

Foundations of Representation Theory

Lecture Notes in the Winter Term 2018/19

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INTRODUCTION

These are my personal lecture notes for the lecture *Foundations of Representation Theory* held by Dr. Hans Franzen at the University of Bonn in the winter term 2018/19.

I try to update them on my website, <https://pankratius.github.io>.
I label my own comments and additions in purple.

The book [Alu09] is used for further references, and highly recommended.
In addition, I want to point out that there is another student who is publishing his notes for this course (which are way better than mine). These are available under ciox.github.io.

Notation

I try to follow Dr. Franzens notation closely. Some deviations are:

- Categories: e.g.: \mathcal{C} , \mathcal{D} .
- functors: e.g.: $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ are functors from the category \mathcal{C} to the category \mathcal{D} .
- Natural transformations: When I want to put more data into the depiction of a natural transformation $\eta : \mathcal{F} \rightarrow \mathcal{G}$ (where $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ are functors), I follow the notation introduced in [Rie17]:

$$\eta : \mathcal{C} \begin{array}{c} \xrightarrow{\mathcal{F}} \\ \Downarrow \\ \xrightarrow{\mathcal{G}} \end{array} \mathcal{D}$$

(There are still some TikZ-related issues however).

1. ALGEBRAS AND MODULES

Conventions: In this lecture, rings will always be unital, and ring homomorphisms f always fulfill $f(1) = 1$. Rings do *not* have to be commutative.

1.1 Algebras - Basics

Let k be a commutative ring.

Definition 1.1.1. A k -**algebra** A is a ring A , together with a structure of a k -module on A , such that

$$\text{for all } a, b \in A, \lambda \in K : (\lambda a)b = a(\lambda b) = \lambda(ab) \quad (*)$$

Definition 1.1.2. Let A, B be k -algebras. A **homomorphism of algebras** is a map $f : A \rightarrow B$ that is both k -linear and a ring homomorphism.

Remark 1.1.3. Let A be a ring. Define

$$Z(A) := \{a \in A \mid \forall b \in A : ab = ba\},$$

which is a commutative subring and is called the **center** of A .

Remark 1.1.4. Let A be a ring. Giving a k -algebra structure on A is the same as giving a ring homomorphism $k \rightarrow Z(A)$. More precisley:

- i) If A is a k -algebra, then $p : k \rightarrow A, \lambda \mapsto \lambda 1$ satisfies $\text{Im } p \subseteq Z(A)$ and is a ring homomorphism. (the first statement follows from $(*)$ and the second one from the fact that A has a k -module structure).
- ii) Let $\varphi : k \rightarrow Z(A)$ be a ring homomorphism. Define

$$\lambda a := \varphi(\lambda)a,$$

for all $\lambda \in k$. This defines a k -algebra structure on A ($(*)$ holds since $\text{Im}(\varphi) \subseteq Z(A)$).

- iii) Let A, B be k -algebras and $f : A \rightarrow B$ a homomorphism of rings. Then f is a homomorphism of k -algebras if and only if the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow & & \uparrow \\ Z(A) & & Z(B) \\ \swarrow \phi_A & & \searrow \phi_B \\ & k & \end{array} .$$

Example 1.1.5. i) Let V be a k -module. Then $\text{End}_k(V)$ has a ring structure given by

$$\text{End}_k(V) \times \text{End}_k(V) \rightarrow \text{End}_k(V), (\phi, \psi) \mapsto \phi \circ \psi.$$

Then $\text{End}_k(V)$ is both a ring and a k -module, and becomes a k -algebra via

$$\varphi : k \rightarrow \text{End}_k(V), \lambda \mapsto \lambda \text{id}.$$

Note that $\text{Im } \varphi \subseteq Z(A)$. If k is a field, then $Z(\text{End}_k(V)) = \{\lambda \text{id} \mid \lambda \in k\}$.

ii) Take $V = k^n$ (free module of rank n). Then $\text{End}_k(V) \cong M_n(k)$. Define

$$T_u := \{\varphi \in M_n(k) \mid \varphi \text{ is upper triangular}\},$$

i.e. T_u preserves flags in k^n . Then T_u is a **subalgebra** of $M_n(k)$, i.e. is both a subring and a k -submodule of the original algebra.

iii) Let G be a group. Define the **group algebra** $k[G]$ of G as follows:

- As k -module, is defined as the free module on G ,

$$k[G] := k^{(G)} = \left\{ \sum_{g \in G} \lambda_g g \mid \lambda_g \in k, \lambda_g \neq 0 \text{ for only finitely many } g \in G \right\}.$$

- Multiplication: Let $a := \sum \lambda_g g$, $b = \sum \mu_h h$ and define:

$$ab := \sum_{g \in G, h \in G} \lambda_g \mu_h (gh) = \sum_{j \in G} \left(\sum_{gh=j} \lambda_g \mu_h \right) j.$$


This multiplication is associative, k -bilinear, distributive and $1|_{k[G]} = e$. In addition, $(*)$ is satisfied.

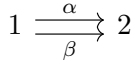
1.2 Quivers - Basics

Definition 1.2.1. A **quiver** is a „directed graph“. Formally, a quiver is a quadruple (Q_0, Q_1, s, t) consisting of sets Q_0 (**vertices**) and Q_1 (**arrows**) and maps $s : Q_1 \rightarrow Q_0$, $t : Q_1 \rightarrow Q_0$. For $\alpha \in Q_1$, we call $s(\alpha)$ the **source** of α and $t(\alpha)$ the **target** of α :

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

Example 1.2.2. i) $Q = (\{1\}, \emptyset, \dots)$ is visualized as: 1

ii) $Q = (\{1\}, \{\alpha\}, \dots)$ is visualized as 

iii) $Q = (\{1, 2\}, \{\alpha, \beta\}, s(\alpha) = s(\beta) = 1, t(\alpha) = t(\beta) = 2)$ is visualized as 

Definition 1.2.3. Let Q be a quiver such that both Q_0 and Q_1 are finite.

- i) Let $\ell \in \mathbb{Z}_{\geq 1}$. A **path** of length ℓ is a sequence $\alpha_\ell, \dots, \alpha_1$ of arrows, such that $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq \ell - 1$,

$$\circ \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_2} \circ \dots \xrightarrow{\alpha_\ell} \circ.$$

Define Q_ℓ to be the set of all paths of length ℓ .

Let $p : \alpha_\ell \dots \alpha_1$ be a path. Define $s(p) := s(\alpha_1)$ and $t(p) := s(\alpha_\ell)$.

Formally define Q_0 to be the set of all paths of length zero. Denote by ε_i for $i \in Q_0$ the constant path at i . ε_i is called a **lazy path**. We set $s(\varepsilon_i) = t(\varepsilon_i) := i$.

- ii) Let $p = \alpha_\ell \dots \alpha_i$ and $q = \beta_m \dots \beta_1$ be paths of length ℓ and m respectively, with $\ell, m \geq 1$. If $t(p) = s(q)$, then set $q \circ p := \beta_m \dots \beta_1 \alpha_\ell \dots \alpha_1$. This is a path of length $\ell + m$. For p a path of length ≥ 0 and ε_i a lazy path:

- if $t(p) = i$, set $\varepsilon_i \circ p := p$,
- if $s(p) = i$, set $p \circ \varepsilon_i := p$.

In all other cases, the composition is not defined.

- iii) Define

$$Q_* := \bigcup_{\ell \geq 0} Q_\ell,$$

the set of all paths. Define the **path-algebra** kQ :

- As a k -module, $kQ := k^{(Q_*)}$.
- Multiplication: Let $a = \sum \lambda_p p$, $b = \sum \mu_q q$. Define

$$ab := \sum_{p, q \in Q_*} \lambda_p \mu_q (p \cdot q),$$

where

$$p \cdot q := \begin{cases} p \circ q, & \text{if it is defined, i.e } t(q) = s(p) \\ 0, & \text{else} \end{cases}.$$

The multiplication is associative (due to the associativity of the composition of paths) and k -bilinear by definition. In addition, distributivity and $(*)$ are fulfilled.

- The identity is given by $\sum \varepsilon_i$.

Example 1.2.4. i) $Q = 1$, then $kQ = k$.

- ii) $Q = 1 \xrightarrow{\quad} 1$, then $Q_* = \{\alpha^n \mid n \geq 0\}$ and $kQ = k[t]$.

- iii) $Q = 1 \xrightleftharpoons[\beta]{\alpha} 2$. Then $Q_* = \{\varepsilon_1, \varepsilon_2, \alpha^n, \beta^n \mid n \geq 0\}$ and

$$kQ = k\varepsilon_1 \oplus k\varepsilon_2 \oplus k\alpha \oplus k\beta.$$

A multiplication table is given by

	ε_1	ε_2	α	β
ε_1	ε_1	0	0	0
ε_2	0	ε_2	α	β
α	α	0	0	0
β	β	0	0	0

Lemma 1.2.5. *Let Q be a finite quiver, k a field. Then the following are equivalent:*

- i) Q contains no cycles.*
- ii) $\dim_k kQ < \infty$.*

Lemma 1.2.6. *Let k be a field, A a k -algebra and $n := \dim(A) < \infty$. Then there exists an injective homomorphism of k -algebras $\varphi : A \rightarrow M_n(k)$.*

Proof. By choosing a basis of A , we get an isomorphism $\text{End}_k(A) \cong M_n(k)$. So it suffices to find an injective homomorphism of k -algebras $\varphi : A \rightarrow \text{End}_k(A)$.

Consider

$$\varphi : A \rightarrow \text{End}_k(A), \quad \varphi(a) : A \rightarrow A, b \mapsto ab.$$

- $\varphi(a)$ is k -linear for all a by the distributivity in A and the condition (*).
- φ is k -linear by the distributivity in A and the condition (*).
- Let $a, a' \in A$. Then

$$\varphi(aa')(b) = (aa')(b) = a(a'b) = (\varphi(a) \circ \varphi(a'))(b).$$

Hence φ is indeed a homomorphism of k -algebras.

To show that φ is injective, let $a \in \ker \varphi$, hence $ab = 0$ for all $b \in A$. But in particular, $0 = a1 = a$. □

End of Lecture 1

Definition 1.2.7. Let A be a k -algebra. Then the **opposite algebra** A^{op} is A (as a k -module), and the multiplication is defined as

$$a \cdot_{A^{\text{op}}} b = b \cdot_A a.$$

Example 1.2.8. Let Q be a quiver, and define $Q^{\text{op}} := (Q_0, Q_1, s^{\text{op}}, t^{\text{op}})$, where $s^{\text{op}}(\alpha) := t(\alpha)$ and $t^{\text{op}}(\alpha) := s(\alpha)$. Then $kQ^{\text{op}} = k(Q^{\text{op}})$

1.3 Modules - Basics

Definition 1.3.1. Let A be a k -algebra. A **left A -module** M is a k -module M together with a map $A \times M \rightarrow M, (a, x) \mapsto ax$, such that:

$$a(x + y) = ax + ay \quad (\text{L1})$$

$$(a + b)x = ax + bx \quad (\text{L2})$$

$$a(bx) = (ab)x \quad (\text{L3})$$

$$1_A x = x \quad (\text{L4})$$

$$(\lambda a)x = \lambda(ax) = a(\lambda x), \quad (\text{L5})$$

for all $a, b \in A, x, y \in M$ and $\lambda \in k$. If A is a left A -module, we denote this as ${}_A M$. A **right A -module** is defined analogous, where (L3) becomes $(xa)b = x(ab)$. If A is a right A -module, we denote this by A_M .

