

DG-Enhancement of Triangulated Categories

Problems with
Triangulated and Derived Categories
and What to do About It

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For this talk chain complexes has differential of degree $+1$.

1 Problems with Triangulated Categories

1.1 The Abelianess of Triangulated Categories

Definition 1. An abelian category \mathcal{A} is **semisimple abelian** or simply **semisimple** if every short exact sequence in \mathcal{A} splits.

Lemma 2. In a triangulated category every epimorphism splits.

Proof. Let $f: x \rightarrow y$ be an epimorphism in a triangulated category \mathcal{T} . We may complete f to a distinguished triangle

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma(x).$$

The composition of any two morphisms in a distinguished triangle vanishes, so $gf = 0$. It follows that $g = 0$ since f is an epimorphism. By applying the homological functor $\mathcal{T}(y, -)$ to this distinguished triangle we arrive at the following long exact sequence:

$$\cdots \rightarrow \mathcal{T}(y, x) \xrightarrow{f_*} \mathcal{T}(y, y) \xrightarrow{0} \mathcal{T}(y, z) \rightarrow \cdots$$

We find that for $\text{id}_y \in \mathcal{T}(y, y)$ there exists some $s \in \mathcal{T}(y, x)$ with $\text{id}_y = f_*(s) = fs$. \square

Corollary 3. A triangulated category that is abelian is already semisimple. \square

We see from Corollary 3 that most triangulated categories are not abelian.

Proposition 4. For an abelian category \mathcal{A} the following conditions on \mathcal{A} and its derived category $\mathbf{D}(\mathcal{A})$ are equivalent:

- (1) The derived category $\mathbf{D}(\mathcal{A})$ is abelian.
- (2) The derived category $\mathbf{D}(\mathcal{A})$ is semisimple abelian.
- (3) The abelian category \mathcal{A} is semisimple.

If these equivalent conditions are satisfied then $\mathbf{D}(\mathcal{A}) \simeq \mathcal{A}^{\mathbb{Z}}$ via the homology functor $H^*: \mathbf{D}(\mathcal{A}) \rightarrow \mathcal{A}^{\mathbb{Z}}$ with quasi-inverse $\mathcal{A}^{\mathbb{Z}} \rightarrow \mathbf{Ch}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$.

Proof.

(3) \implies (1): We realize $\mathbf{D}(\mathcal{A})$ by first passing from the category of chain complexes $\mathbf{Ch}(\mathcal{A})$ to its homotopy category $\mathbf{K}(\mathcal{A})$ and then localizing at the class of quasi-isomorphisms.

Every chain complex $X \in \text{Ob}(\mathbf{Ch}(\mathcal{A}))$ splits¹ since \mathcal{A} is semisimple and can thus be decomposed as $X = X' \oplus X''$ where X' is split acyclic and X'' has zero differential. In the homotopy category $\mathbf{K}(\mathcal{A})$ the chain complex X' becomes zero as it is split acyclic and hence contractible. Every isomorphism class in $\mathbf{K}(\mathcal{A})$ is therefore represented by a chain complex with zero differential, i.e. an object of $\mathcal{A}^{\mathbb{Z}}$. No two morphisms between such chain complexes become identified in $\mathbf{K}(\mathcal{A})$ so that the categories $\mathbf{K}(\mathcal{A})$ and $\mathcal{A}^{\mathbb{Z}}$ are equivalent. We note that this equivalence is indeed given by $H^*: \mathbf{K}(\mathcal{A}) \rightarrow \mathcal{A}^{\mathbb{Z}}$.

We also see that a quasi-isomorphism between chain complexes with zero differentials must already be an isomorphism. We therefore see that $\mathbf{D}(\mathcal{A})$ is just $\mathbf{K}(\mathcal{A})$ again.

(1) \implies (2): This follows from Corollary 3.

