The Black-Litterman Approach:
Original Model and Extensions[1]

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This version: October 12 2010
Last version available at `http://ssrn.com/abstract=1117574`

Abstract

We walk the reader through the Black-Litterman approach, providing all the proofs. We show how minor modifications of the original model greatly improve its range of applications. We discuss full generalizations of this and related models. Code is available at MATLAB Central File Exchange.

JEL Classification: *C1, G11*

[1]A shorter version of this article appears as Meucci A., *The Black-Litterman Approach: Original Model and Extensions*, The Encyclopedia of Quantitative Finance, Wiley (2010). Please cite as such.

[2]The author gratefully acknowledges the very helpful feedback from Bob Litterman and Jay Walters

1 Introduction

At a time when portfolio optimization used to take as inputs only the expectations and the covariances of a set of assets computed from a given reference econometric model, the pathbreaking technique by Black and Litterman (1990) (BL in the sequel) provided a framework in which more satisfactory results could be obtained from a larger set of inputs: view portfolios, the expected returns on those portfolios, the confidence in the view portfolios and the uncertainty on the reference model. Using BL, a portfolio manager could process those inputs, blend them into the reference return distribution, and obtain an optimal allocation that reflected the views in a consistent way without corner solutions.

In Section 2 we review the original BL methodology: the reference model is normal, centered around the CAPM equilibrium; the views are normal; and the distribution that blends these two inputs is obtained analytically using Bayes' formula.

In Section 3 we rephrase BL in terms of views on the market, instead of the market parameters: this market-based version is more parsimonious and it allows for the inclusion of scenario analysis as a special case.

In Section 4 we review the literature related to BL and its extensions: ranking views, stress-test of correlations and volatilities, views on generalized risk factors rather than returns, views on external factors that influence the p&l indirectly through correlation, non-normal markets, multiple users, markets of complex derivatives.

2 The original model

Here we follow He and Litterman (2002), see also Satchell and Scowcroft (2000), Idzorek (2004) and Walters (2008).

The market model

We consider a market of N securities or asset classes, whose returns are normally distributed:

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{1}$$

The covariance $\boldsymbol{\Sigma}$ is estimated by exponential smoothing of the past return realizations. To specify $\boldsymbol{\mu}$, BL acknowledge and address the issue estimation risk: since $\boldsymbol{\mu}$ cannot be known with certainty, it is modeled as a random variable whose dispersion represents the possible estimation error. In particular, BL state that $\boldsymbol{\mu}$ is normally distributed

$$\boldsymbol{\mu} \sim \mathrm{N}(\boldsymbol{\pi}, \tau\boldsymbol{\Sigma}), \tag{2}$$

where $\boldsymbol{\pi}$ represents the best guess for $\boldsymbol{\mu}$ and $\tau\boldsymbol{\Sigma}$ the uncertainty on this guess.

To set $\boldsymbol{\pi}$, BL invoke an equilibrium argument. Assuming there is no estimation error, i.e. $\tau \equiv 0$ in (2), the reference model (1) becomes

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\pi}, \boldsymbol{\Sigma}). \tag{3}$$

Assume that, consistently with this normal market, all investors maximize a mean-variance trade-off and that the optimization is unconstrained:

$$\mathbf{w}_{\lambda} \equiv \underset{\mathbf{w}}{\operatorname{argmax}} \left\{ \mathbf{w}'\boldsymbol{\pi} - \lambda \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \right\}. \tag{4}$$

Setting to zero the derivative with respect to $\mathbf{w}$ of the term in curly brackets we can solve explicitly this problem, obtaining the relationship between the equilibrium portfolio $\widetilde{\mathbf{w}}$ which stems from an average risk-aversion level $\overline{\lambda}$ and the reference expected returns:

$$\boldsymbol{\pi} \equiv 2\overline{\lambda}\boldsymbol{\Sigma}\widetilde{\mathbf{w}}. \tag{5}$$

Therefore, $\boldsymbol{\pi}$ can be set in terms of $\widetilde{\mathbf{w}}$, where BL set exogenously $\overline{\lambda} \approx 1.2$. Giacometti, Bertocchi, Rachev, and Fabozzi (2007) generalize this argument to stable-distributed markets.

Notice that historical information does not play a direct role in the determination of $\boldsymbol{\pi}$: this is an instance of the shrinkage approach to estimation risk. Indeed, portfolio optimization is extremely sensitive to the input parameters, in particular the expected values. In BL the risk drivers $\mathbf{X}$ represent the returns of a broad market that are independently and identically normally distributed across time as in (3). The standard estimator of the expectations in this context is the sample mean

$$\widehat{\boldsymbol{\mu}} \equiv \frac{1}{T} \sum_{t=1}^{T} \mathbf{X}_t \sim \mathrm{N}\left(\boldsymbol{\pi}, \frac{\boldsymbol{\Sigma}}{T}\right), \tag{6}$$

where T is the length of the available time series. The sample mean is a very inefficient estimator: therefore, the "optimal" allocations based on (6) vary wildly when different time series are fed into the allocation process, see Jobson and Korkie (1980), Best and Grauer (1991), Green and Hollifield (1992), Chopra and Ziemba (1993), Britten-Jones (1999) and Meucci (2005) for a review. One way to cope with this issue is to use as in Stein (1955) more efficient "shrinkage" estimators:

$$\boldsymbol{\mu}^{(s)} \equiv (1-s)\,\widehat{\boldsymbol{\mu}} + s\boldsymbol{\pi}^0, \tag{7}$$

where $\boldsymbol{\pi}^0$ is a generic shrinkage target and $0 \leq s \leq 1$ is the amount of shrinkage. Therefore, the BL prior can be seen as an extreme case of shrinkage toward the theoretical expectations (5) implied by equilibrium. Modifications of BL with different targets were considered early on by the authors, see Section 4.

To calibrate the overall uncertainty level τ in the reference model, we can compare the specification (2) with the uncertainty on the sample estimator (6) and set

$$\tau \approx \frac{1}{T}. \tag{8}$$

Satchell and Scowcroft (2000) propose an ingenious model where τ is stochastic, but extra parameters need to be calibrated. In practice, a tailor-made calibration that spans the interval $(0, 1)$ is called for in most applications, see also the discussion in Walters (2008).