Remark 1.3.2. A right A -module is the same as a left A^{op} -module.

Definition 1.3.3. Let A be a k -algebra, and M, N left A -modules. A **homomorphism of left A -modules** $f : M \rightarrow N$ is a k -linear map such that

$$f(ax) = af(x)$$

for all $a \in A$ and $x \in M$.

Define the set of all left A -algebra homomorphisms as

$$\text{hom}_A(M, N) := \text{hom}_A({}_A M, {}_A M) := \{f : M \rightarrow N \mid f \text{ is a homomorphism of left } A\text{-modules}\}.$$

A homomorphism of left A -modules is an **isomorphism** if it is a bijective homomorphism of left A -modules.

Homomorphism of right A -modules are defined analogous.

Remark 1.3.4. Let M, N be left A -modules. Then

- i) $\text{hom}_A(M, N)$ has a k -module structure given by

$$\lambda f : M \rightarrow N, \quad x \mapsto \lambda f(x) = f(\lambda x).$$

This is well defined, as k lies in the center of A .

- ii) In general, $\text{hom}_A(M, N)$ has neither a left nor a right A -module structure.
- iii) f is an isomorphism if and only if there is a homomorphism of left A -modules $g : N \rightarrow M$ such that

$$g \circ f = \text{id}_M \quad \text{and} \quad f \circ g = \text{id}_N.$$

- iv) Let $f : M \rightarrow M'$ and $g : N \rightarrow N'$ be **homomorphisms of left A -modules**. Then we obtain k -linear maps

$$\begin{aligned} f^* : \text{hom}_A(M', N) &\rightarrow \text{hom}_A(M, N), \quad h \mapsto h \circ f \\ g_* : \text{hom}_A(M, N) &\rightarrow \text{hom}_A(M, N'), \quad h \mapsto g \circ h. \end{aligned}$$

Remark 1.3.5. Let A be a k -algebra and M, N left A -modules.

- i) A subset $M' \subseteq M$ is called a **submodule** if

$$(SM1) \quad 0 \in M'$$

$$(SM2) \quad x, x' \in M' \implies x + x' \in M'$$

$$(SM3) \quad a \in A, x \in M' \implies ax \in M'.$$

In particular, submodules of A -modules are submodules of the underlying k -module, as follows using (L4)

- ii) Let M be a submodule. Then the **quotient** has a left A -module structure in the obvious way. The projection

$$\pi : M \rightarrow M'$$

is a homomorphism of left A -modules.

- iii) A **left ideal** is left A -submodule of ${}_A A$. Similar, a **right ideal** is right A -submodule of A_A . For a left ideal $I \subseteq A$, the quotient A/I is a left A -module, but in general not an algebra.
- iv) A **two-sided ideal** $I \subset A$ is both a left- and a right-ideal of A . Then A/I has an algebra structure, by setting

$$(x + I)(y + I) := (xy) + I.$$

In general, this is only well-defined if I is a two-sided ideal of A .

- v) Let $f : M \rightarrow N$ be a homomorphism of left A -modules. Then we obtain left A -modules:

$$\ker f, \text{Im } f, \text{coker } f := N/\text{Im } f, \text{coim } f := M/\ker f.$$

In particular, f factors uniquely as

$$\begin{array}{ccccc} & & f & & \\ & \searrow & & \nearrow & \\ M & \longrightarrow & \text{coim } f & \xrightarrow[\cong]{\exists!} & \text{Im } f \longrightarrow N \end{array} \quad (F)$$

- vi) Let $\{M_i \subset M \mid i \in I\}$ be a family of left A -submodules, for some index set I . Then

$$\bigcap_{i \in I} M_i \text{ and } \sum_{i \in I} M_i$$

are left A -modules.

vii) Let $x \in M$. Define

$$Ax := \{ax \mid a \in A\},$$

which is a left A -submodule. Similar, for $x \in M_A$, define $xA := \{xa \mid a \in A\}$, which is a right A -submodule. For a subset $E \subset M$,

$$\sum_{x \in E} Ax = \bigcap_{\substack{E \subseteq M' \subseteq M \\ M' \text{ submodule}}} M'.$$

M is called **finitely generated**, if there are $x_1, \dots, x_n \in M$, such that

$$M = \sum_{i=1}^n Ax_i.$$

viii) Let $\{M_i \mid i \in I\}$ be a family of left A -modules. Then

$$\prod_{i \in I} M_i := \{(x_i)_{i \in I} \mid x_i \in M_i\}$$

is called the **product**, and

$$\bigoplus_{i \in I} M_i := \{(x_i)_{i \in I} \mid x_i \in M_i, x_i \neq 0 \text{ for only finitely many } i\}$$

is called the **coproduct**. They are both left A -modules. The **projection**

$$\pi_j : \prod_{i \in I} M_i \rightarrow M_j, (x_i)_{i \in I} \mapsto x_j$$

and the **inclusion**

$$\iota_j : \bigoplus_{i \in I} x_j \mapsto (\delta_{ij} x_j)_{i \in I}$$

are morphism of left A -modules.

ix) A left A -module M is finitely generated if and only if there is a surjective homomorphism of left A -modules

$$A^n := \bigoplus_{i=1}^n A \longrightarrow M$$

for some $n \geq 1$. A is called **finitely presented**, if there is an exact sequence of left A -modules

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

for some $m, n \geq 1$.

Proposition 1.3.6. *Let*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0 \quad (*)$$

and

$$0 \longrightarrow N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \quad (**)$$

be sequences of left A -modules.

i) The following are equivalent:

a) $(*)$ is exact.

b) For all left A -modules N , the sequence

$$0 \longrightarrow \operatorname{hom}_A(M_3, N) \xrightarrow{f_2^*} \operatorname{hom}_A(M_2, N) \xrightarrow{f_1^*} \operatorname{hom}_A(M, N)$$

is exact.

ii) The following are equivalent:

a) $(**)$ is exact.

b) For all left A -modules M , the sequence

$$0 \longrightarrow \operatorname{hom}_A(M, N_1) \xrightarrow{g_{1,*}} \operatorname{hom}_A(M, N_2) \xrightarrow{g_{2,*}} \operatorname{hom}_A(M, N_3)$$

is exact.

Proof. We will only prove $a) \implies b)$ of ii).

Lemma 1.3.7. Let K, M, N be left A -modules, and $\zeta : K \rightarrow M$, $\varphi : M \rightarrow N$ be homomorphisms of left A -modules, such that $\varphi \circ \zeta = 0$. Then there is a unique homomorphism $\bar{\zeta}$, such that

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \curvearrowright & \swarrow & \\ K & \xrightarrow{\zeta} & M & \xrightarrow{\varphi} & N \\ & \searrow \exists! \bar{\zeta} & \uparrow & & \\ & & \ker \varphi & & \end{array}$$

commutes.

- $g_{1,*}$ injective: Let $h \in \ker(g_{1,*})$. Then

$$g_1 \circ h : M \xrightarrow{h} N_1 \xrightarrow{g_1} N_2$$

and since g_1 is injective, it follows $h = 0$.

- $\operatorname{Im} g_{1,*} \subseteq \ker g_{2,*}$: Since $(**)$ is exact, it follows that $g_2 \circ g_1 = 0$. For $h \in \operatorname{Im} g_{1,*}$ there exists an $h' : M \rightarrow N_1$ such that $h = g_1 \circ h'$, and hence $g_2 \circ h = g_2 \circ g_1 \circ h' = 0$.
- $\ker g_{2,*} \subseteq \operatorname{Im} g_{1,*}$: As $(**)$ is exact, $\ker g_2 = \operatorname{Im} g_1$ holds.
Let $h : M \rightarrow N_2 \in \ker g_{2,*}$, i.e. $g_2 \circ h = 0$:

$$\begin{array}{ccccccc} & & M & & & & \\ & & \downarrow h & \searrow 0 & & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 \end{array} .$$

By lemma 1.3.7, h factors uniquely through $\ker g_2 = \operatorname{Im} g_1$:

$$\begin{array}{ccccccc}
 & & & M & & & \\
 & & & \downarrow h & \searrow 0 & & \\
 0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 \ . \\
 & & & \uparrow \iota & \nwarrow & & \\
 & & & \operatorname{Im} g_1 & & &
 \end{array}$$

But since g_1 is injective, (F) implies that there is a uniquely determined isomorphism $\tilde{g}_1 : N_1 \longrightarrow \operatorname{Im} g_1$.

Putting everything together, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & M & & & \\
 & & & \downarrow h & \searrow 0 & & \\
 0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 \ . \\
 & & & \uparrow \iota & \nwarrow & & \\
 & & & \operatorname{Im} g_1 & & & \\
 & \swarrow \exists! \tilde{g}_1 & & & & &
 \end{array}$$

Setting $h' := \tilde{g}_1^{-1} \circ h$, we obtain

$$g_1 \circ h' = \iota \circ \tilde{g}_1 \circ \tilde{g}_1^{-1} \circ h = \iota \circ h = h.$$

□

Proposition 1.3.8. *Let A be a k -algebra. To give a left A -module structure is the same as to give a k -module structure V together with a homomorphism $\varphi : A \rightarrow \operatorname{End}_k(V)$ of k -algebras. To give a right A -module structure is the same as giving a k -module structure V together with a homomorphism $\varphi : A \rightarrow \operatorname{End}_k(V)^{\operatorname{op}}$.*

1.4 Representation of quivers

Let k be a field and Q be a quiver.

Definition 1.4.1. A **representation** X of Q over k consists of

- a k -vector space X_i for all $i \in Q_0$,
- a k -linear map

$$X_\alpha : X_{s(\alpha)} \rightarrow X_{t(\alpha)}$$

for each $\alpha \in Q_1$

Example 1.4.2 (Continue example 1.2.4). i) Let $Q = \cdot$. Then a representation of Q is simply a k -vector space.

- ii) Let $Q = 1 \overset{\curvearrowright}{\leftarrow}$. Then a representation of Q is a k -vector space V together with an endomorphism $\varphi \in \text{End}_k(V)$:

$$Q = V \overset{f}{\curvearrowright}.$$

- iii) Let $Q = 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$, the **Kronecker Quiver**. Then a representation of Q is a pair of vector spaces V, W and two linear maps $f, g \in \text{hom}_K(V, W)$:

$$Q = V \overset{f}{\underset{g}{\rightrightarrows}} W$$

Definition 1.4.3. Take X, Y to be two representations of Q over k . A **homomorphism of representations** $f : X \rightarrow Y$ is a tuple $(f_i)_{i \in Q_0}$ of linear maps $f_i : X_i \rightarrow Y_i$, such that for all $\alpha \in Q_1$ the diagram

$$\begin{array}{ccc} X_{s(\alpha)} & \xrightarrow{f_{s(\alpha)}} & Y_{s(\alpha)} \\ X_\alpha \downarrow & & \downarrow Y_\alpha \\ X_{t(\alpha)} & \xrightarrow{f_{t(\alpha)}} & Y_{t(\alpha)} \end{array}$$

commutes.

Example 1.4.4 (Continue example 1.4.2). i) Homomorphisms of representations are k -linear maps $X \rightarrow Y$.

- ii) Homomorphisms of representations (V, φ) and (W, ψ) are k -linear maps $f : V \rightarrow W$, such that

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \varphi \downarrow & & \downarrow \psi \\ V & \xrightarrow{f} & W \end{array}$$

commutes

- iii) Homomorphisms of representations (V_1, V_2, A, B) and (W_1, W_2, C, D) are pairs (f_1, f_2) of linear maps $f_1 : V_1 \rightarrow W_1$ and $f_2 : V_2 \rightarrow W_2$, such that $A \circ f_1 = f_2 \circ A$ and $B \circ f_1 = f_2 \circ B$.

