Exercises for Part II

Exercise 1 Discrete forward rates

Show using static replication that today's discrete forward rates – either riskless or defaultable – depend on discount factors in the following way:

$$F(0, T_1, T_2) = \frac{1}{T_2 - T_1}\left(\frac{P(0, T_1)}{P(0, T_2)} - 1\right)$$

Exercise 2 Discrete forward rates

US Treasury Bills with maturities 1 and 2 years are quoted in the market at 97.39/97.45 and 95.75/95.92 respectively. Determine the bid-ask spread of the arbitrage-free forward rate that the market should expect for the second year from now.
(Hint: Take a closer look at the replication strategy.)

Exercise 3 Discrete forward rates

The following forward rates are quoted in the market:

$$F(0, 3M, 6M) = 2.16\% \quad \text{and} \quad F(0, 3M, 9M) = 2.23\%.$$

Calculate the $F(0, 6M, 9M)$ forward rate.

Exercise 4 Instantaneous forward rates

An analyst needs a quick-and-dirty model of ZCB prices and decides to model instantaneous forward rates as linear functions of maturity:

$$f(0, T) = \alpha_0 + 2\alpha_1 T \quad \text{und} \quad f^*(0, T) = \beta_0 + 2\beta_1 T.$$

1. How are the discount factors $P(0, T)$ und $P^*(0, T)$ determined in this model?
2. Which functional form does the instantaneous hazard rate exhibit?

Exercise 5 Survival probabilities

For a certain obligor you obtained the following column of numbers from a colleague, ambigously titled 'Survival Prob':

T [Y]	Survival Prob
1	0.9953
2	0.9941
3	0.9877
4	0.9901
5	0.9825

1. What kind of survival probabilities are these? Why?
2. Calculate the other kind of survival probabilities as well as the hazard rates!

Exercise 6 Discrete hazard rates

The price of a 1-year riskless ZCB is 0.9697, that of a 3-year riskless ZCB 0.9293. The price of a 1-year ZCB of the firm *Toast Ltd* is 0.9192, that of their 3-year ZCB 0.7878. You are considering an internship at this firm, so you are wondering about the possibility of the firm going bankrupt, though you are hoping that it will not default during your internship. To minimize this probability, would you rather do the internship during the first year or during the following two years from now?

Exercise 7 Discrete hazard rates

A firm issued a 2-year bond with a 12% p.a. semiannual coupon which currently quotes at 114.11. To obtain more short-term funding the firm wants to issue a 1-year bond with a 5% p.a. annual coupon. Your quant group is assuming a constant hazard rate for this firm in the next 2 years, i.e. constant and equal semiannual hazard rates as seen from today. A colleague has already told you the exact value but you don't remember whether it was 0.02 or 0.03. The continuously compounded riskless short rate is 2.5% p.a. What is your model price for the bond to be issued?

Exercise 8 Variants of coupon bonds

How would you adapt the valuation formula for a defaultable coupon bond if it had the following additional features?

1. amortizing principal
2. collateralized principal
   (e.g. so-called Principal Protected Notes with fully collateralized principal)
3. interest payment holidays
4. deterministic changes in coupon size

Exercise 9 Interest rate swaps

Show using static replication that today's swap rates depend on discount factors in the following way:

$$S(0,T_N) = \frac{1 - P(0,T_N)}{\sum_{i=1}^{N} P(0,T_i)}$$

The fixed-rate payment frequency is assumed to be annual.

Exercise 10 Interest rate swaps

Deduce the following alternative pricing formula for interest rate swaps, which shows swap rates as a weighted sum of discrete forward rates, rather than using discount factors:

$$S(0,T_N) = \sum_{n=1}^{N} w_n\, F(0,T_{n-1},T_n), \quad \text{where} \quad w_n = \frac{P(0,T_n)}{\sum_{i=1}^{N} P(0,T_i)}.$$

Compare this representation with the CDS pricing formula. Interpret your observations!

Exercise 11 Interest rate swaps

Generalize both pricing formulas for interest rate swaps to the case of a forward-starting swap!

Exercise 12 Credit default swaps

Generalize the CDS pricing formula to the case of a forward-starting CDS!

Exercise 13 Homogeneous intensity model

An obligor's default intensity amounts to 0.5% p.a. The continuously compounded riskless short rate is constant at 5% p.a. Price the following instruments:

1. a bond issued by this obligor with 2 years to maturity, a 4% p.a. semiannual coupon, and expected recovery of 57%;
2. a 2-year CDS and a 2-year Digital Default Swap on this obligor.

Exercise 14 Homogeneous intensity model

Your bank has a corporate bond in its portfolio with a 6% p.a. semiannual coupon, 6 months to maturity, and a price of 95. The risk manager is afraid that the obligor might soon default and wants to transfer its credit risk via a CDS to the market. Your quant group is using a constant-intensity model for pricing. Determine the model spread on a 1-year CDS with quarterly payments. The expected recovery is 10%, and the continuously compounded riskless short rate is constant at 2% p.a.

Exercise 15 Homogeneous intensity model

Your bank has a corporate bond in its portfolio with a 6% p.a. semiannual coupon, 1 year to maturity, and a price of 100. The risk manager is afraid that the obligor might soon default and wants to transfer its credit risk via a CDS to the market. Your quant group is using a constant-intensity model for pricing. Determine the model spread on a 1-year CDS with semiannual payments. The expected recovery is 30%, and the continuously compounded riskless short rate is constant at 3% p.a.

