Foundations of Representation Theory

Lecture Notes in the Winter Term 2018/19

CONTENTS

These are my personal lecture notes for the lecture *Foundations of Representation Theory* held by Dr. Hans Franzen at the University of Bonn in the winter term 2018/19.

I try to update them on my website, https://pankratius.github.io.
I label my own comments and additions in purple.

The book [**aluffi**] is used for further references, and highly recommended.
In addition, I want to point out that there is another student who is publishing his notes for this course (which are way better than mine). These are available under cionx.github.io.

Notation

I try to follow Dr. Franzens notation closley. Some deviations are:

- Categories: e.g.: $\mathcal{C}$, $\mathcal{D}$.
- functors: e.g: $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathcal{D}$ are functors from the category $\mathcal{C}$ to the category $\mathcal{D}$.
- Natural transformations: When I want do put more data into the depiction of a natural transformation $\eta : \mathscr{F} \to \mathscr{G}$ (where $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathcal{D}$ are functors), I follow the notation introduced in [**context**]:

$$\eta : \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathscr{F}"] \arrow[r, bend right, "\mathscr{G}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D} \end{tikzcd}$$

(There are still some TikZ-related issues however).

Conventions: In this lecture, rings will always be unital, and ring homomorphisms f always fulfill $f(1) = 1$. Rings do *not* have to be commutative.

1.1 Algebras - Basics

Let k be a commutative ring.

Definition 1.1.1. A k-**algebra** A is a ring A, together with a structure of a k-module on A, such that

$$\text{for all } a, b \in A, \lambda \in K: \ (\lambda a)b = a(\lambda b) = \lambda(ab) \tag{$*$}$$

Definition 1.1.2. Let A, B be k-algebras. A **homomorphism of algebras** is a map $f : A \to B$ that is both k-linear and a ring homomorphism.

Remark 1.1.3. Let A be a ring. Define

$$Z(A) := \{a \in A \mid \forall b \in A : ab = ba\},$$

which is a commutative subring and is called the **center** of A.

Remark 1.1.4. Let A be a ring. Giving a k-algebra structure on A is the same as giving a ring homomorphism $k \to Z(A)$. More precisley:

i) If A is a k-algebra, then $p : k \to A$, $\lambda \mapsto \lambda 1$ satisfies $\operatorname{Im} p \subseteq Z(A)$ and is a ring homomorphism. (the first statement follows from ($*$) and the second one from the fact that A has a k-module structure).

ii) Let $\varphi : k \to Z(A)$ be a ring homomorphism. Define

$$\lambda a := \varphi(\lambda)a,$$

for all $\lambda \in k$. This defines a k-algebra structure on A (($*$) holds since $\operatorname{Im}(\varphi) \subseteq Z(A)$).

iii) Let A, B be k-algebras and $f : A \to B$ a homomorphism of rings. Then f is a homomorphism of k-algebras if and only if the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "f"] & & B \\
Z(A) \arrow[u, hook] & & Z(B) \arrow[u, hook] \\
& k \arrow[ul, "\phi_A"] \arrow[ur, "\phi_B"'] &
\end{tikzcd} .$$

Example 1.1.5. i) Let V be a k-module. Then $\operatorname{End}_k(V)$ has a ring structure given by

$$\operatorname{End}_k(V) \times \operatorname{End}_k(V) \to \operatorname{End}_k(V), \ (\phi, \psi) \mapsto \varphi \circ \psi.$$

Then $\mathrm{End}_k(V)$ is both a ring and a k-module, and becomes a k-algebra via

$$\varphi : k \to \mathrm{End}_k(V),\ \lambda \mapsto \lambda \,\mathrm{id}\,.$$

Note that $\mathrm{Im}\,\varphi \subseteq Z(A)$. If k is a field, then $Z(\mathrm{End}_k(V)) = \{\lambda\,\mathrm{id} \mid \lambda \in k\}$.

ii) Take $V = k^n$ (free module of rank n). Then $\mathrm{End}_k(V) \cong \mathrm{M}_n(k)$. Define

$$T_u := \{\varphi \in \mathrm{M}_n(k) \mid \varphi \text{ is upper triangular}\},$$

i.e. T_u presevers flags in k^n. Then T_u is a **subalgebra** of $\mathrm{M}_n(k)$, i.e. is both a subring and a k-submodule of the original algebra.

iii) Let G be a group. Define the **group algebra** $k[G]$ of G as follows:

- As k-module, is defined as the free module on G,

$$k[G] := k^{(G)} = \{\sum_{g\in G} \lambda_g g \mid \lambda_g \in k,\ \lambda_g \neq 0 \text{ for only finitely many } g \in G\}.$$

- Multiplication: Let $a := \sum \lambda_g g$, $b = \sum \mu_g g$ and define:

$$ab := \sum_{g\in G, h\in G} \lambda_g \mu_h (gh) = \sum_{j\in G} \left(\sum_{gh=j} \lambda_g \mu_h \right) j.$$

This multiplication is associative, k-bilinear, distributive and $1|_{k[G]} = e$. In addition, $(*)$ is satisfied.

1.2 Quivers - Basics

Definition 1.2.1. A **quiver** is a „directed graph“. Formally, a quiver is a quadruple (Q_0, Q_1, s, t) consiting of sets Q_0 (**vertices**) and Q_1 (**arrows**) and maps $s : Q_1 \to Q_0$, $t : Q_1 \to Q_0$. For $\alpha \in Q_1$, we call $s(\alpha)$ the **source** of α and $t(\alpha)$ the **target** of α:

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

Example 1.2.2. i) $Q = (\{1\}, \emptyset, ...)$ is visualized as: 1

ii) $Q = (\{1\}, \{\alpha\}, ...)$ is visualized as $\begin{tikzcd} 1 \arrow[loop above] \end{tikzcd}$

iii) $Q = (\{1, 2\}, \{\alpha, \beta\}, s(\alpha) = s(\beta) = 1, t(\alpha) = t(\beta) = 2)$ is visualized as $\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}$

Definition 1.2.3. Let Q be a quiver sucht that both Q_0 and Q_1 are finite.

i) Let $\ell \in \mathbb{Z}_{\geq 1}$. A **path** of length ℓ is a sequence $\alpha_\ell ..., \alpha_1$ of arrows, such that $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq \ell - 1$,

$$\circ \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_2} \circ \ldots \xrightarrow{\alpha_\ell} \circ.$$

Define Q_ℓ to be the set of all paths of length ℓ.
Let $p : \alpha_\ell ... \alpha_1$ be a path. Define $s(p) := s(\alpha_1)$ and $t(p) := s(\alpha_\ell)$.
Formally define Q_0 to be the set of all paths of length zero. Denote by ε_i for $i \in Q_0$ the constant path at i. ε_i is called a **lazy path**. We set $s(\varepsilon_i) = t(\varepsilon_i) := i$.

ii) Let $p = \alpha_\ell ... \alpha_i$ and $q = \beta_m ... \beta_1$ be paths of length ℓ and m respectivley, with $\ell, m \geq 1$. If $t(p) = s(q)$, then set $q \circ p := \beta_m .... \beta_1 \alpha_\ell ... \alpha_1$. This is a path of length $\ell + m$.
For p a path of length ≥ 0 and ε_i a lazy path:

- if $t(p) = i$, set $\varepsilon_i \circ p := p$,
- if $s(p) = i$, set $p \circ \varepsilon_i := p$.

In all others cases, the composition is not defined.

iii) Define

$$Q_* := \bigcup_{\ell \geq 0} Q_\ell,$$

the set of all paths. Define the **path-algebra** kQ:

- As a k-module, $kQ := k^{(Q_*)}$.
- Multiplication: Let $a = \sum \lambda_p p, \ b = \sum \mu_p p$. Define

$$ab := \sum_{p,q \in Q_*} \lambda_p \mu_q (p \cdot q),$$

where

$$p \cdot q := \begin{cases} p \circ q, & \text{if it is defined, i.e } t(q) = s(p) \\ 0, & \text{else} \end{cases}.$$

The multiplication is associative (due to the associativity of the composition of paths) and k-bilinear by definition. In addition, distributivity and $(*)$ are fulfilled.

- The identity is given by $\sum \varepsilon_i$.

Example 1.2.4. i) $Q = 1$, then $kQ = k$.

ii) $Q = \begin{tikzcd} 1 \arrow[loop above] \end{tikzcd}$, then $Q_* = \{\alpha^n \mid n \geq 0\}$ and $kQ = k[t]$.

iii) $Q = \begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}$. Then $Q_* = \{\varepsilon_1, \varepsilon_2, \alpha^n, \beta^n \mid n \geq 0\}$ and

$$kQ = k\varepsilon_1 \oplus k\varepsilon_2 \oplus k\alpha \oplus k\beta.$$

A multiplication table is given by

	ε_1	ε_2	α	β
ε_1	ε_1	0	0	0
ε_2	0	ε_2	α	β
α	α	0	0	0
β	β	0	0	0

Lemma 1.2.5. *Let Q be a finite quiver, k a field. Then the following are equivalent:*

i) Q contains no cycles.

ii) $\dim_k kQ < \infty$.

Lemma 1.2.6. *Let k be a field, A a k-algebra and $n := \dim(A) < \infty$. Then there exists an injective homomorphism of k-algebras $\varphi : A \to \mathrm{M}_n(k)$.*

Proof. By choosing a basis of A, we get an isomorphism $\mathrm{End}_k(A) \cong \mathrm{M}_n(k)$. So it suffices to find an injective homomorphism of k-algebras $\varphi : A \to \mathrm{End}_k(A)$.
Consider

$$\varphi : A \to \mathrm{End}_k(A), \ \varphi(a) : A \to A, b \mapsto ab.$$

- $\varphi(a)$ is k-linear for all a by the distributivity in A and the condition $(*)$.
- φ is k-linear by the distributivity in A and the condition $(*)$.
- Let $a, a' \in A$. Then

$$\varphi(aa')(b) = (aa')(b) = a(a'b) = \big(\varphi(a) \circ \varphi(a')\big)(b).$$

Hence φ is indeed a homomorphism of k-algebras.
To show that φ is injective, let $a \in \ker \varphi$, hence $ab = 0$ for all $b \in A$. But in particular, $0 = a1 = a$. □

End of Lecture 1

Definition 1.2.7. Let A be a k-algebra. Then the **opposite algebra** A^{op} is A (as a k-module), and the multiplication is defined as

$$a \cdot_{A^{\mathrm{op}}} b = b \cdot_A b.$$

Example 1.2.8. Let Q be a quiver, and define $Q^{\mathrm{op}} := (Q_0, Q_1, s^{\mathrm{op}}, t^{\mathrm{op}})$, where $s^{\mathrm{op}}(\alpha) := t(\alpha)$ and $t^{\mathrm{op}}(\alpha) := s(\alpha)$. Then $kQ^{\mathrm{op}} = k(Q^{\mathrm{op}})$

1.3 Modules - Basics

Definition 1.3.1. Let A be a k-algebra. A **left A-module** M is a k-module M together with a map $A \times M \to M, (a,x) \mapsto ax$, such that:

$$a(x+y) = ax + ay \tag{L1}$$

$$(a+b)x = ax + bx \tag{L2}$$

$$a(bx) = (ab)x \tag{L3}$$

$$1_A x = x \tag{L4}$$

$$(\lambda a)x = \lambda(ax) = a(\lambda x), \tag{L5}$$

for all $a, b \in A$, $x, y \in M$ and $\lambda \in k$. If A is a left A-module, we denote this as ${}_AM$. A **right A-module** is defined analogous, where (L3) becomes $(xa)b = x(ab)$. If A is a right A-module, we denote this by A_M.

Remark 1.3.2. A right A-module is the same as a left A^{op}-module.

Definition 1.3.3. Let A be a k-algebra, and M, N left A-modules. A **homomorphism of left A-modules** $f : M \to N$ is a k-linear map such that

$$f(ax) = af(x)$$

for all $a \in A$ and $x \in M$.
Define the set of all left A-algebra homomorphisms as

$$\hom_A(M,N) := \hom_A({}_AM, {}_AM) := \{f : M \to N \mid f \text{ is a homomorphism of left } A\text{-modules}\}.$$

A homomorphism of left A-modules is an **isomorphism** if it is a bijective homomorphism of left A-modules.
Homomorphism of right A-modules are defined analogous.

Remark 1.3.4. Let M, N be left A-modules. Then

i) $\hom_A(M,N)$ has a k-module structure given by

$$\lambda f : M \to N,\ x \mapsto \lambda f(x) = f(\lambda x).$$

This is well defined, as k lies in the center of A.

ii) In general, $\hom_A(M,N)$ has neither a left nor a right A-module structure.

iii) f is an isomorphism if and only if there is a homomorphism of left A-modules $g : N \to M$ such that

$$g \circ f = \mathrm{id}_M \text{ and } f \circ g = \mathrm{id}_N\,.$$

iv) Let $f : M \to M'$ and $g : N \to N'$ be homomorphisms of left A-modules. Then we obtain k-linear maps

$$\begin{aligned} f^* &: \hom_A(M', N) \to \hom_A(M, N),\ h \mapsto h \circ f \\ g_* &: \hom_A(M, N) \to \hom_A(M, N'),\ h \mapsto g \circ h. \end{aligned}$$

Remark 1.3.5. Let A be a k-algebra and M, N left A-modules.

i) A subset $M' \subseteq M$ is called a **submodule** if

(SM1) $0 \in M'$

(SM2) $x, x' \in M' \implies x + x' \in M'$

(SM3) $a \in A, x \in M' \implies ax \in M'$.

