

Foundations of Representation Theory

Lecture Notes in the Winter Term 2018/19

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INTRODUCTION

These are my personal lecture notes for the lecture *Foundations of Representation Theory* held by Dr. Hans Franzen at the University of Bonn in the winter term 2018/19.

I try to update them on my website, <https://pankratius.github.io>.
The authors labels his own comments and additions in purple.

The book [Alu09] is used by the author for further references, and highly recommended.

1. ALGEBRAS AND MODULES

Conventions: In this lecture, rings will always be unital, and ring homomorphisms f always fulfill $f(1) = 1$. Rings do *not* have to be commutative.

1.1 Algebras - Basics

Let k be a commutative ring.

Definition 1.1.1. A k -**algebra** A is a ring A , together with a structure of a k -module on A , such that

$$\text{for all } a, b \in A, \lambda \in K : (\lambda a)b = a(\lambda b) = \lambda(ab) \quad (*)$$

Definition 1.1.2. Let A, B be k -algebras. A **homomorphism of algebras** is a map $f : A \rightarrow B$ that is both k -linear and a ring homomorphism.

Remark 1.1.3. Let A be a ring. Define

$$Z(A) := \{a \in A \mid \forall b \in A : ab = ba\},$$

which is a commutative subring and is called the **center** of A .

Remark 1.1.4. Let A be a ring. Giving a k -algebra structure on A is the same as giving a ring homomorphism $k \rightarrow Z(A)$. More precisley:

- i) If A is a k -algebra, then $p : k \rightarrow A, \lambda \mapsto \lambda 1$ satisfies $\text{Im } p \subseteq Z(A)$ and is a ring homomorphism. (the first statement follows from $(*)$ and the second one from the fact that A has a k -module structure).
- ii) Let $\varphi : k \rightarrow Z(A)$ be a ring homomorphism. Define

$$\lambda a := \varphi(\lambda)a,$$

for all $\lambda \in k$. This defines a k -algebra structure on A ($(*)$ holds since $\text{Im}(\varphi) \subseteq Z(A)$).

- iii) Let A, B be k -algebras and $f : A \rightarrow B$ a homomorphism of rings. Then f is a homomorphism of k -algebras if and only if the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow & & \uparrow \\ Z(A) & & Z(B) \\ \swarrow \phi_A & & \searrow \phi_B \\ & k & \end{array} .$$

Example 1.1.5. i) Let V be a k -module. Then $\text{End}_k(V)$ has a ring structure given by

$$\text{End}_k(V) \times \text{End}_k(V) \rightarrow \text{End}_k(V), (\phi, \psi) \mapsto \phi \circ \psi.$$

Then $\text{End}_k(V)$ is both a ring and a k -module, and becomes a k -algebra via

$$\varphi : k \rightarrow \text{End}_k(V), \lambda \mapsto \lambda \text{id}.$$

Note that $\text{Im } \varphi \subseteq Z(A)$. If k is a field, then $Z(\text{End}_k(V)) = \{\lambda \text{id} \mid \lambda \in k\}$.

ii) Take $V = k^n$ (free module of rank n). Then $\text{End}_k(V) \cong M_n(k)$. Define

$$T_u := \{\varphi \in M_n(k) \mid \varphi \text{ is upper triangular}\},$$

i.e. T_u preserves flags in k^n . Then T_u is a **subalgebra** of $M_n(k)$, i.e. is both a subring and a k -submodule of the original algebra.

iii) Let G be a group. Define the **group algebra** $k[G]$ of G as follows:

- As k -module, is defined as the free module on G ,

$$k[G] := k^{(G)} = \left\{ \sum_{g \in G} \lambda_g g \mid \lambda_g \in k, \lambda_g \neq 0 \text{ for only finitely many } g \in G \right\}.$$

- Multiplication: Let $a := \sum \lambda_g g$, $b = \sum \mu_h h$ and define:

$$ab := \sum_{g \in G, h \in G} \lambda_g \mu_h (gh) = \sum_{j \in G} \left(\sum_{gh=j} \lambda_g \mu_h \right) j.$$


This multiplication is associative, k -bilinear, distributive and $1|_{k[G]} = e$. In addition, $(*)$ is satisfied.

