tau = \sqrt{\pi}.$

In some applications, the transform pair is defined as

$$\mathscr{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = F(\omega) \tag{9.120}$$

and

$$\mathscr{F}^{-1}[F(\omega)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega = f(t). \tag{9.121}$$

Amplitude spectrum The graph of $(\omega, |F(\omega)|)$ is called the amplitude spectrum of $f(t)$. ω is called the frequency of the transform.

Example 9.25 Find the amplitude spectrum of the function

$$f(t) = \begin{cases} 5, & -2 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

Solution We have

$$F(\omega) = \mathscr{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_{-2}^{2} 5e^{-i\omega t}\, dt$$

$$= -5\left[\frac{e^{-i\omega t}}{i\omega}\right]_{-2}^{2} = -\frac{5}{i\omega}\left[e^{-2i\omega} - e^{2i\omega}\right] = \frac{10}{\omega}\sin 2\omega.$$

The graph of ω versus $|F(\omega)|$ is the amplitude spectrum of $f(t)$.

Linearity of Fourier transform

$$\mathscr{F}[af(t) + bg(t)] = a\,\mathscr{F}[f(t)] + b\,\mathscr{F}[g(t)]$$

provided the Fourier transforms of $f(t)$ and $g(t)$ exist.

We now present the shift theorems analogous to the shift theorems of Laplace transforms.

Theorem 9.9 **(Shifting on t-axis)** If $\mathscr{F}[f(t)] = F(\omega)$ and t_0 is any real number then

$$\mathscr{F}[f(t - t_0)] = F(\omega)\, e^{-i\omega t_0}. \tag{9.122}$$

Proof From the definition, we get

$$\mathscr{F}[f(t - t_0)] = \int_{-\infty}^{\infty} f(t - t_0)\, e^{-i\omega t}\, dt = e^{-i\omega t_0} \int_{-\infty}^{\infty} f(t - t_0)\, e^{-i\omega(t - t_0)}\, dt.$$

Let $t - t_0 = \tau$. Then,

$$\mathscr{F}[f(t - t_0)] = e^{-i\omega t_0} \int_{-\infty}^{\infty} f(\tau)\, e^{-i\omega \tau}\, d\tau = e^{-i\omega t_0}\, F(\omega).$$

Remark 4

$$\mathscr{F}^{-1}[e^{-i\omega t_0}\, F(\omega)] = f(t - t_0). \tag{9.123}$$

Example 9.26 Find $\mathscr{F}^{-1}\left[\dfrac{e^{4i\omega}}{3+i\omega}\right]$.

Solution From Eq. (9.119), we have $\mathscr{F}[e^{-3t}\,u_0(t)] = \dfrac{1}{3+i\omega}$

or
$$\mathscr{F}^{-1}\left[\frac{1}{3+i\omega}\right] = e^{-3t}\,u_0(t) = f(t).$$

Using the shift theorem, we get

$$\mathscr{F}^{-1}\left[\frac{e^{-(-4i\omega)}}{3+i\omega}\right] = f(t-(-4)) = e^{-3(t+4)}\,u_{-4}(t) = \begin{cases} 0, & t < -4 \\ e^{-3(t+4)}, & t \geq -4. \end{cases}$$

Theorem 9.10 (Frequency shifting) If $\mathscr{F}[f(t)] = F(\omega)$ and ω_0 is any real number, then

$$\mathscr{F}[e^{i\omega_0 t} f(t)] = F(\omega - \omega_0). \tag{9.124}$$

Proof From the definition, we get

$$\mathscr{F}[e^{i\omega_0 t} f(t)] = \int_{-\infty}^{\infty} e^{i\omega_0 t}\, e^{-i\omega t} f(t)\,dt = \int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)t} f(t)\,dt = F(\omega-\omega_0).$$

Remark 5

$$\mathscr{F}^{-1}[F(\omega-\omega_0)] = e^{i\omega_0 t} f(t). \tag{9.125}$$

Theorem 9.11 (Modulation theorem) If $\mathscr{F}[f(t)] = F(\omega)$ and ω_0 is any real number, then

$$\mathscr{F}[f(t)\cos(\omega_0 t)] = \frac{1}{2}[F(\omega+\omega_0) + F(\omega-\omega_0)] \tag{9.126}$$

and
$$\mathscr{F}[f(t)\sin(\omega_0 t)] = \frac{i}{2}[F(\omega+\omega_0) - F(\omega-\omega_0)]. \tag{9.127}$$

These results can be proved by using Eq. (9.124)

Fourier transforms of derivatives

Let $f(t)$ be continuous and $f^{(k)}(t)$, $k = 1, 2, \ldots, n$ be piecewise continuous on every interval $[-l, l]$ and $\int_{-\infty}^{\infty} |f^{(k-1)}(t)|\,dt$, $k = 1, 2, \ldots, n$ converge. Let $f^{(k)}(t) \to 0$ as $t \to \infty$ for $k = 0, 1, \ldots, n-1$. If $\mathscr{F}[f(t)] = F(\omega)$, then

$$\mathscr{F}[f^{(n)}(t)] = (i\omega)^n F(\omega) \tag{9.128}$$

where $f(x)$ and all its derivatives vanish at infinity.
For example,

$$\mathscr{F}[f'(t)] = \int_{-\infty}^{\infty} f'(t)\,e^{-i\omega t}\,dt$$

$$= [f(t)\,e^{-i\omega t}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} i\omega f(t) e^{-i\omega t}\,dt = i\omega F(\omega).$$

$$\mathscr{F}[f''(t)] = -\omega^2 F(\omega).$$

Example 9.27 Find the solution of the differential equation

$$y' - 2y = H(t)\, e^{-2t}, = -\infty < t < \infty.$$

using Fourier transforms, where $H(t) = u_0(t)$ is the unit step function.

Solution Applying the Fourier transform to the differential equation, we get

$$\mathscr{F}[y'] - 2\,\mathscr{F}[y] = \mathscr{F}[H(t)\, e^{-2t}]$$

or

$$i\omega Y(\omega) - 2Y(\omega) = \frac{1}{2 + i\omega} \qquad \text{(using Eq. (9.119))}$$

or

$$Y(\omega) = -\frac{1}{(2 + i\omega)(2 - i\omega)} = -\frac{1}{4 + \omega^2}$$

where $\mathscr{F}[y(t)] = Y(\omega)$.

Therefore,

$$y(t) = \mathscr{F}^{-1}\left[-\frac{1}{4 + \omega^2}\right] = -\frac{1}{4}e^{-2|t|}. \qquad \text{(using Example 9.23)}$$

The solution can also be written as

$$y(t) = \begin{cases} -\frac{1}{4}e^{2t}, & t < 0, \\ -\frac{1}{4}e^{-2t}, & t > 0, \end{cases}$$

Symmetry property of Fourier transforms

Let $\mathscr{F}[f(t)] = F(\omega)$. Then

$$\mathscr{F}[F(t)] = 2\pi f(-\omega). \qquad (9.129)$$

The result can be proved from the definition. From Eq. (9.118), we have

$$f(t) = \mathscr{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{ist}\, ds$$

since ω is a dummy variable of integration. Hence, setting $t = -\omega$, we get

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(s)\, e^{-i\omega s}\, ds = \int_{-\infty}^{\infty} F(t)\, e^{-i\omega t}\, dt = \mathscr{F}[F(t)].$$

Example 9.28 Find the Fourier transform of $f(t) = 1/(5 + it)$.