Exercise 16 Homogeneous intensity model

1. An obligor has a bond outstanding with a 6% p.a. semiannual coupon and 1 year to maturity. Its price is 98.91. To obtain additional capital the obligor wants to issue another bond with a 5% p.a. semiannual coupon and a 2-year maturity. Your quant team is using a constant-intensity model for pricing. A colleague was mentioning 2%, but you don't remember whether he was talking about the default intensity or about the (conditional semiannual) default probability. The continuously compounded riskless short rate is 3% p.a. No recovery is expected at default. What is the model value of the bond to be issued?
2. Your bank sold a CDS on this obligor some time ago for 125 bp. The CDS has semiannual payments, 2 years to maturity and a notional value of 10 million. The portfolio is marked to market and you are supposed to determine the current market value of this CDS.

Exercise 17 Inhomogeneous intensity model

An obligor has 2 bonds outstanding: a 1-year bond with a 3% p.a. coupon and a price of 100.5, and a 2-year bond with a 3.5% p.a. coupon p.a. and a price of 102.2, both with annual coupons. Price a 2-year CDS with semiannual payments on this obligor. The expected recovery is 40%, and the continuously compounded riskless short rate is 2% p.a.

Exercise 18 Inhomogeneous intensity model

An obligor has 2 bonds outstanding: Bond A has a 5% p.a. semiannual coupon and 6 months to maturity, and bond B has a 4% p.a. semiannual coupon and 1 year to maturity. The prices of bonds A and B are 100 and 99, respectively. To obtain additional capital the obligor wants to issue a 2-year bond with a 3% p.a. semiannual coupon. Your quant team is pricing in an intensity model using a step-function with semiannual steps. The change in the intensity expected for the 3rd and 4th half-years from now is assumed equal to the change from the 1st to the 2nd half-year. No recovery is expected in default. The continuously compounded riskless short rate is 4% p.a.

1. What is your model price for the bond to be issued?
2. You want to insure against default risk of the obligor for the term of the new bond. Which premium are you ready to pay? Assume semiannual payments.

Exercise 19 Inhomogeneous intensity model

An obligor has 2 bonds outstanding: Bond A has a 5% p.a. semiannual coupon and 6 months to maturity, and bond B has a 4% p.a. semiannual coupon and 1 year to maturity. The prices of bonds A and B are 100 and 99, respectively. To obtain additional capital the obligor wants to issue a 2-year bond with a 3% p.a. semiannual coupon. Your quant team is pricing in an intensity model using a linear intensity. No recovery is expected in default. The continuously compounded riskless short rate is 4% p.a.

1. What is your model price for the bond to be issued?
2. You want to insure against default risk of the obligor for the term of the new bond. Which premium are you ready to pay? Assume semiannual payments.

Exercise 20 OU process

The SDE for an OU process Y is more commonly given by

$$dY_t = \kappa(\theta - Y_t)dt + \sigma dW_t\,.$$

Derive the expressions for its expected value and variance at time t:

$$\mathbb{E}[Y_t|Y_0] = \theta + (Y_0 - \theta)\,e^{-\kappa t} \quad \text{and} \quad \mathbb{V}[Y_t|Y_0] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right).$$

Exercise 21 OU process

Determine the following limits for an OU process Y and give a short intuitive explanation:

1. $\lim_{t\to 0} \mathbb{E}[Y_t|Y_0]$
2. $\lim_{t\to\infty} \mathbb{E}[Y_t|Y_0]$

3. $\lim_{t\to 0} \mathbb{V}[Y_t|Y_0]$
4. $\lim_{t\to \infty} \mathbb{V}[Y_t|Y_0]$

Exercise 22 OU process

Determine the following limits for an OU process Y and give a short intuitive explanation:

1. $\lim_{\kappa\to 0} \mathbb{E}[Y_t|Y_0]$
2. $\lim_{\kappa\to \infty} \mathbb{E}[Y_t|Y_0]$
3. $\lim_{\kappa\to 0} \mathbb{V}[Y_t|Y_0]$
4. $\lim_{\kappa\to \infty} \mathbb{V}[Y_t|Y_0]$

(Hints: $\frac{c}{\infty}$ is defined as 0. $\frac{0}{0}$ is not defined. In this case, the limit is determined using L'Hôpital's rule.)

Exercise 23 Riskless Vasiček model

A 1-factor Vasiček short-rate model is used for riskless bond pricing. The parameters were calibrated to market prices of liquid interest-rate derivatives: $\kappa_r = 0.3$, $\theta_r = 5\%$ and $\sigma_r = 10\%$ p.a. The current short rate equals 5.5%. Price a 4-year riskless bond with a 5% p.a. annual coupon

Exercise 24 Riskless Vasiček model

A 2-factor Vasiček short-rate model is used for riskless bond pricing. The 2 factors for the level and spread of the short rate are assumed independent. The parameters were calibrated to market prices of liquid interest-rate derivatives:

$$\kappa_1 = 0.1, \quad \theta_1 = 5\% \quad \text{and} \quad \sigma_1 = 5\%,$$
$$\kappa_2 = 5, \quad \theta_2 = 2\% \quad \text{and} \quad \sigma_2 = 25\%,$$

The current level is 4.5% and the spread is 2.5%. Price a 4-year riskless bond with a 6% p.a. annual coupon

Exercise 25 Defaultable Vasiček model

A 2-factor Vasiček model is used for corporate bond pricing. The parameters were determined as:

$$r_0 = 4.5\%, \quad \kappa_r = 0.3, \quad \theta_r = 5\% \quad \text{and} \quad \sigma_r = 10\%,$$
$$\lambda_0 = 0.5\%, \quad \kappa_\lambda = 1, \quad \theta_\lambda = 2\% \quad \text{and} \quad \sigma_\lambda = 15\%,$$

Price a 4-year corporate bond with a 4% p.a. annual coupon

1. assuming independence between the short rate and the intensity;
2. assuming a correlation of 20% between the noise terms driving the short rate and the intensity.

