Statistics 2 Unit 2

Team 8

March 20, 2018

Contents

1 Task 18

Let's consider two cases: when $\lambda = 0 \wedge \lambda \neq 0$. Also note that X_i are i.i.d, therefore their joint density is the product of their marginal densities, which is the likelihood function. The we only need to take logarithm.

$$f(x_1, ..., x_n|\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} exp - \frac{1}{2}(\frac{(log(X_i) - \mu)}{\sigma})^2 \text{ , when } \lambda = 0$$

By taking the logarithm we find the log likelihood:

$$l(\mu, \sigma^2) = -\frac{n}{2}log(2\pi) - nlog(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(log(X_i) - \mu)^2.$$

$$f(x_1, ..., x_n|\mu, \sigma^2, \lambda) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\frac{x^\lambda - 1}{\lambda} + \mu)^2}{2\sigma^2}} \text{ , when } \lambda \neq 0$$

By taking the logarithm we find the log likelihood:

$$l(\mu, \sigma^2) = -\frac{n}{2}log(2\pi) - nlog(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}((\frac{X_i^\lambda - 1}{\lambda}) - \mu)^2.$$

2 Task 19

Assume F is differentiable at x_i, then by definition of differentiation:

$lim_{h\to 0}\frac{F_0(x_i+h|\gamma) - F_0(x_i-h|\gamma)}{2h} = f_0(x_i|\gamma)$

$L(\theta|x_1, ..., x_n) = \lim_{h\to 0} \prod_{i=1}^{n} \frac{F_0(x_i+h|p_0,p_1,\gamma) - F_0(x_i-h|p_0,p_1,\gamma)}{2h} = f_0(x_i|p_0, p_1, \gamma)$

Now, let's use the equation from the task:
$F(x|p_0, p_1, \gamma) = p_0 I(0 \leq x) + (1 - p_0 - p_1)F_0(x|\gamma) + p_1 I(x \geq 1)$

$L(\theta|x_1, ..., x_n) =$
$\lim_{h\to 0} \prod_{i=1}^{n} \frac{p_0 I(0\leq x_i+h) + (1-p_0-p_1)F_0(x_i+h|\gamma) + p_1 I(x_i+h\geq 1) - p_0 I(0\leq x_i-h) - (1-p_0-p_1)F_0(x_i-h|\gamma) - p_1 I(x_i-h\geq 1)}{2h}$

It is given that n_0 values are 0 and n_1 values are 1. Therefore:

$p_0 I(0 \leq 0 - h) = 0$
$p_0 I(0 \leq 0 + h) = p_0$
$p_1 I(1 + h \geq 1) = p_1$
$p_1 I(1 - h \geq 1) = 0$

$\Rightarrow$

$L(\theta|x_1, ..., x_n) = \prod_{i:0 \leq x_i \leq 1} (1 - p_0 - p_1) lim_{h \to 0} \frac{F_0(x_i + h|\gamma) - F_0(x_i - h|\gamma)}{2h} \prod_{n_0} p_0 \prod_{n_1} p_1 =$

$(1 - p_0 - p_1)^{n - n_0 - n_1} p_0^{n_0} p_1^{n_1} \prod_{i:0 \leq x_i \leq 1} f_0(x_i|\gamma)$

3 Task 20

Let X be a serial numbers, which follow a **discrete** uniform distribution from 1 to N. Thus

$$P(X = x) = \frac{1}{N}.$$

Method of Moments
Let's find the first moment:

$$\mathbb{E}(X_1) = \sum_{x=1}^{N} x P(X_1 = x) = \sum_{x=1}^{N} x \frac{1}{N} = \frac{N+1}{2}.$$

Let's find N:

$$N = 2\mu - 1$$

where μ is the first moment.
We can rewrite it using the sample moment

$$\hat{N} = 2\bar{X} - 1.$$

We know that $\bar{X} = 888 \Rightarrow \hat{N} = 2 \cdot 888 - 1 = 1775.$

Maximum Likelihood Estimation

Assume X_i is iid $\Rightarrow$

$$lik(N) = \prod_{i=1}^{m} f(X_i|N) = \prod_{i=1}^{m} \frac{1}{N}\mathbb{I}(X_i \in \{1, ..., N\}) = \frac{1}{N^m}\mathbb{I}\{X_{(n)} \leq N\}$$

where $X_{(n)}$ is the maximum.

