Foundations of Representation Theory

Lecture Notes in the Winter Term 2018/19

CONTENTS

INTRODUCTION

These are my personal lecture notes for the lecture *Foundations of Representation Theory* held by Dr. Hans Franzen at the University of Bonn in the winter term 2018/19.

I try to update them on my website, https://pankratius.github.io.
The author labels his own comments and additions in purple.

The book [Alu09] is used by the author for further references, and highly recommended.

Conventions: In this lecture, rings will always be unital, and ring homomorphisms f always fulfill $f(1) = 1$. Rings do *not* have to be commutative.

1.1 Algebras - Basics

Let k be a commutative ring.

Definition 1.1.1. A k-**algebra** A is a ring A, together with a structure of a k-module on A, such that

$$\text{for all } a, b \in A, \lambda \in K: \ (\lambda a)b = a(\lambda b) = \lambda(ab) \tag{$*$}$$

Definition 1.1.2. Let A, B be k-algebras. A **homomorphism of algebras** is a map $f : A \to B$ that is both k-linear and a ring homomorphism.

Remark 1.1.3. Let A be a ring. Define

$$Z(A) := \{a \in A \mid \forall b \in A : ab = ba\},$$

which is a commutative subring and is called the **center** of A.

Remark 1.1.4. Let A be a ring. Giving a k-algebra structure on A is the same as giving a ring homomorphism $k \to Z(A)$. More precisley:

i) If A is a k-algebra, then $p : k \to A$, $\lambda \mapsto \lambda 1$ satisfies $\operatorname{Im} p \subseteq Z(A)$ and is a ring homomorphism. (the first statement follows from ($*$) and the second one from the fact that A has a k-module structure).

ii) Let $\varphi : k \to Z(A)$ be a ring homomorphism. Define

$$\lambda a := \varphi(\lambda)a,$$

for all $\lambda \in k$. This defines a k-algebra structure on A (($*$) holds since $\operatorname{Im}(\varphi) \subseteq Z(A)$).

iii) Let A, B be k-algebras and $f : A \to B$ a homomorphism of rings. Then f is a homomorphism of k-algebras if and only if the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "f"] & & B \\
Z(A) \arrow[u, hook] & & Z(B) \arrow[u, hook] \\
& k \arrow[ul, "\phi_A"] \arrow[ur, "\phi_B"'] &
\end{tikzcd} .$$

Example 1.1.5. i) Let V be a k-module. Then $\operatorname{End}_k(V)$ has a ring structure given by

$$\operatorname{End}_k(V) \times \operatorname{End}_k(V) \to \operatorname{End}_k(V), \ (\phi, \psi) \mapsto \varphi \circ \psi.$$

Then $\mathrm{End}_k(V)$ is both a ring and a k-module, and becomes a k-algebra via

$$\varphi : k \to \mathrm{End}_k(V),\ \lambda \mapsto \lambda \,\mathrm{id}\,.$$

Note that $\mathrm{Im}\,\varphi \subseteq Z(A)$. If k is a field, then $Z(\mathrm{End}_k(V)) = \{\lambda \,\mathrm{id} \mid \lambda \in k\}$.

ii) Take $V = k^n$ (free module of rank n). Then $\mathrm{End}_k(V) \cong \mathrm{M}_n(k)$. Define

$$T_u := \{\varphi \in \mathrm{M}_n(k) \mid \varphi \text{ is upper triangular}\},$$

i.e. T_u presevers flags in k^n. Then T_u is a **subalgebra** of $\mathrm{M}_n(k)$, i.e. is both a subring and a k-submodule of the original algebra.

iii) Let G be a group. Define the **group algebra** $k[G]$ of G as follows:

- As k-module, is defined as the free module on G,

$$k[G] := k^{(G)} = \{\sum_{g\in G} \lambda_g g \mid \lambda_g \in k,\ \lambda_g \neq 0 \text{ for only finitely many } g \in G\}.$$

- Multiplication: Let $a := \sum \lambda_g g$, $b = \sum \mu_g g$ and define:

$$ab := \sum_{g\in G, h\in G} \lambda_g \mu_h (gh) = \sum_{j\in G} \left(\sum_{gh=j} \lambda_g \mu_h \right) j.$$

This multiplication is associative, k-bilinear, distributive and $1|_{k[G]} = e$. In addition, $(*)$ is satisfied.

1.2 Quivers - Basics

Definition 1.2.1. A **quiver** is a „directed graph“. Formally, a quiver is a quadruple (Q_0, Q_1, s, t) consiting of sets Q_0 (**vertices**) and Q_1 (**arrows**) and maps $s : Q_1 \to Q_0$, $t : Q_1 \to Q_0$. For $\alpha \in Q_1$, we call $s(\alpha)$ the **source** of α and $t(\alpha)$ the **target** of α:

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

Example 1.2.2. i) $Q = (\{1\}, \emptyset, ...)$ is visualized as: 1

ii) $Q = (\{1\}, \{\alpha\}, ...)$ is visualized as
$$\begin{tikzcd} 1 \arrow[loop above] \end{tikzcd}$$

iii) $Q = (\{1, 2\}, \{\alpha, \beta\}, s(\alpha) = s(\beta) = 1, t(\alpha) = t(\beta) = 2)$ is visualized as $1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$

Definition 1.2.3. Let Q be a quiver sucht that both Q_0 and Q_1 are finite.

i) Let $\ell \in \mathbb{Z}_{\geq 1}$. A **path** of length ℓ is a sequence $\alpha_\ell ..., \alpha_1$ of arrows, such that $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq \ell - 1$,

$$\circ \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_2} \circ \ldots \xrightarrow{\alpha_\ell} \circ.$$

Define Q_ℓ to be the set of all paths of length ℓ.
Let $p : \alpha_\ell ... \alpha_1$ be a path. Define $s(p) := s(\alpha_1)$ and $t(p) := s(\alpha_\ell)$.
Formally define Q_0 to be the set of all paths of length zero. Denote by ε_i for $i \in Q_0$ the constant path at i. ε_i is called a **lazy path**. We set $s(\varepsilon_i) = t(\varepsilon_i) := i$.

ii) Let $p = \alpha_\ell ... \alpha_i$ and $q = \beta_m ... \beta_1$ be paths of length ℓ and m respectivley, with $\ell, m \geq 1$. If $t(p) = s(q)$, then set $q \circ p := \beta_m .... \beta_1 \alpha_\ell ... \alpha_1$. This is a path of length $\ell + m$.
For p a path of length ≥ 0 and ε_i a lazy path:

- if $t(p) = i$, set $\varepsilon_i \circ p := p$,
- if $s(p) = i$, set $p \circ \varepsilon_i := p$.

In all others cases, the composition is not defined.

iii) Define

$$Q_* := \bigcup_{\ell \geq 0} Q_\ell,$$

the set of all paths. Define the **path-algebra** kQ:

- As a k-module, $kQ := k^{(Q_*)}$.
- Multiplication: Let $a = \sum \lambda_p p,\ b = \sum \mu_p p$. Define

$$ab := \sum_{p,q \in Q_*} \lambda_p \mu_q (p \cdot q),$$

where

$$p \cdot q := \begin{cases} p \circ q, & \text{if it is defined, i.e } t(q) = s(p) \\ 0, & \text{else} \end{cases}.$$

The multiplication is associative (due to the associativity of the composition of paths) and k-bilinear by definition. In addition, distributivity and ($*$) are fulfilled.
- The identity is given by $\sum \varepsilon_i$.

Example 1.2.4. i) $Q = 1$, then $kQ = k$.

ii) $Q = \begin{tikzcd} 1 \arrow[loop above] \end{tikzcd}$, then $Q_* = \{\alpha^n \mid n \geq 0\}$ and $kQ = k[t]$.

iii) $Q = 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$. Then $Q_* = \{\varepsilon_1, \varepsilon_2, \alpha^n, \beta^n \mid n \geq 0\}$ and

$$kQ = k\varepsilon_1 \oplus k\varepsilon_2 \oplus k\alpha \oplus k\beta.$$

A multiplication table is given by

	ε_1	ε_2	α	β
ε_1	ε_1	0	0	0
ε_2	0	ε_2	α	β
α	α	0	0	0
β	β	0	0	0

Lemma 1.2.5. *Let Q be a finite quiver, k a field. Then the following are equivalent:*

i) *Q contains no cycles.*

ii) $\dim_k kQ < \infty$.

Lemma 1.2.6. *Let k be a field, A a k-algebra and $n := \dim(A) < \infty$. Then there exists an injective homomorphism of k-algebras $\varphi : A \to \mathrm{M}_n(k)$.*

Proof. By choosing a basis of A, we get an isomorphism $\mathrm{End}_k(A) \cong \mathrm{M}_n(k)$. So it suffices to find an injective homomorphism of k-algebras $\varphi : A \to \mathrm{End}_k(A)$.
Consider

$$\varphi : A \to \mathrm{End}_k(A),\ \ \varphi(a) : A \to A, b \mapsto ab.$$

- $\varphi(a)$ is k-linear for all a by the distributivity in A and the condition $(*)$.
- φ is k-linear by the distributivity in A and the condition $(*)$.
- Let $a, a' \in A$. Then

$$\varphi(aa')(b) = (aa')(b) = a(a'b) = \big(\varphi(a) \circ \varphi(a')\big)(b).$$

Hence φ is indeed a homomorphism of k-algebras.
To show that φ is injective, let $a \in \ker \varphi$, hence $ab = 0$ for all $b \in A$. But in particular, $0 = a1 = a$. ☐

End of Lecture 1

Definition 1.2.7. Let A be a k-algebra. Then the **opposite algebra** A^{op} is A (as a k-module), and the multiplication is defined as

$$a \cdot_{A^{\mathrm{op}}} b = b \cdot_A b.$$

Example 1.2.8. Let Q be a quiver, and define $Q^{\mathrm{op}} := (Q_0, Q_1, s^{\mathrm{op}}, t^{\mathrm{op}})$, where $s^{\mathrm{op}}(\alpha) := t(\alpha)$ and $t^{\mathrm{op}}(\alpha) := s(\alpha)$. Then $kQ^{\mathrm{op}} = k(Q^{\mathrm{op}})$

1.3 Modules - Basics

Definition 1.3.1. Let A be a k-algebra. A **left** A**-module** M is a k-module M together with a map $A \times M \to M, (a, x) \mapsto ax$, such that:

$$a(x + y) = ax + ay \tag{L1}$$

$$(a + b)x = ax + bx \tag{L2}$$

$$a(bx) = (ab)x \tag{L3}$$

$$1_A x = x \tag{L4}$$

$$(\lambda a)x = \lambda(ax) = a(\lambda x), \tag{L5}$$

for all $a, b \in A$, $x, y \in M$ and $\lambda \in k$. If A is a left A-module, we denote this as ${}_A M$. A **right** A**-module** is defined analogous, where (L3) becomes $(xa)b = x(ab)$. If A is a right A-module, we denote this by A_M.

Remark 1.3.2. A right A-module is the same as a left A^{op}-module.

Definition 1.3.3. Let A be a k-algebra, and M, N left A-modules. A **homomorphism of left** A**-modules** $f : M \to N$ is a k-linear map such that

$$f(ax) = af(x)$$

for all $a \in A$ and $x \in M$.
Define the set of all left A-algebra homomorphisms as

$$\hom_A(M, N) := \hom_A({}_A M, {}_A M) := \{f : M \to N \mid f \text{ is a homomorphism of left } A\text{-modules}\}.$$

A homomorphism of left A-modules is an **isomorphism** if it is a bijective homomorphism of left A-modules.
Homomorphism of right A**-modules** are defined analogous.