In particular, submodules of A-modules are submodules of the underlying k-module, as follows using (L4)

ii) Let M be a submodule. Then the **quotient** has a left A-module structure in the obvious way. The projection

$$\pi : M \to M'$$

is a homomorphism of left A-modules.

iii) A **left ideal** is left A-submodule of ${}_AA$. Similar, a **right ideal** is right A-submodule of A_A. For a left ideal $I \subseteq A$, the quotient A/I is a left A-module, but in general not an algebra.

iv) A **two-sided ideal** $I \subset A$ is both a left- and a right-ideal of A. Then A/I has an algebra structure, by setting

$$(x + I)(y + I) := (xy) + I.$$

In general, this is only well-defined if I is a two-sided ideal of A.

v) Let $f : M \to N$ be a homomorphism of left A-modules. Then we obtain left A-modules:

$$\ker f,\ \operatorname{Im} f,\ \operatorname{coker} f := N/\operatorname{Im} f,\ \operatorname{coim} f := M/\ker f.$$

In particular, f factors uniquely as

$$\begin{tikzcd} M \arrow[r] \arrow[rrr, bend left, "f"] & \operatorname{coim} f \arrow[r, "\exists!", "\cong"'] & \operatorname{Im} f \arrow[r] & N \end{tikzcd} . \tag{F}$$

vi) Let $\{M_i \subset M \mid i \in I\}$ be a family of left A-submodules, for some index set I. Then

$$\bigcap_{i \in I} M_i \text{ and } \sum_{i \in I} M_i$$

are left A-modules.

vii) Let $x \in M$. Define

$$Ax := \{ax \mid a \in A\},$$

which is a left A-submodule. Similar, for $x \in M_A$, define $xA := \{xa \mid a \in A\}$, which is a right A-submodule. For a subset $E \subset M$,

$$\sum_{x \in E} Ax = \bigcap_{\substack{E \subseteq M' \subseteq M \\ M' \text{ submodule}}} M'.$$

M is called **finitely generated**, if there are $x_1, ..., x_n \in M$, such that

$$M = \sum_{i=1}^{n} Ax_i.$$

viii) Let $\{M_i \mid i \in I\}$ be a family of left A-modules. Then

$$\prod_{i \in I} M_i := \{(x_i)_{i \in I} \mid x_i \in M_i\}$$

is called the **product** , and

$$\bigoplus_{i \in I} M_i := \{(x_i)_{i \in I} \mid x_i \in M_i,\ x_i \neq 0 \text{ for only finitely many } i\}$$

is called the **coproduct** .They are both left A-modules. The **projection**

$$\pi_j : \prod_{i \in I} M_i \to M_j,\ (x_i)_{i \in I} \mapsto x_j$$

and the **inclusion**

$$\iota_j : \bigoplus_{i \in I} x_j \mapsto (\delta_{ij} x_j)_{i \in I}$$

are morphism of left A-modules.

ix) A left A-module M is finitely generated if and only if there is a surjective homomorphism of left A-modules

$$A^n := \bigoplus_{i=1}^{n} A \longrightarrow\!\!\!\!\to M$$

for some $n \geq 1$. A is called **finitely presented**, if there is an exact sequence of left A-modules

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

for some $m, n \geq 1$.

Proposition 1.3.6. *Let*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0 \tag{$*$}$$

and

$$0 \longrightarrow N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \tag{$**$}$$

be sequences of left A-modules.

i) The following are equivalent:

a) $(*)$ *is exact.*

b) For all left A-modules N, the sequence

$$0 \longrightarrow \hom_A(M_3, N) \xrightarrow{f_2^*} \hom_A(M_2, N) \xrightarrow{f_1^*} \hom_A(M, N)$$

is exact.

ii) The following are equivalent:

a) $(**)$ *is exact.*

b) For all left A-modules M, the sequence

$$0 \longrightarrow \hom_A(M, N_1) \xrightarrow{g_{1,*}} \hom_A(M, N_2) \xrightarrow{g_{2,*}} \hom_A(M, N_3)$$

is exact.

Proof. We will only prove *a)* $\implies$ *b)* of *ii)*.

Lemma 1.3.7. *Let K, M, N be left A-modules, and $\zeta : K \to M$, $\varphi : M \to N$ be homomorphisms of left A-modules, such that $\varphi \circ \zeta = 0$. Then there is a unique homomorphism $\overline{\zeta}$, such that*

$$\begin{tikzcd} K \arrow[r, "\zeta"] \arrow[rr, bend left, "0"] \arrow[dr, "\exists!\overline{\zeta}"'] & M \arrow[r, "\varphi"] & N \\ & \ker\varphi \arrow[u, hook] & \end{tikzcd}$$

commutes.

- $g_{1,*}$ injective: Let $h \in \ker(g_{1,*})$. Then

$$\begin{tikzcd} g_1 \circ h : \; M \arrow[r, "h"] \arrow[rr, bend left, "0"] & N_1 \arrow[r, "g_1"] & N_2 \end{tikzcd}$$

  and since g_1 is injective, it follows $h = 0$.

- $\operatorname{Im} g_{1,*} \subseteq \ker g_{2,*}$: Since $**$ is exact, it follows that $g_2 \circ g_1 = 0$. For $h \in \operatorname{Im} g_{1,*}$ there exists an $h' : M \to N_1$ such that $h = g_1 \circ h'$, and hence $g_2 \circ h = g_2 \circ g_1 \circ h' = 0$.

- $\ker g_{2,*} \subseteq \operatorname{Im} g_{1,*}$: As $(**)$ is exact, $\ker g_2 = \operatorname{Im} g_1$ holds.
  Let $h : M \to N_2 \in \ker g_{2,*}$, i.e. $g_2 \circ h = 0$:

$$\begin{tikzcd} & & M \arrow[d, "h"'] \arrow[dr, "0"] & \\ 0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \end{tikzcd} .$$

By lemma 1.3.7, h factors uniquely through $\ker g_2 = \operatorname{Im} g_1$:

$$\begin{tikzcd}
& & M \arrow[d, "h"'] \arrow[dr, "0"] \arrow[dd, dashed, bend left] & \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \, . \\
& & \operatorname{Im} g_1 \arrow[u, hook, "\iota"] &
\end{tikzcd}$$

But since g_1 is injective, (F) implies that there is a uniquely determined isomorphism $\tilde{g}_1 : N_1 \longrightarrow \operatorname{Im} g_1$.

Putting everything together, we obtain the following commutative diagram:

$$\begin{tikzcd}
& & M \arrow[d, "h"'] \arrow[dr, "0"] \arrow[dd, dashed, bend left] & \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] \arrow[dr, dashed, bend right, "\exists! \tilde{g}_1"] & N_2 \arrow[r, "g_2"] & N_3 \, . \\
& & \operatorname{Im} g_1 \arrow[u, hook, "\iota"] &
\end{tikzcd}$$

Setting $h' := \tilde{g}_1^{-1} \circ h'$, we obtain

$$g_1 \circ h' = \iota \circ \tilde{g}_1 \circ \tilde{g}_1^{-1} \circ \tilde{h} = \iota \circ \tilde{h} = h.$$

□

Proposition 1.3.8. *Let A be a k-algebra. To give a left A-module structur is the same as to give a k-module structure V together with a homomorphism $\varphi : A \to \operatorname{End}_k(V)$ of k-algebras. To give a right A-module structure is the same as giving a k-module structure V together with a homomorphism $\varphi : A \to \operatorname{End}_k(V)^{\mathrm{op}}$.*

1.4 Representation of quivers

Let k be a field and Q be a quiver.

Definition 1.4.1. A **representation** X of Q over k consists of

- a k-vector space X_i for all $i \in Q_0$,
- a k-linear map
$$X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)}$$
for each $\alpha \in Q_1$

Example 1.4.2 (Continue example 1.2.4). i) Let $Q = \cdot$. Then a representation of Q is simply a k-vector space.

ii) Let $Q = \begin{tikzcd} 1 \arrow[loop right] \end{tikzcd}$. Then a representation of Q is a k-vector space V together with an endomorphism $\varphi \in \mathrm{End}_k(V)$:

$$Q = \begin{tikzcd} V \arrow[loop right, "f"] \end{tikzcd} .$$

iii) Let $Q = \begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}$, the **Kronecker Quiver**. Then a representation of Q is a pair of vector spaces V, W and two linear maps $f, g \in \hom_K(V, W)$:

$$Q = \begin{tikzcd} V \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & W \end{tikzcd}$$

Definition 1.4.3. Take X, Y to be two representations of Q over k. A **homomorphism of representations** $f : X \to Y$ is a tupel $(f_i)_{i \in Q_0}$ of linear maps $f_i : X_i \to Y_i$, such that for all $\alpha \in Q_1$ the diagram

$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "f_{s(\alpha)}"] \arrow[d, "X_\alpha"'] & Y_{s(\alpha)} \arrow[d, "Y_\alpha"] \\ X_{t(\alpha)} \arrow[r, "f_{t(\alpha)}"'] & Y_{t(\alpha)} \end{tikzcd}$$

commutes.

Example 1.4.4 (Continue example 1.4.2). i) Homomorphisms of representations are k-linear maps $X \to Y$.

ii) Homomorphisms of representations (V, φ) and (W, ψ) are k-linear maps $f : V \to W$, such that

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi"'] & W \arrow[d, "\psi"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

commutes

iii) Homomorphisms of representations (V_1, V_2, A, B) and (W_1, W_2, C, D) are pairs (f_1, f_2) of linear maps $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$, such that $A \circ f_1 = f_2 \circ A$ and $B \circ f_1 = f_2 \circ B$.

Definition 1.4.5. An **isomorphism of representations** $f : X \to Y$ is a homomorphism of representations, such that there exists $g : Y \to X$ homomorphism of representations satisfying

$$g \circ f = \mathrm{id}_X \text{ and } f \circ g = \mathrm{id}_y .$$

An isomorphism of representations is a homomorphism of representations such that each map f_i is bijective.

End of Lecture 2

Let Q be a quiver over a field k.

Remark 1.4.6. i) Let X be a representation of Q over k. Associate a left kQ-module $M = F(X)$ as follows:

- As k-vector space, let
$$M := \bigoplus_{i \in Q_0} X_i.$$
- Define the action of kQ on M by an action of paths. Let p be a path of length ≥ 1. Define
$$X_p : X_{s(p)} \to X_{t(p)} \text{ given by } X_p = X_{\alpha_\ell} \circ \ldots \circ X_{\alpha_i},$$
with $X_p \in \hom_k(X_{s(p)}, X_{t(p)})$. Use this to define a k-linear map $\tilde{X}_p : M \to M$ as composition:
$$\tilde{X}_p : M = \bigoplus_{i \in Q_0} X_i \xrightarrow{\pi_{s(p)}} X_{s(p)} \xrightarrow{X_p} X_{t(p)} \xhookrightarrow{\iota_{t(p)}} \bigoplus_{i \in Q_0} X_i$$
If the length of $p = 0$, then p is a lazy part at some $i \in Q_0$, and we set
$$X_{\varepsilon_i} := \mathrm{id}_{X_i},$$
and $\tilde{X}_{\varepsilon_i}$ like $\tilde{X}_p$.
Now these k-linear endomorphisms define a kQ-module structure on M, given by:
$$kQ \times M \to M, \left(a := \sum_{p \in Q_*} \lambda_p \cdot p, (x_i)_i =: x \right) \mapsto a.x := \sum_{p \in Q_*} \lambda_p \cdot \tilde{X}_p(x)$$
$$= \sum_{p \in Q_*} \lambda_p \cdot (\iota_{t(p)} X_p(x_{s(p)})),$$
where we denote an element in M by a sequence $(x_i)_i$ with $x_i \in X_i$.
- We check that this actually defines a kQ-module structure:
  - (L3): Assume that $a, b \in kQ$. By the bilinearity of the multiplication, we can assume that $a = p$ and $b = q$ are both paths in Q_*. Then
  $$a.(b.x) = \tilde{X}_p \left(\tilde{X}_q(x) \right)$$
  $$= \iota_{t(p)} X_p \underbrace{\pi_{s(p)} \iota_{t(q)}} X_q(x_{s(q)}),$$
  where
  $$\pi_{s(p)} \iota_{t(q)} = \begin{cases} \mathrm{id}_{X_q}, & \text{if } t(q) = s(p) \\ 0, & \text{otherwise} \end{cases}.$$
  This gives
  $$a.(b.x) = \begin{cases} \iota_{t(p)} X_p X_q(x_{s(q)}), & \text{if } t(q) = s(p) \\ 0 & \text{otherwise} \end{cases}.$$
  Additionally,
  $$(a.b).x = \begin{cases} \tilde{X}_{p \circ q} & \text{if } f(q) = s(p), \\ 0 & \text{otherwise} \end{cases}.$$

But in the case $f(q) = s(p)$,

$$\tilde{X}_{p\circ q}(x) = \iota_{t(p)} \circ X_p X_q(x_{s(q)}).$$

The construction F is functorial, i.e. for $f : X \to Y$ a homomorphism of representations, F induces a homomorphism of kQ-algebras

$$Ff : F(X) \to F(Y) \text{ by } (Ff)((x_i)_i) := (f_i(x_i))_i.$$

ii) Let M be a left kQ-module. Define a representation $X := G(M)$ as follows

- As k-vector spaces, set
$$X_i := \varepsilon_i M.$$
- For $\alpha \in Q_*$, set
$$X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)},\ \varepsilon_{s(\alpha)}x \mapsto \alpha\varepsilon_{s(\alpha)}x = \alpha x = \varepsilon_{t(\alpha)}\alpha x \in X_{t(\alpha)},$$
as $\varepsilon_{t(\alpha)}\alpha = \alpha\varepsilon_{s(\alpha)}$.
So $X := ((X_i)_{i\in Q_0}, (X_\alpha)_{\alpha\in Q_*})$ is a representation of Q.
- This construction is also functorial: take $g : M \to N$ a homomorphism of left kQ-modules. Define $G(g) : X \to Y$, with $X := G(M)$ and $Y := G(N)$. Set
$$(Gg)_i := X_i \to Y_i,\ \varepsilon_i x \mapsto g(\varepsilon_i x) = \varepsilon_i g(x) \text{ with } X_i := \varepsilon_i M \text{ and } Y_i := \varepsilon_i N.$$
This is indeed a homomorphism of representations: Let $\alpha \in Q_1$ be arbitrary, and consider
$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "(Gg)_{s(\alpha)}"] \arrow[d, "X_\alpha"'] & Y_{s(\alpha)} \arrow[d, "Y_\alpha"] \\ X_{t(\alpha)} \arrow[r, "(Gg)_{t(\alpha)}"'] & Y_{t(\alpha)} \end{tikzcd}.$$
Then
$$y_\alpha(Gg)_{\varepsilon_{s(\alpha)}}x = Y_\alpha(g(\varepsilon_{s(\alpha)}x)) = \alpha g(\varepsilon_{s(\alpha)}x)$$
and
$$(Gg)_{\varepsilon_{t(\alpha)}}(X_\alpha(\varepsilon_{s(\alpha)}x)) = g(\alpha(\varepsilon_{s(\alpha)}x)) = \alpha g(\varepsilon_{s(\alpha)}x),$$
hence the diagram commutes.