(2) \implies (3) We first observe that an abelian category \mathcal{B} is semisimple if and only if every morphism $f: x \rightarrow y$ in \mathcal{B} admits a **pseudoinverse** $g: y \rightarrow x$ satisfying $f g f = f$ and $g f g = g$: If every morphism f in \mathcal{B} admits such a pseudoinverse g then it follows for every epimorphism f in \mathcal{B} from $f g f = f$ that $f g = \text{id}$ so that f splits. If on the other hand \mathcal{B} is semisimple and $f: x \rightarrow y$ is any morphism in \mathcal{B} then we have decompositions $x = x' \oplus \ker(f)$ and $y = y' \oplus \text{im}(f)$ with f inducing an isomorphism $x' \rightarrow \text{im}(f)$. The inverse $\text{im}(f) \rightarrow x'$ composed with the projection $y \rightarrow \text{im}(f)$ and the inclusion $x' \rightarrow x$ then give the desired pseudoinverse $g: y \rightarrow x$.

By assumption every morphism in $\mathbf{D}(\mathcal{A})$ admits a pseudoinverse. Every morphism f in \mathcal{A} hence admits a pseudoinverse g in $\mathbf{D}(\mathcal{A})$ (where we regard \mathcal{A} as chain complexes concentrated in degree 0) which becomes the pseudoinverse $H_0(g)$ to f in \mathcal{A} . This shows that every morphism in \mathcal{A} admits a pseudoinverse, so that \mathcal{A} is semisimple. \square

¹Recall that a chain complex X is **split** if it can be (up to isomorphism) degreewise decomposed as $X^n = B^n \oplus H^n \oplus B^{n+1}$ such that the differential of X is with respect to this decomposition given by $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $Z^n(X) = B^n \oplus H^n$, $B^n(X) = B^n$ and $H^n(X) \cong H^n$. The claimed decomposition $X = X' \oplus X''$ is then given degreewise by $(X')^n = B^n \oplus B^{n+1}$ with differential $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $(X'')^n = H^n$.

1.2 Non-Functoriality of Cones

Triangulated categories do in general not admit functorial cones. As a consequence of this we see that for a triangulated category \mathcal{T} and a (small) index category I the diagram category \mathcal{T}^I does in general not inherit a triangulated structure from \mathcal{T} :

Given a morphism $f: D \rightarrow D'$ in \mathcal{T}^I we would otherwise like to compute its cone for the inherited triangulated structure of \mathcal{T}^I pointwise, i.e. for any morphism $e: i \rightarrow j$ in I we want the following commutative diagram:

$$\begin{array}{ccccc} D(i) & \xrightarrow{f_i} & D'(i) & \longrightarrow & \text{cone}_{\mathcal{T}}(f)(i) \\ D(e) \downarrow & & \downarrow D'(e) & & \downarrow \text{cone}(f)(e) \\ D(j) & \xrightarrow{f_j} & D'(j) & \longrightarrow & \text{cone}_{\mathcal{T}}(f)(j) \end{array}$$

The vertical dashed arrow comes from (TR3). But by the missing functoriality of the cone in \mathcal{T} these diagrams do not assemble into a functor $\text{cone}(f): I \rightarrow \mathcal{T}$.

The missing functoriality of the cone is in general unfixable, as the following result asserts:

Proposition 5. Let \mathcal{T} be an idempotent complete triangulated category. If \mathcal{T} admits functorial cones then \mathcal{T} is abelian and semisimple.

1.3 Difference between $\mathbf{D}(\mathcal{A})^I$ and $\mathbf{D}(\mathcal{A}^I)$

One might suspect that for $\mathcal{T} = \mathbf{D}(\mathcal{A})$, where \mathcal{A} is some abelian category, the above problems can be fixed by using an equivalence $\mathbf{D}(\mathcal{A})^I \simeq \mathbf{D}(\mathcal{A}^I)$. (Note that the category \mathcal{A}^I is again abelian.) The problem is that there is in general no such equivalence.

Take for example $\mathcal{A} = k\text{-}\mathbf{Vect}$ and let $I = (\bullet \rightarrow \bullet)$ be the arrow category. The abelian category \mathcal{A} is semisimple whence $\mathbf{D}(\mathcal{A})$ and then also $\mathbf{D}(\mathcal{A})^I$ is again abelian by Proposition 4. But the functor category \mathcal{A}^I is equivalent to the category of representations of the quiver $\bullet \rightarrow \bullet$ and therefore not abelian. (The path algebra of this quiver is isomorphic to the algebra of upper triangular matrices of size 2, which is not semisimple.) The derived category $\mathbf{D}(\mathcal{A}^I)$ is thus not again abelian. This shows that $\mathbf{D}(\mathcal{A})^I$ is abelian but $\mathbf{D}(\mathcal{A}^I)$ is not abelian, which entails that these categories are not equivalent.

