Statistics 1 Unit 1

Team 9

December 6, 2017

Contents

1 Task 1

1.1 a

```
> #Solution 1
> Hilbert <- function(n) {
+   Hilbert_matrix <- matrix(0, n, n)
+   for(i in 1:n) {
+     Hilbert_matrix[i, ] <- c(i + 1:n - 1)
+   }
+   for(j in 1:n) {
```

```
+     Hilbert_matrix[ ,j] <- c(1:n + j - 1)
+   }
+   1 / Hilbert_matrix
+ }
> #Solution 2
> Hilbert2 <- function(n) {
+   i <- 1:n
+   1 / outer(i - 1, i, "+")
+ }
```

1.2 b, c

```
> for(i in 1:11) {
+   #print(solve(Hilbert2(i)))
+ }
> for(i in 1:6) {
+   #print(qr.solve(Hilbert2(i)))
+ }
```

There is no problems for the first 11 inverses for "solve" and for 6 inverses for "qr.solve" (default tolerance for qr.solve is higher), but then the is an error in R. However, as the inverse of Hilbert matrix can be represented explicitly, it always exists and contains integers only, but the determinant of Hilbert matrices with high dimensions converges to zero. That is why R gives us error.

2 Task 2

To find the coefficients of the quintic polynomial, we basically need to solve the system of equations. The result represents the coefficients.

```
> x <- 10:15
> p_x <- c(25, 16, 26, 19, 21, 20)
> matrix_of_coef_before_alphas <- outer(x, 0:5, "^")
> alphas <- solve(matrix_of_coef_before_alphas, p_x)
> alphas
```

```
[1]  2.536100e+05 -1.025510e+05  1.650092e+04 -1.320667e+03  5.258333e+01
[6] -8.333333e-01
```

3 Task 3

First, let's create the required random matrix:

```
> matrix_X <- matrix(runif(5*3), 5, 3)
```

3.1 a

Now, let's calculate $H = X(X'X)^{-1}X'$:

```
> matrix_H <- matrix_X %*% solve((t(matrix_X) %*% matrix_X)) %*% t(matrix_X)
> matrix_H
```

```
           [,1]        [,2]        [,3]       [,4]        [,5]
[1,]  0.4217027  0.31499654  0.23703859  0.2304760 -0.18798993
[2,]  0.3149965  0.81265935 -0.04468126 -0.2129909  0.07523136
[3,]  0.2370386 -0.04468126  0.45058841  0.3739505  0.22256639
[4,]  0.2304760 -0.21299093  0.37395052  0.4229783 -0.07579140
[5,] -0.1879899  0.07523136  0.22256639 -0.0757914  0.89207129
```

Eigenvalues and corresponding eigenvectors of H are:

```
> eigen(matrix_H)
```

```
eigen() decomposition
$values
[1]  1.000000e+00  1.000000e+00  1.000000e+00  4.440892e-16 -6.661338e-16

$vectors
           [,1]       [,2]       [,3]       [,4]       [,5]
[1,] 0.00000000  0.6493864  0.0000000  0.7604586  0.0000000
[2,] 0.02220984  0.4850680  0.7595230 -0.4142192 -0.1255513
[3,] 0.47312433  0.3650193 -0.3057820 -0.3117048  0.6724966
[4,] 0.18532844  0.3549136 -0.5125115 -0.3030750 -0.6965395
[5,] 0.86099562 -0.2894886  0.2587551  0.2472060 -0.2163745
```

3.2 b

```
> trace_H <- sum(diag(matrix_H))
> trace_H
```

```
[1] 3
```

```
> # by default tolerance = sqrt(.Machine$double.eps)
> all.equal(trace_H, sum(eigen(matrix_H)$values))
```

```
[1] TRUE
```

As we can see they are the same.