Theorem 1.4.7. *i) Let M be a left kQ-module. Then $FG(M) \cong M$ as left kQ-modules.*

ii) Let X be a representation of Q over k. Then $GF(X) \cong X$ as representations of Q.

Proof. i) Denote $X := G(M)$. Then

$$F(X) = \bigoplus_{i\in Q_0} X_i = \bigoplus_{i\in Q_0} \varepsilon_i M$$

as a k-vector space. Observe

- The identity in kQ is given by

$$\mathrm{id}_{kQ} = \sum \varepsilon_i.$$

Hence, for all x in X:

$$x = \mathrm{id}_{kQ}\, x = \left(\sum \varepsilon_i\right) x = \sum (\varepsilon_i x_i) \in \sum \varepsilon_i M.$$

- For all $i \neq j$, $\varepsilon_i \varepsilon_j = 0$ holds. So for

$$x \in X_i = \varepsilon_i M \bigcap \sum_{j \neq i} \varepsilon_j M_j \implies x = \sum_{j \neq i} \varepsilon_j x_j \text{ for some } x_j \in M.$$

But as $x \in \varepsilon_i M$, $\varepsilon_i x = x$. So

$$x = \varepsilon_i x = \varepsilon_i \left(\sum_{j \neq i} \varepsilon_j x_j\right) = \sum_{j \neq i} \varepsilon_i \varepsilon_j x_j = 0.$$

These observations show that

$$\varphi : F(X) = \bigoplus_{i \in Q_0} \varepsilon_i M \longrightarrow M \;,\; (\varepsilon_i x_i)_{i \in Q_0} \mapsto \sum \varepsilon_i x_i$$

is an isomorphism of k-vector spaces.
Show that φ is isomorphism of left kQ-modules:
Without loss generaly, assume that $a = p$ is a path in Q (regarded as element of kQ), and let $x \in M$. Then

$$\begin{aligned} \varphi(a.x) = \varphi\left(\iota_{t(p)} X_p(x_{s(p)})\right) &= \varphi\left(\iota_{t(p)}(p x_{s(p)})\right) \\ &= p x_{s(p)}, \end{aligned}$$

and

$$a.\varphi(x) = a.\sum x_i = a.\sum \epsilon_i x_i = p x_{s(p)}.$$

ii) Let $M := F(X)$ be the left kQ-module associated with X. Then

$$G(M)_i = \varepsilon_i M = \varepsilon_i \bigoplus_{j \in Q_0} X_j = X_i,$$

and

$$\left(G(M)\right)_\alpha (x_{s(\alpha)}) = \alpha . x_{s(\alpha)} = X_\alpha(x_{s(\alpha)}).$$

Be careful! $X_i \subset \oplus X_j$ is still different from X_i as part of the representation, because one is a subspace and one is just a space. So the appropriate isomorphism in this case would be

$$G(M)_i = X_i \xrightarrow{x_i \mapsto x_i} X_i,$$

which is a morphism of representations, as for any $\alpha \in Q_1$

$$\begin{tikzcd} G(M)_{s(\alpha)} \arrow[d, equal] & \\ X_{s(\alpha)} \arrow[r] \arrow[d, "(G(M))_\alpha"'] & X_{s(\alpha)} \arrow[d, "X_\alpha"] \\ X_{t(\alpha)} \arrow[r] \arrow[d, equal] & X_{t(\alpha)} \\ G(M)_{t(\alpha)} & \end{tikzcd}$$

commutes.

□

Remark 1.4.8. Let M be a left kQ-module, with Q finite and k a field.

i) $\dim_k M = \sum_{i \in Q_0} \dim_k X_i$ where $X = G(M)$,where G is the functor from remark 1.4.6

ii) $\dim_K kQ < \infty \iff Q$ contains no **oriented cycles** (a path p of length ≥ 1, such that $s(p) = t(p)$)

iii) If Q has no oriented cycle, then the following are equivalent:

  (a) M is a finitely generated kQ-module.

  (b) $\dim_k X_i < \infty$.

Proof. (a) $\implies$ (b):(b) implies in particular, that M is finitely generated as a k-module. But as $k \subset kQ$, (a) follows immediatley.

(b) $\implies$ (a): Set $A := kQ$,and let $x_1, ..., x_n \in kQ$ generate M as a left kQ-module. Then there is a kQ-linear surjection

$$A^n \longrightarrow\!\!\!\!\!\rightarrow M$$

given by $e_i \mapsto x_i$, where the $(e_i)_{1 \leq i \leq n}$ are a basis of A^n. As this is in particular k-linear, we have that

$$\dim_k M \leq \dim_k(A^n) = n \dim_k A < \infty,$$

as Q contains no cycle. □

iv) Under G, the notion of a „left submodule“ corresponds to **subrepresentations** of Q, i.e. a tupel of subspaces $Y_i \subset X_i$ for all $i \in Q_0$ such that $X_\alpha(Y_{s(\alpha)}) \subset Y_{t(\alpha)}$ for all $\alpha \in Q_1$.

v) Under G, a direct sum of modules corresponds to **direct sum of representations**: Given X, Y two representations of Q, define a new representation $X \oplus Y$ where the vector spaces are given by

$$(X \oplus Y)_i := (X \oplus Y)_i$$

and the k-linear maps

$$(X \oplus Y)_\alpha : X_{s(\alpha)} \oplus Y_{s(\alpha)} \longrightarrow X_{t(\alpha)} \oplus Y_{t(\alpha)}$$

given by

$$\left(\begin{array}{c|c} X_\alpha & \\ \hline & Y_\alpha \end{array}\right).$$

1.5 Bimodules and tensor products

Definition 1.5.1. Let A, B be k-algebras. A A-B-**bimodule** M is a set M, together with maps:

$$A \times M \longrightarrow M, (a, x) \longmapsto ax$$

$$M \times B \longrightarrow M, (x, b) \longmapsto xb,$$

such that

i) M is a left A-module

ii) M is a right B-module

iii) for all $a \in A, b \in B$ and $x \in M$, the relation

$$(ax)b = a(xb)$$

holds.

We denote a A-B-bimodule by

$${}_AM_B.$$

Lemma 1.5.2. *Let A, B, C be k-algebras, and consider ${}_AM_B$ and ${}_AN_C$, a A-B-bimodule and a A-C-bimodule respectivley. Then $\hom_A(M, N)$ becomes a B-C-bimodule via*

- $B \times \hom_A(M, B) \longrightarrow \hom_A(M, N), \qquad (b, f) \longmapsto bf : M \to N, x \mapsto f(xb)$
- $\hom_A(M, N) \times C \longrightarrow \hom_A(M, N), \qquad (f, c) \longmapsto fc : M \to N, x \mapsto f(cx)$

Proof.

- well-defined:

$$bf(ax) = f((ax)b) = f(a(xb)) = af(xb) = a(bf)(x)$$

- $\hom_A(M,N)$ is a left B-module: Show e.g. (L3):

$$\begin{aligned}((bb')f)(x) = (x(bb')) &= f((xb)b') \\ &= b'(f(xb)) \\ &= b((b'f)(x))\end{aligned}$$

- compatibility:

$$((af)b)(x) = f((ax)b) = f(a(xb)) = (a(fb))(x).$$

□

End of Lecture 3

Definition 1.5.3. Let A be a k-algebra, and M_A be a right A-module, ${}_AN$ a left A-module.

i) Let P be a k-module. A map
$$\varphi : M \times N \to P$$
is called **A-balanced** if
   - $\varphi(x + x', y) = \varphi(x, y) + \varphi(x', y)$
   - $\varphi(x, y + y') = \varphi(x, y) + \varphi(x, y')$
   - $\varphi(xa, y) = \varphi(x, ay)$
   - φ is k-linear.

ii) A pair (T, τ) where T is a k-module and τ a A-balanced map $M \times N \to T$ is called a **tensor product** of M with N over A, if the follwing universal property holds: For all A-balanced maps $\varphi : M \times N \to P$, where P is any k-module, there is a unique k-linear map f, such that
$$\begin{tikzcd} M \times N \arrow[r, "\tau"] \arrow[d, "\varphi"'] & T \arrow[dl, dashed, "\exists! f\ k\text{-linear}"] \\ P & \end{tikzcd}$$
commutes.

Lemma 1.5.4. *Let A be a k-algebra, M_A a right A-module, ${}_AN$ a left A-module.*

i) There exists a tensor product (T, τ) of M with N over A.

ii) This tensor product is unique up to unique isomorphism. More precisely, if (T', τ') is any other tensor product, then there exists a unique isomorphism of k-modules $f : T \to T'$, such that $f \circ \tau = \tau'$.

Proof. i) *existence:* Let F be the free k-module with basis $M \times N$, and U the submodule generated by elements of the form

- $(x + x', y) - (x, y) - (x', y)$

- $(x, y + y') - (x, y) - (x, y')$
- $(xa, y) - (x, ay)$
- $(\lambda x, y) - \lambda(x, y)$
- $(x, \lambda y) - \lambda(x, y)$

Then F/U is a k-module, and

$$\tau : M \times N \longrightarrow F \longrightarrow F/U$$

is A-balanced by definition. We set

$$a \otimes b := \tau((a, b)) \text{ and } M \otimes_A N := F/U.$$

The pair $(M \otimes_A N, \otimes)$ satisfies the universal property of a tensor product:
Let $\varphi : M\times \to P$ be A-balanced. Then there exists a unique $\hat{\varphi}$ such that the following diagram commutes, as F is free with basis $M \times N$:

$$\begin{tikzcd} M \times N \arrow[r, hook] \arrow[d, "\varphi"'] & F \arrow[dl, dashed, "\exists!\hat{\varphi}"] \\ P & \end{tikzcd}.$$

Since φ is A-balanced, $\hat{\varphi}$ factors through F/U, i.e.:

$$\begin{tikzcd} M \times N \arrow[r, hook] \arrow[d, "\varphi"'] & F \arrow[r, "\pi"] \arrow[dl, "\hat{\varphi}"'] & F/U \arrow[dll, dashed, "\exists!\overline{\hat{\varphi}}"] \\ P & & \end{tikzcd}.$$

Now set $\tau := \overline{\hat{\varphi}}$.

ii) later...

□

Lemma 1.5.5. *Let A, B, C be k-algebras, ${}_AM_B$ and ${}_BN_C$ bimodules. Then $M \otimes_B N$ is a A-C-bimodule, via*

- $a(x \otimes y) = (ax) \otimes y$
- $(x \otimes y)c = x \otimes (yc),$

for all $x \in X, y \in Y, a \in A, c \in C$.