In our sample we have only one draw, 888, this is also the maximum and the maximum likelihood estimation.

4 Task 21

The outcomes of trials are i.i.d and follow the Binomial distribution.
We are looking for a θ, which is a probability that a coin comes up head.
As x_i are i.i.d, their mutual CDF is a product of their marginals:

$$\begin{aligned} L_G(\theta|X_1, X_2, X_3) &= \prod_{i=1}^{n} \theta^{x_i}(1\text{-}\ \theta)^{1-x_i} \\ &= \theta^{\sum x_i}(1-\theta)^{\sum 1-x_i} \end{aligned}$$

In case of Hilary we can use Geometric distribution (stating the number of failres before success)

$$L_H(\theta|X_1, X_2, X_3, X_4) = \theta(1-\theta)^{m-1}$$

Where $x_i = 1$ (head) or $x_i = 0$ (tail),
$n =$ number of George's trials, $m =$ number of Hilary's trials.
Then:

$$L_{G+H}(\theta|X) = \theta^{\sum_1^n (x_i)+1}(1-\theta)^{\sum_1^n (1-x_i)+m-1}$$

To find an MLE of θ we need to take the log of the likelihood function and maximize it with respect to θ. We can do it by taking the derivative w.r.t θ:

$$\begin{aligned} (\log L_{G+H}(\theta|X))' &= ((\sum_1^n (x_i) + 1)\log\theta + (\sum_1^n (1-x_i) + m - 1)\log{(1-\theta)})' \\ &= \frac{\sum_1^n (x_i) + 1}{\theta} + \frac{\sum_1^n (1-x_i) + m - 1}{1-\theta} = 0 \end{aligned}$$

$\Rightarrow \hat{\theta} = \frac{\sum_1^n (x_i)+1}{n+m} = \frac{1}{7}$

5 Task 23

5.1 Method of moments

Let's calculate the first two moments of the shifted exponential distribution:

$$\mu_1 = \int_{\mu}^{\infty} x f(x) dx = \frac{1}{\sigma} \int_{\mu}^{\infty} x e^{-(x-\mu)/\sigma} dx = \tag{1}$$

Let $u = (x-\mu)/\sigma$. Then $du = dx/\sigma$. So the integral is calculated as follows:

$$= \int_0^{\infty} (u\sigma + \mu) e^{-u} du = \sigma \int_0^{\infty} u e^{-u} du + \mu \int_0^{\infty} e^{-u} du = \sigma\Gamma(2) + \mu\Gamma(1) = \sigma + \mu \tag{2}$$

Therefore:

$$\mu_1 = E(X) = \mu + \sigma \tag{3}$$

The second moment:

$$\mu_2 = \int_{\mu}^{\infty} x^2 f(x) dx = \frac{1}{\sigma} \int_{\mu}^{\infty} x^2 e^{-(x-\mu)/\sigma} dx = \int_0^{\infty} ((u\sigma + \mu))^2 e^{-u} du \tag{4}$$

$$= \sigma^2 \int_0^{\infty} u^2 e^{-u} du + \mu^2 \int_0^{\infty} e^{-u} du + 2\mu\sigma \int_0^{\infty} u e^{-u} du \tag{5}$$

$$= \sigma^2 \Gamma(3) + \mu^2 + 2\mu\sigma\Gamma(1) = 2\sigma^2 + \mu^2 + 2\mu\sigma \tag{6}$$

Therefore:

$$\mu_2 = E(X^2) = 2\sigma^2 + 2\mu\sigma + \mu^2 \tag{7}$$

Therefore the variance is:

$$(X) = E(X^2) - E(X)^2 = 2\sigma^2 + 2\mu\sigma + \mu^2 - (\mu + \sigma)^2 = \sigma^2 \tag{8}$$

And the sample moments are:

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{9}$$

$$m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \tag{10}$$

Let's solve the quations $m_1 = \mu_1$ and $m_2 = \mu_2$:

$$m_1 = \mu_1 \tag{11}$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \mu + \sigma \tag{12}$$

$$m_2 = \mu_2 \tag{13}$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = 2\sigma^2 + 2\mu\sigma + \mu^2 = (\mu + \sigma)^2 + \sigma^2 \tag{14}$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = (\frac{1}{n}\sum_{i=1}^{n} X_i)^2 + \sigma^2 \tag{15}$$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2 \tag{16}$$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \tag{17}$$