3.3 c

```
> det_H <- det(matrix_H)
> all.equal(det_H, prod(eigen(matrix_H)$values))
```

```
[1] TRUE
```

So, the same result: determinant is equal to product of eigenvalues

3.4 d

To verify that the columns of X are eigenvectors of H, we need to check the main property of eigenvector: Hx = λx:

```
> for(i in 1:ncol(matrix_X)){
+   v <- matrix_H %*% matrix_X[,i] / matrix_X[,i]
+   cat("Multiplicators are:",v,'\n')
+ }
```

```
Multiplicators are: 1 1 1 1 1
Multiplicators are: 1 1 1 1 1
Multiplicators are: 1 1 1 1 1
```

This means that all 3 vectors of X are eigenvecotrs of H with corresponding eigenvalue $\lambda = 1$

4 Task 4

Hilbert 6 by 6 matrix:

```
> Hilbert_6 <- Hilbert2(6)
> Hilbert_6
```

```
         [,1]      [,2]      [,3]      [,4]      [,5]       [,6]
[1,] 1.0000000 0.5000000 0.3333333 0.2500000 0.2000000 0.16666667
[2,] 0.5000000 0.3333333 0.2500000 0.2000000 0.1666667 0.14285714
[3,] 0.3333333 0.2500000 0.2000000 0.1666667 0.1428571 0.12500000
[4,] 0.2500000 0.2000000 0.1666667 0.1428571 0.1250000 0.11111111
[5,] 0.2000000 0.1666667 0.1428571 0.1250000 0.1111111 0.10000000
[6,] 0.1666667 0.1428571 0.1250000 0.1111111 0.1000000 0.09090909
```

It's eigenvalues and eigenvectors are:

```
> eigen(Hilbert_6)
```

```
eigen() decomposition
$values
[1] 1.618900e+00 2.423609e-01 1.632152e-02 6.157484e-04 1.257076e-05
[6] 1.082799e-07

$vectors
           [,1]       [,2]       [,3]        [,4]        [,5]         [,6]
[1,] -0.7487192  0.6145448 -0.2403254 -0.06222659  0.01114432 -0.001248194
[2,] -0.4407175 -0.2110825  0.6976514  0.49083921 -0.17973276  0.035606643
[3,] -0.3206969 -0.3658936  0.2313894 -0.53547692  0.60421221 -0.240679080
[4,] -0.2543114 -0.3947068 -0.1328632 -0.41703769 -0.44357472  0.625460387
[5,] -0.2115308 -0.3881904 -0.3627149  0.04703402 -0.44153664 -0.689807199
[6,] -0.1814430 -0.3706959 -0.5027629  0.54068156  0.45911482  0.271605453
```

Inverse matrix is:

```
> Hilbert_6_inverse <- solve(Hilbert_6)
```

Let's find inverse eigenvalues and vectors:

```
> eigen(Hilbert_6_inverse)$values
```

```
[1] 9.235320e+06 7.954970e+04 1.624040e+03 6.126880e+01 4.126079e+00
[6] 6.177034e-01
```

```
> eigen(Hilbert_6)$values
```

```
[1] 1.618900e+00 2.423609e-01 1.632152e-02 6.157484e-04 1.257076e-05
[6] 1.082799e-07
```

```
> rev(1 / eigen(Hilbert_6_inverse)$values)
```

```
[1] 1.618900e+00 2.423609e-01 1.632152e-02 6.157484e-04 1.257076e-05
[6] 1.082799e-07
```

Check that eigenvalues of $A^{-1} = \lambda$

```
> all.equal(eigen(Hilbert_6)$values, rev(1 / eigen(Hilbert_6_inverse)$values))
```

```
[1] TRUE
```

The result is expected, since:
$A\mathbf{v} = \lambda\mathbf{v} \implies A^{-1}A\mathbf{v} = \lambda A^{-1}\mathbf{v} \implies A^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v}$

5 Task 5

```
> Circulant <- function(x) {
+   n <- length(x)
+   matrix_C <- matrix(NA, n, n)
+   for (i in 1:n) {
+     matrix_C[i, ] <- c(x[-(1:(n + 1 - i))], x[1:(n + 1 - i)])
+     }
+   return(matrix_C)
+ }
> mat_P <- Circulant(c(0.1, 0.2, 0.3, 0.4))
```

5.1 a

```
> rowSums(mat_P)
```

```
[1] 1 1 1 1
```

As we can see, the sums are equal to 1.