Solution We shall use the symmetry property to find the Fourier transform. We know that $1/(5 + i\omega)$ is the Fourier transform $H(t)\, e^{-5t}$ (see Example 9.22). Now, let $b(t) = H(t)e^{-5t}$. Then

$$\mathscr{F}[b(t)] = \mathscr{F}[H(t)e^{-5t}] = \frac{1}{5 + i\omega} = B(\omega).$$

Using the symmetry result, we obtain

$$\mathscr{F}[B(t)] = 2\pi b(-\omega), \text{ or } \mathscr{F}[B(t)] = \mathscr{F}\left[\frac{1}{5 + it}\right] = 2\pi\, b(-\omega) = 2\pi H(-\omega)\, e^{5\omega}.$$

Therefore,

$$\mathscr{F}[f(t)] = F(\omega) = \begin{cases} 2\pi e^{5\omega}, & \omega \le 0 \\ 0, & \omega > 0. \end{cases}$$

Differentiation with respect to frequency ω

Theorem 9.12 Let $f(t)$ be piecewise continuous on every interval $[-l, l]$. Let $\int_{-\infty}^{\infty} |t^n f(t)| \, dt$ converge. Then

$$\mathscr{F}[t^n f(t)] = i^n F^{(n)}(\omega). \tag{9.130}$$

Proof From the definition, we have (assuming that integration and differentiation can be interchanged)

$$F'(\omega) = \frac{d}{d\omega} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} \, dt = \int_{-\infty}^{\infty} \frac{d}{d\omega} [f(t) e^{-i\omega t}] \, dt$$

$$= -i \int_{-\infty}^{\infty} [t f(t)] \, e^{-i\omega t} \, dt = -i \mathscr{F}[t f(t)].$$

$$F''(\omega) = -i \frac{d}{d\omega} \int_{-\infty}^{\infty} t f(t)\, e^{-i\omega t} \, dt = -i \int_{-\infty}^{\infty} \frac{d}{d\omega} [t f(t)\, e^{-i\omega t}] \, dt$$

$$= (-i)^2 \int_{-\infty}^{\infty} t^2 f(t)\, e^{-i\omega t} \, dt = (-i)^2 \mathscr{F}[t^2 f(t)].$$

By induction we have the result.

In particular $\quad \mathscr{F}[t f(t)] = i F'(\omega)$ and $\mathscr{F}[t^2 f(t)] = -F''(\omega)$.

Remark 6
From Eq. (9.130), we have

$$\mathscr{F}^{-1}[F^{(n)}(\omega)] = (-i)^n t^n f(t). \tag{9.131}$$

Fourier transform of an integral

Theorem 9.13 Let $f(t)$ be piecewise continuous an every interval $[-l, l]$ and $\int_{-\infty}^{\infty} |f(t)| \, dt$ converge. Let $\mathscr{F}[f(t)] = F(\omega)$ and $F(\omega)$ satisfies $F(0) = 0$. Then

$$\mathscr{F}\left[\int_{-\infty}^{t} f(\tau)\, d\tau\right] = \frac{1}{i\omega} F(\omega). \tag{9.132}$$

Example 9.29 Find the inverse Fourier transform of $(\sqrt{\pi}\, \omega e^{-\omega^2/8})/(4\sqrt{2}\, i)$.

Solution We have $\dfrac{\sqrt{\pi}\, \omega}{4\sqrt{2}\, i} e^{-\omega^2/8} = -\dfrac{\sqrt{\pi}}{\sqrt{2}\, i} \dfrac{d}{d\omega} [e^{-\omega^2/8}].$

Therefore, $\mathscr{F}^{-1}\left[\dfrac{\sqrt{\pi}\omega}{4\sqrt{2}i}e^{-\omega^2/8}\right] = -\dfrac{\sqrt{\pi}}{i\sqrt{2}}\mathscr{F}^{-1}\left[\{e^{-\omega^2/8}\}'\right] = -\dfrac{\sqrt{\pi}}{i\sqrt{2}}\mathscr{F}^{-1}[F'(\omega)]$

where $F(\omega) = e^{-\omega^2/8}$. Hence, using Eq. (9.130), we obtain

$$\mathscr{F}^{-1}\left[\frac{\sqrt{\pi}\omega}{4\sqrt{2}i}e^{-\omega^2/8}\right] = -\frac{\sqrt{\pi}}{i\sqrt{2}}[-itf(t)]$$

where

$$f(t) = \mathscr{F}^{-1}[F(\omega)] = \mathscr{F}^{-1}[e^{-\omega^2/8}] = \frac{\sqrt{2}}{\sqrt{\pi}}e^{-2t^2}.$$

(see Example 9.24). Therefore,

$$\mathscr{F}^{-1}\left[\frac{\sqrt{\pi}\omega}{4\sqrt{2}i}e^{-\omega^2/8}\right] = te^{-2t^2}.$$

Convolution

Theorem 9.14 Let $f(t)$, $g(t)$ be piecewise continuous on every interval $[-l, l]$ and let

$$\int_{-\infty}^{\infty}|f(t)|\,dt,\quad \int_{-\infty}^{\infty}|g(t)|\,dt$$

converge. Denote $\mathscr{F}[f(t)] = F(\omega)$ and $\mathscr{F}[g(t)] = G(\omega)$. Then

$$\mathscr{F}[\{f * g\}(t)] = F(\omega)\,G(\omega) \qquad \text{(convolution with respect to time)} \tag{9.133}$$

and

$$\mathscr{F}[f(t)\,g(t)] = \frac{1}{2\pi}[F * G](\omega) \quad \text{(convolution with respect to frequency)} \tag{9.134}$$

where the convolution $(f * g)(t)$ is defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)\,g(t-\tau)\,d\tau = (g * f)(t).$$

The inverse transforms are given by

$$\mathscr{F}^{-1}[F(\omega)\,G(\omega)] = (f * g)(t) \tag{9.135}$$

and

$$\mathscr{F}^{-1}[(F * G)(\omega)] = 2\pi f(t)\,g(t). \tag{9.136}$$

Example 9.30 Using convolution find $\mathscr{F}^{-1}[1/(12 + 7i\omega - \omega^2)]$.

Solution We have $12 + 7i\omega - \omega^2 = (4 + i\omega)(3 + i\omega)$. Therefore,

$$\mathscr{F}^{-1}\left[\frac{1}{(4+i\omega)(3+i\omega)}\right] = \mathscr{F}^{-1}\left[\frac{1}{4+i\omega}\cdot\frac{1}{3+i\omega}\right].$$

But $\mathscr{F}^{-1}\left[\dfrac{1}{4+i\omega}\right] = e^{-4t}H(t)$ and $\mathscr{F}^{-1}\left[\dfrac{1}{3+i\omega}\right] = e^{-3t}H(t)$ (see Example 9.22), where $H(t)$ is the unit step function. Using convolution, we obtain

$$\mathcal{F}^{-1}\left[\frac{1}{(4+i\omega)(3+i\omega)}\right] = [e^{-4t} H(t)] * [e^{-3t} H(t)]$$

$$= \int_{-\infty}^{\infty} e^{-4\tau} H(\tau)\, e^{-3(t-\tau)} H(t-\tau)\, d\tau = e^{-3t} \int_{-\infty}^{\infty} e^{-\tau} H(\tau) H(t-\tau)\, d\tau$$

But
$$H(\tau) H(t-\tau) = \begin{cases} 0, & \text{for } \tau < 0 \text{ and } \tau > t. \\ 1, & \text{for } 0 < \tau < t. \end{cases}$$

Therefore,
$$\mathcal{F}^{-1}\left[\frac{1}{(4+i\omega)(3+i\omega)}\right] = e^{-3t} \int_0^t e^{-\tau}\, d\tau = e^{-3t}\, [1 - e^{-t}], t \geq 0.$$

Fourier transform of the Dirac-delta function

In section 8.5, we have shown that the Laplace transform of the Dirac-delta function is $\mathcal{L}[\delta(t)] = 1$. We have also proved the filtering property of Dirac-delta function as

$$\int_0^{\infty} f(t)\, \delta(t-a)\, dt = f(a). \tag{9.137}$$

Similar results can be proved for the Fourier transform of the Dirac-delta function. We have defined the delta function as (see Eq. (8.29))

$$\delta(t) = \lim_{k \to 0} \frac{1}{k} [H(t) - H(t-k)]$$

where $H(t)$ is the unit step function.