The expected recovery equals 40%.

Exercise 26 Defaultable Vasiček model

A 2-factor Vasiček model is used for corporate bond pricing. The parameters were determined as:

$$r_0 = 5.5\%, \quad \kappa_r = 0.3, \quad \theta_r = 5\% \quad \text{and} \quad \sigma_r = 15\%,$$
$$\lambda_0 = 2\%, \quad \kappa_\lambda = 0.75, \quad \theta_\lambda = 2\% \quad \text{and} \quad \sigma_\lambda = 20\%.$$

Determine the premium on a 1-year CDS with semiannual payments assuming independence between the short rate and the intensity. The expected recovery equals 50%.

Exercise 27 Defaultable Vasiček model

In a stochastic-intensity model a defaultable ZCB has the following value:

$$P^*(0,T) = \mathbb{E}\left[e^{-\int_0^T (r_s+\lambda_s)ds}\right].$$

The riskless short rate is modeled by 2 independent Vasiček factors for level and spread. The intensity is modeled by a single Vasiček process independent of the riskless short rate. Write out the concrete closed-form solution to the expression above.

Exercise 28 Dependence between short rate and intensity

In a stochastic-intensity model a defaultable ZCB has the following value:

$$P^*(0,T) = \mathbb{E}\left[e^{-\int_0^T (r_s+\lambda_s)ds}\right].$$

We want to model dependence between the riskless short rate r_t and the intensity λ_t, and define the intensity as a linear function of the riskless short rate:

$$\lambda_t = \alpha + \beta r_t\,.$$

The riskless short rate is modeled by a CIR process. Write out the concrete closed-form solution to the expression above.

Exercise 29 Multiple defaults

Derive the price of a defaultable ZCB in the MD recovery model assuming a constant intensity (and a constant riskless rate).

(Hints: $\mathbb{E}[f(X)] = \sum_i f(x_i)\, p_i$ and $e^x = \sum_{n\geq 0} \frac{1}{n!}\, x^n$)

Exercise 30 Multiple defaults

Your quant team is using the MD recovery model for pricing. With this formulation a defaultable ZCB has the following value:

$$P^*(0,T) = \mathbb{E}\left[e^{-\int_0^T (r_s + q\lambda_s)ds}\right].$$

Write out the concrete closed-form solution to the expression above if both the riskless short rate and the intensity are modeled as independent OU processes.

Exercises for Part III

Exercise 31 Discrete-time Markov chain

Consider a model with a constant riskless (continuously compounded) interest rate, in which the transitions between 2 non-default rating classes A and B and an absorbing default class D are described by the following 1-year transition matrix:

$$Q(0,1) = \begin{pmatrix} 0.90 & 0.08 & 0.02 \\ 0.10 & 0.85 & 0.05 \\ 0 & 0 & 1 \end{pmatrix}$$

Assume that the dynamics specified in $Q(0,1)$ are those used also for pricing purposes, either because of an assumption of risk neutrality or because the matrix describes the evolution after a calibration to observed prices has been performed.

1. Given the dynamics specified by $Q(0,1)$, what is the probability that a firm starting in A is in state B after 2 years? What is the probability of a firm starting in B being in the default state at time 2?

2. Assumming a recovery (of treasury) of 0.4, i.e. a payment in the event of default of 0.4 at maturity (instead of a promised payment of 1) and assuming a 5% p.a. constant riskless rate, what is the price of 1-year, 2-year, and 3-year ZCBs issued by firms in categories A and B?

3. What is the fair premium on a CDS, protecting up to an including time 3, against default on a 3-year ZCB issued by a firm rated A at time 0? Assume that the swap pays, at the default date, 0.6 (regardless of the value on the default date of the defaulted bond). Also, assume that no premium is paid at date 0 and defaults only happen at discrete time points 1, 2 or 3 according to the dynamics specified by $Q(0,1)$. If the default date is i, there is no premium paid on the swap at date i, but the protection buyer receives the payment from the protection seller on that same date.

Exercise 32 Estimation comparison

Consider a rating system consisting of 2 non-default rating categories A and B and a default category D. Assume that we observe over 1 year the history of 20 firms, of which 10 start in category A and 10 in category B. Assume that over the year of observation, 1 A-rated firm changes its rating to category B after 1 month and stays there for the rest of the year. Assume that over the same period, 1 B-rated firm is upgraded after 2 months and remains in A for the rest of the period, and a firm which started in B defaults after 6 months and stays there for the remaining part of the period.

1. Determine the transition probability matrix via cohort estimation.
2. Determine the transition probability matrix via generator estimation.