Therefore:

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} \tag{18}$$

Substituting the result in $m_1 = \mu_1$, we get

$$\hat{\mu} = \bar{X} - \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} \tag{19}$$

5.2 Method of maximum likelihood

the likelihood function is:

$$l(\mu, \sigma) = \frac{1}{\sigma^n} exp(-\frac{1}{\sigma}\sum_{i=1}^{n}(x_i - \mu)) \tag{20}$$

from which, the log-likelihood:

$$log(l(\mu, \sigma)) = -n \log \sigma - \frac{1}{\sigma}\sum_{i=1}^{n}(x_i - \mu) \tag{21}$$

If μ is known, we set the first derivative of log-likelihood with respect to σ to 0 in order to calculate the mle of σ:

$$\frac{\partial}{\partial \sigma} = 0 \rightarrow \hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)) \tag{22}$$

If μ is not known, we take into account that $x \geq \mu$, therefore we get:

$$\hat{\mu} = min(X_1, X_2, ...) \tag{23}$$

So in this case σ is estimated as follows:

$$\frac{\partial}{\partial \sigma} = 0 \rightarrow \hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n}(X_i - min(X_i)) \tag{24}$$

6 Task 24

We need to prove that:

$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \log(1+X_i/\lambda)}$

MLE method:
$lik(\alpha, \lambda) = \prod_{i=1}^{n} f(X_i|\alpha, \lambda)$

We will maximize natural logarithm of $lik(\alpha, \lambda)$
$l(\alpha, \lambda) = \sum_{i=1}^{n} \log(f(X_i|\alpha, \lambda))$

Lomax density: $f(x) = \frac{\alpha}{\lambda}(1 + \frac{x}{\lambda})^{-(\alpha+1)}$

$l(\alpha, \lambda) = \sum_{i=1}^{n} \log(\frac{\alpha}{\lambda}(1 + \frac{x_i}{\lambda})^{-(\alpha+1)}) = n\log(\frac{\alpha}{\lambda}) - (\alpha+1)\sum_{i=1}^{n} \log(1 + \frac{x_i}{\lambda})$

Take a derivative wrt to α and equal it to 0!
$\frac{n\lambda}{\alpha\lambda} - \sum_{i=1}^{n} \log(1 + \frac{x_i}{\lambda}) = 0$

$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \log(1+\frac{x_i}{\lambda})}$

$\hat{\alpha}$ is also random variable! We know from theory that $E(\hat{\alpha}) = 0$ and that $Var(\hat{\alpha})$ is equal to Fisher Information. So, the standard error will be square root from Fisher Information.

$I(\alpha) = E_{\alpha}(\frac{\log f(X|\alpha)'}{\alpha})^2 = -E_{\alpha}(\frac{\log f(X|\alpha)''}{\alpha})$

So, we need to take the second derivative wrt α.

$(\frac{n}{\alpha} - \sum_{i=1}^{n} \log(1 + \frac{x_i}{\lambda}))' = -E_{\alpha}(-n/\alpha^2) = E_{\alpha}(n/\alpha^2)$

The maximum likelihood estimator $\hat{\alpha}$ is asymptotically distributed as: $\hat{\alpha} \sim N(\alpha, 1/nI_{\alpha})$

Plug in the I_{α}:
$\hat{\alpha} \sim N(\alpha, 1/(n/\alpha)^2)$

From here we derive: $sd(\hat{\alpha}) = \sqrt{(\frac{\alpha}{n})^2} = \alpha/n$

Next, we need to prove that if $U \sim U(0,1)$, $\lambda(U^{-1/\alpha} - 1) \sim$ Lomax (α, λ).

Here we start: $F_X(x) = P(X \leq x) = P(\lambda(U^{-1/\alpha} - 1) \leq x) = P(U^{-1/\alpha} \leq \frac{X+\lambda}{\lambda}) = P(U \geq (\frac{x+\lambda}{\lambda})^{-\alpha}) = 1 - P(U \leq (\frac{x+\lambda}{\lambda})^{-\alpha}) = 1 - (1 + \frac{x}{\lambda})^{-\alpha}$.

