Foundations of Representation Theory

Lecture Notes in the Winter Term 2018/19

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These are my personal lecture notes for the lecture *Foundations of Representation Theory* held by Dr. Hans Franzen at the University of Bonn in the winter term 2018/19.

I try to update them on my website, https://pankratius.github.io.
The authors labels his own comments and additions in purple.

The book [Alu09] is used by the author for further references, and highly recommended.

Conventions: In this lecture, rings will always be unital, and ring homomorphisms f always fulfill $f(1) = 1$. Rings do *not* have to be commutative.

1.1 Algebras - Basics

Let k be a commutative ring.

Definition 1.1.1. A k-**algebra** A is a ring A, together with a structure of a k-module on A, such that

$$\text{for all } a, b \in A, \lambda \in K: \ (\lambda a)b = a(\lambda b) = \lambda(ab) \tag{$*$}$$

Definition 1.1.2. Let A, B be k-algebras. A **homomorphism of algebras** is a map $f : A \to B$ that is both k-linear and a ring homomorphism.

Remark 1.1.3. Let A be a ring. Define

$$Z(A) := \{a \in A \mid \forall b \in A : ab = ba\},$$

which is a commutative subring and is called the **center** of A.

Remark 1.1.4. Let A be a ring. Giving a k-algebra structure on A is the same as giving a ring homomorphism $k \to Z(A)$. More precisley:

i) If A is a k-algebra, then $p : k \to A$, $\lambda \mapsto \lambda 1$ satisfies $\operatorname{Im} p \subseteq Z(A)$ and is a ring homomorphism. (the first statement follows from ($*$) and the second one from the fact that A has a k-module structure).

ii) Let $\varphi : k \to Z(A)$ be a ring homomorphism. Define

$$\lambda a := \varphi(\lambda) a,$$

for all $\lambda \in k$. This defines a k-algebra structure on A (($*$) holds since $\operatorname{Im}(\varphi) \subseteq Z(A)$).

iii) Let A, B be k-algebras and $f : A \to B$ a homomorphism of rings. Then f is a homomorphism of k-algebras if and only if the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "f"] & & B \\
Z(A) \arrow[u, hook] & & Z(B) \arrow[u, hook] \\
& k \arrow[ul, "\phi_A"] \arrow[ur, "\phi_B"'] &
\end{tikzcd} .$$

Example 1.1.5. i) Let V be a k-module. Then $\operatorname{End}_k(V)$ has a ring structure given by

$$\operatorname{End}_k(V) \times \operatorname{End}_k(V) \to \operatorname{End}_k(V), \ (\phi, \psi) \mapsto \varphi \circ \psi.$$

Then $\mathrm{End}_k(V)$ is both a ring and a k-module, and becomes a k-algebra via

$$\varphi : k \to \mathrm{End}_k(V),\ \lambda \mapsto \lambda\,\mathrm{id}\,.$$

Note that $\mathrm{Im}\,\varphi \subseteq Z(A)$. If k is a field, then $Z(\mathrm{End}_k(V)) = \{\lambda\,\mathrm{id} \mid \lambda \in k\}$.

ii) Take $V = k^n$ (free module of rank n). Then $\mathrm{End}_k(V) \cong \mathrm{M}_n(k)$. Define

$$T_u := \{\varphi \in \mathrm{M}_n(k) \mid \varphi \text{ is upper triangular}\},$$

i.e. T_u presevers flags in k^n. Then T_u is a **subalgebra** of $\mathrm{M}_n(k)$, i.e. is both a subring and a k-submodule of the original algebra.

iii) Let G be a group. Define the **group algebra** $k[G]$ of G as follows:

- As k-module, is defined as the free module on G,

$$k[G] := k^{(G)} = \{\sum_{g\in G} \lambda_g g \mid \lambda_g \in k,\ \lambda_g \neq 0 \text{ for only finitely many } g \in G\}.$$

- Multiplication: Let $a := \sum \lambda_g g$, $b = \sum \mu_g g$ and define:

$$ab := \sum_{g\in G, h\in G} \lambda_g \mu_h (gh) = \sum_{j\in G} \left(\sum_{gh=j} \lambda_g \mu_h \right) j.$$

This multiplication is associative, k-bilinear, distributive and $1|_{k[G]} = e$. In addition, $(*)$ is satisfied.

1.2 Quivers - Basics

Definition 1.2.1. A **quiver** is a „directed graph". Formally, a quiver is a quadruple (Q_0, Q_1, s, t) consiting of sets Q_0 (**vertices**) and Q_1 (**arrows**) and maps $s : Q_1 \to Q_0$, $t : Q_1 \to Q_0$. For $\alpha \in Q_1$, we call $s(\alpha)$ the **source** of α and $t(\alpha)$ the **target** of α:

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

Example 1.2.2. i) $Q = (\{1\}, \emptyset, ...)$ is visualized as: 1

ii) $Q = (\{1\}, \{\alpha\}, ...)$ is visualized as
$$\begin{tikzcd} 1 \arrow[loop above] \end{tikzcd}$$

iii) $Q = (\{1,2\}, \{\alpha, \beta\}, s(\alpha) = s(\beta) = 1, t(\alpha) = t(\beta) = 2)$ is visualized as $1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$

Definition 1.2.3. Let Q be a quiver sucht that both Q_0 and Q_1 are finite.

i) Let $\ell \in \mathbb{Z}_{\geq 1}$. A **path** of length ℓ is a sequence $\alpha_\ell ..., \alpha_1$ of arrows, such that $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq \ell - 1$,

$$\circ \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_2} \circ \ldots \xrightarrow{\alpha_\ell} \circ.$$