Remark 1.3.4. Let M, N be left A-modules. Then

i) $\hom_A(M, N)$ has a k-module structure given by

$$\lambda f : M \to N, \ x \mapsto \lambda f(x) = f(\lambda x).$$

This is well defined, as k lies in the center of A.

ii) In general, $\hom_A(M, N)$ has neither a left nor a right A-module structure.

iii) f is an isomorphism if and only if there is a homomorphism of left A-modules $g : N \to M$ such that

$$g \circ f = \mathrm{id}_M \ \text{ and } \ f \circ g = \mathrm{id}_N \,.$$

iv) Let $f : M \to M'$ and $g : N \to N'$ be homomorphisms of left A-modules. Then we obtain k-linear maps

$$f^* : \hom_A(M', N) \to \hom_A(M, N), \; h \mapsto h \circ f$$
$$g_* : \hom_A(M, N) \to \hom_A(M, N'), \; h \mapsto g \circ h.$$

Remark 1.3.5. Let A be a k-algebra and M, N left A-modules.

i) A subset $M' \subseteq M$ is called a **submodule** if

(SM1) $0 \in M'$

(SM2) $x, x' \in M' \implies x + x' \in M'$

(SM3) $a \in A, x \in M' \implies ax \in M'$.

In particular, submodules of A-modules are submodules of the underlying k-module, as follows using (L4)

ii) Let M be a submodule. Then the **quotient** has a left A-module structure in the obvious way. The projection

$$\pi : M \to M'$$

is a homomorphism of left A-modules.

iii) A **left ideal** is left A-submodule of ${}_AA$. Similar, a **right ideal** is right A-submodule of A_A. For a left ideal $I \subseteq A$, the quotient A/I is a left A-module, but in general not an algebra.

iv) A **two-sided ideal** $I \subset A$ is both a left- and a right-ideal of A. Then A/I has an algebra structure, by setting

$$(x + I)(y + I) := (xy) + I.$$

In general, this is only well-defined if I is a two-sided ideal of A.

v) Let $f : M \to N$ be a homomorphism of left A-modules. Then we obtain left A-modules:

$$\ker f, \; \operatorname{Im} f, \; \operatorname{coker} f := N/\operatorname{Im} f, \; \operatorname{coim} f := M/\ker f.$$

In particular, f factors uniquely as

$$\begin{tikzcd} M \arrow[r] \arrow[rrr, bend left, "f"] & \operatorname{coim} f \arrow[r, "\exists!", "\cong"'] & \operatorname{Im} f \arrow[r] & N \end{tikzcd} . \tag{F}$$

vi) Let $\{M_i \subset M \mid i \in I\}$ be a family of left A-submodules, for some index set I. Then

$$\bigcap_{i \in I} M_i \text{ and } \sum_{i \in I} M_i$$

are left A-modules.

vii) Let $x \in M$. Define

$$Ax := \{ax \mid a \in A\},$$

which is a left A-submodule. Similar, for $x \in M_A$, define $xA := \{xa \mid a \in A\}$, which is a right A-submodule. For a subset $E \subset M$,

$$\sum_{x \in E} Ax = \bigcap_{\substack{E \subseteq M' \subseteq M \\ M' \text{ submodule}}} M'.$$

M is called **finitely generated**, if there are $x_1, ..., x_n \in M$, such that

$$M = \sum_{i=1}^{n} Ax_i.$$

viii) Let $\{M_i \mid i \in I\}$ be a family of left A-modules. Then

$$\prod_{i \in I} M_i := \{(x_i)_{i \in I} \mid x_i \in M_i\}$$

is called the **product** , and

$$\bigoplus_{i \in I} M_i := \{(x_i)_{i \in I} \mid x_i \in M_i,\ x_i \neq 0 \text{ for only finitely many } i\}$$

is called the **coproduct** .They are both left A-modules. The **projection**

$$\pi_j : \prod_{i \in I} M_i \to M_j,\ (x_i)_{i \in I} \mapsto x_j$$

and the **inclusion**

$$\iota_j : \bigoplus_{i \in I} x_j \mapsto (\delta_{ij} x_j)_{i \in I}$$

are morphism of left A-modules.

ix) A left A-module M is finitely generated if and only if there is a surjective homomorphism of left A-modules

$$A^n := \bigoplus_{i=1}^{n} A \twoheadrightarrow M$$

for some $n \geq 1$. A is called **finitely presented**, if there is an exact sequence of left A-modules

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

for some $m, n \geq 1$.

Proposition 1.3.6. *Let*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0 \tag{$*$}$$

and

$$0 \longrightarrow N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \tag{$**$}$$

be sequences of left A-modules.

i) The following are equivalent:

a) (∗) is exact.

b) For all left A-modules N, the sequence

$$0 \longrightarrow \hom_A(M_3, N) \xrightarrow{f_2^*} \hom_A(M_2, N) \xrightarrow{f_1^*} \hom_A(M, N)$$

is exact.

ii) The following are equivalent:

a) (∗∗) is exact.

b) For all left A-modules M, the sequence

$$0 \longrightarrow \hom_A(M, N_1) \xrightarrow{g_{1,*}} \hom_A(M, N_2) \xrightarrow{g_{2,*}} \hom_A(M, N_3)$$

is exact.

Proof. We will only prove *a)* $\implies$ *b)* of *ii)*.

Lemma 1.3.7. *Let K, M, N be left A-modules, and $\zeta : K \to M$, $\varphi : M \to N$ be homomorphisms of left A-modules, such that $\varphi \circ \zeta = 0$. Then there is a unique homomorphism $\overline{\zeta}$, such that*

$$\begin{tikzcd} K \arrow[r, "\zeta"] \arrow[rr, bend left, "0"] \arrow[dr, "\exists!\overline{\zeta}"'] & M \arrow[r, "\varphi"] & N \\ & \ker\varphi \arrow[u, hook] & \end{tikzcd}$$

commutes.

- $g_{1,*}$ injective: Let $h \in \ker(g_{1,*})$. Then

$$\begin{tikzcd} g_1 \circ h : \quad M \arrow[r, "h"] \arrow[rr, bend left, "0"] & N_1 \arrow[r, "g_1"] & N_2 \end{tikzcd}$$

  and since g_1 is injective, it follows $h = 0$.

- $\operatorname{Im} g_{1,*} \subseteq \ker g_{2,*}$: Since ∗∗ is exact, it follows that $g_2 \circ g_1 = 0$. For $h \in \operatorname{Im} g_{1,*}$ there exists an $h' : M \to N_1$ such that $h = g_1 \circ h'$, and hence $g_2 \circ h = g_2 \circ g_1 \circ h' = 0$.

- $\ker g_{2,*} \subseteq \operatorname{Im} g_{1,*}$: As (∗∗) is exact, $\ker g_2 = \operatorname{Im} g_1$ holds.
  Let $h : M \to N_2 \in \ker g_{2,*}$, i.e. $g_2 \circ h = 0$:

$$\begin{tikzcd} & & M \arrow[d, "h"'] \arrow[dr, "0"] & \\ 0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \end{tikzcd}.$$

By lemma 1.3.7, h factors uniquely through $\ker g_2 = \operatorname{Im} g_1$:

$$\begin{tikzcd}
 & & M \arrow[d, "h"'] \arrow[dr, "0"] \arrow[dd, dashed, bend left] & \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \, . \\
 & & \operatorname{Im} g_1 \arrow[u, hook, "\iota"] &
\end{tikzcd}$$

But since g_1 is injective, (F) implies that there is a uniquely determined isomorphism $\tilde{g}_1 : N_1 \longrightarrow \operatorname{Im} g_1$.
Putting everything together, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & & M \arrow[d, "h"'] \arrow[dr, "0"] \arrow[dd, dashed, bend left] & \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] \arrow[dr, dashed, bend right, "\exists! \tilde{g}_1"] & N_2 \arrow[r, "g_2"] & N_3 \, . \\
 & & \operatorname{Im} g_1 \arrow[u, hook, "\iota"] &
\end{tikzcd}$$

Setting $h' := \tilde{g}_1^{-1} \circ h'$, we obtain

$$g_1 \circ h' = \iota \circ \tilde{g}_1 \circ \tilde{g}_1^{-1} \circ \tilde{h} = \iota \circ \tilde{h} = h.$$

□

Proposition 1.3.8. *Let A be a k-algebra. To give a left A-module structur is the same as to give a k-module structure V together with a homomorphism $\varphi : A \to \operatorname{End}_k(V)$ of k-algebras. To give a right A-module structure is the same as giving a k-module structure V together with a homomorphism $\varphi : A \to \operatorname{End}_k(V)^{\mathrm{op}}$.*

1.4 Representation of quivers

Let k be a field and Q be a quiver.

Definition 1.4.1. A **representation** X of Q over k consists of

- a k-vector space X_i for all $i \in Q_0$,
- a k-linear map
$$X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)}$$
for each $\alpha \in Q_1$

Example 1.4.2 (Continue example 1.2.4). i) Let $Q = \cdot$. Then a representation of Q is simply a k-vector space.

ii) Let $Q = \begin{tikzcd} 1 \arrow[loop right] \end{tikzcd}$. Then a representation of Q is a k-vector space V together with an endomorphism $\varphi \in \mathrm{End}_k(V)$:

$$Q = \begin{tikzcd} V \arrow[loop right, "f"] \end{tikzcd}.$$

iii) Let $Q = \begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}$, the **Kronecker Quiver**. Then a representation of Q is a pair of vector spaces V, W and two linear maps $f, g \in \hom_K(V, W)$:

$$Q = \begin{tikzcd} V \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & W \end{tikzcd}$$

Definition 1.4.3. Take X, Y to be two representations of Q over k. A **homomorphism of representations** $f : X \to Y$ is a tupel $(f_i)_{i \in Q_0}$ of linear maps $f_i : X_i \to Y_i$, such that for all $\alpha \in Q_1$ the diagram

$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "f_{s(\alpha)}"] \arrow[d, "X_\alpha"'] & Y_{s(\alpha)} \arrow[d, "Y_\alpha"] \\ X_{t(\alpha)} \arrow[r, "f_{t(\alpha)}"'] & Y_{t(\alpha)} \end{tikzcd}$$

commutes.

Example 1.4.4 (Continue example 1.4.2)**.** i) Homomorphisms of representations are k-linear maps $X \to Y$.

ii) Homomorphisms of representations (V, φ) and (W, ψ) are k-linear maps $f : V \to W$, such that

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi"'] & W \arrow[d, "\psi"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

commutes

iii) Homomorphisms of representations (V_1, V_2, A, B) and (W_1, W_2, C, D) are pairs (f_1, f_2) of linear maps $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$, such that $A \circ f_1 = f_2 \circ A$ and $B \circ f_1 = f_2 \circ B$.

Definition 1.4.5. An **isomorphism of representations** $f : X \to Y$ is a homomorphism of representations, such that there exists $g : Y \to X$ homomorphism of representations satisfying

$$g \circ f = \mathrm{id}_X \text{ and } f \circ g = \mathrm{id}_y .$$

An isomorphism of representations is a homomorphism of representations such that each map f_i is bijective.

End of Lecture 2

Let Q be a quiver over a field k.