We have
$$H(t) - H(t-k) = \begin{cases} 0, & t < 0 \\ 1, & 0 \leq t < k \\ 0, & t \geq k. \end{cases}$$

Hence,
$$\mathcal{F}[\delta(t)] = \lim_{k \to 0} \left\{\frac{1}{k} \mathcal{F}[H(t) - H(t-k)]\right\} = \lim_{k \to 0} \left\{\frac{1}{k} \int_0^k e^{-i\omega t}\, dt\right\}$$

$$= \lim_{k \to 0} \frac{1}{k} \left[\frac{1 - e^{-i\omega k}}{i\omega}\right] = 1.$$

Therefore, Laplace transform and Fourier transform of the Dirac-delta function are both equal to 1. Hence, $\mathcal{F}^{-1}[1] = \delta(t)$.

The filtering property given in Eq. (9.137) gets modified as

$$\int_{-\infty}^{\infty} f(t)\, \delta(t-a)\, dt = f(a). \tag{9.138}$$

This result can be proved using the definition of the delta function.

Using the time-shift theorem (see Eq. 9.122), we obtain

$$\mathscr{F}[\delta(t-a)] = e^{-ia\omega}F(\omega) = e^{-ia\omega} \tag{9.139}$$

Using the symmetry result (see Eq. 9.129), we obtain

$$\mathscr{F}[1] = 2\pi\delta(-\omega) = 2\pi\delta(\omega). \tag{9.140}$$

Fourier cosine transform

The Fourier cosine transform of $f(t)$ is defined as

$$\mathscr{F}_c[f(t)] = \int_0^\infty f(t)\cos(\omega t)\,dt = F_c(\omega). \tag{9.141}$$

We can compare Fourier cosine transform with the Fourier cosine integral representation of $f(t)$ on $[0, \infty)$ which is given by (see Eq. (9.40))

$$f(t) = \frac{1}{\pi}\int_0^\infty A(\omega)\cos(\omega t)\,d\omega \tag{9.142}$$

where
$$A(\omega) = 2\int_0^\infty f(t)\cos(\omega t)\,dt. \tag{9.143}$$

Comparing Eqs. (9.141) and (9.143), we obtain $A(\omega) = 2F_c(\omega)$. Substituting in Eq. (9.142), we get

$$f(t) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos(\omega t)\,d\omega. \tag{9.144}$$

This result can be interpreted as the inverse Fourier cosine transform.

Example 9.31 Find the Fourier cosine transform of $f(t)$, where

(i) $f(t) = \begin{cases} 1, & 0 \le t \le l, \\ 0, & t > l. \end{cases}$ (ii) $f(t) = \begin{cases} t, & 0 \le t \le l \\ 0, & t > l. \end{cases}$

Solution

(i)
$$\mathscr{F}_c[f(t)] = \int_0^\infty f(t)\cos(\omega t)\,dt = \int_0^l \cos(\omega t)\,dt = \frac{\sin(\omega l)}{\omega}.$$

(ii)
$$\mathscr{F}_c[f(t)] = \int_0^\infty f(t)\cos(\omega t)\,dt = \int_0^l t\cos(\omega t)\,dt$$

$$= \left[\frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}\right]_0^l = \frac{l}{\omega}\sin(\omega l) + \frac{1}{\omega^2}[\cos(\omega l) - 1].$$

Fourier sine transform

The Fourier sine transform of $f(t)$ is defined as

$$\mathscr{F}_s[f(t)] = \int_0^\infty f(t)\sin(\omega t)\,dt = F_s(\omega). \tag{9.145}$$

Comparing with the Fourier sine integral representation of $f(t)$ (see Eq. (9.42))

$$f(t) = \frac{1}{\pi} \int_0^\infty B(\omega) \sin(\omega t) d\omega, \quad \text{where} \quad B(\omega) = 2 \int_0^\infty f(t) \sin(\omega t)\, dt$$

we obtain $B(\omega) = 2F_s(\omega)$. Therefore,

$$f(t) = \frac{2}{\pi} \int_0^\infty F_s(\omega) \sin(\omega t)\, d\omega. \tag{9.146}$$

This result can be interpreted as the inverse Fourier sine transform.

Example 9.32 Find the Fourier sine transform of $f(t)$, where

(i) $f(t) = \begin{cases} 1, & 0 \le t \le l, \\ 0, & t > l. \end{cases}$ (ii) $f(t) = \begin{cases} t, & 0 \le t \le l, \\ 0, & t > l. \end{cases}$

Solution

(i)

$$\mathcal{F}_s[f(t)] = \int_0^\infty f(t) \sin(\omega t)\, dt = \int_0^l \sin(\omega t)\, dt$$

$$= \left[-\frac{\cos(\omega t)}{\omega}\right]_0^l = \frac{1}{\omega}[1 - \cos(\omega l)].$$

(ii)

$$\mathcal{F}_s[f(t)] = \int_0^\infty f(t) \sin(\omega t)\, dt = \int_0^l t \sin(\omega t)\, dt$$

$$= \left[-\frac{t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}\right]_0^l = \frac{\sin(\omega l)}{\omega^2} - \frac{l\cos(\omega l)}{\omega}.$$

Fourier cosine and sine transforms of derivatives

Let $f(t)$ and $f'(t)$ be continuous on the interval $[0, \infty)$. Let $f(t) \to 0, f'(t) \to 0$ as $t \to \infty$ and $f''(t)$ be piecewise continuous on every subinterval $[0, l]$. Then

$$\mathcal{F}_c[f''(t)] = -\omega^2 \mathcal{F}_c[f(t)] - f'(0) \quad \text{and} \quad \mathcal{F}_s[f''(t)] = -\omega^2 \mathcal{F}_s[f(t)] + \omega f(0).$$

From the definition, we have

$$\mathcal{F}_c[f''(t)] = \int_0^\infty f''(t) \cos(\omega t) dt$$

$$= [f'(t)\cos(\omega t) + \omega f(t) \sin(\omega t)]_0^\infty - \omega^2 \int_0^\infty f(t) \cos(\omega t) dt$$

$$= -\omega^2 \mathcal{F}_c[f(t)] - f'(0) = -\omega^2 F_c(\omega) - f'(0). \tag{9.147}$$

$$\mathscr{F}_s[f''(t)] = \int_0^\infty f''(t)\sin(\omega t)\,dt$$

$$= [f'(t)\sin(\omega t) - \omega f(t)\cos(\omega t)]_0^\infty - \omega^2 \int_0^\infty f(t)\sin(\omega t)\,dt$$

$$= -\omega^2 F_s[f(t)] + \omega f(0) = -\omega^2 F_s(\omega) + \omega f(0). \tag{9.148}$$

Example 9.33 Prove the following

$$\mathscr{F}_c[f'(t)] = \omega F_s(\omega) - f(0) \quad \text{and} \quad \mathscr{F}_s[f'(t)] = -\omega F_c(\omega)$$

assuming that $f(t) \to 0$ as $t \to \infty$.

