Foundations of Representation Theory

Lecture Notes in the Winter Term 2018/19

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These are my personal lecture notes for the lecture *Foundations of Representation Theory* held by Dr. Hans Franzen at the University of Bonn in the winter term 2018/19.

I try to update them on my website, `https://pankratius.github.io`.
The authors labels his own comments and additions in purple.

Conventions: In this lecture, rings will always be unital, and ring homomorphisms f always fulfill $f(1) = 1$. Rings do *not* have to be commutative.

1.1 Algebras - Basics

Let k be a commutative ring.

Definition 1.1.1. A k-**algebra** A is a ring A, togehter with a structure of a k-module on A, such that

$$\text{for all } a, b \in A, \lambda \in K: \ (\lambda a)b = a(\lambda b) = \lambda(ab) \tag{$*$}$$

Definition 1.1.2. Let A, B be k-algebras. A **homomorphism of algebras** is a map $f : A \to B$ that is both k-linear and a ring homomorphism.

Remark 1.1.3. Let A be a ring. Define

$$Z(A) := \{a \in A \mid \forall b \in A : ab = ba\},$$

which is a commutative subring and is called the **center** of A.

Remark 1.1.4. Let A be a ring. Giving a k-algebra structure on A is the same as giving a ring homomorphism $k \to Z(A)$. More precisley:

i) If A is a k-algebra, then $p : k \to A, \ \lambda \mapsto \lambda 1$ satisfies $\operatorname{Im} p \subseteq Z(A)$ and is a ring homomorphism. (the first statement follows from ($*$) and the second one from the fact that A has a k-module structure).

ii) Let $\varphi : k \to Z(A)$ be a ring homomorphism. Define

$$\lambda a := \varphi(\lambda)a,$$

for all $\lambda \in k$. This defines a k-algebra structure on A (Scalar multiplication with elements from k in A follows from the distributivity in A, and ($*$) since $\operatorname{Im}(\varphi) \subseteq Z(A)$).

iii) Let A, B be k-algebras and $f : A \to B$ a homomorphism of rings. Then f is a homomorphism of k-algebras if and only if the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "f"] & & B \\
Z(A) \arrow[u, hook] & & Z(B) \arrow[u, hook] \\
& k \arrow[ul, "\phi_A"] \arrow[ur, "\phi_B"'] &
\end{tikzcd} .$$

Example 1.1.5. i) Let V be a k-module. Consider $\operatorname{End}_k(V)$. This has a ring structure given by

$$\operatorname{End}_k(V) \times \operatorname{End}_k(V) \to \operatorname{End}_k(V), \ (\phi, \psi) \mapsto \varphi \circ \psi.$$

Then $\mathrm{End}_k(V)$ is both a ring and a k-module, and becomes a k-algebra via

$$\varphi : k \to \mathrm{End}_k(V),\ \lambda \mapsto \lambda\,\mathrm{id}\,.$$

Note that $\operatorname{Im}\varphi \subseteq Z(A)$. If k is a field, then $Z(\mathrm{End}_k(V)) = \{\lambda\,\mathrm{id} \mid \lambda \in k\}$.

ii) Take $V = k^n$ (free module of rank n). Then $\mathrm{End}_k(V) \cong \mathrm{M}_n(k)$. Define

$$T_u := \{n\varphi \in \mathrm{M}_n(k) \mid \varphi \text{ is upper triangular}\},$$

i.e. T_u presevers flags in k^n. Then T_u is a **subalgebra** of $\mathrm{M}_n(k)$, i.e. is both a subring and a k-submodule of the original algebra.

iii) Let G be a group. Define to be the **group algebra** of $k[G]$ as follows:

- As k-module, is defined as the free module on G,

$$k[G] := k^{(G)} = \{\sum_{g\in G} \lambda_g g \mid \lambda_g \in k,\ \lambda_g \neq 0 \text{ for only finitely many } g \in G\}.$$

- Multiplication: Let $a := \sum \lambda_g g$, $b = \sum \mu_g g$ and define:

$$ab := \sum_{g\in G, h\in G} \lambda_g \mu_h (gh) = \sum_{j\in G} \left(\sum_{gh=j} \lambda_g \mu_h \right) j.$$

This multiplication is associative, k-bilinear, distributive and $1|_{k[G]} = e$. In addition, $(*)$ is satisfied.