Define Q_ℓ to be the set of all paths of length ℓ.
Let $p : \alpha_\ell ... \alpha_1$ be a path. Define $s(p) := s(\alpha_1)$ and $t(p) := s(\alpha_\ell)$.
Formally define Q_0 to be the set of all paths of length zero. Denote by ε_i for $i \in Q_0$ the constant path at i. ε_i is called a **lazy path**. We set $s(\varepsilon_i) = t(\varepsilon_i) := i$.

ii) Let $p = \alpha_\ell ... \alpha_i$ and $q = \beta_m ... \beta_1$ be paths of length ℓ and m respectivley, with $\ell, m \geq 1$. If $t(p) = s(q)$, then set $q \circ p := \beta_m .... \beta_1 \alpha_\ell ... \alpha_1$. This is a path of length $\ell + m$.
For p a path of length ≥ 0 and ε_i a lazy path:

- if $t(p) = i$, set $\varepsilon_i \circ p := p$,
- if $s(p) = i$, set $p \circ \varepsilon_i := p$.

In all others cases, the composition is not defined.

iii) Define

$$Q_* := \bigcup_{\ell \geq 0} Q_\ell,$$

the set of all paths. Define the **path-algebra** kQ:

- As a k-module, $kQ := k^{(Q_*)}$.
- Multiplication: Let $a = \sum \lambda_p p,\ b = \sum \mu_p p$. Define

$$ab := \sum_{p,q \in Q_*} \lambda_p \mu_q (p \cdot q),$$

where

$$p \cdot q := \begin{cases} p \circ q, & \text{if it is defined, i.e } t(q) = s(p) \\ 0, & \text{else} \end{cases}.$$

The multiplication is associative (due to the associativity of the composition of paths) and k-bilinear by definition. In addition, distributivity and ($*$) are fulfilled.
- The identity is given by $\sum \varepsilon_i$.

Example 1.2.4. i) $Q = 1$, then $kQ = k$.

ii) $Q = $ $\begin{tikzcd} 1 \arrow[loop above] \end{tikzcd}$, then $Q_* = \{\alpha^n \mid n \geq 0\}$ and $kQ = k[t]$.

iii) $Q = $ $\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}$. Then $Q_* = \{\varepsilon_1, \varepsilon_2, \alpha^n, \beta^n \mid n \geq 0\}$ and

$$kQ = k\varepsilon_1 \oplus k\varepsilon_2 \oplus k\alpha \oplus k\beta.$$

A multiplication table is given by

	ε_1	ε_2	α	β
ε_1	ε_1	0	0	0
ε_2	0	ε_2	α	β
α	α	0	0	0
β	β	0	0	0

Lemma 1.2.5. *Let Q be a finite quiver, k a field. Then the following are equivalent:*

i) *Q contains no cycles.*

ii) $\dim_k kQ < \infty$.

Lemma 1.2.6. *Let k be a field, A a k-algebra and $n := \dim(A) < \infty$. Then there exists an injective homomorphism of k-algebras $\varphi : A \to \mathrm{M}_n(k)$.*

Proof. By choosing a basis of A, we get an isomorphism $\mathrm{End}_k(A) \cong \mathrm{M}_n(k)$. So it suffices to find an injective homomorphism of k-algebras $\varphi : A \to \mathrm{End}_k(A)$.
Consider

$$\varphi : A \to \mathrm{End}_k(A), \;\; \varphi(a) : A \to A, b \mapsto ab.$$

- $\varphi(a)$ is k-linear for all a by the distributivity in A and the condition $(*)$.
- φ is k-linear by the distributivity in A and the condition $(*)$.
- Let $a, a' \in A$. Then

$$\varphi(aa')(b) = (aa')(b) = a(a'b) = \big(\varphi(a) \circ \varphi(a')\big)(b).$$

Hence φ is indeed a homomorphism of k-algebras.
To show that φ is injective, let $a \in \ker \varphi$, hence $ab = 0$ for all $b \in A$. But in particular, $0 = a1 = a$. □

End of Lecture 1

Definition 1.2.7. Let A be a k-algebra. Then the **opposite algebra** A^{op} is A (as a k-module), and the multiplication is defined as

$$a \cdot_{A^{\mathrm{op}}} b = b \cdot_A b.$$

Example 1.2.8. Let Q be a quiver, and define $Q^{\mathrm{op}} := (Q_0, Q_1, s^{\mathrm{op}}, t^{\mathrm{op}})$, where $s^{\mathrm{op}}(\alpha) := t(\alpha)$ and $t^{\mathrm{op}}(\alpha) := s(\alpha)$. Then $kQ^{\mathrm{op}} = k(Q^{\mathrm{op}})$

1.3 Modules - Basics

Definition 1.3.1. Let A be a k-algebra. A **left A-module** M is a k-module M together with a map $A \times M \to M, (a, x) \mapsto ax$, such that:

$$a(x + y) = ax + ay \tag{L1}$$

$$(a + b)x = ax + bx \tag{L2}$$

$$a(bx) = (ab)x \tag{L3}$$

$$1_A x = x \tag{L4}$$

$$(\lambda a)x = \lambda(ax) = a(\lambda x), \tag{L5}$$

for all $a, b \in A$, $x, y \in M$ and $\lambda \in k$. If A is a left A-module, we denote this as ${}_A M$. A **right A-module** is defined analogous, where (L3) becomes $(xa)b = x(ab)$. If A is a right A-module, we denote this by A_M.

