Foundations of Representation Theory

Lecture Notes in the Winter Term 2018/19

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These are my personal lecture notes for the lecture *Foundations of Representation Theory* held by Dr. Hans Franzen at the University of Bonn in the winter term 2018/19.

I try to update them on my website, https://pankratius.github.io. The authors labels his own comments and additions in purple.

The book [Alu09] is used by the author for further references, and highly recommended.

Conventions: In this lecture, rings will always be unital, and ring homomorphisms f always fulfill $f(1) = 1$. Rings do *not* have to be commutative.

1.1   Algebras - Basics

Let k be a commutative ring.

Definition 1.1. A k-**algebra** A is a ring A, together with a structure of a k-module on A, such that

$$\text{for all } a, b \in A, \lambda \in K : \ (\lambda a)b = a(\lambda b) = \lambda(ab) \tag{$*$}$$

Definition 1.2. Let A, B be k-algebras. A **homomorphism of algebras** is a map $f : A \to B$ that is both k-linear and a ring homomorphism.

Remark 1.3. Let A be a ring. Define

$$Z(A) := \{a \in A \mid \forall b \in A : ab = ba\},$$

which is a commutative subring and is called the **center** of A.

Remark 1.4. Let A be a ring. Giving a k-algebra structure on A is the same as giving a ring homomorphism $k \to Z(A)$. More precisley:

  i) If A is a k-algebra, then $p : k \to A$, $\lambda \mapsto \lambda 1$ satisfies $\operatorname{Im} p \subseteq Z(A)$ and is a ring homomorphism. (the first statement follows from $(*)$ and the second one from the fact that A has a k-module structure).

 ii) Let $\varphi : k \to Z(A)$ be a ring homomorphism. Define

$$\lambda a := \varphi(\lambda)a,$$

   for all $\lambda \in k$. This defines a k-algebra structure on A ($(*)$ holds since $\operatorname{Im}(\varphi) \subseteq Z(A)$).

iii) Let A, B be k-algebras and $f : A \to B$ a homomorphism of rings. Then f is a homomorphism of k-algebras if and only if the following diagram commutes:

Example 1.5.   i) Let V be a k-module. Then $\operatorname{End}_k(V)$ has a ring structure given by

$$\operatorname{End}_k(V) \times \operatorname{End}_k(V) \to \operatorname{End}_k(V), \ (\phi, \psi) \mapsto \varphi \circ \psi.$$

Then $\text{End}_k(V)$ is both a ring and a k-module, and becomes a k-algebra via

$$\varphi : k \to \text{End}_k(V), \ \lambda \mapsto \lambda \, \text{id} \, .$$

Note that $\text{Im} \, \varphi \subseteq Z(A)$. If k is a field, then $Z(\text{End}_k(V)) = \{\lambda \, \text{id} \mid \lambda \in k\}$.

ii) Take $V = k^n$ (free module of rank n). Then $\text{End}_k(V) \cong \text{M}_n(k)$. Define

$$T_u := \{\varphi \in \text{M}_n(k) \mid \varphi \text{ is upper triangular}\},$$

i.e. T_u presevers flags in k^n. Then T_u is a **subalgebra** of $\text{M}_n(k)$, i.e. is both a subring and a k-submodule of the original algebra.

iii) Let G be a group. Define the **group algebra** $k[G]$ of G as follows:

- As k-module, is defined as the free module on G,

$$k[G] := k^{(G)} = \{\sum_{g \in G} \lambda_g g \mid \lambda_g \in k, \ \lambda_g \neq 0 \text{ for only finitely many } g \in G\}.$$

- Multiplication: Let $a := \sum \lambda_g g$, $b = \sum \mu_g g$ and define:

$$ab := \sum_{g \in G, h \in G} \lambda_g \mu_h (gh) = \sum_{j \in G} \left(\sum_{gh=j} \lambda_g \mu_h \right) j.$$

This multiplication is associative, k-bilinear, distributive and $1|_{k[G]} = e$. In addition, $(*)$ is satisfied.
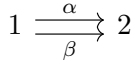
1.2   Quivers - Basics

Definition 1.6. A **quiver** is a „directed graph". Formally, a quiver is a quadruple (Q_0, Q_1, s, t) consiting of sets Q_0 (**vertices**) and Q_1 (**arrows**) and maps $s : Q_1 \to Q_0$, $t : Q_1 \to Q_0$. For $\alpha \in Q_1$, we call $s(\alpha)$ the **source** of α and $t(\alpha)$ the **target** of α:

$$s(\alpha) \xrightarrow{\ \alpha \ } t(\alpha).$$

Example 1.7.   i) $Q = (\{1\}, \emptyset, ...)$ is visualized as: 1

ii) $Q = (\{1\}, \{\alpha\}, ...)$ is visualized as 1 ↶

iii) $Q = (\{1, 2\}, \{\alpha, \beta\}, s(\alpha) = s(\beta) = 1, t(\alpha) = t(\beta) = 2)$ is visualized as $1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$

Definition 1.8. Let Q be a quiver sucht that both Q_0 and Q_1 are finite.

i) Let $\ell \in \mathbb{Z}_{\geq 1}$. A **path** of length ℓ is a sequence $\alpha_\ell ..., \alpha_1$ of arrows, such that $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq \ell - 1$,

$$\circ \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_2} \circ \ldots \xrightarrow{\alpha_\ell} \circ.$$

Define Q_ℓ to be the set of all paths of length ℓ.
Let $p : \alpha_\ell ... \alpha_1$ be a path. Define $s(p) := s(\alpha_1)$ and $t(p) := s(\alpha_\ell)$.
Formally define Q_0 to be the set of all paths of length zero. Denote by ε_i for $i \in Q_0$ the constant path at i. ε_i is called a **lazy path**. We set $s(\varepsilon_i) = t(\varepsilon_i) := i$.

ii) Let $p = \alpha_\ell ... \alpha_i$ and $q = \beta_m ... \beta_1$ be paths of length ℓ and m respectivley, with $\ell, m \geq 1$. If $t(p) = s(q)$, then set $q \circ p := \beta_m .... \beta_1 \alpha_\ell ... \alpha_1$. This is a path of length $\ell + m$.
For p a path of length ≥ 0 and ε_i a lazy path:

- if $t(p) = i$, set $\varepsilon_i \circ p := p$,

- if $s(p) = i$, set $p \circ \varepsilon_i := p$.

In all others cases, the composition is not defined.

iii) Define
$$Q_* := \bigcup_{\ell \geq 0} Q_\ell,$$

the set of all paths. Define the **path-algebra** kQ:

- As a k-module, $kQ := k^{(Q_*)}$.

