Portfolio Variance Proof

panko.aliaksandr

October 2020

1 Introduction

We all know that portfolio variance is $\sigma^2 = w\Sigma w$, but actually why?

First of all, let us say that, in other words, the task is to prove that multivariate variance can be derived using this formula.

2 The Proof

By definition variance is:

$\sigma^2 = E[(X - E(X))^2] = E[X^2 - 2X * E(X) + E^2(X)] =$
$= E(X^2) - 2E(X) * E(X) + E^2(X)] = E(X^2) - E^2(X)$

Now if X consists of several assets (let's proof for 2). For the proof initial definition $\sigma^2 = E[(X - E(X))^2]$ is required.

$$E(X) = w_1X_1 + w_2X_2$$

$$\sigma^2_{portf} = E[(X_{portf} - E(X_{portf}))^2] = E[(w_1X_1 + w_2X_2 - w_1E(X_1) - w_2E(X_2))^2] =$$

$$= E[(w_1(X_1 - E(X_1)) + w_2(X_2 - E(X_2)))^2] =$$

$$= E[w_1^2(X_1 - E(X_1))^2 + w_2^2(X_2 - E(X_2))^2 + 2w_1(X_1 - E(X_1))w_2(X_2 - E(X_2))] =$$

$$= w_1^2E[(X_1 - E(X_1))^2] + w_2^2E[(X_2 - E(X_2))^2] + 2w_1w_2E[(X_1 - E(X_1)(X_2 - E(X_2))] =$$

$$= [def] = w_1^2\sigma^2_{X_1} + w_2^2\sigma^2_{X_2} + 2w_1w_2\sigma_{X_1}\sigma_{X_2}$$

It is fairly easy to notice that if there are more assets it will be

$$\sigma^2 = \sum_i w_i^2\sigma_i^2 + 2\sum_{ij} Cov_{ij}w_iw_j; i \neq j$$

Ok let's prove it also:

$$\sigma = E[\sum_i w_iX_i - \sum_i w_iE(X_i)]^2 =$$

$$= E[\sum_i w_i(X_i - E(X_i))]^2 = E[\sum_i [w_i(X_i - E(X_i))]^2 + 2\sum_{i\neq j}[w_i(X_i - E(X_i))w_j(X_j - E(X_j))]] =$$

$$= \sum_i w_i^2 E[(X_i - E(X_i))^2] + 2 \sum_{i \neq j} w_i w_j E[(X_i - E(X_i))(X_j - E(X_j))] = [def] =$$

$$= [def] = \sum_i w_i^2 \sigma_i^2 + 2 \sum_{i \neq j} w_i w_j Cov_{ij} = w \Sigma w$$