1.2 Quivers - Basics

Definition 1.2.1. A **quiver** is a „directed graph“. Formally, a quiver is a quadruple (Q_0, Q_1, s, t) consiting of sets Q_0 (**vertices**) and Q_1 (**arrows**) and maps $s : Q_1 \to Q_0$, $t : Q_1 \to Q_0$. For $\alpha \in Q_1$, we call $s(\alpha)$ the **source** of α and $t(\alpha)$ the **target** of α:

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

Example 1.2.2. i) $Q = (\{1\}, \emptyset, ...)$ is visualized as: 1

ii) $Q = (\{1\}, \{\alpha\}, ...)$ is visualized as $\begin{tikzcd} 1 \arrow[loop above] \end{tikzcd}$

iii) $Q = (\{1, 2\}, \{\alpha, \beta\}, s(\alpha) = s(\beta) = 1, t(\alpha) = t(\beta) = 2)$ is visualized as $\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}$

Definition 1.2.3. Let Q be a quiver sucht that both Q_0 and Q_1 are finite.

i) Let $\ell \in \mathbb{Z}_{\geq 1}$. A **path** of length ℓ is a sequence $\alpha_\ell ..., \alpha_1$ of arrows, such that $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq \ell - 1$,

$$\circ \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_2} \circ \ldots \xrightarrow{\alpha_\ell} \circ.$$

Define Q_ℓ to be the set of all paths of length ℓ.
Let $p : \alpha_\ell ... \alpha_1$ be a path. Define $s(p) := s(\alpha_1)$ and $t(p) := s(\alpha_\ell)$.
Formally define Q_0 to be the set of all paths of length zero. Denote by ε_i for $i \in Q_0$ the constant path at i. ε_i is called a **lazy path**. We set $s(\varepsilon_i) = t(\varepsilon_i) := i$.

ii) Let $p = \alpha_\ell ... \alpha_i$ and $q = \beta_m ... \beta_1$ be paths of length ℓ and m respectivley, with $\ell, m \geq 1$. If $t(p) = s(q)$, then set $q \circ p := \beta_m .... \beta_1 \alpha_\ell ... \alpha_1$. This is a path of length $\ell + m$.
For p a path of length ≥ 0 and ε_i a lazy path:

- if $t(p) = i$, set $\varepsilon_i \circ p := p$,
- if $s(p) = i$, set $p \circ \varepsilon_i := p$.

In all others cases, the composition is not defined.

iii) Define

$$Q_* := \bigcup_{\ell \geq 0} Q_\ell,$$

the set of all paths. Define the **path-algebra** kQ:

- As a k-module, $kQ := k^{(Q_*)}$.
- Multiplication: Let $a = \sum \lambda_p p,\ b = \sum \mu_p p$. Define

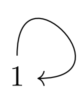
$$ab := \sum_{p,q \in Q_*} \lambda_p \mu_q (p \cdot q),$$

where

$$p \cdot q := \begin{cases} p \circ q, & \text{if it is defined, i.e } t(q) = s(p) \\ 0, & \text{else} \end{cases}.$$

The multiplication is associative (due to the associativity of the composition of paths) and k-bilinear by definition. In addition, distributivity and ($*$) are fulfilled.
- The identity is given by $\sum \varepsilon_i$.

Example 1.2.4. i) $Q = 1$, then $kQ = k$.

ii) $Q =$  , then $Q_* = \{\alpha^n \mid n \geq 0\}$ and $kQ = k[t]$.

iii) $Q = 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$. Then $Q_* = \{\varepsilon_1, \varepsilon_2, \alpha^n, \beta^n \mid n \geq 0\}$ and

$$kQ = k\varepsilon_1 \oplus k\varepsilon_2 \oplus k\alpha \oplus k\beta.$$

A multiplication table is given by

	ε_1	ε_2	α	β
ε_1	ε_1	0	0	0
ε_2	0	ε_2	α	β
α	α	0	0	0
β	β	0	0	0

Lemma 1.2.5. *Let k be a field, A a k-algebra and $n := \dim(A) < \infty$. Then there exists an injective homomorphism of k-algebras $\varphi : A \to \mathrm{M}_n(k)$.*

Proof. By choosing a basis of A, we get an isomorphism $\mathrm{End}_k(A) \cong \mathrm{M}_n(k)$. So it suffices to find an injective homomorphism of k-algebras $\varphi : A \to \mathrm{End}_k(A)$.
Consider

$$\varphi : A \to \mathrm{End}_k(A),\ \varphi(a) : A \to A, b \mapsto ab.$$

- $\varphi(a)$ is k-linear for all a by the distributivity in A and the condition $(*)$.
- φ is k-linear by the distributivity in A and the condition $(*)$.
- Let $a, a' \in A$. Then

$$\varphi(aa')(b) = (aa')(b) = a(a'b) = \left(\varphi(a) \circ \varphi(a')\right)(b).$$