- Multiplication: Let $a = \sum \lambda_p p$, $b = \sum \mu_p p$. Define

$$ab := \sum_{p,q \in Q_*} \lambda_p \mu_q (p \cdot q),$$

where
$$p \cdot q := \begin{cases} p \circ q, & \text{if it is defined, i.e } t(q) = s(p) \\ 0, & \text{else} \end{cases}.$$

The multiplication is associative (due to the associativity of the composition of paths) and k-bilinear by definition. In addition, distributivity and $(*)$ are fulfilled.

- The identity is given by $\sum \varepsilon_i$.

Example 1.9. i) $Q = 1$, then $kQ = k$.

ii) $Q = \; 1 \circlearrowleft$, then $Q_* = \{\alpha^n \mid n \geq 0\}$ and $kQ = k[t]$.

iii) $Q = \; 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$. Then $Q_* = \{\varepsilon_1, \varepsilon_2, \alpha^n, \beta^n \mid n \geq 0\}$ and

$$kQ = k\varepsilon_1 \oplus k\varepsilon_2 \oplus k\alpha \oplus k\beta.$$

A multiplication table is given by

	ε_1	ε_2	α	β
ε_1	ε_1	0	0	0
ε_2	0	ε_2	α	β
α	α	0	0	0
β	β	0	0	0

Lemma 1.10. *Let Q be a finite quiver, k a field. Then the following are equivalent:*

i) Q contains no cycles.

ii) $\dim_k kQ < \infty$.

Lemma 1.11. *Let k be a field, A a k-algebra and $n := \dim(A) < \infty$. Then there exists an injective homomorphism of k-algebras $\varphi : A \to M_n(k)$.*

Proof. By choosing a basis of A, we get an isomorphism $\mathrm{End}_k(A) \cong M_n(k)$. So it suffices to find an injective homomorphism of k-algebras $\varphi : A \to \mathrm{End}_k(A)$.
Consider

$$\varphi : A \to \mathrm{End}_k(A), \ \varphi(a) : A \to A, b \mapsto ab.$$

- $\varphi(a)$ is k-linear for all a by the distributivity in A and the condition $(*)$.

- φ is k-linear by the distributivity in A and the condition $(*)$.

- Let $a, a' \in A$. Then

$$\varphi(aa')(b) = (aa')(b) = a(a'b) = \big(\varphi(a) \circ \varphi(a')\big)(b).$$

Hence φ is indeed a homomorphism of k-algebras.
To show that φ is injective, let $a \in \ker \varphi$, hence $ab = 0$ for all $b \in A$. But in particular, $0 = a1 = a$. $\qquad\square$

─────────────── End of Lecture 1 ───────────────

Definition 1.12. Let A be a k-algebra. Then the **opposite algebra** A^{op} is A (as a k-module), and the multiplication is defined as

$$a \cdot_{A^{\mathrm{op}}} b = b \cdot_A b.$$

Example 1.13. Let Q be a quiver, and define $Q^{\mathrm{op}} := (Q_0, Q_1, s^{\mathrm{op}}, t^{\mathrm{op}})$, where $s^{\mathrm{op}}(\alpha) := t(\alpha)$ and $t^{\mathrm{op}}(\alpha) := s(\alpha)$. Then $kQ^{\mathrm{op}} = k(Q^{\mathrm{op}})$

1.3   Modules - Basics

Definition 1.14. Let A be a k-algebra. A **left A-module** M is a k-module M together with a map $A \times M \to M, (a, x) \mapsto ax$, such that:

$$a(x + y) = ax + ay \tag{L1}$$

$$(a + b)x = ax + bx \tag{L2}$$

$$a(bx) = (ab)x \tag{L3}$$

$$1_A x = x \tag{L4}$$

$$(\lambda a)x = \lambda(ax) = a(\lambda x), \tag{L5}$$

for all $a, b \in A$, $x, y \in M$ and $\lambda \in k$. If A is a left A-module, we denote this as $_A M$. A **right A-module** is defined analogous, where (L3) becomes $(xa)b = x(ab)$. If A is a right A-module, we denote this by A_M.

Remark 1.15. A right A-module is the same as a left A^{op}-module.

Definition 1.16. Let A be a k-algebra, and M, N left A-modules. A **homomorphism of left A-modules** $f : M \to N$ is a k-linear map such that

$$f(ax) = af(x)$$

for all $a \in A$ and $x \in M$.
Define the set of all left A-algebra homomorphisms as

$\hom_A(M, N) := \hom_A(_A M, _A M) := \{f : M \to N \mid f \text{ is a homomorphism of left } A\text{-modules}\}.$

A homomorphism of left A-modules is an **isomorphism** if it is a bijective homomorphism of left A-modules.
Homomorphism of right A-modules are defined analogous.

Remark 1.17. Let M, N be left A-modules. Then

i) $\hom_A(M, N)$ has a k-module structure given by

$$\lambda f : M \to N, \ x \mapsto \lambda f(x) = f(\lambda x).$$

This is well defined, as k lies in the center of A.

ii) In general, $\hom_A(M, N)$ has neither a left nor a right A-module structure.

iii) f is an isomorphism if and only if there is a homomorphism of left A-modules $g : N \to M$ such that

$$g \circ f = \mathrm{id}_M \text{ and } f \circ g = \mathrm{id}_N.$$

iv) Let $f : M \to M'$ and $g : N \to N'$ be homomorphisms of left A-modules. Then we obtain k-linear maps

$$f^* : \hom_A(M', N) \to \hom_A(M, N), \ h \mapsto h \circ f$$
$$g_* : \hom_A(M, N) \to \hom_A(M, N'), \ h \mapsto g \circ h.$$

Remark 1.18. Let A be a k-algebra and M, N left A-modules.

i) A subset $M' \subseteq M$ is called a **submodule** if

(SM1) $0 \in M'$

(SM2) $x, x' \in M' \implies x + x' \in M'$

(SM3) $a \in A, x \in M' \implies ax \in M'$.

