Statistics 1 Unit 6

Group 8

January 31, 2018

Contents

1 Task 96

1.1 a)

The d-dimensional independence copula C_0 is the CDF of d **mutually independent Uniform(0,1)** random variables. Thus, $C_0\left(u_1,\ldots,u_d\right)=u_1\cdots u_d$

1.2 b)

The co-monotonicity copula C_+ is the CDF of $U=(U,...,U)$
Thus, $C_+\left(u_1,\ldots,u_d\right) = P\left(U\leq u_1,\ldots,U\leq u_d\right) = P\left\{U\leq\min\left(u_1,\ldots,u_d\right)\right\} = \min\left(u_1,\ldots,u_d\right)$

1.3 c)

The two-dimensional counter-monotonicity copula C_- is CDF of $(U,1-U)$
Thus, $C_-(u_1,u_2)=P(U\leq u_1,1-U\leq u_2)=P(1-u_2\leq U\leq u_1)=\max(u_1+u_2-1,0)$
The last equation holds, since, if $1-u_2>u_1$, then $1-u_2\geq U\leq u_1$ is impossible $\Rightarrow$ the probability $=0$. Otherwise, the probability is the length of the interval $(1-u_2,u_1)$, which is u_1+u_2-1

2 Task 97

Proof. First, let's prove the RHS of the inequality:
$\bigcap\limits_{1\leq j\leq d}\{U_j \leq u_j\} \subseteq \{U_i \leq u_i\}\,\forall i \in \{1,\ldots,d\} \overset{1}{\Rightarrow} C\left(u_1,\ldots,u_d\right) \leq \min\left\{u_1,\ldots,u_d\right\}$
Now let's prove the LHS of the inequality:

$$\begin{aligned}\mathrm{C}(u_1,\ldots,u_d) = P&\left(\bigcap_{1\leq i\leq d}\{U_i \leq u_i\}\right)\\ =&1-P\left(\bigcup_{1\leq i\leq d}\{U_i > u_i\}\right)\\ \geq&1-\sum_{i=1}^{d}P\left(U_i > u_i\right) \overset{2}{=} 1-d+\sum_{i=1}^{d}u_i\end{aligned}$$

$\Rightarrow C\left(u_1,\ldots,u_d\right) \geq \max\left\{\sum_{i=1}^{d}u_i + 1 - d, 0\right\}$
Combining RHS and LHS we get:
$\max\left\{\sum_{i=1}^{d}u_i + 1 - d, 0\right\} \leq C\left(u_1,\ldots,u_d\right) \leq \min\left\{u_1,\ldots,u_d\right\}$ □

1. $P\left(\bigcap\limits_{1\leq j\leq d}\{U_j \leq u_j\}\right) \leq P\left(\{U_i \leq u_i\}\right)\forall i \in \{1,\ldots,d\}$
2. $\sum_{i=1}^{d}P\left(U_i > u_i\right) = \sum_{i=1}^{d}\left(1 - P\left(U_i \leq u_i\right)\right)$

3 Task 98

First let's compute marginal distributions:

$X_1 \downarrow X_2 \rightarrow$	0	1	$P(X_1 = x) \downarrow$
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$
1	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{5}{8}$
$P(X_2 = x) \rightarrow$	$\frac{3}{8}$	$\frac{5}{8}$	1

Therefore, $P(X_1 = 0) = P(X_2 = 0) = \frac{3}{8}$ and marginal distributions F_1 of X_1 and F_2 of X_2 are the same. From Sklar's Theorem it is known that:

$$P(X_1 \leq x, X_2 \leq x) = C(P(X_1 \leq x), P(X_2 \leq x))$$

for all x_1, x_2 and some copula C. Since $RanF_1 = RanF_2 = \{0, \frac{3}{8}, 1\}$, the only constraint on C is that $C(\frac{3}{8}, \frac{3}{8}) = \frac{1}{8}$. So, any copula fulfilling this constraint is a copula of (X_1, X_2), and there are infinitely many such copulas.

4 Task 99

4.1 a)

Proof. $P(\max(X,Y) \leq t) = P(X \leq t, Y \leq t) \stackrel{1}{=} C(F_x(t), F_y(t))$ □

4.2 b)

Proof. $P(\min(X,Y) \leq t) = P(X \leq t \cup Y \leq t) = P(X \leq t) + P(Y \leq t) - P(X \leq t \cap Y \leq t) \stackrel{1}{=} F_x(t) + F_y(t) - C(F_x(t), F_y(t))$ □

1. Use Sklar's Theorem:

$F(x_1, \ldots, x_d) = P(X_1 \leq x_1, \ldots, X_d \leq x_d) = C(F_1(x_1), \ldots, F_d(x_d))$

5 Task 102

Please see this Task on separate pdf file, since to whatever reason its content was out of border here.

6 Task 103

Let, $X_1 \sim N(0,1), X_2 \sim N(0,1)$ and $\rho(X_1, X_2) = 0$.
Model A: X_1 and X_2 are independent.
Model B: $(X_1, X_2) = (Z, \epsilon Z)$ where $Z \sim N(0,1)$ and ϵ takes the values 1 and -1 with probabilties $1/2$ and is independent of Z.

6.1 a)

Verification that for model B we have standard normal margins and zero correlation:
$X_1 \sim N(0,1)$ as given. Let's prove that $X_2 \sim N(0,1)$:

$$= F_{X_2}(x) = P(X_2 \leq x) = P(\epsilon Z \leq x) = P(\epsilon = 1)P(Z \leq x) + P(\epsilon = -1)P(-Z \leq x)$$

$$= 1/2\Phi_Z(x) + 1/2P(Z \geq -x) = 1/2\Phi_Z(x) + 1/2(1 - P(Z \leq -x) = 1/2\Phi_Z(x) + 1/2(1 - \Phi_Z(-x))$$
$$= 1/2\Phi_Z(x) + 1/2(1 - (1 - \Phi_Z(x))) = 1/2\Phi_Z(x) + 1/2\Phi_Z(x) = \Phi_Z(x).$$

Let's prove that $\rho(X_1, X_2) = 0$.
Preliminary step:

$$E(\epsilon) = 1P(\epsilon = 1) + (-1)P(\epsilon = -1) = 1/2 + (-1/2) = 0.$$

It follows that
$$E(\epsilon) = 0.$$

In addition, if T, R are independent, then $g(T)$ and $f(R)$ are independent as well. Thus ϵ and Z^2 are independent. Therefore:

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} = \frac{Cov(X_1, X_2)}{1} = E(X_1X_2) - E(X_1)E(X_2)$$
$$= E(Z\epsilon Z) - E(Z)E(\epsilon Z) = E(\epsilon Z^2) - E(\epsilon)E(Z)E(Z)$$
$$= E(\epsilon)E(Z^2) - E(\epsilon)E(Z)E(Z) = 0.$$