Proof. $M \otimes_B N$ is already a k-module. For all $a \in A$ define

$$\tau_a : M \times N \longrightarrow M \times N \xrightarrow{\otimes} M \otimes_B N$$

$$(x, y) \longmapsto (ax, y) \longmapsto (ax) \otimes y$$

τ_a is A-balanced:

- $\tau_a((x+x',y)) - \tau_a((x,y)) - \tau_a((x',y)) = (a(x+x')) \otimes y - (ax) \otimes y - (ax') \otimes y = 0$
- $\tau_a((xb,y)) - \tau_a((x,by)) = (a(xb)) \otimes y - (ax) \otimes (by) = ((ax)b) \otimes y - (ax) \otimes (by) = 0$

So it factors through $M \otimes_B N$ as follows:

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes"] \arrow[d, "\tau_a"] & M \otimes_B N \arrow[dl, "\exists! f_a \text{ bilinear}"] \\
M \otimes_B N &
\end{tikzcd}$$

The map

$$A \to \mathrm{End}_k(M \otimes_B N),\ a \mapsto f_a$$

is a homomorphism of k-algebras. So by proposition 1.3.8, $M \otimes_B N$ is a left A-algebra.
If we consinder the map

$$\begin{aligned}
\tau_c : M \times N &\longrightarrow M \times N \longrightarrow M \otimes_B N \\
(x,y) &\longmapsto (x,yc) \longmapsto x \otimes (yc)
\end{aligned}$$

$\square$

Remark 1.5.6. Let $\varphi : A \to B$ be a k-algebra homomorphism, ${}_BN$ a left B-module. Then $M := A$ is a A-B-bimodule via

$$a.x.b := ax\varphi(b) \text{ for all } a \in A, x \in M, b \in B.$$

Then by lemma 1.5.5, $A \otimes_B N$ is a left A-module, where we think of B as a right k-module. This construction is sometimes called **extension of scalars** or **induction of B by A**.

Lemma 1.5.7. *Let A, B, C, D be k-algebras and consider*

$${}_AM_B,\ {}_A(M_i)_B\ (i \in I),\ {}_BN_C,\ {}_B(N_j)_C\ (j \in J).$$

Then there are isomorphisms:

i)

$$\begin{aligned}
\left(\bigoplus_{i\in I} M_i\right) \otimes_B N &\longrightarrow \bigoplus_{i \in I} (M_i \otimes_B N) \\
(x_i) \otimes y &\longmapsto (x_i \otimes y)
\end{aligned}$$

of A-C-bimodules.

ii)

$$M \otimes_B \left(\bigoplus_{j \in J} N_j \right) \longrightarrow \bigoplus_{j \in J} (M \otimes_B N_j)$$

$$x \otimes (y_j) \longmapsto (x \otimes y_j)$$

of A-C-bimodules.

iii)

$$(M \otimes_B N) \otimes_C P \longrightarrow M \otimes_B (N \otimes_C P)$$

$$(x \otimes y) \otimes z \longmapsto x \otimes (y \otimes z)$$

of A-B-bimodules.

iv)

$$A \otimes_A M \longrightarrow M$$

$$a \otimes x \longmapsto ax$$

$$M \otimes_B \longrightarrow M$$

$$x \otimes b \longmapsto xb$$

of A-B-bimodules.

Proof. This is supposed to beexactly the same as [**franzen**]. □

Proposition 1.5.8. *Let A, B be k-algebras, and M_A, ${}_AN_B$ and P_b (bi)-modules. The map*

$$\hom_B(M \otimes_A N, P) \to \hom_A(M, \hom_B(N, P)), \; f \mapsto \Phi(f) \text{ where } \Phi(f)(x)(y) \mapsto f(x \otimes y),$$

for $x \in M$ and $y \in N$.
is a well-defined isomorphism of k-modules, natural in M, N, P.

Proof.

- $\varphi(f)(x)$ is right B-module homomorphism:

$$\Phi(f)(x)(yb) = f(x \otimes yb) = f((x \otimes y)b) = f(x \otimes y)b$$

- $\varphi(f)$ is right A-module homomorphism:

$$\Phi(f)(xa)(y) = f(xa \otimes y) = f(a \otimes ay)$$
$$(\Phi(f)(x)a)(y) = \Phi(f)(x)(ay) = f(a \otimes ay)$$

- Φ is k-linear.

So Φ is a well-defined map of k-modules.

Φ has inverse: Let $g \in \hom_A(M, \hom_B(N, P))$. Define

$$\psi(g) : M \times N \to P, \ (x, y) \mapsto g(x)(y).$$

Then $\psi(g)$ is A-balanced, so it factors through the tensor product:

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[d, "\psi(g)"'] & M \otimes_A N \arrow[dl, "\exists! \hat{\psi}(g) \ k\text{-linear}"] \\ P & \end{tikzcd}$$

The map $\hat{\varphi}(g)$ is also as right B-module homomorphism, $g(x) \in \hom_B(N, P)$.

But the map
$$\hat{\psi} : \hom_A(M, \hom_B(N, P)) \to \hom_B(M \otimes N, P)$$
is the inverse of φ, as

- $\hat{\psi}(\Phi(f))(x \otimes y) = \Phi(f)(x)(y) = f(x \otimes y)$
- $\Phi(\hat{\psi}(g))(x)(y) = \hat{\psi}(g)(x \otimes y) = g(x)(y)$

for all $x \in M$ and $y \in N$. $\square$

Definition 2.0.1. A category $\mathcal{C}$ consists of:

- A class $\mathrm{Ob}(\mathcal{C})$, whose elements are called the **objects** of $\mathcal{C}$.
- For all $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathcal{C}(X, Y)$. An element of $\mathcal{C}(X, Y)$ is called a **morphism** from X to Y as is denoted by
$$f : X \to Y.$$
- For all $X, Y, Z \in \mathrm{Ob}\,(\mathcal{C})$, a map
$$\varphi(x, y) \times \varphi(y, z) \to \varphi(x, z),\ (f, g) \mapsto g \circ f.$$

These should satisfy:

- (L1): Associativity: For all $X, Y, Z, W \in \mathrm{Ob}\,(\mathcal{C})$, and morphisms
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W,$$
we have
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
- (L2): Identity: For all $X \in \mathrm{Ob}\,(\mathcal{C})$, there is a morphism $\mathrm{id}_X \in \mathcal{C}(X, X)$, such that for any object $Y \in \mathrm{Ob}\,(\mathcal{C})$:
$$f \circ \mathrm{id}_X = f \text{ and } \mathrm{id}_X \circ g = g \text{ for all } f : X \to Y, g : Y \to X$$
holds.

Remark 2.0.2. i) $\varphi(X, Y) = \emptyset$ can happen if $X \neq Y$

ii) id_X is unique, as $\mathrm{id}_X = \mathrm{id}_X \circ \mathrm{id}'_X = \mathrm{id}'_X$

Remark 2.0.3. We sometimes want to consider categories whose objects are all sets (with additional conditions). But this can cause logical problems. As a solution, we introduce so called universes. We will always fix a universe, such that sets are elements of this universe, and classes are subsets of this universe. Consider [**catwork**] for further reference.

End of Lecture 4

Example 2.0.4. i) The category $\mathcal{S}et$ of all sets, with

- $\mathrm{Ob}(\mathcal{S}et)$ are all sets in the given universe
- $\mathcal{S}et(X, Y) = \{\text{maps } f : X \to Y\}$

ii) The category $\mathcal{G}rp$ of groups, with group homomorphism as morphisms.

iii) Let A be a k-algebra. Let $A-\mathcal{M}od$ be the category of left A-modules, and $\mathcal{M}od-A$ the category of right A-modules.

iv) The category $\mathcal{T}op$ of topological spaces, with

- $\mathrm{Ob}(\mathcal{T}op)$ the set of all topological spaces,
- $\mathcal{T}op(X, Y) = \{f : X \to Y \mid f \text{ is continous}\}$

v) Let G be a group. Let $\mathcal{C}$ be the category defined as

- $\mathrm{Ob}(\mathcal{C}) = \{*\}$
- $\mathcal{C}(*, *) = G$, with composition defined as $h \circ g := hg$.

vi) Let Q be a quiver. Let $\mathcal{Q}_*$ be the **category of paths** of Q, defined as

- $\mathrm{Ob}(\mathcal{Q}_*) = Q_*$
- for $i, j \in Q_*$, let $\mathcal{Q}_*(i, j) := \{\text{paths } p \text{ in } Q \mid s(p) = i, t(p) = j\}$,
- composition is given by concatination of paths.

This is a category, as composition is associative, and $\mathrm{id}_i = \varepsilon_i$ (the lazy path at i).

Definition 2.0.5. Let $\mathcal{C}$ be a category. The **opposite category** $\mathcal{C}^{\mathrm{op}}$ is defined as

- $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$
- for all $x, y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, the morphisms are defined as $\mathcal{C}^{\mathrm{op}}(X, Y) := \mathcal{C}(Y, X)$
- for $f \in \mathcal{C}^{\mathrm{op}}(X, Y), y \in \mathcal{C}^{\mathrm{op}}(Y, Z)$, set

$$g \circ_{\mathcal{C}^{\mathrm{op}}} f := f \circ_{\mathcal{C}} g$$

2.1 Functors

Definition 2.1.1. Let $\mathcal{C}, \mathcal{D}$ be two categories. A **functor** $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ consists of the following:

- a map
$$\mathscr{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(D),\ X \mapsto F(X)$$
- for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, a map
$$\mathcal{C}(X, Y) \to \mathcal{C}(\mathscr{F}(X), \mathscr{F}(Y)),\ f \mapsto \mathscr{F}(f),$$

such that

- (F1): $\mathscr{F}(\mathrm{id}_X) = \mathrm{id}_{\mathscr{F}(X)}$
- (F2): for all sequences
$$X \xrightarrow{f} Y \xrightarrow{g} Z$$
in $\mathcal{C}$, the relation
$$\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f),$$

holds for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

Remark 2.1.2. What we call a functor is sometimes called a *covariant functor*. A *contravariant functor* is a (covariant) functor $\mathscr{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Remark 2.1.3. i) Let $\mathcal{C}$ be a category. The **identical functor** is given by

$$\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}, \begin{cases} x \mapsto x & \text{on objects} \\ f \mapsto f & \text{on morphism} \end{cases}$$

ii) If
$$\mathcal{C} \xrightarrow{\mathscr{F}} \mathcal{D} \xrightarrow{\mathscr{G}} \mathcal{E}$$
are two functors, then their composition is also a functor.

Example 2.1.4. i) Consider the category $\mathcal{S}et$, and let $P(X)$ denote the power set of a set X. Define

$$\mathscr{P}_* : \mathcal{S}et \to \mathcal{S}et, \begin{cases} x \mapsto P(X) \\ (X \xrightarrow{f} Y) \mapsto P_*(f) : P(X) \to P(Y), A \mapsto f(A) \end{cases}$$

which is a covariant functor, and

$$\mathscr{P}^* : \mathcal{S}et \to \mathcal{S}et, \begin{cases} x \mapsto P(X) \\ (X \xrightarrow{f} Y) \mapsto P^*(f) : P(Y) \to P(X), B \mapsto f^{-1}(B) \end{cases},$$

which is a contravariant functor.

ii) Consider the functors

$$-^* : k\text{−}\mathcal{A}lg \to \mathcal{G}rp, \begin{cases} A \mapsto A^\times \\ (A \xrightarrow{f} B) \mapsto A^* \xrightarrow{f^\times} B^\times \end{cases}$$

and
$$k[-] : \mathcal{G}rp \to k\text{−}\mathcal{A}lg, \begin{cases} G \mapsto k[G] \\ (G \xrightarrow{\varphi} H) \mapsto k[G] \xrightarrow{\varphi} k[H] \end{cases}$$

iii) The functor
$$\mathcal{G}rp \to \mathcal{S}et, \begin{cases} G \mapsto G \\ f \mapsto f \end{cases}$$
is called a **forgetful functor** . Other examples of forgetful functors are

- $\mathcal{T}op \to \mathcal{S}et$
- $A{-}\mathcal{M}od \to k{-}\mathcal{M}od$

Example 2.1.5. Let $\mathcal{C}$ be a category and $X \in \mathrm{Ob}(\mathcal{C})$ an object in $\mathcal{C}$. Consider

i)
$$H^X : \mathcal{C} \to \mathcal{S}et, \begin{cases} Y \mapsto \mathcal{C}(X,Y) \\ (Y \xrightarrow{f} Y') \mapsto H^X(f) : \mathcal{C}(X,Y) \to \mathcal{C}(X,Y'),\ g \mapsto f \circ g \end{cases}.$$

We also denote this as $H^X =: \mathcal{C}(X,-)$. This is a covariant functor.

ii)
$$H_X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et, \begin{cases} Z \mapsto \mathcal{C}(Z,X) \\ (Z \xrightarrow{f} Z') \mapsto H_X(f) : \mathcal{C}(Z',X) \to \mathcal{C}(Z,X),\ g \mapsto g \circ f \end{cases}.$$

We also denote this as $H_X =: \mathcal{C}(-,X)$. This is a contravariant functor.

Definition 2.1.6. Let $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ and consider the induced map

$$\mathcal{C}(X,Y) \to \mathcal{D}(\mathscr{F}(X), \mathscr{F}(Y)).$$

i) If this map is injective, then F is called **faithful**.

ii) If this map is surjective, then F is called **full**.

iii) F is **fully faithful**, if F is full and faithful.

iv) F is **dense** or **essentially surjective**, if for any $Y \in \mathcal{D}$, there is an object $X \in \mathcal{C}$, such that $\mathscr{F}(X) \cong Y$

2.2 Isomorphism

Definition 2.2.1. Let $\mathcal{C}$ be a category. A morphism $f : X \to Y$ in $\mathcal{C}$ is called an **isomorphism**, if there is a $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Remark 2.2.2. i) Identities are isomorphism.

ii) The morphism g (if it exists) is uniquely determined by f. We therefore call $g =: f^{-1}$.

iii) If $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is a functor and f an isomorphism in $\mathcal{C}$, then $\mathscr{F}(f)$ is an isomorphism in $\mathcal{D}$.