In particular, submodules of A-modules are submodules of the underlying k-module, as follows using (L4)

ii) Let M be a submodule. Then the **quotient** has a left A-module structure in the obvious way. The projection

$$\pi : M \to M'$$

is a homomorphism of left A-modules.

iii) A **left ideal** is left A-submodule of $_A A$. Similar, a **right ideal** is right A-submodule of A_A. For a left ideal $I \subseteq A$, the quotient A/I is a left A-module, but in general not an algebra.

iv) A **two-sided ideal** $I \subset A$ is both a left- and a right-ideal of A. Then A/I has an algebra structure, by setting

$$(x + I)(y + I) := (xy) + I.$$

In general, this is only well-defined if I is a two-sided ideal of A.

v) Let $f : M \to N$ be a homomorphism of left A-modules. Then we obtain left A-modules:

$$\ker f, \ \operatorname{Im} f, \ \operatorname{coker} f := N / \operatorname{Im} f, \ \operatorname{coim} f := M / \ker f.$$

In particular, f factors uniquely as

$$M \xrightarrow{} \operatorname{coim} f \xrightarrow[\cong]{\exists !} \operatorname{Im} f \xrightarrow{} N \ . \tag{F}$$

vi) Let $\{M_i \subset M \mid i \in I\}$ be a family of left A-submodules, for some index set I. Then

$$\bigcap_{i \in I} M_i \text{ and } \sum_{i \in I} M_i$$

are left A-modules.

vii) Let $x \in M$. Define
$$Ax := \{ax \mid a \in A\},$$

which is a left A-submodule. Similar, for $x \in M_A$, define $xA := \{xa \mid a \in A\}$, which is a right A-submodule. For a subset $E \subset M$,

$$\sum_{x \in E} Ax = \bigcap_{\substack{E \subseteq M' \subseteq M \\ M' \text{ submodule}}} M'.$$

M is called **finitely generated**, if there are $x_1, ..., x_n \in M$, such that

$$M = \sum_{i=1}^{n} Ax_i.$$

viii) Let $\{M_i \mid i \in I\}$ be a family of left A-modules. Then

$$\prod_{i \in I} M_i := \{(x_i)_{i \in I} \mid x_i \in M_i\}$$

is called the **product** , and

$$\bigoplus_{i \in I} M_i := \{(x_i)_{i \in I} \mid x_i \in M_i, \ x_i \neq 0 \text{ for only finitely many } i\}$$

is called the **coproduct** .They are both left A-modules. The **projection**

$$\pi_j : \prod_{i \in I} M_i \to M_j, \ (x_i)_{i \in I} \mapsto x_j$$

and the **inclusion**

$$\iota_j : \bigoplus_{i \in I} x_j \mapsto (\delta_{ij} x_j)_{i \in I}$$

are morphism of left A-modules.

ix) A left A-module M is finitely generated if and only if there is a surjective homomorphism of left A-modules
$$A^n := \bigoplus_{i=1}^{n} A \longrightarrow\!\!\!\!\rightarrow M$$

for some $n \geq 1$. A is called **finitely presented**, if there is an exact sequence of left A-modules
$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

for some $m, n \geq 1$.

Proposition 1.19. *Let*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0 \qquad\qquad (*)$$

and

$$0 \longrightarrow N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \qquad\qquad (**)$$

be sequences of left A-modules.

    i) The following are equivalent:

       a) $(*)$ is exact.

       b) For all left A-modules N, the sequence

$$0 \longrightarrow \hom_A(M_3, N) \xrightarrow{f_2^*} \hom_A(M_2, N) \xrightarrow{f_1^*} \hom_A(M, N)$$

     is exact.

   ii) The following are equivalent:

       a) $(**)$ is exact.

       b) For all left A-modules M, the sequence

$$0 \longrightarrow \hom_A(M, N_1) \xrightarrow{g_{1,*}} \hom_A(M, N_2) \xrightarrow{g_{2,*}} \hom_A(M, N_3)$$

     is exact.

Proof. We will only prove *a) $\implies$ b)* of *ii)*.

Lemma 1.20. *Let K, M, N be left A-modules, and $\zeta : K \to M$, $\varphi : M \to N$ be homomorphisms of left A-modules, such that $\varphi \circ \zeta = 0$. Then there is a unique homomorphism $\overline{\zeta}$, such that*

commutes.

- $g_{1,*}$ injective: Let $h \in \ker(g_{1,*})$. Then

$$g_1 \circ h : \quad M \xrightarrow{h} N_1 \xrightarrow{g_1} N_2$$

  and since g_1 is injective, it follows $h = 0$.

- $\operatorname{Im} g_{1,*} \subseteq \ker g_{2,*}$: Since $**$ is exact, it follows that $g_2 \circ g_1 = 0$. For $h \in \operatorname{Im} g_{1,*}$ there exists an $h' : M \to N_1$ such that $h = g_1 \circ h'$, and hence $g_2 \circ h = g_2 \circ g_1 \circ h' = 0$.

- $\ker g_{2,*} \subseteq \operatorname{Im} g_{1,*}$: As $(**)$ is exact, $\ker g_2 = \operatorname{Im} g_1$ holds.
  Let $h : M \to N_2 \in \ker g_{2,*}$, i.e. $g_2 \circ h = 0$:

By lemma 1.20, h factors uniquely through $\ker g_2 = \operatorname{Im} g_1$:

$$
\begin{array}{ccccccc}
& & & & M & & \\
& & & & \downarrow h \quad \searrow^{0} & & \\
0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 \quad. \\
& & & & \uparrow \iota & & \\
& & & & \operatorname{Im} g_1 & &
\end{array}
$$

But since g_1 is injective, (F) implies that there is a uniquely determined isomorphism $\tilde{g}_1 : N_1 \longrightarrow \operatorname{Im} g_1$.

Putting everything together, we obtain the following commutative diagram:

$$
\begin{array}{ccccccc}
& & & & M & & \\
& & & & \downarrow h \quad \searrow^{0} & & \\
0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 \quad. \\
& & & \searrow_{\exists! \tilde{g}_1} & \uparrow \iota & & \\
& & & & \operatorname{Im} g_1 & &
\end{array}
$$

Setting $h' := \tilde{g}_1^{-1} \circ h'$, we obtain

$$
g_1 \circ h' = \iota \circ \tilde{g}_1 \circ \tilde{g}_1^{-1} \circ \tilde{h} = \iota \circ \tilde{h} = h.
$$

$\square$

Proposition 1.21. *Let A be a k-algebra. To give a left A-module structur is the same as to give a k-module structure V together with a homomorphism $\varphi : A \to \operatorname{End}_k(V)$ of k-algebras. To give a right A-module structure is the same as giving a k-module structure V together with a homomorphism $\varphi : A \to \operatorname{End}_k(V)^{\mathrm{op}}$.*

1.4 Representation of quivers

Let k be a field and Q be a quiver.