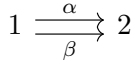
1.2 Quivers - Basics

Definition 1.2.1. A **quiver** is a „directed graph“. Formally, a quiver is a quadruple (Q_0, Q_1, s, t) consisting of sets Q_0 (**vertices**) and Q_1 (**arrows**) and maps $s : Q_1 \rightarrow Q_0$, $t : Q_1 \rightarrow Q_0$. For $\alpha \in Q_1$, we call $s(\alpha)$ the **source** of α and $t(\alpha)$ the **target** of α :

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

Example 1.2.2. i) $Q = (\{1\}, \emptyset, \dots)$ is visualized as: 1

ii) $Q = (\{1\}, \{\alpha\}, \dots)$ is visualized as 

iii) $Q = (\{1, 2\}, \{\alpha, \beta\}, s(\alpha) = s(\beta) = 1, t(\alpha) = t(\beta) = 2)$ is visualized as 

Definition 1.2.3. Let Q be a quiver such that both Q_0 and Q_1 are finite.

- i) Let $\ell \in \mathbb{Z}_{\geq 1}$. A **path** of length ℓ is a sequence $\alpha_\ell, \dots, \alpha_1$ of arrows, such that $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq \ell - 1$,

$$\circ \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_2} \circ \dots \xrightarrow{\alpha_\ell} \circ.$$

Define Q_ℓ to be the set of all paths of length ℓ .

Let $p : \alpha_\ell \dots \alpha_1$ be a path. Define $s(p) := s(\alpha_1)$ and $t(p) := s(\alpha_\ell)$.

Formally define Q_0 to be the set of all paths of length zero. Denote by ε_i for $i \in Q_0$ the constant path at i . ε_i is called a **lazy path**. We set $s(\varepsilon_i) = t(\varepsilon_i) := i$.

- ii) Let $p = \alpha_\ell \dots \alpha_i$ and $q = \beta_m \dots \beta_1$ be paths of length ℓ and m respectively, with $\ell, m \geq 1$. If $t(p) = s(q)$, then set $q \circ p := \beta_m \dots \beta_1 \alpha_\ell \dots \alpha_1$. This is a path of length $\ell + m$. For p a path of length ≥ 0 and ε_i a lazy path:

- if $t(p) = i$, set $\varepsilon_i \circ p := p$,
- if $s(p) = i$, set $p \circ \varepsilon_i := p$.

In all others cases, the composition is not defined.

- iii) Define

$$Q_* := \bigcup_{\ell \geq 0} Q_\ell,$$

the set of all paths. Define the **path-algebra** kQ :

- As a k -module, $kQ := k^{(Q_*)}$.
- Multiplication: Let $a = \sum \lambda_p p$, $b = \sum \mu_q q$. Define

$$ab := \sum_{p, q \in Q_*} \lambda_p \mu_q (p \cdot q),$$

where

$$p \cdot q := \begin{cases} p \circ q, & \text{if it is defined, i.e } t(q) = s(p) \\ 0, & \text{else} \end{cases}.$$

The multiplication is associative (due to the associativity of the composition of paths) and k -bilinear by definition. In addition, distributivity and $(*)$ are fulfilled.

- The identity is given by $\sum \varepsilon_i$.

Example 1.2.4. i) $Q = 1$, then $kQ = k$.

- ii) $Q = 1 \xrightarrow{\quad} 1$, then $Q_* = \{\alpha^n \mid n \geq 0\}$ and $kQ = k[t]$.

- iii) $Q = 1 \xrightleftharpoons[\beta]{\alpha} 2$. Then $Q_* = \{\varepsilon_1, \varepsilon_2, \alpha^n, \beta^n \mid n \geq 0\}$ and

$$kQ = k\varepsilon_1 \oplus k\varepsilon_2 \oplus k\alpha \oplus k\beta.$$

A multiplication table is given by

	ε_1	ε_2	α	β
ε_1	ε_1	0	0	0
ε_2	0	ε_2	α	β
α	α	0	0	0
β	β	0	0	0

Lemma 1.2.5. *Let k be a field, A a k -algebra and $n := \dim(A) < \infty$. Then there exists an injective homomorphism of k -algebras $\varphi : A \rightarrow M_n(k)$.*

Proof. By choosing a basis of A , we get an isomorphism $\text{End}_k(A) \cong M_n(k)$. So it suffices to find an injective homomorphism of k -algebras $\varphi : A \rightarrow \text{End}_k(A)$.