Exercise 33 Continuous-time Markov chain

Consider a continuous-time Markov chain on the state space A, B, D corresponding to 2 non-default ratings and a default rating. Assume that the generator matrix is given as follows:

$$\begin{pmatrix} -\lambda_A & \lambda_{AB} & \lambda_{AD} \\ \lambda_{BA} & -\lambda_B & \lambda_{BD} \\ 0 & 0 & 0 \end{pmatrix}$$

1. Explain the restrictions on the parameters which must hold for the matrix to be a generator. Given that the chain starts in state A, what is the distribution of the time the chain spends in A until the 1st rating transition? What is the probability that the chain jumps to state B at the time of its 1st jump away from A?
2. Consider 20 company histories seen over 1 year. 10 firms start out in class A and 10 in class B. Of the 10 firms in A, 1 firm is downgraded after 3 months to B but returns to A after 9 months (6 months after the downgrade). 1 firm defaults after 6 months from class A. Of the 10 firms initially in B, 1 firm defaults after 2 months, 1 firm defaults after 10 months, and 2 firms are upgraded after 4 and 8 months respectively. How would you estimate the generator matrix based on these data?

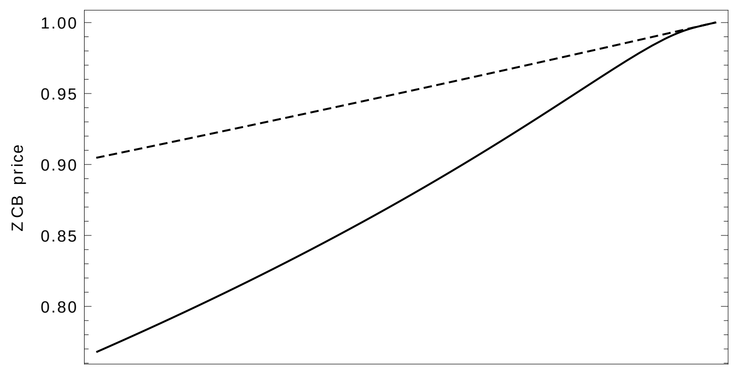
Exercises for Part IV

Exercise 34

A firm is financed by an 8-year ZCB with 110m notional. The current asset value equals 220m and its volatility 50% p.a. The riskless short rate is 3.1% p.a. Determine the current market values of equity and debt for this firm!

Exercise 35

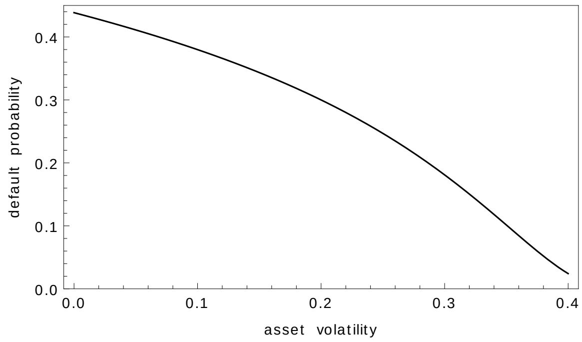
On the vertical axis, the following figure shows prices of a riskless and a defaultable ZCB with notionals 1, where the Merton model was used to determine the price of the defaultable ZCB. Unfortunately, the author forgot to label the horizontal axis. Which of the following are possible labels?

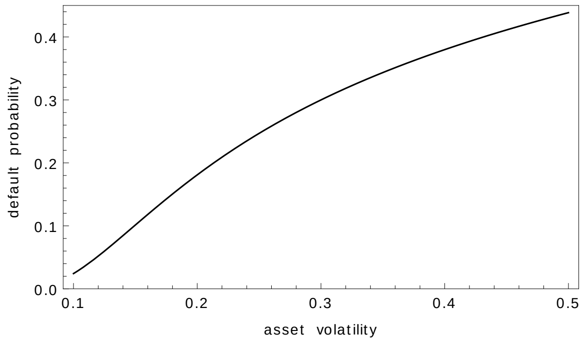


1. (time to) maturity increasing
2. (time to) maturity decreasing
3. asset volatility increasing
4. asset volatility decreasing
5. initial asset value increasing
6. initial asset value decreasing
7. ZCB notional increasing
8. ZCB notional decreasing
9. riskless short rate increasing
10. riskless short rate decreasing

Exercise 36

One of the following figures shows how the default probability of a firm depends on its asset volatility, where the firm is represented by Merton's model. Which of the figures is correct and why?





Exercise 37

1. A firm's current asset value equals 175m. The firm is financed by a 5-year ZCB with 133m notional. The riskless short rate is 3.8% p.a. The current market value of debt is estimated at 100m. Is the asset volatility underlying this estimation higher or lower than 24% p.a.?

2. A firm's current asset value equals 310m. The firm is financed by a 5-year ZCB with 250m notional. The riskless short rate is 3.5% p.a. The current market value of equity is estimated at 110m. Is the asset volatility underlying this estimation higher or lower than 20% p.a.?

3. A firm's current asset value equals 130m. The firm is financed by a 1-year ZCB with 70m notional. The riskless short rate is 3% p.a. The current default probability is estimated at 5%. Is the asset volatility underlying this estimation higher or lower than 30% p.a.?

Exercise 38

A firm's current market cap equals 50m with an equity volatility of 25% p.a. The firm is financed by a 5-year ZCB with 120m notional. The riskless short rate is 4.5% p.a. Which of the following combinations for the firm's current asset value and its volatility is more plausible?