Hence φ is indeed a homomorphism of k-algebras.
To show that φ is injective, let $a \in \ker\varphi$, hence $ab = 0$ for all $b \in A$. But in particular, $0 = a1 = a$. □

End of Lecture 1

Definition 1.2.6. Let A be a k-algebra. Then the **opposite algebra** A^{op} is A (as a k-module), and the multiplication is defined as

$$a \cdot_{A^{\mathrm{op}}} b = b \cdot_A b.$$

Example 1.2.7. Let Q be a quiver, and define $Q^{\mathrm{op}} := (Q_0, Q_1, s^{\mathrm{op}}, t^{\mathrm{op}})$, where $s^{\mathrm{op}}(\alpha) := t(\alpha)$ and $t^{\mathrm{op}}(\alpha) := s(\alpha)$. Then $kQ^{\mathrm{op}} = k(Q^{\mathrm{op}})$

1.3 Modules - Basics

Definition 1.3.1. Let A be a k-algebra. A **left A-module** M is a k-module M together with a map $A \times M \to M, (a, x) \mapsto ax$, such that:

$$a(x + y) = ax + ay \tag{L1}$$

$$(a + b)x = ax + bx \tag{L2}$$

$$a(bx) = (ab)x \tag{L3}$$

$$1_A x = x \tag{L4}$$

$$(\lambda a)x = \lambda(ax) = a(\lambda x), \tag{L5}$$

for all $a, b \in A$, $x, y \in M$ and $\lambda \in k$. If A is a left A-module, we denote this as ${}_AM$. A **right A-module** is defined analogous, where (L3) becomes $(xa)b = x(ab)$. If A is a right A-module, we denote this by A_M.

Remark 1.3.2. A right A-module is the same as a left A^{op}-module.

Definition 1.3.3. Let A be a k-algebra, and M, N left A-modules. A **homomorphism of left A-modules** $f : M \to N$ is a k-linear map such that

$$f(ax) = af(x)$$

for all $a \in A$ and $x \in M$.
Define the set of all left A-algebra homomorphisms as

$$\hom_A(M, N) := \hom_A({}_AM, {}_AM) := \{f : M \to N \mid f \text{ is a homomorphism of left } A\text{-modules}\}.$$

A homomorphism of left A-modules is an **isomorphism** if it is a bijective homomorphism of left A-modules.
Homomorphism of right A-modules are defined analogous.

Remark 1.3.4. Let M, N be left A-modules. Then

i) $\hom_A(M, N)$ has a k-module structure given by

$$\lambda f : M \to N,\ x \mapsto \lambda f(x) = f(\lambda x).$$

This is well defined, as k lies in the center of A.

ii) In general, $\hom_A(M, N)$ has neither a left nor a right A-module structure.

iii) f is an isomorphism if and only if there is a homomorphism of left A-modules $g : N \to M$ such that

$$g \circ f = \mathrm{id}_M \text{ and } f \circ g = \mathrm{id}_N .$$

iv) Let $f : M \to M'$ and $g : N \to N'$ be homomorphisms of left A-modules. Then we obtain k-linear maps

$$f^* : \hom_A(M', N) \to \hom_A(M, N),\ h \mapsto h \circ f$$
$$g_* : \hom_A(M, N) \to \hom_A(M, N'),\ h \mapsto g \circ h.$$

Remark 1.3.5. Let A be a k-algebra and M, N left A-modules.

i) A subset $M' \subseteq M$ is called a **submodule** if

(SM1) $0 \in M'$

(SM2) $x, x' \in M' \implies x + x' \in M'$

(SM3) $a \in A, x \in M' \implies ax \in M'$.

In particular, submodules of A-modules are submodules of the underlying k-module, as follows using (L4)

ii) Let M be a submodule. Then the **quotient** has a left A-module structure in the obvious way. The projection

$$\pi : M \to M'$$

is a homomorphism of left A-modules.

iii) A **left ideal** is left A-submodule of ${}_AA$. Similar, a **right ideal** is right A-submodule of A_A. For a left ideal $I \subseteq A$, the quotient A/I is a left A-module, but in general not an algebra.

iv) A **two-sided ideal** $I \subset A$ is both a left- and a right-ideal of A. Then A/I has an algebra structure, by setting

$$(x + I)(y + I) := (xy) + I.$$

In general, this is only well-defined if I is a two-sided ideal of A.

v) Let $f : M \to N$ be a homomorphism of left A-modules. Then we obtain left A-modules:

$$\ker f,\ \operatorname{Im} f,\ \operatorname{coker} f := N/\operatorname{Im} f,\ \operatorname{coim} f := M/\ker f.$$