1.4 (Non-)Existence of Limits and Colimits

A triangulated category \mathcal{T} is in general neither complete nor cocomplete.

We see for example that in the category $\mathbf{D}(\mathbb{Z}) = \mathbf{D}(\mathbf{Mod}_{\mathbb{Z}})$ the nonzero morphism $f: \mathbb{Z}/2 \rightarrow \mathbb{Z}/4$ does not admit a cokernel: Suppose otherwise that $c: \mathbb{Z}/4 \rightarrow C$ is a such a cokernel. Then c is a split epimorphism in $\mathbf{D}(\mathbb{Z})$ by Lemma 2. It follows that $H^0(c): \mathbb{Z}/4 \rightarrow H^0(C)$ is a split epimorphism in $\mathbf{Mod}_{\mathbb{Z}}$. But $\mathbb{Z}/4$ is indecomposable, so it follows that $H^0(c) = 0$ or $H^0(c)$ is an isomorphism.

If $H^0(c) = 0$ then we consider the nonzero morphism $g: \mathbb{Z}/4 \rightarrow \mathbb{Z}/2$. It follows from $g \circ f = 0$ that g factors through c in $\mathbf{D}(\mathbb{Z})$ and therefore factors through $H^0(c)$ in $\mathbf{Mod}_{\mathbb{Z}}$. But this is not possible since $H^0(c) = 0$ while $g \neq 0$.

Suppose now that $H^0(c)$ is an isomorphism. It follows from $c \circ f = 0$ in $\mathbf{D}(\mathbb{Z})$ that $H^0(c) \circ f = 0$ in $\mathbf{Mod}_{\mathbb{Z}}$. It now further follows that $f = 0$ which is wrong.

2 Solution: Working Derived

We give a first approach to circumventing the above problems. We denote for any category \mathcal{C} by $\mathbf{Mor}(\mathcal{C})$ its morphism category. We have $\mathbf{Mor}(\mathcal{C}) \cong \mathcal{C}^I$ for the arrow category $I = (\bullet \rightarrow \bullet)$.

2.1 Functorial Cones

We have seen that for a triangulated category \mathcal{T} there exists in general no cone functor $\mathbf{Mor}(\mathcal{T}) \rightarrow \mathcal{T}$, i.e. no cone functor $\mathcal{T}^I \rightarrow \mathcal{T}$ for $I = (\bullet \rightarrow \bullet)$. But if $\mathcal{T} = \mathbf{D}(\mathcal{A})$ for some abelian category \mathcal{A} then we have seen above that the categories $\mathcal{T}^I = \mathbf{D}(\mathcal{A})^I$ and $\mathbf{D}(\mathcal{A}^I)$ are in general not equivalent. We can therefore instead try to construct a cone functor cone: $\mathbf{D}(\mathcal{A}^I) \rightarrow \mathbf{D}(\mathcal{A})$.

This is indeed possible: We start with the usual cone functor

$$C: \mathbf{Mor}(\mathbf{Ch}(\mathcal{A})) \rightarrow \mathbf{Ch}(\mathcal{A})$$

which assigns to each morphism $f: X \rightarrow Y$ in $\mathbf{Ch}(\mathcal{A})$ the usual mapping cone

$$C(f) = \left(X[1] \oplus Y, \begin{bmatrix} -d_X & 0 \\ f & d_Y \end{bmatrix} \right)$$

and to each morphism $(g, h): f \rightarrow f'$ in $\mathbf{Mor}(\mathcal{A})$, i.e. every commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{h} & Y' \end{array}$$

in $\mathbf{Ch}(\mathcal{A})$ the induced morphism

$$C(g, h): C(f) \xrightarrow{\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}} C(f').$$