6.2 b)

$$C(u_1, u_2) = C(F_{X_1}(x_1), F_{X_2}(x_2)) = P(X_1 \le x_1, X_2 \le x_2) = P(Z \le x_1, \epsilon Z \le x_2)$$
$$= P(\epsilon = 1)P(Z \le x_1, 1Z \le x_2) + P(\epsilon = -1)P(Z \le x_1, -Z \le x_2)$$
$$= \frac{1}{2}P(Z \le x_1, Z \le x_2) + \frac{1}{2}P(Z \le x_1, Z \ge -x_2))$$
$$= \frac{1}{2}(P(Z \le min(x_1, x_2)) + P(-x_2 \le Z \le x_1)$$
$$= \frac{1}{2}(\Phi(min(x_1, x_2)) + max(\Phi(x_1) - \Phi(-x_2), 0))$$
$$= \frac{1}{2}(\Phi(min(x_1, x_2)) + max(\Phi(x_1) - (1 - \Phi(x_2)), 0))$$
$$= \frac{1}{2}(\Phi(min(x_1, x_2)) + max(\Phi(x_1) + \Phi(x_2) - 1, 0))$$
$$= \frac{min(u_1, u_2) + max(u_1 + u_2 - 1, 0)}{2}.$$
$$= \frac{1}{2}max(u_1 + u_2 - 1, 0) + \frac{1}{2}min(u_1, u_2).$$

6.3 c)

For multivariate normal distribution VaR is as follows:

$$VaR = x^t\mu + \phi^{-1}(\alpha)\sqrt{x^t\Sigma x}$$

where where $\phi^{-1}(\alpha)$ is the α-quantile of the standard normal distribution N(0,1) and x is vector of coefficients. In model A X_1 and X_2 are standard normal and uncorrelated, and $\mu = 0$. Therefore

$$VaR = \phi^{-1}(\alpha)\sqrt{(1,1)^t I(1,1)}$$

where I is 2×2 identity matrix (since X_1 and X_2 are uncorrelated). Therefore

$$VaR = \phi^{-1}(\alpha)\sqrt{2}$$

For $\alpha = 0.01$ quantile of the standard normal distribution is -2.3277. Therefore theoretical VaR in model A is

```
VaR <- -2.3277*sqrt(2)

VaR

## [1] -3.291865
```

Let's test model A:

```
number <- 10000

x1 <- rnorm(number)

x2 <- rnorm(number)

correlationA <- cor(x1, x2)

correlationA

## [1] 0.01327564

alpha <- 0.01

VaRcalcA <- quantile(x1+x2, alpha)

VaRcalcA

##        1%
## -3.37147
```

There is a slight difference, supposedly because generated X_1 and X_2 are not uncorrelated.
Let's test model B, starting by generating $X_2 = \epsilon Z$, where ϵ and Z are distributed as defined by the task.

```
P <- c(0.5, 1)

X <- c(-1, 1)

Z <- rnorm(number)
```

```
X_2 <- c()

for (i in (1:number)){
counter <- 1
r <- runif(1)
r
while(r > P[counter]) {
counter <- counter + 1
end
}
Epsilon <- X[counter]

X_2[i] <- Epsilon*Z[i]
counter <- 1
}


correlationB <- cor(Z, X_2)

correlationB

## [1] -0.03368592

alpha <- 0.01

VaRcalcB <- quantile(Z + X_2, alpha)

VaRcalcB

##      1%
## -4.021
```

In model B VaR which is larger by absolute value, than in model A.

7 Task 104

From the copula definition, we know: a copula is the joint distribution of random variables U_i, each of them is marginally uniformly distributed as $U \sim (0,1)$.
$C(u_1, ..., u_n) = P(U_1 \leq u_1, ..., U_n \leq u_n)$
Every $F_i(x)$ is continuous and has a inverse F_i^{-1} such that $F_i(F_i^{-1}(u)) = u$ for all $u \in [0,1]$.
If $U_i = F_i(x_i)$, then U_i has uniform distribution on $(0,1)$.
$P(U_i \leq u) = P(F_i(x_i) \leq u) = P(X_i \leq F_i^{-1}(u)) = F_i(F_i^{-1}(u)) = u$.
The copula in this case will be:
$F(x_1, ..., x_n) = P(X_1 \leq x_1, ..., X_n \leq x_n) = P(U_1 \leq F_1(x_1), ..., U_n \leq F_n(x_n)) = C(F_1(x_1), ..., F_n(x_n)) = C(u_1, ..., u_n)$.
This result is known as the Sklar's Theorem.

8 Task 105

Solution for the task is taken from Springer Series in Statistics Roger B. Nelson book: "An Introduction to Copulas" Second Edition 2006 Springer Science+Business Media, Inc

Lemma 4.1.2. *Let* φ *be a continuous, strictly decreasing function from* $\mathbf{I}$ *to* $[0,\infty]$ *such that* $\varphi(1) = 0$, *and let* $\varphi^{[-1]}$ *be the pseudo-inverse of* φ *defined by* (4.1.2). *Let* C *be the function from* $\mathbf{I}^2$ *to* $\mathbf{I}$ *given by*

$$C(u,v) = \varphi^{[-1]}(\varphi(u)+\varphi(v)). \tag{4.1.3}$$

Then C *satisfies the boundary conditions* (2.2.2a) *and* (2.2.2b) *for a copula.*

$$C(u,0) = 0 = C(0,v) \tag{2.2.2a}$$

$$C(u,1) = u \text{ and } C(1,v) = v; \tag{2.2.2b}$$

Proof. $C(u,0) = \varphi^{[-1]}(\varphi(u)+\varphi(0)) = 0$, and $C(u,1) = \varphi^{[-1]}(\varphi(u)+\varphi(1)) = \varphi^{[-1]}(\varphi(u)) = u$. By symmetry, $C(0,v) = 0$ and $C(1,v) = v$. ☐

In the following lemma, we obtain a necessary and sufficient condition for the function C in (4.1.3) to be 2-increasing.

Lemma 4.1.3. *Let* φ, $\varphi^{[-1]}$ *and* C *satisfy the hypotheses of Lemma* 4.1.2. *Then* C *is* 2*-increasing if and only if for all* v *in* **I**, *whenever* $u_1 \le u_2$,

$$C(u_2,v) - C(u_1,v) \le u_2 - u_1. \tag{4.1.4}$$

Proof. Because (4.1.4) is equivalent to $V_C([u_1,u_2]\times[v,1]) \ge 0$, it holds whenever C is 2-increasing. Hence assume that C satisfies (4.1.4). Choose v_1, v_2 in **I** such that $v_1 \le v_2$, and note that $C(0,v_2) = 0 \le v_1 \le v_2 = C(1,v_2)$. But C is continuous (because φ and $\varphi^{[-1]}$ are), and thus there is a t in **I** such that $C(t, v_2) = v_1$, or $\varphi(v_2) + \varphi(t) = \varphi(v_1)$. Hence

$$\begin{aligned} C(u_2,v_1) - C(u_1,v_1) &= \varphi^{[-1]}(\varphi(u_2)+\varphi(v_1)) - \varphi^{[-1]}(\varphi(u_1)+\varphi(v_1)), \\ &= \varphi^{[-1]}(\varphi(u_2)+\varphi(v_2)+\varphi(t)) - \varphi^{[-1]}(\varphi(u_1)+\varphi(v_2)+\varphi(t)), \end{aligned}$$

$$= C\left(C(u_2,v_2),t\right) - C\left(C(u_1,v_2),t\right),$$
$$\leq C(u_2,v_2) - C(u_1,v_2),$$

so that C is 2-increasing. □

We are now ready to state and prove the main result of this section.