1. 145.5m and 8.7% p.a.

2. 160m and 12.5% p.a.

Exercise 39

The following table lists all input quantities to the Merton model. Indicate how an increase in each of these quantities affects the model probabilities of default (PD) and survival (PS). (Use ↑ or + for an increase, ↓ or − for a decrease.)

input quantity	effect on PD	effect on PS
initial asset value		
ZCB notional		
riskless short rate		
(time to) maturity		
asset volatility		

Exercise 40

The following table lists all input quantities to the Merton model. Indicate how an increase in each of these quantities affects the market values of equity (E) and debt (D). (Use ↑ or + for an increase, ↓ or − for a decrease.)

input quantity	effect on E	effect on D
initial asset value		
ZCB notional		
riskless short rate		
(time to) maturity		
asset volatility		

Exercise 41

A firm's empirical probability of default equals 10%. Determine the implied distance to default!

Exercise 42

A firm with 44m market cap is financed by a 5-year ZCB with 90m motional. The Merton model yields a sensitivity of the firm's equity to changes in asset value of 0.8046.

Its equity volatility is twice as high as its asset volatility. The riskless short rate equals 3% p.a. What is the firm's 5-year default probability in the Merton model?

Exercise 43

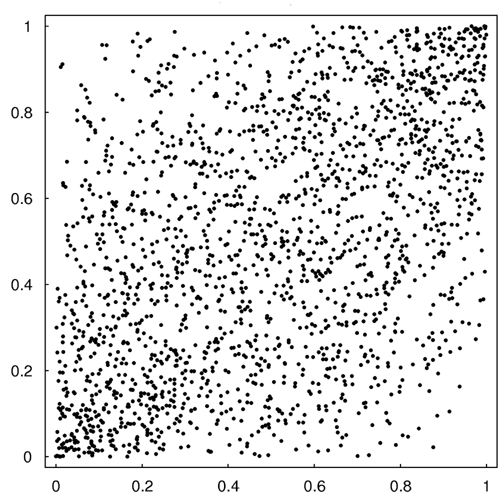
A firm is financed by a 5-year ZCB with 90m notional. The Merton model yields a default probability of 13.57% and a sensitivity of the firm's equity to changes in asset value of 93.91%. The riskless short rate equals 3% p.a. What is the firm's implied equity volatility in the Merton model?

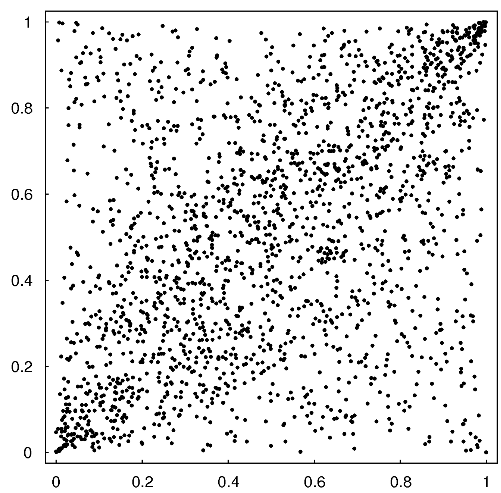
Exercises for Part V

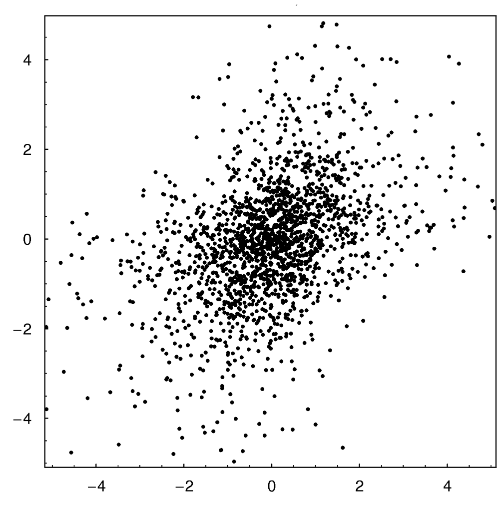
Exercise 44

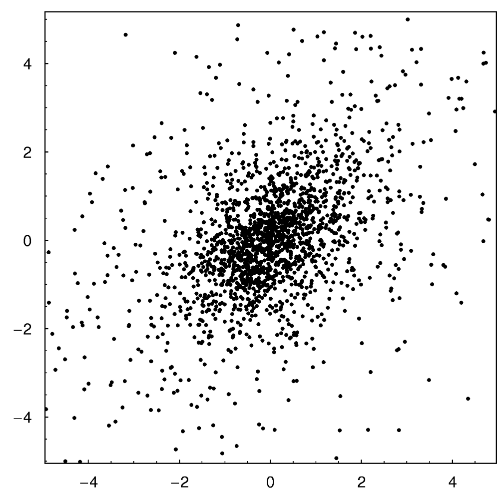
Match the figures and the descriptions!