Definition 1.4.5. An **isomorphism of representations** $f : X \rightarrow Y$ is a homomorphism of representations, such that there exists $g : Y \rightarrow X$ homomorphism of representations satisfying

$$g \circ f = \text{id}_X \text{ and } f \circ g = \text{id}_Y.$$

An isomorphism of representations is a homomorphism of representations such that each map f_i is bijective.

End of Lecture 2

Let Q be a quiver over a field k .

Remark 1.4.6. i) Let X be a representation of Q over k . Associate a left kQ -module $M = F(X)$ as follows:

- As k -vector space, let

$$M := \bigoplus_{i \in Q_0} X_i.$$

- Define the action of kQ on M by an action of paths. Let p be a path of length ≥ 1 . Define

$$X_p : X_{s(p)} \rightarrow X_{t(p)} \text{ given by } X_p = X_{\alpha_\ell} \circ \dots \circ X_{\alpha_1},$$

with $X_p \in \text{hom}_k(X_{s(p)}, X_{t(p)})$. Use this to define a k -linear map $\tilde{X}_p : M \rightarrow M$ as composition:

$$\tilde{X}_p : M = \bigoplus_{i \in Q_0} X_i \xrightarrow{\pi_{s(p)}} X_{s(p)} \xrightarrow{X_p} X_{t(p)} \xrightarrow{\iota_{t(p)}} \bigoplus_{i \in Q_0} X_i$$

If the length of $p = 0$, then p is a lazy part at some $i \in Q_0$, and we set

$$X_{\varepsilon_i} := \text{id}_{X_i},$$

and $\tilde{X}_{\varepsilon_i}$ like \tilde{X}_p .

Now these k -linear endomorphisms define a kQ -module structure on M , given by:

$$\begin{aligned} kQ \times M \rightarrow M, \left(a := \sum_{p \in Q_*} \lambda_p \cdot p, (x_i)_i =: x \right) &\mapsto a.x := \sum_{p \in Q_*} \lambda_p \cdot \tilde{X}_p(x) \\ &= \sum_{p \in Q_*} \lambda_p \cdot (\iota_{t(p)} X_p(x_{s(p)})), \end{aligned}$$

where we denote an element in M by a sequence $(x_i)_i$ with $x_i \in X_i$.

- We check that this actually defines a kQ -module structure:
 - (L3): Assume that $a, b \in kQ$. By the bilinearity of the multiplication, we can assume that $a = p$ and $b = q$ are both paths in Q_* . Then

$$\begin{aligned} a.(b.x) &= \tilde{X}_p(\tilde{X}_q(x)) \\ &= \iota_{t(p)} X_p \underbrace{\pi_{s(p)} \iota_{t(q)}}_{\pi_{s(p)} \iota_{t(q)}} X_q(x_{s(q)}), \end{aligned}$$

where

$$\pi_{s(p)} \iota_{t(q)} = \begin{cases} \text{id}_{X_q}, & \text{if } t(q) = s(p) \\ 0, & \text{otherwise} \end{cases}.$$

This gives

$$a.(b.x) = \begin{cases} \iota_{t(p)} X_p X_q(x_{s(q)}), & \text{if } t(q) = s(p) \\ 0 & \text{otherwise} \end{cases}.$$

Additionally,

$$(a.b).x = \begin{cases} \tilde{X}_{p \circ q} & \text{if } t(q) = s(p), \\ 0 & \text{otherwise} \end{cases}.$$

But in the case $f(q) = s(p)$,

$$\tilde{X}_{p \circ q}(x) = \iota_{t(p)} \circ X_p X_q(x_{s(q)}).$$

The construction F is functorial, i.e. for $f : X \rightarrow Y$ a homomorphism of representations, F induces a homomorphism of kQ -algebras

$$Ff : F(X) \rightarrow F(Y) \text{ by } (Ff)((x_i)_i) := (f_i(x_i))_i.$$

ii) Let M be a left kQ -module. Define a representation $X := G(M)$ as follows

- As k -vector spaces, set

$$X_i := \varepsilon_i M.$$

- For $\alpha \in Q_*$, set

$$X_\alpha : X_{s(\alpha)} \rightarrow X_{t(\alpha)}, \quad \varepsilon_{s(\alpha)} x \mapsto \alpha \varepsilon_{s(\alpha)} x = \alpha x = \varepsilon_{t(\alpha)} \alpha x \in X_{t(\alpha)},$$

as $\varepsilon_{t(\alpha)} \alpha = \alpha \varepsilon_{s(\alpha)}$.

So $X := ((X_i)_{i \in Q_0}, (X_\alpha)_{\alpha \in Q_*})$ is a representation of Q .

- This construction is also functorial: take $g : M \rightarrow N$ a homomorphism of left kQ -modules. Define $G(g) : X \rightarrow Y$, with $X := G(M)$ and $Y := G(N)$. Set

$$(Gg)_i := X_i \rightarrow Y_i, \quad \varepsilon_i x \mapsto g(\varepsilon_i x) = \varepsilon_i g(x) \text{ with } X_i := \varepsilon_i M \text{ and } Y_i := \varepsilon_i N.$$

This is indeed a homomorphism of representations: Let $\alpha \in Q_1$ be arbitrary, and consider

$$\begin{array}{ccc} X_{s(\alpha)} & \xrightarrow{(Gg)_{s(\alpha)}} & Y_{s(\alpha)} \\ X_\alpha \downarrow & & \downarrow Y_\alpha \\ X_{t(\alpha)} & \xrightarrow{(Gg)_{t(\alpha)}} & Y_{t(\alpha)} \end{array}.$$

Then

$$y_\alpha (Gg)_{\varepsilon_{s(\alpha)}} x = Y_\alpha (g(\varepsilon_{s(\alpha)} x)) = \alpha g(\varepsilon_{s(\alpha)} x)$$

and

$$(Gg)_{\varepsilon_{t(\alpha)}} (X_\alpha(\varepsilon_{s(\alpha)} x)) = g(\alpha(\varepsilon_{s(\alpha)} x)) = \alpha g(\varepsilon_{s(\alpha)} x),$$

hence the diagram commutes.

Theorem 1.4.7. *i) Let M be a left kQ -module. Then $FG(M) \cong M$ as left kQ -modules.*

ii) Let X be a representation of Q over k . Then $GF(X) \cong X$ as representations of Q .

Proof. i) Denote $X := G(M)$. Then

$$F(X) = \bigoplus_{i \in Q_0} X_i = \bigoplus_{i \in Q_0} \varepsilon_i M$$

as a k -vector space. Observe

- The identity in kQ is given by

$$\text{id}_{kQ} = \sum \varepsilon_i.$$

Hence, for all x in X :

$$x = \text{id}_{kQ} x = \left(\sum \varepsilon_i \right) x = \sum (\varepsilon_i x_i) \in \sum \varepsilon_i M.$$

- For all $i \neq j$, $\varepsilon_i \varepsilon_j = 0$ holds. So for

$$x \in X_i = \varepsilon_i M \cap \sum_{j \neq i} \varepsilon_j M_j \implies x = \sum_{j \neq i} \varepsilon_j x_j \text{ for some } x_j \in M.$$

But as $x \in \varepsilon_i M$, $\varepsilon_i x = x$. So

$$x = \varepsilon_i x = \varepsilon_i \left(\sum_{j \neq i} \varepsilon_j x_j \right) = \sum_{j \neq i} \varepsilon_i \varepsilon_j x_j = 0.$$

These observations show that

$$\varphi : F(X) = \bigoplus_{i \in Q_0} \varepsilon_i M \longrightarrow M, \quad (\varepsilon_i x_i)_{i \in Q_0} \mapsto \sum \varepsilon_i x_i$$

is an isomorphism of k -vector spaces.

Show that φ is isomorphism of left kQ -modules:

Without loss generally, assume that $a = p$ is a path in Q (regarded as element of kQ), and let $x \in M$. Then

$$\begin{aligned} \varphi(a.x) &= \varphi(\iota_{t(p)} X_p(x_{s(p)})) = \varphi(\iota_{t(p)}(px_{s(p)})) \\ &= px_{s(p)}, \end{aligned}$$

and

$$a.\varphi(x) = a. \sum x_i = a. \sum \varepsilon_i x_i = px_{s(p)}.$$

- ii) Let $M := F(X)$ be the left kQ -module associated with X . Then

$$G(M)_i = \varepsilon_i M = \varepsilon_i \bigoplus_{j \in Q_0} X_j = X_i,$$

and

$$(G(M))_\alpha(x_{s(\alpha)}) = \alpha.x_{s(\alpha)} = X_\alpha(x_{s(\alpha)}).$$

Be careful! $X_i \subset \bigoplus X_j$ is still different from X_i as part of the representation, because one is a subspace and one is just a space. So the appropriate isomorphism in this case would be

$$G(M)_i = X_i \xrightarrow{x_i \mapsto x_i} X_i,$$

which is a morphism of representations, as for any $\alpha \in Q_1$

$$\begin{array}{ccc}
 & G(M)_{s(\alpha)} & \\
 & \parallel & \\
 & X_{s(\alpha)} & \longrightarrow X_{s(\alpha)} \\
 (G(M))_\alpha \downarrow & & \downarrow X_\alpha \\
 & X_{t(\alpha)} & \longrightarrow X_{t(\alpha)} \\
 & \parallel & \\
 & G(M)_{t(\alpha)} &
 \end{array}$$

commutes.

□

Remark 1.4.8. Let M be a left kQ -module, with Q finite and k a field.

- i) $\dim_k M = \sum_{i \in Q_0} \dim_k X_i$ where $X = G(M)$, where G is the functor from remark 1.4.6
- ii) $\dim_K kQ < \infty \iff Q$ contains no **oriented cycles** (a path p of length ≥ 1 , such that $s(p) = t(p)$)
- iii) If Q has no oriented cycle, then the following are equivalent:
 - (a) M is a finitely generated kQ -module.
 - (b) $\dim_k X_i < \infty$.

Proof. (a) \implies (b): (b) implies in particular, that M is finitely generated as a k -module. But as $k \subset kQ$, (a) follows immediatley.

(b) \implies (a): Set $A := kQ$, and let $x_1, \dots, x_n \in kQ$ generate M as a left kQ -module. Then there is a kQ -linear surjection

$$A^n \twoheadrightarrow M$$

given by $e_i \mapsto x_i$, where the $(e_i)_{1 \leq i \leq n}$ are a basis of A^n . As this is in particular k -linear, we have that

$$\dim_k M \leq \dim_k(A^n) = n \dim_k A < \infty,$$

as Q contains no cycle.

□

- iv) Under G , the notion of a „left submodule“ corresponds to **subrepresentations** of Q , i.e. a tupel of subspaces $Y_i \subset X_i$ for all $i \in Q_0$ such that $X_\alpha(Y_{s(\alpha)}) \subset Y_{t(\alpha)}$ for all $\alpha \in Q_1$.

- v) Under **G**, a direct sum of modules corresponds to **direct sum of representations**: Given X, Y two representations of Q , define a new representation $X \oplus Y$ where the vector spaces are given by

$$(X \oplus Y)_i := (X \oplus Y)_i$$

and the k -linear maps

$$(X \oplus Y)_\alpha : X_{s(\alpha)} \oplus Y_{s(\alpha)} \longrightarrow X_{t(\alpha)} \oplus Y_{t(\alpha)}$$

given by

$$\left(\begin{array}{c|c} X_\alpha & \\ \hline & Y_\alpha \end{array} \right).$$

1.5 Bimodules and tensor products

Definition 1.5.1. Let A, B be k -algebras. A A - B -bimodule M is a set M , together with maps:

$$A \times M \longrightarrow M, (a, x) \longmapsto ax$$

$$M \times B \longrightarrow M, (x, b) \longmapsto xb,$$

such that

- i) M is a left A -module
- ii) M is a right B -module
- iii) for all $a \in A, b \in B$ and $x \in M$, the relation

$$(ax)b = a(xb)$$

holds.