5.2 b

```
> for(i in c(2, 3, 5, 10)){
+ print(Reduce("%*%" , rep(list(mat_P), i)))
+ }
```

```
     [,1] [,2] [,3] [,4]
[1,] 0.26 0.28 0.26 0.20
[2,] 0.20 0.26 0.28 0.26
[3,] 0.26 0.20 0.26 0.28
```

```
[4,] 0.28 0.26 0.20 0.26
      [,1]  [,2]  [,3]  [,4]
[1,] 0.256 0.244 0.240 0.260
[2,] 0.260 0.256 0.244 0.240
[3,] 0.240 0.260 0.256 0.244
[4,] 0.244 0.240 0.260 0.256
        [,1]    [,2]    [,3]    [,4]
[1,] 0.25056 0.25072 0.24928 0.24944
[2,] 0.24944 0.25056 0.25072 0.24928
[3,] 0.24928 0.24944 0.25056 0.25072
[4,] 0.25072 0.24928 0.24944 0.25056
          [,1]      [,2]      [,3]      [,4]
[1,] 0.2500000 0.2500016 0.2500000 0.2499983
[2,] 0.2499983 0.2500000 0.2500016 0.2500000
[3,] 0.2500000 0.2499983 0.2500000 0.2500016
[4,] 0.2500016 0.2500000 0.2499983 0.2500000
```

Yes, there is a pattern. All the elements tend to converge to 0.25. That is the sum of any row of circulant matrix divided by number of rows/columns

5.3 c

```
> solve(t(mat_P) - diag(4), cbind(rep(0, 4)), tol = 1e-17)
```

```
     [,1]
[1,]    0
[2,]    0
[3,]    0
[4,]    0
```

```
> eigen(t(mat_P))
```

```
eigen() decomposition
$values
[1]  1.0+0.0i -0.2+0.2i -0.2-0.2i -0.2+0.0i

$vectors
        [,1]      [,2]      [,3]    [,4]
[1,] -0.5+0i  0.0+0.5i  0.0-0.5i -0.5+0i
[2,] -0.5+0i -0.5+0.0i -0.5-0.0i  0.5+0i
[3,] -0.5+0i  0.0-0.5i  0.0+0.5i -0.5+0i
[4,] -0.5+0i  0.5+0.0i  0.5+0.0i  0.5+0i
```

Both these results and the fact that we are searching basically for such eigen-vector of P^T which has eigen value 1. That is the one which has all his elemets equal to each other. $P^T x = \lambda x$, where $\lambda = 1$. Therefore, our vector x of dim 4×1 will have 4 elements $= 0.25$.

```
> all.equal(t(mat_P) %*% cbind(rep(0.25, 4)), cbind(rep(0.25, 4)))
```

```
[1] TRUE
```

The pattern is following: Elements of vector x equal to the elemets to which matrix P^{10} converges.

8 Task 8

Supposing element in 1st row 2d col is zero (similiar to ex 22). If not the algorithm is basically the same.
To solve a partitioned system of equations of such form, one can easily find x and then y.
$x = L_1^{-1} b \Rightarrow y = L_2^{-1}(c - Bx) = L_2^{-1}(c - BL_1^{-1} b)$

```
> # x <- solve(L_1, b)
> # y <- solve(L_2, c - B * x)
```

9 Task 9

9.1 a

For every elementary matrix if we apply r and then r^{-1} (reversed operation), we will return to original elementary matrix. As our matrix is elementary it is always invertible. Also its determinant equals to 1.