Theorem 4.1.4. *Let φ be a continuous, strictly decreasing function from* $\mathbf{I}$ *to* $[0,\infty]$ *such that* $\varphi(1) = 0$, *and let* $\varphi^{[-1]}$ *be the pseudo-inverse of* φ *defined by* (4.1.2). *Then the function C from* $\mathbf{I}^2$ *to* $\mathbf{I}$ *given by* (4.1.3) *is a copula if and only if* φ *is convex.*

Proof (Alsina et al. 2005). We have already shown that C satisfies the boundary conditions for a copula, and as a consequence of the preceding lemma, we need only prove that (4.1.4) holds if and only if φ is convex [note that φ is convex if and only if $\varphi^{[-1]}$ is convex]. Observe that (4.1.4) is equivalent to

$$u_1 + \varphi^{[-1]}\left(\varphi(u_2) + \varphi(v)\right) \leq u_2 + \varphi^{[-1]}\left(\varphi(u_1) + \varphi(v)\right)$$

for $u_1 \leq u_2$, so that if we set $a = \varphi(u_1)$, $b = \varphi(u_2)$, and $c = \varphi(v)$, then (4.1.4) is equivalent to

$$\varphi^{[-1]}(a) + \varphi^{[-1]}(b+c) \leq \varphi^{[-1]}(b) + \varphi^{[-1]}(a+c), \qquad (4.1.5)$$

where $a \geq b$ and $c \geq 0$. Now suppose (4.1.4) holds, i.e., suppose that $\varphi^{[-1]}$ satisfies (4.1.5). Choose any s, t in $[0,\infty]$ such that $0 \leq s < t$. If we set $a = (s+t)/2$, $b = s$, and $c = (t-s)/2$ in (4.1.5), we have

$$\varphi^{[-1]}\left(\frac{s+t}{2}\right) \leq \frac{\varphi^{[-1]}(s) + \varphi^{[-1]}(t)}{2}. \tag{4.1.6}$$

Thus $\varphi^{[-1]}$ is midconvex, and because $\varphi^{[-1]}$ is continuous it follows that $\varphi^{[-1]}$ is convex.

In the other direction, assume $\varphi^{[-1]}$ is convex. Fix a, b, and c in $\mathbf{I}$ such that $a \geq b$ and $c \geq 0$; and let $\gamma = (a-b)/(a-b+c)$. Now $a = (1-\gamma)b + \gamma(a+c)$ and $b+c = \gamma b + (1-\gamma)(a+c)$, and hence

$$\varphi^{[-1]}(a) \leq (1-\gamma)\varphi^{[-1]}(b) + \gamma\varphi^{[-1]}(a+c)$$

and

$$\varphi^{[-1]}(b+c) \leq \gamma\varphi^{[-1]}(b) + (1-\gamma)\varphi^{[-1]}(a+c).$$

Adding these inequalities yields (4.1.5), which completes the proof. □

9 Task 106

The Gumbel family of copulas has

$$C_\theta^{Gu}(u,v) = exp(-((-ln(u))^\theta + (-ln(v))^\theta)^{\frac{1}{\theta}})$$

where $\theta \geq 1$.
Let $\psi(t) = e^{-t^{\frac{1}{\theta}}}$. Therefore for $0 < u \leq 1$, $\psi^{-1}(u)$ in the sense of definition $\psi^{-1}(u) = inf\{t : \psi(t) = u\}$ is $\psi^{-1}(u) = (-ln(u))^\theta$. Therefore $C_\theta^{Gu}(u,v)$ can be expressed in the form

$$C(u,v) = \psi(\psi^{-1}(u) + \psi^{-1}(v))$$

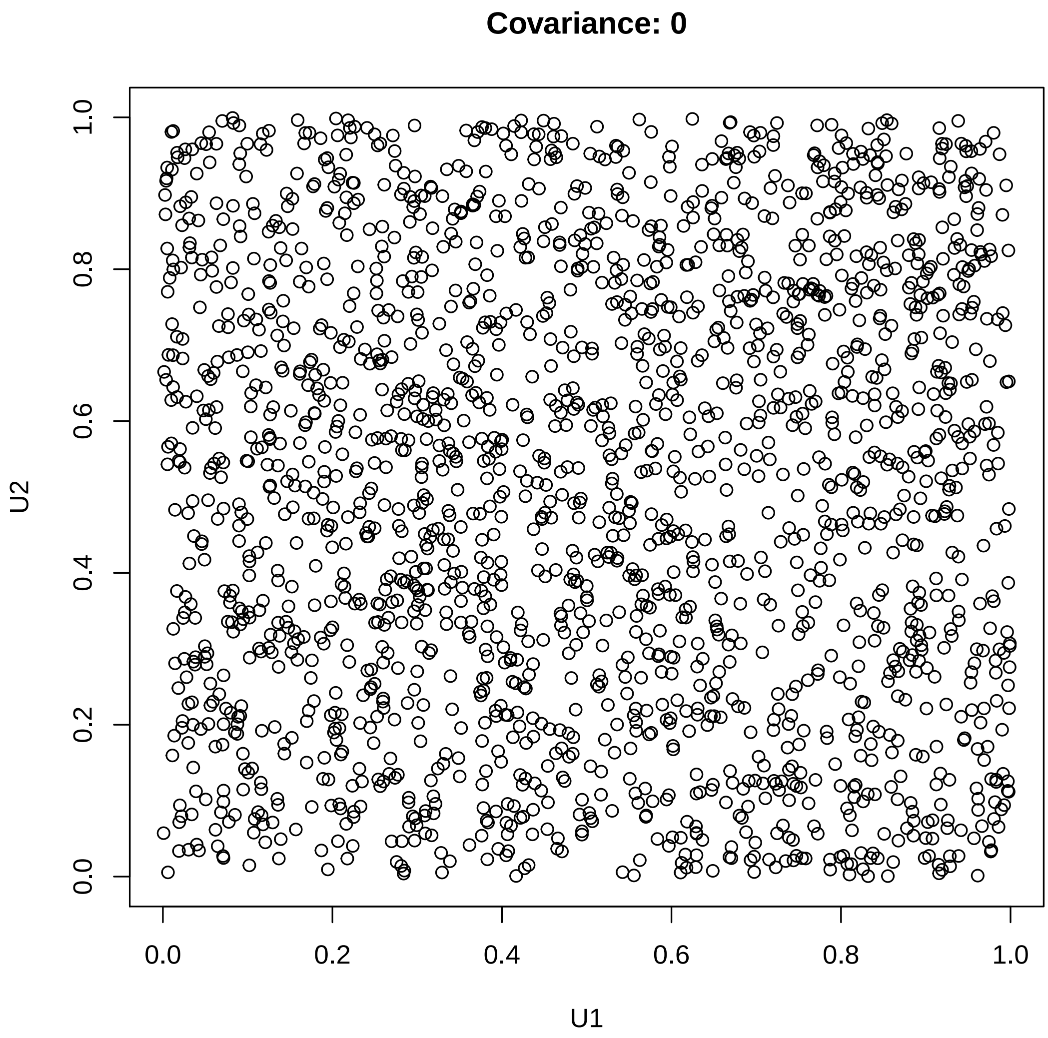
where $\psi(t) = e^{-t^{\frac{1}{\theta}}}$ is convex, for $t \geq 0$ maps to $[0, 1]$, continious and non-increasing with $\psi(0) = 1$ and $\lim_{x\to\infty} \psi(t) = 0$.
Therefore Gumbel family of copulas is Archimedean copula with generator $\psi = e^{-t^{\frac{1}{\theta}}}$.
If $\theta \to 1$, $C_\theta^{Gu}(u,v)$ converges to