We denote a A - B -bimodule by

$${}_A M_B.$$

Lemma 1.5.2. Let A, B, C be k -algebras, and consider ${}_A M_B$ and ${}_A N_C$, a A - B -bimodule and a A - C -bimodule respectively. Then $\text{hom}_A(M, N)$ becomes a B - C -bimodule via

- $B \times \text{hom}_A(M, N) \longrightarrow \text{hom}_A(M, N), \quad (b, f) \longmapsto bf : M \rightarrow N, x \mapsto f(xb)$
- $\text{hom}_A(M, N) \times C \longrightarrow \text{hom}_A(M, N), \quad (f, c) \longmapsto fc : M \rightarrow N, x \mapsto f(cx)$

Proof. • well-defined:

$$bf(ax) = f((ax)b) = f(a(xb)) = af(xb) = a(bf)(x)$$

- $\text{hom}_A(M, N)$ is a left B -module: Show e.g. (L3):

$$\begin{aligned} ((bb')f)(x) &= (x(bb')) = f((xb)b') \\ &= b'(f(xb)) \\ &= b((b'f)(x)) \end{aligned}$$

- compatibility:

$$((af)b)(x) = f((ax)b) = f(a(xb)) = (a(fb))(x).$$

□

End of Lecture 3

Definition 1.5.3. Let A be a k -algebra, and M_A be a right A -module, ${}_A N$ a left A -module.

- i) Let P be a k -module. A map

$$\varphi : M \times N \rightarrow P$$

is called **A -balanced** if

- $\varphi(x + x', y) = \varphi(x, y) + \varphi(x', y)$
 - $\varphi(x, y + y') = \varphi(x, y) + \varphi(x, y')$
 - $\varphi(xa, y) = \varphi(x, ay)$
 - φ is k -linear.
- ii) A pair (T, τ) where T is a k -module and τ a A -balanced map $M \times N \rightarrow T$ is called a **tensor product** of M with N over A , if the following universal property holds: For all A -balanced maps $\varphi : M \times N \rightarrow P$, where P is any k -module, there is a unique k -linear map f , such that

$$\begin{array}{ccc} M \times N & \xrightarrow{\tau} & T \\ \varphi \downarrow & \swarrow \exists! f \text{ } k\text{-linear} & \\ P & & \end{array}$$

commutes.

Lemma 1.5.4. Let A be a k -algebra, M_A a right A -module, ${}_A N$ a left A -module.

- i) There exists a tensor product (T, τ) of M with N over A .
- ii) This tensor product is unique up to unique isomorphism. More precisely, if (T', τ') is any other tensor product, then there exists a unique isomorphism of k -modules $f : T \rightarrow T'$, such that $f \circ \tau = \tau'$.

Proof. i) *existence:* Let F be the free k -module with basis $M \times N$, and U the submodule generated by elements of the form

$$(x + x', y) - (x, y) - (x', y)$$

- $(x, y + y') - (x, y) - (x, y')$
- $(xa, y) - (x, ay)$
- $(\lambda x, y) - \lambda(x, y)$
- $(x, \lambda y) - \lambda(x, y)$

Then F/U is a k -module, and

$$\tau : M \times N \longrightarrow F \longrightarrow F/U$$

is A -balanced by definition. We set

$$a \otimes b := \tau((a, b)) \text{ and } M \otimes_A N := F/U.$$

The pair $(M \otimes_A N, \otimes)$ satisfies the universal property of a tensor product:

Let $\varphi : M \times N \rightarrow P$ be A -balanced. Then there exists a unique $\hat{\varphi}$ such that the following diagram commutes, as F is free with basis $M \times N$:

$$\begin{array}{ccc} M \times N & \hookrightarrow & F \\ \varphi \downarrow & \nearrow \exists! \hat{\varphi} & \\ P & & \end{array}.$$

Since φ is A -balanced, $\hat{\varphi}$ factors through F/U , i.e.:

$$\begin{array}{ccccc} M \times N & \hookrightarrow & F & \xrightarrow{\pi} & F/U \\ \varphi \downarrow & \nearrow \hat{\varphi} & & \nearrow \exists! \bar{\varphi} & \\ P & & & & \end{array}.$$

Now set $\tau := \bar{\varphi}$.

ii) later...

□

Lemma 1.5.5. *Let A, B, C be k -algebras, ${}_A M_B$ and ${}_B N_C$ bimodules. Then $M \otimes_B N$ is a A - C -bimodule, via*

- $a(x \otimes y) = (ax) \otimes y$
- $(x \otimes y)c = x \otimes (yc),$

for all $x \in X, y \in Y, a \in A, c \in C$.

Proof. $M \otimes_B N$ is already a k -module. For all $a \in A$ define

$$\tau_a : M \times N \longrightarrow M \times N \xrightarrow{\otimes} M \otimes_B N$$

$$(x, y) \longmapsto (ax, y) \longmapsto (ax) \otimes y$$

τ_a is A -balanced:

- $\tau_a((x + x', y)) - \tau_a((x, y)) - \tau_a((x', y)) = (a(x + x')) \otimes y - (ax) \otimes y - (ax') \otimes y = 0$
- $\tau_a((xb, y)) - \tau_a((x, by)) = (a(xb)) \otimes y - (ax) \otimes (by) = ((ax)b) \otimes y - (ax) \otimes (by) = 0$

So it factors through $M \otimes_B N$ as follows:

$$\begin{array}{ccc} M \times N & \xrightarrow{\otimes} & M \otimes_B N \\ \downarrow \tau_a & \swarrow \exists! f_a \text{ bilinear} & \\ M \otimes_B N & & \end{array}$$

The map

$$A \rightarrow \text{End}_k(M \otimes_B N), \quad a \mapsto f_a$$

is a homomorphism of k -algebras. So by proposition 1.3.8, $M \otimes_B N$ is a left A -algebra. If we consider the map

$$\tau_c : M \times N \longrightarrow M \times N \longrightarrow M \otimes_B N$$

$$(x, y) \longmapsto (x, yc) \longmapsto x \otimes (yc)$$

□

Remark 1.5.6. Let $\varphi : A \rightarrow B$ be a k -algebra homomorphism, ${}_B N$ a left B -module. Then $M := A$ is a A - B -bimodule via

$$a.x.b := ax\varphi(b) \text{ for all } a \in A, x \in M, b \in B.$$

Then by lemma 1.5.5, $A \otimes_B N$ is a left A -module, where we think of B as a right k -module. This construction is sometimes called **extension of scalars** or **induction of B by A** .

Lemma 1.5.7. Let A, B, C, D be k -algebras and consider

$${}_A M_B, {}_A (M_i)_B \ (i \in I), {}_B N_C, {}_B (N_j)_C \ (j \in J).$$

Then there are isomorphisms:

i)

$$\left(\bigoplus_{i \in I} M_i \right) \otimes_B N \longrightarrow \bigoplus_{i \in I} (M_i \otimes_B N)$$

$$(x_i) \otimes y \longmapsto (x_i \otimes y)$$

of A - C -bimodules.

ii)

$$M \otimes_B \left(\bigoplus_{j \in J} N_j \right) \longrightarrow \bigoplus_{j \in J} (M \otimes_B N_j)$$

$$x \otimes (y_j) \longmapsto (x \otimes y_j)$$

of A - C -bimodules.

iii)

$$(M \otimes_B N) \otimes_C P \longrightarrow M \otimes_B (N \otimes_C P)$$

$$(x \otimes y) \otimes z \longmapsto x \otimes (y \otimes z)$$

of A - B -bimodules.

iv)

$$A \otimes_A M \longrightarrow M$$

$$a \otimes x \longmapsto ax$$

$$M \otimes_B M \longrightarrow M$$

$$x \otimes b \longmapsto xb$$

of A - B -bimodules.

Proof. This is **supposed to be** exactly the same as [Fra18, 2.27]. □

Proposition 1.5.8. *Let A, B be k -algebras, and $M_A, {}_A N_B$ and P_b (bi)-modules. The map*

$$\text{hom}_B(M \otimes_A N, P) \rightarrow \text{hom}_A(M, \text{hom}_B(N, P)), \quad f \mapsto \Phi(f) \text{ where } \Phi(f)(x)(y) \mapsto f(x \otimes y),$$

for $x \in M$ and $y \in N$.

is a well-defined isomorphism of k -modules, natural in M, N, P .

Proof. • $\varphi(f)(x)$ is right B -module homomorphism:

$$\Phi(f)(x)(yb) = f(x \otimes yb) = f((x \otimes y)b) = f(x \otimes y)b$$

• $\varphi(f)$ is right A -module homomorphism:

$$\Phi(f)(xa)(y) = f(xa \otimes y) = f(a \otimes ay)$$

$$(\Phi(f)(x)a)(y) = \Phi(f)(x)(ay) = f(a \otimes ay)$$

- Φ is k -linear.

So Φ is a well-defined map of k -modules.

Φ has inverse: Let $g \in \text{hom}_A(M, \text{hom}_B(N, P))$. Define

$$\psi(g) : M \times N \rightarrow P, (x, y) \mapsto g(x)(y).$$

Then $\psi(g)$ is A -balanced, so it factors through the tensor product:

$$\begin{array}{ccc} M \times N & \xrightarrow{\otimes} & M \otimes_A N \\ \psi(g) \downarrow & \swarrow \exists! \hat{\psi}(g) \text{ } k\text{-linear} & \\ P & & \end{array}$$

The map $\hat{\psi}(g)$ is also as right B -module homomorphism, $g(x) \in \text{hom}_B(N, P)$.

But the map

$$\hat{\psi} : \text{hom}_A(M, \text{hom}_B(N, P)) \rightarrow \text{hom}_B(M \otimes N, P)$$

is the inverse of φ , as

- $\hat{\psi}(\Phi(f))(x \otimes y) = \Phi(f)(x)(y) = f(x \otimes y)$
- $\Phi(\hat{\psi}(g))(x)(y) = \hat{\psi}(g)(x \otimes y) = g(x)(y)$

for all $x \in M$ and $y \in N$. □

2. CATEGORIES AND FUNCTORS

Definition 2.0.1. A category \mathcal{C} consists of:

- A class $\text{Ob}(\mathcal{C})$, whose elements are called the **objects** of \mathcal{C} .
- For all $X, Y \in \text{Ob}(\mathcal{C})$, a set $\mathcal{C}(X, Y)$. An element of $\mathcal{C}(X, Y)$ is called a **morphism** from X to Y as is denoted by

$$f : X \rightarrow Y.$$

- For all $X, Y, Z \in \text{Ob}(\mathcal{C})$, a map

$$\varphi(x, y) \times \varphi(y, z) \rightarrow \varphi(x, z), (f, g) \mapsto g \circ f.$$

These should satisfy:

- (L1): Associativity: For all $X, Y, Z, W \in \text{Ob}(\mathcal{C})$, and morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W,$$

we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (L2): Identity: For all $X \in \text{Ob}(\mathcal{C})$, there is a morphism $\text{id}_X \in \mathcal{C}(X, X)$, such that for any object $Y \in \text{Ob}(\mathcal{C})$:

$$f \circ \text{id}_X = f \text{ and } \text{id}_Y \circ g = g \text{ for all } f : X \rightarrow Y, g : Y \rightarrow X$$

holds.