Solution From the definition, we have

$$\mathscr{F}_c[f'(t)] = \int_0^\infty f'(t)\cos(\omega t)\,dt = [f(t)\cos(\omega t)]_0^\infty + \omega \int_0^\infty f(t)\sin(\omega t)\,dt$$

$$= \omega F_s(\omega) - f(0).$$

$$\mathscr{F}_s[f'(t)] = \int_0^\infty f'(t)\sin(\omega t)dt = [f(t)\sin(\omega t)]_0^\infty - \omega \int_0^\infty f(t)\cos(\omega t)\,dt$$

$$= -\omega F_c(\omega).$$

Example 9.34 Find the Fourier cosine and sine transforms of

$$f(t) = e^{-\alpha t},\ t \ge 0,\ \alpha > 0.$$

Solution We shall use the formulas for derivatives to find the Fourier cosine and sine transforms of $f(t)$. Since $f(t) = e^{-\alpha t}$, we have $f'(t) = -\alpha e^{-\alpha t}$ and $f''(t) = \alpha^2 e^{-\alpha t}$. Using Eq. (9.147), we have

$$\mathscr{F}_c[f''(t)] = \mathscr{F}_c[\alpha^2 e^{-\alpha t}] = \alpha^2 \mathscr{F}_c[e^{-\alpha t}] = \alpha^2 F_c(\omega).$$

Also,

$$\mathscr{F}_c[f''(t)] = -\omega^2 F_c(\omega) - f'(0) = -\omega^2 F_c(\omega) + \alpha.$$

Therefore,

$$\alpha^2 F_c(\omega) = -\omega^2 F_c(\omega) + \alpha, \text{ or } F_c(\omega) = \frac{\alpha}{\alpha^2 + \omega^2}.$$

Now,

$$\mathscr{F}_s[f''(t)] = \alpha^2 \mathscr{F}_s[e^{-\alpha t}] = \alpha^2 F_s(\omega).$$

Also,

$$\mathscr{F}_s[f''(t)] = -\omega^2 F_s(\omega) + \omega f(0) = -\omega^2 F_s(\omega) + \omega.$$

Therefore,

$$\alpha^2 F_s(\omega) = -\omega^2 F_s(\omega) + \omega, \quad \text{or} \quad F_s(\omega) = \frac{\omega}{\alpha^2 + \omega^2}.$$

Finite Fourier cosine transform

The Fourier cosine and sine transforms are defined on $[0, \infty)$ and were obtained from Fourier cosine and sine integral representations of functions. However, in many applications, we are to deal with problems defined on finite intervals. In this case, we define the finite Fourier cosine and sine transforms and are obtained from the Fourier cosine and sine series.

Let the given function $f(t)$ be piecewise continuous on $[0, \pi]$. Then, the *finite Fourier cosine transform* of $f(t)$ is defined by

$$F_c(n) = \int_0^{\pi} f(t)\cos(nt)\,dt \tag{9.149}$$

where n is an non-negative integer, $n = 0, 1, 2, \ldots$. This transform is also denoted by $C_n[f(t)]$.

The Fourier cosine half-range series is defined by (see Eq. (9.25))

$$f(t) = \frac{1}{\pi}\int_0^{\pi} f(t^*)dt^* + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\int_0^{\pi} f(t^*)\cos(nt^*)\,dt^*\right]\cos(nt).$$

In terms of the finite Fourier cosine transform, we can write this equation as

$$f(t) = \frac{1}{\pi}\left[F_c(0) + 2\sum_{n=1}^{\infty} F_c(n)\cos(nt)\right]. \tag{9.150}$$

This result can be interpreted as the inverse finite Fourier cosine transform.

Formula for the second derivative

We assume that $f(t)$ and $f'(t)$ are continuous and $f''(t)$ is piecewise continuous on $[0, \pi]$. Then

$$C_n[f''(t)] = -n^2F_c(n) - f'(0) + (-1)^n f'(\pi),\ n = 1, 2, \ldots.$$

This result can be proved using the definition of $C_n[f(t)]$. We have

$$C_n[f''(t)] = \int_0^{\pi} f''(t)\cos(nt)\,dt$$

$$= [f'(t)\cos(nt) + nf(t)\sin(nt)]_0^{\pi} - n^2\int_0^{\pi} f(t)\cos(nt)\,dt$$

$$= -n^2F_c(n) - f'(0) + (-1)^n f'(\pi).$$

Finite Fourier sine transform

Let the function $f(t)$ be piecewise continuous on $[0, \pi]$. Then, the *finite Fourier sine transform* of $f(t)$ is defined as

$$F_s(n) = \int_0^{\pi} f(t)\sin(nt)dt \tag{9.151}$$

where n is an integer and $n = 1, 2, \ldots$. This transform is also denoted by $S_n[f(t)]$.

The Fourier sine half-range series is defined by (see Eq. (9.26))

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\int_0^{\pi} f(t^*)\sin(nt^*)\right]\sin(nt).$$

In terms of the finite Fourier sine transform, we can write this equation as

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n)\sin(nt). \tag{9.152}$$

This result can be interpreted as the inverse finite Fourier sine transform.

Formula for the second derivative

We assume that $f(t)$ and $f'(t)$ are continuous and $f''(t)$ is piecewise continuous on $[0, \pi]$. Then

$$S_n[f''(t)] = -n^2 F_s(n) + nf(0) - n(-1)^n f(\pi), \quad n = 1, 2, \ldots. \tag{9.153}$$

Using the definition, we have

$$S_n[f''(t)] = \int_0^\pi f''(t) \sin(nt)\,dt$$

$$= \left[f'(t) \sin(nt) - nf(t) \cos(nt)\right]_0^\pi - n^2 \int_0^\pi f(t) \sin(nt)\,dt$$

$$= -n^2 F_s(n) + nf(0) - n(-1)^n f(\pi).$$

Example 9.35 Find the finite Fourier sine transform of

$$f(t) = \begin{cases} 0, & 0 \le x < \pi/2 \\ 1, & \pi/2 \le x \le \pi. \end{cases}$$

Solution We have

$$S_n[f(t)] = \int_0^\pi f(t) \sin(nt)\,dt = \int_{\pi/2}^\pi \sin(nt)\,dt = -\frac{1}{n}\left[\cos(nt)\right]_{\pi/2}^\pi$$

$$= -\frac{1}{n}[\cos n\pi - \cos(n\pi/2)] = -\frac{1}{n}[(-1)^n - \cos(n\pi/2)].$$

9.6.1 Fourier Transform Solution of Some Partial Differential Equations

In section 8.7, we have studied Laplace transform methods for the solution of some partial differential equations and in section 9.5 we have studied the Fourier series and Fourier integral solution of heat equation, wave equation and Laplace equation. In this section, we shall discuss the Fourier transform solution of some of these partial differential equations.

If the Fourier transform is applied with respect to one of the variables in the differential equation, then we obtain an ordinary differential equation in terms of the other variable. We solve this differential equation. The solution of the given *bvp* is then obtained by taking the inverse Fourier transform.

Example 9.36 The temperature distribution $u(x, t)$ in a thin, homogeneous, infinite bar can be modelled by the initial boundary value problem

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, t > 0$$

$$u(x, 0) = f(x), \quad u(x, t) \text{ is finite as } x \to \pm\infty.$$

Find $u(x, t)$, $t > 0$.