Remark 1.4.6. i) Let X be a representation of Q over k. Associate a left kQ-module $M = F(X)$ as follows:

- As k-vector space, let
$$M := \bigoplus_{i \in Q_0} X_i.$$
- Define the action of kQ on M by an action of paths. Let p be a path of length ≥ 1. Define
$$X_p : X_{s(p)} \to X_{t(p)} \text{ given by } X_p = X_{\alpha_\ell} \circ \ldots \circ X_{\alpha_i},$$
with $X_p \in \hom_k(X_{s(p)}, X_{t(p)})$. Use this to define a k-linear map $\tilde{X}_p : M \to M$ as composition:
$$\tilde{X}_p : M = \bigoplus_{i \in Q_0} X_i \xrightarrow{\pi_{s(p)}} X_{s(p)} \xrightarrow{X_p} X_{t(p)} \xhookrightarrow{\iota_{t(p)}} \bigoplus_{i \in Q_0} X_i$$
If the length of $p = 0$, then p is a lazy part at some $i \in Q_0$, and we set
$$X_{\varepsilon_i} := \mathrm{id}_{X_i},$$
and $\tilde{X}_{\varepsilon_i}$ like $\tilde{X}_p$.
Now these k-linear endomorphisms define a kQ-module structure on M, given by:
$$\begin{aligned} kQ \times M \to M, \left(a := \sum_{p \in Q_*} \lambda_p \cdot p, (x_i)_i =: x \right) \mapsto a.x :&= \sum_{p \in Q_*} \lambda_p \cdot \tilde{X}_p(x) \\ &= \sum_{p \in Q_*} \lambda_p \cdot (\iota_{t(p)} X_p(x_{s(p)})), \end{aligned}$$
where we denote an element in M by a sequence $(x_i)_i$ with $x_i \in X_i$.
- We check that this actually defines a kQ-module structure:
  - (L3): Assume that $a, b \in kQ$. By the bilinearity of the multiplication, we can assume that $a = p$ and $b = q$ are both paths in Q_*. Then
$$\begin{aligned} a.(b.x) &= \tilde{X}_p\left(\tilde{X}_q(x)\right) \\ &= \iota_{t(p)} X_p \underbrace{\pi_{s(p)} \iota_{t(q)}} X_q(x_{s(q)}), \end{aligned}$$
where
$$\pi_{s(p)} \iota_{t(q)} = \begin{cases} \mathrm{id}_{X_q}, & \text{if } t(q) = s(p) \\ 0, & \text{otherwise} \end{cases}.$$
This gives
$$a.(b.x) = \begin{cases} \iota_{t(p)} X_p X_q(x_{s(q)}), & \text{if } t(q) = s(p) \\ 0 & \text{otherwise} \end{cases}.$$
Additionally,
$$(a.b).x = \begin{cases} \tilde{X}_{p \circ q} & \text{if } f(q) = s(p), \\ 0 & \text{otherwise} \end{cases}.$$

But in the case $f(q) = s(p)$,

$$\tilde{X}_{p\circ q}(x) = \iota_{t(p)} \circ X_p X_q(x_{s(q)}).$$

The construction F is functorial, i.e. for $f : X \to Y$ a homomorphism of representations, F induces a homomorphism of kQ-algebras

$$Ff : F(X) \to F(Y) \text{ by } (Ff)((x_i)_i) := (f_i(x_i))_i.$$

ii) Let M be a left kQ-module. Define a representation $X := G(M)$ as follows

- As k-vector spaces, set
$$X_i := \varepsilon_i M.$$
- For $\alpha \in Q_*$, set
$$X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)},\ \varepsilon_{s(\alpha)}x \mapsto \alpha\varepsilon_{s(\alpha)}x = \alpha x = \varepsilon_{t(\alpha)}\alpha x \in X_{t(\alpha)},$$
as $\varepsilon_{t(\alpha)}\alpha = \alpha\varepsilon_{s(\alpha)}$.
So $X := ((X_i)_{i\in Q_0}, (X_\alpha)_{\alpha\in Q_*})$ is a representation of Q.
- This construction is also functorial: take $g : M \to N$ a homomorphism of left kQ-modules. Define $G(g) : X \to Y$, with $X := G(M)$ and $Y := G(N)$. Set
$$(Gg)_i := X_i \to Y_i,\ \varepsilon_i x \mapsto g(\varepsilon_i x) = \varepsilon_i g(x) \text{ with } X_i := \varepsilon_i M \text{ and } Y_i := \varepsilon_i N.$$
This is indeed a homomorphism of representations: Let $\alpha \in Q_1$ be arbitrary, and consider
$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "(Gg)_{s(\alpha)}"] \arrow[d, "X_\alpha"'] & Y_{s(\alpha)} \arrow[d, "Y_\alpha"] \\ X_{t(\alpha)} \arrow[r, "(Gg)_{t(\alpha)}"'] & Y_{t(\alpha)} \end{tikzcd}.$$
Then
$$y_\alpha(Gg)_{\varepsilon_{s(\alpha)}}x = Y_\alpha(g(\varepsilon_{s(\alpha)}x)) = \alpha g(\varepsilon_{s(\alpha)}x)$$
and
$$(Gg)_{\varepsilon_{t(\alpha)}}(X_\alpha(\varepsilon_{s(\alpha)}x)) = g(\alpha(\varepsilon_{s(\alpha)}x)) = \alpha g(\varepsilon_{s(\alpha)}x),$$
hence the diagram commutes.

Theorem 1.4.7. *i) Let M be a left kQ-module. Then $FG(M) \cong M$ as left kQ-modules.*

ii) Let X be a representation of Q over k. Then $GF(X) \cong X$ as representations of Q.

Proof. i) Denote $X := G(M)$. Then

$$F(X) = \bigoplus_{i\in Q_0} X_i = \bigoplus_{i\in Q_0} \varepsilon_i M$$

as a k-vector space. Observe

- The identity in kQ is given by

$$\mathrm{id}_{kQ} = \sum \varepsilon_i.$$

Hence, for all x in X:

$$x = \mathrm{id}_{kQ}\, x = \left(\sum \varepsilon_i\right) x = \sum (\varepsilon_i x_i) \in \sum \varepsilon_i M.$$

- For all $i \neq j$, $\varepsilon_i \varepsilon_j = 0$ holds. So for

$$x \in X_i = \varepsilon_i M \bigcap \sum_{j \neq i} \varepsilon_j M_j \implies x = \sum_{j \neq i} \varepsilon_j x_j \text{ for some } x_j \in M.$$

But as $x \in \varepsilon_i M$, $\varepsilon_i x = x$. So

$$x = \varepsilon_i x = \varepsilon_i \left(\sum_{j \neq i} \varepsilon_j x_j \right) = \sum_{j \neq i} \varepsilon_i \varepsilon_j x_j = 0.$$

These observations show that

$$\varphi : F(X) = \bigoplus_{i \in Q_0} \varepsilon_i M \longrightarrow M \;,\; (\varepsilon_i x_i)_{i \in Q_0} \mapsto \sum \varepsilon_i x_i$$

is an isomorphism of k-vector spaces.
Show that φ is isomorphism of left kQ-modules:
Without loss generaly, assume that $a = p$ is a path in Q (regarded as element of kQ), and let $x \in M$. Then

$$\begin{aligned}\varphi(a.x) = \varphi\left(\iota_{t(p)} X_p(x_{s(p)})\right) &= \varphi\left(\iota_{t(p)}(p x_{s(p)})\right) \\ &= p x_{s(p)},\end{aligned}$$

and

$$a.\varphi(x) = a. \sum x_i = a. \sum \epsilon_i x_i = p x_{s(p)}.$$

ii) Let $M := F(X)$ be the left kQ-module associated with X. Then

$$G(M)_i = \varepsilon_i M = \varepsilon_i \bigoplus_{j \in Q_0} X_j = X_i,$$

and

$$(G(M))_\alpha (x_{s(\alpha)}) = \alpha . x_{s(\alpha)} = X_\alpha(x_{s(\alpha)}).$$

Be careful! $X_i \subset \oplus X_j$ is still different from X_i as part of the representation, because one is a subspace and one is just a space. So the appropriate isomorphism in this case would be

$$G(M)_i = X_i \xrightarrow{x_i \mapsto x_i} X_i,$$

which is a morphism of representations, as for any $\alpha \in Q_1$

$$\begin{tikzcd}
G(M)_{s(\alpha)} \arrow[d, equal] & \\
X_{s(\alpha)} \arrow[r] \arrow[d, "(G(M))_\alpha"'] & X_{s(\alpha)} \arrow[d, "X_\alpha"] \\
X_{t(\alpha)} \arrow[r] \arrow[d, equal] & X_{t(\alpha)} \\
G(M)_{t(\alpha)} &
\end{tikzcd}$$

commutes.

□

Remark 1.4.8. Let M be a left kQ-module, with Q finite and k a field.

i) $\dim_k M = \sum_{i \in Q_0} \dim_k X_i$ where $X = G(M)$,where G is the functor from remark 1.4.6

ii) $\dim_K kQ < \infty \iff Q$ contains no **oriented cycles** (a path p of length ≥ 1, such that $s(p) = t(p)$)

iii) If Q has no oriented cycle, then the following are equivalent:
   (a) M is a finitely generated kQ-module.
   (b) $\dim_k X_i < \infty$.

Proof. (a) $\implies$ (b):(b) implies in particular, that M is finitely generated as a k-module. But as $k \subset kQ$, (a) follows immediatley.

(b) $\implies$ (a): Set $A := kQ$,and let $x_1, ..., x_n \in kQ$ generate M as a left kQ-module. Then there is a kQ-linear surjection

$$A^n \twoheadrightarrow M$$

given by $e_i \mapsto x_i$, where the $(e_i)_{1 \leq i \leq n}$ are a basis of A^n. As this is in particular k-linear, we have that

$$\dim_k M \leq \dim_k(A^n) = n \dim_k A < \infty,$$

as Q contains no cycle. □

iv) Under G, the notion of a „left submodule“ corresponds to **subrepresentations** of Q, i.e. a tupel of subspaces $Y_i \subset X_i$ for all $i \in Q_0$ such that $X_\alpha(Y_{s(\alpha)}) \subset Y_{t(\alpha)}$ for all $\alpha \in Q_1$.

v) Under G, a direct sum of modules corresponds to **direct sum of representations**: Given X, Y two representations of Q, define a new representation $X \oplus Y$ where the vector spaces are given by

$$(X \oplus Y)_i := (X \oplus Y)_i$$

and the k-linear maps

$$(X \oplus Y)_\alpha : X_{s(\alpha)} \oplus Y_{s(\alpha)} \longrightarrow X_{t(\alpha)} \oplus Y_{t(\alpha)}$$

given by

$$\left(\begin{array}{c|c} X_\alpha & \\ \hline & Y_\alpha \end{array}\right).$$

1.5 Bimodules and tensor products

Definition 1.5.1. Let A, B be k-algebras. A A-B**-bimodule** M is a set M, together with maps:

$$A \times M \longrightarrow M, (a, x) \longmapsto ax$$

$$M \times B \longrightarrow M, (x, b) \longmapsto xb,$$

such that

i) M is a left A-module

ii) M is a right B-module

iii) for all $a \in A, b \in B$ and $x \in M$, the relation

$$(ax)b = a(xb)$$

holds.