This functor is additive, and under the identification

$$\mathbf{Mor}(\mathbf{Ch}(\mathcal{A})) \cong \mathbf{Ch}(\mathcal{A})^I \cong \mathbf{Ch}(\mathcal{A}^I) \cong \mathbf{Ch}(\mathbf{Mor}(\mathcal{A}))$$

this cone functor C respect quasi-isomorphisms: The quasi-isomorphism in $\mathbf{Ch}(\mathcal{A}^I)$ correspond to a pointwise quasi-isomorphism in $\mathbf{Ch}(\mathcal{A})^I$. If $(g, h): f \rightarrow f'$ in $\mathbf{Ch}(\mathcal{A})^I$ is such a pointwise quasi-isomorphism, i.e. if both g and h are quasi-isomorphism, then

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} & C(f) & \xrightarrow{[-1 \ 0]} & X[1] \longrightarrow 0 \\ & & \downarrow h & & \downarrow C(g, h) & & \downarrow g \\ 0 & \longrightarrow & Y' & \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} & C(f') & \xrightarrow{[-1 \ 0]} & X'[1] \longrightarrow 0 \end{array}$$

is a commutative diagram with short exact rows. We get an induced ladder diagram whose rows are the long exact cone sequences and where h and g induce vertical isomorphisms. It follows from the five lemma that $C(g, h)$ also induces isomorphisms in homology, i.e. is a quasi-isomorphism. It follows that the cone functor C descends to an additive functor

$$C: \mathbf{D}(\mathbf{Mor}(\mathcal{A})) \rightarrow \mathbf{D}(\mathcal{A})$$

where $\mathbf{D}(\mathcal{A}^I) = \mathbf{D}(\mathbf{Mor}(\mathcal{A}))$.

2.2 Colimits and Limits

Let I be an index category and let $\Delta: \mathcal{C} \rightarrow \mathcal{C}^I$ be the constant diagram functor. Recall that the existence of (co)limits of I -shaped diagrams in \mathcal{C} is equivalent to Δ admitting adjoints

$$\operatorname{colim}_I \dashv \Delta \dashv \lim_I .$$

That a derived category $\mathbf{D}(\mathcal{A})$ does in general not admit (co)limits can be circumvented by considering **homotopy (co)limits** instead:

The constant diagram functor $\Delta: \mathbf{Ch}(\mathcal{A}) \rightarrow \mathbf{Ch}(\mathcal{A})^I \cong \mathbf{Ch}(\mathcal{A}^I)$ is additive and respects quasi-isomorphisms and hence descends to an additive functor

$$\Delta: \mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A}^I) .$$

If \mathcal{A} admits all (co)limits of shape I then it follows that the functor Δ admits adjoints

$$\operatorname{hocolim}_I \dashv \Delta \dashv \operatorname{holim}_I .$$

2.3 Description as Total Derived Functors

The above constructions can be understood in terms of total derived functors:

Let \mathcal{A} and \mathcal{B} be abelian categories and let $F: \mathbf{Ch}(\mathcal{A}) \rightarrow \mathbf{Ch}(\mathcal{B})$ be an additive functor, so that we have the following:

$$\begin{array}{ccc} \mathbf{Ch}(\mathcal{A}) & \xrightarrow{F} & \mathbf{Ch}(\mathcal{B}) \\ \gamma \downarrow & & \downarrow \gamma \\ \mathbf{D}(\mathcal{A}) & & \mathbf{D}(\mathcal{B}) \end{array}$$

A **total left derived functor** of the functor F is a pair (LF, ε) consisting of a functor $LF: \mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{B})$ together with a natural transformation $\varepsilon: LF\gamma \Rightarrow \gamma F$ which is terminal with this properties (in a suitable sense).

$$\begin{array}{ccc} \mathbf{Ch}(\mathcal{A}) & \xrightarrow{F} & \mathbf{Ch}(\mathcal{B}) \\ \gamma \downarrow & \nearrow \varepsilon & \downarrow \gamma \\ \mathbf{D}(\mathcal{A}) & \xrightarrow{LF} & \mathbf{D}(\mathcal{B}) \end{array}$$

By replacing “terminal” with “inital” we arrive at the definition of a **total right derived functor** RF . If F respects quasi-isomorphisms, i.e. is exact, then LF (resp. RF) is simply the induced functor $\mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{B})$ and ε is the identity transformation.