Example 2.2.3. i) In $\mathcal{S}et$, $\mathcal{G}rp$, $A{-}\mathcal{M}od$, the following are equivalent:

- f is an isomorphism
- f is bijective.

ii) In $\mathcal{T}op$, not all bijective maps are isomorphism.

iii) In $\mathcal{Q}_*$, the only isomorphisms are the lazy paths, because lengths of paths are additive.

2.3 Natural transformations

Definition 2.3.1. Let $\mathscr{F}, \mathscr{G} : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors. A **natural transformation**

$$\eta : \mathscr{F} \to \mathscr{G}$$

is a family of morphisms

$$\{\eta_X\}_{X \in \mathrm{Ob}(\mathcal{C})} : \mathscr{F}X \to \mathscr{G}X$$

in $\mathcal{D}$, such that for all $X, Y \in \mathcal{C}$ and morphisms $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes

$$\begin{tikzcd} \mathscr{F}X \arrow[r, "\eta_X"] \arrow[d, "\mathscr{F}f"'] & \mathscr{G}X \arrow[d, "\mathscr{G}f"] \\ \mathscr{F}Y \arrow[r, "\eta_Y"'] & \mathscr{G}Y. \end{tikzcd}$$

Remark 2.3.2. i) For two natural transformations

$$\mathscr{F} \xrightarrow{\eta} \mathscr{G} \xrightarrow{\xi} \mathscr{H}, \ (\mathscr{F}, \mathscr{G}, \mathscr{H} : \mathcal{C} \to \mathcal{D}),$$

we can define the composition

$$\xi \circ \eta : \mathscr{F} \to \mathscr{H}$$

by

$$(\xi \circ \eta)_X \mathscr{F}X \to \mathscr{H}X, \ (\xi \circ \eta)_X := \xi_X \circ \eta_X.$$

ii) For $\mathscr{F} : \mathcal{C} \to \mathcal{D}$, we have **identical transformation**

$$\mathrm{id}_{\mathscr{F}} \text{ given by } (\mathrm{id}_{\mathscr{F}})_X := \mathrm{id}_{\mathscr{F}X}$$

the part about the natural transformations on the exe-sheets is still missing.

End of Lecture 5

Definition 2.3.3. Let $\mathscr{F}, \mathscr{G} : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors. A natural transformation $\eta : \mathscr{F} \to \mathscr{G}$ is called a **natural isomorphism** if for all $x \in \mathcal{C}$, η_x is an isomorphism in $\mathcal{D}$.
η is a natural transformation if and only iff there is a natural transformation $\zeta : \mathscr{G} \to \mathscr{F}$, such that $\zeta \circ \eta = \mathrm{id}_{\mathscr{F}}$ and $\eta \circ \zeta = \mathrm{id}_{\mathscr{G}}$.
If η is a natural isomorphism, we write $\eta : F \xrightarrow{\cong} G$. If there is a natural transformation between two functors, we denote this by $\mathscr{F} \cong \mathscr{G}$.

Being naturally isomorphic defines an equivalence relation on the functor catogory

Definition 2.3.4. A functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories**, if there is a functor $\mathscr{G} : \mathcal{D} \to \mathcal{C}$, such that

$$\mathscr{G} \circ \mathscr{F} \cong \mathrm{id}_{\mathcal{C}} \text{ and } \mathscr{F} \circ \mathscr{G} \cong \mathrm{id}_{\mathcal{D}}\,.$$

If for two categories $\mathcal{C}, \mathcal{D}$ an equivalence of categories $\mathcal{C} \to \mathcal{D}$ exists, we say that $\mathcal{C}$ and $\mathcal{D}$ are **equivalent**, and write $\mathcal{C} \simeq \mathcal{D}$.

Example 2.3.5. Some examples for equivalences of categories:

i) Let Q be a finite quiver. Then theorem 1.4.7 shows that

$$\mathcal{R}ep_k(Q) \simeq kQ - \mathcal{M}od.$$

Proof. For the functors

$$\mathscr{F} : \mathcal{R}ep_k(Q) \to kQ - \mathcal{M}od \text{ and } \mathscr{G} : kQ - \mathcal{M}od \to \mathcal{R}ep_k(Q)$$

constructed in remark 1.4.6 the relationships $\mathscr{G}\mathscr{F}(X) \cong X$ and $\mathscr{F}\mathscr{G}(M) \cong M$ hold for any representation X and kQ-modules M, by theorem 1.4.7. So it suffice to check naturallity:
Let $M, N \in kQ - \mathcal{M}od$ be two kQ-modules. Recall that as k-vector space,

$$\mathscr{F}\mathscr{G}(M) = \bigoplus_{i \in Q_0} \varepsilon_i M$$

where ε_i denotes the lazy path at $i \in Q_0$, and the isomorphism is given by

$$\varphi : \mathscr{F}\mathscr{G}(M) = \bigoplus_{i \in Q_0} \varepsilon_i M \longrightarrow M \ , \ (\varepsilon_i x_i)_{i \in Q_0} \mapsto \sum \varepsilon_i x_i.$$

Now let $\alpha \in kQ - \mathcal{M}od(M, N)$ be a homomorphism of left kQ-modules M, N. We need to show that the diagram

$$\begin{tikzcd}
\mathscr{F}\mathscr{G}(M) \arrow[d, equal] & \\
\bigoplus_{i \in Q_0} \varepsilon_i M \arrow[r, "\varphi"] \arrow[d, "\mathscr{F}\mathscr{G}(\alpha)"] & MM \arrow[d, "\alpha"'] \\
\bigoplus_{i \in Q_0} \varepsilon_i N \arrow[r, "\varphi"] \arrow[d, equal] & N \\
\mathscr{F}\mathscr{G}(N) &
\end{tikzcd}$$

commutes, i.e. that φ is actually a natural isomorphism:

$$\begin{aligned}
\alpha\left(\varphi(\varepsilon_i x_i)\right) &= \alpha\left(\sum_{i \in Q_0} \varepsilon_i x_i\right) \\
&= \sum_{i \in Q_0} \varepsilon_i \alpha(x_i),
\end{aligned}$$

as α is kQ-linear. Furthermore:

$$\begin{aligned}
\varphi\left(\mathscr{F}\mathscr{G}(M)\right) &= \varphi\left((\varepsilon_i \alpha(x_i)\right) \\
&= \sum_{i \in Q_0} \varepsilon_i \alpha(x_i)
\end{aligned}$$

So φ is indeed a natural transformation. The other map is also a natural transformation. This follows from the fact that it is an isomorphism of representations, as was shown in the proof of theorem 1.4.7. $\square$

ii) Let G be a group. A **representation of** G is pair (V, ρ) consisting of a k-vector space V and a group homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

A morphism of (group)representations $f : (V, \rho) \to (W, \sigma)$ is a k-linear map $f : V \to W$, such that for all $g \in G$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "f"] & W \end{tikzcd}.$$

We will show maybe on the next sheet, who knows...that we can define an equivalence of categories

$$\mathcal{R}ep_k(G) \simeq k[G] - \mathcal{M}od.$$

iii) The category of k-$\mathcal{A}lg$ is equivalent to the category $\mathcal{C}$ of pairs (A, φ), where A is a ring and $\varphi : k \to Z(A)$ is a ring homomorphism and morphisms correspond to ring homomorphism $f : A \to B$, such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \\ Z(A) \arrow[u, hook] & & Z(B) \arrow[u, hook] \\ & k \arrow[ul, "\varphi_A"] \arrow[ur, "\varphi_B"'] & \end{tikzcd}.$$

show that this is actually all natural and well-defined

iv) Let A be a k-algebra. The category $A - \mathcal{M}od$ of left A-modules is equivalent the category $\mathcal{D}$ of pairs (V, ρ) of k-vector spaces V and homomorphisms of k-algebras:

$$\varphi : A \to \mathrm{End}_k(V).$$

Morphisms $(V, \varphi) \to (W, \psi)$ in $\mathcal{D}$ are given by k-linear maps $f : V \to W$, such that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi(a)"'] & W \arrow[d, "\psi"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

commutes for all $a \in A$.

2.4 Functor Categories

Definition 2.4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Define the **functor category** $\mathcal{F}un(\mathcal{C},\mathcal{D})$ by

- objects are all functors $\mathcal{C} \to \mathcal{D}$; i.e. $\mathrm{Ob}(\mathcal{F}un(\mathcal{C},\mathcal{D})) := \{\mathscr{F} : \mathcal{C} \to \mathcal{D} \mid \mathscr{F} \text{ is a functor}\}$
- morphism between functors are natural transformations; i.e. for $\mathscr{F},\mathscr{G} : \mathcal{C} \rightrightarrows \mathcal{D}$, set $\mathcal{F}un(\mathscr{F},\mathscr{G})(\mathcal{C},\mathcal{D}) := \{\eta : \mathscr{F} \to \mathscr{G} \mid \eta \text{ is a natural transformation}\}$
- Composition of morphisms is given by composition of natural transformations.

Remark 2.4.2. We are running into set-theoretic issues again. If $\mathcal{C}$ and $\mathcal{D}$ are categories in a fixed universe U (i.e. $\mathrm{Ob}(\mathcal{C}), \mathrm{Ob}(\mathcal{D}) \subseteq U$) then $\mathrm{Ob}(\mathcal{F}un(\mathcal{C},\mathcal{D}))$ might not be a subset of U any longer. As a solution, we choose another universe V, s.t. $U \in V$. Then $\mathrm{Ob}(\mathcal{F}un(\mathcal{C},\mathcal{D})) \subseteq V$ and $\mathcal{F}un(\mathcal{C},\mathcal{D})$ is a category in V.

Example 2.4.3. Let Q be a quiver, k a field. Consider the functor category $\mathcal{F}un(\mathcal{Q}_*, k{-}\mathcal{M}od)$. For $\mathscr{V} \in \mathrm{Ob}(\mathcal{F}un(\mathcal{Q}_*, k - \mathcal{M}od))$, $\mathscr{V}$ is a functor

$$\mathscr{V} : \mathcal{Q}_* \to k{-}\mathcal{M}od, \begin{cases} \mathrm{Ob}(\mathcal{Q}_*) = Q_* \ni i \mapsto V(i) & \text{a vector space} \\ \mathcal{Q}_*(i,j) \ni p \to V(p) : V(i) \to V(p) & \text{a } k\text{-linear map} \end{cases}$$

We now have a forgetful functor

$$\mathscr{F} : \mathcal{F}un(\mathcal{Q}_*, k{-}\mathcal{M}od) \to \mathcal{R}ep_k(Q),$$

forgetting all paths of length > 1.

Conversley, let X be a representation of Q over k. Define a functor

$$\mathscr{G}X : \mathcal{Q}_* \to k{-}\mathcal{M}od, \begin{cases} i \mapsto X_i \\ p = \alpha_\ell \circ \ldots \circ X_{\alpha_1} \end{cases}.$$

This yields a functor

$$\mathscr{G} : \mathcal{R}ep_k(Q) \to \mathcal{F}un(\mathcal{Q}_*, k - \mathcal{M}od)$$

We see that

$$\mathscr{G} \circ \mathscr{F} \cong \mathrm{id}_{\mathcal{F}un(\mathcal{Q}_*, k-\mathcal{M}od)} \text{ and } \mathscr{F} \circ \mathscr{G} \cong \mathrm{id}_{\mathcal{R}ep_k(Q)}$$

.

Definition 2.4.4. Let $\mathcal{C}, \mathcal{D}$ be two categories, and $X \in \mathcal{C}$ an object in $\mathcal{C}$. Define

$$\mathrm{ev}_X : \mathcal{F}un(\mathcal{C},\mathcal{D}) \to \mathcal{D}$$

by

- $\mathrm{ev}_X(\mathscr{F}) := \mathscr{F}(X)$

- $\mathrm{ev}_X(\mathscr{F}) \xrightarrow{\eta} \mathscr{G} := \eta_X : \mathscr{F}(X) \to \mathscr{G}(X)$.

ev_X is called the **evaluation at** X.

Remark 2.4.5. Let $\mathcal{C}, \mathcal{D}$ be categories, and $X \in \mathcal{C}$ an object. Then ev_X is indeed a functor, as the associativity of composition of natural transformations is inherited from the associativity of composition in $\mathcal{D}$.