Definition 1.22. A **representation** X of Q over k consists of

- a k-vector space X_i for all $i \in Q_0$,

- a k-linear map

$$
X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)}
$$

for each $\alpha \in Q_1$

Example 1.23 (Continue example 1.9). i) Let $Q = \cdot$. Then a representation of Q is simply a k-vector space.

ii) Let $Q = 1 \circlearrowleft$. Then a representation of Q is a k-vector space V together with an endomorphism $\varphi \in \mathrm{End}_k(V)$:

$$Q = V \overset{f}{\circlearrowleft} \quad .$$

iii) Let $Q = 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$, the **Kronecker Quiver**. Then a representation of Q is a pair of vector spaces V, W and two linear maps $f, g \in \hom_K(V, W)$:

$$Q = V \underset{g}{\overset{f}{\rightrightarrows}} W$$

Definition 1.24. Take X, Y to be two representations of Q over k. A **homomorphism of representations** $f : X \to Y$ is a tupel $(f_i)_{i \in Q_0}$ of linear maps $f_i : X_i \to Y_i$, such that for all $\alpha \in Q_1$ the diagram

$$
\begin{array}{ccc}
X_{s(\alpha)} & \xrightarrow{f_{s(\alpha)}} & Y_{s(\alpha)} \\
\downarrow{\scriptstyle X_\alpha} & & \downarrow{\scriptstyle Y_\alpha} \\
X_{t(\alpha)} & \xrightarrow[f_{t(\alpha)}]{} & Y_{t(\alpha)}
\end{array}
$$

commutes.

Example 1.25 (Continue example 1.23).   i) Homomorphisms of representations are k-linear maps $X \to Y$.

ii) Homomorphisms of representations (V, φ) and (W, ψ) are k-linear maps $f : V \to W$, such that

$$
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\
V & \xrightarrow{f} & W
\end{array}
$$

commutes

iii) Homomorphisms of representations (V_1, V_2, A, B) and (W_1, W_2, C, D) are pairs (f_1, f_2) of linear maps $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$, such that $A \circ f_1 = f_2 \circ A$ and $B \circ f_1 = f_2 \circ B$.

Definition 1.26. An **isomorphism of representations** $f : X \to Y$ is a homomorphism of representations, such that there exists $g : Y \to X$ homomorphism of representations satisfying

$$g \circ f = \mathrm{id}_X \text{ and } f \circ g = \mathrm{id}_y .$$

An isomorphism of representations is a homomorphism of representations such that each map f_i is bijective.

Let Q be a quiver over a field k.

Remark 1.27.   i) Let X be a representation of Q over k. Associate a left kQ-module $M = F(X)$ as follows:

- As k-vector space, let
$$M := \bigoplus_{i \in Q_0} X_i.$$

- Define the action of kQ on M by an action of paths. Let p be a path of length ≥ 1. Define
$$X_p : X_{s(p)} \to X_{t(p)} \text{ given by } X_p = X_{\alpha_\ell} \circ \ldots \circ X_{\alpha_i},$$
with $X_p \in \hom_k(X_{s(p)}, X_{t(p)})$. Use this to define a k-linear map $\tilde{X}_p : M \to M$ as composition:
$$\tilde{X}_p : M = \bigoplus_{i \in Q_0} X_i \xrightarrow{\pi_{s(p)}} X_{s(p)} \xrightarrow{X_p} X_{t(p)} \xhookrightarrow{\iota_{t(p)}} \bigoplus_{i \in Q_0} X_i$$

If the length of $p = 0$, then p is a lazy part at some $i \in Q_0$, and we set
$$X_{\varepsilon_i} := \mathrm{id}_{X_i},$$
and $\tilde{X}_{\varepsilon_i}$ like $\tilde{X}_p$.
Now these k-linear endomorphisms define a kQ-module structure on M, given by:
$$kQ \times M \to M, \left(a := \sum_{p \in Q_*} \lambda_p \cdot p, (x_i)_i =: x \right) \mapsto a.x := \sum_{p \in Q_*} \lambda_p \cdot \tilde{X}_p(x)$$
$$= \sum_{p \in Q_*} \lambda_p \cdot (\iota_{t(p)} X_p(x_{s(p)})),$$

where we denote an element in M by a sequence $(x_i)_i$ with $x_i \in X_i$.

- We check that this actually defines a kQ-module structure:
  - (L3): Assume that $a, b \in kQ$. By the bilinearity of the multiplication, we can assume that $a = p$ and $b = q$ are both paths in Q_*. Then
$$a.(b.x) = \tilde{X}_p \left(\tilde{X}_q(x) \right)$$
$$= \iota_{t(p)} X_p \underbrace{\pi_{s(p)} \iota_{t(q)}}_{} X_q(x_{s(q)}),$$
    where
$$\pi_{s(p)} \iota_{t(q)} = \begin{cases} \mathrm{id}_{X_q}, & \text{if } t(q) = s(p) \\ 0, & \text{otherwise} \end{cases}.$$
    This gives
$$a.(b.x) = \begin{cases} \iota_{t(p)} X_p X_q(x_{s(q)}), & \text{if } t(q) = s(p) \\ 0 & \text{otherwise} \end{cases}.$$
    Additionally,
$$(a.b).x = \begin{cases} \tilde{X}_{p \circ q} & \text{if } f(q) = s(p), \\ 0 & \text{otherwise} \end{cases}.$$

But in the case $f(q) = s(p)$,

$$\tilde{X}_{p \circ q}(x) = \iota_{t(p)} \circ X_p X_q(x_{s(q)}).$$

The construction F is factorial, i.e. for $f : X \to Y$ a homomorphism of representations, F induces a homomorphism of kQ-algebras

$$Ff : F(X) \to F(Y) \text{ by } (Ff)((x_i)_i) := (f_i(x_i))_i.$$

ii) Let M be a left kQ-module. Define a representation $X := G(M)$ as follows

- As k-vector spaces, set
$$X_i := \varepsilon_i M.$$

- For $\alpha \in Q_*$, set

$$X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)}, \ \varepsilon_{s(\alpha)} x \mapsto \alpha \varepsilon_{s(\alpha)} x = \alpha x = \varepsilon_{t(\alpha)} \alpha x \in X_{t(\alpha)},$$

as $\varepsilon_{t(\alpha)} \alpha = \alpha \varepsilon_{s(\alpha)}$.
So $X := ((X_i)_{i \in Q_0}, (X_\alpha)_{\alpha \in Q_*})$ is a representation of Q.