$$exp(-(-ln(u)) + (-ln(v))) = exp(-(-ln(u))) * exp((-ln(v)))$$

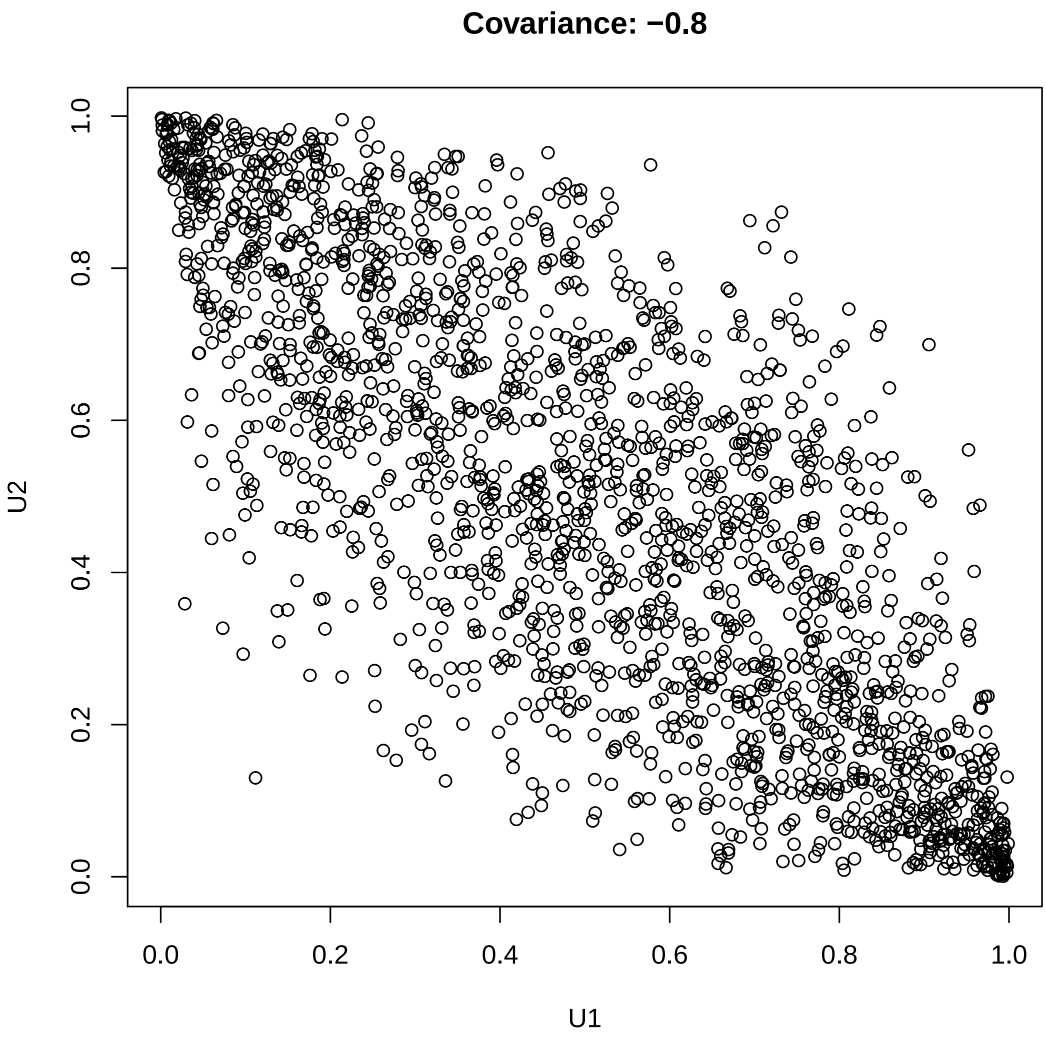
which is independence copula.
If $\theta \to \infty$, $C_\theta^{Gu}(u,v)$ converges to the comonotonicity copula.

10 Task 107

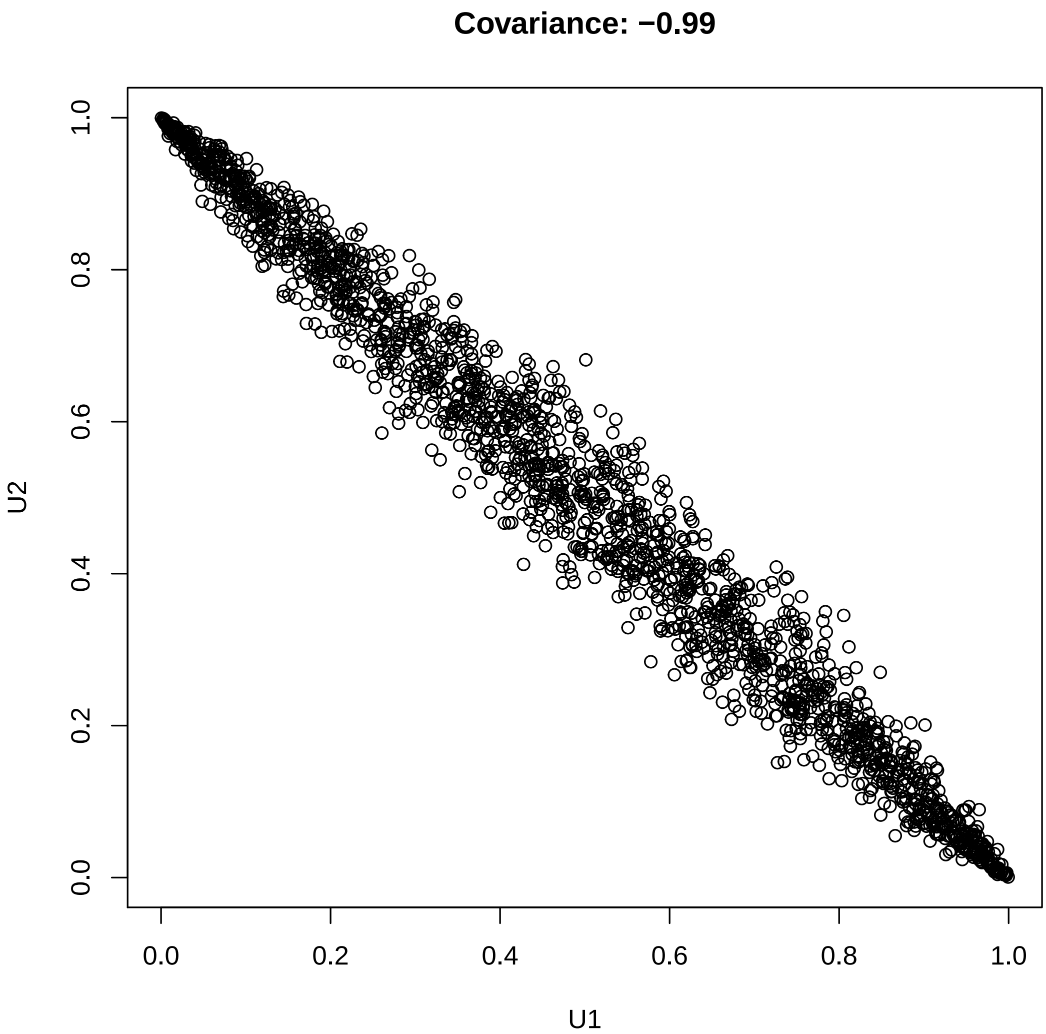
```
# Task 107

library("MASS")

simulate_from_gaussian_copula <- function(
   number_of_simulations, mu, Sigma)
{
  multivar_norm_dist_matrix <- mvrnorm(
     number_of_simulations, mu, Sigma)
  U <- pnorm(multivar_norm_dist_matrix)
  title <- paste('Covariance:', Sigma[1,2])
  plot(U[,1], U[,2], xlab = 'U1', ylab = 'U2', main =
     title)
  return(U)

}


number_of_simulations <- 2000


# Cov = 0

mu <- c(0,0)

Sigma <- diag(2)

U <- simulate_from_gaussian_copula(number_of_simulations,
   mu, Sigma)
```




```
# Cov = -0.8

Sigma <- matrix(c(1, -0.8, -0.8, 1), nrow = 2, ncol = 2)

U <- simulate_from_gaussian_copula(number_of_simulations,
   mu, Sigma)
```




```
# Cov = -0.99

Sigma <- matrix(c(1, -0.99, -0.99, 1), nrow = 2, ncol = 2)

U <- simulate_from_gaussian_copula(number_of_simulations,
   mu, Sigma)
```