Here we recognise the CDF of Lomax distribution with parameters α, λ

Also, we need to prove that $\hat{(\alpha)}/\alpha \sim Gamma(n, n)$. Let's have a look at $\hat{\alpha}/\alpha$.

$\hat{\alpha}/\alpha = \frac{n}{\alpha \sum_{i=1}^{n} \log(1+\frac{x_i}{\lambda})}$ Let's find out the distribution of $Z = \log(1 + \frac{x}{\lambda})$

$F_Z(z) = P(\log(1 + \frac{x}{\lambda}) \leq x) = P((1 + \frac{x}{\lambda}) \leq e^x) = P(X \leq \lambda(\exp^x - 1)) = F_X(\lambda(\exp^x - 1))$

Recall $X \sim Lomax(\alpha, \lambda)$. Thus, we obtain the folowing: $F_Z(z) = 1 - (1 + \frac{\lambda(\exp^x - 1))}{\lambda})^{-\alpha} = 1 - \exp^{-x\alpha}$

From here we conclude that $Z \sim E(\alpha)$.

We know the characteristic function of Exponentional distribution: $\phi_z(t) = \frac{\alpha}{\alpha - it}$

But we need the characteristic function of the sum of i.i.d RVs $\sim E(\alpha)$. $\phi_{z_1+...+z_n} = (1 - \frac{it}{\alpha})^{-n}$ due to independence of $Z_1, ..., Z_n$ characteristic function of sum is the product of characteristic functions. And here we recognize the characteristic function of $Gamma(\alpha, n)$. But the parameter α is the rate. To have a scale parameter, we need inverse of Gamma, i.e. $Gamma(1/\alpha, n)$.

We know the property of the Gamma distribution:

if $X \sim Gamma(n, 1/\alpha)$, then $bX \sim Gamma(n, b/\alpha)$. From MLE method of α estimation we can see that $n = \hat{\alpha}$*some constant b. So, we have just proved that $\hat{\alpha}/\alpha \sim Gamma(n, n)$ (where n is scale and shape parameter!).

7 Task 25

This is an example of excess loss reinsurance. For W we are in particular interested in the case where $X > 40000$. So let's denote this positive random variable with W_E.

7.1 a)

The distribution of W can be viewed as a mixture distribution. The insurance company pays nothing with probability $F_X(40000)$ and the excess with probability $\bar{F}_X(40000)$ (often called the survival function, just the counter probability). Let's find the excess distribution:

$$\bar{F}_{W_E} = P(X > 40000 + x | X > 40000) = \frac{\bar{F}_X(40000 + w)}{\bar{F}_X(40000)}.$$

The density then can be found by differentiating:

$$f_{W_E} = \frac{f_X(40000 + x)}{\bar{F}_X(40000)}, \quad x > 0$$

for $X > 40000$ and $W_E = 0$ else.

By rearranging the given density function of X a bit, we can see that this is just the Lomax distribution (with $\lambda = 20000$).
The cumulative distribution function of the Lomax distribution is

$$F(x|\alpha) = 1 \; - \; \left(1 + \frac{x}{20000}\right)^{-\alpha}.$$

The desired density for W then yields

$$f_{W_E} = \frac{\frac{\alpha 2^{\alpha} 10^{4\alpha}}{(20000+(40000+x))^{\alpha+1}}}{\left(1 + \frac{40000}{20000}\right)^{-\alpha}} = \frac{\alpha 20000^{\alpha} \; (1+2)^{\alpha}}{(60000+x)^{\alpha+1}} = \frac{\alpha\,(60000)^{\alpha}}{(60000+x)^{\alpha+1}}$$

for $X > 40000$ and 0 else.
This is again a Lomax distribution with parameter $\lambda = 60000$.
Now let us find the mean. One can think of W as a mixture of the excess distribution and 0. So we can calculate the mean by weighting according to the probability:

$$\mathbb{E}(W) = 0 + \mathbb{E}(W_E)P(X > 40000) = \mathbb{E}(W_E)\bar{F}_X(40000) = \frac{60000}{\alpha - 1}\frac{1}{3^{\alpha}}.$$

W has only variability, if $X > 40000$. We can calculate the variance again by seeing it as a mixture:

$$\mathbb{V}(W) = 0 + \mathbb{V}(W_E)P(X > 40000) = \frac{\alpha 60000^2}{(\alpha-1)^2(\alpha-2)}\frac{1}{3^{\alpha}}.$$

Note that this is just the variance for the Lomax distribution with $\lambda = 60000$ multiplied with the corresponding probability.