To illustrate, we consider the oversimplified case of an international stock fund that invests in the following six stock market national indexes: Italy, Spain, Switzerland, Canada, US and Germany. The covariance matrix of daily returns on the above classes $\boldsymbol{\Sigma}$ is estimated as follows in terms of the (annualized) volatilities $\boldsymbol{\sigma} \approx (21\%, 24\%, 24\%, 25\%, 29\%, 31\%)$ and the correlation matrix

$$\mathbf{C} \approx \begin{pmatrix} 1 & 54\% & 62\% & 25\% & 41\% & 59\% \\ \cdot & 1 & 69\% & 29\% & 36\% & 83\% \\ \cdot & \cdot & 1 & 15\% & 46\% & 65\% \\ \cdot & \cdot & \cdot & 1 & 47\% & 39\% \\ \cdot & \cdot & \cdot & \cdot & 1 & 38\% \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \tag{9}$$

To determine the prior expectation $\boldsymbol{\pi}$ we start from the market-weighted portfolio $\widetilde{\mathbf{w}} \approx (4\%, 4\%, 5\%, 8\%, 71\%, 8\%)'$ and obtain from (5) the annualized expected returns $\boldsymbol{\pi} \approx (6\%, 7\%, 9\%, 8\%, 17\%, 10\%)'$. Finally, we set $\tau \approx 0.4$ in (2).

The views

A view is a statement on the market that can potentially clash with the reference market model (1). For instance, the portfolio manager might say that the third asset class will outperform the second, in which case the view is $X_3 - X_2 \geq 0$. Another view could be that the fourth asset class experiences twice to three times the volatility predicted by the model: $4\Sigma_{44} \leq \mathbb{V}ar\{X_4\} \leq 9\Sigma_{44}$.

BL consider views on expectations. In the normal market (1), this corresponds to statements on the parameter $\boldsymbol{\mu}$. Furthermore, BL focus on linear views: K views are represented by a $K \times N$ "pick" matrix $\mathbf{P}$, whose generic k-th row determines the relative weight of each expected return in the respective view. In order to associate uncertainty with the views, BL use a normal model:

$$\mathbf{P}\boldsymbol{\mu} \sim \mathrm{N}(\mathbf{v}, \boldsymbol{\Omega}), \tag{10}$$

where the meta-parameters $\mathbf{v}$ and $\boldsymbol{\Omega}$ quantify views and uncertainty thereof respectively.

If the user has only qualitative views, it is convenient to set the entries of $\mathbf{v}$ in terms of the volatility induced by the market:

$$v_k \equiv (\mathbf{P}\boldsymbol{\pi})_k + \eta_k \sqrt{(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')_{k,k}}, \quad k = 1, \ldots, K, \tag{11}$$

where $\eta_k \in \{-\beta, -\alpha, +\alpha, +\beta\}$ defines "very bearish", "bearish", "bullish" and "very bullish" views respectively. Typical choices for these parameters are $\alpha \equiv 1$ and $\beta \equiv 2$. Also, it is convenient to set as in Meucci (2005)

$$\boldsymbol{\Omega} \equiv \frac{1}{c}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}', \tag{12}$$

where the scatter structure of uncertainty is inherited from the market volatilities and correlations and $c \in (0, \infty)$ represents an overall level of confidence in the views. Also, in order to assign a scale-independent, relative uncertainty level to the different views, it is convenient to modify (12) as follows:

$$\boldsymbol{\Omega} \equiv \frac{1}{c} \operatorname{diag}(\mathbf{u}) \, \mathbf{P} \boldsymbol{\Sigma} \mathbf{P}' \operatorname{diag}(\mathbf{u}), \tag{13}$$

where $\mathbf{u} \in (0, \infty)^K$.

To continue with our example, the manager might assess two views: the Spanish index will rise by 12% on an annualized basis, and the spread US-Germany will experience a negative annualized change of 10%. Therefore the pick matrix reads

$$\mathbf{P} \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}; \tag{14}$$

and the annualized views vector becomes $\mathbf{v} \equiv (12\%, -10\%)'$. We set the uncertainty in the views of the same order of magnitude as the market, i.e. $c \equiv 1$ in (12).

The posterior

In Appendix 5.1 we show how to obtain the distribution of $\boldsymbol{\mu}$ given the views using Bayes' formula:

$$\boldsymbol{\mu} | \mathbf{v}; \boldsymbol{\Omega} \sim \mathrm{N}\left(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL}^{\boldsymbol{\mu}}\right), \tag{15}$$

where

$$\boldsymbol{\mu}_{BL} \equiv \left((\tau \boldsymbol{\Sigma})^{-1} + \mathbf{P}' \boldsymbol{\Omega}^{-1} \mathbf{P} \right)^{-1} \left((\tau \boldsymbol{\Sigma})^{-1} \boldsymbol{\pi} + \mathbf{P}' \boldsymbol{\Omega}^{-1} \mathbf{v} \right) \tag{16}$$

and

$$\boldsymbol{\Sigma}_{BL}^{\boldsymbol{\mu}} \equiv \left((\tau \boldsymbol{\Sigma})^{-1} + \mathbf{P}' \boldsymbol{\Omega}^{-1} \mathbf{P} \right)^{-1}. \tag{17}$$

However, we are interested in the distribution of the risk factors $\mathbf{X}$. To compute this distribution we rewrite the reference model (1) as $\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \mathbf{Z}$, where $\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma})$. Therefore the posterior market model is $\mathbf{X} | \mathbf{v}; \boldsymbol{\Omega} \stackrel{d}{=} \boldsymbol{\mu} | \mathbf{v}; \boldsymbol{\Omega} + \mathbf{Z}$ or

$$\mathbf{X} | \mathbf{v}; \boldsymbol{\Omega} \sim \mathrm{N}\left(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL}\right), \tag{18}$$

where $\boldsymbol{\mu}_{BL}$ is defined in (16) and $\boldsymbol{\Sigma}_{BL}$ follows from (17) by assuming that $\boldsymbol{\mu}$ and $\mathbf{Z}$ are independent:

$$\boldsymbol{\Sigma}_{BL} \equiv \boldsymbol{\Sigma} + \boldsymbol{\Sigma}_{BL}^{\boldsymbol{\mu}}. \tag{19}$$