In particular, f factors uniquely as

$$\begin{tikzcd} M \arrow[r] \arrow[rrr, bend left, "f"] & \operatorname{coim} f \arrow[r, "\exists!", "\cong"'] & \operatorname{Im} f \arrow[r] & N \end{tikzcd} . \tag{F}$$

vi) Let $\{M_i \subset M \mid i \in I\}$ be a family of left A-submodules, for some index set I. Then

$$\bigcap_{i \in I} M_i \text{ and } \sum_{i \in I} M_i$$

are left A-modules.

vii) Let $x \in M$. Define

$$Ax := \{ax \mid a \in A\},$$

which is a left A-submodule. Similar, for $x \in M_A$, define $xA := \{xa \mid a \in A\}$, which is a right A-submodule. For a subset $E \subset M$,

$$\sum_{x \in E} Ax = \bigcap_{\substack{E \subseteq M' \subseteq M \\ M' \text{ submodule}}} M'.$$

M is called **finitely generated**, if there are $x_1, ..., x_n \in M$, such that

$$M = \sum_{i=1}^{n} Ax_i.$$

viii) Let $\{M_i \mid i \in I\}$ be a family of left A-modules. Then

$$\prod_{i\in I} M_i := \{(x_i)_{i\in I} \mid x_i \in M_i\}$$

is called the **product** , and

$$\bigoplus_{i\in I} M_i := \{(x_i)_{i\in I} \mid x_i \in M_i,\ x_i \neq 0 \text{ for only finitely many } i\}$$

is called the **coproduct** .They are both left A-modules. The **projection**

$$\pi_j : \prod_{i\in I} M_i \to M_j,\ (x_i)_{i\in I} \mapsto x_j$$

and the **inclusion**

$$\iota_j : \bigoplus_{i\in I} x_j \mapsto (\delta_{ij} x_j)_{i\in I}$$

are morphism of left A-modules.

ix) A left A-module M is finitely generated if and only if there is a surjective homomorphism of left A-modules

$$A^n := \bigoplus_{i=1}^n A \longrightarrow\!\!\!\!\rightarrow M$$

for some $n \geq 1$. A is called **finitely presented**, if there is an exact sequence of left A-modules

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

for some $m, n \geq 1$.

Proposition 1.3.6. *Let*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0 \tag{$*$}$$

and

$$0 \longrightarrow N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \tag{$**$}$$

be sequences of left A-modules.

i) The following are equivalent:
- *a) $(*)$ is exact.*
- *b) For all left A-modules N, the sequence*

$$0 \longrightarrow \hom_A(M_3, N) \xrightarrow{f_2^*} \hom_A(M_2, N) \xrightarrow{f_1^*} \hom_A(M, N)$$

is exact.

ii) The following are equivalent:
- *a) $(**)$ is exact.*
- *b) For all left A-modules M, the sequence*

$$0 \longrightarrow \hom_A(M, N_1) \xrightarrow{g_{1,*}} \hom_A(M, N_2) \xrightarrow{g_{2,*}} \hom_A(M, N_3)$$

is exact.

Proof. We will only prove *a)* $\implies$ *b)* of *ii)*.

Lemma 1.3.7. *Let K, M, N be left A-modules, and $\zeta : K \to M$, $\varphi : M \to N$ be homomorphisms of left A-modules, such that $\varphi \circ \zeta = 0$. Then there is a unique homomorphism $\overline{\zeta}$, such that*

$$\begin{tikzcd}
K \arrow[r, "\zeta"] \arrow[rr, bend left, "0"] \arrow[dr, "\exists!\overline{\zeta}"'] & M \arrow[r, "\varphi"] & N \\
& \ker\varphi \arrow[u, hook] &
\end{tikzcd}$$

commutes.

- $g_{1,*}$ injective: Let $h \in \ker(g_{1,*})$. Then

$$\begin{tikzcd}
g_1 \circ h : \quad M \arrow[r, "h"] \arrow[rr, bend left, "0"] & N_1 \arrow[r, "g_1"] & N_2
\end{tikzcd}$$

  and since g_1 is injective, it follows $h = 0$.

- $\operatorname{Im} g_{1,*} \subseteq \ker g_{2,*}$: Since $**$ is exact, it follows that $g_2 \circ g_1 = 0$. For $h \in \operatorname{Im} g_{1,*}$ there exists an $h' : M \to N_1$ such that $h = g_1 \circ h'$, and hence $g_2 \circ h = g_2 \circ g_1 \circ h' = 0$.