Moreover, for $f : X \to Y$ a morphism in $\mathcal{C}$, we can define a natural transformation between the functors

$$\mathrm{ev}_f : \mathrm{ev}_X \to \mathrm{ev}_Y,$$

which is, for a functor $\mathscr{F}$, just a map $\mathrm{ev}_X(\mathscr{F}) \to \mathrm{ev}_Y(\mathscr{F})$; by considering

$$\begin{tikzcd} \mathrm{ev}_X(\mathscr{F}) \arrow[r, "(\mathrm{ev}_f)_F"] \arrow[d, equal] & \mathrm{ev}_y(F) \arrow[d, equal] \\ \mathscr{F}X \arrow[r, "Ff"] & \mathscr{F}Y \end{tikzcd} \quad \text{and setting } (\mathrm{ev}_f)_F := \mathscr{F}f,$$

i.e. $(\mathrm{ev}_f)_{\mathscr{F}}$ is induced the maps which are induced by $\mathscr{F}$. To show that the (ev_f) define indeed a natural transformation of functors

$$\begin{tikzcd} \mathcal{F}un(\mathcal{C}, \mathcal{D}) \arrow[r, bend left, "\mathrm{ev}_X"] \arrow[r, bend right, "\mathrm{ev}_Y"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D} \end{tikzcd},$$

we need to show that for all maps (i.e. natural transformations) $\eta : \mathscr{F} \to \mathscr{G}$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{ev}_X(\mathscr{F}) \arrow[r, "\mathrm{ev}_X(\eta)"] \arrow[d, "(\mathrm{ev}_f)_{\mathscr{F}}"'] & \mathrm{ev}_X(\mathscr{G}) \arrow[d, "(\mathrm{ev}_f)_{\mathscr{G}}"] \\ \mathrm{ev}_Y(\mathscr{F}) \arrow[r, "\mathrm{ev}_Y \eta"] & \mathrm{ev}_Y(\mathscr{G}) \end{tikzcd}.$$

But this is just inherited, in following way:
Consider the extended diagram:

$$\begin{tikzcd} \mathscr{F}(X) \arrow[r, "\eta_X"] \arrow[d, equal] \arrow[ddd, bend right=60, "\mathscr{F}(f)"'] & \mathscr{G}(X) \arrow[d, equal] \arrow[ddd, bend left=60, "\mathscr{G}(f)"] \\ \mathrm{ev}_X(\mathscr{F}) \arrow[r, "\mathrm{ev}_X(\eta)"] \arrow[d, "(\mathrm{ev}_f)_{\mathscr{F}}"] & \mathrm{ev}_X(\mathscr{G}) \arrow[d, "(\mathrm{ev}_f)_{\mathscr{G}}"'] \\ \mathrm{ev}_Y(\mathscr{F}) \arrow[r, "\mathrm{ev}_Y \eta"] \arrow[d, equal] & \mathrm{ev}_Y(\mathscr{G}) \arrow[d, equal] \\ \mathscr{F}(Y) \arrow[r, "\eta_Y"'] & \mathscr{G}(Y) \end{tikzcd}.$$

As η is a natural transformation, the outer diagram commutes. But this already implies that the inner one does as well.

This enables us to define another functor:

$$\mathrm{ev} : \mathcal{C} \to \mathcal{F}un\left(\mathcal{F}un\left(\mathcal{C}, \mathcal{D}\right), \mathcal{D}\right), \quad \begin{cases} X \mapsto \mathrm{ev}_X \\ f \mapsto \mathrm{ev}_f \end{cases}$$

2.5 Representable functors

We now consider functors of the form

$$\mathcal{C} \to \mathcal{S}et \text{ and } \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et,$$

for an arbitrary category $\mathcal{C}$.

Lemma 2.5.1 (Yoneda)**.** *Let $X \in Ob(\mathcal{C}) = Ob(\mathcal{C}^{\mathrm{op}})$ be an object of $\mathcal{C}$.*

i) Let $\mathscr{F} : \mathcal{C} \to \mathcal{S}et$ be a (covariant) functor. The map

$$Y^{\mathscr{F},X} : \quad \begin{array}{rcl} \left(\mathcal{F}un(\mathcal{C}, \mathcal{S}et)\right)\left(h^X, \mathscr{F}\right) & \longrightarrow & \mathscr{F}(X) \\ \left(\eta : h^X \to \mathscr{F}\right) & \longmapsto & \eta_X(\mathrm{id}_X) \end{array},$$

is a bijection, where

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "h^X"] \arrow[r, bend right, "\mathscr{F}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{S}et \end{tikzcd},$$

is a natural transformation, and

$$\eta_X : \mathcal{C}(X, X) = h^X(X) \to F(X)$$

is just a map.

ii) Let $\mathscr{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ be a (contravariant) functor. The map:

$$Y_{\mathscr{G},X} : \quad \begin{array}{rcl} \left(\mathcal{F}un(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)\right)\left(h_X, \mathscr{G}\right) & \longrightarrow & \mathscr{G}(X) \\ \left(\zeta : h_X \to \mathscr{G}\right) & \longmapsto & \zeta_X(\mathrm{id}_X) \end{array},$$

is a bijection.

Proof. i) Assume that ii) holds, then this follows, as $\mathcal{C} = (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ and

$$h^X_{\mathcal{C}} = \mathcal{C}(-, X) = \mathcal{C}^{\mathrm{op}}(X, -) = h_X^{\mathcal{C}^{\mathrm{op}}}.$$

ii) $Y_{\mathscr{G},X}$ is injective: Let $\xi, \eta \in \left(\mathcal{F}un(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)\right)\left(h_X, \mathscr{G}\right)$ be two natural transformations

$$\xi, \eta : \begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, bend left, "h_X"] \arrow[r, bend right, "\mathscr{G}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{S}et \end{tikzcd}$$

and suppose that
$$\xi_X(\mathrm{id}_X) = \eta_X(\mathrm{id}_X).$$
We need to show that this implies $\xi = \eta$, i.e. $\xi_Y = \eta_Y$ for all $Y \in \mathrm{Ob}\,(\mathcal{C})$. As these are maps of sets, it suffices to show
$$\xi_Y(f) = \eta_Y(f) \text{ for all } f \in h_X(Y) = \mathcal{C}^{\mathrm{op}}(X,Y).$$
As ξ, η are natural transformations, the diagrams
$$\begin{tikzcd} h_X(X) \arrow[d, equal] & \\ \mathcal{C}(X,X)^{\mathrm{op}} \arrow[r, shift left, "\eta_X"] \arrow[r, shift right, "\xi_X"'] \arrow[d, "h_X(f)"'] & \mathscr{G}(X) \arrow[d, "\mathscr{G}(f)"] \\ \mathcal{C}(X,Y)^{\mathrm{op}} \arrow[r, shift left, "\eta_Y"] \arrow[r, shift right, "\xi_X"'] \arrow[d, equal] & \mathscr{G}(Y) \\ h_X(Y) & \end{tikzcd} \tag{D1}$$
and commute. This implies
$$\begin{array}{l} \mathscr{G}(f)(\eta_X(\mathrm{id}_X)) \overset{(\mathrm{D1})}{=} \eta_Y(h_X(f)(\mathrm{id}_X)) = \eta_Y(f) \\ \qquad \| \\ \mathscr{G}(f)(\xi_X(\mathrm{id}_X)) \overset{(\mathrm{D1})}{=} \xi_Y(h_X(f)(\mathrm{id}_X)) = \xi_Y(f), \end{array}$$
so $Y_{\mathscr{G},X}$ is injective.

$Y_{\mathscr{G},X}$ is surjective: Let $z \in \mathscr{G}(X)$ be arbitrary. We need to find a natural transformation
$$\zeta : \begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, bend left, "h_X"] \arrow[r, bend right, "\mathscr{G}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{S}et \end{tikzcd},$$
such that $\zeta_X(\mathrm{id}_X) = z$. Define for $y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$ a map
$$\zeta_Y : \mathcal{C}(X,Y) = h_X(Y) \to \mathscr{G}(Y),\ f \mapsto \mathscr{G}f(z).$$
Show that ζ is indeed a natural transformation: Let $g : Y \to Y'$ be a morphism in $\mathcal{C}$, i.e. $g \in \mathcal{C}^{\mathrm{op}}(Y', Y)$. We have to show that
$$\begin{tikzcd} h_X(Y') \arrow[d, equal] & \\ \mathcal{C}(Y',X) \arrow[r, "zeta_{Y'}"] \arrow[d, "h_X(f)"'] & \mathscr{G}(Y') \arrow[d, "\mathscr{G}(g)"] \\ \mathcal{C}(Y,X) \arrow[r, "\zeta_Y"] \arrow[d, equal] & \mathscr{G}(Y) \\ h_X(Y) & \end{tikzcd}$$

commutes.
Let $u \in \mathcal{C}(Y', X)$. Then

$$\begin{aligned}\mathscr{G}(g)(\zeta_{Y'}(u)) &= \mathscr{G}(g)(\mathscr{G}(u)(z)) \\ &= \mathscr{G}(u \circ g)(z)\end{aligned}$$

and

$$\begin{aligned}\zeta_Y((h_x(g))(u)) &= \zeta_Y(u \circ g) \\ &= \mathscr{G}(u \circ g)(z)\end{aligned}$$

Hence ζ defines a natural transformation, and

$$\zeta_X(\mathrm{id}_X) = (\mathscr{G}(\mathrm{id}_x))(z) = (\mathrm{id}_(\mathscr{G}X))(z) = z$$

□

End of Lecture 6

the recollection of Yonnedas Lemma is ommited.

Theorem 2.5.2 (**Yonneda Embedding**). *Let $\mathcal{C}$ be a category.*

i) The functor

$$h^- : \mathcal{C}^{\mathrm{op}}\mathcal{F}un(\mathcal{C}, \mathcal{S}et), \begin{cases} X & \mapsto h^X = \mathcal{C}(X, -) \\ \mathcal{C}(Y, X) \ni X \to Y & \mapsto h^g : \mathcal{C}(X, -) \to \mathcal{C}(Y, -)\end{cases}$$

is fully faithful (i.e. induces a bijection on morphisms).

ii) The functor

$$h_- : \mathcal{C} \to \mathcal{F}un(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et), \begin{cases} X & \mapsto h_X = \mathcal{C}(-, X) \\ \mathcal{C}(X, Y) \ni f & \mapsto h_f : \mathcal{C}(-, X) \to \mathcal{C}(-, Y)\end{cases}$$

is fully faithful.

Proof. Recall from **??**, that we have to show:
For all $X, Y \in \mathcal{C}$ objecgts, the map

$$\mathcal{C}(X, Y) \to \mathscr{F}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(h_X, h_Y), \; f \mapsto h_f$$

is a bijection.
Here we can use Yonnedas Lemma (lemma 2.5.1), by applying it to the functor $\mathscr{G} := h_Y$. Then we get a bijection

$$\mathscr{F}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(h_X, h_Y) \to h_Y(X) = \mathcal{C}(X, Y), \; \zeta \mapsto \zeta_X(\mathrm{id}_X).$$

So we already have a bijection, but in the opposite direction. Now consider the natural transformation $\zeta := h_f$, then

$$(h_f)X : \mathcal{C}(X, X) \to \mathcal{C}(X, Y) \; g \mapsto X \xrightarrow{g} X \xrightarrow{f} Y,$$

and in particular, $\zeta_X(\mathrm{id}_X) = f$. □

Definition 2.5.3. A functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}et$ is called **representable**, if it is naturally isomorphic to the functor h^X for an object $X \in \mathcal{C}$. A (contravariant) functor $\mathcal{C} \to \mathcal{S}et$ is called representable, if it is naturally isomorphic to h_X.
In both cases, X is called an **representing object** of $\mathscr{F}$ or $\mathscr{G}$ respectivley.

2.6 Equivalence of categories revisted

Dr. Franzen pointed out that none of the topics discussed in the preceding section is relevant for this topic, and that we could have shown this theorem right after the definitions.

Theorem 2.6.1. *Let $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ be a functor. Then the following are equivalent:*

i) $\mathscr{F}$ is an equivalence of categories.

ii) $\mathscr{F}$ is fully faithful and dense (c.f. **??** *for the definition)*

Proof. i) $\implies$ ii): If $\mathscr{F}$ is an equivalence of categories, then we find a functor

$$\mathscr{G} : \mathcal{D} \to \mathcal{C} \text{ such that } \mathscr{G} \circ \mathscr{F} \cong \mathrm{id}_{\mathcal{C}} \text{ and } \mathscr{F} \circ \mathscr{G} \cong \mathrm{id}_{\mathcal{D}}\,.$$

Dense Let $Y \in \mathcal{D}$. Then $\mathscr{F}(\mathscr{G}(Y)) \cong Y$.
fully faithful Let $X, X' \in \mathcal{C}$ be two objects. We want to show that the induced map

$$\mathcal{C}(X, X') \to \mathcal{D}(\mathscr{F}(X), \mathscr{F}(X'))$$

is bijective.
Consider the natural isomorphism $\eta : \mathrm{id}_{\mathcal{C}} \to \mathscr{G}\mathscr{F}$. Applied to objects, this means that for all maps $f : X \to X'$, the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \mathscr{G}\mathscr{F}(X) \arrow[d, "\mathscr{G}\mathscr{F}(f)"] \\ X' \arrow[r, "\eta_{X'}"'] & \mathscr{G}\mathscr{F}X' \end{tikzcd}$$

commutes, and $\eta_X, \eta_{X'}$ are isomorphisms in $\mathcal{C}$. In particular, this implies that η induces a bijection

$$\mathcal{C}(X, X') \to \mathcal{C}(\mathscr{G}\mathscr{F}X, \mathscr{G}\mathscr{F}X'),\ f \mapsto \eta_{X'}^{-1} \circ \mathscr{G}\mathscr{F}f \circ \eta_X, andhence:$$

We have a commutative diagram

$$\begin{tikzcd} \mathcal{C}(X, X') \arrow[r, "\mathscr{F}"] \arrow[d, equal] & \mathcal{D}(\mathscr{F}X', \mathscr{F}X) \arrow[d, "\mathscr{G}"] \\ \mathcal{C}(X, X') \arrow[r, leftrightarrow] & \mathcal{C}(\mathscr{G}\mathscr{F}X', \mathscr{G}\mathscr{F}X) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{D}(\mathscr{F}X, \mathscr{F}X') \arrow[r, "\mathscr{G}"] \arrow[d, equal] & \mathcal{C}(\mathscr{G}\mathscr{F}X', \mathscr{G}\mathscr{F}X) \arrow[d, leftrightarrow] \\ \mathcal{D}(\mathscr{F}X', \mathscr{F}X) & \mathcal{C}(X, X') \arrow[l, "\mathscr{F}"'] \end{tikzcd},$$

so $\mathscr{F}$ is indeed a bijection on morphisms.