9.2 b

$$M_k = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -\mu_{k+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu_n & 0 & \cdots & 1 \end{bmatrix}$$

$$I - m_k e_k' = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mu_{k+1} \\ \vdots \\ \mu_n \end{bmatrix} \begin{bmatrix} 0 & \cdots & \begin{matrix} e_k \\ \text{where } e_k = 1 \end{matrix} & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & -\mu_{k+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu_n & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -\mu_{k+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu_n & 0 & \cdots & 1 \end{bmatrix} = M_k$$

9.3 c

To prove that $M_k^{-1} = I + m_k e_k'$ we need to check that $M_k^{-1} M_k = I$

$$M_k^{-1} M_k = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -\mu_{k+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu_n & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & \mu_{k+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n & 0 & \cdots & 1 \end{bmatrix}$$

Actually it is crear from the block structure:
$\begin{bmatrix} I & 0 \\ -M & I \end{bmatrix} \begin{bmatrix} I & 0 \\ M & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -M + M & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

9.4 d

$$M_k M_l = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -\mu_{k+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu_n & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -\mu_l & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu_n & 0 & \cdots & 1 \end{bmatrix}$$

Let's have a look at the block structure:
$\begin{bmatrix} I & 0 \\ -M_k & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -M_l & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -M_k - M_l & I \end{bmatrix}$ This can be represented as:

$$\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mu_{k+1} \\ \vdots \\ \mu_n \end{bmatrix} \begin{bmatrix} 0 & \cdots & \begin{matrix} e_k \\ \text{where } e_k = 1 \end{matrix} & \cdots & 0 \end{bmatrix} -$$

$$- \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mu_l \\ \vdots \\ \mu_n \end{bmatrix} \begin{bmatrix} 0 & \cdots & \begin{matrix} e_l \\ \text{where } e_l = 1 \end{matrix} & \cdots & 0 \end{bmatrix} = I - m_k e_k' - m_l e_l'$$

10 Task 10

First of all, let's compute determinant of A:
$det(A) = 0 * 0 - 1 * 1 = -1$

If $A = LU$, then $a_{11} = l_{11} * u_{11}$.
Since $a_{11} = 0$, then $l_{11} = 0$ or $u_{11} = 0$.
Therefore, $det(L) = 0$ or $det(U) = 0$ (since determinant of a lower or upper triangular matrix equals product of the elements of its main diagonal, and here the first element of the main diagonal of at least one matrix of L and U is zero).
So, $det(A) = det(LU) = det(L) * det(U) = 0$.
However, it was calculated before, det(A) = -1.
Therefore, we have a contradiction.
So matrix A has no LU factorization.
Q.E.D.

11 Exercise 11

a) Show that if $n \times n$ matrix A has rank 1, then there are non-zero n-vectors u and v such that $A = uv^t$.
Let A be a $n \times n$ matrix with rank 1. Therefore, it is equivalent to the form:

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where at least one of the numbers $v_1...v_n$ must be non-zero. This matrix is in turn equivalent to:

$$\begin{pmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & \cdots & u_1 \cdot v_n \\ u_2 \cdot v_1 & u_2 \cdot v_2 & \cdots & u_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n \cdot v_1 & u_n \cdot v_2 & \cdots & u_n \cdot v_n \end{pmatrix}$$

where at least one of the numbers $u_1...u_n$ must be non-zero.
Therefore, the latter matrix can be expressed as an outer product uv^t of non-zero n-vectors u and v:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

So, direct statement is proved.

b) Show that if there are non-zero n-vectors u and v such that $A = uv^t$, then $n \times n$ matrix A has rank 1.
Let u and v be non-zero n-vectors such that $A = uv^t$.
Let's calculate $A = uv^t$:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & \cdots & u_1 \cdot v_n \\ u_2 \cdot v_1 & u_2 \cdot v_2 & \cdots & u_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n \cdot v_1 & u_n \cdot v_2 & \cdots & u_n \cdot v_n \end{pmatrix}$$

This is an $n \times n$ matrix, and its every row is a multiple (and therefore linear combination) of non-zero n-vector v. Since we have here all possible

combinations of u and v and both vectors are non-zero, exists at least one element that is different from zero. Hence, the rank of the matrix must be 1. So, converse statement is proved.
Q.E.D.

12 Task 12

$A = I - uv'$ - elementary matrix.