Solution Since the domain of the bar is $-\infty < x < \infty$, we use the Fourier transform with respect to x. Denote $\mathcal{F}[u(x, t)] = F(\omega, t)$. Taking Fourier transform of the differential equation, we obtain

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}\left[c^2 \frac{\partial^2 u}{\partial x^2}\right] = c^2 \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right]. \tag{9.154}$$

Now,

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\omega x}\, dx = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u(x, t) e^{-i\omega x}\, dx = \frac{\partial F(\omega, t)}{\partial t} = \frac{dF(\omega, t)}{dt}$$

assuming ω as a parameter, and

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -\omega^2 F(\omega, t). \qquad \text{(from Eq. (9.128))}$$

Substituting in Eq. (9.154), we obtain

$$\frac{dF(\omega, t)}{dt} = -c^2\omega^2 F(\omega, t).$$

The solution of this linear, first order differential equation is $F(\omega, t) = ke^{-c^2\omega^2 t}$ where k is a parameter (function of ω) to be determined. Writing the transform of the initial condition, we obtain

$$\mathcal{F}[u(x, 0)] = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = G(\omega).$$

Therefore, $\quad F(\omega, 0) = k = G(\omega)$ and $F(\omega, t) = G(\omega) e^{-c^2\omega^2 t}$.

The solution $u(x, t)$ of the *bvp* is the inverse transform of $F(\omega, t)$. The inverse can be computed in a number of ways. We can obtain the inverse directly, using the definition (see Eq. (9.118)). We have

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega, t) e^{i\omega x}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{-c^2\omega^2 t} e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi}\, d\xi\right] e^{-c^2\omega^2 t} e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega(\xi - x)} e^{-c^2\omega^2 t}\, d\xi\, d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) \cos \omega(\xi - x) e^{-c^2\omega^2 t}\, d\xi\, d\omega\right.$$

$$\left. - i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) \sin \omega(\xi - x) e^{-c^2\omega^2 t}\, d\xi\, d\omega\right] = P - iQ$$

where P and Q are the integrals on the left hand side. The integrand of Q is an odd function of ω. Therefore,

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\cos\omega(\xi - x)\, e^{-c^2\omega^2 t}\, d\xi\, d\omega.$$

Example 9.37 Find the solution in Example 9.36, if

$$f(x) = \begin{cases} v, & -l < x < l \\ 0, & \text{otherwise, } v \text{ a constant.} \end{cases}$$

Solution We have

$$G(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = v\int_{-l}^{l} e^{-i\omega x}\, dx$$

$$= \frac{v}{i\omega}[e^{i\omega l} - e^{-i\omega l}] = \frac{2v}{\omega}\sin(\omega l).$$

Therefore,

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2v}{\omega}\sin(\omega l)\, e^{-c^2\omega^2 t}\, e^{i\omega x}\, d\omega$$

$$= \frac{v}{\pi}\int_{-\infty}^{\infty}\frac{1}{\omega}\sin(\omega l)\,[\cos\omega x + i\sin\omega x]\, e^{-c^2\omega^2 t}\, d\omega$$

$$= \frac{v}{\pi}\int_{-\infty}^{\infty}\frac{1}{\omega}\sin(\omega l)\cos\omega x\, e^{-c^2\omega^2 t}\, d\omega \tag{9.155}$$

since $[\sin(\omega l)\sin(\omega x)]/\omega$ is an odd function of ω. If the solution of the *bvp* was obtained by the Laplace transform method, then the solution can be expressed in terms of error function. Therefore, Eq. (9.155) gives another form of the solution.

Example 9.38 The temperature distribution $u(x, t)$ in a thin, homogeneous semi-infinite bar can be modelled by the initial boundary value problem

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2},\ 0 < x < \infty,\ t > 0$$

$$u(x, 0) = f(x),\ x > 0;\ u(0, t) = 0,\ t > 0.$$

Find the temperature distribution $u(x, t)$, $t > 0$, $0 < x < \infty$.

Solution The domain of definition of x is $0 < x < \infty$. Hence, Fourier transform cannot be used. We can explore the possibility of using Fourier cosine or sine transforms. Application of Fourier cosine transform requires the information of a derivative (see Eq. (9.147)). Therefore, we can attempt finding solution by Fourier sine transform. Applying the sine transform to the differential equation, we obtain

$$\mathcal{F}_s\left[\frac{\partial u}{\partial t}\right] = c^2\,\mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}\right].$$

Using Eq. (9.148), we obtain

$$\frac{d}{dt} F_s(\omega, t) = c^2[-\omega^2 F_s(\omega, t) + \omega u(0, t)] = -c^2\omega^2 F_s(\omega, t).$$

The solution of this linear, first order equation is

$$F_s(\omega, t) = k(\omega) e^{-c^2\omega^2 t} \tag{9.156}$$

where $k(\omega)$ is any arbitrary function of ω.

Taking sine transform of the boundary condition, we get

$$\mathcal{F}_s[u(x, 0)] = F_s(\omega, 0) = \mathcal{F}_s[f(x)] = G(\omega).$$

Substituting in Eq. (9.156), we get $k(\omega) = G(\omega)$. Therefore,

$$F_s(\omega, t) = G(\omega)\, e^{-c^2\omega^2 t}.$$

The inverse transform is given by (see Eq. (9.146))

$$u(x, t) = \frac{2}{\pi}\int_0^\infty F_s(\omega, t)\sin(\omega x)\,d\omega = \frac{2}{\pi}\int_0^\infty G(\omega) e^{-c^2\omega^2 t}\sin(\omega x)\,d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(\xi)\sin(\omega\xi)\,d\xi\right] e^{-c^2\omega^2 t}\sin(\omega x)\,d\omega.$$

Example 9.39 The steady state temperature distribution $u(x, y)$ in a thin, homogeneous semi-infinite plate is governed by the *bvp*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < l,\ 0 < y < \infty$$

$$u(0, y) = e^{-2y},\ u(l, y) = 0,\ y > 0;\ \left(\frac{\partial u}{\partial y}\right)(x, 0) = 0,\ 0 < x < l.$$

Find the temperature distribution $u(x, y)$, $0 < x < l$, $y > 0$.

Solution Since the domain of x is finite and the domain of y is $0 < y < \infty$, we can attempt to use Fourier cosine or sine transform (with respect to the variable y). Since the derivative $(\partial u/\partial y)(x, 0)$ is prescribed, we can use the cosine transform. Applying the cosine transform to the differential equation, we obtain

$$\mathcal{F}_c\left[\frac{\partial^2 u}{\partial x^2}\right] + \mathcal{F}_c\left[\frac{\partial^2 u}{\partial y^2}\right] = 0$$

where the transform is taken with respect to the variable y, that is

$$\mathcal{F}_c[u(x, y)] = \int_0^\infty u(x, y)\cos(\omega y)\,dy = F(x, \omega).$$

Hence, we obtain (using Eq. (9.147))

$$\frac{d^2}{dx^2} F(x, \omega) - \omega^2 F(x, \omega) - \frac{\partial u}{\partial y}(x, 0) = 0 \quad \text{or} \quad \frac{d^2 F}{dx^2} - \omega^2 F = 0.$$

The solution of this ordinary differential equation is

$$F(x, \omega) = A \cosh(\omega x) + B \sinh(\omega x). \tag{9.157}$$

Taking cosine transform of the other boundary conditions, we obtain

$$\mathcal{F}_c[u(0, y)] = F(0, \omega) = \mathcal{F}_c[e^{-2y}] = \frac{2}{4 + \omega^2} \quad \text{(see Example 9.34)}$$

and

$$\mathcal{F}_c[u(l, y)] = \mathcal{F}_c[0] = 0 = F(l, \omega).$$

Using these conditions in Eq. (9.157), we get

$$F(0, \omega) = A = \frac{2}{4 + \omega^2} \quad \text{and} \quad F(l, \omega) = A \cosh(\omega l) + B \sinh(\omega l) = 0.$$