Consider

$$\varphi : A \rightarrow \text{End}_k(A), \quad \varphi(a) : A \rightarrow A, b \mapsto ab.$$

- $\varphi(a)$ is k -linear for all a by the distributivity in A and the condition (*).
- φ is k -linear by the distributivity in A and the condition (*).
- Let $a, a' \in A$. Then

$$\varphi(aa')(b) = (aa')(b) = a(a'b) = (\varphi(a) \circ \varphi(a'))(b).$$

Hence φ is indeed a homomorphism of k -algebras.

To show that φ is injective, let $a \in \ker \varphi$, hence $ab = 0$ for all $b \in A$. But in particular, $0 = a1 = a$. □

End of Lecture 1

Definition 1.2.6. Let A be a k -algebra. Then the **opposite algebra** A^{op} is A (as a k -module), and the multiplication is defined as

$$a \cdot_{A^{\text{op}}} b = b \cdot_A a.$$

Example 1.2.7. Let Q be a quiver, and define $Q^{\text{op}} := (Q_0, Q_1, s^{\text{op}}, t^{\text{op}})$, where $s^{\text{op}}(\alpha) := t(\alpha)$ and $t^{\text{op}}(\alpha) := s(\alpha)$. Then $kQ^{\text{op}} = k(Q^{\text{op}})$

1.3 Modules - Basics

Definition 1.3.1. Let A be a k -algebra. A **left A -module** M is a k -module M together with a map $A \times M \rightarrow M, (a, x) \mapsto ax$, such that:

$$a(x + y) = ax + ay \tag{L1}$$

$$(a + b)x = ax + bx \tag{L2}$$

$$a(bx) = (ab)x \quad (\text{L3})$$

$$1_A x = x \quad (\text{L4})$$

$$(\lambda a)x = \lambda(ax) = a(\lambda x), \quad (\text{L5})$$

for all $a, b \in A$, $x, y \in M$ and $\lambda \in k$. If A is a left A -module, we denote this as ${}_A M$. A **right A -module** is defined analogous, where (L3) becomes $(xa)b = x(ab)$. If A is a right A -module, we denote this by A_M .

Remark 1.3.2. A right A -module is the same as a left A^{op} -module.

Definition 1.3.3. Let A be a k -algebra, and M, N left A -modules. A **homomorphism of left A -modules** $f : M \rightarrow N$ is a k -linear map such that

$$f(ax) = af(x)$$

for all $a \in A$ and $x \in M$.

Define the set of all left A -algebra homomorphisms as

$$\text{hom}_A(M, N) := \text{hom}_A({}_A M, {}_A M) := \{f : M \rightarrow N \mid f \text{ is a homomorphism of left } A\text{-modules}\}.$$

A homomorphism of left A -modules is an **isomorphism** if it is a bijective homomorphism of left A -modules.

Homomorphism of right A -modules are defined analogous.

Remark 1.3.4. Let M, N be left A -modules. Then

i) $\text{hom}_A(M, N)$ has a k -module structure given by

$$\lambda f : M \rightarrow N, \quad x \mapsto \lambda f(x) = f(\lambda x).$$

This is well defined, as k lies in the center of A .

ii) In general, $\text{hom}_A(M, N)$ has neither a left nor a right A -module structure.

iii) f is an isomorphism if and only if there is a homomorphism of left A -modules $g : N \rightarrow M$ such that

$$g \circ f = \text{id}_M \quad \text{and} \quad f \circ g = \text{id}_N.$$

iv) Let $f : M \rightarrow M'$ and $g : N \rightarrow N'$ be **homomorphisms of left A -modules**. Then we obtain k -linear maps

$$\begin{aligned} f^* : \text{hom}_A(M', N) &\rightarrow \text{hom}_A(M, N), \quad h \mapsto h \circ f \\ g_* : \text{hom}_A(M, N) &\rightarrow \text{hom}_A(M, N'), \quad h \mapsto g \circ h. \end{aligned}$$

Remark 1.3.5. Let A be a k -algebra and M, N left A -modules.

i) A subset $M' \subseteq M$ is called a **submodule** if

$$(\text{SM1}) \quad 0 \in M'$$

$$(SM2) \quad x, x' \in M' \implies x + x' \in M'$$

$$(SM3) \quad a \in A, x \in M' \implies ax \in M'.$$

In particular, submodules of A -modules are submodules of the underlying k -module, as follows using (L4)

- ii) Let M be a submodule. Then the **quotient** has a left A -module structure in the obvious way. The projection

$$\pi : M \rightarrow M'$$

is a homomorphism of left A -modules.

