Solution 1.3. i) φ_m is k-linear:

- $\varphi(a+b)x = (a+b)x \overset{(L1)}{=} ax + bx = \varphi(a)x + \varphi(b)x$
- $\varphi(\lambda a) = (\lambda a)x \overset{(L5)}{=} \lambda(ax) = \lambda\varphi(a)x.$

φ_m is ring homomorphism:

- $\varphi(ab) = (ab)x \overset{(L3)}{=} a(bx) = \varphi(a)\varphi(b)x$
- $\varphi(1_A) \overset{(L4)}{=} (1_A)x = x.$

As these relations hold for all x, the assertion follows.

ii) V_φ is already a k-module.

- (L1) $a(x+y) = (\varphi(a))(x+y) \overset{\varphi\in\mathrm{End}_k(V)}{=} (\varphi(a))(x) + (\varphi(a))(y) = ax + ay$
- (L2) $(a+b)x = (\varphi(a+b))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\varphi(a)+\varphi(b))x = \varphi(a)x + \varphi(b)x = ax + bx$
- (L3) $(ab)x = (\varphi(ab))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\varphi(a)\varphi(b))x = a(bx)$
- (L4) $1_a x = (\varphi(1_a))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} \mathrm{id}\, x = x$
- (L5) $(\lambda a)(x) = ((\varphi(\lambda a))x \overset{\varphi \text{ homo of } k-\text{algebras}}{=} (\lambda\varphi(a))x = \lambda(ax)$ and
  $a(\lambda x) = (\varphi(a)(\lambda x)) \overset{\varphi(a) \text{ endo of } k-\text{module}}{=} \lambda(\varphi(a))x = \lambda(ax),$

for all $a, b \in A, x, y \in V$ and $\lambda \in k$.

iii) We regard V and W as A-modules in the sense of part ii). Assume that $\psi(a) \circ f = f \circ \varphi(a)$ $(*)$ for all $a \in A$. Then

$$f(ax) = f((\varphi(a))x) \overset{*}{=} (\psi(a))(f(x)) = af(x),$$

for all $x \in X$. Hence f is an A-module homomorphism.
Assume that f is a a-module homomorphism, then

$$(\psi(a))(f(x)) = a(f(x)) = f(ax) = f((\varphi(a))x)$$

for all $x \in V$. Hence $\psi(a) \circ f - f \circ \varphi(a) = 0$ and so $(*)$ holds.

Solution 1.4. i) Assume that I is a non-zero ideal of A. Let $a = (a)_{ij} \neq 0$ be an arbitrary matrix in I. Then there exist permutation $\sigma, \pi \in \mathrm{GL}_n(K)$ matrices, such that $(\sigma a \pi)_{11} \neq 0$, which is in a, as I is a two-sided ideal. So without loss of generality, suppose $a_{11} \neq 0$.
Define

$$b \in \mathrm{M}_n(k), (b)_{ij} := \begin{cases} 1, & \text{if } i = j = 1 \\ 0, & \text{else} \end{cases}$$

and E_n as the identity of $\mathrm{M}_n(k)$. Then we get

$$\left(\frac{1}{a_{11}} E_n\right) \cdot b \cdot a \cdot b = b.$$

By repeatetly using permutation matrixes, it is possible to write any matrix as sum of products of a, b and permutation matrices on the left- and right. As I is a two-sided ideal, a all of these combinations are in I as well. Hence a generates all of A, and $I = A$.

ii) Consider A as a k-vector space, then $\dim_K A = n^2$. Let M be any left A-module. As shown in task 3, there is a homomorphism of k-algebras

$$\varphi A \to \mathrm{End}_k(M),\ a \mapsto a : (x \mapsto ax),$$

which is in particular a homomorphism of k-vector spaces. The kernel of φ is a two-sided ideal of A, as

$$a0x = 0ax = 0$$

for all $a \in A$ and $x \in M$.
Now i) implies that $\ker \varphi$ is either zero or $\ker \varphi = A$. But since $\varphi(E_n) = \mathrm{id}_M$, the latter one is not possible. Hence φ is injective, and in particular $\dim A \leq \dim \mathrm{End}_k(V)$, so $n \leq m$.