1. t copula with t margins
2. Gaussian copula
3. t copula
4. Gaussian copula with t margins









Exercise 45

A simplified rating-migration model distinguishes between 4 states: *good, normal, bad,* and *defaulted.* Assume an obligor is rated *normal* at the beginning of the year. The migration probabilities per year for the *normal* state are historically estimated to be:

end state	*good*	*normal*	*bad*	*defaulted*
mig. prob. (%)	7	80	10	3

Assume the obligor's creditworthiness directly depends on its equity return, and assume it to be standard normally distributed. The first 5 simulations yield the outcomes 1.5, 0.3, –2.1, –0.8, and –0.1. Which end-year states do these simulated return values correspond to?

Exercise 46

An obligor exhibits a BB rating with the following historical migration probabilities per year:

end rating	AAA	AA	A	BBB	BB	B	CCC	D
mig. prob. (%)	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06

Assume the obligor's creditworthiness directly depends on its yearly equity return, and assume it to be normally distributed with mean 1.26% and standard deviation 18.02%. Determine the migration thresholds in the equity returns.

Exercise 47

An analyst assumes that an obligor's creditworthiness directly depends on its yearly equity return, so he publishes the following migration thresholds for the equity returns of a certain obligor in an info leaflet:

end rating	AAA	AA	A	BBB	BB	B	CCC	D
threshold (%)	$+\infty$	48.53	39.62	27.63	-23.63	-30.90	-39.07	-44.53

The 1st row indicates the worse of the 2 adjacent ratings. Assume the yearly equity return of this obligor is normally distributed with mean 2% and standard deviation 20%. Determine the implied migration probabilities for this obligor! Which rating does it exhibit right now?

Exercise 48

A simplified rating-migration model distinguishes between 4 states: *good, normal, bad,* and *defaulted.* Assume 2 obligors are rated *bad* at the beginning of the year. Their joint migration probabilities per year are given by one of the following matrices:

Matrix A:

	good	*normal*	*bad*	*defaulted*
good	5	2	1	0
normal	1	11	14	2
bad	1	16	37	2
defaulted	0	1	3	4

Matrix B:

	good	*normal*	*bad*	*defaulted*
good	3	2	2	1
normal	1	7	16	2
bad	2	18	34	4
defaulted	1	3	3	1

1. What is the respective probability that both obligors exhibit the same rating at the end of the year?
2. What is the respective probability that one (and only one) of the obligors defaults until the end of the year?
3. Which matrix describes stronger positive dependence, and why?

Exercise 49

A simplified rating-migration model distinguishes between 4 states: A, B, C, and D. Assume 2 obligors are rated C at the beginning of the year. Their joint migration probabilities per year are given by one of the following matrices:

Matrix A:

	A	B	C	D
A	4	3	2	1
B	2	7	15	2
C	2	17	33	4
D	1	3	3	1

Matrix B:

	A	B	C	D
A	6	3	1	0
B	2	11	13	2
C	1	15	36	2
D	0	1	3	4

1. What is the respective probability that both obligors exhibit at least rating B at the end of the year?
2. What is the respective probability that only one of the obligors stays rated C until the end of the year?
3. Which matrix describes stronger positive dependence, and why?

Exercise 50

A 5-year credit derivative is written on an underlying portfolio of 5 bonds issued by different obligors. The instrument is priced in Li's framework by simulating default arrivals. The 1st draw from an appropriate copula function yields the following random numbers:

$$(0.43, 0.04, 0.30, 0.17, 0.10).$$

Differently than in Li's original model, the individual default probabilities are given by stepwise intensity functions with step lengths of 1 year:

obligor	1. year	step height
1	1%	+0.5%
2	2%	+0.4%
3	3%	–0.3%
4	4%	+0.2%
5	5%	–0.1%

Which of the obligors default in this scenario before the maturity of the credit derivative? Determine the exact default time of these obligors!

Exercise 51

A credit derivative is written on an underlying portfolio of 5 bonds issued by different obligors. The instrument is priced in Li's framework by simulating default arrivals. The

1st draw from an appropriate copula function yields the following random numbers:

$$(0.43, 0.04, 0.30, 0.17, 0.10).$$

Differently than in Li's original model, the individual default probabilities are given by linear intensity functions:

$$\begin{aligned}
\lambda_1(t) &= 0.01\,t + 0.001 \\
\lambda_2(t) &= 0.02\,t + 0.002 \\
\lambda_3(t) &= 0.03\,t + 0.003 \\
\lambda_4(t) &= 0.04\,t + 0.004 \\
\lambda_5(t) &= 0.05\,t + 0.005
\end{aligned}$$

Determine the exact simulated default time of each obligor in this scenario!

Exercise 52

In the course of pricing a CDO your task is to simulate the arrival of portfolio defaults using Li's approach. The portfolio underlying the CDO contains 5 obligors with equal volumes of 10 million each. The expected LGD is equal for all obligors and amounts to 60%. The CDO's maturity is 5 years. Differently than in Li's original model, default probabilities are given by the following intensity functions:

$$\begin{aligned}
\lambda_1(t) &= \frac{1\%}{1 + 0.5\%t} \\
\lambda_2(t) &= \frac{2\%}{1 + 0.4\%t} \\
\lambda_3(t) &= \frac{3\%}{1 + 0.3\%t} \\
\lambda_4(t) &= \frac{4\%}{1 + 0.2\%t} \\
\lambda_5(t) &= \frac{5\%}{1 + 0.1\%t}
\end{aligned}$$