We have

$$B = -\frac{A \cosh(\omega l)}{\sinh(\omega l)}$$

and

$$F(x, \omega) = A \cosh(\omega x) + B \sinh(\omega x) = A \left[\cosh(\omega x) - \frac{\cosh(\omega l) \sinh(\omega x)}{\sinh(\omega l)}\right]$$

$$= \frac{A}{\sinh(\omega l)} [\cosh(\omega x) \sinh(\omega l) - \cosh(\omega l) \sinh(\omega x)]$$

$$= \frac{2 \sinh[\omega(l - x)]}{(4 + \omega^2) \sinh(\omega l)}.$$

The inverse transform gives

$$u(x, y) = \frac{2}{\pi} \int_0^\infty F(x, \omega) \cos(\omega y)\, d\omega = \frac{4}{\pi} \int_0^\infty \frac{\sinh[\omega(l - x)] \cos(\omega y)}{(4 + \omega^2) \sinh(\omega l)}\, d\omega.$$

Exercise 9.5

Find the Fourier transform of the following functions.

1. $f(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & |t| > 1. \end{cases}$

2. $f(t) = \begin{cases} -(1 + t), & -1 \le t \le 0, \\ t - 1, & 0 < t \le 1, \\ 0, & |t| > 1. \end{cases}$

3. $f(t) = \begin{cases} \cos t, & -l \le t \le l, \\ 0, & |t| > l. \end{cases}$

4. $f(t) = \begin{cases} e^{\alpha t}, & t < 0 \\ 0, & t > 0,\ \alpha > 0. \end{cases}$

5. $f(t) = \begin{cases} -e^{\alpha t}, & t < 0 \\ e^{-\alpha t}, & t > 0,\ \alpha > 0. \end{cases}$

6. $f(t) = H(t - 3)e^{-4t}$.

7. $f(t) = e^{-a|t+1|}$, $a > 0$.

8. $f(t) = 1/(1 + t^2)$.

9. $f(t) = 2e^{-3|t|} \sin(4t)$.

Find the inverse Fourier transform of the following functions.

10. $\dfrac{e^{-i\omega}}{2(1+i\omega)}$. **11.** $\dfrac{2i\omega}{(3+i\omega)}$. **12.** $\dfrac{e^{-(1-i\omega)}}{3+i\omega}$. **13.** $\dfrac{e^{-2i\omega}}{2+3i\omega}$.

14. $\dfrac{1}{6+5i\omega-\omega^2}$. **15.** $\dfrac{i\omega}{(i\omega+2)(i\omega+3)}$. **16.** $\dfrac{e^{-2i\omega}}{4+\omega^2}$. **17.** $\dfrac{1}{a^4+\omega^4}$, $a > 0$.

Using frequency convolution show that

18. $\displaystyle\int_{-\infty}^{\infty} \frac{d\tau}{(2-i\tau+i\omega)(2+i\tau)} = \frac{2\pi}{4+i\omega}$. **19.** $\displaystyle\int_{-\infty}^{\infty} \frac{d\tau}{(4-i\tau)(4-i\tau+i\omega)} = 0$.

Use the time convolution to find the inverse of the following functions.

20. $\dfrac{1}{(i\omega+k)^2}$, $k > 0$. **21.** $\dfrac{1}{(i\omega+k)^3}$, $k > 0$.

22. Let $F(\omega)$, $G(\omega)$ be the Fourier transforms of $f(x)$ and $g(x)$ respectively. Then, show that

(i) $\displaystyle\int_{-\infty}^{\infty} F(\omega)\overline{G}(\omega)\, d\omega = 2\pi \int_{-\infty}^{\infty} f(x)\,\overline{g}(x)\, dx$, (ii) $\displaystyle\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega = 2\pi \int_{-\infty}^{\infty} |f(x)|^2\, dx$

(**Parseval's identities** in Fourier transforms)

Using Fourier transforms, find the solution of the following differential equations.

23. $y' - 4y = H(t)\, e^{-4t}$, $-\infty < t < \infty$. **24.** $y' + 3y = H(t)\, e^{-2t}$, $-\infty < t < \infty$.

25. $y'' + 5y' + 4y = \delta(t-2)$. **26.** $y'' + 3y' + 2y = \delta(t-3)$.

Find the Fourier cosine and sine transforms of the following functions.

27. $f(t) = \begin{cases} \sin t, & 0 \le t \le l, \\ 0, & t > l. \end{cases}$ **28.** $f(t) = \begin{cases} \cos t, & 0 \le t \le l \\ 0, & t > l. \end{cases}$

29. $f(t) = \begin{cases} 1+t, & 0 \le t \le l, \\ 0, & t > l. \end{cases}$ **30.** $f(t) = \begin{cases} e^{2t} - e^{-2t}, & 1 \le t < 2, \\ 0, & \text{otherwise.} \end{cases}$

The following functions are defined on $[0, \pi]$. Find the finite Fourier cosine and sine transform of the following functions.

31. $f(t) = t$. **32.** $f(t) = \sin(at)$, $a > 0$.

33. $f(t) = e^{-t}$. **34.** $f(t) = \sinh(at)$, $a > 0$.

Using the Fourier integral transforms, solve the following initial boundary value problems.

35. $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$,

$u(x, 0) = e^{-2|x|}$, $-\infty < x < \infty$.

36. $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$,

$$u(x, 0) = \begin{cases} 1, & -1 < x < 0 \\ -1, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

37. $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}, \; -\infty < x < \infty, t > 0.$

$u(x, 0) = e^{-4x^2}, -\infty < x < \infty.$

It is given that $\mathscr{F}[e^{-at^2}] = \sqrt{\dfrac{\pi}{a}}\, e^{-\omega^2/(4a)}.$

38. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \; 0 < x < \infty, t > 0,$

$u(x, 0) = \begin{cases} 1, & 0 < x \le l, \\ 0, & x > l. \end{cases}$

$u(0, t) = 0, t > 0.$

39. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \; 0 < x < \pi, y > 0.$

$u(0, y) = 0, u(\pi, y) = 0, y > 0.$

$u(x, 0) = \sin x, 0 < x < \pi.$

40. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty, 0 < y < \pi,$

$u(x, 0) = e^{-2x} H(x),$

$u(x, \pi) = 0, -\infty < x < \infty.$

41. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty, 0 < y < 1,$

$\dfrac{\partial u}{\partial y}(x, 0) = 0, u(x, 1) = e^{-2|x|}, -\infty < x < \infty.$

9.7 Answers and Hints

Exercise 9.1

In the following problems denote $p_n = 1 - \cos n\pi = 1 - (-1)^n$. The summations are all from $n = 1$ to ∞, except where it is specifically mentioned.