- This construction is also functorial: take $g : M \to N$ a homomorphism of left kQ-modules. Define $G(g) : X \to Y$, with $X := G(M)$ and $Y := G(N)$. Set

$$(Gg)_i := X_i \to Y_i, \ \varepsilon_i x \mapsto g(\varepsilon_i x) = \varepsilon_i g(x) \text{ with } X_i := \varepsilon_i M \text{ and } Y_i := \varepsilon_i N.$$

This is indeed a homomorphism of representations: Let $\alpha \in Q_1$ be arbitrary, and consider

$$\begin{array}{ccc} X_{s(\alpha)} & \xrightarrow{(Gg)_{s(\alpha)}} & Y_{s(\alpha)} \\ {\scriptstyle X_\alpha}\downarrow & & \downarrow{\scriptstyle Y_\alpha} \\ X_{t(\alpha)} & \xrightarrow[(Gg)_{t(\alpha)}]{} & Y_{t(\alpha)} \end{array} \ .$$

Then

$$y_\alpha (Gg)_{\varepsilon_{s(\alpha)}} x = Y_\alpha(g(\varepsilon_{s(\alpha)} x)) = \alpha g(\varepsilon_{s(\alpha)} x)$$

and

$$(Gg)_{\varepsilon_{t(\alpha)}}(X_\alpha(\varepsilon_{s(\alpha)} x)) = g(\alpha(\varepsilon_{s(\alpha)} x)) = \alpha g(\varepsilon_{s(\alpha)} x),$$

hence the diagram commutes.

Theorem 1.28. *i) Let M be a left kQ-module. Then $FG(M) \cong M$ as left kQ-modules.*

ii) Let X be a representation of Q over k. Then $GF(X) \cong X$ as representations of Q.

Proof. i) Denote $X := G(M)$. Then

$$F(X) = \bigoplus_{i \in Q_0} X_i = \bigoplus_{i \in Q_0} \varepsilon_i M$$

as a k-vector space. Observe

- The identity in kQ is given by

$$\text{id}_{kQ} = \sum \varepsilon_i.$$

Hence, for all x in X:

$$x = \text{id}_{kQ}\, x = \left(\sum \varepsilon_i\right) x = \sum \left(\varepsilon_i x_i\right) \in \sum \varepsilon_i M.$$

- For all $i \neq j$, $\varepsilon_i \varepsilon_j = 0$ holds. So for

$$x \in X_i = \varepsilon_i M \bigcap \sum_{j \neq i} \varepsilon_j M_j \implies x = \sum_{j \neq i} \varepsilon_j x_j \text{ for some } x_j \in M.$$

But as $x \in \varepsilon_i M$, $\varepsilon_i x = x$. So

$$x = \varepsilon_i x = \varepsilon_i \left(\sum_{j \neq i} \varepsilon_j x_j\right) = \sum_{j \neq i} \varepsilon_i \varepsilon_j x_j = 0.$$

These observations show that

$$\varphi : F(X) = \bigoplus_{i \in Q_0} \varepsilon_i M \longrightarrow M \ , \ (\varepsilon_i x_i)_{i \in Q_0} \mapsto \sum \varepsilon_i x_i$$

is an isomorphism of k-vector spaces.
Show that φ is isomorphism of left kQ-modules:
Without loss generaly, assume that $a = p$ is a path in Q (regarded as element of kQ), and let $x \in M$. Then

$$\varphi(a.x) = \varphi\left(\iota_{t(p)} X_p(x_{s(p)})\right) = \varphi\left(\iota_{t(p)}(p x_{s(p)})\right)$$
$$= p x_{s(p)},$$

and

$$a.\varphi(x) = a.\sum x_i = a.\sum \epsilon_i x_i = p x_{s(p)}.$$

ii) Let $M := F(X)$ be the left kQ-module associated with X. Then

$$G(M)_i = \varepsilon_i M = \varepsilon_i \bigoplus_{j \in Q_0} X_j = X_i,$$

and

$$(G(M))_\alpha (x_{s(\alpha)}) = \alpha.x_{s(\alpha)} = X_\alpha(x_{s(\alpha)}).$$

$\square$

Remark 1.29. Let M be a left kQ-module, with Q finite and k a field.

i) $\dim_k M = \sum_{i \in Q_0} \dim_k X_i$ where $X = G(M)$, where G is the functor from remark 1.27

ii) $\dim_K kQ < \infty \Longleftrightarrow Q$ contains no **oriented cycles** (a path p of length ≥ 1, such that $s(p) = t(p)$)

iii) If Q has no oriented cycle, then the following are equivalent:
   (a) M is a finitely generated kQ-module.

   (b) $\dim_k X_i < \infty$.

   Proof. (a) $\Longrightarrow$ (b):(b) implies in particular, that M is finitely generated as a k-module. But as $k \subset kQ$, (a) follows immediatley.

   (b) $\Longrightarrow$ (a): Set $A := kQ$,and let $x_1, ..., x_n \in kQ$ generate M as a left kQ-module. Then there is a kQ-linear surjection

   $$A^n \longrightarrow\!\!\!\!\!\rightarrow M$$

   given by $e_i \mapsto x_i$, where the $(e_i)_{1 \leq i \leq n}$ are a basis of A^n. As this is in particular k-linear, we have that
   $$\dim_k M \leq \dim_k(A^n) = n \dim_k A < \infty,$$

   as Q contains no cycle. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

iv) Under G, the notion of a „left submodule" corresponds to **subrepresentations** of Q, i.e. a tupel of subspaces $Y_i \subset X_i$ for all $i \in Q_0$ such that $X_\alpha(Y_{s(\alpha)}) \subset Y_{t(\alpha)}$ for all $\alpha \in Q_1$.

v) Under G, a direct sum of modules corresponds to **direct sum of representations**: Given X, Y two representations of Q, define a new representation $X \oplus Y$ where the vector spaces are given by
   $$(X \oplus Y)_i := (X \oplus Y)_i$$

   and the k-linear maps

   $$(X \oplus Y)_\alpha : X_{s(\alpha)} \oplus Y_{s(\alpha)} \longrightarrow X_{t(\alpha)} \oplus Y_{t(\alpha)}$$

   given by

   $$\left(\begin{array}{c|c} X_\alpha & \\ \hline & Y_\alpha \end{array} \right).$$

1.5   Bimodules and tensor products

Definition 1.30. Let A, B be k-algebras. A **A-B-bimodule** M is a set M, together with maps:

$$A \times M \longrightarrow M, (a, x) \longmapsto ax$$

$$M \times B \longrightarrow M, (x, b) \longmapsto xb,$$

such that

i) M is a left A-module

ii) M is a right B-module

iii) for all $a \in A, b \in B$ and $x \in M$, the relation

$$(ax)b = a(xb)$$

holds.