- iii) A **left ideal** is left A -submodule of ${}_A A$. Similar, a **right ideal** is right A -submodule of A_A . For a left ideal $I \subseteq A$, the quotient A/I is a left A -module, but in general not an algebra.

- iv) A **two-sided ideal** $I \subset A$ is both a left- and a right-ideal of A . Then A/I has an algebra structure, by setting

$$(x + I)(y + I) := (xy) + I.$$

In general, this is only well-defined if I is a two-sided ideal of A .

- v) Let $f : M \rightarrow N$ be a homomorphism of left A -modules. Then we obtain left A -modules:

$$\ker f, \operatorname{Im} f, \operatorname{coker} f := N/\operatorname{Im} f, \operatorname{coim} f := M/\ker f.$$

In particular, f factors uniquely as

$$\begin{array}{c} \xrightarrow{\quad f \quad} \\ M \longrightarrow \operatorname{coim} f \xrightarrow[\cong]{\exists!} \operatorname{Im} f \longrightarrow N \end{array} \quad (F)$$

- vi) Let $\{M_i \subset M \mid i \in I\}$ be a family of left A -submodules, for some index set I . Then

$$\bigcap_{i \in I} M_i \text{ and } \sum_{i \in I} M_i$$

are left A -modules.

- vii) Let $x \in M$. Define

$$Ax := \{ax \mid a \in A\},$$

which is a left A -submodule. Similar, for $x \in M_A$, define $xA := \{xa \mid a \in A\}$, which is a right A -submodule. For a subset $E \subset M$,

$$\sum_{x \in E} Ax = \bigcap_{\substack{E \subseteq M' \subseteq M \\ M' \text{ submodule}}} M'.$$

M is called **finitely generated**, if there are $x_1, \dots, x_n \in M$, such that

$$M = \sum_{i=1}^n Ax_i.$$

viii) Let $\{M_i \mid i \in I\}$ be a family of left A -modules. Then

$$\prod_{i \in I} M_i := \{(x_i)_{i \in I} \mid x_i \in M_i\}$$

is called the **product**, and

$$\bigoplus_{i \in I} M_i := \{(x_i)_{i \in I} \mid x_i \in M_i, x_i \neq 0 \text{ for only finitely many } i\}$$

is called the **coproduct**. They are both left A -modules. The **projection**

$$\pi_j : \prod_{i \in I} M_i \rightarrow M_j, (x_i)_{i \in I} \mapsto x_j$$

and the **inclusion**

$$\iota_j : \bigoplus_{i \in I} x_j \mapsto (\delta_{ij} x_j)_{i \in I}$$

are morphism of left A -modules.

ix) A left A -module M is finitely generated if and only if there is a surjective homomorphism of left A -modules

$$A^n := \bigoplus_{i=1}^n A \longrightarrow M$$

for some $n \geq 1$. A is called **finitely presented**, if there is an exact sequence of left A -modules

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

for some $m, n \geq 1$.

Proposition 1.3.6. *Let*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0 \quad (*)$$

and

$$0 \longrightarrow N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \quad (**)$$

be sequences of left A -modules.

i) The following are equivalent:

a) $(*)$ is exact.

b) For all left A -modules N , the sequence

$$0 \longrightarrow \text{hom}_A(M_3, N) \xrightarrow{f_2^*} \text{hom}_A(M_2, N) \xrightarrow{f_1^*} \text{hom}_A(M, N)$$

is exact.

ii) The following are equivalent:

a) $(**)$ is exact.

b) For all left A -modules M , the sequence

$$0 \longrightarrow \text{hom}_A(M, N_1) \xrightarrow{g_{1,*}} \text{hom}_A(M, N_2) \xrightarrow{g_{2,*}} \text{hom}_A(M, N_3)$$

is exact.

Proof. We will only prove $a) \implies b)$ of ii).