Remark 1.3.2. A right A-module is the same as a left A^{op}-module.

Definition 1.3.3. Let A be a k-algebra, and M, N left A-modules. A **homomorphism of left A-modules** $f : M \to N$ is a k-linear map such that

$$f(ax) = af(x)$$

for all $a \in A$ and $x \in M$.
Define the set of all left A-algebra homomorphisms as

$$\hom_A(M, N) := \hom_A({}_A M, {}_A M) := \{f : M \to N \mid f \text{ is a homomorphism of left } A\text{-modules}\}.$$

A homomorphism of left A-modules is an **isomorphism** if it is a bijective homomorphism of left A-modules.
Homomorphism of right A-modules are defined analogous.

Remark 1.3.4. Let M, N be left A-modules. Then

i) $\hom_A(M, N)$ has a k-module structure given by

$$\lambda f : M \to N,\ x \mapsto \lambda f(x) = f(\lambda x).$$

This is well defined, as k lies in the center of A.

ii) In general, $\hom_A(M, N)$ has neither a left nor a right A-module structure.

iii) f is an isomorphism if and only if there is a homomorphism of left A-modules $g : N \to M$ such that

$$g \circ f = \mathrm{id}_M \text{ and } f \circ g = \mathrm{id}_N\,.$$

iv) Let $f : M \to M'$ and $g : N \to N'$ be homomorphisms of left A-modules. Then we obtain k-linear maps

$$f^* : \hom_A(M', N) \to \hom_A(M, N),\ h \mapsto h \circ f$$
$$g_* : \hom_A(M, N) \to \hom_A(M, N'),\ h \mapsto g \circ h.$$

Remark 1.3.5. Let A be a k-algebra and M, N left A-modules.

i) A subset $M' \subseteq M$ is called a **submodule** if

(SM1) $0 \in M'$

(SM2) $x, x' \in M' \implies x + x' \in M'$

(SM3) $a \in A, x \in M' \implies ax \in M'$.

In particular, submodules of A-modules are submodules of the underlying k-module, as follows using (L4)

ii) Let M be a submodule. Then the **quotient** has a left A-module structure in the obvious way. The projection

$$\pi : M \to M'$$

is a homomorphism of left A-modules.

iii) A **left ideal** is left A-submodule of ${}_AA$. Similar, a **right ideal** is right A-submodule of A_A. For a left ideal $I \subseteq A$, the quotient A/I is a left A-module, but in general not an algebra.

iv) A **two-sided ideal** $I \subset A$ is both a left- and a right-ideal of A. Then A/I has an algebra structure, by setting

$$(x + I)(y + I) := (xy) + I.$$

In general, this is only well-defined if I is a two-sided ideal of A.

v) Let $f : M \to N$ be a homomorphism of left A-modules. Then we obtain left A-modules:

$$\ker f,\ \operatorname{Im} f,\ \operatorname{coker} f := N/\operatorname{Im} f,\ \operatorname{coim} f := M/\ker f.$$

In particular, f factors uniquely as

$$\begin{tikzcd} M \arrow[r] \arrow[rrr, bend left, "f"] & \operatorname{coim} f \arrow[r, "\exists!", "\cong"'] & \operatorname{Im} f \arrow[r] & N \end{tikzcd} . \tag{F}$$

vi) Let $\{M_i \subset M \mid i \in I\}$ be a family of left A-submodules, for some index set I. Then

$$\bigcap_{i \in I} M_i \text{ and } \sum_{i \in I} M_i$$

are left A-modules.

vii) Let $x \in M$. Define
$$Ax := \{ax \mid a \in A\},$$
which is a left A-submodule. Similar, for $x \in M_A$, define $xA := \{xa \mid a \in A\}$, which is a right A-submodule. For a subset $E \subset M$,
$$\sum_{x\in E} Ax = \bigcap_{\substack{E\subseteq M'\subseteq M \\ M' \text{ submodule}}} M'.$$
M is called **finitely generated**, if there are $x_1, ..., x_n \in M$, such that
$$M = \sum_{i=1}^{n} Ax_i.$$

viii) Let $\{M_i \mid i \in I\}$ be a family of left A-modules. Then
$$\prod_{i\in I} M_i := \{(x_i)_{i\in I} \mid x_i \in M_i\}$$
is called the **product** , and
$$\bigoplus_{i\in I} M_i := \{(x_i)_{i\in I} \mid x_i \in M_i,\ x_i \neq 0 \text{ for only finitely many } i\}$$
is called the **coproduct** .They are both left A-modules. The **projection**
$$\pi_j : \prod_{i\in I} M_i \to M_j,\ (x_i)_{i\in I} \mapsto x_j$$
and the **inclusion**
$$\iota_j : \bigoplus_{i\in I} x_j \mapsto (\delta_{ij}x_j)_{i\in I}$$
are morphism of left A-modules.

ix) A left A-module M is finitely generated if and only if there is a surjective homomorphism of left A-modules
$$A^n := \bigoplus_{i=1}^{n} A \longrightarrow\!\!\!\!\!\rightarrow M$$
for some $n \geq 1$. A is called **finitely presented**, if there is an exact sequence of left A-modules
$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$
for some $m, n \geq 1$.