We denote a A-B-bimodule by

$$_A M_B.$$

Lemma 1.31. *Let A, B, C be k-algebras, and consider $_A M_B$ and $_A N_C$, a A-B-bimodule and a A-C-bimodule respectivley. Then $\hom_A(M, N)$ becomes a B-C-bimodule via*

- $B \times \hom_A(M, B) \longrightarrow \hom_A(M, N), \qquad (b, f) \longmapsto bf : M \to N, x \mapsto f(xb)$

- $\hom_A(M, N) \times C \longrightarrow \hom_A(M, N), \qquad (f, c) \longmapsto fc : M \to N, x \mapsto f(cx)$

Proof.
- well-defined:

$$bf(ax) = f((ax)b) = f(a(xb)) = af(xb) = a(bf)(x)$$

- $\hom_A(M, N)$ is a left B-module: Show e.g. (L3):

$$
\begin{aligned}
((bb')f)(x) = (x(bb')) = f((xb)b') \\
= b'(f(xb)) \\
= b((b'f)(x))
\end{aligned}
$$

- compatibility:
$$((af)b)(x) = f((ax)b) = f(a(xb)) = (a(fb))(x).$$

$\square$

$-$ End of Lecture 3 $-$

Definition 1.32. Let A be a k-algebra, and M_A be a right A-module, $_A M$ a left A-module.

i) Let P be a k-module. A map
$$\varphi : M \times N \to P$$
is called A-**balanced** if

- $\varphi(x + x', y) = \varphi(x, y) + \varphi(x', y)$
- $\varphi(x, y + y') = \varphi(x, y) + \varphi(x, y')$
- $\varphi(xa, y) = \varphi(x, ay)$

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- φ is k-linear.

ii) A pair (T, τ) where T is a k-module and τ a A-balanced map $M \times N \to T$ is called a **tensor product** of M with N over A, if the follwing universal property holds: For all A-balanced maps $\varphi : M \times N \to P$, where P is any k-module, there is a unique k-linear map f, such that

$$
\begin{array}{ccc}
M \times N & \xrightarrow{\ \tau\ } & T \\
{\scriptstyle \varphi}\downarrow & \ \ \ \ \ \ \ & \\
P & & \exists! f\ k\text{-linear}
\end{array}
$$

commutes.

Lemma 1.33. *Let A be a k-algebra, M_A a right A-module, $_A N$ a left A-module.*

i) There exists a tensor product (T, τ) of M with N over A.

ii) This tensor product is unique up to unique isomorphism. More precisely, if (T', τ') is any other tensor product, then there exists a unique isomorphism of k-modules $f : T \to T'$, such that $f \circ \tau = \tau'$.

Proof.   i) *existence:* Let F be the free k-module with basis $M \times N$, and U the submodule generated by elements of the form

- $(x + x', y) - (x, y) - (x', y)$
- $(x, y + y') - (x, y) - (x, y')$
- $(xa, y) - (x, ay)$
- $(\lambda x, y) - \lambda(x, y)$
- $(x, \lambda y) - \lambda(x, y)$

Then F/U is a k-module, and

$$
\tau : M \times N \longrightarrow F \longrightarrow F/U
$$

is A-balanced by definition. We set

$$
a \otimes b := \tau((a, b)) \text{ and } M \otimes_A N := F/U.
$$

The pair $(M \otimes_A N, \otimes)$ satisfies the universal property of a tensor product:
Let $\varphi : M \times \to P$ be A-balanced. Then there exists a unique $\hat{\varphi}$ such that the following diagram commutes, as F is free with basis $M \times N$:

$$
\begin{array}{ccc}
M \times N & \hookrightarrow & F \\
{\scriptstyle \varphi}\downarrow & & \\
P & & \exists! \hat{\varphi}
\end{array}
$$

Since φ is A-balanced, $\hat{\varphi}$ factors through F/U, i.e.:

$$M \times N \lhook\joinrel\longrightarrow F \xrightarrow{\ \pi\ } F/U$$

Now set $\tau := \overline{\hat{\varphi}}$.

ii) later...

$\square$

Lemma 1.34. *Let A, B, C be k-algebras, $_AM_B$ and $_BN_C$ bimodules. Then $M \otimes_B N$ is a A-C-bimodule, via*

- $a(x \otimes y) = (ax) \otimes y$

- $(x \otimes y)c = x \otimes (yc)$,

for all $x \in X, y \in Y, a \in A, c \in C$.

Proof. $M \otimes_B N$ is already a k-module. For all $a \in A$ define

$$\tau_a : M \times N \longrightarrow M \times N \xrightarrow{\ \otimes\ } M \otimes_B N$$

$$(x, y) \longmapsto (ax, y) \longmapsto (ax) \otimes y$$

τ_a is A-balanced:

- $\tau_a((x + x', y)) - \tau_a((x, y)) - \tau_a((x', y)) = (a(x + x')) \otimes y - (ax) \otimes y - (ax') \otimes y = 0$

- $\tau_a((xb, y)) - \tau_a((x, by)) = (a(xb)) \otimes y - (ax) \otimes (by) = ((ax)b) \otimes y - (ax) \otimes (by) = 0$

So it factors through $M \otimes_B N$ as follows:

The map

$$A \to \mathrm{End}_k(M \otimes_B N), \ a \mapsto f_a$$

is a homomorphism of k-algebras. So by proposition 1.21, $M \otimes_B N$ is a left A-algebra. If we consinder the map

$$\tau_c : M \times N \longrightarrow M \times N \longrightarrow M \otimes_B N$$

$$(x, y) \longmapsto (x, yc) \longmapsto x \otimes (yc)$$

$\square$

Remark 1.35. Let $\varphi : A \to B$ be a k-algebra homomorphism, $_BN$ a left B-module. Then $M := A$ is a A-B-bimodule via

$$a.x.b := ax\varphi(b) \text{ for all } a \in A, x \in M, b \in B.$$

Then by lemma 1.34, $A \otimes_B N$ is a left A-module, where we think of B as a right k-module. This construction is sometimes called **extension of scalars** or **induction of B by A**.