Lemma 1.3.7. *Let K, M, N be left A -modules, and $\zeta : K \rightarrow M$, $\varphi : M \rightarrow N$ be homomorphisms of left A -modules, such that $\varphi \circ \zeta = 0$. Then there is a unique homomorphism $\bar{\zeta}$, such that*

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \text{---} & \nearrow & \\ K & \xrightarrow{\zeta} & M & \xrightarrow{\varphi} & N \\ & \searrow \exists! \bar{\zeta} & \uparrow & & \\ & & \ker \varphi & & \end{array}$$

commutes.

- $g_{1,*}$ injective: Let $h \in \ker(g_{1,*})$. Then

$$g_1 \circ h : M \xrightarrow{h} N_1 \xrightarrow{g_1} N_2$$

and since g_1 is injective, it follows $h = 0$.

- $\text{Im } g_{1,*} \subseteq \ker g_{2,*}$: Since $**$ is exact, it follows that $g_2 \circ g_1 = 0$. For $h \in \text{Im } g_{1,*}$ there exists an $h' : M \rightarrow N_1$ such that $h = g_1 \circ h'$, and hence $g_2 \circ h = g_2 \circ g_1 \circ h' = 0$.
- $\ker g_{2,*} \subseteq \text{Im } g_{1,*}$: As $(**)$ is exact, $\ker g_2 = \text{Im } g_1$ holds.
Let $h : M \rightarrow N_2 \in \ker g_{2,*}$, i.e. $g_2 \circ h = 0$:

$$\begin{array}{ccccccc} & & M & & & & \\ & & \downarrow h & \searrow 0 & & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 \end{array}$$

By lemma 1.3.7, h factors uniquely through $\ker g_2 = \text{Im } g_1$:

$$\begin{array}{ccccccc} & & M & & & & \\ & & \downarrow h & \searrow 0 & & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 \\ & & & & \uparrow \iota & & \\ & & & & \text{Im } g_1 & & \end{array}$$

But since g_1 is injective, (F) implies that there is a uniquely determined isomorphism $\tilde{g}_1 : N_1 \longrightarrow \text{Im } g_1$.

Putting everything together, we obtain the following commutative diagram:

$$\begin{array}{ccccccc} & & M & & & & \\ & & \downarrow h & \searrow 0 & & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 \\ & & \searrow \exists! \tilde{g}_1 & & \uparrow \iota & & \\ & & & & \text{Im } g_1 & & \end{array}$$

Setting $h' := \tilde{g}_1^{-1} \circ h'$, we obtain

$$g_1 \circ h' = \iota \circ \tilde{g}_1 \circ \tilde{g}_1^{-1} \circ \tilde{h} = \iota \circ \tilde{h} = h.$$

□

Proposition 1.3.8. *Let A be a k -algebra. To give a left A -module structure is the same as to give a k -module structure V together with a homomorphism $\varphi : A \rightarrow \text{End}_k(V)$ of k -algebras. To give a right A -module structure is the same as giving a k -module structure V together with a homomorphism $\varphi : A \rightarrow \text{End}_k(V)^{\text{op}}$.*

1.4 Representation of quivers

Let k be a field and Q be a quiver.

Definition 1.4.1. A **representation** over k consists of

- a k -vector space X_i for all $i \in Q_0$,
- a k -linear map

$$X_\alpha : X_{s(\alpha)} \rightarrow X_{t(\alpha)}$$

for each $\alpha \in Q_1$

Example 1.4.2 (Continue example 1.2.4). i) Let $Q = \cdot$. Then a representation of Q is simply a k -vector space.

- ii) Let $Q = 1 \curvearrowright$. Then a representation of Q is a k -vector space V together with an endomorphism $\varphi \in \text{End}_k(V)$:

$$Q = V \curvearrowright^f.$$

- iii) Let $Q = 1 \begin{smallmatrix} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{smallmatrix} 2$, the **Kronecker Quiver**. Then a representation of Q is a pair of vector spaces V, W and two linear maps $f, g \in \text{hom}_K(V, W)$:

$$Q = V \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} W$$

Definition 1.4.3. Take X, Y to be two representations of Q over k . A **homomorphism of representations** $f : X \rightarrow Y$ is a tuple $(f_i)_{i \in Q_0}$ of linear maps $f_i : X_i \rightarrow Y_i$, such that for all

$\alpha \in Q_1$ the diagram

$$\begin{array}{ccc} X_{s(\alpha)} & \xrightarrow{f_{s(\alpha)}} & Y_{s(\alpha)} \\ X_\alpha \downarrow & & \downarrow Y_\beta \\ X_{t(\alpha)} & \xrightarrow{f_{t(\alpha)}} & Y_{t(\alpha)} \end{array}$$

commutes.