Proposition 1.3.6. *Let*
$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0 \tag{$*$}$$
and
$$0 \longrightarrow N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \tag{$**$}$$
be sequences of left A-modules.

i) The following are equivalent:

a) $(*)$ *is exact.*

b) For all left A*-modules* N*, the sequence*

$$0 \longrightarrow \hom_A(M_3, N) \xrightarrow{f_2^*} \hom_A(M_2, N) \xrightarrow{f_1^*} \hom_A(M, N)$$

is exact.

ii) The following are equivalent:

a) $(**)$ *is exact.*

b) For all left A*-modules* M*, the sequence*

$$0 \longrightarrow \hom_A(M, N_1) \xrightarrow{g_{1,*}} \hom_A(M, N_2) \xrightarrow{g_{2,*}} \hom_A(M, N_3)$$

is exact.

Proof. We will only prove *a)* $\Longrightarrow$ *b)* of *ii)*.

Lemma 1.3.7. *Let* K, M, N *be left* A*-modules, and* $\zeta : K \to M$*,* $\varphi : M \to N$ *be homomorphisms of left* A*-modules, such that* $\varphi \circ \zeta = 0$*. Then there is a unique homomorphism* $\overline{\zeta}$*, such that*

$$\begin{tikzcd}
K \arrow[r, "\zeta"] \arrow[rr, bend left, "0"] \arrow[dr, "\exists!\overline{\zeta}"'] & M \arrow[r, "\varphi"] & N \\
& \ker\varphi \arrow[u, hook] &
\end{tikzcd}$$

commutes.

- $g_{1,*}$ injective: Let $h \in \ker(g_{1,*})$. Then

$$\begin{tikzcd}
g_1 \circ h : \; M \arrow[r, "h"] \arrow[rr, bend left, "0"] & N_1 \arrow[r, "g_1"] & N_2
\end{tikzcd}$$

  and since g_1 is injective, it follows $h = 0$.

- $\operatorname{Im} g_{1,*} \subseteq \ker g_{2,*}$: Since $**$ is exact, it follows that $g_2 \circ g_1 = 0$. For $h \in \operatorname{Im} g_{1,*}$ there exists an $h' : M \to N_1$ such that $h = g_1 \circ h'$, and hence $g_2 \circ h = g_2 \circ g_1 \circ h' = 0$.

- $\ker g_{2,*} \subseteq \operatorname{Im} g_{1,*}$: As $(**)$ is exact, $\ker g_2 = \operatorname{Im} g_1$ holds.
  Let $h : M \to N_2 \in \ker g_{2,*}$, i.e. $g_2 \circ h = 0$:

$$\begin{tikzcd}
& & M \arrow[d, "h"'] \arrow[dr, "0"] & \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3
\end{tikzcd} .$$

By lemma 1.3.7, h factors uniquely through $\ker g_2 = \operatorname{Im} g_1$:

$$\begin{tikzcd}
 & & M \arrow[d, "h"'] \arrow[dr, "0"] \arrow[dd, dashed, bend left] & \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \; . \\
 & & \operatorname{Im} g_1 \arrow[u, hook, "\iota"] &
\end{tikzcd}$$

But since g_1 is injective, (F) implies that there is a uniquely determined isomorphism $\tilde{g}_1 : N_1 \longrightarrow \operatorname{Im} g_1$.
Putting everything together, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & & M \arrow[d, "h"'] \arrow[dr, "0"] \arrow[dd, dashed, bend left] & \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] \arrow[dr, dashed, bend right, "\exists ! \tilde{g}_1"] & N_2 \arrow[r, "g_2"] & N_3 \; . \\
 & & \operatorname{Im} g_1 \arrow[u, hook, "\iota"] &
\end{tikzcd}$$

Setting $h' := \tilde{g}_1^{-1} \circ h'$, we obtain

$$g_1 \circ h' = \iota \circ \tilde{g}_1 \circ \tilde{g}_1^{-1} \circ \tilde{h} = \iota \circ \tilde{h} = h.$$

□

Proposition 1.3.8. *Let A be a k-algebra. To give a left A-module structur is the same as to give a k-module structure V together with a homomorphism $\varphi : A \to \operatorname{End}_k(V)$ of k-algebras. To give a right A-module structure is the same as giving a k-module structure V together with a homomorphism $\varphi : A \to \operatorname{End}_k(V)^{\mathrm{op}}$.*

1.4 Representation of quivers

Let k be a field and Q be a quiver.

Definition 1.4.1. A **representation** X of Q over k consists of

- a k-vector space X_i for all $i \in Q_0$,
- a k-linear map
$$X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)}$$
for each $\alpha \in Q_1$

Example 1.4.2 (Continue example 1.2.4). i) Let $Q = \cdot$. Then a representation of Q is simply a k-vector space.

ii) Let $Q = \begin{tikzcd} 1 \arrow[loop right] \end{tikzcd}$. Then a representation of Q is a k-vector space V together with an endomorphism $\varphi \in \operatorname{End}_k(V)$:

$$Q = \begin{tikzcd} V \arrow[loop right, "f"] \end{tikzcd} .$$

iii) Let $Q = \begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}$, the **Kronecker Quiver**. Then a representation of Q is a pair of vector spaces V, W and two linear maps $f, g \in \hom_K(V, W)$:

$$Q = \begin{tikzcd} V \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & W \end{tikzcd}$$

Definition 1.4.3. Take X, Y to be two representations of Q over k. A **homomorphism of representations** $f : X \to Y$ is a tupel $(f_i)_{i \in Q_0}$ of linear maps $f_i : X_i \to Y_i$, such that for all $\alpha \in Q_1$ the diagram

$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "f_{s(\alpha)}"] \arrow[d, "X_\alpha"'] & Y_{s(\alpha)} \arrow[d, "Y_\alpha"] \\ X_{t(\alpha)} \arrow[r, "f_{t(\alpha)}"'] & Y_{t(\alpha)} \end{tikzcd}$$

commutes.