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U <- simulate_from_gaussian_copula(number_of_simulations,
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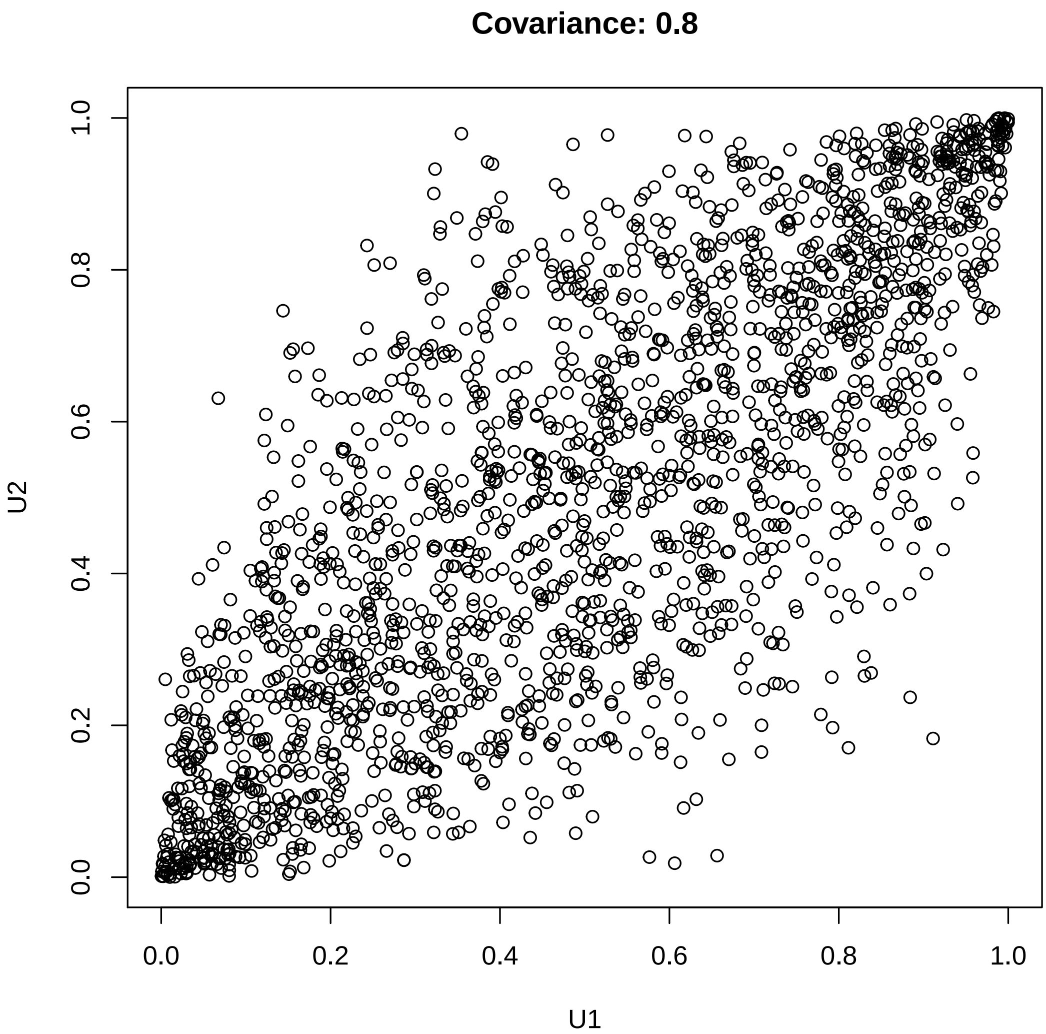
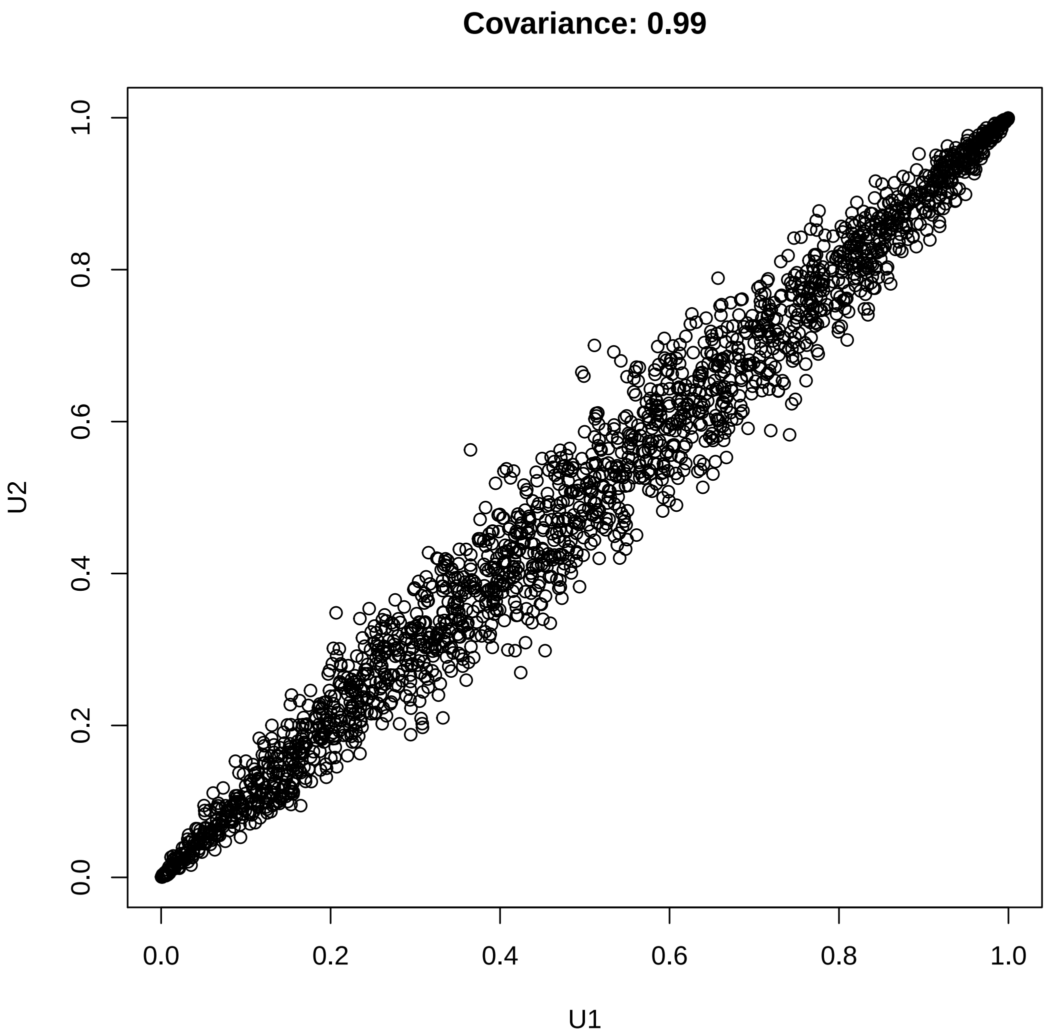