Remark 2.0.2. i) $\varphi(X, Y) = \emptyset$ can happen if $X \neq Y$

ii) id_X is unique, as $\text{id}_X = \text{id}_X \circ \text{id}'_X = \text{id}'_X$

Remark 2.0.3. We sometimes want to consider categories whose objects are all sets (with additional conditions). But this can cause logical problems. As a solution, we introduce so called universes. We will always fix a universe, such that sets are elements of this universe, and classes are subsets of this universe. Consider [Lan98, 1.6] for further reference.

End of Lecture 4

Example 2.0.4. i) The category *Set* of all sets, with

- $\text{Ob}(\text{Set})$ are all sets in the given universe
- $\text{Set}(X, Y) = \{\text{maps } f : X \rightarrow Y\}$

- ii) The category $\mathcal{G}rp$ of groups, with group homomorphism as morphisms.
- iii) Let A be a k -algebra. Let $A\text{-Mod}$ be the category of left A -modules, and $\text{Mod-}A$ the category of right A -modules.
- iv) The category $\mathcal{T}op$ of topological spaces, with
 - $\text{Ob}(\mathcal{T}op)$ the set of all topological spaces,
 - $\mathcal{T}op(X, Y) = \{f : X \rightarrow Y \mid f \text{ is continuous}\}$
- v) Let G be a group. Let \mathcal{C} be the category defined as
 - $\text{Ob}(\mathcal{C}) = \{*\}$
 - $\mathcal{C}(*, *) = G$, with composition defined as $h \circ g := hg$.
- vi) Let Q be a quiver. Let \mathcal{Q}_* be the **category of paths** of Q , defined as
 - $\text{Ob}(\mathcal{Q}_*) = Q_*$
 - for $i, j \in Q_*$, let $\mathcal{Q}_*(i, j) := \{\text{paths } p \text{ in } Q \mid s(p) = i, t(p) = j\}$,
 - composition is given by concatenation of paths.

This is a category, as composition is associative, and $\text{id}_i = \varepsilon_i$ (the lazy path at i).

Definition 2.0.5. Let \mathcal{C} be a category. The **opposite category** \mathcal{C}^{op} is defined as

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$
- for all $x, y \in \text{Ob}(\mathcal{C}^{\text{op}})$, the morphisms are defined as $\mathcal{C}^{\text{op}}(X, Y) := \mathcal{C}(Y, X)$
- for $f \in \mathcal{C}^{\text{op}}(X, Y), g \in \mathcal{C}^{\text{op}}(Y, Z)$, set

$$g \circ_{\mathcal{C}^{\text{op}}} f := f \circ_{\mathcal{C}} g$$

2.1 Functors

Definition 2.1.1. Let \mathcal{C}, \mathcal{D} be two categories. A **functor** $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ consists of the following:

- a map

$$\mathcal{F} : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D}), \quad X \mapsto \mathcal{F}(X)$$

- for all $X, Y \in \text{Ob}(\mathcal{C})$, a map

$$\mathcal{C}(X, Y) \rightarrow \mathcal{C}(\mathcal{F}(X), \mathcal{F}(Y)), \quad f \mapsto \mathcal{F}(f),$$

such that

- (F1): $\mathcal{F}(\text{id}_X) = \text{id}_{\mathcal{F}(X)}$
- (F2): for all sequences

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in \mathcal{C} , the relation

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f),$$

holds for all $X, Y, Z \in \text{Ob}(\mathcal{C})$.

Remark 2.1.2. What we call a functor is sometimes called a *covariant functor*. A *contravariant functor* is a (covariant) functor $\mathcal{G} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$.

Remark 2.1.3. i) Let \mathcal{C} be a category. The **identical functor** is given by

$$\text{Id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}, \begin{cases} x \mapsto x & \text{on objects} \\ f \mapsto f & \text{on morphism} \end{cases}$$

ii) If

$$\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{D} \xrightarrow{\mathcal{G}} \mathcal{E}$$

are two functors, then their composition is also a functor.

Example 2.1.4. i) Consider the category Set , and let $P(X)$ denote the power set of a set X . Define

$$\mathcal{P}_* : \text{Set} \rightarrow \text{Set}, \begin{cases} x \mapsto P(X) \\ (X \xrightarrow{f} Y) \mapsto P_*(f) : P(X) \rightarrow P(Y), A \mapsto f(A) \end{cases}$$

which is a covariant functor, and

$$\mathcal{P}^* : \text{Set} \rightarrow \text{Set}, \begin{cases} x \mapsto P(X) \\ (X \xrightarrow{f} Y) \mapsto P^*(f) : P(Y) \rightarrow P(X), B \mapsto f^{-1}(B) \end{cases},$$

which is a contravariant functor.

ii) Consider the functors

$$-^* : k\text{-Alg} \rightarrow \text{Grp}, \begin{cases} A \mapsto A^\times \\ (A \xrightarrow{f} B) \mapsto A^* \xrightarrow{f^\times} B^\times \end{cases}$$

and

$$k[-] : \text{Grp} \rightarrow k\text{-Alg}, \begin{cases} G \mapsto k[G] \\ (G \xrightarrow{\varphi} H) \mapsto k[G] \xrightarrow{\varphi} k[H] \end{cases}$$

iii) The functor

$$\text{Grp} \rightarrow \text{Set}, \begin{cases} G \mapsto G \\ f \mapsto f \end{cases}$$

is called a **forgetful functor**. Other examples of forgetful functors are

- $\mathcal{T}op \rightarrow \mathcal{S}et$
- $A\text{-Mod} \rightarrow k\text{-Mod}$

Example 2.1.5. Let \mathcal{C} be a category and $X \in \text{Ob}(\mathcal{C})$ an object in \mathcal{C} . Consider

i)

$$H^X : \mathcal{C} \rightarrow \mathcal{S}et, \begin{cases} Y \mapsto \mathcal{C}(X, Y) \\ (Y \xrightarrow{f} Y') \mapsto H^X(f) : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Y'), g \mapsto f \circ g \end{cases}.$$

We also denote this as $H^X =: \mathcal{C}(X, -)$. **This is a covariant functor.**

ii)

$$H_X : \mathcal{C}^{\text{op}} \rightarrow \mathcal{S}et, \begin{cases} Z \mapsto \mathcal{C}(Z, X) \\ (Z \xrightarrow{f} Z') \mapsto H_X(f) : \mathcal{C}(Z', X) \rightarrow \mathcal{C}(Z, X), g \mapsto g \circ f \end{cases}.$$

We also denote this as $H_X =: \mathcal{C}(-, X)$. **This is a contravariant functor.**

Definition 2.1.6. Let $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ and consider the induced map

$$\mathcal{C}(X, Y) \rightarrow \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y)).$$

- i) If this map is injective, then F is called **faithful**.
- ii) If this map is surjective, then F is called **full**.
- iii) F is **fully faithful**, if F is full and faithful.
- iv) F is **dense** or **essentially surjective**, if for any $Y \in \mathcal{D}$, there is an object $X \in \mathcal{C}$, such that $\mathcal{F}(X) \cong Y$

2.2 Isomorphism

Definition 2.2.1. Let \mathcal{C} be a category. A morphism $f : X \rightarrow Y$ in \mathcal{C} is called an **isomorphism**, if there is a $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

Remark 2.2.2. i) Identities are isomorphism.

- ii) The morphism g (**if it exists**) is uniquely determined by f . We therefore call $g =: f^{-1}$.
- iii) If $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is a functor and f an isomorphism in \mathcal{C} , then $\mathcal{F}(f)$ is an isomorphism in \mathcal{D} .

Example 2.2.3. i) In $\mathcal{S}et$, $\mathcal{G}rp$, $A\text{-Mod}$, the following are equivalent:

- f is an isomorphism
- f is bijective.

ii) In $\mathcal{T}op$, not all bijective maps are isomorphism.

iii) In \mathcal{Q}_* , the only isomorphisms are the lazy paths, **because lengths of paths are additive.**

2.3 Natural transformations

Definition 2.3.1. Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors. A **natural transformation**

$$\eta : \mathcal{F} \rightarrow \mathcal{G}$$

is a family of morphisms

$$\{\eta_X\}_{X \in \text{Ob}(\mathcal{C})} : \mathcal{F}X \rightarrow \mathcal{G}X$$

in \mathcal{D} , such that for all $X, Y \in \mathcal{C}$ and morphisms $f : X \rightarrow Y$ in \mathcal{C} , the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{\eta_X} & \mathcal{G}X \\ \mathcal{F}f \downarrow & & \downarrow \mathcal{G}f \\ \mathcal{F}Y & \xrightarrow{\eta_Y} & \mathcal{G}Y. \end{array}$$

Remark 2.3.2. i) For two natural transformations

$$\mathcal{F} \xrightarrow{\eta} \mathcal{G} \xrightarrow{\xi} \mathcal{H}, (\mathcal{F}, \mathcal{G}, \mathcal{H} : \mathcal{C} \rightarrow \mathcal{D}),$$

we can define the composition

$$\xi \circ \eta : \mathcal{F} \rightarrow \mathcal{H}$$

by

$$(\xi \circ \eta)_X \mathcal{F}X \rightarrow \mathcal{H}X, (\xi \circ \eta)_X := \xi_X \circ \eta_X.$$

ii) For $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$, we have **identical transformation**

$$\text{id}_{\mathcal{F}} \text{ given by } (\text{id}_{\mathcal{F}})_X := \text{id}_{\mathcal{F}X}$$

the part about the natural transformations on the exe-sheets is still missing.

End of Lecture 5

Definition 2.3.3. Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors. A natural transformation $\eta : \mathcal{F} \rightarrow \mathcal{G}$ is called a **natural isomorphism** if for all $x \in \mathcal{C}$, η_x is an isomorphism in \mathcal{D} .

η is a natural transformation if and only iff there is a natural transformation $\zeta : \mathcal{G} \rightarrow \mathcal{F}$, such that $\zeta \circ \eta = \text{id}_{\mathcal{F}}$ and $\eta \circ \zeta = \text{id}_{\mathcal{G}}$.

If η is a natural isomorphism, we write $\eta : \mathcal{F} \xrightarrow{\cong} \mathcal{G}$. If there is a natural transformation between two functors, we denote this by $\mathcal{F} \cong \mathcal{G}$.

Definition 2.3.4. A functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is called an **equivalence of categories**, if there is a functor $\mathcal{G} : \mathcal{D} \rightarrow \mathcal{C}$, such that

$$\mathcal{G} \circ \mathcal{F} \cong \text{id}_{\mathcal{C}} \text{ and } \mathcal{F} \circ \mathcal{G} \cong \text{id}_{\mathcal{D}}.$$

If for two categories \mathcal{C}, \mathcal{D} an equivalence of categories $\mathcal{C} \rightarrow \mathcal{D}$ exists, we say that \mathcal{C} and \mathcal{D} are **equivalent**, and write $\mathcal{C} \simeq \mathcal{D}$.