Example 1.4.4 (Continue example 1.4.2)**.** i) Homomorphisms of representations are k-linear maps $X \to Y$.

ii) Homomorphisms of representations (V, φ) and (W, ψ) are k-linear maps $f : V \to W$, such that

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi"'] & W \arrow[d, "\psi"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

commutes

iii) Homomorphisms of representations (V_1, V_2, A, B) and (W_1, W_2, C, D) are pairs (f_1, f_2) of linear maps $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$, such that $A \circ f_1 = f_2 \circ A$ and $B \circ f_1 = f_2 \circ B$.

Definition 1.4.5. An **isomorphism of representations** $f : X \to Y$ is a homomorphism of representations, such that there exists $g : Y \to X$ homomorphism of representations satisfying

$$g \circ f = \mathrm{id}_X \text{ and } f \circ g = \mathrm{id}_y .$$

An isomorphism of representations is a homomorphism of representations such that each map f_i is bijective.

End of Lecture 2

Let Q be a quiver over a field k.

Remark 1.4.6. i) Let X be a representation of Q over k. Associate a left kQ-module $M = F(X)$ as follows:

- As k-vector space, let
$$M := \bigoplus_{i \in Q_0} X_i.$$
- Define the action of kQ on M by an action of paths. Let p be a path of length ≥ 1. Define
$$X_p : X_{s(p)} \to X_{t(p)} \text{ given by } X_p = X_{\alpha_\ell} \circ \ldots \circ X_{\alpha_i},$$
with $X_p \in \hom_k(X_{s(p)}, X_{t(p)})$. Use this to define a k-linear map $\tilde{X}_p : M \to M$ as composition:
$$\tilde{X}_p : M = \bigoplus_{i \in Q_0} X_i \xrightarrow{\pi_{s(p)}} X_{s(p)} \xrightarrow{X_p} X_{t(p)} \xrightarrow{\iota_{t(p)}} \bigoplus_{i \in Q_0} X_i$$
If the length of $p = 0$, then p is a lazy part at some $i \in Q_0$, and we set
$$X_{\varepsilon_i} := \mathrm{id}_{X_i},$$
and $\tilde{X}_{\varepsilon_i}$ like $\tilde{X}_p$.
Now these k-linear endomorphisms define a kQ-module structure on M, given by:
$$kQ \times M \to M, \left(a := \sum_{p \in Q_*} \lambda_p \cdot p, (x_i)_i =: x\right) \mapsto a.x := \sum_{p \in Q_*} \lambda_p \cdot \tilde{X}_p(x)$$
$$= \sum_{p \in Q_*} \lambda_p \cdot (\iota_{t(p)} X_p(x_{s(p)})),$$
where we denote an element in M by a sequence $(x_i)_i$ with $x_i \in X_i$.
- We check that this actually defines a kQ-module structure:
  - (L3): Assume that $a, b \in kQ$. By the bilinearity of the multiplication, we can assume that $a = p$ and $b = q$ are both paths in Q_*. Then
$$a.(b.x) = \tilde{X}_p\left(\tilde{X}_q(x)\right)$$
$$= \iota_{t(p)} X_p \underbrace{\pi_{s(p)} \iota_{t(q)}} X_q(x_{s(q)}),$$
where
$$\pi_{s(p)} \iota_{t(q)} = \begin{cases} \mathrm{id}_{X_q}, & \text{if } t(q) = s(p) \\ 0, & \text{otherwise} \end{cases}.$$
This gives
$$a.(b.x) = \begin{cases} \iota_{t(p)} X_p X_q(x_{s(q)}), & \text{if } t(q) = s(p) \\ 0 & \text{otherwise} \end{cases}.$$
Additionally,
$$(a.b).x = \begin{cases} \tilde{X}_{p \circ q} & \text{if } f(q) = s(p), \\ 0 & \text{otherwise} \end{cases}.$$

But in the case $f(q) = s(p)$,

$$\tilde{X}_{p \circ q}(x) = \iota_{t(p)} \circ X_p X_q(x_{s(q)}).$$

The construction F is factorial, i.e. for $f : X \to Y$ a homomorphism of representations, F induces a homomorphism of kQ-algebras