The 1st draw from an appropriate copula function yields the following random numbers:

$$(0.43, 0.04, 0.30, 0.17, 0.10).$$

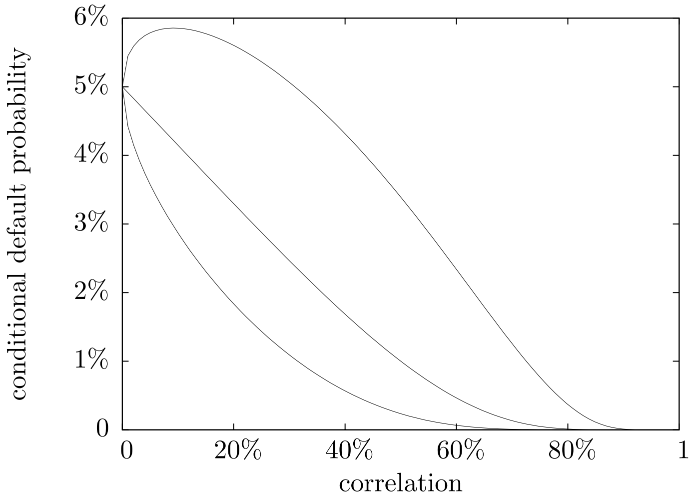
1. Determine the exact simulated default time of each obligor in this scenario!
2. Determine the cumulative portfolio loss through time in this scenario!
3. Determine the notional of the equity tranche through time in this scenario! The detachment point of the equity tranche equals 10% of total portfolio volume.

Exercises for Part VI

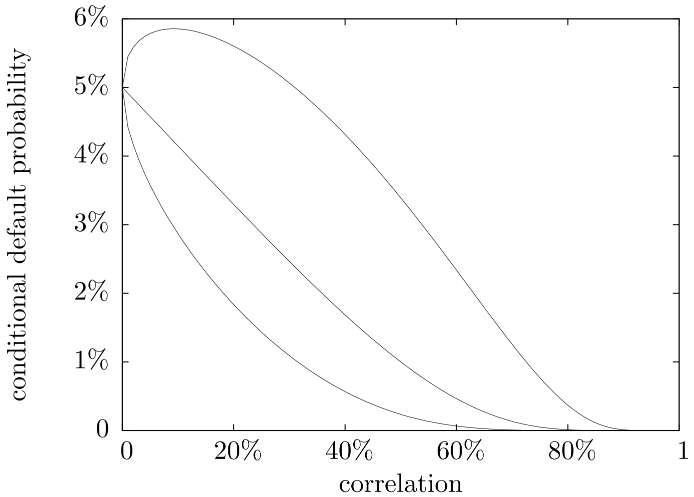
Exercise 53

The figure shows the dependence of conditional default probability on correlation in a 1-factor model. An unconditional default probability of 5% underlies all 3 curves; they differ in the realization of the macrofactor.

1. The values of the macrofactor underlying the curves are 0, 0.5, and –0.5. Which line does each value belong to?



2. Draw, label, and describe the approximate curves for realizations of the macrofactor of 0.6 and –0.6!



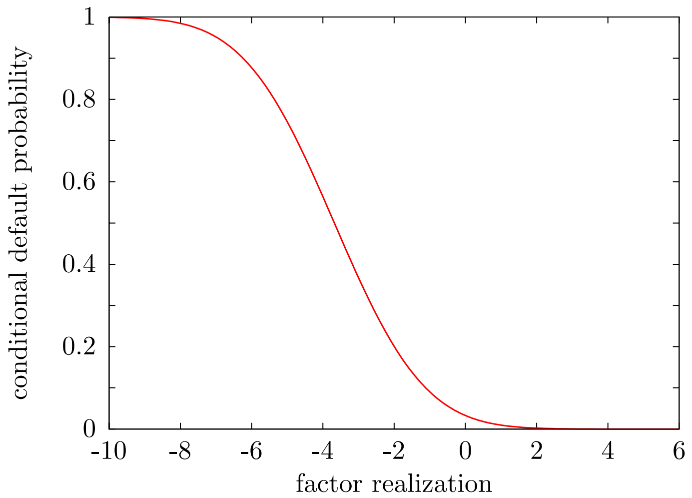
Exercise 54

The following figure shows the dependence of the conditional default probability on the realization of the macrofactor in a 1-factor model, i.e.:

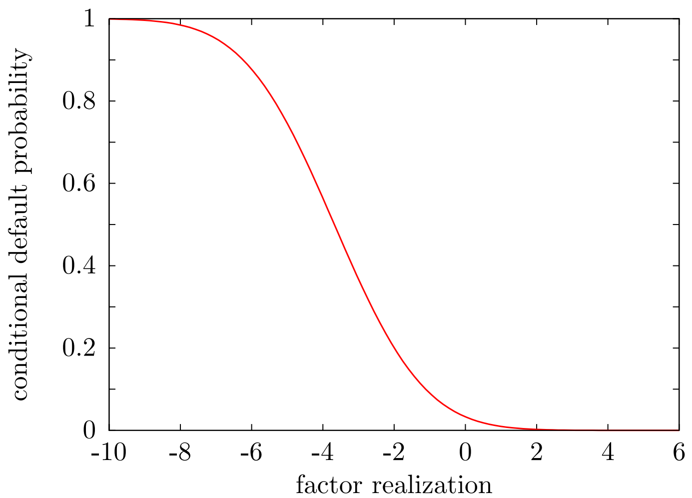
$$V_n = \sqrt{\rho_n} F + \sqrt{1 - \rho_n} \varepsilon_n$$

with 5% unconditional default probability and 20% correlation.