ii) $\implies$ i): For every $Y \in \mathcal{D}$, there is an $X \in \mathcal{C}$ such that $\mathscr{F}X \cong Y$ (by density). Choose for any $Y \in \mathcal{D}$ such an object, and call it $X =: \mathscr{G}Y \in \mathcal{C}$. In addition, fix an isomorphism:

$$\xi_Y : \mathscr{F}(\mathscr{G}X) \xrightarrow{\cong} Y$$

Now let $g : Y \to Y'$ be any morphism in $\mathcal{D}$, and consider the morphism

$$\begin{tikzcd} \mathscr{F}\mathscr{G}(Y) \arrow[r, "\cong", "\xi_Y"'] \arrow[rrr, bend left, "\xi_{Y'}^{-1} \circ g \circ \xi_Y"] & Y \arrow[r, "g"] & Y' \arrow[r, "\cong", "\xi_{Y'}^{-1}"'] & \mathscr{F}\mathscr{G}(Y') \end{tikzcd}$$

As $\mathscr{F}$ is fully faithful, there exists a unique

$$\mathscr{G}(g) \in \mathcal{C}(\mathscr{G}(Y), \mathscr{G}(Y')) \text{ such that } \mathscr{F}\mathscr{G}(g) = \xi_{Y'} - 1 \circ g \circ \xi_Y.$$

Now this assignement

$$\begin{tikzcd} \mathcal{D} \arrow[rr] & & \mathcal{C} \\ Y \arrow[d, "g"'] & \mapsto & \mathscr{G}(Y) \arrow[d, "\mathscr{G}(g)"] \\ {} & \mapsto & {} \\ Y' & \mapsto & \mathscr{G}(Y') \end{tikzcd}$$

is functorial. In addition, the $\xi := (\xi_Y)$ define a natural transformation

$$\xi : \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\mathscr{F} \circ \mathscr{G}"] \arrow[r, bend right, "\mathrm{id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D} \end{tikzcd},$$

as for all $Y, Y' \in \mathcal{D}$, the diagram

$$\begin{tikzcd} \mathscr{F}\mathscr{G}(Y) \arrow[r, "\xi_Y"] \arrow[d, "\mathscr{F}\mathscr{G}(g)"'] & Y \arrow[d, "g"] \\ \mathscr{F}\mathscr{G}(Y') \arrow[r, "\xi_{Y'}"'] & Y' \end{tikzcd} \tag{$*$}$$

commutes by the definition of $\mathscr{F}\mathscr{G}(g)$. By construction, ξ is even a natural isomorphism, as all ξ_Y are isomorphisms.
We now show that for this functor $\mathscr{G}$, $\mathscr{G} \circ \mathscr{F} \cong \mathrm{id}_{\mathcal{C}}$ holds as well. Let $X \in \mathcal{C}$ be an object, and consider the induced isomorphism

$$\xi_{\mathscr{F}X} : \mathscr{F}\mathscr{G}\mathscr{F}(X) \xrightarrow{\cong} \mathscr{F}X.$$

As $\mathscr{F}$ is a fully faithful functor, there is exactly one morphism

$$\zeta_X \in \mathcal{C}(\mathscr{G}\mathscr{F}(X), X) \text{ such that } \mathscr{F}(\zeta_X) = \xi_{\mathscr{F}(X)}. \tag{$**$}$$

We now show that the collection $\zeta = (\zeta_X)$ defines indeed a natural transformation

$$\zeta : \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathscr{G}\circ\mathscr{F}"] \arrow[r, bend right, "\mathrm{id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{C} \end{tikzcd} :$$

Let $f : X \to X'$ be any morphism in $\mathcal{C}$. We need to show that the diagram

$$\begin{tikzcd} \mathscr{G}\mathscr{F}(X) \arrow[r, "\zeta_X"] \arrow[d, "\mathscr{G}\mathscr{F}(f)"'] & X \arrow[d, "f"] \\ \mathscr{G}\mathscr{F}(X') \arrow[r, "\zeta_{X'}"'] & X' \end{tikzcd}$$

does commute. For that, we apply $\mathscr{F}$ to the two branches of the diagram:

$$\begin{aligned} \mathscr{F}(f \circ \zeta_X) &= \mathscr{F}(f) \circ \mathscr{F}(\zeta) \\ &\overset{(**)}{=} \mathscr{F}(f) \circ \xi_{\mathscr{F}(X)} \\ &\overset{(*)}{=} \eta_{\mathscr{F}(X')} \circ \mathscr{F}\mathscr{G}\mathscr{F}(f) \\ &= \mathscr{F}(\zeta_X \circ \mathscr{G}\mathscr{F} f). \end{aligned}$$

As $\mathscr{F}$ is faithfull,

$$f \circ \zeta_X = \zeta_{X'} \circ \mathscr{G}\mathscr{F}(f)$$

follows. Furthermore, all ζ_X are isomorphisms, as $\mathscr{F}$ is fully faithful.

□

2.7 Adjunction

Definition 2.7.1. Let $\mathcal{C}, \mathcal{D}$ be two categories. An **adjunction** from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(\mathscr{F}, \mathscr{G}, \varphi)$, consiting of

- two functors $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ and $\mathscr{G} : \mathcal{D} \to \mathcal{C}$
- a family $\varphi = (\varphi_{X,Y})$, indexed by the objects $X \in \mathcal{C}, Y \in \mathcal{D}$, of bijections

  $$\varphi_{X,Y} : \mathcal{D}(\mathscr{F}X, Y) \to \mathcal{C}(X, \mathscr{G}Y)$$

  which are **natural** in X, Y. In this context, naturality is defined as follows:
  For any morphisms $f : X \to X'$ and $g : Y \to Y'$, the diagram

  $$\begin{tikzcd} \mathcal{D}(\mathscr{F}X', Y) \arrow[r, "\varphi_{X',Y}"] \arrow[d, "h_Y(\mathscr{F}(f))"'] & \mathcal{C}(X', \mathscr{G}Y) \arrow[d, "h_{\mathscr{G}Y}(f)"] \\ \mathcal{D}(\mathscr{F}X, Y) \arrow[r, "\varphi_{X,Y}"'] & \mathcal{C}(X, \mathscr{G}y) \end{tikzcd}$$

  commutes. the particular interpretation is still missingIn this case, we call $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ a **left-adjoint** and $\mathscr{G} : \mathcal{D} \to \mathcal{C}$ a **right-adjoint**.

Example 2.7.2. The following are adjoint pairs, where $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is a left-adjoint and $\mathscr{G} : \mathcal{D} \to \mathcal{C}$ is a right-adjoint.

i) $\mathcal{C} = \mathcal{S}et$, $\mathcal{D} = \mathcal{G}rp$. Where $\mathscr{F}X$ is the free group generated by X, and $\mathscr{G} = \mathscr{V}$ is the forgetful functor.

ii) $\mathcal{C} = \mathcal{S}et$, $\mathcal{D} = A - \mathcal{M}od$, and again $\mathscr{F}X$ is the free A-module over X, and $\mathscr{G} = \mathscr{V}$ is the forgetful functor.

iii) Let $\mathcal{C} = \mathcal{G}rp$, $\mathcal{D} = k - \mathcal{A}lg$. Set $\mathscr{F} = k[-]$ and $\mathscr{G} = -^*$. It was shown on Sheet I that this is indeed an adjunction.

iv) Let $\mathcal{C} = \mathcal{S}et$, $\mathcal{D} = k{-}\mathcal{C}omm - \mathcal{A}lg$ the category of commutative algebras. For a set I, $Ft_{i i\in I} := k[t_i]_{i\in I}$, and again $\mathscr{G} = \mathscr{V}$.

v) Let A, B be two k-algebras, and ${}_AN_B$ a A-B-bimodule. Let $\mathcal{C} = \mathcal{M}od - A$ (the category of right A-modules) and $\mathcal{D} = \mathcal{M}od - B$. Set $\mathscr{F}(M_A) := M \otimes_A N$ and $\mathscr{G}(P_B) := \hom_B(N, P)$. In **??**, we showded that this is indeed an adjunction.

vi) Let $\varphi : A \to B$ be a morphism of k-algebras, and set $\mathcal{C} = A{-}\mathcal{M}od$, $\mathcal{D} = B{-}\mathcal{M}od$, and consider $\mathscr{F}({}_AM) := B \otimes_A M$ and $\mathscr{G}$ is restriction of scalars.

vii) $\mathcal{C}$ is given as follows:

- for objects:

$$\mathrm{Ob}(\mathcal{C}) := \{(A, S) \mid A \text{ commutative ring and } S \subset A \text{ multiplicativley clossed}\}$$

- for morphisms:

$$\mathcal{C}\left((A, S), (B, T)\right) := \{f : A \to B \mid f(S) \subseteq t, f \text{ homomorphism of rings}\}$$

Set now
$$\mathscr{F}(A, S) := S^{-1}A \text{ and } \mathscr{G}(B) := (B, B^\times)$$

Remark 2.7.3. Let $(\mathscr{F}, \mathscr{G}, \varphi)$ be an adjunction from $\mathcal{C}$ to $\mathcal{D}$.

i) Define a natural transformation

$$\eta : \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{id}_{\mathcal{C}}"] \arrow[r, bend right, "\mathscr{G}\circ\mathscr{F}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{C} \end{tikzcd},$$

as follows:
Let $X \in \mathcal{C}$ be an object, and set $\eta_X := \varphi_{X,\mathscr{F}X}(\mathrm{id}_{\mathscr{F}X})$. This is indeed a natural transformation:
Let $f : X \to X'$ be a morphism. We need to show that

A.0 Sheet 0

Definition A.0.1. Let X be any set, and k any commutative ring with unit. Define the **free algebra** generated by X:

i) As k-module, set $k\langle X\rangle$ as the free k-module generated by X.

ii) Define the multiplication of two words as the concatination.

Then $k\langle X\rangle$ satisfies the following universal property: Let B be any k-algebra, and $f : X \to B$ a homomorphism of sets. There there is a unique homomorphism of k-algebras $k\langle X\rangle$, such that the following diagram commutes:

$$\begin{tikzcd} k\langle X\rangle \arrow[r, dashed, "\exists! f"] & B \\ X \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}.$$

Proposition A.0.2. *Consider the forgetful functor*

$$\mathscr{F} : k\text{-}\mathcal{A}\} \to \mathcal{S},\ B \mapsto B$$

and

$$\mathscr{K}\langle -\rangle : \mathcal{S} \to k\text{-}\mathcal{A}\},\ X \mapsto k\langle X\rangle.$$

Then for all sets X and k-algebras,

$$\hom_{\mathcal{S}}(X, F(B)) \cong \hom_{k\text{-}\mathcal{A}\}}(k\langle X\rangle, B)$$

holds.

We say that F is **right-adjoint** to $\mathscr{K}\langle -\rangle$.

Problem A.0.1. Consider the k-algebra

$$A := k\langle x, y\rangle/(\langle xy - yx - 1\rangle)$$

over a fiel k with $\operatorname{char} k = 0$. Show that there are no non-zero, finite-dimensional representations of A.

Definition A.0.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A covariant functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is **faithful**, if for all objects A, B of $\mathcal{C}$, the induced function of sets

$$\hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(\mathscr{F}(A), \mathscr{F}(B))$$

is injective; it is **full**, if this function is surjective for all objects A, B of $\mathcal{C}$.

Definition A.0.4. A covariant functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories**, if it is fully faithful (i.e. bijective on hom-sets) and **essentially surjective**, i,e, for every object Y of $\mathcal{D}$, there is an object X of $\mathcal{C}$ such that $\mathscr{F}(X) \cong Y$.

Some people write natural transformations in the follwing way:

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

A.1 Sheet 1

Solution A.1.1. i) φ_m is k-linear:

- $\varphi(a+b)x = (a+b)x \overset{(L1)}{=} ax + bx = \varphi(a)x + \varphi(b)x$
- $\varphi(\lambda a) = (\lambda a)x \overset{(L5)}{=} \lambda(ax) = \lambda\varphi(a)x.$

φ_m is ring homomorphism:

- $\varphi(ab) = (ab)x \overset{(L3)}{=} a(bx) = \varphi(a)\varphi(b)x$
- $\varphi(1_A) \overset{(L4)}{=} (1_A)x = x.$

As these relations hold for all x, the assertion follows.

ii) V_φ is already a k-module.

- (L1) $a(x+y) = (\varphi(a))(x+y) \overset{\varphi \in \mathrm{End}_k(V)}{=} (\varphi(a))(x) + (\varphi(a))(y) = ax + ay$
- (L2) $(a+b)x = (\varphi(a+b))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\varphi(a) + \varphi(b))x = \varphi(a)x + \varphi(b)x = ax + bx$
- (L3) $(ab)x = (\varphi(ab))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\varphi(a)\varphi(b))x = a(bx)$
- (L4) $1_a x = (\varphi(1_a))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} \mathrm{id}\, x = x$
- (L5) $(\lambda a)(x) = ((\varphi(\lambda a))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\lambda\varphi(a))x = \lambda(ax)$ and $a(\lambda x) = (\varphi(a)(\lambda x)) \overset{\varphi(a) \text{ endo of } k-\text{module}}{=} \lambda(\varphi(a))x = \lambda(ax),$

for all $a, b \in A, x, y \in V$ and $\lambda \in k$.

iii) We regard V and W as A-modules in the sense of part ii). Assume that $\psi(a) \circ f = f \circ \varphi(a)$ $(*)$ for all $a \in A$. Then

$$f(ax) = f((\varphi(a))x) \overset{*}{=} (\psi(a))(f(x)) = af(x),$$

for all $x \in X$. Hence f is an A-module homomorphism.
Assume that f is a a-module homomorphism, then

$$(\psi(a))(f(x)) = a(f(x)) = f(ax) = f((\varphi(a))x)$$

for all $x \in V$. Hence $\psi(a) \circ f - f \circ \varphi(a) = 0$ and so $(*)$ holds.