As we prove in Appendix 5.2, an equivalent, computationally more stable representation of the posterior parameters (16) and (19) reads:

$$\boldsymbol{\mu}_{BL} = \boldsymbol{\pi} + \tau \boldsymbol{\Sigma} \mathbf{P}' \left(\tau \mathbf{P} \boldsymbol{\Sigma} \mathbf{P}' + \boldsymbol{\Omega} \right)^{-1} (\mathbf{v} - \mathbf{P} \boldsymbol{\pi}) \tag{20}$$

$$\boldsymbol{\Sigma}_{BL} = (1 + \tau) \boldsymbol{\Sigma} - \tau^2 \boldsymbol{\Sigma} \mathbf{P}' \left(\tau \mathbf{P} \boldsymbol{\Sigma} \mathbf{P}' + \boldsymbol{\Omega} \right)^{-1} \mathbf{P} \boldsymbol{\Sigma}. \tag{21}$$

The normal posterior distribution (18) with (20) and (21) represents the modification of the reference model (1) that incorporates the views (10).

The allocation
With the posterior distribution it is now possible to set and solve a mean-variance optimization, possibly under a set of linear constraints, such as boundaries on securities/asset classes, or a budget constraint. This quadratic programming problem can be easily solved numerically. The ensuing efficient frontier represents a gentle twist to equilibrium that reflects the views without extreme corner solutions.

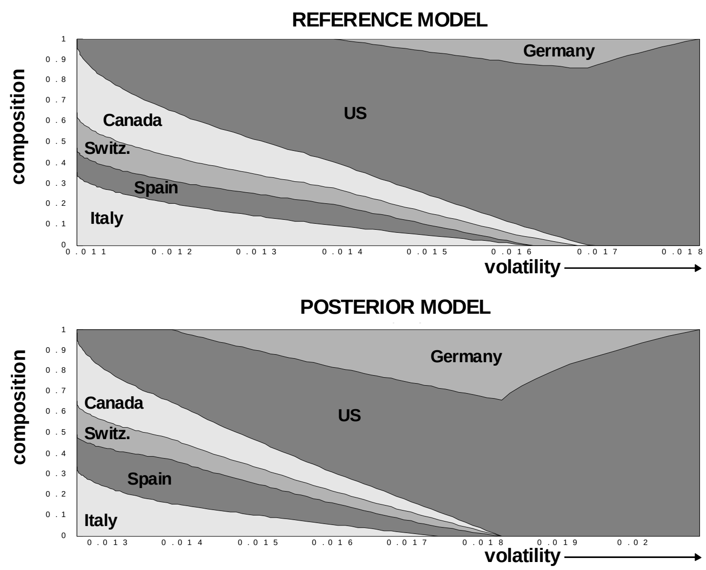

Figure 1: BL: efficient frontier twisted according to the views

In our example we assume the standard long-only and budget constraints, i.e. $\mathbf{w} \geq \mathbf{0}$ and $\mathbf{w}'\mathbf{1} \equiv 1$. In Figure 1 we plot the efficient frontier from the reference model (3) and from the posterior model (18). Consistently with the views, the exposure to the Spanish market increases for lower risk values; the exposure to Germany increases across all levels of risk aversion; and the exposure to the US market decreases.

3 The market formulation

The BL posterior distribution (18) presents two puzzles.

On one extreme, when the views are uninformative, i.e. $\boldsymbol{\Omega} \to \infty$ in (10), one would expect the posterior to equal the reference model (3), which we report here:

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\pi}, \boldsymbol{\Sigma}). \tag{22}$$

This condition is important to ensure that, when the user does not have strong views, the model does not depart from the reference prior. In BL, in this limit the posterior becomes

$$\mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\pi}, (1+\tau)\,\boldsymbol{\Sigma}\right). \tag{23}$$

In other words, the covariance of the uninformative posterior model appears distorted, unless $\tau \equiv 0$. This result is actually fully consistent. Indeed, the reference model (22) was derived by assuming no estimation risk, i.e. $\tau \equiv 0$. Without that assumption, the reference model would coincide exactly with the uninformative model (23). However, the covariance of (23) is not as easy to interpret because it contains, in addition to the pure volatility-correlation structure of the market $\boldsymbol{\Sigma}$, the additional term $\tau\boldsymbol{\Sigma}$, which represents the estimation error on $\boldsymbol{\mu}$.

On the other extreme, when the confidence in the views $\mathbf{v}$ is full, i.e. $\boldsymbol{\Omega} \to \mathbf{0}$, one would expect the posterior to become the reference model (22) conditioned on the specific views, which, as proved in Meucci (2005), is also normal:

$$\mathbf{X}|\mathbf{v} \sim \mathrm{N}\left(\boldsymbol{\mu}|\mathbf{v}, \boldsymbol{\Sigma}|\mathbf{v}\right), \tag{24}$$

where

$$\boldsymbol{\mu}|\mathbf{v} \equiv \boldsymbol{\pi} + \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}(\mathbf{v} - \mathbf{P}\boldsymbol{\pi}) \tag{25}$$

$$\boldsymbol{\Sigma}|\mathbf{v} \equiv \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{26}$$

The conditional distribution (24) is the core of scenario analysis: the user selects a set of deterministic scenarios $\mathbf{v} \equiv (v_1, \ldots, v_K)'$ for the combinations of factor realizations that are assumed to take place and analyzes their effect on the reference model. Indeed, (25)-(26) generalize the classical regression-like result utilized e.g. in Mina and Xiao (2001).

In BL, the full-confidence posterior becomes

$$\mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\mu}_{BL}^{\boldsymbol{\Omega}=\mathbf{0}}, \boldsymbol{\Sigma}_{BL}^{\boldsymbol{\Omega}=\mathbf{0}}\right), \tag{27}$$

where

$$\boldsymbol{\mu}_{BL}^{\boldsymbol{\Omega}=\mathbf{0}} = \boldsymbol{\pi} + \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}(\mathbf{v} - \mathbf{P}\boldsymbol{\pi}) \tag{28}$$

$$\boldsymbol{\Sigma}_{BL}^{\boldsymbol{\Omega}=\mathbf{0}} = (1+\tau)\,\boldsymbol{\Sigma} - \tau\boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{29}$$

Therefore, the expectation of the full-confidence posterior (28) equals the conditional expectation of scenario analysis (25), but the full-confidence posterior covariance (29) never yields the conditional covariance of scenario analysis (26): the volatility-correlation portion $\boldsymbol{\Sigma}$ of the covariance in (19) prevents the BL full-confidence posterior covariance from ever becoming degenerate. Again, this apparent paradox can be explained: in BL, the views (10) are expressed on the parameter $\boldsymbol{\mu}$, not on the market $\mathbf{X}$: therefore, the confidence in the views is only supposed to affect the estimation risk part of the covariance in (23), not its volatility-correlation component $\boldsymbol{\Sigma}$.