We denote a A-B-bimodule by

$${}_AM_B.$$

Lemma 1.5.2. *Let A, B, C be k-algebras, and consider ${}_AM_B$ and ${}_AN_C$, a A-B-bimodule and a A-C-bimodule respectivley. Then* $\hom_A(M, N)$ *becomes a B-C-bimodule via*

- $B \times \hom_A(M, B) \longrightarrow \hom_A(M, N), \qquad (b, f) \longmapsto bf : M \to N, x \mapsto f(xb)$
- $\hom_A(M, N) \times C \longrightarrow \hom_A(M, N), \qquad (f, c) \longmapsto fc : M \to N, x \mapsto f(cx)$

Proof. • well-defined:

$$bf(ax) = f((ax)b) = f(a(xb)) = af(xb) = a(bf)(x)$$

- $\hom_A(M,N)$ is a left B-module: Show e.g. (L3):
$$\begin{aligned}((bb')f)(x) = (x(bb')) &= f((xb)b') \\ &= b'(f(xb)) \\ &= b((b'f)(x))\end{aligned}$$

- compatibility:
$$((af)b)(x) = f((ax)b) = f(a(xb)) = (a(fb))(x).$$

$\square$

End of Lecture 3

Definition 1.5.3. Let A be a k-algebra, and M_A be a right A-module, ${}_AN$ a left A-module.

i) Let P be a k-module. A map
$$\varphi : M \times N \to P$$
is called **A-balanced** if
   - $\varphi(x + x', y) = \varphi(x, y) + \varphi(x', y)$
   - $\varphi(x, y + y') = \varphi(x, y) + \varphi(x, y')$
   - $\varphi(xa, y) = \varphi(x, ay)$
   - φ is k-linear.

ii) A pair (T, τ) where T is a k-module and τ a A-balanced map $M \times N \to T$ is called a **tensor product** of M with N over A, if the follwing universal property holds: For all A-balanced maps $\varphi : M \times N \to P$, where P is any k-module, there is a unique k-linear map f, such that
$$\begin{tikzcd} M \times N \arrow[r, "\tau"] \arrow[d, "\varphi"'] & T \arrow[dl, dashed, "\exists! f\ k\text{-linear}"] \\ P & \end{tikzcd}$$
commutes.

Lemma 1.5.4. *Let A be a k-algebra, M_A a right A-module, ${}_AN$ a left A-module.*

i) There exists a tensor product (T, τ) of M with N over A.

ii) This tensor product is unique up to unique isomorphism. More precisely, if (T', τ') is any other tensor product, then there exists a unique isomorphism of k-modules $f : T \to T'$, such that $f \circ \tau = \tau'$.

Proof. i) *existence:* Let F be the free k-module with basis $M \times N$, and U the submodule generated by elements of the form

- $(x + x', y) - (x, y) - (x', y)$

- $(x, y + y') - (x, y) - (x, y')$
- $(xa, y) - (x, ay)$
- $(\lambda x, y) - \lambda(x, y)$
- $(x, \lambda y) - \lambda(x, y)$

Then F/U is a k-module, and

$$\tau : M \times N \longrightarrow F \longrightarrow F/U$$

is A-balanced by definition. We set

$$a \otimes b := \tau((a, b)) \text{ and } M \otimes_A N := F/U.$$

The pair $(M \otimes_A N, \otimes)$ satisfies the universal property of a tensor product:
Let $\varphi : M\times \to P$ be A-balanced. Then there exists a unique $\hat{\varphi}$ such that the following diagram commutes, as F is free with basis $M \times N$:

$$\begin{tikzcd} M \times N \arrow[r, hook] \arrow[d, "\varphi"'] & F \arrow[dl, dashed, "\exists ! \hat{\varphi}"] \\ P & \end{tikzcd}.$$

Since φ is A-balanced, $\hat{\varphi}$ factors through F/U, i.e.:

$$\begin{tikzcd} M \times N \arrow[r, hook] \arrow[d, "\varphi"'] & F \arrow[r, "\pi"] \arrow[dl, "\hat{\varphi}"'] & F/U \arrow[dll, dashed, "\exists ! \overline{\hat{\varphi}}"] \\ P & & \end{tikzcd}.$$

Now set $\tau := \overline{\hat{\varphi}}$.

ii) later...

$\square$

Lemma 1.5.5. *Let A, B, C be k-algebras, ${}_AM_B$ and ${}_BN_C$ bimodules. Then $M \otimes_B N$ is a A-C-bimodule, via*

- $a(x \otimes y) = (ax) \otimes y$
- $(x \otimes y)c = x \otimes (yc)$,

for all $x \in X, y \in Y, a \in A, c \in C$.

Proof. $M \otimes_B N$ is already a k-module. For all $a \in A$ define

$$\tau_a : M \times N \longrightarrow M \times N \xrightarrow{\otimes} M \otimes_B N$$

$$(x, y) \longmapsto (ax, y) \longmapsto (ax) \otimes y$$

τ_a is A-balanced:

- $\tau_a((x+x',y)) - \tau_a((x,y)) - \tau_a((x',y)) = (a(x+x')) \otimes y - (ax) \otimes y - (ax') \otimes y = 0$
- $\tau_a((xb,y)) - \tau_a((x,by)) = (a(xb)) \otimes y - (ax) \otimes (by) = ((ax)b) \otimes y - (ax) \otimes (by) = 0$

So it factors through $M \otimes_B N$ as follows:

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[d, "\tau_a"] & M \otimes_B N \arrow[dl, "\exists! f_a \text{ bilinear}"] \\ M \otimes_B N & \end{tikzcd}$$

The map

$$A \to \mathrm{End}_k(M \otimes_B N), \ a \mapsto f_a$$

is a homomorphism of k-algebras. So by proposition 1.3.8, $M \otimes_B N$ is a left A-algebra.
If we consinder the map

$$\tau_c : M \times N \longrightarrow M \times N \longrightarrow M \otimes_B N$$

$$(x,y) \longmapsto (x, yc) \longmapsto x \otimes (yc)$$

□

Remark 1.5.6. Let $\varphi : A \to B$ be a k-algebra homomorphism, ${}_BN$ a left B-module. Then $M := A$ is a A-B-bimodule via

$$a.x.b := ax\varphi(b) \text{ for all } a \in A, x \in M, b \in B.$$

Then by lemma 1.5.5, $A \otimes_B N$ is a left A-module, where we think of B as a right k-module. This construction is sometimes called **extension of scalars** or **induction of B by A**.

Lemma 1.5.7. *Let A, B, C, D be k-algebras and consider*

$${}_AM_B, \ {}_A(M_i)_B \ (i \in I), \ {}_BN_C, \ {}_B(N_j)_C \ (j \in J).$$

Then there are isomorphisms:

i)

$$\left(\bigoplus_{i \in I} M_i\right) \otimes_B N \longrightarrow \bigoplus_{i \in I} (M_i \otimes_B N)$$

$$(x_i) \otimes y \longmapsto (x_i \otimes y)$$

of A-C-bimodules.

ii)

$$\begin{aligned} M \otimes_B \left(\bigoplus_{j \in J} N_j \right) &\longrightarrow \bigoplus_{j \in J} (M \otimes_B N_j) \\ x \otimes (y_j) &\longmapsto (x \otimes y_j) \end{aligned}$$

of A-C-bimodules.

iii)

$$\begin{aligned} (M \otimes_B N) \otimes_C P &\longrightarrow M \otimes_B (N \otimes_C P) \\ (x \otimes y) \otimes z &\longmapsto x \otimes (y \otimes z) \end{aligned}$$

of A-B-bimodules.

iv)

$$\begin{aligned} A \otimes_A M &\longrightarrow M \\ a \otimes x &\longmapsto ax \\ M \otimes_B &\longrightarrow M \\ x \otimes b &\longmapsto xb \end{aligned}$$

of A-B-bimodules.

Proof. This is supposed to beexactly the same as [Fra18, 2.27]. □

Proposition 1.5.8. *Let A, B be k-algebras, and M_A, ${}_AN_B$ and P_b (bi)-modules. The map*

$$\hom_B(M \otimes_A N, P) \to \hom_A(M, \hom_B(N, P)), \ f \mapsto \Phi(f) \text{ where } \Phi(f)(x)(y) \mapsto f(x \otimes y),$$

for $x \in M$ and $y \in N$.
is a well-defined isomorphism of k-modules, natural in M, N, P.

Proof.

- $\varphi(f)(x)$ is right B-module homomorphism:

$$\Phi(f)(x)(yb) = f(x \otimes yb) = f((x \otimes y)b) = f(x \otimes y)b$$

- $\varphi(f)$ is right A-module homomorphism:

$$\begin{aligned} \Phi(f)(xa)(y) &= f(xa \otimes y) = f(a \otimes ay) \\ (\Phi(f)(x)a)(y) &= \Phi(f)(x)(ay) = f(a \otimes ay) \end{aligned}$$

- Φ is k-linear.

So Φ is a well-defined map of k-modules.

Φ has inverse: Let $g \in \hom_A(M, \hom_B(N, P))$. Define

$$\psi(g) : M \times N \to P,\ (x, y) \mapsto g(x)(y).$$

Then $\psi(g)$ is A-balanced, so it factors through the tensor product:

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[d, "\psi(g)"'] & M \otimes_A N \arrow[dl, "\exists! \hat{\psi}(g)\ k\text{-linear}"] \\ P & \end{tikzcd}$$

The map $\hat{\varphi}(g)$ is also as right B-module homomorphism, $g(x) \in \hom_B(N, P)$.

But the map

$$\hat{\psi} : \hom_A(M, \hom_B(N, P)) \to \hom_B(M \otimes N, P)$$

is the inverse of φ, as

- $\hat{\psi}(\Phi(f))(x \otimes y) = \Phi(f)(x)(y) = f(x \otimes y)$
- $\Phi(\hat{\psi}(g))(x)(y) = \hat{\psi}(g)(x \otimes y) = g(x)(y)$

for all $x \in M$ and $y \in N$. $\square$

Definition 2.0.1. A category $\mathcal{C}$ consists of:

- A class $\mathrm{Ob}(\mathcal{C})$, whose elements are called the **objects** of $\mathcal{C}$.
- For all $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathcal{C}(X, Y)$. An element of $\mathcal{C}(X, Y)$ is called a **morphism** from X to Y as is denoted by
$$f : X \to Y.$$
- For all $X, Y, Z \in \mathrm{Ob}\,(\mathcal{C})$, a map
$$\varphi(x, y) \times \varphi(y, z) \to \varphi(x, z),\ (f, g) \mapsto g \circ f.$$

These should satisfy:

- (L1): Associativity: For all $X, Y, Z, W \in \mathrm{Ob}\,(\mathcal{C})$, and morphisms
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W,$$
we have
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
- (L2): Identity: For all $X \in \mathrm{Ob}\,(\mathcal{C})$, there is a morphism $\mathrm{id}_X \in \mathcal{C}(X, X)$, such that for any object $Y \in \mathrm{Ob}\,(\mathcal{C})$:
$$f \circ \mathrm{id}_X = f \text{ and } \mathrm{id}_X \circ g = g \text{ for all } f : X \to Y, g : Y \to X$$
holds.

Remark 2.0.2. i) $\varphi(X, Y) = \emptyset$ can happen if $X \neq Y$

ii) id_X is unique, as $\mathrm{id}_X = \mathrm{id}_X \circ \mathrm{id}'_X = \mathrm{id}'_X$

Remark 2.0.3. We sometimes want to consider categories whose objects are all sets (with additional conditions). But this can cause logical problems. As a solution, we introduce so called universes. We will always fix a universe, such that sets are elements of this universe, and classes are subsets of this universe. Consider [Lan98, 1.6] for further reference.