Example 1.4.4 (Continue example 1.4.2). i) Homomorphisms of representations are k -linear maps $X \rightarrow Y$.

ii) Homomorphisms of representations (V, φ) and (W, ψ) are k -linear maps $f : V \rightarrow W$, such that

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \varphi \downarrow & & \downarrow \psi \\ V & \xrightarrow{f} & W \end{array}$$

commutes

iii) Homomorphisms of representations (V_1, V_2, A, B) and (W_1, W_2, C, D) are pairs (f_1, f_2) of linear maps $f_1 : V_1 \rightarrow W_1$ and $f_2 : V_2 \rightarrow W_2$, such that $A \circ f_1 = f_2 \circ A$ and $B \circ f_1 = f_2 \circ B$.

Definition 1.4.5. An **isomorphism of representations** $f : X \rightarrow Y$ is a homomorphism of representations, such that there exists $g : Y \rightarrow X$ homomorphism of representations satisfying

$$g \circ f = \text{id}_X \text{ and } f \circ g = \text{id}_Y.$$

An isomorphism of representations is a homomorphism of representations such that each map f_i is bijective.

End of Lecture 2

A.1 Sheet 1

Solution A.1.1. i) φ_m is k -linear:

- $\varphi(a+b)x = (a+b)x \stackrel{(L1)}{=} ax + bx = \varphi(a)x + \varphi(b)x$
- $\varphi(\lambda a) = (\lambda a)x \stackrel{(L5)}{=} \lambda(ax) = \lambda\varphi(a)x$.

φ is ring homomorphism:

- $\varphi(ab) = (ab)x \stackrel{(L3)}{=} a(bx) = \varphi(a)\varphi(b)x$
- $\varphi(1_A) \stackrel{(L4)}{=} (1_A)x = x$.

As these relations hold for all x , the assertion follows.

ii) V_φ is already a k -module.

- (L1) $a(x+y) = (\varphi(a))(x+y) \stackrel{\varphi \in \text{End}_k(V)}{=} (\varphi(a))(x) + (\varphi(a))(y) = ax + ay$
- (L2) $(a+b)x = (\varphi(a+b))x \stackrel{\varphi \text{ homo of } k\text{-algebras}}{=} (\varphi(a) + \varphi(b))x = \varphi(a)x + \varphi(b)x = ax + bx$
- (L3) $(ab)x = (\varphi(ab))x \stackrel{\varphi \text{ homo of } k\text{-algebras}}{=} (\varphi(a)\varphi(b))x = a(bx)$
- (L4) $1_ax = (\varphi(1_a))x \stackrel{\varphi \text{ homo of } k\text{-algebras}}{=} \text{id } x = x$
- (L5) $(\lambda a)(x) = ((\varphi(\lambda a))x) \stackrel{\varphi \text{ homo of } k\text{-algebras}}{=} (\lambda\varphi(a))x = \lambda(ax)$ and
 $a(\lambda x) = (\varphi(a)(\lambda x)) \stackrel{\varphi(a) \text{ endo of } k\text{-module}}{=} \lambda(\varphi(a)x) = \lambda(ax),$

for all $a, b \in A, x, y \in V$ and $\lambda \in k$.

iii) We regard V and W as A -modules in the sense of part ii). Assume that $\psi(a) \circ f = f \circ \varphi(a) (*)$ for all $a \in A$. Then

$$f(ax) = f((\varphi(a))x) \stackrel{*}{=} (\psi(a))(f(x)) = af(x),$$

for all $x \in X$. Hence f is an A -module homomorphism.

Assume that f is a a -module homomorphism, then

$$(\psi(a))(f(x)) = a(f(x)) = f(ax) = f((\varphi(a))x)$$

for all $x \in V$. Hence $\psi(a) \circ f - f \circ \varphi(a) = 0$ and so $(*)$ holds.