Example 2.3.5. Some examples for equivalences of categories:

i) Let Q be a finite quiver. Then theorem 1.4.7 shows that

$$\mathcal{R}ep_k(Q) \simeq kQ - \mathcal{M}od.$$

Proof. For the functors

$$\mathcal{F} : \mathcal{R}ep_k(Q) \rightarrow kQ - \mathcal{M}od \text{ and } \mathcal{G} : kQ - \mathcal{M}od \rightarrow \mathcal{R}ep_k(Q)$$

constructed in remark 1.4.6 the relationships $\mathcal{G}\mathcal{F}(X) \cong X$ and $\mathcal{F}\mathcal{G}(M) \cong M$ hold for any representation X and kQ -modules M , by theorem 1.4.7. So it suffice to check naturallity:

Let $M, N \in kQ - \mathcal{M}od$ be two kQ -modules. Recall that as k -vector space,

$$\mathcal{F}\mathcal{G}(M) = \bigoplus_{i \in Q_0} \varepsilon_i M$$

where ε_i denotes the lazy path at $i \in Q_0$, and the isomorphism is given by

$$\varphi : \mathcal{F}\mathcal{G}(M) = \bigoplus_{i \in Q_0} \varepsilon_i M \longrightarrow M, \quad (\varepsilon_i x_i)_{i \in Q_0} \mapsto \sum \varepsilon_i x_i.$$

Now let $\alpha \in kQ - \mathcal{M}od(M, N)$ be a homomorphism of left kQ -modules M, N . We need to show that the diagram

$$\begin{array}{ccc} \mathcal{F}\mathcal{G}(M) & & \\ \parallel & & \\ \bigoplus_{i \in Q_0} \varepsilon_i M & \xrightarrow{\varphi} & M \\ \downarrow \mathcal{F}\mathcal{G}(\alpha) & & \downarrow \alpha \\ \bigoplus_{i \in Q_0} \varepsilon_i N & \xrightarrow{\varphi} & N \\ \parallel & & \\ \mathcal{F}\mathcal{G}(N) & & \end{array}$$

commutes, i.e. that φ is actually a natural isomorphism:

$$\begin{aligned} \alpha(\varphi(\varepsilon_i x_i)) &= \alpha\left(\sum_{i \in Q_0} \varepsilon_i x_i\right) \\ &= \sum_{i \in Q_0} \varepsilon_i \alpha(x_i), \end{aligned}$$

as α is kQ -linear. Furthermore:

$$\begin{aligned} \varphi(\mathcal{F}\mathcal{G}(M)) &= \varphi((\varepsilon_i \alpha(x_i))) \\ &= \sum_{i \in Q_0} \varepsilon_i \alpha(x_i) \end{aligned}$$

So φ is indeed a natural transformation. The other map is also a natural transformation. This follows from the fact that it is an isomorphism of representations, as was shown in the proof of theorem 1.4.7. \square

- ii) Let G be a group. A **representation of G** is pair (V, ρ) consisting of a k -vector space V and a group homomorphism

$$\rho : G \rightarrow \mathrm{GL}(V)$$

A morphism of (group)representations $f : (V, \rho) \rightarrow (W, \sigma)$ is a k -linear map $f : V \rightarrow W$, such that for all $g \in G$, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \rho(g) \downarrow & & \downarrow \sigma(g) \\ V & \xrightarrow{f} & W \end{array}$$

We will show maybe on the next sheet, who knows...that we can define an equivalence of categories

$$\mathcal{R}ep_k(G) \simeq k[G] - \mathrm{Mod}.$$

- iii) The category of $k\text{-Alg}$ is equivalent to the category \mathcal{C} of pairs (A, φ) , where A is a ring and $\varphi : k \rightarrow Z(A)$ is a ring homomorphism and morphisms correspond to ring homomorphism $f : A \rightarrow B$, such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow & & \uparrow \\ Z(A) & & Z(B) \\ & \swarrow \varphi_A \quad \searrow \varphi_B & \\ & k & \end{array}$$

show that this is actually all natural and well-defined

- iv) Let A be a k -algebra. The category $A\text{-Mod}$ of left A -modules is equivalent the category \mathcal{D} of pairs (V, ρ) of k -vector spaces V and homomorphisms of k -algebras:

$$\varphi : A \rightarrow \mathrm{End}_k(V).$$

Morphisms $(V, \varphi) \rightarrow (W, \psi)$ in \mathcal{D} are given by k -linear maps $f : V \rightarrow W$, such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \varphi(a) \downarrow & & \downarrow \psi \\ V & \xrightarrow{f} & W \end{array}$$

commutes for all $a \in A$.

2.4 Functor Categories

Definition 2.4.1. Let \mathcal{C} and \mathcal{D} be categories. Define the **functor category** $\mathcal{F}un(\mathcal{C}, \mathcal{D})$ by

- objects are all functors $\mathcal{C} \rightarrow \mathcal{D}$; i.e. $\text{Ob}(\mathcal{F}un(\mathcal{C}, \mathcal{D})) := \{\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D} \mid \mathcal{F} \text{ is a functor}\}$
- morphism between functors are natural transformations; i.e. for $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightrightarrows \mathcal{D}$, set $\mathcal{F}un(\mathcal{F}, \mathcal{G})(\mathcal{C}, \mathcal{D}) := \{\eta : \mathcal{F} \rightarrow \mathcal{G} \mid \eta \text{ is a natural transformation}\}$
- Composition of morphisms is given by composition of natural transformations.

Remark 2.4.2. We are running into set-theoretic issues again. If \mathcal{C} and \mathcal{D} are categories in a fixed universe U (i.e. $\text{Ob}(\mathcal{C}), \text{Ob}(\mathcal{D}) \subseteq U$) then $\text{Ob}(\mathcal{F}un(\mathcal{C}, \mathcal{D}))$ might not be a subset of U any longer. As a solution, we choose another universe V , s.t. $U \in V$. Then $\text{Ob}(\mathcal{F}un(\mathcal{C}, \mathcal{D})) \subseteq V$ and $\mathcal{F}un(\mathcal{C}, \mathcal{D})$ is a category in V .

Example 2.4.3. Let Q be a quiver, k a field. Consider the functor category $\mathcal{F}un(\mathcal{Q}_*, k\text{-Mod})$. For $\mathcal{V} \in \text{Ob}(\mathcal{F}un(\mathcal{Q}_*, k\text{-Mod}))$, \mathcal{V} is a functor

$$\mathcal{V} : \mathcal{Q}_* \rightarrow k\text{-Mod}, \begin{cases} \text{Ob}(\mathcal{Q}_*) = \mathcal{Q}_* \ni i \mapsto V(i) & \text{a vector space} \\ \mathcal{Q}_*(i, j) \ni p \mapsto V(p) : V(i) \rightarrow V(j) & \text{a } k\text{-linear map} \end{cases}$$

We now have a forgetful functor

$$\mathcal{F} : \mathcal{F}un(\mathcal{Q}_*, k\text{-Mod}) \rightarrow \text{Rep}_k(Q),$$

forgetting all paths of length > 1 .

Conversely, let X be a representation of Q over k . Define a functor

$$\mathcal{G}X : \mathcal{Q}_* \rightarrow k\text{-Mod}, \begin{cases} i \mapsto X_i \\ p = \alpha_\ell \circ \dots \circ \alpha_1 \mapsto X_{\alpha_1} \end{cases}.$$

This yields a functor

$$\mathcal{G} : \text{Rep}_k(Q) \rightarrow \mathcal{F}un(\mathcal{Q}_*, k\text{-Mod})$$

We see that

$$\mathcal{G} \circ \mathcal{F} \cong \text{id}_{\mathcal{F}un(\mathcal{Q}_*, k\text{-Mod})} \quad \text{and} \quad \mathcal{F} \circ \mathcal{G} \cong \text{id}_{\text{Rep}_k(Q)}$$

.

Definition 2.4.4. Let \mathcal{C}, \mathcal{D} be two categories, and $X \in \mathcal{C}$ an object in \mathcal{C} . Define

$$\text{ev}_X : \mathcal{F}un(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}$$

by

- $\text{ev}_X(\mathcal{F}) := \mathcal{F}(X)$

- $\text{ev}_X(\mathcal{F}) \xrightarrow{\eta} \mathcal{G} := \eta_X : \mathcal{F}(X) \rightarrow \mathcal{G}(X)$.

ev_X is called the **evaluation at X** .

Remark 2.4.5. Let \mathcal{C}, \mathcal{D} be categories, and $X \in \mathcal{C}$ an object. Then ev_X is indeed a functor, as the associativity of composition of natural transformations is inherited from the associativity of composition in \mathcal{D} .

Moreover, for $f : X \rightarrow Y$ a morphism in \mathcal{C} , we can define a natural transformation between the functors

$$\text{ev}_f : \text{ev}_X \rightarrow \text{ev}_Y,$$

which is, for a functor \mathcal{F} , just a map $\text{ev}_X(\mathcal{F}) \rightarrow \text{ev}_Y(\mathcal{F})$; by considering

$$\begin{array}{ccc} \text{ev}_X(\mathcal{F}) & \xrightarrow{(\text{ev}_f)_F} & \text{ev}_Y(F) \\ \parallel & & \parallel \\ \mathcal{F}X & \xrightarrow{Ff} & \mathcal{F}Y \end{array} \quad \text{and setting } (\text{ev}_f)_F := \mathcal{F}f,$$

i.e. $(\text{ev}_f)_{\mathcal{F}}$ is induced the maps which are induced by \mathcal{F} . To show that the (ev_f) define indeed a natural transformation of functors

$$\begin{array}{ccc} & \xrightarrow{\text{ev}_X} & \mathcal{D} \\ \text{Fun}(\mathcal{C}, \mathcal{D}) & \Downarrow & \\ & \xleftarrow{\text{ev}_Y} & \end{array}$$

we need to show that for all maps (i.e. natural transformations) $\eta : \mathcal{F} \rightarrow \mathcal{G}$ the following diagram commutes:

$$\begin{array}{ccc} \text{ev}_X(\mathcal{F}) & \xrightarrow{\text{ev}_X(\eta)} & \text{ev}_X(\mathcal{G}) \\ (\text{ev}_f)_{\mathcal{F}} \downarrow & & \downarrow (\text{ev}_f)_{\mathcal{G}} \\ \text{ev}_Y(\mathcal{F}) & \xrightarrow{\text{ev}_Y \eta} & \text{ev}_Y(\mathcal{G}) \end{array}$$

But this is just inherited, in following way:

Consider the extended diagram:

$$\begin{array}{ccccc} & \mathcal{F}(X) & \xrightarrow{\eta_X} & \mathcal{G}(X) & \\ & \parallel & & \parallel & \\ \mathcal{F}(f) \swarrow & \text{ev}_X(\mathcal{F}) & \xrightarrow{\text{ev}_X(\eta)} & \text{ev}_X(\mathcal{G}) & \searrow \mathcal{G}(f) \\ & \downarrow (\text{ev}_f)_{\mathcal{F}} & & \downarrow (\text{ev}_f)_{\mathcal{G}} & \\ & \text{ev}_Y(\mathcal{F}) & \xrightarrow{\text{ev}_Y \eta} & \text{ev}_Y(\mathcal{G}) & \\ & \parallel & & \parallel & \\ & \mathcal{F}(Y) & \xrightarrow{\eta_Y} & \mathcal{G}(Y) & \end{array}$$

As η is a natural transformation, the outer diagram commutes. But this already implies that the inner one does as well.

This enables us to define another functor:

$$\text{ev} : \mathcal{C} \rightarrow \text{Fun}(\text{Fun}(\mathcal{C}, \mathcal{D}), \mathcal{D}), \quad \begin{cases} X \mapsto \text{ev}_X \\ f \mapsto \text{ev}_f \end{cases}$$

2.5 Representable functors

We now consider functors of the form

$$\mathcal{C} \rightarrow \text{Set} \text{ and } \mathcal{C}^{\text{op}} \rightarrow \text{Set},$$

for an arbitrary category \mathcal{C} .