12.1 Task (a)

If A is elementary, what condition on u and v ensures that A is non-singular? To be non-singular $det(A) \neq 0$ Considering task 24:

$$det(I + uv') = 1 + v'u \tag{1}$$

Now using (1) let's find $det(A = I - uv')$:

$$det(A = I - uv') = 1 - v'u \tag{2}$$

This gives the condition:

$$v'u \neq 1 \tag{3}$$

12.2 Task (b)

Prove that A^{-1} is also elementary: To prove this fact let's use the Sherman-Morrison formula:

$$(A - uv')^{-1} = A^{-1} + A^{-1}u(1 - v'A^{-1}u)^{-1}v'A^{-1} \tag{4}$$

Applying this gives us:

$$(I - uv')^{-1} = I^{-1} + I^{-1}u(1 - v'I^{-1}u)^{-1}v'I^{-1} = I - \frac{1}{v'u - 1}uv' \tag{5}$$

So $A^{-1} = I - \sigma uv'$, where $\sigma = \frac{1}{v'u-1}$. It is also elementary.

12.3 Task (c)

Are elementary elimination matrices elementary? Yes, they are. There are 3 cases.

12.3.1 Case 1: Row-switching transformations

A is the matrix produced by exchanging row i and row j of A. To have this type of transformation using $A = (I - uv')$, we need to choose uv' such that:
$\begin{cases} u_i v_i = 1 \\ u_j v_j = 1 \\ u_i v_j = -1 \\ u_j v_i = -1 \end{cases}$ Assuming that i and j are switched. All others elements of u and v = 0. This gives us next result for σ (only 2 sums are different from 0):

$$\sigma = \frac{1}{v'u - 1} = \frac{1}{v_i u_i + v_j u_j - 1} = 1 \tag{6}$$

12.3.2 Case 2: Row-multiplying transformations

The corresponding elementary matrix is a diagonal matrix, with diagonal entries 1 everywhere except in the i th position, where it is m: To have such structure $v'u$ must have 1-m element of i th row, all others are zeros. So:
$\begin{cases} u_i v_i = 1 - m \\ \sigma = \frac{1}{\sum u_k v_k - 1} = -\frac{1}{m} \end{cases}$

12.3.3 Case 3: Row-addition transformations

The corresponding elementary matrix is the identity matrix but with an m in the (i,j) position $\begin{cases} u_i v_j = -m \\ \sigma = \frac{1}{\sum u_k v_k - 1} = -1 \end{cases}$ Since $\sum u_k v_k = 0$

13 Exercise 13

If we mulitply the left-hand side of the formula from any side by matrix $A - uv^t$, we get the identity matrix, since it is multiplication of inverse matrices:
$(A - uv^t)(A - uv^t)^{-1} = I$
$(A - uv^t)^{-1}(A - uv^t) = I$
To prove that the formula is true, we must verify that the right-hand side of the formula becomes identity matrix if it is multiplied by $A - uv^t$ both from the left side and from the right side.
a) multiplication from the left:

$= (A - uv^t)(A^{-1} + \frac{A^{-1}uv^tA^{-1}}{1 - v^tA^{-1}u)})$

$= AA^{-1} - uv^tA^{-1} + \frac{AA^{-1}uv^tA^{-1} - uv^tA^{-1}uv^tA^{-1}}{1-v^tA^{-1}u}$

$= I - uv^tA^{-1} + \frac{uv^tA^{-1} - uv^tA^{-1}uv^tA^{-1}}{1-v^tA^{-1}u}$

$= I - uv^tA^{-1} + \frac{(u - uv^tA^{-1}u)v^tA^{-1}}{1-v^tA^{-1}u}$

$= I - uv^tA^{-1} + \frac{u(1-v^tA^{-1}u)v^tA^{-1}}{1-v^tA^{-1}u}$

$(1 - v^tA^{-1}u)$ can be cancelled out since it is a scalar. Therefore:

$= I - uv^tA^{-1} + uv^tA^{-1}$

$= I$

b) multiplication from the right:

$= (A^{-1} + \frac{A^{-1}uv^tA^{-1}}{1-v^tA^{-1}u)})(A - uv^t)$

$= A^{-1}A - A^{-1}uv^t + \frac{A^{-1}uv^tA^{-1}A - A^{-1}uv^tA^{-1}uv^t}{1-v^tA^{-1}u}$