1. Draw and describe the approximate curve for a correlation of 40%!

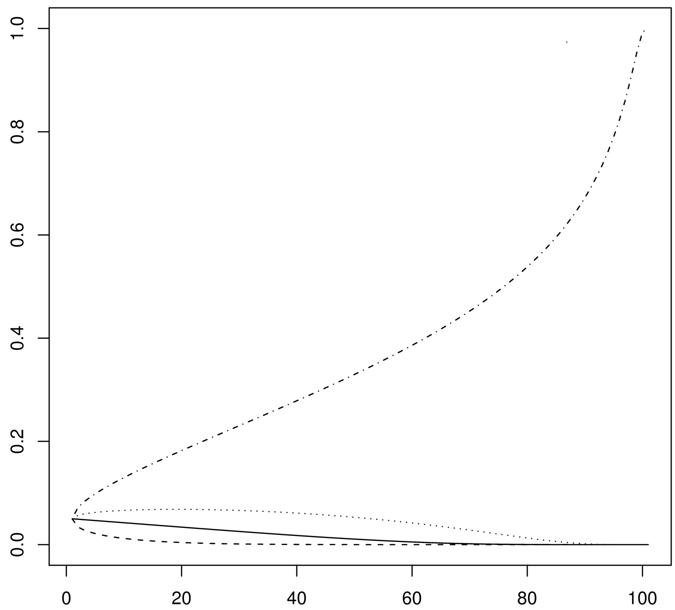


2. Draw and describe the approximate curve for a default probability of 10%!



Exercise 55

The figure shows the dependence of conditional default probability on correlation in a 1-factor model. An unconditional default probability of 5% underlies all 4 curves; they differ in the realization of the macrofactor.



1. The values of the macrofactor underlying the curves are 0, 1.7, –0.7, and –1.9. Which of the lines does each value belong to?
2. Explain the behavior of the conditional default probability for correlations of 0 and 1, both economically and mathematically!

Exercise 56

A large homogeneous portfolio is modeled by an exchangeable linear factor model with a single macrofactor and equal (pairwise/factor) correlations, meaning that the asset values V_n of the individual obligors n are given by

$$V_n = \sqrt{\rho}F + \sqrt{1-\rho}\varepsilon_n$$

where F denotes a standard normally distributed macrofactor, ρ the correlation, and ε_n the standard normally distributed firm-specific factors mutually independent and independent of the macrofactor F. Determine the 95% and 99% VaR for the average proportion of defaults in such a large portfolio if the unconditional default probability equals 10% and the correlation 25%!

Exercise 57

The asset value V_n of each obligor n in a portfolio is assumed to depend on 2 factors: its sector $S(n)$ and its region $R(n)$ affiliation:

$$V_n = \beta_n g_n (F_{S(n)} + F_{R(n)}) + \sqrt{1-\beta_n^2}\, \varepsilon_n \,.$$

$F_{S(n)}$ and $F_{R(n)}$ denote the sector and region factors, respectively, corresponding to obligor n. The parameter β_n represents the sensitivity of its asset value to systematic effects. The firm-specific risk factor ε_n is assumed independent of the systematic factors and standard normally distributed. There are 4 sectors and 4 regions in the portfolio, and these factors are assumed to be standard normally distributed with correlation matrix:

	R_1	R_2	R_3	R_4	S_1	S_2	S_3	S_4
R_1	1	0.70	0.50	0.65	0.10	0.15	0.10	0.05
R_2		1	0.60	0.80	0.20	0.10	0.30	0.05
R_3			1	0.75	0.05	0.10	0.20	0.25
R_4				1	0.15	0.20	0.05	0.10
S_1					1	0.70	0.35	0.55
S_2						1	0.75	0.25
S_3							1	0.30
S_4								1

You are given the following information about the portfolio:

obligor	region	sector	sensitivity
1	2	3	65%
2	1	1	70%
3	4	2	80%
4	3	4	75%

1. Determine the standardizing constant g_n for all obligors in the portfolio!
2. Determine the correlation matrix of the obligors in the portfolio!

Exercise 58

You are modeling in Li's framework. As in Li's original model, the individual default probabilities are given by the following constant intensities:

$$\lambda_1 = 1\%, \quad \lambda_2 = 2\%, \quad \lambda_3 = 3\%, \quad \lambda_4 = 4\%, \quad \lambda_5 = 5\%.$$

Differently than in Li's original model, the Gaussian copula of the default times is extracted from a linear factor model with 1 macrofactor and equal correlations:

$$V_n = \sqrt{\rho}F + \sqrt{1-\rho}\varepsilon_n$$

You simulate the standard normally distributed macrofactor and obtain 1.05937 in the 1st draw, as well as

$$(0.38412, 0.83065, -1.53183, 0.96034, 0.20243)$$

for the firm-specific factors, individually standard normally distributed as well. A correlation of 30% is assumed. Determine the simulated default times of the obligors!

Exercise 59

The loss distribution of a homogeneous credit portfolio with 50 exposures is modeled by a 1-factor model with equal correlations. The individual obligors default with an unconditional probability of 6% and exhibit a correlation of 20% with the macrofactor, which equals -0.5 at the moment. Determine the 95% and 99% VaR in terms of the number of portfolio defaults!