End of Lecture 4

Example 2.0.4. i) The category $\mathcal{S}et$ of all sets, with

- $\mathrm{Ob}(\mathcal{S}et)$ are all sets in the given universe
- $\mathcal{S}et(X, Y) = \{\text{maps } f : X \to Y\}$

ii) The category $\mathcal{G}rp$ of groups, with group homomorphism as morphisms.

iii) Let A be a k-algebra. Let $A{-}\mathcal{M}od$ be the category of left A-modules, and $\mathcal{M}od{-}A$ the category of right A-modules.

iv) The category $\mathcal{T}op$ of topological spaces, with

- $\mathrm{Ob}(\mathcal{T}op)$ the set of all topological spaces,
- $\mathcal{T}op(X, Y) = \{f : X \to Y \mid f \text{ is continous}\}$

v) Let G be a group. Let $\mathcal{C}$ be the category defined as

- $\mathrm{Ob}(\mathcal{C}) = \{*\}$
- $\mathcal{C}(*, *) = G$, with composition defined as $h \circ g := hg$.

vi) Let Q be a quiver. Let $\mathcal{Q}_*$ be the **category of paths** of Q, defined as

- $\mathrm{Ob}(\mathcal{Q}_*) = Q_*$
- for $i, j \in Q_*$, let $\mathcal{Q}_*(i, j) := \{\text{paths } p \text{ in } Q \mid s(p) = i, t(p) = j\}$,
- composition is given by concatination of paths.

This is a category, as composition is associative, and $\mathrm{id}_i = \varepsilon_i$ (the lazy path at i).

Definition 2.0.5. Let $\mathcal{C}$ be a category. The **opposite category** $\mathcal{C}^{\mathrm{op}}$ is defined as

- $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$
- for all $x, y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, the morphisms are defined as $\mathcal{C}^{\mathrm{op}}(X, Y) := \mathcal{C}(Y, X)$
- for $f \in \mathcal{C}^{\mathrm{op}}(X, Y), y \in \mathcal{C}^{\mathrm{op}}(Y, Z)$, set
$$g \circ_{\mathcal{C}^{\mathrm{op}}} f := f \circ_{\mathcal{C}} g$$

2.1 Functors

Definition 2.1.1. Let $\mathcal{C}, \mathcal{D}$ be two categories. A **functor** $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ consists of the following:

- a map
$$\mathscr{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(D),\ X \mapsto F(X)$$
- for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, a map
$$\mathcal{C}(X, Y) \to \mathcal{C}(\mathscr{F}(X), \mathscr{F}(Y)),\ f \mapsto \mathscr{F}(f),$$

such that

- (F1): $\mathscr{F}(\mathrm{id}_X) = \mathrm{id}_{\mathscr{F}(X)}$
- (F2): for all sequences
$$X \xrightarrow{f} Y \xrightarrow{g} Z$$
in $\mathcal{C}$, the relation
$$\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f),$$

holds for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

Remark 2.1.2. What we call a functor is sometimes called a *covariant functor*. A *contravariant functor* is a (covariant) functor $\mathscr{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Remark 2.1.3. i) Let $\mathcal{C}$ be a category. The **identical functor** is given by
$$\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}, \begin{cases} x \mapsto x & \text{on objects} \\ f \mapsto f & \text{on morphism} \end{cases}$$

ii) If
$$\mathcal{C} \xrightarrow{\mathscr{F}} \mathcal{D} \xrightarrow{\mathscr{G}} \mathcal{E}$$
are two functors, then their composition is also a functor.

Example 2.1.4. i) Consider the category $\mathcal{S}et$, and let $P(X)$ denote the power set of a set X. Define
$$\mathscr{P}_* : \mathcal{S}et \to \mathcal{S}et, \begin{cases} x \mapsto P(X) \\ (X \xrightarrow{f} Y) \mapsto P_*(f) : P(X) \to P(Y), A \mapsto f(A) \end{cases}$$
which is a covariant functor, and
$$\mathscr{P}^* : \mathcal{S}et \to \mathcal{S}et, \begin{cases} x \mapsto P(X) \\ (X \xrightarrow{f} Y) \mapsto P^*(f) : P(Y) \to P(X), B \mapsto f^{-1}(B) \end{cases},$$
which is a contravariant functor.

ii) Consider the functors
$$-^* : k\text{--}\mathcal{A}lg \to \mathcal{G}rp, \begin{cases} A \mapsto A^\times \\ (A \xrightarrow{f} B) \mapsto A^* \xrightarrow{f^\times} B^\times \end{cases}$$
and
$$k[-] : \mathcal{G}rp \to k\text{--}\mathcal{A}lg, \begin{cases} G \mapsto k[G] \\ (G \xrightarrow{\varphi} H) \mapsto k[G] \xrightarrow{\varphi} k[H] \end{cases}$$

iii) The functor
$$\mathcal{G}rp \to \mathcal{S}et, \begin{cases} G \mapsto G \\ f \mapsto f \end{cases}$$
is called a **forgetful functor** . Other examples of forgetful functors are

- $\mathcal{T}op \to \mathcal{S}et$
- $A{-}\mathcal{M}od \to k{-}\mathcal{M}od$

Example 2.1.5. Let $\mathcal{C}$ be a category and $X \in \mathrm{Ob}(\mathcal{C})$ an object in $\mathcal{C}$. Consider

i)
$$H^X : \mathcal{C} \to \mathcal{S}et, \begin{cases} Y \mapsto \mathcal{C}(X,Y) \\ (Y \xrightarrow{f} Y') \mapsto H^X(f) : \mathcal{C}(X,Y) \to \mathcal{C}(X,Y'),\ g \mapsto f \circ g \end{cases}.$$

We also denote this as $H^X =: \mathcal{C}(X,-)$. This is a covariant functor.

ii)
$$H_X : \mathcal{C} \to \mathcal{S}et, \begin{cases} Z \mapsto \mathcal{C}(Z,X) \\ (Z \xrightarrow{f} Z') \mapsto H_X(f) : \mathcal{C}(Z',X) \to \mathcal{C}(Z,X),\ g \mapsto g \circ f \end{cases}.$$

We also denote this as $H_X =: \mathcal{C}(-,X)$. This is a contravariant functor.

Definition 2.1.6. Let $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ and consider the induced map

$$\mathcal{C}(X,Y) \to \mathcal{D}(\mathscr{F}(X), \mathscr{F}(Y)).$$

i) If this map is injective, then F is called **faithful**.

ii) If this map is surjective, then F is called **full**.

iii) F is **fully faithful**, if F is full and faithful.

iv) F is **dense** or **essentially surjective**, if for any $Y \in \mathcal{D}$, there is an object $X \in \mathcal{C}$, such that $\mathscr{F}(X) \cong Y$

2.2 Isomorphism

Definition 2.2.1. Let $\mathcal{C}$ be a category. A morphism $f : X \to Y$ in $\mathcal{C}$ is called an **isomorphism**, if there is a $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Remark 2.2.2. i) Identities are isomorphism.

ii) The morphism g (if it exists) is uniquely determined by f. We therefore call $g =: f^{-1}$.

iii) If $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is a functor and f an isomorphism in $\mathcal{C}$, then $\mathscr{F}(f)$ is an isomorphism in $\mathcal{D}$.

Example 2.2.3. i) In $\mathcal{S}et$, $\mathcal{G}rp$, $A{-}\mathcal{M}od$, the following are equivalent:

- f is an isomorphism
- f is bijective.

ii) In $\mathcal{T}op$, not all bijective maps are isomorphism.

iii) In $\mathcal{Q}_*$, the only isomorphisms are the lazy paths, because lengths of paths are additive.

2.3 Natural transformations

Definition 2.3.1. Let $\mathscr{F},\mathscr{G} : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors. A **natural transformation**

$$\eta : \mathscr{F} \to \mathscr{G}$$

is a family of morphisms

$$\{\eta_X\}_{X\in \mathrm{Ob}(\mathcal{C})} : \mathscr{F}X \to \mathscr{G}X$$

in $\mathcal{D}$, such that for all $X, Y \in \mathcal{C}$ and morphisms $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes

$$\begin{tikzcd} \mathscr{F}X \arrow[r, "\eta_X"] \arrow[d, "\mathscr{F}f"'] & \mathscr{G}X \arrow[d, "\mathscr{G}f"] \\ \mathscr{F}Y \arrow[r, "\eta_Y"'] & \mathscr{G}Y. \end{tikzcd}$$

Remark 2.3.2. i) For two natural transformations

$$\mathscr{F} \xrightarrow{\eta} \mathscr{G} \xrightarrow{\xi} \mathscr{H}, \ (\mathscr{F},\mathscr{G},\mathscr{H} : \mathcal{C} \to \mathcal{D}),$$

we can define the composition

$$\xi \circ \eta : \mathscr{F} \to \mathscr{H}$$

by

$$(\xi \circ \eta)_X \mathscr{F}X \to \mathscr{H}X, \ (\xi \circ \eta)_X := \xi_X \circ \eta_X.$$

ii) For $\mathscr{F} : \mathcal{C} \to \mathcal{D}$, we have **identical transformation**

$$\mathrm{id}_{\mathscr{F}} \text{ given by } (\mathrm{id}_{\mathscr{F}})_X := \mathrm{id}_{\mathscr{F}X}$$

the part about the natural transformations on the exe-sheets is still missing.

End of Lecture 5

Definition 2.3.3. Let $\mathscr{F},\mathscr{G} : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors. A natural transformation $\eta : \mathscr{F} \to \mathscr{G}$ is called a **natural isomorphism** if for all $x \in \mathcal{C}$, η_x is an isomorphism in $\mathcal{D}$.
η is a natural transformation if and only iff there is a natural transformation $\zeta : \mathscr{G} \to \mathscr{F}$, such that $\zeta \circ \eta = \mathrm{id}_{\mathscr{F}}$ and $\eta \circ \zeta = \mathrm{id}_{\mathscr{G}}$.
If η is a natural isomorphism, we write $\eta : F \xrightarrow{\cong} G$. If there is a natural transformation between two functors, we denote this by $\mathscr{F} \cong \mathscr{G}$.

Definition 2.3.4. A functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories**, if there is a functor $\mathscr{G} : \mathcal{D} \to \mathcal{C}$, such that

$$\mathscr{G} \circ \mathscr{F} \cong \mathrm{id}_{\mathcal{C}} \text{ and } \mathscr{F} \circ \mathscr{G} \cong \mathrm{id}_{\mathcal{D}}.$$

If for two categories $\mathcal{C}, \mathcal{D}$ an equivalence of categories $\mathcal{C} \to \mathcal{D}$ exists, we say that $\mathcal{C}$ and $\mathcal{D}$ are **equivalent**, and write $\mathcal{C} \simeq \mathcal{D}$.