1. $\dfrac{k}{2} - \dfrac{k}{\pi} \Sigma \left[\dfrac{1}{n} p_n \sin(nx)\right].$

2. $\dfrac{3\pi}{4} + \Sigma \left[\dfrac{1}{\pi n^2} p_n \cos(nx) + \dfrac{1}{n} \cos(n\pi) \sin(nx)\right].$

3. $\dfrac{k}{2} + \dfrac{2k}{\pi} \Sigma \left[\dfrac{1}{n} \sin\left(\dfrac{n\pi}{2}\right) \cos(nx)\right].$

4. $\dfrac{2k}{\pi} \Sigma \left[\dfrac{1}{n} p_n \sin(nx)\right].$

5. $3 + \dfrac{2}{\pi} \Sigma \left[\dfrac{1}{n} p_n \sin(nx)\right].$

6. $-\dfrac{\pi}{2} - \dfrac{2}{\pi} \Sigma \left[\dfrac{1}{n^2} p_n \cos(nx)\right].$

7. $\dfrac{1}{2}(2 - \pi) + \dfrac{2}{\pi} \Sigma \left[\dfrac{1}{n^2} p_n \cos(nx)\right].$

8. $\dfrac{\pi^2}{3} + 4 \Sigma \left[\dfrac{1}{n^2} \cos(n\pi) \cos(nx)\right].$

9. $\dfrac{2}{\pi} \Sigma \left[\left(\dfrac{6\pi}{n^3} - \dfrac{\pi^3}{n}\right) \cos(n\pi) \sin(nx)\right].$

10. $\dfrac{2}{\pi} \Sigma \left[\dfrac{\pi^2}{n} \cos n\pi + \dfrac{2}{n^3} p_n\right] \sin(nx).$

11. $\cos x + \dfrac{8}{\pi} \sum_{n=2}^{\infty} \left[\dfrac{n \cos(n\pi)}{(4n^2 - 1)} \sin(nx)\right].$

12. $\dfrac{1}{\pi} + \dfrac{1}{2} \sin x + \dfrac{1}{\pi} \sum_{n=2}^{\infty} \left[\dfrac{(-1)^{n-1} - 1}{(n^2 - 1)} \cos nx\right].$

13. $\dfrac{1}{4} + \dfrac{1}{2\pi^2}(4 + \pi^2) \cos x + \dfrac{1}{\pi} \sin x$

$+ \dfrac{1}{\pi^2} \sum_{n=2}^{\infty} \left[\dfrac{1}{n^2} p_n \cos(nx) + \left\{\dfrac{n\pi}{(n^2 - 1)} ((-1)^{n-1} - 1) + \dfrac{\pi}{n}\right\} \sin(nx)\right]$

50. $\frac{1}{2}A_0 y + \Sigma\left[A_n \cos\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\right], A_0 = \frac{2}{ab}\int_0^a f(x)dx,$

$$A_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx.$$

$$A_0 = \frac{a}{2b}, A_n = T\left(\frac{a}{n\pi}\right)^2\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - \cos n\pi\right], T = \frac{2}{a\sinh(n\pi b/a)}.$$

51. $\Sigma A_n\left[\cosh\left(\frac{n\pi y}{a}\right) - \coth\left(\frac{n\pi b}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\right]\sin\left(\frac{n\pi x}{a}\right), A_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx.$

52. $\Sigma A_n \sinh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right), A_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx.$

53. $\Sigma A_n \cosh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right), A_n = \frac{2}{a\cosh(n\pi b/a)}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx,$

54. $\frac{x}{2} + \frac{2a}{\pi^2\sinh(n\pi)}\Sigma\frac{1}{n^2}p_n\sinh\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right), p_n = 1-(-1)^n.$

55. $\Sigma\,[A_n\cosh(n\pi x) + B_n\sinh(n\pi x)]\sin(n\pi y)$

$$A_n = -\frac{2}{n\pi}\cos(n\pi), B_n = \frac{-1}{n\pi\cosh(n\pi)}\left[\frac{10}{(n\pi)^2}p_n + A_n\sinh(n\pi)\right], p_n = 1-(-1)^n.$$

Exercise 9.5

1. $2\sin(\omega)/\omega.$
2. $2\,[\cos\omega - 1]/\omega^2.$
3. $2\,[\omega\cos l\sin(\omega l) - \sin l\cos(\omega l)]/(\omega^2 - 1).$
4. $1/(\alpha - i\omega).$
5. $-2i\omega/(a^2 + \omega^2).$
6. $e^{-3(4+i\omega)}/(4 + i\omega).$
7. $2ae^{i\omega}/(a^2 + \omega^2).$
8. Write $\frac{1}{1+t^2} = \frac{1}{2}\left[\frac{1}{1+it} + \frac{1}{1-it}\right]$. From Example 9.28, $\mathscr{F}\left[\frac{1}{1+it}\right] = 2\pi e^{\omega}H(-\omega).$ Define $b(t) = H(-t)\,e^t$, so that $\mathscr{F}[b(t)] = 1/(1 - i\omega) = B(\omega)$. Use the symmetry property to show $\mathscr{F}\left[\frac{1}{1-it}\right] = 2\pi e^{-\omega}H(\omega)$. Hence, $\mathscr{F}[1/(1 + t^2)] = \pi e^{-|\omega|}.$
9. Use frequency modulation theorem, $(6i)\left[\frac{1}{9+(\omega+4)^2} - \frac{1}{9+(\omega-4)^2}\right]$
10. Use shift theorem, $e^{-(t-1)}H(t-1)/2.$
11. $F(\omega) = 2 - \frac{6}{3+i\omega}, 2\delta(t) - 6e^{-3t}H(t).$
12. $e^{-(3t+4)}H(t+1).$
13. $e^{-2(t-2)/3}H(t-2)/3.$
14. $e^{-2t}(1 - e^{-t})\,H(t).$
15. $e^{-2t}(3e^{-t} - 2)\,H(t).$
16. $e^{-2|t-2|}/4.$
17. Roots of $\omega^4 + a^4 = 0$ are $a_1(\pm 1 \pm i), a_1 = a/\sqrt{2}$. Combine two roots each and write

$$F(\omega)=\frac{1}{2\sqrt{2}a^3}\left[\frac{\sqrt{2}\,a-\omega}{(\omega-a_1)^2+a_1^2}+\frac{\sqrt{2}a+\omega}{(\omega+a_1)^2+a_1^2}\right]=\frac{1}{2\sqrt{2}a^3}\left[\frac{a_1-(\omega-a_1)}{(\omega-a_1)^2+a_1^2}+\frac{a_1+(\omega+a_1)}{(\omega+a_1)^2+a_1^2}\right]$$

$$=\frac{1}{2\sqrt{2}a^3}\left[\left\{\frac{a_1}{(\omega-a_1)^2+a_1^2}+\frac{a_1}{(\omega+a_1)^2+a_1^2}\right\}+\frac{1}{2i}\left\{\frac{2i(\omega+a_1)}{(\omega+a_1)^2+a_1^2}-\frac{2i(\omega-a_1)}{(\omega-a_1)^2+a_1^2}\right\}\right]$$

$\mathcal{F}^{-1}[F(\omega)]=\dfrac{1}{2\sqrt{2}a^3}\left[e^{-a_1|t|}\cos(a_1 t)+g(t)\sin(a_1 t)\right]$ (see Problem 5 and use modulation Theorem) where $g(t)=-\,e^{a_1 t}$ for $t<0$, and $e^{-a_1 t}$ for $t>0$. Therefore,

$$\mathcal{F}^{-1}[F(\omega)]=\frac{1}{2\sqrt{2}\,a^3}e^{a_1 t}(\cos(a_1 t)-\sin(a_1 t)),\text{ for } t<0$$

$$=\frac{1}{2\sqrt{2}\,a^3}e^{-a_1 t}(\cos(a_1 t)+\sin(a_1 t)),\text{ for } t>0.$$

18. $F(\tau)=1/(2+i\tau)$, $G(\omega-\tau)=1/[2+i(\omega-\tau)]$. That is $G(t)=1/(2+it)$.
$I=2\pi\,\mathcal{F}[f(t)\,g(t)]=2\pi\,\mathcal{F}[e^{-4t}H(t)]=2\pi/(4+i\omega)$.