$= I - A^{-1}uv^t + \frac{A^{-1}uv^t - A^{-1}uv^tA^{-1}uv^t}{1-v^tA^{-1}u}$

$= I - A^{-1}uv^t + \frac{(A^{-1}u - A^{-1}uv^tA^{-1}u)v^t}{1-v^tA^{-1}u}$

$= I - A^{-1}uv^t + \frac{A^{-1}u(1-v^tA^{-1}u)v^t}{1-v^tA^{-1}u}$

$(1 - v^tA^{-1}u)$ can be cancelled out since it is a scalar. Therefore:

$= I - A^{-1}uv^t + A^{-1}uv^t$

$= I$

Q.E.D.

14 Task 14

We need to prove that:

$$(A - UV')^{-1} = A^{-1} + A^{-1}U(I - U'A^{-1}U)^{-1}V'A^{-1} \tag{7}$$

To do this let's multiply both sides by (A - UV') LHS gives us I. So, let's prove that RHS gives I too.

$$(A - UV') \times (A^{-1} + A^{-1}U(I - U'A^{-1}U)^{-1}V'A^{-1}) =$$

Open the brackets and use $AA^{-1} = I$:

$$= I + U(I - V'A^{-1}U)^{-1}V'A^{-1} - UV'A^{-1} - UV'A^{-1}U(I - V'A^{-1}U)^{-1}V'A^{-1} =$$

regroup 1 + 3 and 2 + 4 and factor out:

$$= I - UV'A^{-1} + U(I - V'A^{-1}U)(I - V'A^{-1}U)^{-1}V'A^{-1} =$$

Finally, we have:

$$= I - UV'A^{-1} + UV'A^{-1} = I$$

Q.E.D.

15 Task 22

15.1 Task (a)

Let's prove that if λ is eigenvalue of A_{11} than λ is eigenvalue of A with corresponding eigenvector $(u', 0')'$.

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{12} \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} A_{11}u \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda u \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} u \\ 0 \end{pmatrix} \tag{8}$$

$\begin{pmatrix} u \\ 0 \end{pmatrix} \neq 0$, since $u \neq 0$ (it is eigenvector of A_{11}). Hence, λ is eigenvalue of A.

15.2 Task (b)

Let's prove that $det(A - \lambda I) = 0$. This will prove that λ is an eigenvalue of A. Take into consideration that $\left|A_{22} - \lambda I\right| = 0$ since λ is eigenvalue of A_{22}

$$\left|A - \lambda I\right| = \begin{vmatrix} A_{11} - \lambda I & A_{12} \\ 0 & A_{22} - \lambda I \end{vmatrix} = \left|A_{11} - \lambda I\right| \left|A_{22} - \lambda I\right| = 0 \tag{9}$$

15.3 Task(c)

We are given that:

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{12} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} \tag{10}$$

After multiplication we have a system:

$$\begin{cases} A_{11}u + A_{12}v = \lambda u \\ A_{22}v = \lambda v \end{cases} \tag{11}$$

This system gives us 2 cases:

1. $v \neq 0$ Then from $A_{22}v = \lambda v$ we have that λ is eigenvalue of A_{22} with corresponding eigenvector v.

2. $v = 0$ Than from the 1 equation we have $A_{11}u = \lambda u$, hence λ is eigenvalue of A_{11} with corresponding eigenvector u.

15.4 Task(d)

From sections (a) and (b) follows that if λ is eigenvalue of A_{11} or A_{22} than λ is eigenvalue of A. From section (c) follows that if λ is eigenvalue of A than λ is eigenvalue of A_{11} or A_{22}. This proves required statement.

16 Task 24

We need to prove:

$$det(I + uv') = 1 + u'v \tag{12}$$

In our case A = I, so the required result follows from the equality:

$$\begin{pmatrix} I & 0 \\ V' & 1 \end{pmatrix} \begin{pmatrix} I + uv' & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ -v' & 1 \end{pmatrix} = \begin{pmatrix} I & u \\ 0 & 1 + v'u \end{pmatrix} \tag{13}$$

The determinant of the left hand side is the product of the determinants of the three matrices. Since the first and third matrix are triangle matrices with unit diagonal, their determinants are just 1. The determinant of the middle matrix is our desired value. The determinant of the right hand side is simply $(1 + v'u)$. So we have the result:

$$det(I + uv') = (1 + v'u) \tag{14}$$

17 Task 25

The Householder transformation is a linear transformation that describes a reflection about a plane or hyperplane containing the origin.