Example 2.3.5. Some examples for equivalences of categories:

i) Let Q be a finite quiver. Then theorem 1.4.7 shows that
$$\mathcal{R}ep_k(Q) \simeq kQ - \mathcal{M}od.$$

Proof. For the functors
$$\mathscr{F} : \mathcal{R}ep_k(Q) \to kQ - \mathcal{M}od \text{ and } \mathscr{G} : kQ - \mathcal{M}od \to \mathcal{R}ep_k(Q)$$
constructed in remark 1.4.6 the relationships $\mathscr{G}\mathscr{F}(X) \cong X$ and $\mathscr{F}\mathscr{G}(M) \cong M$ hold for any representation X and kQ-modules M, by theorem 1.4.7. So it suffice to check naturallity:
Let $M, N \in kQ - \mathcal{M}od$ be two kQ-modules. Recall that as k-vector space,
$$\mathscr{F}\mathscr{G}(M) = \bigoplus_{i \in Q_0} \varepsilon_i M$$
where ε_i denotes the lazy path at $i \in Q_0$, and the isomorphism is given by
$$\varphi : \mathscr{F}\mathscr{G}(M) = \bigoplus_{i \in Q_0} \varepsilon_i M \longrightarrow M \ , \ (\varepsilon_i x_i)_{i \in Q_0} \mapsto \sum \varepsilon_i x_i.$$
Now let $\alpha \in kQ - \mathcal{M}od(M, N)$ be a homomorphism of left kQ-modules M, N. We need to show that the diagram
$$\begin{tikzcd}
\mathscr{F}\mathscr{G}(M) \arrow[d, equal] & \\
\bigoplus_{i \in Q_0} \varepsilon_i M \arrow[r, "\varphi"] \arrow[d, "\mathscr{F}\mathscr{G}(\alpha)"] & MM \arrow[d, "\alpha"'] \\
\bigoplus_{i \in Q_0} \varepsilon_i N \arrow[r, "\varphi"] \arrow[d, equal] & N \\
\mathscr{F}\mathscr{G}(N) &
\end{tikzcd}$$
commutes, i.e. that φ is actually a natural isomorphism:
$$\begin{aligned}
\alpha\left(\varphi(\varepsilon_i x_i)\right) &= \alpha\left(\sum_{i \in Q_0} \varepsilon_i x_i\right) \\
&= \sum_{i \in Q_0} \varepsilon_i \alpha(x_i),
\end{aligned}$$
as α is kQ-linear. Furthermore:
$$\begin{aligned}
\varphi\left(\mathscr{F}\mathscr{G}(M)\right) &= \varphi\left((\varepsilon_i \alpha(x_i)\right) \\
&= \sum_{i \in Q_0} \varepsilon_i \alpha(x_i)
\end{aligned}$$

So φ is indeed a natural transformation. The other map is also a natural transformation. This follows from the fact that it is an isomorphism of representations, as was shown in the proof of theorem 1.4.7. □

ii) Let G be a group. A **representation of** G is pair (V, ρ) consisting of a k-vector space V and a group homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

A morphism of (group)representations $f : (V, \rho) \to (W, \sigma)$ is a k-linear map $f : V \to W$, such that for all $g \in G$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\sigma(g)"] \\ V \arrow[r, "f"] & W \end{tikzcd} .$$

We will show maybe on the next sheet, who knows...that we can define an equivalence of categories

$$\mathcal{R}ep_k(G) \simeq k[G] - \mathcal{M}od.$$

iii) The category of k-$\mathcal{A}lg$ is equivalent to the category $\mathcal{C}$ of pairs (A, φ), where A is a ring and $\varphi : k \to Z(A)$ is a ring homomorphism and morphisms correspond to ring homomorphism $f : A \to B$, such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \\ Z(A) \arrow[u, hook] & & Z(B) \arrow[u, hook] \\ & k \arrow[ul, "\varphi_A"] \arrow[ur, "\varphi_B"'] & \end{tikzcd} .$$

show that this is actually all natural and well-defined

iv) Let A be a k-algebra. The category $A - \mathcal{M}od$ of left A-modules is equivalent the category $\mathcal{D}$ of pairs (V, ρ) of k-vector spaces V and homomorphisms of k-algebras:

$$\varphi : A \to \mathrm{End}_k(V).$$

Morphisms $(V, \varphi) \to (W, \psi)$ in $\mathcal{D}$ are given by k-linear maps $f : V \to W$, such that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi(a)"'] & W \arrow[d, "\psi"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

commutes for all $a \in A$.

2.4 Functor Categories

Definition 2.4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Define the **functor category** $\mathcal{F}un(\mathcal{C}, \mathcal{D})$ by

- objects are all functors $\mathcal{C} \to \mathcal{D}$; i.e. $\mathrm{Ob}(\mathcal{F}un(\mathcal{C}, \mathcal{D})) := \{\mathscr{F} : \mathcal{C} \to \mathcal{D} \mid \mathscr{F} \text{ is a functor}\}$
- morphism between functors are natural transformations; i.e. for $\mathscr{F}, \mathscr{G} : \mathcal{C} \rightrightarrows \mathcal{D}$, set $\mathcal{F}un(\mathscr{F}, \mathscr{G})(\mathcal{C}, \mathcal{D}) := \{\eta : \mathscr{F} \to \mathscr{G} \mid \eta \text{ is a natural transformation}\}$
- Composition of morphisms is given by composition of natural transformations.

Remark 2.4.2. We are running into set-theoretic issues again. If $\mathcal{C}$ and $\mathcal{D}$ are categories in a fixed universe U (i.e. $\mathrm{Ob}(\mathcal{C}), \mathrm{Ob}(\mathcal{D}) \subseteq U$) then $\mathrm{Ob}(\mathcal{F}un(\mathcal{C}, \mathcal{D}))$ might not be a subset of U any longer. As a solution, we choose another universe V, s.t. $U \in V$. Then $\mathrm{Ob}(\mathcal{F}un(\mathcal{C}, \mathcal{D})) \subseteq V$ and $\mathcal{F}un(\mathcal{C}, \mathcal{D})$ is a category in V.

Example 2.4.3. Let Q be a quiver, k a field. Consider the functor category $\mathcal{F}un(\mathcal{Q}_*, k{-}\mathcal{M}od)$. For $\mathscr{V} \in \mathrm{Ob}(\mathcal{F}un(\mathcal{Q}_*, k - \mathcal{M}od))$, $\mathscr{V}$ is a functor

$$\mathscr{V} : \mathcal{Q}_* \to k{-}\mathcal{M}od, \begin{cases} \mathrm{Ob}(\mathcal{Q}_*) = Q_* \ni i \mapsto V(i) & \text{a vector space} \\ \mathcal{Q}_*(i,j) \ni p \to V(p) : V(i) \to V(p) & \text{a } k\text{-linear map} \end{cases}$$

We now have a forgetful functor

$$\mathscr{F} : \mathcal{F}un(\mathcal{Q}_*, k{-}\mathcal{M}od) \to \mathcal{R}ep_k(Q),$$

forgetting all paths of length > 1.

Conversley, let X be a representation of Q over k. Define a functor

$$\mathscr{G}X : \mathcal{Q}_* \to k{-}\mathcal{M}od, \begin{cases} i \mapsto X_i \\ p = \alpha_\ell \circ \ldots \circ X_{\alpha_1} \end{cases} .$$

This yields a functor

$$\mathscr{G} : \mathcal{R}ep_k(Q) \to \mathcal{F}un(\mathcal{Q}_*, k - \mathcal{M}od)$$

We see that

$$\mathscr{G} \circ \mathscr{F} \cong \mathrm{id}_{\mathcal{F}un(\mathcal{Q}_*, k-\mathcal{M}od)} \text{ and } \mathscr{F} \circ \mathscr{G} \cong \mathrm{id}_{\mathcal{R}ep_k(Q)}$$

.

Definition 2.4.4. Let $\mathcal{C}, \mathcal{D}$ be two categories, and $X \in \mathcal{C}$ an object in $\mathcal{C}$. Define

$$\mathrm{ev}_X : \mathcal{F}un(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$

by

- $\mathrm{ev}_X(\mathscr{F}) := \mathscr{F}(X)$

- $\mathrm{ev}_X(\mathscr{F}) \xrightarrow{\eta} \mathscr{G} := \eta_X : \mathscr{F}(X) \to \mathscr{G}(X)$.

ev_X is called the **evaluation at** X.

Remark 2.4.5. Let $\mathcal{C}, \mathcal{D}$ be categories, and $X \in \mathcal{C}$ an object. Then ev_X is indeed a functor, as the associativity of composition of natural transformations is inherited from the associativity of composition in $\mathcal{D}$.

Moreover, for $f : X \to Y$ a morphism in $\mathcal{C}$, we can define a natural transformation between the functors

$$\mathrm{ev}_f : \mathrm{ev}_X \to \mathrm{ev}_Y,$$

which is, for a functor $\mathscr{F}$, just a map $\mathrm{ev}_X(\mathscr{F}) \to \mathrm{ev}_Y(\mathscr{F})$; by considering

$$\begin{tikzcd}
\mathrm{ev}_X(\mathscr{F}) \arrow[r, "(\mathrm{ev}_f)_F"] \arrow[d, equal] & \mathrm{ev}_y(F) \arrow[d, equal] \\
\mathscr{F}X \arrow[r, "Ff"] & \mathscr{F}Y
\end{tikzcd} \quad \text{and setting } (\mathrm{ev}_f)_F := \mathscr{F}f,$$

i.e. $(\mathrm{ev}_f)_{\mathscr{F}}$ is induced the maps which are induced by $\mathscr{F}$. To show that the (ev_f) define indeed a natural transformation of functors

$$\begin{tikzcd}
\mathcal{F}un(\mathcal{C}, \mathcal{D}) \arrow[r, bend left, "\mathrm{ev}_X"] \arrow[r, bend right, "\mathrm{ev}_Y"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D}
\end{tikzcd},$$

we need to show that for all maps (i.e. natural transformations) $\eta : \mathscr{F} \to \mathscr{G}$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{ev}_X(\mathscr{F}) \arrow[r, "\mathrm{ev}_X(\eta)"] \arrow[d, "(\mathrm{ev}_f)_{\mathscr{F}}"'] & \mathrm{ev}_X(\mathscr{G}) \arrow[d, "(\mathrm{ev}_f)_{\mathscr{G}}"] \\
\mathrm{ev}_Y(\mathscr{F}) \arrow[r, "\mathrm{ev}_Y \eta"] & \mathrm{ev}_Y(\mathscr{G})
\end{tikzcd}.$$

But this is just inherited, in following way:
Consider the extended diagram:

$$\begin{tikzcd}
\mathscr{F}(X) \arrow[r, "\eta_X"] \arrow[d, equal] \arrow[ddd, bend right=60, "\mathscr{F}(f)"'] & \mathscr{G}(X) \arrow[d, equal] \arrow[ddd, bend left=60, "\mathscr{G}(f)"] \\
\mathrm{ev}_X(\mathscr{F}) \arrow[r, "\mathrm{ev}_X(\eta)"] \arrow[d, "(\mathrm{ev}_f)_{\mathscr{F}}"] & \mathrm{ev}_X(\mathscr{G}) \arrow[d, "(\mathrm{ev}_f)_{\mathscr{G}}"'] \\
\mathrm{ev}_Y(\mathscr{F}) \arrow[r, "\mathrm{ev}_Y \eta"] \arrow[d, equal] & \mathrm{ev}_Y(\mathscr{G}) \arrow[d, equal] \\
\mathscr{F}(Y) \arrow[r, "\eta_Y"'] & \mathscr{G}(Y)
\end{tikzcd}.$$

As η is a natural transformation, the outer diagram commutes. But this already implies that the inner one does as well.

This enables us to define another functor:

$$\mathrm{ev} : \mathcal{C} \to \mathcal{F}un\left(\mathcal{F}un\left(\mathcal{C}, \mathcal{D}\right), \mathcal{D}\right), \begin{cases} X \mapsto \mathrm{ev}_X \\ f \mapsto \mathrm{ev}_f \end{cases}$$

2.5 Representable functors

We now consider functors of the form

$$\mathcal{C} \to \mathcal{S}et \text{ and } \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et,$$

for an arbitrary category $\mathcal{C}$.

Lemma 2.5.1 (Yoneda)**.** *Let $X \in Ob(\mathcal{C}) = Ob(\mathcal{C}^{\mathrm{op}})$ be an object of $\mathcal{C}$.*

i) Let $\mathscr{F} : \mathcal{C} \to \mathcal{S}et$ be a (covariant) functor. The map

$$Y^{\mathscr{F},X} : \quad \begin{array}{ccc} \left(\mathcal{F}un(\mathcal{C}, \mathcal{S}et)\right)\left(h^X, \mathscr{F}\right) & \longrightarrow & \mathscr{F}(X) \\ \left(\eta : h^X \to \mathscr{F}\right) & \longmapsto & \eta_X(\mathrm{id}_X) \end{array},$$

is a bijection, where

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "h^X"] \arrow[r, bend right, "\mathscr{F}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{S}et \end{tikzcd},$$

is a natural transformation, and

$$\eta_X : \mathcal{C}(X, X) = h^X(X) \to F(X)$$

is just a map.

ii) Let $\mathscr{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ be a (contravariant) functor. The map:

$$Y_{\mathscr{G},X} : \quad \begin{array}{ccc} \left(\mathcal{F}un(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)\right)\left(h_X, \mathscr{G}\right) & \longrightarrow & \mathscr{G}(X) \\ \left(\zeta : h_X \to \mathscr{G}\right) & \longmapsto & \zeta_X(\mathrm{id}_X) \end{array},$$

is a bijection.