- $\ker g_{2,*} \subseteq \operatorname{Im} g_{1,*}$: As $(**)$ is exact, $\ker g_2 = \operatorname{Im} g_1$ holds.
  Let $h : M \to N_2 \in \ker g_{2,*}$, i.e. $g_2 \circ h = 0$:

$$\begin{tikzcd}
& & M \arrow[d, "h"'] \arrow[dr, "0"] & \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3
\end{tikzcd} .$$

  By lemma 1.3.7, h factors uniquely through $\ker g_2 = \operatorname{Im} g_1$:

$$\begin{tikzcd}
& & M \arrow[d, "h"'] \arrow[dr, "0"] \arrow[dd, dashed, bend left] & \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \\
& & \operatorname{Im} g_1 \arrow[u, hook, "\iota"] &
\end{tikzcd} .$$

  But since g_1 is injective, (F) implies that there is a uniquely determined isomorphism $\tilde{g}_1 : N_1 \longrightarrow \operatorname{Im} g_1$.
  Putting everything together, we obtain the following commutative diagram:

$$\begin{tikzcd}
& & M \arrow[d, "h"'] \arrow[dr, "0"] \arrow[dd, dashed, bend left] & \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] \arrow[dr, dashed, bend right, "\exists!\tilde{g}_1"] & N_2 \arrow[r, "g_2"] & N_3 \\
& & \operatorname{Im} g_1 \arrow[u, hook, "\iota"] &
\end{tikzcd} .$$

Setting $h' := \tilde{g}_1^{-1} \circ h'$, we obtain

$$g_1 \circ h' = \iota \circ \tilde{g}_1 \circ \tilde{g}_1^{-1} \circ \tilde{h} = \iota \circ \tilde{h} = h.$$

$\square$

Proposition 1.3.8. *Let A be a k-algebra. To give a left A-module structur is the same as to give a k-module structure V together with a homomorphism $\varphi : A \to \operatorname{End}_k(V)$ of k-algebras. To give a right A-module structure is the same as giving a k-module structure V together with a homomorphism $\varphi : A \to \operatorname{End}_k(V)^{\mathrm{op}}$.*

1.4 Representation of quivers

Let k be a field and Q be a quiver.

Definition 1.4.1. A **representation** over k consists of

- a k-vector space X_i for all $i \in Q_0$,
- a k-linear map

  $$X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)}$$

  for each $\alpha \in Q_1$

Example 1.4.2 (Continue example 1.2.4)**.** i) Let $Q = \cdot$. Then a representation of Q is simply a k-vector space.

ii) Let $Q = \begin{tikzcd} 1 \arrow[loop right] \end{tikzcd}$. Then a representation of Q is a k-vector space V together with an endomorphism $\varphi \in \operatorname{End}_k(V)$:

$$Q = \begin{tikzcd} V \arrow[loop right, "f"] \end{tikzcd} .$$

iii) Let $Q = \begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}$, the **Kronecker Quiver**. Then a representation of Q is a pair of vector spaces V, W and two linear maps $f, g \in \hom_K(V, W)$:

$$Q = \begin{tikzcd} V \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & W \end{tikzcd}$$

Definition 1.4.3. Take X, Y to be two representations of Q over k. A **homomorphism of representations** $f : X \to Y$ is a tupel $(f_i)_{i \in Q_0}$ of linear maps $f_i : X_i \to Y_i$, such that for all

$\alpha \in Q_1$ the diagram

$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "f_{s(\alpha)}"] \arrow[d, "X_\alpha"'] & Y_{s(\alpha)} \arrow[d, "Y_\beta"] \\ X_{t(\alpha)} \arrow[r, "f_{t(\alpha)}"'] & Y_{t(\alpha)} \end{tikzcd}$$

commutes.

Example 1.4.4 (Continue example 1.4.2). i) Homomorphisms of representations are k-linear maps $X \to Y$.

ii) Homomorphisms of representations (V, φ) and (W, ψ) are k-linear maps $f : V \to W$, such that

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi"'] & W \arrow[d, "\psi"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

commutes

iii) Homomorphisms of representations (V_1, V_2, A, B) and (W_1, W_2, C, D) are pairs (f_1, f_2) of linear maps $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$, such that $A \circ f_1 = f_2 \circ A$ and $B \circ f_1 = f_2 \circ B$.

Definition 1.4.5. An **isomorphism of representations** $f : X \to Y$ is a homomorphism of representations, such that there exists $g : Y \to X$ homomorphism of representations satisfying

$$g \circ f = \mathrm{id}_X \text{ and } f \circ g = \mathrm{id}_y .$$

An isomorphism of representations is a homomorphism of representations such that each map f_i is bijective.

End of Lecture 2

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