$$Ff : F(X) \to F(Y) \text{ by } (Ff)((x_i)_i) := (f_i(x_i))_i.$$

ii) Let M be a left kQ-module. Define a representation $X := G(M)$ as follows

- As k-vector spaces, set
$$X_i := \varepsilon_i M.$$
- For $\alpha \in Q_*$, set
$$X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)},\ \varepsilon_{s(\alpha)} x \mapsto \alpha \varepsilon_{s(\alpha)} x = \alpha x = \varepsilon_{t(\alpha)} \alpha x \in X_{t(\alpha)},$$
as $\varepsilon_{t(\alpha)} \alpha = \alpha \varepsilon_{s(\alpha)}$.
So $X := ((X_i)_{i \in Q_0}, (X_\alpha)_{\alpha \in Q_*})$ is a representation of Q.
- This construction is also functorial: take $g : M \to N$ a homomorphism of left kQ-modules. Define $G(g) : X \to Y$, with $X := G(M)$ and $Y := G(N)$. Set
$$(Gg)_i := X_i \to Y_i,\ \varepsilon_i x \mapsto g(\varepsilon_i x) = \varepsilon_i g(x) \text{ with } X_i := \varepsilon_i M \text{ and } Y_i := \varepsilon_i N.$$
This is indeed a homomorphism of representations: Let $\alpha \in Q_1$ be arbitrary, and consider
$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "(Gg)_{s(\alpha)}"] \arrow[d, "X_\alpha"'] & Y_{s(\alpha)} \arrow[d, "Y_\alpha"] \\ X_{t(\alpha)} \arrow[r, "(Gg)_{t(\alpha)}"'] & Y_{t(\alpha)} \end{tikzcd}.$$
Then
$$y_\alpha (Gg)_{\varepsilon_{s(\alpha)}} x = Y_\alpha(g(\varepsilon_{s(\alpha)} x)) = \alpha g(\varepsilon_{s(\alpha)} x)$$
and
$$(Gg)_{\varepsilon_{t(\alpha)}}(X_\alpha(\varepsilon_{s(\alpha)} x)) = g(\alpha(\varepsilon_{s(\alpha)} x)) = \alpha g(\varepsilon_{s(\alpha)} x),$$
hence the diagram commutes.

Theorem 1.4.7. *i) Let M be a left kQ-module. Then $FG(M) \cong M$ as left kQ-modules.*

ii) Let X be a representation of Q over k. Then $GF(X) \cong X$ as representations of Q.

Proof. i) Denote $X := G(M)$. Then

$$F(X) = \bigoplus_{i \in Q_0} X_i = \bigoplus_{i \in Q_0} \varepsilon_i M$$

as a k-vector space. Observe

- The identity in kQ is given by

$$\mathrm{id}_{kQ} = \sum \varepsilon_i.$$

Hence, for all x in X:

$$x = \mathrm{id}_{kQ}\, x = \left(\sum \varepsilon_i\right) x = \sum (\varepsilon_i x_i) \in \sum \varepsilon_i M.$$

- For all $i \neq j$, $\varepsilon_i \varepsilon_j = 0$ holds. So for

$$x \in X_i = \varepsilon_i M \bigcap \sum_{j \neq i} \varepsilon_j M_j \implies x = \sum_{j \neq i} \varepsilon_j x_j \text{ for some } x_j \in M.$$

But as $x \in \varepsilon_i M$, $\varepsilon_i x = x$. So

$$x = \varepsilon_i x = \varepsilon_i \left(\sum_{j \neq i} \varepsilon_j x_j\right) = \sum_{j \neq i} \varepsilon_i \varepsilon_j x_j = 0.$$

These observations show that

$$\varphi : F(X) = \bigoplus_{i \in Q_0} \varepsilon_i M \longrightarrow M \ , \ (\varepsilon_i x_i)_{i \in Q_0} \mapsto \sum \varepsilon_i x_i$$

is an isomorphism of k-vector spaces.
Show that φ is isomorphism of left kQ-modules:
Without loss generaly, assume that $a = p$ is a path in Q (regarded as element of kQ), and let $x \in M$. Then

$$\begin{aligned}\varphi(a.x) = \varphi\left(\iota_{t(p)} X_p(x_{s(p)})\right) &= \varphi\left(\iota_{t(p)}(p x_{s(p)})\right) \\ &= p x_{s(p)},\end{aligned}$$

and

$$a.\varphi(x) = a. \sum x_i = a. \sum \epsilon_i x_i = p x_{s(p)}.$$

ii) Let $M := F(X)$ be the left kQ-module associated with X. Then

$$G(M)_i = \varepsilon_i M = \varepsilon_i \bigoplus_{j \in Q_0} X_j = X_i,$$

and

$$(G(M))_\alpha \left(x_{s(\alpha)}\right) = \alpha . x_{s(\alpha)} = X_\alpha(x_{s(\alpha)}).$$

$\square$

Remark 1.4.8. Let M be a left kQ-module, with Q finite and k a field.

i) $\dim_k M = \sum_{i \in Q_0} \dim_k X_i$ where $X = G(M)$,where G is the functor from remark 1.4.6

ii) $\dim_K kQ < \infty \iff Q$ contains no **oriented cycles** (a path p of length ≥ 1, such that $s(p) = t(p)$)

iii) If Q has no oriented cycle, then the following are equivalent:
   (a) M is a finitely generated kQ-module.
   (b) $\dim_k X_i < \infty$.

Proof. (a) $\implies$ (b):(b) implies in particular, that M is finitely generated as a k-module. But as $k \subset kQ$, (a) follows immediatley.

(b) $\implies$ (a): Set $A := kQ$,and let $x_1, ..., x_n \in kQ$ generate M as a left kQ-module. Then there is a kQ-linear surjection

$$A^n \twoheadrightarrow M$$

given by $e_i \mapsto x_i$, where the $(e_i)_{1\leq i\leq n}$ are a basis of A^n. As this is in particular k-linear, we have that

$$\dim_k M \leq \dim_k(A^n) = n \dim_k A < \infty,$$

as Q contains no cycle. □

iv) Under G, the notion of a „left submodule" corresponds to **subrepresentations** of Q, i.e. a tupel of subspaces $Y_i \subset X_i$ for all $i \in Q_0$ such that $X_\alpha(Y_{s(\alpha)}) \subset Y_{t(\alpha)}$ for all $\alpha \in Q_1$.

v) Under G, a direct sum of modules corresponds to **direct sum of representations**: Given X, Y two representations of Q, define a new representation $X \oplus Y$ where the vector spaces are given by

$$(X \oplus Y)_i := (X \oplus Y)_i$$

and the k-linear maps

$$(X \oplus Y)_\alpha : X_{s(\alpha)} \oplus Y_{s(\alpha)} \longrightarrow X_{t(\alpha)} \oplus Y_{t(\alpha)}$$

given by

$$\left(\begin{array}{c|c} X_\alpha & \\ \hline & Y_\alpha \end{array}\right).$$

1.5 Bimodules and tensor products

Definition 1.5.1. Let A, B be k-algebras. A A-B-**bimodule** M is a set M, together with maps:

$$A \times M \longrightarrow M, (a, x) \longmapsto ax$$

$$M \times B \longrightarrow M, (x, b) \longmapsto xb,$$

such that

i) M is a left A-module

ii) M is a right B-module

iii) for all $a \in A, b \in B$ and $x \in M$, the relation

$$(ax)b = a(xb)$$

holds.