Solution A.1.2. i) Assume that I is a non-zero ideal of A . Let $a = (a)_{ij} \neq 0$ be an arbitrary matrix in I . Then there exist elementary matrixes, such that $a_{11} \neq 0$. Define

$$b \in M_n(k), (b)_{ij} := \begin{cases} 1 & \text{if } i = j = 1 \\ 0 & \text{else} \end{cases}$$

and E_n as the identity matrix. Then we get

$$\left(\frac{1}{a_{11}}E_n\right) \cdot b \cdot a \cdot b = b.$$

By repeatedly using permutation matrixes, it is possible to write any matrix as sum of products of a with permutation matrix on the left- and right. Hence a generates all of A , and $I = A$.

- ii) As k is a field, $M_n(k) \cong \text{End}_k(k^n)$ holds for a choosen basis of k^n , and $\dim \text{End}_k(k^n) = n^2$. Hence $\dim M_n(k) = n^2$ as a k -vector space. In addition, $\dim \text{End}_k(M) = m^2$, where $m := \dim M$ is the dimension of M as a k -vector space.

Consider now the map

$$\varphi : M_n(k) \rightarrow \text{End}_k(M), \quad a \mapsto a : (x \mapsto ax),$$

which maps a matrix a to the linear map induced by the $M_n(K)$ -algebra structure on M . This is a homomorphism of k -algebras, and in particular, the kernel of φ is a two-sided ideal of $M_n(k)$, as

$$a0x = 0ax = 0$$

holds for any $a \in M_n(k)$. Now i) implies that either $\ker \varphi = 0$ or $\ker \varphi = M_n(k)$. But since $\varphi(E_n) = \text{id}_M$, where E_n is the $n \times n$ identity matrix, the latter one is not possible. Hence $\ker \varphi = 0$, and φ is an injective map of k -vector spaces. But this implies $\dim A \leq \dim \text{End}_K(V)$, so $m \geq n$.

Proposition A.1.1. *Let k be a field, $k[X]$ the polynomial ring and $p \in k[X]$ a polynomial with $\deg p = n$. Then*

$$k[X]/(p)$$

is a n -dimensional k vector space, and a basis is given by

$$\{1, x, \dots, x^{n-1}\}.$$

The following propositions are taken from [Alu09].
Let R be any commutative ring.

Proposition A.1.2. *Let I_1, \dots, I_k be ideals of R such that $I_i + I_j = (1)$ for all $i \neq j$. Then the natural homomorphism*

$$\varphi : R \rightarrow R/I_1 \times \dots \times R/I_k$$

is surjective and induces an isomorphism

$$\frac{R}{I_1 \dots I_k} \rightarrow R/I_1 \times \dots \times R/I_k$$

Corollary A.1.3 (Chinese remainder theorem). *Let R be a PID and $a_1, \dots, a_k \in R$ be elements such that $\gcd(a_i, a_j) = 1$ for all $i \neq j$. Let $a = a_1 \dots a_k$. Then the function*

$$\varphi : R/(a) \rightarrow R/(a_1) \times \dots \times R/(a_k).$$

Proposition A.1.4 (Yoneda Lemma). *Let \mathcal{C} be a category, X an object of \mathcal{C} and consider the contravariant functor*

$$h_X := \text{hom}_{\mathcal{C}}(-, X).$$

Then for every contravariant functor $\mathcal{F} : \mathcal{C} \rightarrow \mathbf{Set}$, there is a bijection between the set of natural transformations $h_X \rightsquigarrow \mathcal{F}$ and $\mathcal{F}(X)$.

Definition A.1.5 ([ASS06]). The (Jacobson) **radical** $\text{rad } A$ of a K -algebra A is the intersection of all maximal right ideals in A . It is the same as the intersection of all left-sided maximal right ideals in A . Furthermore, $\text{rad } A$ is a two-sided ideal.

Definition A.1.6. Let $f, g : X \rightarrow Y$ be morphisms in a category \mathcal{C} . Then a morphism $e : E \rightarrow X$ is called **equalizer** of f and g if $f \circ e = g \circ e$ and for all other morphisms $o : O \rightarrow X$, such that $f \circ o = g \circ o$, there is a unique morphism $O \rightarrow E$, such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{e} & X \rightrightarrows X \\ \uparrow \exists! & \nearrow o & \\ O & & \end{array} .$$

Proposition A.1.7. *Equalizers exist in abelian categories.*

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