To summarize, although the null- and full-confidence limits of the BL posterior distribution are fully consistent, they might clash with the practitioner's intuition. To fix this problem we can proceed as in Meucci (2005) and rephrase BL in terms of views directly on the market $\mathbf{X}$, instead of the parameter $\boldsymbol{\mu}$.

We start from the reference model (1). However, we do *not* consider $\boldsymbol{\mu}$ as a random variable. Therefore we set $\boldsymbol{\mu} \equiv \boldsymbol{\pi}$ as in (3) to obtain the reference model without estimation risk (22). The user has views on linear functions of the *market* $\mathbf{V} \equiv \mathbf{PX}$, where $\mathbf{P}$ is a pick matrix as in (10). The distribution of the views in general clashes with the distribution implied by the reference model (22): as a random variable, the realization of $\mathbf{V}$ according to the views could turn out larger or smaller than the realization of $\mathbf{PX}$ according to the reference model. Therefore, the views $\mathbf{V}$ as a random variable are a perturbation of the outcome implied by the reference model and as such they are modeled as a conditional distribution

$$\mathbf{V}|\mathbf{x} \sim \mathrm{N}\left(\mathbf{Px}, \boldsymbol{\Omega}\right), \tag{30}$$

where $\boldsymbol{\Omega}$ represents the uncertainty on the views as in (10).

Once the model is set up, the user quantifies his views, choosing a specific value $\mathbf{v}$ for the random variable $\mathbf{V}$. As we show in Appendix 5.3, applying Bayes' rule as in the original BL we obtain the posterior distribution of the market given the views, which is also normal

$$\mathbf{X}|\mathbf{v}; \boldsymbol{\Omega} \sim \mathrm{N}\left(\boldsymbol{\mu}_{BL}^{m}, \boldsymbol{\Sigma}_{BL}^{m}\right), \tag{31}$$

where

$$\boldsymbol{\mu}_{BL}^{m} \equiv \boldsymbol{\pi} + \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}\left(\mathbf{v} - \mathbf{P}\boldsymbol{\pi}\right) \tag{32}$$

$$\boldsymbol{\Sigma}_{BL}^{m} \equiv \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{33}$$

These market-posterior formulas are very similar to their counterparts (20)-(21) in the original BL model. However, in the first place the parameter τ in (2) never appears here. Second, as it turns out, in the market based-specification there is no need to add the original covariance to the posterior as in (19).

As a result, the market specification is consistent with both the reference model and with scenario analysis. Indeed, it is immediate to check that if the confidence is null, i.e. $\boldsymbol{\Omega} \to \infty$, then the posterior (31) equals the prior (22). At the other extreme, if the confidence in the views is full, i.e. $\boldsymbol{\Omega} \to \mathbf{0}$, then the posterior (31) equals the conditional distribution (24).

$$\begin{array}{lcll}
 & & \mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\pi}, \boldsymbol{\Sigma}\right) & (\text{no confidence: } \boldsymbol{\Omega} \to \infty) \\
 & \nearrow & \text{prior} & \\
\mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\mu}_{BL}^{m}, \boldsymbol{\Sigma}_{BL}^{m}\right) & & & \\
\text{mkt-posterior} & \searrow & & \\
 & & \mathbf{X} \sim \mathrm{N}\left(\boldsymbol{\mu}|\mathbf{v}, \boldsymbol{\Sigma}|\mathbf{v}\right) & (\text{full confidence: } \boldsymbol{\Omega} \to \mathbf{0}) \\
 & & \text{conditional} &
\end{array}$$

4 Related literature

Scenario analysis (24) allows the practitioner to explore the implications on a given portfolio of a set of subjective views on possible market realizations, see e.g. Mina and Xiao (2001).

BL adds uncertainty on the views by means of Bayesian formulas, although this is not the standard Bayesian framework of decision theory, see Bawa, Brown, and Klein (1979) and Goel and Zellner (1986). In particular, in BL there is no historical updating of the estimate of $\boldsymbol{\mu}$, which does not depend on the time series of the past returns.

Under the normality assumption, Qian and Gorman (2001) process views on expectations of a set of portfolios and covariances of the same portfolios; Pezier (2007) processes full and partial views on expectations and covariances by least discrimination; Almgren and Chriss (2006) provide a framework to express ranking, "lax" views on expectations.

The equilibrium-based prior expectations (5) in the original BL and in the above literature appears to restrict the potential applications of these approaches to the tactical management of a global diversified fund. However, the posterior formulas (20)-(21) and their modified versions (32)-(33), as well as the formulas in the above literature can be applied to any normal distribution, not necessarily equilibrium. Indeed, active management was among the first applications of BL by their authors, where the prior expectation was assumed null.

Accordingly, Meucci (2009) applies the above approaches to fully generic risk factors that map non-linearly into the final p&l, instead of securities returns, thereby handling views and stress-testing in derivatives markets.

Further generalizations of BL should handle non-normal reference risk models; non-linear views; views on generalized features such as medians, ranges, tail behaviors, etc.; equality and inequality (ranking) views; and simultaneous inputs from multiple users with different hierarchical confidence levels.

The Bayesian formalism of BL cannot achieve this: aside from the insurmountable computational problems when leaving the normal assumption, the Bayesian approach in BL acts on the parameter $\boldsymbol{\mu}$ of a normal distribution: this is correct, because the practitioner wishes to express views on the expectations of the market and, under the normal assumption, these are represented by $\boldsymbol{\mu}$. However, in non-normal markets with views on features other than expectations one should develop a different approach for each different possible market parametrization and for each possible feature on which the practitioner wishes to express views.