Lemma 1.36. *Let A, B, C, D be k-algebras and consider*

$$_AM_B, \ _A(M_i)_B \ (i \in I), \ _BN_C, \ _B(N_j)_C \ (j \in J).$$

Then there are isomorphisms:

   i)

$$\left(\bigoplus_{i \in I} M_i \right) \otimes_B N \longrightarrow \bigoplus_{i \in I} (M_i \otimes_B N)$$

$$(x_i) \otimes y \longmapsto (x_i \otimes y)$$

      of A-C-bimodules.

   ii)

$$M \otimes_B \left(\bigoplus_{j \in J} N_j \right) \longrightarrow \bigoplus_{j \in J} (M \otimes_B N_j)$$

$$x \otimes (y_j) \longmapsto (x \otimes y_j)$$

      of A-C-bimodules.

   iii)

$$(M \otimes_B N) \otimes_C P \longrightarrow M \otimes_B (N \otimes_C P)$$

$$(x \otimes y) \otimes z \longmapsto x \otimes (y \otimes z)$$

      of A-B-bimodules.

iv)

$$A \otimes_A M \longrightarrow M$$

$$a \otimes x \longmapsto ax$$

$$M \otimes_B \longrightarrow M$$

$$x \otimes b \longmapsto xb$$

of *A-B-bimodules.*

Proof. This is supposed to beexactly the same as [Fra18, p. 2.27]. □

Proposition 1.37. *Let A, B be k-algebras, and M_A, $_A N_B$ and P_b (bi)-modules. The map*

$$\hom_B(M \otimes_A N, P) \to \hom_A (M, \hom_B (N, P)) , \ f \mapsto \Phi(f) \ where \ \Phi(f)(x)(y) \mapsto f(x \otimes y),$$

for $x \in M$ and $y \in N$.
is a well-defined isomorphism of k-modules, natural in M, N, P.

Proof. • $\varphi(f)(x)$ is right B-module homomorphism:

$$\Phi(f)(x)(yb) = f(x \otimes yb) = f((x \otimes y)b) = f(x \otimes y)b$$

  • $\varphi(f)$ is right A-module homomorphism:

$$\Phi(f)(xa)(y) = f(xa \otimes y) = f(a \otimes ay)$$
$$(\Phi(f)(x)a)(y) = \Phi(f)(x)(ay) = f(a \otimes ay)$$

  • Φ is k-linear.

So Φ is a well-defined map of k-modules.

Φ has inverse: Let $g \in \hom_A(M, \hom_B(N, P))$. Define

$$\psi(g) : M \times N \to P, \ (x, y) \mapsto g(x)(y).$$

Then $\psi(g)$ is A-balanced, so it factors through the tensor product:

$$\begin{array}{ccc} M \times N & \xrightarrow{\ \otimes\ } & M \otimes_A N \\ {\scriptstyle \exists! \hat{\psi}(g)} {\scriptstyle \hat{\psi}(g)} \downarrow {\scriptstyle k\text{-linear}} & \swarrow & \\ P & & \end{array}$$

The map $\hat{\varphi}(g)$ is also as right B-module homomorphism, $g(x) \in \hom_B(N, P)$.

But the map
$$\hat{\psi} : \hom_A(M, \hom_B(N, P)) \to \hom_B(M \otimes N, P)$$
is the inverse of φ, as

- $\hat{\psi}(\Phi(f))(x \otimes y) = \Phi(f)(x)(y) = f(x \otimes y)$
- $\Phi(\hat{\psi}(g))(x)(y) = \hat{\psi}(g)(x \otimes y) = g(x)(y)$

for all $x \in M$ and $y \in N$. $\qquad\square$

Definition 2.1. A category $\mathcal{C}$ consists of:

- A class $\mathrm{Ob}\,(\mathcal{C})$, whose elements are called the **objects** of $\mathcal{C}$.

- For all $X, Y \in \mathrm{Ob}\,(\mathcal{C})$, a set $\mathcal{C}(X, Y)$. An element of $\mathcal{C}(X, Y)$ is called a **morphism** from X to Y as is denoted by
$$f : X \to Y.$$

- For all $X, Y, Z \in \mathrm{Ob}\,(\mathcal{C})$, a map
$$\varphi(x, y) \times \varphi(y, z) \to \varphi(x, z), \ (f, g) \mapsto g \circ f.$$

These should satisfy:

- (L1): Associativity: For all $X, Y, Z, W \in \mathrm{Ob}\,(\mathcal{C})$, and morphisms
$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \xrightarrow{\ h\ } W,$$

  we have
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (L2): Identity: For all $X \in \mathrm{Ob}\,(\mathcal{C})$, there is a morphism $\mathrm{id}_X \in \mathcal{C}(X, X)$, such that for any object $Y \in \mathrm{Ob}\,(\mathcal{C})$:
$$f \circ \mathrm{id}_X = f \text{ and } \mathrm{id}_X \circ g = g \text{ for all } f : X \to Y, g : Y \to X$$

  holds.

Remark 2.2.   i) $\varphi(X, Y) = \emptyset$ can happen if $X \neq Y$

  ii) id_X is unique, as $\mathrm{id}_X = \mathrm{id}_X \circ \mathrm{id}'_X = \mathrm{id}'_X$

Remark 2.3. We sometimes want to consider categories whose objects are all sets (with additional conditions). But this can cause logical problems. As a solution, we introduce so called universes. We will always fix a universe, such that sets are elements of this universe, and classes are subsets of this universe. Consider [Lan98, p. 1.6] for further reference.

A.0 Sheet 0

Definition A.1. Let X be any set, and k any commutative ring with unit. Define the **free algebra** generated by X:

   i) As k-module, set $k\langle X \rangle$ as the free k-module generated by X.

   ii) Define the multiplication of two words as the concatination.

Then $k\langle X \rangle$ satisfies the following universal property: Let B be any k-algebra, and $f : X \to B$ a homomorphism of sets. There there is a unique homomorphism of k-algebras $k\langle X \rangle$, such that the following diagram commutes:

$$
\begin{array}{ccc}
k\langle X \rangle & \xdashrightarrow{\ \exists! f\ } & B \\
\uparrow & \nearrow & \\
X & {\scriptstyle f} &
\end{array}
\quad .
$$

Proposition A.2. *Consider the forgetful functor*

$$\mathscr{F} : k\text{-}\mathcal{A}\} \to \mathcal{S}, \ \ B \mapsto B$$

and

$$\mathscr{K}\langle - \rangle : \mathcal{S} \to k\text{-}\mathcal{A}\}, \ \ X \mapsto k\langle X \rangle.$$

Then for all sets X and k-algebras,

$$\hom_{\mathcal{S}}(X, F(B)) \cong \hom_{k\text{-}\mathcal{A}\}}(k\langle X \rangle, B)$$

holds.

We say that F is **right-adjoint** to $\mathscr{K}\langle - \rangle$.

Problem A.0.1. Consider the k-algebra

$$A := k\langle x, y \rangle / (\langle xy - yx - 1 \rangle)$$

over a fiel k with char $k = 0$. Show that there are no non-zero, finite-dimensional representations of A.

Definition A.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A covariant functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is **faithful**, if for all objects A, B of $\mathcal{C}$, the induced function of sets

$$\hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(\mathscr{F}(A), \mathscr{F}(B))$$

is injective; it is **full**, if this function is surjective for all objects A, B of $\mathcal{C}$.