One can rephase the above constructions in the language of total derived functors:

- (1) The cone functor $C: \mathbf{D}(\mathbf{Mor}(\mathcal{A})) \rightarrow \mathbf{D}(\mathcal{A})$ constructed above is the total left derived of the cokernel functor

$$\text{coker}: \mathbf{Ch}(\mathbf{Mor}(\mathcal{A})) \cong \mathbf{Mor}(\mathbf{Ch}(\mathcal{A})) \rightarrow \mathbf{Ch}(\mathcal{A}).$$

- (2) Let I be an index category and suppose that \mathcal{A} admits all (co)limits of shape I . Then the constant diagram functor $\Delta_{\mathbf{Ch}(\mathcal{A})}: \mathbf{Ch}(\mathcal{A}) \rightarrow \mathbf{Ch}(\mathcal{A})^I$ admits the adjoints

$$\text{colim}_I \dashv \Delta \dashv \lim_I.$$

The functors

$$\text{colim}_I, \lim_I: \mathbf{Ch}(\mathcal{A}^I) \cong \mathbf{Ch}(\mathcal{A})^I \rightarrow \mathbf{Ch}(\mathcal{A})$$

admit total derived functors

$$\text{Lcolim}_I, \text{Rlim}_I: \mathbf{D}(\mathcal{A}^I) \rightarrow \mathbf{D}(\mathcal{A}).$$

These are precisely the functors hocolim_I and holim_I introduced above. Hence the adjunction

$$\text{colim}_I \dashv \Delta \dashv \lim_I.$$

descends to a derived adjunction

$$\text{hocolim}_I = \text{Lcolim}_I \dashv \Delta \dashv \text{Rlim}_I = \text{holim}_I.$$

3 Solution: DG-Enhancement

In the following we denote by k some commutative ring.

A dg enhancement of category \mathcal{T} is, roughly speaking, a dg-category \mathcal{A} together with an equivalence $\mathcal{T} \simeq \mathbf{H}^0(\mathcal{A})$. If the category \mathcal{T} carries additional structures which we want to be respected by this equivalence, then we need to make sure that the homotopy category $\mathbf{H}^0(\mathcal{A})$ does carry such a structure itself. If \mathcal{T} is a triangulated category then this leads us to the notion of a pretriangulated dg-category.

3.1 Review on DG-Categories

A dg-category \mathcal{A} consists of a class of objects $\text{Ob}(\mathcal{A})$, for any two objects $x, y \in \mathcal{A}$ a chain complex $\mathcal{A}(x, y)$, for any three objects $x, y, z \in \text{Ob}(\mathcal{A})$ a morphism of chain complexes

$$(-) \circ (-): \mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \rightarrow \mathcal{A}(x, z)$$

satisfying the usual associativity diagram, and for every object $x \in \text{Ob}(\mathcal{A})$ an element $1_x \in Z^0(\mathcal{A}(x, y))$ satisfying the usual identity diagrams. As an example we enrich $\mathbf{Ch}(k)$ into a dg-category $\underline{\mathbf{Ch}}(k)$ with $\underline{\mathbf{Ch}}(k)(X, Y) = \underline{\text{Hom}}(X, Y)$ being the usual Hom-chain complex.

To any dg-category \mathcal{A} we can associated the k -linear categories $Z^0(\mathcal{A})$ and $H^0(\mathcal{A})$ that are given by $\text{Ob}(Z^0(\mathcal{A})) = \text{Ob}(H^0(\mathcal{A})) = \text{Ob}(\mathcal{A})$ and

$$Z^0(\mathcal{A})(x, y) = Z^0(\mathcal{A}(x, y)) \quad \text{and} \quad H^0(\mathcal{A})(x, y) = H^0(\mathcal{A}(x, y)).$$

As an example we have $Z^0(\underline{\mathbf{Ch}}(k)) = \mathbf{Ch}(k)$ and $H^0(\underline{\mathbf{Ch}}(k)) = \mathbf{K}(k)$.

This examples motivates that one should think about $Z^0(\mathcal{A})$ as the **underlying k -linear category** of \mathcal{A} , and as the elements of $Z^0(\mathcal{A}(x, y))$ as the “actual morphisms” from x to y . The category $H^0(\mathcal{A})$ is the **homotopy category** of \mathcal{A} .