We denote a A-B-bimodule by

$$_AM_B.$$

Lemma 1.5.2. *Let A, B, C be k-algebras, and consider $_AM_B$ and $_AN_C$, a A-B-bimodule and a A-C-bimodule respectivley. Then $\hom_A(M, N)$ becomes a B-C-bimodule via*

- $B \times \hom_A(M, B) \longrightarrow \hom_A(M, N), \qquad (b, f) \longmapsto bf : M \to N, x \mapsto f(xb)$
- $\hom_A(M, N) \times C \longrightarrow \hom_A(M, N), \qquad (f, c) \longmapsto fc : M \to N, x \mapsto f(cx)$

Proof.

- well-defined:

$$bf(ax) = f((ax)b) = f(a(xb)) = af(xb) = a(bf)(x)$$

- $\hom_A(M, N)$ is a left B-module: Show e.g. (L3):

$$\begin{aligned}((bb')f)(x) = (x(bb')) &= f((xb)b') \\ &= b'(f(xb)) \\ &= b((b'f)(x))\end{aligned}$$

- compatibility:

$$((af)b)(x) = f((ax)b) = f(a(xb)) = (a(fb))(x).$$

□

End of Lecture 3

A.0 Sheet 0

Definition A.0.1. Let X be any set, and k any commutative ring with unit. Define the **free algebra** generated by X:

i) As k-module, set $k\langle X\rangle$ as the free k-module generated by X.

ii) Define the multiplication of two words as the concatination.

Then $k\langle X\rangle$ satisfies the following universal property: Let B be any k-algebra, and $f : X \to B$ a homomorphism of sets. There there is a unique homomorphism of k-algebras $k\langle X\rangle$, such that the following diagram commutes:

$$\begin{tikzcd} k\langle X\rangle \arrow[r, dashed, "\exists! f"] & B \\ X \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}.$$

Proposition A.0.2. *Consider the forgetful functor*

$$\mathcal{F} : k\text{-}\mathsf{Alg} \to \mathsf{Sets},\ \ B \mapsto B$$

and

$$\mathcal{K}\langle - \rangle : \mathsf{Sets} \to k\text{-}\mathsf{Alg},\ \ X \mapsto k\langle X\rangle.$$

Then for all sets X and k-algebras,

$$\hom_{\mathsf{Sets}}(X, F(B)) \cong \hom_{k\text{-}\mathsf{Alg}}(k\langle X\rangle, B)$$

holds.

We say that F is **right-adjoint** to $\mathcal{K}\langle - \rangle$.

Problem A.0.1. Consider the k-algebra

$$A := k\langle x, y\rangle/(\langle xy - yx - 1\rangle)$$

over a fiel k with $\operatorname{char} k = 0$. Show that there are no non-zero, finite-dimensional representations of A.

Definition A.0.3. Let C and D be two categories. A covariant functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ is **faithful**, if for all objects A, B of C, the induced function of sets

$$\hom_{\mathsf{C}}(A, B) \to \hom_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(B))$$

is injective; it is **full**, if this function is surjective for all objects A, B of C.

Definition A.0.4. A covariant functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ is an **equivalence of categories**, if it is fully faithful (i.e. bijective on hom-sets) and **essentially surjective**, i,e, for every object Y of D, there is an object X of C such that $\mathcal{F}(X) \cong Y$.

Some people write natural transformations in the follwing way:

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

A.1 Sheet 1

Solution A.1.1. i) φ_m is k-linear:

- $\varphi(a+b)x = (a+b)x \overset{(L1)}{=} ax + bx = \varphi(a)x + \varphi(b)x$
- $\varphi(\lambda a) = (\lambda a)x \overset{(L5)}{=} \lambda(ax) = \lambda\varphi(a)x.$

φ_m is ring homomorphism:

- $\varphi(ab) = (ab)x \overset{(L3)}{=} a(bx) = \varphi(a)\varphi(b)x$
- $\varphi(1_A) \overset{(L4)}{=} (1_A)x = x.$

As these relations hold for all x, the assertion follows.

ii) V_φ is already a k-module.