Definition A.4. A covariant functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories**, if it is fully faithful (i.e. bijective on hom-sets) and **essentially surjective**, i,e, for every object Y of $\mathcal{D}$, there is an object X of $\mathcal{C}$ such that $\mathscr{F}(X) \cong Y$.

Some people write natural transformations in the follwing way:

$$A \quad\Downarrow\quad B$$

A.1 Sheet 1

Solution A.1.1. i) φ_m is k-linear:

- $\varphi(a+b)x = (a+b)x \overset{(L1)}{=} ax + bx = \varphi(a)x + \varphi(b)x$
- $\varphi(\lambda a) = (\lambda a)x \overset{(L5)}{=} \lambda(ax) = \lambda\varphi(a)x.$

φ_m is ring homomorphism:

- $\varphi(ab) = (ab)x \overset{(L3)}{=} a(bx) = \varphi(a)\varphi(b)x$
- $\varphi(1_A) \overset{(L4)}{=} (1_A)x = x.$

As these relations hold for all x, the assertion follows.

ii) V_φ is already a k-module.

- (L1) $a(x+y) = (\varphi(a))(x+y) \overset{\varphi \in \mathrm{End}_k(V)}{=} (\varphi(a))(x) + (\varphi(a))(y) = ax + ay$
- (L2) $(a+b)x = (\varphi(a+b))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\varphi(a) + \varphi(b))x = \varphi(a)x + \varphi(b)x = ax + bx$
- (L3) $(ab)x = (\varphi(ab))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\varphi(a)\varphi(b))x = a(bx)$
- (L4) $1_a x = (\varphi(1_a))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} \mathrm{id}\, x = x$
- (L5) $(\lambda a)(x) = ((\varphi(\lambda a))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\lambda\varphi(a))x = \lambda(ax)$ and
  $a(\lambda x) = (\varphi(a)(\lambda x)) \overset{\varphi(a) \text{ endo of } k-\text{module}}{=} \lambda(\varphi(a))x = \lambda(ax),$

for all $a, b \in A, x, y \in V$ and $\lambda \in k$.

iii) We regard V and W as A-modules in the sense of part ii). Assume that $\psi(a) \circ f = f \circ \varphi(a)$ ($*$) for all $a \in A$. Then

$$f(ax) = f((\varphi(a))x) \overset{*}{=} (\psi(a))(f(x)) = af(x),$$

for all $x \in X$. Hence f is an A-module homomorphism.
Assume that f is a a-module homomorphism, then

$$(\psi(a))(f(x)) = a(f(x)) = f(ax) = f((\varphi(a))x)$$

for all $x \in V$. Hence $\psi(a) \circ f - f \circ \varphi(a) = 0$ and so ($*$) holds.

Solution A.1.2. i) Assume that I is a non-zero ideal of A. Let $a = (a)_{ij} \neq 0$ be an arbitrary matrix in I. Then there exist permutation $\sigma, \pi \in \mathrm{GL}_n(K)$ matrices, such that $(\sigma a \pi)_{11} \neq 0$, which is in a, as I is a two-sided ideal. So without loss of generality, suppose $a_{11} \neq 0$.
Define

$$b \in \mathrm{M}_n(k), (b)_{ij} := \begin{cases} 1, & \text{if } i = j = 1 \\ 0, & \text{else} \end{cases}$$

and E_n as the identity of $\mathrm{M}_n(k)$. Then we get

$$\left(\frac{1}{a_{11}} E_n \right) \cdot b \cdot a \cdot b = b.$$

By repeatetly using permutation matrixes, it is possible to write any matrix as sum of products of a, b and permutation matrices on the left- and right. As I is a two-sided ideal, a all of these combinations are in I as well. Hence a generates all of A, and $I = A$.

ii) Consider A as a k-vector space, then $\dim_K A = n^2$. Let M be any left A-module. As shown in task 3, there is a homomorphism of k-algebras

$$\varphi A \to \mathrm{End}_k(M), \ a \mapsto a : (x \mapsto ax),$$

which is in particular a homomorphism of k-vector spaces. The kernel of φ is a two-sided ideal of A, as

$$a0x = 0ax = 0$$

for all $a \in A$ and $x \in M$.
Now i) implies that $\ker \varphi$ is either zero or $\ker \varphi = A$. But since $\varphi(E_n) = \mathrm{id}_M$, the latter one is not possible. Hence φ is injective, and in particular $\dim A \leq \dim \mathrm{End}_k(V)$, so $n \leq m$.

Proposition A.5. *Let k be a field, $k[X]$ the polynomial ring and $p \in k[X]$ a polynomial with $\deg p = n$. Then*

$$k[X]/(p)$$

is a n-dimensional k vector space, and a basis is given by

$$\{1, x, ..., x^{n-1}\}.$$

The following propositions are taken from [Alu09].
Let R be any commutative ring.

Proposition A.6. *Let $I_1, ..., I_k$ be ideals of R such that $I_i + I_j = (1)$ for all $i \neq j$. Then the natural homomorphism*

$$\varphi : R \to R/I_1 \times ... \times R/I_k$$

is surjective and induces an isomorphism

$$\frac{R}{I_1 \ldots I_k} \to R/I_1 \times ... \times R/I_k$$

Corollary A.7 (Chinese remainder theorem)**.** *Let R be a PID and $a_1, ..., a_k \in R$ be elements such that $\gcd(a_i, a_j) = 1$ for all $i \neq j$. Let $a = a_1 \ldots a_k$. Then the function*

$$\varphi : R/(a) \to R/(a_1) \times \ldots \times R/(a_k).$$

Proposition A.8 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category, X an object of $\mathcal{C}$ and consider the contravariant functor*

$$h_X := \hom_{\mathcal{C}}(-, X).$$

Then for every contravariant functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$, there is a bijection between the set of natural transformations $h_x \rightsquigarrow \mathscr{F}$ and (X).

Definition A.9 ([ASS06])**.** The (Jacobson) **radical** rad A of a K-algebra A is the intersection of all maximal right ideals in A. It is the same as the intersetion of all left-sided maximal right ideals in A. Furthermore, rad A is a two-sided ideal.

Definition A.10. Let $f, g : X \to Y$ be morphisms in a category $\mathcal{C}$. Then a morphism $e : E \to X$ is called **equalizer** of f and g if $f \circ e = g \circ e$ and for all other morphisms $o : O \to X$, such that $f \circ o = g \circ o$, there is a unique morphis $O \to E$, such that the following diagram commutes:

$$E \xrightarrow{e} X \rightrightarrows X$$

Proposition A.11. *Equalizers exists in abelian categories.*

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