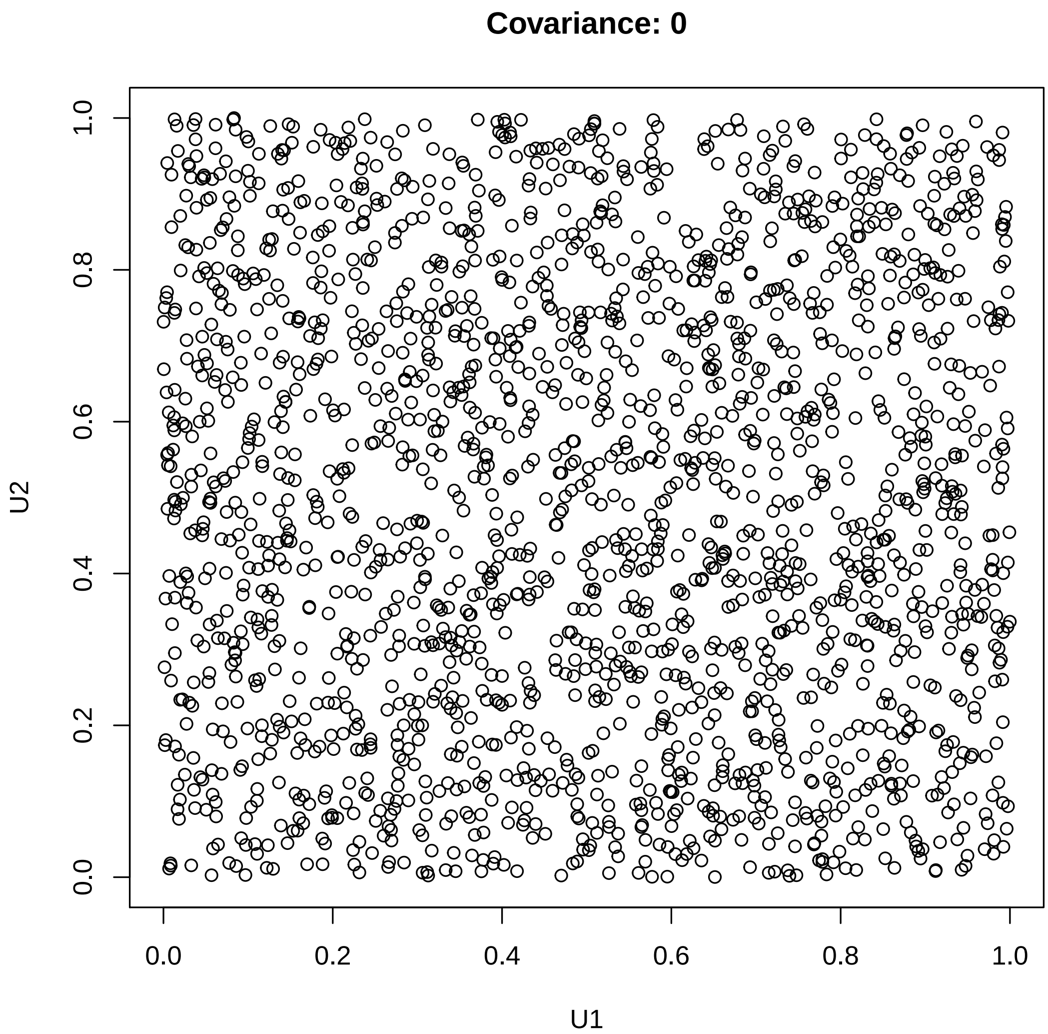

```
# Cov = 0.99

Sigma <- matrix(c(1, 0.99, 0.99, 1), nrow = 2, ncol = 2)

U <- simulate_from_gaussian_copula(number_of_simulations,
   mu, Sigma)
```