Instead, it is more natural to input views directly on the *market*, instead of the combinations of *parameters* that correspond to those features. The COP approach in Meucci (2006) proceeds in this direction, although it does not cover all the above desired applications. More importantly, the COP relies on ad-hoc manipulations.

The entropy pooling approach in Meucci (2008) solves the above issues: a posterior consistent with the most general views is obtained naturally by entropy minimization, whereas opinion pooling accounts for different confidence levels and multiple users. Furthermore, the posterior is represented in terms of the same Monte Carlo scenarios as the reference model, but with different probabilities: therefore, even the most complex derivatives can be handled, as no costly repricing is ever necessary.

We summarize in the table below the capabilities of the original BL and its market formulation in Section 3, Almgren and Chriss (2006), Qian and Gorman (2001), Pezier (2007), Meucci (2009), the COP in Meucci (2006) and the entropy-pooling approach in Meucci (2008).

	BL	AC	QG	P	M	COP	EP
normal market & linear views	✓	·	✓	✓	✓	✓	✓
scenario analysis	✓	·	✓	✓	✓	✓	✓
correlation stress-test	·	·	✓	✓	✓	·	✓
trading desk: non-linear pricing	·	·	·	·	✓	✓	✓
external factors: macro, etc.	·	·	·	·	✓	✓	✓
partial specifications	·	·	·	✓	✓	·	✓
non-normal market	·	·	·	·	·	✓	✓
multiple users	·	·	·	·	·	✓	✓
non-linear views	·	·	·	·	·	·	✓
trading desk: costly pricing	·	·	·	·	·	·	✓
lax constraints: ranking	·	✓	·	·	·	·	✓

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5 Technical Appendix

In this appendix we discuss some technical results that can be skipped at first reading.

5.1 Derivation of BL

Here we adapt from Satchell and Scowcroft (2000). The pdf of (2) is

$$f_{\boldsymbol{\mu}}(\boldsymbol{\mu}) \equiv \frac{|(\tau\boldsymbol{\Sigma})|^{-\frac{1}{2}}}{(2\pi)^{\frac{N}{2}}} e^{-\frac{1}{2}(\boldsymbol{\mu}-\boldsymbol{\pi})'(\tau\boldsymbol{\Sigma})^{-1}(\boldsymbol{\mu}-\boldsymbol{\pi})}. \tag{34}$$

As for (10) we write it as

$$\mathbf{v} \overset{d}{=} \mathbf{P}\boldsymbol{\mu} + \mathbf{Z}, \tag{35}$$

where $\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Omega})$. Therefore, we can model $\mathbf{v}$ as a random variable $\mathbf{V}$ whose distribution, conditioned on the realization of $\boldsymbol{\mu}$ reads:

$$\mathbf{V}|\boldsymbol{\mu} \sim \mathrm{N}(\mathbf{P}\boldsymbol{\mu}, \boldsymbol{\Omega}). \tag{36}$$

The conditional pdf of this variable is thus

$$f_{\mathbf{V}|\boldsymbol{\mu}}(\mathbf{v}) \equiv \frac{|\boldsymbol{\Omega}|^{-\frac{1}{2}}}{(2\pi)^{\frac{K}{2}}} e^{-\frac{1}{2}(\mathbf{v}-\mathbf{P}\boldsymbol{\mu})'\boldsymbol{\Omega}^{-1}(\mathbf{v}-\mathbf{P}\boldsymbol{\mu})}. \tag{37}$$

To determine the posterior of $\boldsymbol{\mu}$ given $\mathbf{V}$ we apply Bayes rule

$$f_{\boldsymbol{\mu}|\mathbf{v}}(\boldsymbol{\mu}) = \frac{f_{\boldsymbol{\mu},\mathbf{V}}(\boldsymbol{\mu}, \mathbf{v})}{f_{\mathbf{V}}(\mathbf{v})} = \frac{f_{\mathbf{V}|\boldsymbol{\mu}}(\mathbf{v}) f_{\boldsymbol{\mu}}(\boldsymbol{\mu})}{\int f_{\mathbf{V}|\boldsymbol{\mu}}(\mathbf{v}) f_{\boldsymbol{\mu}}(\boldsymbol{\mu}) d\boldsymbol{\mu}}. \tag{38}$$

Our strategy will be to prove that the numerator can be written as a conditional pdf $f_{\mathbf{V}|\boldsymbol{\mu}}$ times a factor: the former will be the required pdf; we do not care about the latter.

The numerator in (38), i.e. the joint pdf of $\boldsymbol{\mu}$ and $\mathbf{V}$, reads:

$$\begin{aligned} f_{\boldsymbol{\mu},\mathbf{V}}(\boldsymbol{\mu}, \mathbf{v}) &= f_{\mathbf{V}|\boldsymbol{\mu}}(\mathbf{v}) f_{\boldsymbol{\mu}}(\boldsymbol{\mu}) \qquad (39) \\ &\propto |(\tau\boldsymbol{\Sigma})|^{-\frac{1}{2}} |\boldsymbol{\Omega}|^{-\frac{1}{2}} e^{-\frac{1}{2}\left[(\boldsymbol{\mu}-\boldsymbol{\pi})'(\tau\boldsymbol{\Sigma})^{-1}(\boldsymbol{\mu}-\boldsymbol{\pi})+(\mathbf{v}-\mathbf{P}\boldsymbol{\mu})'\boldsymbol{\Omega}^{-1}(\mathbf{v}-\mathbf{P}\boldsymbol{\mu})\right]}. \end{aligned}$$

This formula is exactly like (53) below, which refers to the BL_m model, once the following substitutions are made:

$$\begin{array}{ccc} \mathrm{BL} & & \mathrm{BL}_m \\ \boldsymbol{\mu} & \longleftrightarrow & \mathbf{X} \\ \boldsymbol{\pi} & \longleftrightarrow & \boldsymbol{\mu} \\ \boldsymbol{\Sigma}\tau & \longleftrightarrow & \boldsymbol{\Sigma} \end{array} \tag{40}$$

Therefore from (58), (55) and (59) we obtain

$$\boldsymbol{\mu}|\mathbf{v};\boldsymbol{\Omega} \sim \mathrm{N}(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}^{\boldsymbol{\mu}}_{BL}), \tag{41}$$

where

$$\boldsymbol{\mu}_{BL} \equiv \left((\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \left((\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v}\right) \tag{42}$$

$$\boldsymbol{\Sigma}^{\boldsymbol{\mu}}_{BL} \equiv \left((\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1}; \tag{43}$$

