Homework 7

Team 8

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1 Task 102

$X, \epsilon_1, \epsilon_2$ are independent. $X \sim N(0,1), P(\epsilon_i = -1) = P(\epsilon_i = 1) = 1/2$. $X_1 = \epsilon_1 X, X_2 = \epsilon_1 X$.

1.1 a)

Prove $X_1 \sim N(0,1), X_2 \sim N(0,1), \rho(X_1, X_2) = 0$.
$X_1 \sim N(0,1):$

$$F_{X_1}(x) = P(X_1 \leq x) = P(\epsilon_1 X \leq x) = P(\epsilon_1 = 1)P(X \leq x) + P(\epsilon_1 = -1)P(-X \leq x)$$

$$= 1/2\Phi_X(x) + 1/2P(X \geq -x) = 1/2\Phi_X(x) + 1/2(1 - P(X \leq -x) = 1/2\Phi_X(x) + 1/2(1 - \Phi_X(-x))$$
$$= 1/2\Phi_X(x) + 1/2(1 - (1 - \Phi_X(x))) = 1/2\Phi_X(x) + 1/2\Phi_X(x) = \Phi_X(x).$$

We see $X_1 \sim N(0,1)$ and the same is true for X_2.

Let's prove that $\rho(X_1, X_2) = 0$.
Preliminary step:

$$E(\epsilon_1) = 1P(\epsilon_1 = 1) + (-1)P(\epsilon_1 = -1) = 1/2 + (-1/2) = 0.$$

It follows that
$$E(\epsilon_1) = E(\epsilon_2) = 0.$$

In addition, if Y, Z are independent, then $g(Y)$ and $f(Z)$ are independent as well. Thus ϵ_1, ϵ_2 and X^2 are independent. Therefore:

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} = \frac{Cov(X_1, X_2)}{1} = E(X_1X_2) - E(X_1)E(X_2)$$

$$= E(\epsilon_1 X \epsilon_2 X) - E(\epsilon_1 X)E(\epsilon_2 X) = E(\epsilon_1\epsilon_2 X^2) - E(\epsilon_1)E(\epsilon_2)E(X)E(X)$$
$$= E(\epsilon_1)E(\epsilon_2)E(X^2) - E(\epsilon_1)E(\epsilon_2)E(X)E(X) = 0.$$

1.2 b)

We can apply the following theorem:
X_1 and X_2 are independent, iff $P(X_1|X_2) = P(X_1)$.
We will find a counterexample which proves dependency.

$$P(X_1 < 0|X_2 = 0) = P((X_1 < 0|X_2 = 0)|\epsilon_1 = 1)P(\epsilon_1 = 1) + P((X_1 < 0|X_2 = 0)|\epsilon_1 = -1)P(\epsilon_1 = -1))$$
$$0.5 * P(X < 0|\epsilon_2 X = 0) + 0.5 * P(-X < 0|\epsilon_2 X = 0)$$
$$= 0.5 * P(X < 0|X = 0) + 0.5 * P(X > 0|X = 0) = 0 + 0 = 0.$$

(We use the fact that since ϵ_2 equals 1 or -1, $\epsilon_2 X = 0$ iff $X = 0$, and value of ϵ_2 has no influence on it).
But
$$P(X_1 < 0) = \Phi(0) = 1/2.$$
We see that here $P(X_1|X_2) \neq P(X_1)$.
Thus X_1 and X_2 are not independent!

1.3 c)

$$C(u_1, u_2) = C(F_{X_1}(x_1), F_{X_2}(x_2)) = P(X_1 \le x_1, X_2 \le x_2) = P(\epsilon_1 X \le x_1, \epsilon_2 X \le x_2)$$
$$= P(\epsilon_1 = 1)P(\epsilon_2 = 1)P(1X \le x_1, 1X \le x_2) + P(\epsilon_1 = -1)P(\epsilon_2 = -1)P(-X \le x_1, -X \le x_2)$$
$$+P(\epsilon_1 = 1)P(\epsilon_2 = -1)P(1X \le x_1, -X \le x_2) + P(\epsilon_1 = -1)P(\epsilon_2 = 1)P(-X \le x_1, 1X \le x_2)$$
$$= \frac{1}{4}P(1X \le x_1, 1X \le x_2) + \frac{1}{4}P(-X \le x_1, -X \le x_2) + \frac{1}{4}P(1X \le x_1, -X \le x_2) + \frac{1}{4}P(-X \le x_1, 1X \le x_2)$$

$$= \frac{1}{4}(P(X \le min(x_1, x_2)) + P(X \ge -x_1, X \ge -x_2) + P(X \le x_1, X \ge -x_2) + P(X \ge -x_1, X \le x_2))$$

$$= \frac{1}{4}(\Phi(min(x_1, x_2)) + P(X \ge max(-x_1, -x_2)) + P(-x_2 \le X \le x_1) + P(-x_1 \le X \le x_2))$$

$$= \frac{1}{4}(\Phi(min(x_1, x_2)) + (1 - P(X \le max(-x_1, -x_2))) + max(\Phi(x_1) - \Phi(-x_2), 0) + max(\Phi(x_2) - \Phi(-x_1), 0))$$

$$= \frac{1}{4}(\Phi(min(x_1, x_2)) + 1 - \Phi(max(-x_1, -x_2)) + max(\Phi(x_1) - (1 - \Phi(x_2)), 0) + max(\Phi(x_2) - (1 - \Phi(x_1)), 0))$$

$$= \frac{1}{4}(\Phi(min(x_1, x_2)) + 1 - (1 - \Phi(-max(-x_1, -x_2))) + max(\Phi(x_1) - 1 + \Phi(x_2), 0) + max(\Phi(x_2) - 1 + \Phi(x_1), 0))$$

$$= \frac{1}{4}(\Phi(min(x_1, x_2)) + 1 - 1 + \Phi(-max(-x_1, -x_2)) + max(\Phi(x_1) + \Phi(x_2) - 1, 0) + max(\Phi(x_1) + \Phi(x_2) - 1, 0))$$

$$= \frac{1}{4}(\Phi(min(x_1, x_2)) + \Phi(-max(-x_1, -x_2)) + 2max(\Phi(x_1) + \Phi(x_2) - 1, 0))$$

$$= \frac{1}{4}(\Phi(min(x_1, x_2)) + \Phi(min(x_1, x_2)) + 2max(\Phi(x_1) + \Phi(x_2) - 1, 0))$$

$$= \frac{1}{4}(2\Phi(min(x_1, x_2)) + 2max(\Phi(x_1) + \Phi(x_2) - 1, 0))$$

$$= \frac{min(u_1, u_2) + max(u_1 + u_2 - 1, 0)}{2}.$$