Lemma 2.5.1 (Yoneda). *Let $X \in \text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C}^{\text{op}})$ be an object of \mathcal{C} .*

i) *Let $\mathcal{F} : \mathcal{C} \rightarrow \text{Set}$ be a (covariant) functor. The map*

$$\begin{aligned} Y^{\mathcal{F}, X} : \quad & (\text{Fun}(\mathcal{C}, \text{Set})) (h^X, \mathcal{F}) \longrightarrow \mathcal{F}(X) \\ & (\eta : h^X \rightarrow \mathcal{F}) \longmapsto \eta_X(\text{id}_X) \end{aligned},$$

is a bijection, where

$$\begin{array}{ccc} \mathcal{C} & \begin{array}{c} \xrightarrow{h^X} \\ \Downarrow \\ \xrightarrow{\mathcal{F}} \end{array} & \text{Set} \end{array},$$

is a natural transformation, and

$$\eta_X : \mathcal{C}(X, X) = h^X(X) \rightarrow \mathcal{F}(X)$$

is just a map.

ii) *Let $\mathcal{G} : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ be a (contravariant) functor. The map:*

$$\begin{aligned} Y_{\mathcal{G}, X} : \quad & (\text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})) (h_X, \mathcal{G}) \longrightarrow \mathcal{G}(X) \\ & (\zeta : h_X \rightarrow \mathcal{G}) \longmapsto \zeta_X(\text{id}_X) \end{aligned},$$

is a bijection.

Proof. i) Assume that ii) holds, then this follows, as $\mathcal{C} = (\mathcal{C}^{\text{op}})^{\text{op}}$ and

$$h_{\mathcal{C}}^X = \mathcal{C}(-, X) = \mathcal{C}^{\text{op}}(X, -) = h_X^{\mathcal{C}^{\text{op}}}.$$

ii) $Y_{\mathcal{G}, X}$ is injective: Let $\xi, \eta \in (\text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})) (h_X, \mathcal{G})$ be two natural transformations

$$\xi, \eta : \begin{array}{ccc} \mathcal{C}^{\text{op}} & \begin{array}{c} \xrightarrow{h_X} \\ \Downarrow \\ \xrightarrow{\mathcal{G}} \end{array} & \text{Set} \end{array}$$

and suppose that

$$\xi_X(\text{id}_X) = \eta_X(\text{id}_X).$$

We need to show that this implies $\xi = \eta$, i.e. $\xi_Y = \eta_Y$ for all $Y \in \text{Ob}(\mathcal{C})$. As these are maps of sets, it suffices to show

$$\xi_Y(f) = \eta_Y(f) \text{ for all } f \in h_X(Y) = \mathcal{C}^{\text{op}}(X, Y).$$

As ξ, η are natural transformations, the diagrams

$$\begin{array}{ccc} h_X(X) & & \\ \parallel & & \\ \mathcal{C}(X, X)^{\text{op}} & \xrightarrow[\xi_X]{\eta_X} & \mathcal{G}(X) \\ h_X(f) \downarrow & & \downarrow \mathcal{G}(f) \\ \mathcal{C}(X, Y)^{\text{op}} & \xrightarrow[\xi_Y]{\eta_Y} & \mathcal{G}(Y) \\ \parallel & & \\ h_X(Y) & & \end{array} \quad (\text{D1})$$

and commute. This implies

$$\begin{aligned} \mathcal{G}(f)(\eta_X(\text{id}_X)) &\stackrel{(\text{D1})}{=} \eta_Y(h_X(f)(\text{id}_X)) = \eta_Y(f) \\ \parallel \\ \mathcal{G}(f)(\xi_X(\text{id}_X)) &\stackrel{(\text{D1})}{=} \xi_Y(h_X(f)(\text{id}_X)) = \xi_Y(f), \end{aligned}$$

so $Y_{\mathcal{G}, X}$ is injective.

$Y_{\mathcal{G}, X}$ is surjective: Let $z \in \mathcal{G}(X)$ be arbitrary. We need to find a natural transformation

$$\zeta : \mathcal{C}^{\text{op}} \begin{array}{c} \xrightarrow{h_X} \\ \Downarrow \mathcal{G} \\ \xrightarrow{\quad} \end{array} \text{Set},$$

such that $\zeta_X(\text{id}_X) = z$. Define for $y \in \text{Ob}(\mathcal{C}^{\text{op}})$ a map

$$\zeta_Y : \mathcal{C}(X, Y) = h_X(Y) \rightarrow \mathcal{G}(Y), \quad f \mapsto \mathcal{G}f(z).$$

Show that ζ is indeed a natural transformation: Let $g : Y \rightarrow Y'$ be a morphism in \mathcal{C} , i.e. $g \in \mathcal{C}^{\text{op}}(Y', Y)$. We have to show that

$$\begin{array}{ccc} h_X(Y') & & \\ \parallel & & \\ \mathcal{C}(Y', X) & \xrightarrow{\zeta_{Y'}} & \mathcal{G}(Y') \\ h_X(f) \downarrow & & \downarrow \mathcal{G}(g) \\ \mathcal{C}(Y, X) & \xrightarrow{\zeta_Y} & \mathcal{G}(Y) \\ \parallel & & \\ h_X(Y) & & \end{array}$$

commutes.

Let $u \in \mathcal{C}(Y', X)$. Then

$$\begin{aligned}\mathcal{G}(g)(\zeta_{Y'}(u)) &= \mathcal{G}(g)(\mathcal{G}(u)(z)) \\ &= \mathcal{G}(u \circ g)(z)\end{aligned}$$

and

$$\begin{aligned}\zeta_Y((h_x(g))(u)) &= \zeta_Y(u \circ g) \\ &= \mathcal{G}(u \circ g)(z)\end{aligned}$$

Hence ζ defines a natural transformation, and

$$\zeta_X(\text{id}_X) = (\mathcal{G}(\text{id}_x))(z) = (\text{id}_(\mathcal{G}X))(z) = z$$

□

End of Lecture 6

A.0 Sheet 0

Definition A.0.1. Let X be any set, and k any commutative ring with unit. Define the **free algebra** generated by X :

- i) As k -module, set $k\langle X \rangle$ as the free k -module generated by X .
- ii) Define the multiplication of two words as the concatenation.

Then $k\langle X \rangle$ satisfies the following universal property: Let B be any k -algebra, and $f : X \rightarrow B$ a homomorphism of sets. There there is a unique homomorphism of k -algebras $k\langle X \rangle$, such that the following diagram commutes:

$$\begin{array}{ccc} k\langle X \rangle & \xrightarrow{\exists! f} & B \\ \uparrow & \nearrow f & \\ X & & \end{array} .$$

Proposition A.0.2. Consider the forgetful functor

$$\mathcal{F} : k\text{-}\mathcal{A} \rightarrow \mathcal{S}, \quad B \mapsto B$$

and

$$\mathcal{K}\langle - \rangle : \mathcal{S} \rightarrow k\text{-}\mathcal{A}, \quad X \mapsto k\langle X \rangle.$$

Then for all sets X and k -algebras,

$$\text{hom}_{\mathcal{S}}(X, F(B)) \cong \text{hom}_{k\text{-}\mathcal{A}}(k\langle X \rangle, B)$$

holds.

We say that F is **right-adjoint** to $\mathcal{K}\langle - \rangle$.

Problem A.0.1. Consider the k -algebra

$$A := k\langle x, y \rangle / (\langle xy - yx - 1 \rangle)$$

over a field k with $\text{char } k = 0$. Show that there are no non-zero, finite-dimensional representations of A .

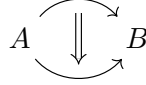
Definition A.0.3. Let \mathcal{C} and \mathcal{D} be two categories. A covariant functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is **faithful**, if for all objects A, B of \mathcal{C} , the induced function of sets

$$\text{hom}_{\mathcal{C}}(A, B) \rightarrow \text{hom}_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(B))$$

is injective; it is **full**, if this function is surjective for all objects A, B of \mathcal{C} .

Definition A.0.4. A covariant functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is an **equivalence of categories**, if it is fully faithful (i.e. bijective on hom-sets) and **essentially surjective**, i.e, for every object Y of \mathcal{D} , there is an object X of \mathcal{C} such that $\mathcal{F}(X) \cong Y$.

Some people write natural transformations in the following way:



A.1 Sheet 1

Solution A.1.1. i) φ_m is k -linear:

- $\varphi(a+b)x = (a+b)x \stackrel{(L1)}{=} ax + bx = \varphi(a)x + \varphi(b)x$
- $\varphi(\lambda a) = (\lambda a)x \stackrel{(L5)}{=} \lambda(ax) = \lambda\varphi(a)x$.

φ_m is ring homomorphism:

- $\varphi(ab) = (ab)x \stackrel{(L3)}{=} a(bx) = \varphi(a)\varphi(b)x$
- $\varphi(1_A) \stackrel{(L4)}{=} (1_A)x = x$.

As these relations hold for all x , the assertion follows.

ii) V_φ is already a k -module.

- (L1) $a(x+y) = (\varphi(a))(x+y) \stackrel{\varphi \in \text{End}_k(V)}{=} (\varphi(a))(x) + (\varphi(a))(y) = ax + ay$
- (L2) $(a+b)x = (\varphi(a+b))x \stackrel{\varphi \text{ homo of } k\text{-algebras}}{=} (\varphi(a) + \varphi(b))x = \varphi(a)x + \varphi(b)x = ax + bx$
- (L3) $(ab)x = (\varphi(ab))x \stackrel{\varphi \text{ homo of } k\text{-algebras}}{=} (\varphi(a)\varphi(b))x = a(bx)$
- (L4) $1_ax = (\varphi(1_a))x \stackrel{\varphi \text{ homo of } k\text{-algebras}}{=} \text{id } x = x$
- (L5) $(\lambda a)(x) = ((\varphi(\lambda a))x \stackrel{\varphi \text{ homo of } k\text{-algebras}}{=} (\lambda\varphi(a))x = \lambda(ax)$ and $a(\lambda x) = (\varphi(a)(\lambda x)) \stackrel{\varphi(a) \text{ endo of } k\text{-module}}{=} \lambda(\varphi(a)x) = \lambda(ax)$,

for all $a, b \in A, x, y \in V$ and $\lambda \in k$.

iii) We regard V and W as A -modules in the sense of part ii). Assume that $\psi(a) \circ f = f \circ \varphi(a) (*)$ for all $a \in A$. Then

$$f(ax) = f((\varphi(a))x) \stackrel{*}{=} (\psi(a))(f(x)) = af(x),$$

for all $x \in X$. Hence f is an A -module homomorphism.

Assume that f is a \mathfrak{a} -module homomorphism, then

$$(\psi(a))(f(x)) = af(x) = f(ax) = f((\varphi(a))x)$$

for all $x \in V$. Hence $\psi(a) \circ f - f \circ \varphi(a) = 0$ and so $(*)$ holds.

Solution A.1.2. i) Assume that I is a non-zero ideal of A . Let $a = (a)_{ij} \neq 0$ be an arbitrary matrix in I . Then there exist permutation matrices $\sigma, \pi \in \text{GL}_n(K)$, such that $(\sigma a \pi)_{11} \neq 0$, which is in I , as I is a two-sided ideal. So without loss of generality, suppose $a_{11} \neq 0$.