5.2 Reshuffling BL expressions

Using the following matrix identity ($\mathbf{A}$ and $\mathbf{D}$ invertible, $\mathbf{B}$ and $\mathbf{C}$ generic, conformable, potentially of different size)

$$\left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C}\mathbf{A}^{-1}\mathbf{B} - \mathbf{D}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} \tag{44}$$

we can write (16) as

$$\begin{aligned}
\boldsymbol{\mu}_{BL} &\equiv \left((\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \left((\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v}\right). \qquad (45)\\
&= \left((\tau\boldsymbol{\Sigma}) - (\tau\boldsymbol{\Sigma})\mathbf{P}'\left(\mathbf{P}(\tau\boldsymbol{\Sigma})\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}\mathbf{P}(\tau\boldsymbol{\Sigma})\right)\left((\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v}\right)\\
&= \boldsymbol{\pi} + (\tau\boldsymbol{\Sigma})\mathbf{P}'\left(\boldsymbol{\Omega}^{-1} - \left(\mathbf{P}(\tau\boldsymbol{\Sigma})\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}\mathbf{P}(\tau\boldsymbol{\Sigma})\mathbf{P}'\boldsymbol{\Omega}^{-1}\right)\mathbf{v}\\
&\quad - (\tau\boldsymbol{\Sigma})\mathbf{P}'\left(\mathbf{P}(\tau\boldsymbol{\Sigma})\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\pi}.
\end{aligned}$$

Noticing that

$$\boldsymbol{\Omega}^{-1} - \left(\mathbf{P}(\tau\boldsymbol{\Sigma})\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}\mathbf{P}(\tau\boldsymbol{\Sigma})\mathbf{P}'\boldsymbol{\Omega}^{-1} = \left(\mathbf{P}(\tau\boldsymbol{\Sigma})\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}, \tag{46}$$

which can be easily checked left-multiplying both sides by $\left(\mathbf{P}(\tau\boldsymbol{\Sigma})\mathbf{P}' + \boldsymbol{\Omega}\right)$, we can write (45) as

$$\boldsymbol{\mu}_{BL} = \boldsymbol{\pi} + (\tau\boldsymbol{\Sigma})\mathbf{P}'\left(\mathbf{P}(\tau\boldsymbol{\Sigma})\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}(\mathbf{v} - \mathbf{P}\boldsymbol{\pi}) \tag{47}$$

Similarly, using (44) we write (19) as

$$\begin{aligned}
\boldsymbol{\Sigma}_{BL} &\equiv \boldsymbol{\Sigma} + \left((\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \qquad (48)\\
&= (1+\tau)\boldsymbol{\Sigma} - (\tau\boldsymbol{\Sigma})\mathbf{P}'\left(\mathbf{P}(\tau\boldsymbol{\Sigma})\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}\mathbf{P}(\tau\boldsymbol{\Sigma})
\end{aligned}$$

5.3 Derivation of BL$_m$

This proof is adapted from Meucci (2005). We report here the reference model (22) for the market

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\pi}, \boldsymbol{\Sigma}). \tag{49}$$

Therefore

$$f_{\mathbf{X}}(\mathbf{x}) \equiv \frac{|\boldsymbol{\Sigma}|^{-\frac{1}{2}}}{(2\pi)^{\frac{N}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\pi})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\pi})}. \tag{50}$$

Also, from (30) we obtain

$$f_{\mathbf{V}|\mathbf{x}}(\mathbf{v}) \equiv \frac{|\mathbf{\Omega}|^{-\frac{1}{2}}}{(2\pi)^{\frac{K}{2}}} e^{-\frac{1}{2}(\mathbf{v}-\mathbf{P}\mathbf{x})'\mathbf{\Omega}^{-1}(\mathbf{v}-\mathbf{P}\mathbf{x})}. \tag{51}$$

To determine the posterior of the market given the views we apply Bayes' rule

$$f_{\mathbf{X}|\mathbf{v};\mathbf{\Omega}}(\mathbf{x}) = \frac{f_{\mathbf{X},\mathbf{V}}(\mathbf{x},\mathbf{v})}{f_{\mathbf{V}}(\mathbf{v})} = \frac{f_{\mathbf{V}|\mathbf{x}}(\mathbf{v}) f_{\mathbf{X}}(\mathbf{x})}{\int f_{\mathbf{V}|\mathbf{x}}(\mathbf{v}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}}. \tag{52}$$

Our strategy will be to prove that the numerator can be written as a conditional pdf $f_{\mathbf{X}|\mathbf{v}}$, times a factor that only depends on $\mathbf{v}$: the former will be the required pdf; we do not care about the latter.

The numerator in (52), i.e. the joint pdf of $\mathbf{V}$ and $\mathbf{X}$, reads:

$$\begin{aligned} f_{\mathbf{X},\mathbf{V}}(\mathbf{x},\mathbf{v}) &= f_{\mathbf{V}|\mathbf{x}}(\mathbf{v}) f_{\mathbf{X}}(\mathbf{x}) \\ &\propto |\mathbf{\Sigma}|^{-\frac{1}{2}} |\mathbf{\Omega}|^{-\frac{1}{2}} e^{-\frac{1}{2}\left[(\mathbf{x}-\boldsymbol{\pi})'\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\pi})+(\mathbf{v}-\mathbf{P}\mathbf{x})'\mathbf{\Omega}^{-1}(\mathbf{v}-\mathbf{P}\mathbf{x})\right]}. \end{aligned} \tag{53}$$