19. $I=2\pi\,\mathcal{F}[\{H(-t)\,e^{4t}\}\,\{e^{-4t}H(t)\}]=0$.

20. $(f*g)(t)=e^{-kt}\displaystyle\int_{-\infty}^{\infty}H(\tau)H(t-\tau)\,d\tau=te^{-kt}H(t)$, since $H(\tau)H(t-\tau)=0$ for $\tau<0$ and $\tau>t$, and 1 for $0<\tau<t$.

21. Use the result of Problem 20, $t^2e^{-kt}H(t)/2$.

22. (i) $g(x)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}G(w)e^{iwx}dw$; $\bar{g}(x)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\bar{G}(w)e^{-iwx}dw$

$$\int_{-\infty}^{\infty}f(x)\bar{g}(x)\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)\left[\int_{-\infty}^{\infty}\bar{G}(w)\,e^{-iwx}\,dw\right]dx$$

$$=\int_{-\infty}^{\infty}\bar{G}(w)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{-iwx}\,dx\right]dw=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(w)\,\bar{G}(w)\,dw$$

Hence, $\displaystyle\int_{-\infty}^{\infty}F(w)\,\bar{G}(w)\,dw=2\pi\int_{-\infty}^{\infty}f(x)\,\bar{g}(x)\,dx$

(ii) set $f(x)=g(x)$, $F(w)=G(w)$

$$\int_{-\infty}^{\infty}|F(w)|^2\,dw=2\pi\int|f(x)|^2\,dx$$

23. $F(\omega)=-\dfrac{1}{(4+i\omega)(4-i\omega)]}$, $y(t)=-\dfrac{1}{8}[e^{-4t}H(t)+e^{4t}H(-t)]\begin{cases}=-\dfrac{1}{8}e^{4t}, & t<0\\ =-\dfrac{1}{8}e^{-4t}, & t>0.\end{cases}$

24. $F(\omega)=1/[(2+i\omega)(3+i\omega)]$, $y(t)=e^{-2t}(1-e^{-t})\,H(t)$.

25. $F(\omega)=e^{-2i\omega}/[(4+i\omega)(1+i\omega)]$, $y(t)=\dfrac{1}{3}[e^{-(t-2)}-e^{-4(t-2)}]\,H(t-2)$.

26. $F(\omega)=e^{-3i\omega}/[(2+i\omega)(1+i\omega)]$, $y(t)=[e^{-(t-3)}-e^{-2(t-3)}]\,H(t-3)$.

27. $\dfrac{1}{1-\omega^2}+\dfrac{\cos(\omega-1)\,l}{2(\omega-1)}-\dfrac{\cos(\omega+1)l}{2(\omega+1)}$; for $\omega=1$, $\dfrac{1}{4}(1-\cos(2l))$.

$\frac{1}{2}\left[\frac{\sin(\omega-1)l}{\omega-1}-\frac{\sin(\omega+1)l}{\omega+1}\right]$; for $\omega=1$, $\frac{1}{4}(2l-\sin(2l)$.

28. $\frac{1}{2}\left[\frac{\sin(\omega+1)l}{\omega+1}+\frac{\sin(\omega-1)l}{\omega-1}\right]$; for $\omega=1$, $\frac{1}{4}(2l+\sin(2l))$.

$\frac{1}{2}\left[\frac{2\omega}{\omega^2-1}-\frac{\cos(\omega+1)l}{\omega+1}-\frac{\cos(\omega-1)l}{\omega-1}\right]$; for $\omega=1$, $\frac{1}{4}(1-\cos(2l))$.

29. $\left[\frac{1}{\omega}(1+l)\sin(\omega l)+\frac{1}{\omega^2}(\cos\omega l-1)\right]$, $\left[\frac{1}{\omega}+\frac{\sin(\omega l)}{\omega^2}-\frac{1}{\omega}(1+l)\cos(\omega l)\right]$.

30. $A[2\cos(2\omega)\cosh(4)+\omega\sin(2\omega)\sinh(4)-2\cos\omega\cosh(2)-\omega\sin\omega\sinh(2)]$,
$A[2\sin(2\omega)\cosh(4)-\omega\cos 2\omega\sinh(4)-2\sin\omega\sinh(2)+\omega\cos\omega\sinh(2)]$, $A=2/(4+\omega^2)$.

31. $C_0=\pi^2/2$, $C_n=(\cos(n\pi)-1)/n^2$, $S_n=-\pi\cos n\pi/n$.

32. $C_0=(1-\cos a\pi)/a$, $C_n=\frac{1}{2(n-a)}[\cos\{(n-a)\pi\}-1]-\frac{1}{2(n+a)}[\cos\{(n+a)\pi\}-1]$,

$C_n=0$ if $n=a$ integer. $S_n=\left[\frac{\sin(n-a)\pi}{2(n-a)}-\frac{\sin(n+a)\pi}{2(n+a)}\right]$, $S_n=\frac{\pi}{2}$ if $n=a$ integer.

33. $C_0=1-e^{-\pi}$, $C_n=[1-e^{-\pi}\cos(n\pi)]/(1+n^2)$. $S_n=n[1-e^{-\pi}\cos(n\pi)]/(1+n^2)$.

34. $C_0=[\cosh(a\pi)-1]/a$, $C_n=a[\cos(n\pi)\cosh(a\pi)-1]/(a^2+n^2)$.
$S_n=-n\cos(n\pi)\sinh(a\pi)/(a^2+n^2)$.

35. $u(x,t)=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{(4+\omega^2)}e^{i\omega x-c^2\omega^2 t}\,d\omega=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{(4+\omega^2)}\cos(\omega x)e^{-c^2\omega^2 t}\,d\omega$

(imaginary part is an odd function of ω).

36. $u(x,t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\omega}(\cos\omega-1)\sin(\omega x)e^{-c^2\omega^2 t}\,d\omega$, (imaginary part is an odd function of ω).

37. $F(\omega,0)=\frac{\sqrt{\pi}}{2}e^{-\omega^2/16}$

$u(x,t)=\frac{1}{4\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\omega^2[(1/16)+c^2t]}\cos\omega x\,d\omega$, (imaginary part is an odd function of ω).

38. Use Fourier sine transform with respect to x.

$F_s(\omega,t)=\frac{1}{\omega}(1-\cos(\omega l))e^{-\omega^2 t}$, $u(x,t)=\frac{2}{\pi}\int_0^{\infty}\frac{1}{\omega}(1-\cos(\omega l))\sin(\omega x)e^{-\omega^2 t}\,d\omega$.

39. Use finite Fourier sine transform with respect to x. $F_s(n,y)=\pi e^{-y}/2$, $u(x,y)=e^{-y}\sin x$.

40. Use Fourier transform with respect to x. $F(\omega,y)=\frac{\sinh[\omega(\pi-y)]}{(2+i\omega)\sinh(\omega\pi)}$,

$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sinh[\omega(\pi-y)]}{(4+\omega^2)\sinh(\omega\pi)}[2\cos(\omega x)+\omega\sin(\omega x)]\,d\omega$

(imaginary part is an odd function of ω).

41. Use Fourier transform with respect to x.

$F(\omega,y)=\frac{4\cosh(\omega y)}{(4+\omega^2)\cosh\omega}$, $u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\cosh(\omega y)}{(4+\omega^2)\cosh\omega}\cos(\omega x)\,d\omega$

(imaginary part is an odd function of ω).