- (L1) $a(x+y) = (\varphi(a))(x+y) \overset{\varphi \in \mathrm{End}_k(V)}{=} (\varphi(a))(x) + (\varphi(a))(y) = ax + ay$
- (L2) $(a+b)x = (\varphi(a+b))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\varphi(a) + \varphi(b))x = \varphi(a)x + \varphi(b)x = ax + bx$
- (L3) $(ab)x = (\varphi(ab))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\varphi(a)\varphi(b))x = a(bx)$
- (L4) $1_a x = (\varphi(1_a))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} \mathrm{id}\, x = x$
- (L5) $(\lambda a)(x) = ((\varphi(\lambda a))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\lambda\varphi(a))x = \lambda(ax)$ and $a(\lambda x) = (\varphi(a)(\lambda x)) \overset{\varphi(a) \text{ endo of } k-\text{module}}{=} \lambda(\varphi(a))x = \lambda(ax),$

for all $a, b \in A, x, y \in V$ and $\lambda \in k$.

iii) We regard V and W as A-modules in the sense of part ii). Assume that $\psi(a) \circ f = f \circ \varphi(a)$ $(*)$ for all $a \in A$. Then

$$f(ax) = f((\varphi(a))x) \overset{*}{=} (\psi(a))(f(x)) = af(x),$$

for all $x \in X$. Hence f is an A-module homomorphism.
Assume that f is a a-module homomorphism, then

$$(\psi(a))(f(x)) = a(f(x)) = f(ax) = f((\varphi(a))x)$$

for all $x \in V$. Hence $\psi(a) \circ f - f \circ \varphi(a) = 0$ and so $(*)$ holds.

Solution A.1.2. i) Assume that I is a non-zero ideal of A. Let $a = (a)_{ij} \neq 0$ be an arbitrary matrix in I. Then there exist permutation $\sigma, \pi \in \mathrm{GL}_n(K)$ matrices, such that $(\sigma a \pi)_{11} \neq 0$, which is in a, as I is a two-sided ideal. So without loss of generality, suppose $a_{11} \neq 0$.
Define

$$b \in \mathrm{M}_n(k), (b)_{ij} := \begin{cases} 1, & \text{if } i = j = 1 \\ 0, & \text{else} \end{cases}$$

and E_n as the identity of $\mathrm{M}_n(k)$. Then we get

$$\left(\frac{1}{a_{11}} E_n\right) \cdot b \cdot a \cdot b = b.$$

By repeatetly using permutation matrixes, it is possible to write any matrix as sum of products of a, b and permutation matrices on the left- and right. As I is a two-sided ideal, a all of these combinations are in I as well. Hence a generates all of A, and $I = A$.

ii) Consider A as a k-vector space, then $\dim_K A = n^2$. Let M be any left A-module. As shown in task 3, there is a homomorphism of k-algebras

$$\varphi A \to \mathrm{End}_k(M),\ a \mapsto a : (x \mapsto ax),$$

which is in particular a homomorphism of k-vector spaces. The kernel of φ is a two-sided ideal of A, as

$$a0x = 0ax = 0$$

for all $a \in A$ and $x \in M$.
Now $i)$ implies that $\ker \varphi$ is either zero or $\ker \varphi = A$. But since $\varphi(E_n) = \mathrm{id}_M$, the latter one is not possible. Hence φ is injective, and in particular $\dim A \leq \dim \mathrm{End}_k(V)$, so $n \leq m$.

Proposition A.1.1. *Let k be a field, $k[X]$ the polynomial ring and $p \in k[X]$ a polynomial with $\deg p = n$. Then*

$$k[X]/(p)$$

is a n-dimensional k vector space, and a basis is given by

$$\{1, x, ..., x^{n-1}\}.$$

The following propositions are taken from [Alu09].
Let R be any commutative ring.

Proposition A.1.2. *Let $I_1, ..., I_k$ be ideals of R such that $I_i + I_j = (1)$ for all $i \neq j$. Then the natural homomorphism*

$$\varphi : R \to R/I_1 \times \ldots \times R/I_k$$

is surjective and induces an isomorphism

$$\frac{R}{I_1 \ldots I_k} \to R/I_1 \times \ldots \times R/I_k$$

Corollary A.1.3 (Chinese remainder theorem)**.** *Let R be a PID and $a_1, ..., a_k \in R$ be elements such that* $\gcd(a_i, a_j) = 1$ *for all $i \neq j$. Let $a = a_1 \dots a_k$. Then the function*

$$\varphi : R/(a) \to R/(a_1) \times \dots \times R/(a_k).$$

Proposition A.1.4 (Yoneda Lemma)**.** *Let C be a category, X an object of C and consider the contravariant functor*

$$h_X := \hom_{\mathsf{C}}(-, X).$$

Then for every contravariant functor $\mathcal{F} : \mathsf{C} \to \mathsf{Set}$, there is a bijection between the set of natural transformations $h_x \rightsquigarrow \mathcal{F}$ and (X).

Definition A.1.5 ([ASS06])**.** The (Jacobson) **radical** $\operatorname{rad} A$ of a K-algebra A is the intersection of all maximal right ideals in A. It is the same as the intersetion of all left-sided maximal right ideals in A. Furthermore, $\operatorname{rad} A$ is a two-sided ideal.

Definition A.1.6. Let $f, g : X \to Y$ be morphisms in a category C. Then a morphism $e : E \to X$ is called **equalizer** of f and g if $f \circ e = g \circ e$ and for all other morphisms $o : O \to X$, such that $f \circ o = g \circ o$, there is a unique morphis $O \to E$, such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "e"] & X \arrow[r, shift left] \arrow[r, shift right] & X \\
O \arrow[u, dashed, "\exists!"] \arrow[ur, "o"] & &
\end{tikzcd}.$$

Proposition A.1.7. *Equalizers exists in abelian categories.*

[ASS06] I. Assem, D. Simson, and A. Skowronski. **Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory**. Elements of the Representation Theory of Associative Algebras. Cambridge University Press, 2006.

[Alu09] Paolo Aluffi. **Algebra: chapter 0**. Vol. 104. American Mathematical Soc., 2009.