In Appendix 5.4 below we show that

$$\begin{aligned} f_{\mathbf{X},\mathbf{V}}(\mathbf{x},\mathbf{v}) \propto\; & \left|\mathbf{\Sigma}^{-1}+\mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}\right|^{\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m})'\left(\mathbf{\Sigma}^{-1}+\mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}\right)(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m})} \\ & \left|\mathbf{\Omega}+\mathbf{P}\mathbf{\Sigma}\mathbf{P}'\right|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{v}-\widetilde{\mathbf{v}})'\left(\mathbf{\Omega}+\mathbf{P}\mathbf{\Sigma}\mathbf{P}'\right)^{-1}(\mathbf{v}-\widetilde{\mathbf{v}})}, \end{aligned} \tag{54}$$

where

$$\boldsymbol{\mu}_{BL}^{m} \equiv \left(\mathbf{\Sigma}^{-1}+\mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}\right)^{-1}\left(\mathbf{\Sigma}^{-1}\boldsymbol{\pi}+\mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{v}\right). \tag{55}$$

and $\widetilde{\mathbf{v}}$ does not depend on either $\mathbf{x}$ or $\mathbf{v}$. Therefore

$$f_{\mathbf{X},\mathbf{V}}(\mathbf{x},\mathbf{v}) \propto f_{\mathbf{X}|\mathbf{v}}(\mathbf{x}|\mathbf{v})\, g(\mathbf{v}), \tag{56}$$

where the conditional pdf of the market given the views is

$$\begin{aligned} f_{\mathbf{X}|\mathbf{v}}(\mathbf{x}|\mathbf{v}) \propto\; & \left|\mathbf{\Sigma}^{-1}+\mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}\right|^{\frac{1}{2}} \\ & e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m})'\left(\mathbf{\Sigma}^{-1}+\mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}\right)(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m})} \end{aligned} \tag{57}$$

Since (57) is a normal probability density function, we obtain for the posterior market conditioned on the views

$$\mathbf{X}|\mathbf{v} \sim \mathrm{N}\left(\boldsymbol{\mu}_{BL}^{m}, \mathbf{\Sigma}_{BL}^{m}\right), \tag{58}$$

where $\boldsymbol{\mu}_{BL}^{m}(\mathbf{v})$ is defined in (55) and

$$\mathbf{\Sigma}_{BL}^{m} \equiv \left(\mathbf{\Sigma}^{-1}+\mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}\right)^{-1}. \tag{59}$$

Now we proceed to reshuffle the expressions (55) and (59) in a more friendly and computationally efficient way, as we did in Appendix 5.2 for the regular BL.

Using the following matrix identity ($\mathbf{A}$ and $\mathbf{D}$ invertible, $\mathbf{B}$ and $\mathbf{C}$ generic, conformable, potentially of different size)

$$\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1}=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C}\mathbf{A}^{-1}\mathbf{B}-\mathbf{D}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} \tag{60}$$

we can write (55) as

$$\begin{aligned}
\boldsymbol{\mu}_{BL}^{m} &\equiv \left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\pi}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v}\right). \qquad (61)\\
&= \left(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'+\boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}\right)\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\pi}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v}\right)\\
&= \boldsymbol{\pi}+\boldsymbol{\Sigma}\mathbf{P}'\left(\boldsymbol{\Omega}^{-1}-\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'+\boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\boldsymbol{\Omega}^{-1}\right)\mathbf{v}\\
&\quad -\boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'+\boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\pi}
\end{aligned}$$

Noticing that

$$\boldsymbol{\Omega}^{-1}-\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'+\boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\boldsymbol{\Omega}^{-1}=\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'+\boldsymbol{\Omega}\right)^{-1}, \tag{62}$$

which can be easily checked left-multiplying both sides by $\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'+\boldsymbol{\Omega}\right)$, we can write (61) as

$$\boldsymbol{\mu}_{BL}^{m}=\boldsymbol{\pi}+\boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'+\boldsymbol{\Omega}\right)^{-1}\left(\mathbf{v}-\mathbf{P}\boldsymbol{\pi}\right). \tag{63}$$

Similarly, using (60) we write (59) as

$$\begin{aligned}
\boldsymbol{\Sigma}_{BL}^{m} &\equiv \left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \qquad (64)\\
&= \boldsymbol{\Sigma}-\boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'+\boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}.
\end{aligned}$$

5.4 Tedious calculations for the joint distribution of views and market

Expanding the expression in square brackets in (53) we obtain:

$$\begin{aligned}
[\cdots] &= \left(\mathbf{x}-\boldsymbol{\mu}\right)'\boldsymbol{\Sigma}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}\right)+\left(\mathbf{v}-\mathbf{P}\mathbf{x}\right)'\boldsymbol{\Omega}^{-1}\left(\mathbf{v}-\mathbf{P}\mathbf{x}\right) \qquad (65)\\
&= \mathbf{x}'\boldsymbol{\Sigma}^{-1}\mathbf{x}-2\mathbf{x}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}+\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}+\mathbf{v}'\boldsymbol{\Omega}^{-1}\mathbf{v}-2\mathbf{x}'\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v}+\mathbf{x}'\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\mathbf{x}\\
&= \mathbf{x}'\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)\mathbf{x}-2\mathbf{x}'\left[\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v}\right]+\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}+\mathbf{v}'\boldsymbol{\Omega}^{-1}\mathbf{v}
\end{aligned}$$

We define

$$\boldsymbol{\mu}_{BL}^{m}\left(\mathbf{v}\right)\equiv\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v}\right). \tag{66}$$

or equivalently

$$\left[\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v}\right]\equiv\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)\boldsymbol{\mu}_{BL}^{m}. \tag{67}$$

Using (67) we easily re-write (65) as follows:

$$\begin{aligned}
[\cdots] &= \mathbf{x}'\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)\mathbf{x}-2\mathbf{x}'\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)\boldsymbol{\mu}_{BL}^{m} \qquad (68)\\
&\quad +\boldsymbol{\mu}_{BL}^{m\prime}\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)\boldsymbol{\mu}_{BL}^{m}+\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}+\mathbf{v}'\boldsymbol{\Omega}^{-1}\mathbf{v}\\
&\quad -\boldsymbol{\mu}_{BL}^{m\prime}\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)\boldsymbol{\mu}_{BL}^{m}\\
&= \left(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m}\right)'\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)\left(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m}\right)+\alpha.
\end{aligned}$$

where

$$\alpha \equiv \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \mathbf{v}'\boldsymbol{\Omega}^{-1}\mathbf{v} - \boldsymbol{\mu}_{BL}^{m}{}' \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right) \boldsymbol{\mu}_{BL}^{m} \tag{69}$$