Proof. i) Assume that ii) holds, then this follows, as $\mathcal{C} = (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ and

$$h^X_{\mathcal{C}} = \mathcal{C}(-, X) = \mathcal{C}^{\mathrm{op}}(X, -) = h_X^{\mathcal{C}^{\mathrm{op}}}.$$

ii) $Y_{\mathscr{G},X}$ is injective: Let $\xi, \eta \in \left(\mathcal{F}un(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)\right)\left(h_X, \mathscr{G}\right)$ be two natural transformations

$$\xi, \eta : \begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, bend left, "h_X"] \arrow[r, bend right, "\mathscr{G}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{S}et \end{tikzcd}$$

and suppose that

$$\xi_X(\mathrm{id}_X) = \eta_X(\mathrm{id}_X).$$

We need to show that this implies $\xi = \eta$, i.e. $\xi_Y = \eta_Y$ for all $Y \in \mathrm{Ob}\,(\mathcal{C})$. As these are maps of sets, it suffices to show

$$\xi_Y(f) = \eta_Y(f) \text{ for all } f \in h_X(Y) = \mathcal{C}(X, Y).$$

As ξ, η are natural transformations, the diagrams

$$\begin{tikzcd} \mathcal{C}(X,X) \arrow[r, "\eta_X"] \arrow[d, "h_X(f)"'] & \mathscr{G}(X) \arrow[d, "\mathscr{G}(f)"] \\ \mathcal{C}(X,Y) \arrow[r, "\eta_Y"] & \mathscr{G}(Y) \end{tikzcd} \tag{D1}$$

and

$$\begin{tikzcd} h_X(X) \arrow[d, equal] & \\ \mathcal{C}(X,X) \arrow[r, "\xi_X"] \arrow[d, "h_X(f)"'] & \mathscr{G}(X) \arrow[d, "\mathscr{G}(f)"] \\ \mathcal{C}(X,Y) \arrow[r, "\xi_Y"] \arrow[d, equal] & \mathscr{G}(Y) \\ h_X(Y) & \end{tikzcd} \tag{D2}$$

commute. This implies

$$\begin{array}{l} \mathscr{G}(f)(\eta_X(\mathrm{id}_X)) \overset{(D1)}{=} \eta_Y(h_X(f)(\mathrm{id}_X)) = \eta_Y(f) \\ \qquad \| \\ \mathscr{G}(f)(\xi_X(\mathrm{id}_X)) \overset{(D2)}{=} \xi_Y(h_X(f)(\mathrm{id}_X)) = \xi_Y(f), \end{array}$$

so $Y_{\mathscr{G},X}$ is injective.

$Y_{\mathscr{G},X}$ is surjective: Let $z \in \mathscr{G}(X)$ be arbitrary. We need to find a natural transformation

$$\zeta : \begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, bend left, "h_X"] \arrow[r, bend right, "\mathscr{G}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{S}et \end{tikzcd},$$

such that $\zeta_X(\mathrm{id}_X) = z$. Define for $y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$ a map

$$\zeta_Y : \mathcal{C}(X, Y) = h_X(Y) \to \mathscr{G}(Y),\ f \mapsto \mathscr{G}f(z).$$

Show that ζ is indeed a natural transformation: Let $g : Y \to Y'$ be a morphism in $\mathcal{C}$, i.e. $g \in \mathcal{C}^{\mathrm{op}}(Y', Y)$. We have to show that

$$\begin{tikzcd}
h_X(Y') \arrow[d, equal] & \\
\mathcal{C}(Y', X) \arrow[r, "zeta_{Y'}"] \arrow[d, "h_X(f)"'] & \mathscr{G}(Y') \arrow[d, "\mathscr{G}(g)"] \\
\mathcal{C}(Y, X) \arrow[r, "\zeta_Y"] \arrow[d, equal] & \mathscr{G}(Y) \\
h_X(Y) &
\end{tikzcd}$$

commutes.
Let $u \in \mathcal{C}(Y', X)$. Then

$$\begin{aligned}
\mathscr{G}(g)(\zeta_{Y'}(u)) &= \mathscr{G}(g)(\mathscr{G}(u)(z)) \\
&= \mathscr{G}(u \circ g)(z)
\end{aligned}$$

and

$$\begin{aligned}
\zeta_Y((h_x(g))(u)) &= \zeta_Y(u \circ g) \\
&= \mathscr{G}(u \circ g)(z)
\end{aligned}$$

Hence ζ defines a natural transformation, and

$$\zeta_X(\mathrm{id}_X) = (\mathscr{G}(\mathrm{id}_x))(z) = (\mathrm{id}_(\mathscr{G}X))(z) = z$$

$\square$

End of Lecture 6

A.0 Sheet 0

Definition A.0.1. Let X be any set, and k any commutative ring with unit. Define the **free algebra** generated by X:

i) As k-module, set $k\langle X\rangle$ as the free k-module generated by X.

ii) Define the multiplication of two words as the concatination.

Then $k\langle X\rangle$ satisfies the following universal property: Let B be any k-algebra, and $f : X \to B$ a homomorphism of sets. There there is a unique homomorphism of k-algebras $k\langle X\rangle$, such that the following diagram commutes:

$$\begin{tikzcd} k\langle X\rangle \arrow[r, dashed, "\exists! f"] & B \\ X \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd} .$$

Proposition A.0.2. *Consider the forgetful functor*

$$\mathscr{F} : k\text{-}\mathcal{A}\} \to \mathcal{S},\ B \mapsto B$$

and

$$\mathscr{K}\langle -\rangle : \mathcal{S} \to k\text{-}\mathcal{A}\},\ X \mapsto k\langle X\rangle.$$

Then for all sets X and k-algebras,

$$\hom_{\mathcal{S}}(X, F(B)) \cong \hom_{k\text{-}\mathcal{A}\}}(k\langle X\rangle, B)$$

holds.

We say that F is **right-adjoint** to $\mathscr{K}\langle -\rangle$.

Problem A.0.1. Consider the k-algebra

$$A := k\langle x, y\rangle/(\langle xy - yx - 1\rangle)$$

over a fiel k with $\operatorname{char} k = 0$. Show that there are no non-zero, finite-dimensional representations of A.

Definition A.0.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A covariant functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is **faithful**, if for all objects A, B of $\mathcal{C}$, the induced function of sets

$$\hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(\mathscr{F}(A), \mathscr{F}(B))$$

is injective; it is **full**, if this function is surjective for all objects A, B of $\mathcal{C}$.

Definition A.0.4. A covariant functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories**, if it is fully faithful (i.e. bijective on hom-sets) and **essentially surjective**, i,e, for every object Y of $\mathcal{D}$, there is an object X of $\mathcal{C}$ such that $\mathscr{F}(X) \cong Y$.

Some people write natural transformations in the follwing way:

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

A.1 Sheet 1

Solution A.1.1. i) φ_m is k-linear:

- $\varphi(a+b)x = (a+b)x \overset{(L1)}{=} ax + bx = \varphi(a)x + \varphi(b)x$
- $\varphi(\lambda a) = (\lambda a)x \overset{(L5)}{=} \lambda(ax) = \lambda\varphi(a)x.$

φ_m is ring homomorphism:

- $\varphi(ab) = (ab)x \overset{(L3)}{=} a(bx) = \varphi(a)\varphi(b)x$
- $\varphi(1_A) \overset{(L4)}{=} (1_A)x = x.$

As these relations hold for all x, the assertion follows.

ii) V_φ is already a k-module.

- (L1) $a(x+y) = (\varphi(a))(x+y) \overset{\varphi \in \mathrm{End}_k(V)}{=} (\varphi(a))(x) + (\varphi(a))(y) = ax + ay$
- (L2) $(a+b)x = (\varphi(a+b))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\varphi(a)+\varphi(b))x = \varphi(a)x + \varphi(b)x = ax + bx$
- (L3) $(ab)x = (\varphi(ab))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\varphi(a)\varphi(b))x = a(bx)$
- (L4) $1_a x = (\varphi(1_a))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} \mathrm{id}\, x = x$
- (L5) $(\lambda a)(x) = ((\varphi(\lambda a))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\lambda\varphi(a))x = \lambda(ax)$ and $a(\lambda x) = (\varphi(a)(\lambda x)) \overset{\varphi(a) \text{ endo of } k-\text{module}}{=} \lambda(\varphi(a))x = \lambda(ax),$

for all $a, b \in A, x, y \in V$ and $\lambda \in k$.

iii) We regard V and W as A-modules in the sense of part ii). Assume that $\psi(a) \circ f = f \circ \varphi(a)$ $(*)$ for all $a \in A$. Then

$$f(ax) = f((\varphi(a))x) \overset{*}{=} (\psi(a))(f(x)) = af(x),$$

for all $x \in X$. Hence f is an A-module homomorphism.
Assume that f is a a-module homomorphism, then

$$(\psi(a))(f(x)) = a(f(x)) = f(ax) = f((\varphi(a))x)$$

for all $x \in V$. Hence $\psi(a) \circ f - f \circ \varphi(a) = 0$ and so $(*)$ holds.

Solution A.1.2. i) Assume that I is a non-zero ideal of A. Let $a = (a)_{ij} \neq 0$ be an arbitrary matrix in I. Then there exist permutation matrices $\sigma, \pi \in \mathrm{GL}_n(K)$, such that $(\sigma a \pi)_{11} \neq 0$, which is in I, as I is a two-sided ideal. So without loss of generality, suppose $a_{11} \neq 0$.
Define

$$b \in \mathrm{M}_n(k), (b)_{ij} := \begin{cases} 1, & \text{if } i = j = 1 \\ 0, & \text{else} \end{cases}$$

and E_n as the identity of $\mathrm{M}_n(k)$. Then we get

$$\left(\frac{1}{a_{11}} E_n \right) \cdot b \cdot a \cdot b = b.$$

By repeatetly using permutation matrixes, it is possible to write any matrix as sum of products of a, b and permutation matrices on the left- and right. As I is a two-sided ideal, all of these combinations are in I as well. Hence a generates all of A, and $I = A$.

ii) Consider A as a k-vector space, then $\dim_K A = n^2$. Let M be any left A-module. As shown in task 3, there is a homomorphism of k-algebras

$$\varphi : A \to \mathrm{End}_k(M), \ a \mapsto a : (x \mapsto ax),$$

which is in particular a homomorphism of k-vector spaces. The kernel of φ is a two-sided ideal of A, as

$$a0x = 0ax = 0$$

for all $a \in A$ and $x \in M$.
Now i) implies that $\ker \varphi$ is either zero or $\ker \varphi = A$. But since $\varphi(E_n) = \mathrm{id}_M$, the latter one is not possible. Hence φ is injective, and in particular $\dim A \leq \dim \mathrm{End}_k(V)$, so $n \leq m$.

Proposition A.1.1. *Let k be a field, $k[X]$ the polynomial ring and $p \in k[X]$ a polynomial with $\deg p = n$. Then*

$$k[X]/(p)$$

is a n-dimensional k vector space, and a basis is given by

$$\{1, x, ..., x^{n-1}\}.$$

The following propositions are taken from [Alu09].
Let R be any commutative ring.