A **dg-functor** $F: \mathcal{A} \rightarrow \mathcal{B}$ is given by a map $F: \text{Ob}(\mathcal{A}) \rightarrow \text{Ob}(\mathcal{B})$ and morphisms of chain complexes $F_{x,y}: \mathcal{A}(x, y) \rightarrow \mathcal{B}(F(x), F(y))$ for any two objects $x, y \in \text{Ob}(\mathcal{A})$, satisfying the usual axioms. Every dg-functor $F: \mathcal{A} \rightarrow \mathcal{B}$ induces k -linear functors $Z^0(\mathcal{A}) \rightarrow Z^0(\mathcal{B})$ and $H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$.

A dg-natural transformation $\alpha: F \Rightarrow G$ between dg-functors $F, G: \mathcal{A} \rightarrow \mathcal{B}$ assigns to is a family $\alpha = (\alpha_x)_{x \in \text{Ob}(\mathcal{A})}$ of “actual morphisms” $\alpha_x \in Z^0(\mathcal{B}(F(x), G(x)))$ with $\alpha_y \circ F(f) = G(f) \circ \alpha_x$ for every $f \in \mathcal{A}(x, y)$. We get a k -linear abelian category $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$ whose objects are dg-functors $\mathcal{A} \rightarrow \mathcal{B}$ and whose morphisms are dg-natural transformations. This k -linear category can be enriched in a dg-category $\underline{\mathbf{dgFun}}(\mathcal{A}, \mathcal{B})$. Then in particular

$$Z^0(\underline{\mathbf{dgFun}}(\mathcal{A}, \mathcal{B})) = \mathbf{dgFun}(\mathcal{A}, \mathcal{B}),$$

so that $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$ is the underlying k -linear category of $\underline{\mathbf{dgFun}}(\mathcal{A}, \mathcal{B})$.

For a dg-category \mathcal{A} a **(right) dg- \mathcal{A} -module** is a dg-functor $M: \mathcal{A}^{\text{op}} \rightarrow \underline{\mathbf{Ch}}(k)$. This means that at every object $x \in \text{Ob}(\mathcal{A})$ we have a chain complex M_x , and for every $f \in \mathcal{A}(x, y)^i$ we have associated a map $M(f): M(y) \rightarrow M(x)$ of degree i such that $M(g \circ f) = (-1)^{ij} M(f) \circ M(g)$ for all $f \in \mathcal{A}(x, y)^i$ and $g \in \mathcal{A}(y, z)^j$. The category of dg- \mathcal{A} -modules is given by

$$\mathbf{dgMod}_{\mathcal{A}} = \mathbf{dgFun}(\mathcal{A}^{\text{op}}, \underline{\mathbf{Ch}}(k))$$

and its dg-enrichment is given by

$$\underline{\mathbf{dgMod}}_{\mathcal{A}} = \underline{\mathbf{dgFun}}(\mathcal{A}^{\text{op}}, \underline{\mathbf{Ch}}(k)).$$

The category $\mathbf{dgMod}_{\mathcal{A}}$ is abelian, complete and cocomplete, and all (co)limits are computed pointwise.

3.2 Review on Frobenius Exact Structures

The abelian category $\mathbf{Ch}(k)$ admits a Frobenius exact structure \mathcal{S}_k whose associated \mathcal{S}_k -stable triangulated category is precisely $\mathbf{K}(k)$. For every chain complex X its \mathcal{S}_k -**injective envelope** is given by the chain complex

$$IX = \left(X \oplus X[1], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

which fits into the short exact sequence

$$0 \rightarrow X \xrightarrow{i_X} IX \xrightarrow{p_X} X[1] \rightarrow 0 \quad (1)$$

belonging to \mathcal{S}_k , where

$$i_X = \begin{bmatrix} 1 \\ d_X \end{bmatrix} \quad \text{and} \quad p_X = \begin{bmatrix} -d_X & 1 \end{bmatrix}.$$