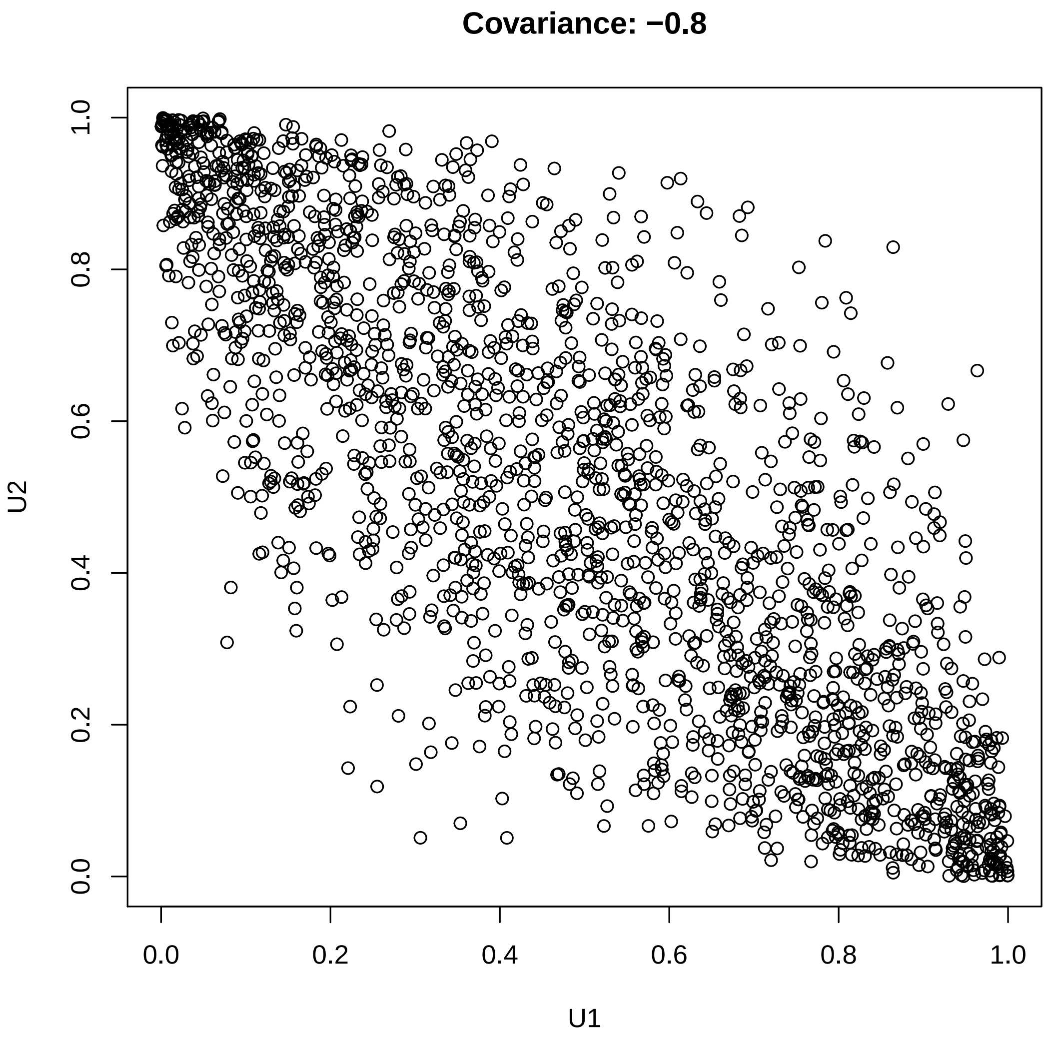
If covariance tends to + 1 or -1 scatter plot tends to a line. Moreover if covariance is positive both U1 and U2 move in the same direction when negative correlation means that U1 and U2 move in opposite directions

11 Task 108

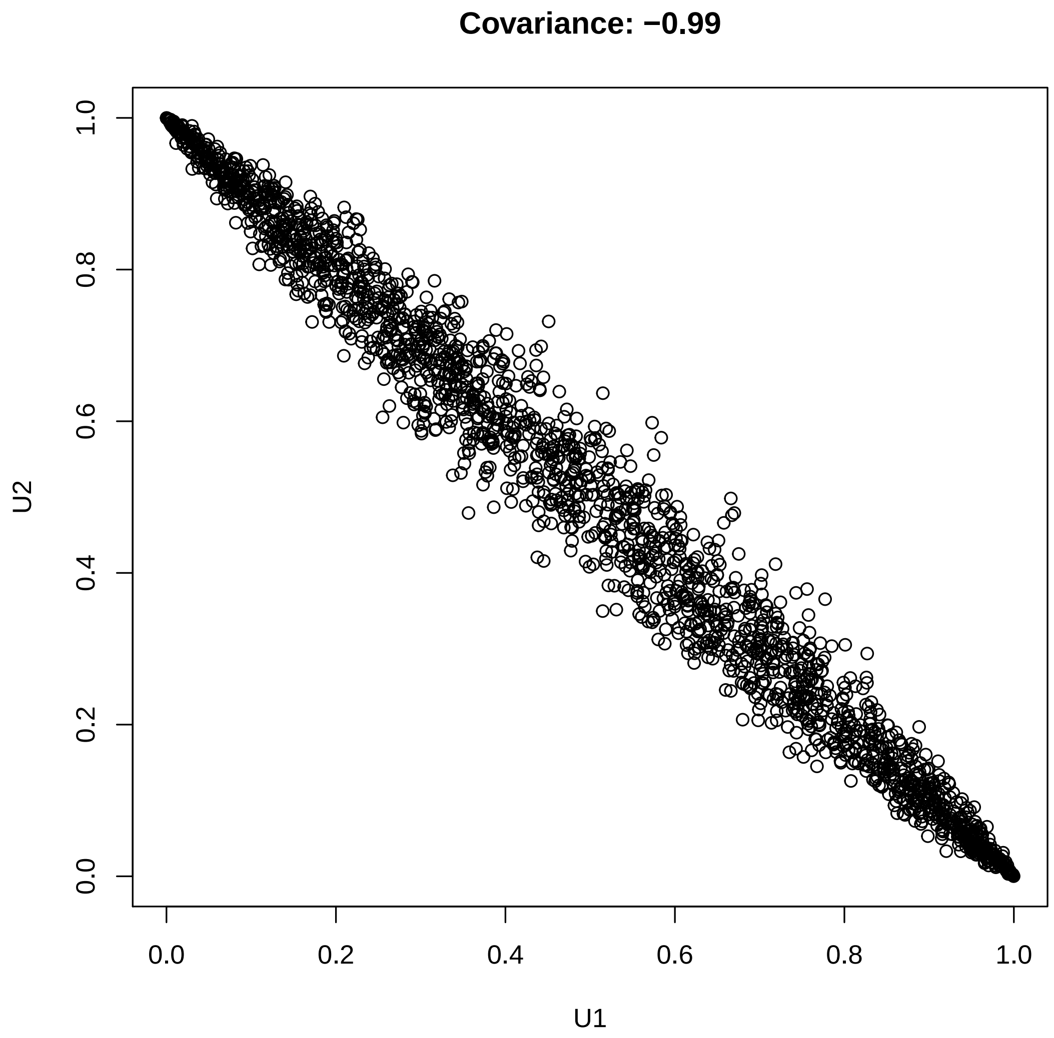
```
# Task 107

library("MASS")

simulate_from_gaussian_copula <- function(
   number_of_simulations, mu, Sigma)
{
  multivar_norm_dist_matrix <- mvrnorm(
     number_of_simulations, mu, Sigma)
  U <- pnorm(multivar_norm_dist_matrix)
  title <- paste('Covariance:', Sigma[1,2])
  plot(U[,1], U[,2], xlab = 'U1', ylab = 'U2', main =
     title)
  return(U)

}


number_of_simulations <- 2000


# Cov = 0

mu <- c(0,0)

Sigma <- diag(2)

U <- simulate_from_gaussian_copula(number_of_simulations,
   mu, Sigma)
```