Define

$$b \in M_n(k), (b)_{ij} := \begin{cases} 1, & \text{if } i = j = 1 \\ 0, & \text{else} \end{cases}$$

and E_n as the identity of $M_n(k)$. Then we get

$$\left(\frac{1}{a_{11}} E_n \right) \cdot b \cdot a \cdot b = b.$$

By repeatedly using permutation matrixes, it is possible to write any matrix as sum of products of a , b and permutation matrices on the left- and right. As I is a two-sided ideal, all of these combinations are in I as well. Hence a generates all of A , and $I = A$.

ii) Consider A as a k -vector space, then $\dim_K A = n^2$. Let M be any left A -module. As shown in task 3, there is a homomorphism of k -algebras

$$\varphi : A \rightarrow \text{End}_k(M), a \mapsto a : (x \mapsto ax),$$

which is in particular a homomorphism of k -vector spaces. The kernel of φ is a two-sided ideal of A , as

$$a0x = 0ax = 0$$

for all $a \in A$ and $x \in M$.

Now i) implies that $\ker \varphi$ is either zero or $\ker \varphi = A$. But since $\varphi(E_n) = \text{id}_M$, the latter one is not possible. Hence φ is injective, and in particular $\dim A \leq \dim \text{End}_k(V)$, so $n \leq m$.

Proposition A.1.1. Let k be a field, $k[X]$ the polynomial ring and $p \in k[X]$ a polynomial with $\deg p = n$. Then

$$k[X]/(p)$$

is a n -dimensional k vector space, and a basis is given by

$$\{1, x, \dots, x^{n-1}\}.$$

The following propositions are taken from [Alu09].
Let R be any commutative ring.

Proposition A.1.2. Let I_1, \dots, I_k be ideals of R such that $I_i + I_j = (1)$ for all $i \neq j$. Then the natural homomorphism

$$\varphi : R \rightarrow R/I_1 \times \dots \times R/I_k$$

is surjective and induces an isomorphism

$$\frac{R}{I_1 \dots I_k} \rightarrow R/I_1 \times \dots \times R/I_k$$

Corollary A.1.3 (Chinese remainder theorem). *Let R be a PID and $a_1, \dots, a_k \in R$ be elements such that $\gcd(a_i, a_j) = 1$ for all $i \neq j$. Let $a = a_1 \dots a_k$. Then the function*

$$\varphi : R/(a) \rightarrow R/(a_1) \times \dots \times R/(a_k).$$

Proposition A.1.4 (Yoneda Lemma). *Let \mathcal{C} be a category, X an object of \mathcal{C} and consider the contravariant functor*

$$h_X := \text{hom}_{\mathcal{C}}(-, X).$$

Then for every contravariant functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{S}$, there is a bijection between the set of natural transformations $h_X \rightsquigarrow \mathcal{F}$ and (X) .

Definition A.1.5 ([ASS06]). The (Jacobson) **radical** $\text{rad } A$ of a K -algebra A is the intersection of all maximal right ideals in A . It is the same as the intersection of all left-sided maximal right ideals in A . Furthermore, $\text{rad } A$ is a two-sided ideal.

Definition A.1.6. Let $f, g : X \rightarrow Y$ be morphisms in a category \mathcal{C} . Then a morphism $e : E \rightarrow X$ is called **equalizer** of f and g if $f \circ e = g \circ e$ and for all other morphisms $o : O \rightarrow X$, such that $f \circ o = g \circ o$, there is a unique morphis $O \rightarrow E$, such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{e} & X \rightrightarrows X \\ \uparrow \exists! & \nearrow o & \\ O & & \end{array}.$$

Proposition A.1.7. *Equalizers exists in abelian categories.*

Task 1

See this as a functors:

$$k\text{-Alg} \xrightleftharpoons[\mathcal{A}]{\mathcal{V}} \text{Grp}$$

Grp. alg. construction \mathcal{A} is left-adjoint to group of units construction \mathcal{V} .

i) Show that there is a natural isomorphism

$$k\text{-Alg}(\mathcal{A}(G), A) \cong \text{Grp}(G, \mathcal{V}(A))$$

for all groups G and k -algebras A .

Task 2

This quiver is called the **linear oriented quiver**. Define

$$\varphi : KQ \rightarrow L_n(k)$$

as linear extension of the k -linear map

$$Q_* \rightarrow L_n(k), p_{ij} \mapsto (E_{ij})_{kl} := \begin{cases} 1 & \text{if } k = i, l = j \\ 0 & \text{otherwise} \end{cases}$$

This is indeed a homomorphism of k -algebras, which sends basis vectors to basis vectors.

Task 3

Let \mathcal{C}, \mathcal{D} be two k -**linear** categories, i.e. $\mathcal{C}(X, Y)$ has the structure of a k -vector space and composition is bilinear. We say that \mathcal{C} is **equivalent** to \mathcal{D} ($\mathcal{C} \simeq \mathcal{D}$) if there are k -**linear** functors

$$\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D} \text{ and } \mathcal{G} : \mathcal{D} \rightarrow \mathcal{C}$$

(i.e. functors that induce k -linear maps $\mathcal{C}(X, Y) \rightarrow \mathcal{D}(\mathcal{F}(X), \mathcal{G}(Y))$, and for \mathcal{G} analogous), such that there are natural isomorphisms

$$\mathcal{G}\mathcal{F} \simeq \text{Id}_{\mathcal{C}} \text{ and } \mathcal{F}\mathcal{G} \simeq \text{Id}_{\mathcal{D}}$$

Theorem A.1.8. *Let \mathcal{C}, \mathcal{D} be k -linear categories. $\mathcal{C} \simeq \mathcal{D}$ if and only if there is a fully faithful, k -linear and dense functor*

$$\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}.$$

Remark A.1.9. The proof is supposed to only invoke the Axiom of Choice, and should work for general hom-Sets.

This task basically shows that there is an equivalence of categories

$$\text{Rep}(A) \simeq k\text{-Mod}(A),$$

where

$$\text{Rep}(A) := \{(V, \varphi) \mid V \text{ a } k\text{-vector-space, } \varphi : A \rightarrow \text{End}_k(V) \text{ a algebra-homomorphism}\}$$

Calvin highly recommends the book [ASS06].

A.2 Sheet 2

Solution A.2.1. Consider the two representations

$$k \xrightarrow[b]{a} k \text{ and } k \xrightarrow[d]{c} k$$

with $a, b, c, d \in k$. Morphism of representations are in this case k -linear maps $k \rightarrow k$, i.e. multiplication by elements $\mu, \nu \in k$, such that the diagrams

$$\begin{array}{ccc} k & \xrightarrow{a} & k \\ \mu \downarrow & & \downarrow \nu \\ k & \xrightarrow{c} & k \end{array} \text{ and } \begin{array}{ccc} k & \xrightarrow{b} & k \\ \mu \downarrow & & \downarrow \nu \\ k & \xrightarrow{d} & k \end{array}$$

commute. This is the case if (μ, ν) satisfies the system of equations

$$\underbrace{\begin{pmatrix} c & -a \\ d & -b \end{pmatrix}}_{=:A} \cdot \begin{pmatrix} \mu \\ \nu \end{pmatrix} = 0 \iff \begin{pmatrix} \mu \\ \nu \end{pmatrix} \in \ker A$$

There are several cases to consider:

- $\det A = ad - bc \neq 0$: As A is invertible in this case, $\ker A$ is trivial, and hence

$$\text{hom}(X_{(a,b)}, X_{(c,d)}) = 0$$

- $\det A = 0; a = b = c = 0$. As $A = 0$, $\ker A = k^2$ holds, and hence

$$\text{hom}(X_{(a,b)}, X_{(c,d)}) = k^2$$

- $\det A = 0; b \neq 0$ In this case, $c = ad/b$ holds.

- $a = 0, d = 0$:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -b \end{pmatrix} \implies \ker A = \text{Lin} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- $a = 0, d \neq 0$

$$A = \begin{pmatrix} 0 & 0 \\ d & -b \end{pmatrix} \implies \ker A = \text{Lin} \begin{pmatrix} b/d \\ 1 \end{pmatrix}$$

- $\det A = 0, b = 0$: Consider the cases:

- $c, d \neq 0, a = 0$:

$$A = \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \implies \ker A = \text{Lin} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $c \neq 0, a, d = 0...$

Solution A.2.2. i) A representation of V consists of a vector space V together with an endomorphism $f \in \text{End}_k V$:

$$X := \begin{array}{c} \text{---} \curvearrowright^f \text{---} \\ V \end{array}.$$

A decomposition of X into subrepresentations would correspond to a decomposition

$$V = V_1 \oplus V_2$$

with subspaces V_1 and V_2 of V , such that both V_1 and V_2 are f -invariant. Now, let X be any representation of Q . Assume first that $n := \dim_k V < \infty$, and f is any. As k is algebraically closed, there is a unique (up to permutation) basis B of V given by a disjoint unions of Jordan-Chains

$$B = \bigcup_{\lambda \in k} \bigcup_{i \in I_\lambda} J(\lambda, \ell_i) \text{ where } \sum_{\lambda \in k} \sum_{i \in I_\lambda} \ell_i = n,$$

I_λ are finite index sets, unequal to zero for only finitely many $\lambda \in k$, and $J(\lambda, \ell_i)$ are Jordan-Chains of f for the eigenvalue λ with length ℓ_i . This basis induces a unique direct sum decomposition

$$V = \bigoplus_{\lambda \in k} \bigoplus_{i \in I_\lambda} \text{Lin } J(\lambda, \ell_i).$$

By construction of the Jordan-Chains, $J(n, \lambda_i)$ can not be decomposed for

Solution A.2.4. i) R is a k -vector space, where the addition and scalar multiplication are defined component-wise. This gives R the structure of a k -vector space, as M , N and X are in particular k -vector spaces.

Consider now the map

$$R \times R \rightarrow R, \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \mapsto \begin{pmatrix} aa' & ax' + xb' \\ 0 & bb' \end{pmatrix},$$

where the operation $ax' + xb'$ is well-defined, as X is an A - B -bimodule. This makes R into a ring, as:

- the unit is given by $\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$:

$$\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$$

- the multiplication is associative, as

$$\begin{pmatrix} a'' & x'' \\ 0 & b'' \end{pmatrix} \cdot \left(\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix} a'' & x'' \\ 0 & b'' \end{pmatrix} \cdot \begin{pmatrix} a'a & a'x + x'b \\ 0 & b'b \end{pmatrix} \\ = (a''(a'a)) \dots$$

- the distributivity holds, as

Extension and restriction of scalars

We do some recap from Algebra 1 (cf. [AM94, 27f.]). For this, we go back to the commutative case: In the following, A, B denote commutative, unital rings.

Proposition A.2.1. *Let $A \xrightarrow{f} B$ be a ring homomorphism and N a B -module. Then N has a A -module structure, given by*

$$A \times N \mapsto N, (a, n) \mapsto f(a)n.$$

Proof. • The addition on N_A is the same as the addition of N_B .

- Associativity: $(ab)n = f(ab)n = (f(a)f(b))n = f(a)(f(b)n)$, as N is B -module
- Unit acts as unit: $1_A n = f(1_A)n = 1_B n = n$, as f is homomorphism of rings.
- Distributivity: $(a+b)n = f(a+b)n = (f(a) + f(b))n = f(a)n + f(b)n$ and $a(n+n') = f(a)(n+n') = f(a)n + f(a)n'$

□

This way of obtaining a A -module structure on N_B is called **restriction of scalars**. In particular, f defines a A -module structure on B in this way.

Proposition A.2.2. *Let M_A be an A -module. Then*

$$M_B := B \otimes_A M$$

carries a B -module structure, and

$$b(b' \otimes x) = (bb') \otimes x$$

*holds for this B -module. We say that the B -module M_B was obtained from M by **extension of scalars***

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