The k -linear situation

$$I, [1]: \mathbf{Ch}(k) \rightarrow \mathbf{Ch}(k), \quad i: \text{id} \Rightarrow I, p: I \Rightarrow [1]$$

extends into the dg-world as

$$I, [1]: \underline{\mathbf{Ch}}(k) \rightarrow \underline{\mathbf{Ch}}(k), \quad i: \text{id} \Rightarrow I, p: I \Rightarrow [1]$$

and by applying $\mathbf{dgFun}(\mathcal{A}^{\text{op}}, -)$ we arrive at the k -linear situation

$$I, [1]: \mathbf{dgMod}_{\mathcal{A}} \rightarrow \mathbf{dgMod}_{\mathcal{A}}, \quad i: \text{id} \Rightarrow I, p: I \Rightarrow [1].$$

We get for every dg- \mathcal{A} -module M a short exact sequence of dg- \mathcal{A} -modules

$$0 \rightarrow M \xrightarrow{i_M} IM \xrightarrow{p_M} M[1] \rightarrow 0 \quad (2)$$

that gives at every point $x \in \text{Ob}(\mathcal{A})$ the short exact sequence of chain complexes

$$0 \rightarrow M_x \xrightarrow{i_{M_x}} IM_x \xrightarrow{p_{M_x}} M_x[1] \rightarrow 0$$

from (1) with $X = M_x$. The short exact sequence (2) belongs to a Frobenius exact structure $\mathcal{S}_{\mathcal{A}}$ on $\mathbf{dgMod}_{\mathcal{A}}$ whose stable triangulated category is precisely the homotopy category $\mathbf{H}^0(\mathbf{dgMod}_{\mathcal{A}})$, with $i_M: M \rightarrow IM$ being an $\mathcal{S}_{\mathcal{A}}$ -injective envelope of M for every dg- \mathcal{A} -module M . This description of $\mathbf{H}^0(\mathbf{dgMod}_{\mathcal{A}})$ and its triangulated structure has two consequences we will need:

- (1) A morphism $f: M \rightarrow N$ in $\mathbf{dgMod}_{\mathcal{A}}$ vanishes in $\mathbf{H}^0(\mathbf{dgMod}_{\mathcal{A}})$ if and only if it factors through some $\mathcal{S}_{\mathcal{A}}$ -injective object I of $\mathbf{dgMod}_{\mathcal{A}}$, i.e. if there exists in $\mathbf{dgMod}_{\mathcal{A}}$ a commutative diagram of the following form:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow & \nearrow \\ & I & \end{array}$$

The morphism $M \rightarrow I$ factors through the morphism $i_M: M \rightarrow IM$ since we have by the definition of an $\mathcal{S}_{\mathcal{A}}$ -injective object the following diagram:

$$\begin{array}{ccccccc} & & I & \xleftarrow{\quad} & & & \\ & \uparrow & & \swarrow & & & \\ 0 & \longrightarrow & M & \xrightarrow{i_M} & IM & \xrightarrow{p_M} & M[1] \longrightarrow 0 \end{array}$$

It follows that f already factors through $i_M: M \rightarrow IM$.

- (2) We get a description of the distinguished triangles in $H^0(\mathbf{dgMod}_{\mathcal{A}})$ from Happel's theorem: Let

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{i_M} & IM & \xrightarrow{p_M} & M[1] \longrightarrow 0 \\ & & \downarrow f & \lrcorner & \downarrow & & \parallel \\ 0 & \longrightarrow & N & \xrightarrow{g} & P & \xrightarrow{h} & M[1] \longrightarrow 0 \end{array}$$

be a commutative diagram in $\mathbf{dgMod}_{\mathcal{A}}$ whose rows are contained in $\mathcal{S}_{\mathcal{A}}$ and where the left hand square is a pushout square. Then the resulting sequence

$$M \xrightarrow{f} N \xrightarrow{g} P \xrightarrow{h} M[1]$$

in $H^0(\mathbf{dgMod}_{\mathcal{A}})$ is a distinguished triangle. (And every distinguished triangle is up to isomorphism of this form.)

3.3 Pretriangulated DG-Categories

Definition 6. A dg-functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is **dg-fully faithful** if for every two objects $x, y \in \mathcal{A}$ the morphism of chain complexes $F_{x,y}: \mathcal{A}(x, y) \rightarrow \mathcal{B}(F(x), F(y))$ is an isomorphism.