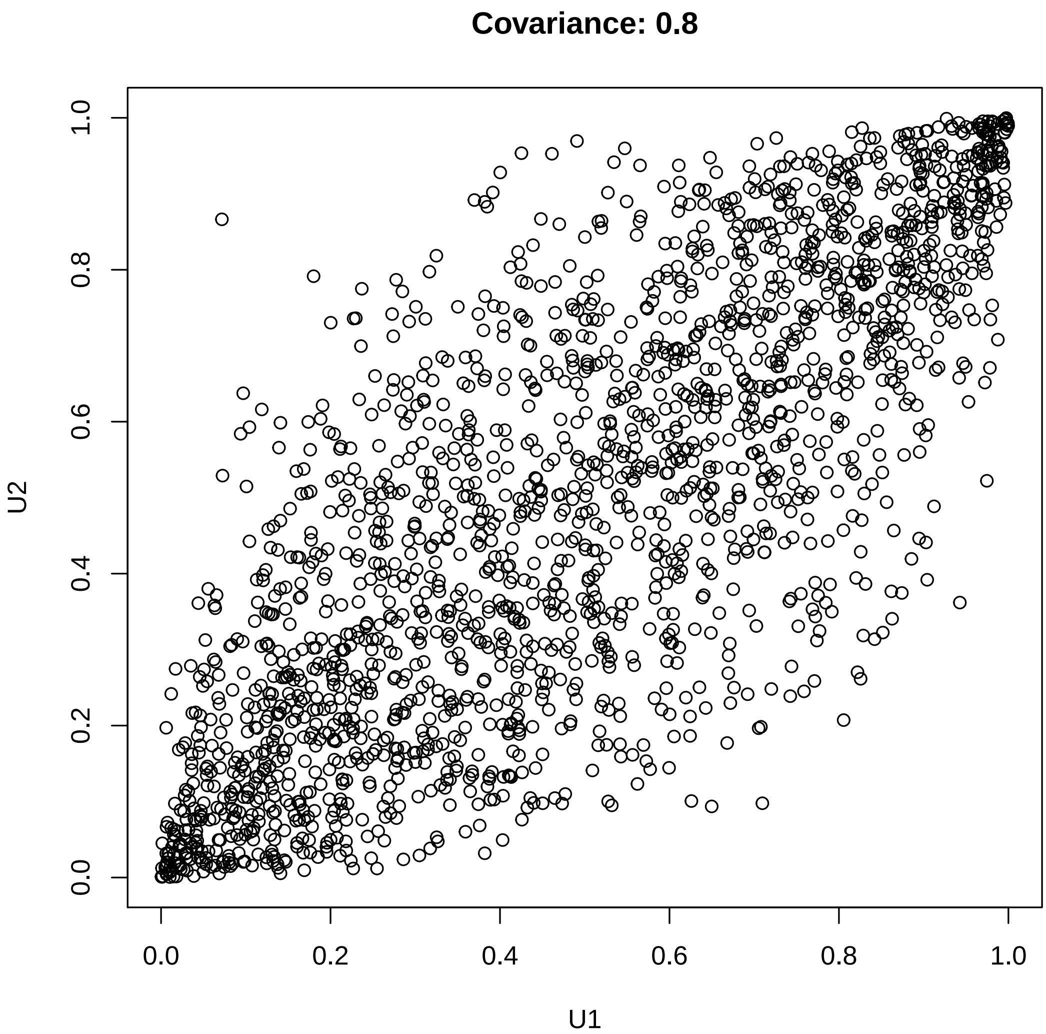

```
# Cov = -0.8

Sigma <- matrix(c(1, -0.8, -0.8, 1), nrow = 2, ncol = 2)

U <- simulate_from_gaussian_copula(number_of_simulations,
   mu, Sigma)
```



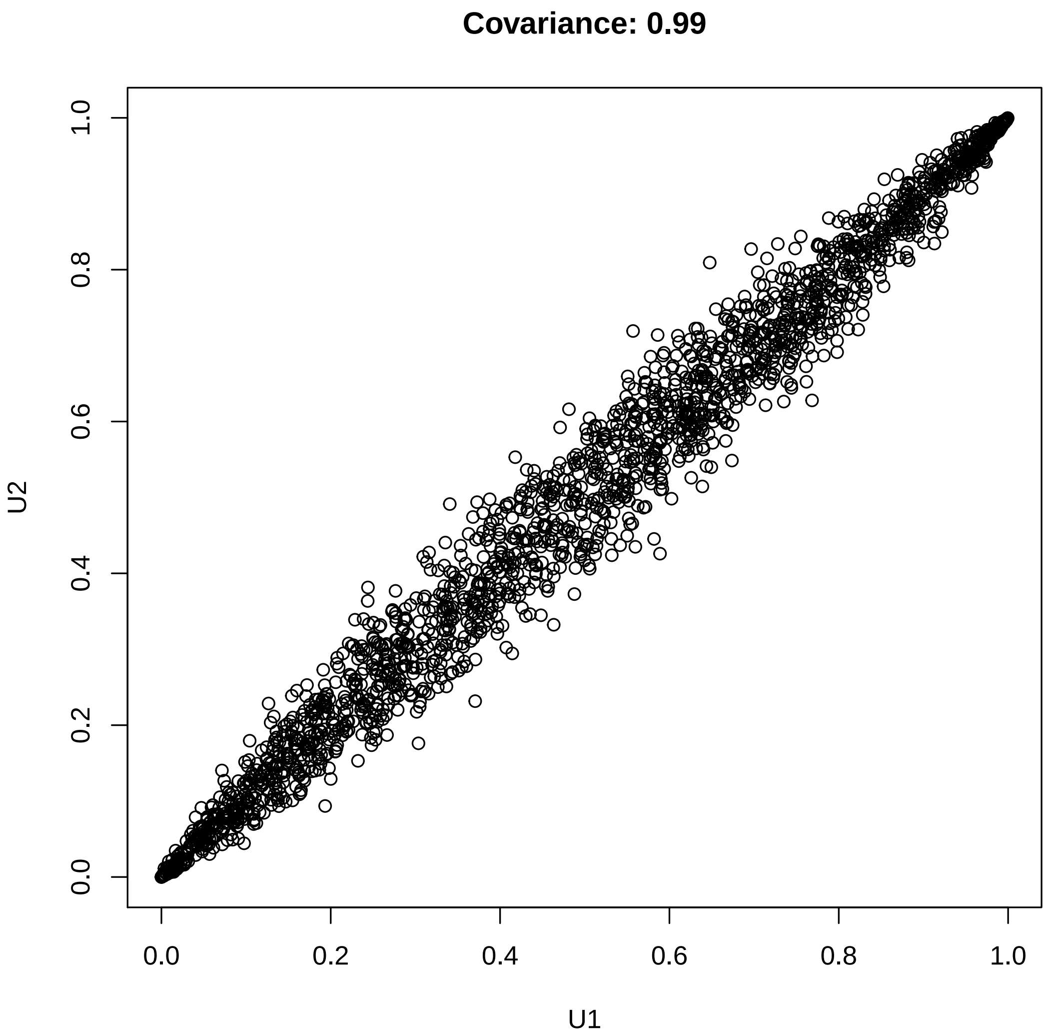

```
# Cov = -0.99

Sigma <- matrix(c(1, -0.99, -0.99, 1), nrow = 2, ncol = 2)

U <- simulate_from_gaussian_copula(number_of_simulations,
   mu, Sigma)
```




```
# Cov = 0.8

Sigma <- matrix(c(1, 0.8, 0.8, 1), nrow = 2, ncol = 2)

U <- simulate_from_gaussian_copula(number_of_simulations,
   mu, Sigma)
```




```
# Cov = 0.99

Sigma <- matrix(c(1, 0.99, 0.99, 1), nrow = 2, ncol = 2)

U <- simulate_from_gaussian_copula(number_of_simulations,
   mu, Sigma)
```



If covariance tends to + 1 or -1 scatter plot tends to a line. Moreover if covariance is positive both U1 and U2 move in the same direction when negative correlation means that U1 and U2 move in opposite directions

Moreover, increasing in degrees of freedom makes the whole picture more similar to the standard normal distribution case. In this case the tendency to a line is more obvious.