Let's prove that eigenvalues of the Householder transformation are -1 and 1. To show it we need to mention 2 points:

- N-dimensional space has n basic orthogonal vectors
- If x and y orthogonal x'y = 0

Using these facts let's apply Householder transformation to n-1 orthogonal vectors x:

$$Hx = x - 2\frac{u^t x}{u^t u}u \tag{15}$$

Using point (2) $Hx = x$ for all n-1 vectors, hence $\lambda = 1$. This λ repeats n-1 times. The last eigenvalue can be received from the other fact. Let's apply Householder transformation to the same vector. In this case we have:

$$Hu = u - 2\frac{u^t u}{u^t u}u = -u \tag{16}$$

This gives us the last eigenvalue $\lambda = -1$. So as a result, Householder transformation has two different eigenvalues: -1 and 1

18 Task 27

Show that $(-1)^n p(z)$ is the characteristic polynomial $det(C - zI)$, where C is:

$$\det(C - zI_n) = \det \begin{pmatrix} -z & 0 & \cdots & 0 & -a_0 \\ 1 & -z & \cdots & 0 & -a_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} - z \end{pmatrix} =$$

$$= -z \cdot \det \begin{pmatrix} -z & 0 & \cdots & 0 & -a_1 \\ 1 & -z & \cdots & 0 & -a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} - z \end{pmatrix} + (-1)^{1+n}(-a_0) \cdot \det \begin{pmatrix} 1 & -z & 0 & \cdots & 0 \\ 0 & 1 & -z & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} =$$

= [the second matrix's determinant is the product of its diagonals = 1 (since it's upper-triangular)] =

$$= -z \cdot \det \begin{pmatrix} -z & 0 & \cdots & 0 & -a_1 \\ 1 & -z & \cdots & 0 & -a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} - z \end{pmatrix} + (-1)^n a_0 =$$

$$= -z \cdot (-z \cdot \det \begin{pmatrix} -z & 0 & \cdots & 0 & -a_2 \\ 1 & -z & \cdots & 0 & -a_3 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} - z \end{pmatrix} + (-1)^{n-1} a_1) + (-1)^n a_0 =$$

= [Using the same approach we can replace the determinant on the left until we finally have:] =

$$= (-1)^n p(z)$$

18.1 Calculations

If $p(z) = 24 - 40x + 35z^2 - 13z^3 + z^4$ The results for both of the calculations are the same and equal to:

1. 0.6839038-0.9409769i
2. 0.6839038+0.9409769i
3. 1.8047699+0.0000000i
4. 9.8274224-0.0000000i

19 Task 29

```
> vec <- function(A)
+ {
+   elements <- c()
+   for(j in 1:ncol(A))
+   {
+     for (i in 1:nrow(A))
+     {
+       elements <- c(elements, A[i,j])
```

```
+     }
+   }
+   return(elements)
+ }
> A <- matrix (c(1,2,3,4,5,6,7,8,9) , nrow = 3, ncol = 3)
> vec(A)

[1] 1 2 3 4 5 6 7 8 9

> B <- matrix (c(1,2,3,4,5,6,7,8,9,10) , nrow = 5, ncol = 2)
> vec(B)

 [1]  1  2  3  4  5  6  7  8  9 10
```

20 Task 30

```
> vech <- function(A)
+ {
+   elements <- c()
+   for(j in 1:ncol(A))
+   {
+     for (i in j:nrow(A))
+     {
+       elements <- c(elements, A[i,j])
+     }
+   }
+   return(elements)
+ }
> A <- matrix (c(3,7,8,7,10,9,8,9,12) , nrow = 3, ncol = 3)
> vech(A)

[1]  3  7  8 10  9 12
```