Substituting the definition (66) in this expression we obtain:

$$\begin{aligned}
\alpha &= \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \mathbf{v}'\boldsymbol{\Omega}^{-1}\mathbf{v} \\
&\quad - \left(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1} + \mathbf{v}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right) \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v}\right) \\
&= \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \mathbf{v}'\boldsymbol{\Omega}^{-1}\mathbf{v} - \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \\
&\quad - \mathbf{v}'\boldsymbol{\Omega}^{-1}\mathbf{P} \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v} \\
&\quad + 2\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{v} \\
&= \mathbf{v}' \left\{\boldsymbol{\Omega}^{-1} - \boldsymbol{\Omega}^{-1}\mathbf{P} \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \mathbf{P}'\boldsymbol{\Omega}^{-1}\right\} \mathbf{v} \\
&\quad + 2\mathbf{v}'\boldsymbol{\Omega}^{-1}\mathbf{P} \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \\
&\quad + \boldsymbol{\mu}' \left(\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \boldsymbol{\Sigma}^{-1}\right) \boldsymbol{\mu}
\end{aligned} \tag{70}$$

Using the following matrix identity ($\mathbf{A}$ and $\mathbf{D}$ invertible, $\mathbf{B}$ and $\mathbf{C}$ generic, conformable, potentially of different size)

$$\left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B} \left(\mathbf{C}\mathbf{A}^{-1}\mathbf{B} - \mathbf{D}\right)^{-1} \mathbf{C}\mathbf{A}^{-1}. \tag{71}$$

we write the expression in curly brackets as follows:

$$\{\cdots\} = \boldsymbol{\Omega}^{-1} - \boldsymbol{\Omega}^{-1}\mathbf{P} \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \mathbf{P}'\boldsymbol{\Omega}^{-1} = \left(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}. \tag{72}$$

Also, we define

$$\widetilde{\mathbf{v}} \equiv -\left(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right) \boldsymbol{\Omega}^{-1}\mathbf{P} \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \tag{73}$$

or equivalently

$$\left(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1} \widetilde{\mathbf{v}} \equiv -\boldsymbol{\Omega}^{-1}\mathbf{P} \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \tag{74}$$

Substituting (72) and (74) back in (70) we obtain

$$\begin{aligned}
\alpha &= \mathbf{v}' \left(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1} \mathbf{v} - 2\mathbf{v}' \left(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1} \widetilde{\mathbf{v}} \\
&\quad + \widetilde{\mathbf{v}}' \left(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1} \widetilde{\mathbf{v}} + \boldsymbol{\mu}' \left(\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \boldsymbol{\Sigma}^{-1}\right) \boldsymbol{\mu} \\
&\quad - \widetilde{\mathbf{v}}' \left(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1} \widetilde{\mathbf{v}} \\
&= \left(\mathbf{v} - \widetilde{\mathbf{v}}\right)' \left(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1} \left(\mathbf{v} - \widetilde{\mathbf{v}}\right) + \phi,
\end{aligned} \tag{75}$$

where

$$\begin{aligned}
\phi &\equiv \boldsymbol{\mu}' \left(\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \boldsymbol{\Sigma}^{-1}\right) \boldsymbol{\mu} \\
&\quad - \widetilde{\mathbf{v}}' \left(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1} \widetilde{\mathbf{v}}
\end{aligned} \tag{76}$$

does not depend on either $\mathbf{v}$ or $\mathbf{x}$. Therefore neither does ϕ in (76). Substituting (75) back in (68) the expression in square brackets in (53) reads

$$\begin{aligned}[\cdots] &= \left(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m}(\mathbf{v})\right)'\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)\left(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m}(\mathbf{v})\right) \\ &\quad +\left(\mathbf{v}-\widetilde{\mathbf{v}}\right)'\left(\boldsymbol{\Omega}+\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}\left(\mathbf{v}-\widetilde{\mathbf{v}}\right)+\phi. \end{aligned} \tag{77}$$

Therefore the joint probability (53) becomes:

$$\begin{aligned} f_{\mathbf{X},\mathbf{V}}(\mathbf{x},\mathbf{v}) &\propto |\boldsymbol{\Sigma}|^{-\frac{1}{2}}|\boldsymbol{\Omega}|^{-\frac{1}{2}}e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m}(\mathbf{v}))'\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m}(\mathbf{v}))} \\ &\quad e^{-\frac{1}{2}(\mathbf{v}-\widetilde{\mathbf{v}})'\left(\boldsymbol{\Omega}+\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}(\mathbf{v}-\widetilde{\mathbf{v}})} \\ &= \left|\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right|^{\frac{1}{2}}e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m}(\mathbf{v}))'\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)(\mathbf{x}-\boldsymbol{\mu}_{BL}^{m}(\mathbf{v}))} \\ &\quad \left|\boldsymbol{\Omega}+\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right|^{-\frac{1}{2}}e^{-\frac{1}{2}(\mathbf{v}-\widetilde{\mathbf{v}})'\left(\boldsymbol{\Omega}+\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)^{-1}(\mathbf{v}-\widetilde{\mathbf{v}})}, \end{aligned} \tag{78}$$

where the last equality follows from

$$\frac{\left|\boldsymbol{\Omega}+\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right|}{|\boldsymbol{\Sigma}|\,|\boldsymbol{\Omega}|\,|\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}|}=1, \tag{79}$$

which in turns follows from

$$\begin{aligned} |\boldsymbol{\Sigma}|\,|\boldsymbol{\Omega}|\left|\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right| &= \left|\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}^{-1}+\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)\right||\boldsymbol{\Omega}| \\ &= \left|\mathbf{I}+\boldsymbol{\Sigma}\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right||\boldsymbol{\Omega}| \\ &= \left|\mathbf{I}+\boldsymbol{\Omega}^{-1}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right||\boldsymbol{\Omega}| = \left|\boldsymbol{\Omega}\left(\mathbf{I}+\boldsymbol{\Omega}^{-1}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right)\right| \\ &= \left|\boldsymbol{\Omega}+\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'\right|, \end{aligned} \tag{80}$$

which in turn follows from this identity ($\mathbf{B}$ and $\mathbf{C}$ generic, conformable, potentially of different size):

$$|\mathbf{I}_J+\mathbf{C}\mathbf{B}|\equiv|\mathbf{I}_K+\mathbf{B}\mathbf{C}|. \tag{81}$$