Solution A.1.2. i) Assume that I is a non-zero ideal of A. Let $a = (a)_{ij} \neq 0$ be an arbitrary matrix in I. Then there exist permutation matrices $\sigma, \pi \in \mathrm{GL}_n(K)$, such that $(\sigma a \pi)_{11} \neq 0$, which is in I, as I is a two-sided ideal. So without loss of generality, suppose $a_{11} \neq 0$.
Define

$$b \in \mathrm{M}_n(k), (b)_{ij} := \begin{cases} 1, & \text{if } i = j = 1 \\ 0, & \text{else} \end{cases}$$

and E_n as the identity of $\mathrm{M}_n(k)$. Then we get

$$\left(\frac{1}{a_{11}} E_n\right) \cdot b \cdot a \cdot b = b.$$

By repeatetly using permutation matrixes, it is possible to write any matrix as sum of products of a, b and permutation matrices on the left- and right. As I is a two-sided ideal, all of these combinations are in I as well. Hence a generates all of A, and $I = A$.

ii) Consider A as a k-vector space, then $\dim_K A = n^2$. Let M be any left A-module. As shown in task 3, there is a homomorphism of k-algebras

$$\varphi : A \to \mathrm{End}_k(M),\ a \mapsto a : (x \mapsto ax),$$

which is in particular a homomorphism of k-vector spaces. The kernel of φ is a two-sided ideal of A, as

$$a0x = 0ax = 0$$

for all $a \in A$ and $x \in M$.
Now i) implies that $\ker \varphi$ is either zero or $\ker \varphi = A$. But since $\varphi(E_n) = \mathrm{id}_M$, the latter one is not possible. Hence φ is injective, and in particular $\dim A \leq \dim \mathrm{End}_k(V)$, so $n \leq m$.

Proposition A.1.1. *Let k be a field, $k[X]$ the polynomial ring and $p \in k[X]$ a polynomial with $\deg p = n$. Then*

$$k[X]/(p)$$

is a n-dimensional k vector space, and a basis is given by

$$\{1, x, ..., x^{n-1}\}.$$

The following propositions are taken from [**aluffi**].
Let R be any commutative ring.

Proposition A.1.2. *Let $I_1, ..., I_k$ be ideals of R such that $I_i + I_j = (1)$ for all $i \neq j$. Then the natural homomorphism*

$$\varphi : R \to R/I_1 \times \ldots \times R/I_k$$

is surjective and induces an isomorphism

$$\frac{R}{I_1 \ldots I_k} \to R/I_1 \times \ldots \times R/I_k$$

Corollary A.1.3 (Chinese remainder theorem)**.** *Let R be a PID and $a_1, ..., a_k \in R$ be elements such that $\gcd(a_i, a_j) = 1$ for all $i \neq j$. Let $a = a_1 \dots a_k$. Then the function*

$$\varphi : R/(a) \to R/(a_1) \times \dots \times R/(a_k).$$

Proposition A.1.4 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category, X an object of $\mathcal{C}$ and consider the contravariant functor*

$$h_X := \hom_{\mathcal{C}}(-, X).$$

Then for every contravariant functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$, there is a bijection between the set of natural transformations $h_x \rightsquigarrow \mathscr{F}$ and (X).

Definition A.1.5 ([**assem1**])**.** The (Jacobson) **radical** $\operatorname{rad} A$ of a K-algebra A is the intersection of all maximal right ideals in A. It is the same as the intersetion of all left-sided maximal right ideals in A. Furthermore, $\operatorname{rad} A$ is a two-sided ideal.

Definition A.1.6. Let $f, g : X \to Y$ be morphisms in a category $\mathcal{C}$. Then a morphism $e : E \to X$ is called **equalizer** of f and g if $f \circ e = g \circ e$ and for all other morphisms $o : O \to X$, such that $f \circ o = g \circ o$, there is a unique morphis $O \to E$, such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left] \arrow[r, shift right] & X \\ O \arrow[u, dashed, "\exists!"] \arrow[ur, "o"] & & \end{tikzcd}.$$

Proposition A.1.7. *Equalizers exists in abelian categories.*

Task 1

See this as a functors:

$$k\text{–}\mathcal{A}lg \underset{\mathscr{A}}{\overset{\mathscr{V}}{\rightleftarrows}} \mathcal{G}rp$$

Grp. alg. construction $\mathscr{A}$ is left-adjoint to group of units construction $\mathscr{V}$.

i) Show that there is a natural isomorphism

$$k\text{–}\mathcal{A}lg(\mathscr{A}(G), A) \cong \mathcal{G}rp(G, \mathscr{V}(A))$$

for all groups G and k-algebras A.

Task 2

This quiver is called the **linear oriented quiver**. Define

$$\varphi : KQ \to \mathrm{L}_n(k)$$

as linear extension of the k-linear map

$$Q_* \to \mathrm{L}_n(k),\ p_{ij} \mapsto (E_{ij})kl := \begin{cases} 1 & \text{if } k = i, l = j \\ 0 & \text{otherwise} \end{cases}$$

This is indeed a homomorphism of k-algebras, which sends basis vectors to basis vectors.

Task 3

Let $\mathcal{C}, \mathcal{D}$ be two k**-linear** categories , i.e. $\mathcal{C}(X,Y)$ has the structure of a k-vector space and composition is bilinear. We say that $\mathcal{C}$ is **equivalent** to $\mathcal{D}$ ($\mathcal{C} \simeq \mathcal{D}$) if there are k**-linear** functors

$$\mathscr{F} : \mathcal{C} \to \mathcal{D} \text{ and } \mathscr{G} : \mathcal{D} \to \mathcal{C}$$

(i.e. functors that induce k-linear maps $\mathcal{C}(X,Y) \to \mathcal{D}(\mathscr{F}(X), \mathscr{G}(Y))$, and for $\mathscr{G}$ analogous), such that there are natural isomorphisms

$$\mathscr{G}\mathscr{F} \simeq \mathrm{Id}_{\mathcal{C}} \text{ and } \mathscr{F}\mathscr{G} \simeq \mathrm{Id}_{\mathcal{D}}$$

Theorem A.1.8. *Let $\mathcal{C}, \mathcal{D}$ be k-linear categories. $\mathcal{C} \simeq \mathcal{D}$ if and only if there is a fully faithful, k-linear and dense functor*

$$\mathscr{F} : \mathcal{C} \to \mathcal{D}.$$

Remark A.1.9. The proof is supposed to only invoke the Axiom of Choice, and should work for general hom-Sets.

This task basically shows that there is an equivalence of categories

$$\mathcal{R}ep(A) \simeq k{-}\mathcal{M}od(A),$$

where

$$\mathcal{R}ep(A) := \{(V, \varphi) \mid V \text{ a } k{-}\text{vector-space}, \varphi : A \to \mathrm{End}_k(V) \text{ a algebra-homomorphism}\}$$

Calvin highly recommends the book [**assem1**].

A.2 Sheet 2

Solution A.2.1. Consider the two representations

$$k \underset{b}{\overset{a}{\rightrightarrows}} k \quad \text{and} \quad k \underset{d}{\overset{c}{\rightrightarrows}} k$$

with $a, b, c, d \in k$. Morphism of representations are in this case k-linear maps $k \to k$, i.e. multiplication by elements $\mu, \nu \in k$, such that the diagrams

$$\begin{tikzcd} k \arrow[r, "a"] \arrow[d, "\mu"'] & k \arrow[d, "\nu"] \\ k \arrow[r, "c"'] & k \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k \arrow[r, "b"] \arrow[d, "\mu"'] & k \arrow[d, "\nu"] \\ k \arrow[r, "d"'] & k \end{tikzcd}$$

commute. This is the case if (μ, ν) satisfies the system of equations

$$\underbrace{\begin{pmatrix} c & -a \\ d & -b \end{pmatrix}}_{=:A} \cdot \begin{pmatrix} \mu \\ \nu \end{pmatrix} = 0 \Longleftrightarrow \begin{pmatrix} \mu \\ \nu \end{pmatrix} \in \ker A$$

There are several cases to consider:

- $\det A = ad - bc \neq 0$: As A is invertible in this case, $\ker A$ is trivial, and hence

$$\hom(X_{(a,b)}, X_{(c,d)}) = 0$$

- $\det A = 0; a = b = c = 0$. As $A = 0$, $\ker A = k^2$ holds, and hence

$$\hom(X_{(a,b)}, X_{(c,d)}) = k^2$$

- $\det A = 0; b \neq 0$ In this case, $c = ad/b$ holds.
  - $a = 0, d = 0$:
  $$A = \begin{pmatrix} 0 & 0 \\ 0 & -b \end{pmatrix} \implies \ker A = \operatorname{Lin} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
  - $a = 0, d \neq 0$
  $$A = \begin{pmatrix} 0 & 0 \\ d & -b \end{pmatrix} \implies \ker A = \operatorname{Lin} \begin{pmatrix} b/d \\ 1 \end{pmatrix}$$
- $\det A = 0, b = 0$: Consider the cases:
  - $c, d \neq 0, a = 0$:
  $$A = \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \implies \ker A = \operatorname{Lin} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
  - $c \neq 0, a, d = 0$...

Solution A.2.2. i) A representation of V consits of a vector space V together with an endomorphism $f \in \operatorname{End}_k V$:

$$X := \begin{tikzcd} V \arrow[loop, "f"] \end{tikzcd} .$$

A decomposition of X into subrepresentations would correspond to a decomposition

$$V = V_1 \oplus V_2$$

with subspaces V_1 and V_2 of V, such that both V_1 and V_2 are f-invariant. Now, let X be any representation of Q. Assume first that $n := \dim_k V < \infty$, and f is any . As k is algebraically closed, there is a unique (up to permutation) basis B of V given by a disjoint unions of Jordan-Chains

$$B = \bigcup_{\lambda \in k} \bigcup_{i \in I_\lambda} J(\lambda, \ell_i) \text{ where } \sum_{\lambda \in k} \sum_{\lambda \in I_\lambda k} \ell_i = n,$$

I_λ are finite index sets, unequal to zero for only finitely many $\lambda \in k$, and $J(\lambda, \ell_i)$ are Jordan-Chains of f for the eigenvalue λ with length ℓ_i. This basis induces a unique direct sum decomposition

$$V = \bigoplus_{\lambda \in k} \bigoplus_{i \in I_\lambda} \operatorname{Lin} J(\lambda, \ell_i).$$

By construction of the Jordan-Chains, $J(n, \lambda_i)$ can not be decomposed for

Solution A.2.4. i) R is a k-vector space,where the addition and scalar multiplication are defined component-wise. This gives R the structure of a k-vector space, as M, N and X are in particual k-vector spaces.
Consider now the map

$$R \times R \to R, \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \mapsto \begin{pmatrix} aa' & ax' + xb' \\ 0 & bb' \end{pmatrix},$$

where the operation $ax' + xb'$ is well-defined, as X is an A-B-bimodule. This makes R into a ring, as:

- the unit is given by $\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$:

$$\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$$

- the multiplication is associative, as

$$\begin{aligned} \begin{pmatrix} a'' & x'' \\ 0 & b'' \end{pmatrix} \cdot \left(\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right) &= \begin{pmatrix} a'' & x'' \\ 0 & b'' \end{pmatrix} \cdot \begin{pmatrix} a'a & a'x + x'b \\ 0 & b'b \end{pmatrix} \\ &= \left(a''(a'a) \right) \ldots \end{aligned}$$

- the distributivity holds, as

Extension and restriction of scalars

We do some recap from Algebra 1 (cf. [**atiyah1994introduction**]). For this, we go back to the commutative case: In the following, A, B denote commutative, untial rings.

Proposition A.2.1. *Let $A \xrightarrow{f} B$ be a ring homomorphism and N a B-module. Then N has a A-module structure, given by*

$$A \times N \mapsto N, \ (a, n) \mapsto f(a)n.$$

Proof.
- The addition on N_A is the same as the addition of N_B.

- Associativity: $(ab)n = f(ab)n = (f(a)f(b))n = f(a)(f(b)n)$, as N is B-module
- Unit acts as unit: $1_A n = f(1_A)n = 1_B n = n$, as f is homomorphism of rings.
- Distributivity: $(a+b)n = f(a+b)n = (f(a)+f(b))n = f(a)n + f(b)n$ and $a(n+n') = f(a)(n+n') = f(a)n + f(a)n'$

□

This way of obtaining a A-module structure on N_B is called **restriction of scalars**. In particular, f defines a A-module structure on B in this way.

Proposition A.2.2. *Let M_A be an A-module. Then*

$$M_B := B \otimes_A M$$

carries a B-module structure, and

$$b(b' \otimes x) = (bb') \otimes x$$

holds for this B-module. We say that the B-module M_B was obtained from M by ***extension of scalars***