Proposition A.1.2. *Let $I_1, ..., I_k$ be ideals of R such that $I_i + I_j = (1)$ for all $i \neq j$. Then the natural homomorphism*

$$\varphi : R \to R/I_1 \times \ldots \times R/I_k$$

is surjective and induces an isomorphism

$$\frac{R}{I_1 \ldots I_k} \to R/I_1 \times \ldots \times R/I_k$$

Corollary A.1.3 (Chinese remainder theorem)**.** *Let R be a PID and $a_1, ..., a_k \in R$ be elements such that $\gcd(a_i, a_j) = 1$ for all $i \neq j$. Let $a = a_1 \dots a_k$. Then the function*

$$\varphi : R/(a) \to R/(a_1) \times \dots \times R/(a_k).$$

Proposition A.1.4 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category, X an object of $\mathcal{C}$ and consider the contravariant functor*

$$h_X := \hom_{\mathcal{C}}(-, X).$$

Then for every contravariant functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$, there is a bijection between the set of natural transformations $h_x \rightsquigarrow \mathscr{F}$ and (X).

Definition A.1.5 ([ASS06])**.** The (Jacobson) **radical** $\operatorname{rad} A$ of a K-algebra A is the intersection of all maximal right ideals in A. It is the same as the intersetion of all left-sided maximal right ideals in A. Furthermore, $\operatorname{rad} A$ is a two-sided ideal.

Definition A.1.6. Let $f, g : X \to Y$ be morphisms in a category $\mathcal{C}$. Then a morphism $e : E \to X$ is called **equalizer** of f and g if $f \circ e = g \circ e$ and for all other morphisms $o : O \to X$, such that $f \circ o = g \circ o$, there is a unique morphis $O \to E$, such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "e"] & X \arrow[r, shift left] \arrow[r, shift right] & X \\
O \arrow[u, dashed, "\exists!"] \arrow[ur, "o"] & &
\end{tikzcd}.$$

Proposition A.1.7. *Equalizers exists in abelian categories.*

Task 1

See this as a functors:

$$\begin{tikzcd}
k\text{–}\mathcal{A}lg \arrow[r, shift left, "\mathscr{V}"] & \mathcal{G}rp \arrow[l, shift left, "\mathscr{A}"]
\end{tikzcd}$$

Grp. alg. construction $\mathscr{A}$ is left-adjoint to group of units construction $\mathscr{V}$.

i) Show that there is a natural isomorphism

$$k\text{–}\mathcal{A}lg(\mathscr{A}(G), A) \cong \mathcal{G}rp(G, \mathscr{V}(A))$$

for all groups G and k-algebras A.

Task 2

This quiver is called the **linear oriented quiver**. Define

$$\varphi : KQ \to \mathrm{L}_n(k)$$

as linear extension of the k-linear map

$$Q_* \to \mathrm{L}_n(k),\ p_{ij} \mapsto (E_{ij})kl := \begin{cases} 1 & \text{if } k = i, l = j \\ 0 & \text{otherwise} \end{cases}$$

This is indeed a homomorphism of k-algebras, which sends basis vectors to basis vectors.

Task 3

Let $\mathcal{C}, \mathcal{D}$ be two k-**linear** categories , i.e. $\mathcal{C}(X,Y)$ has the structure of a k-vector space and composition is bilinear. We say that $\mathcal{C}$ is **equivalent** to $\mathcal{D}$ ($\mathcal{C} \simeq \mathcal{D}$) if there are k-**linear** functors

$$\mathscr{F} : \mathcal{C} \to \mathcal{D} \text{ and } \mathscr{G} : \mathcal{D} \to \mathcal{C}$$

(i.e. functors that induce k-linear maps $\mathcal{C}(X,Y) \to \mathcal{D}(\mathscr{F}(X), \mathscr{G}(Y))$, and for $\mathscr{G}$ analogous), such that there are natural isomorphisms

$$\mathscr{G}\mathscr{F} \simeq \mathrm{Id}_{\mathcal{C}} \text{ and } \mathscr{F}\mathscr{G} \simeq \mathrm{Id}_{\mathcal{D}}$$

Theorem A.1.8. *Let $\mathcal{C}, \mathcal{D}$ be k-linear categories. $\mathcal{C} \simeq \mathcal{D}$ if and only if there is a fully faithful, k-linear and dense functor*

$$\mathscr{F} : \mathcal{C} \to \mathcal{D}.$$

Remark A.1.9. The proof is supposed to only invoke the Axiom of Choice, and should work for general hom-Sets.

This task basically shows that there is an equivalence of categories

$$\mathcal{R}ep(A) \simeq k{-}\mathcal{M}od(A),$$

where

$$\mathcal{R}ep(A) := \{(V, \varphi) \mid V \text{ a } k\text{−vector-space}, \varphi : A \to \mathrm{End}_k(V) \text{ a algebra-homomorphism}\}$$

Calvin highly recommends the book [ASS06].

A.2 Sheet 2

Solution A.2.1. Consider the two representations

$$k \underset{b}{\overset{a}{\rightrightarrows}} k \quad \text{and} \quad k \underset{d}{\overset{c}{\rightrightarrows}} k$$

with $a, b, c, d \in k$. Morphism of representations are in this case k-linear maps $k \to k$, i.e. multiplication by elements $\mu, \nu \in k$, such that the diagrams

$$\begin{tikzcd} k \arrow[r, "a"] \arrow[d, "\mu"'] & k \arrow[d, "\nu"] \\ k \arrow[r, "c"'] & k \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k \arrow[r, "b"] \arrow[d, "\mu"'] & k \arrow[d, "\nu"] \\ k \arrow[r, "d"'] & k \end{tikzcd}$$

commute. This is the case if (μ, ν) satisfies the system of equations

$$\underbrace{\begin{pmatrix} c & -a \\ d & -b \end{pmatrix}}_{=:A} \cdot \begin{pmatrix} \mu \\ \nu \end{pmatrix} = 0 \Longleftrightarrow \begin{pmatrix} \mu \\ \nu \end{pmatrix} \in \ker A$$

There are several cases to consider:

- $\det A = ad - bc \neq 0$: As A is invertible in this case, $\ker A$ is trivial, and hence
$$\hom(X_{(a,b)}, X_{(c,d)}) = 0$$
- $\det A = 0; a = b = c = 0$. As $A = 0$, $\ker A = k^2$ holds, and hence
$$\hom(X_{(a,b)}, X_{(c,d)}) = k^2$$
- $\det A = 0; b \neq 0$ In this case, $c = ad/b$ holds.
  - $a = 0, d = 0$:
$$A = \begin{pmatrix} 0 & 0 \\ 0 & -b \end{pmatrix} \implies \ker A = \operatorname{Lin} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
  - $a = 0, d \neq 0$
$$A = \begin{pmatrix} 0 & 0 \\ d & -b \end{pmatrix} \implies \ker A = \operatorname{Lin} \begin{pmatrix} b/d \\ 1 \end{pmatrix}$$
- $\det A = 0, b = 0$: Consider the cases:
  - $c, d \neq 0, a = 0$:
$$A = \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \implies \ker A = \operatorname{Lin} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
  - $c \neq 0, a, d = 0$...

Solution A.2.2. i) A representation of V consits of a vector space V together with an endomorphism $f \in \operatorname{End}_k V$:

$$X := \begin{tikzcd} V \arrow[loop above, "f"] \end{tikzcd} .$$

A decomposition of X into subrepresentations would correspond to a decomposition
$$V = V_1 \oplus V_2$$
with subspaces V_1 and V_2 of V, such that both V_1 and V_2 are f-invariant. Now, let X be any representation of Q. Assume first that $n := \dim_k V < \infty$, and f is any . As k is algebraically closed, there is a unique (up to permutation) basis B of V given by a disjoint unions of Jordan-Chains
$$B = \bigcup_{\lambda \in k} \bigcup_{i \in I_\lambda} J(\lambda, \ell_i) \text{ where } \sum_{\lambda \in k} \sum_{\lambda \in I_\lambda k} \ell_i = n,$$
I_λ are finite index sets, unequal to zero for only finitely many $\lambda \in k$, and $J(\lambda, \ell_i)$ are Jordan-Chains of f for the eigenvalue λ with length ℓ_i. This basis induces a unique direct sum decomposition
$$V = \bigoplus_{\lambda \in k} \bigoplus_{i \in I_\lambda} \operatorname{Lin} J(\lambda, \ell_i).$$
By construction of the Jordan-Chains, $J(n, \lambda_i)$ can not be decomposed for

Solution A.2.4. i) R is a k-vector space,where the addition and scalar multiplication are defined component-wise. This gives R the structure of a k-vector space, as M, N and X are in particual k-vector spaces.
Consider now the map

$$R \times R \to R, \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \mapsto \begin{pmatrix} aa' & ax' + xb' \\ 0 & bb' \end{pmatrix},$$

where the operation $ax' + xb'$ is well-defined, as X is an A-B-bimodule. This makes R into a ring, as:

- the unit is given by $\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$:

$$\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$$

- the multiplication is associative, as

$$\begin{aligned} \begin{pmatrix} a'' & x'' \\ 0 & b'' \end{pmatrix} \cdot \left(\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right) &= \begin{pmatrix} a'' & x'' \\ 0 & b'' \end{pmatrix} \cdot \begin{pmatrix} a'a & a'x + x'b \\ 0 & b'b \end{pmatrix} \\ &= \big(a''(a'a)\big) \ldots \end{aligned}$$

- the distributivity holds, as

Extension and restriction of scalars

We do some recap from Algebra 1 (cf. [AM94, 27f.]). For this, we go back to the commutative case: In the following, A, B denote commutative, untial rings.

Proposition A.2.1. *Let $A \xrightarrow{f} B$ be a ring homomorphism and N a B-module. Then N has a A-module structure, given by*

$$A \times N \mapsto N, \ (a, n) \mapsto f(a)n.$$

Proof.

- The addition on N_A is the same as the addition of N_B.
- Associativity: $(ab)n = f(ab)n = (f(a)f(b))n = f(a)(f(b)n)$, as N is B-module
- Unit acts as unit: $1_A n = f(1_A)n = 1_B n = n$, as f is homomorphism of rings.
- Distributivity: $(a + b)n = f(a + b)n = (f(a) + f(b))n = f(a)n + f(b)n$ and $a(n + n') = f(a)(n + n') = f(a)n + f(a)n'$

□

This way of obtaining a A-module structure on N_B is called **restriction of scalars**. In particular, f defines a A-module structure on B in this way.

Proposition A.2.2. *Let M_A be an A-module. Then*

$$M_B := B \otimes_A M$$

carries a B-module structure, and

$$b(b' \otimes x) = (bb') \otimes x$$

holds for this B-module. We say that the B-module M_B was obtained from M by ***extension of scalars***

[AM94] M.F. Atiyah and I.G. MacDonald. **Introduction To Commutative Algebra**. Addison-Wesley series in mathematics. Avalon Publishing, 1994.

[Lan98] S.M. Lane. **Categories for the Working Mathematician**. Graduate Texts in Mathematics. Springer New York, 1998.

[ASS06] I. Assem, D. Simson, and A. Skowronski. **Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory**. Elements of the Representation Theory of Associative Algebras. Cambridge University Press, 2006.

[Alu09] Paolo Aluffi. **Algebra: chapter 0**. Vol. 104. American Mathematical Soc., 2009.

[Fra18] Dr. Hans Franzen. **Algebra 1, Lecture Notes**. https://lkempf.github.io/AlgebraStroppel/algebraI.pdf. Accessed: 2018-10-19. 2018.