7.2 b)

For finding the MLE we can use the result from the exercise before where $\lambda = 60000$. So the MLE is

```
> Xi <- c(14000,21000,6000,32000, 2000)
> n <- length(Xi)
> MLElomax <- n/sum(log(1 + Xi/60000))
> MLElomax

[1] 4.693209
```

The standard error can be approximated by drawing a large enough sample where we estimate the MLE for every sample.

```
> require(matrixStats)
> require(Renext)
> Lomaxsample <- matrix(rlomax(length(Xi)*1000,
+          scale = 60000, shape = MLElomax), nrow=length(Xi), ncol=1000)
> LoMLEs <- 5/colSums(apply(Lomaxsample,2,function(x){log(1+x/60000)}))
> alphabar <- rep(mean(LoMLEs),1000)
```

And finally the estimated standard error of $\hat{\alpha}$.

```
> sqrt(sum((LoMLEs - alphabar)^2)/1000)
```

```
[1] 3.139049
```

8 Task 26

After taking derivative wrt x, we can find the PMF for each X and then, as they are i.i.d find the generall PMF:

$$f(X,M) = \prod_{1}^{n} \big(- \frac{\alpha\lambda^{\alpha}}{(\lambda + x_i)^{\alpha+1}} \big) \big(- \frac{\alpha\lambda^{\alpha}}{(\lambda + M)^{\alpha+1}} \big)^{m}$$

Then

$$\begin{aligned}
\log f(X,M) &= \sum_{1}^{n} \big(\log -\alpha\lambda^{\alpha} - (\alpha+1)\log{(\lambda + x_i)} \big) + m \log \big(- \frac{\alpha\lambda^{\alpha}}{(\lambda + M)^{\alpha+1}} \big) \\
&= \sum_{1}^{n} \big(\log -\alpha + \alpha \log \lambda - (\alpha+1)\log{(\lambda + x_i)} \big) + m \log \big(- \frac{\alpha\lambda^{\alpha}}{(\lambda + M)^{\alpha+1}} \big) \\
&= n \log -\alpha + n\alpha \log \lambda - n(\alpha+1)\log{(\lambda + x_i)} \\
&\quad + m \log -\alpha + m\alpha \log \lambda - m(\alpha+1)\log{(\lambda + M)} \\
(\log f(X,M))' &= -\frac{m+n}{\alpha} + m(\log(\frac{\lambda}{\lambda + M})) + \sum_{i=1}^{n} \log(\frac{\lambda}{\lambda + x_i}) = 0 \\
\hat{\alpha}_{Total} &= \frac{m+n}{m \log \frac{\lambda}{\lambda+M} + \sum_{i=1}^{n} \log \frac{\lambda}{\lambda + x_i}} \\
&= \frac{m+n}{-(m \log{(1 + \frac{M}{\lambda})} + \sum_{i=1}^{n} \log{(1 + \frac{x_i}{\lambda})})} \\
\hat{\alpha}_{EireGeneral} &= \frac{n}{\sum_{i=1}^{n} \log{(1 + \frac{x_i}{\lambda})} + m \log{(1 + \frac{M}{\lambda})}}
\end{aligned}$$

In the particular example, our $\hat{\alpha}_{MLE}$ is equal to

```
> amounts <- c(14.9, 775.7, 805.2,
+              993.9, 1127.5, 1602.5, 1998.3, 2000, 2000, 2000)
> ex26 <- function(n,m,lambda,amounts){
+  for(i in 1:n){
+     a <- sum(log(1 + amounts[i]/lambda))
+ }
+ alpha_mle <- n/(a + m*log(1+amounts[length(amounts)]/lambda))
+
+ return(alpha_mle)
+ }
> ex26(7,3,8400,amounts)

[1] 8.195446
```