Proposition 7 (dg-Yoneda embedding). Let \mathcal{A} be a dg-category. Then the mapping

$$\mathcal{A} \rightarrow \mathbf{dgMod}_{\mathcal{A}}, \quad x \mapsto \mathcal{A}(-, x) = \mathcal{A}^{\text{op}}(x, -)$$

extends (in the usual way) to a dg-fully faithful dg-functor.

Definition 8. A dg-category \mathcal{A} is **pretriangulated** if the fully faithful k -linear functor

$$H^0(\mathcal{A}) \rightarrow H^0(\mathbf{dgMod}_{\mathcal{A}})$$

that is induced by the Yoneda embedding $\mathcal{A} \rightarrow \mathbf{dgMod}_{\mathcal{A}}$ identifies \mathcal{A} with a triangulated subcategory of $H^0(\mathbf{dgMod}_{\mathcal{A}})$.

The above definition ensures that for a pretriangulated dg-category \mathcal{A} its homotopy category $H^0(\mathcal{A})$ does carry in a canonical way the structure of a triangulated category.

Definition 9. A **dg-enhancement** of a triangulated category \mathcal{T} is a pretriangulated category \mathcal{A} together with an equivalence of triangulated categories $\mathcal{T} \simeq H^0(\mathcal{A})$.

A dg-enhancement of a triangulated category \mathcal{T} allows us to identify \mathcal{T} with a full triangulated subcategory of $H^0(\mathbf{dgMod}_{\mathcal{A}})$ for some dg-category \mathcal{A} . We can then try to understand the original triangulated category \mathcal{T} through the “higher structure” of the dg-category \mathcal{A} .

3.4 Cones as Derived Cokernels

Recall that if $f: x \rightarrow y$ is a morphism in an (pre)additive category \mathcal{A} then its cokernel (which does not need to exist) can be thought of in two equivalent ways:

- A morphism $\text{coker}(f) \rightarrow z$ into any other object is “the same” as a morphism $g: y \rightarrow z$ with $gf = 0$.
- The cokernel $\text{coker}(f)$ is the pushout of the following diagram:

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \\ 0 & & \end{array}$$

In a triangulated category \mathcal{T} we do in general have neither cokernels nor pushouts. But given a dg-enhancement $\mathcal{T} \simeq H^0(\mathcal{A})$ for some pretriangulated dg-category \mathcal{A} we can identify \mathcal{T} with a full triangulated subcategory of $H^0(\mathbf{dgMod}_{\mathcal{A}})$. We can then do the desired calculations in the abelian category $\mathbf{dgMod}_{\mathcal{A}} = Z^0(\mathbf{dgMod}_{\mathcal{A}})$ which is (co)complete.

More precisely, given a fixed morphism $f: M \rightarrow N$ in $\mathbf{dgMod}_{\mathcal{A}}$ we try to understand all morphisms $g: N \rightarrow P$ in $\mathbf{dgMod}_{\mathcal{A}}$ for which the composition $g \circ f$ vanishes in $H^0(\mathbf{dgMod}_{\mathcal{A}})$. We have previously seen that $g \circ f$ vanishes if and only if $g \circ f$ factors through $i_M: M \rightarrow IM$. We are hence interested in the pushout of the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ i_M \downarrow & & \\ IM & & \end{array}$$

in $\mathbf{dgMod}_{\mathcal{A}}$. This pushout is computed pointwise, hence we want to compute for every $x \in \text{Ob}(\mathcal{A})$ the pushout

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & N_x \\ i_{M_x} \downarrow & & \\ IM_x & & \end{array}$$

in the category $\mathbf{Ch}(k)$ where

$$IM_x = \left(M_x \oplus M_x[1], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \quad \text{and} \quad i_{M_x} = \begin{bmatrix} 1 \\ d_{M_x} \end{bmatrix}.$$

Lemma 10. Every morphism of chain complexes $f: X \rightarrow Y$ results in the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_X \downarrow & \lrcorner & \downarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ IX & \xrightarrow{\begin{bmatrix} -d_X & 1 \\ f & 0 \end{bmatrix}} & C(f) \end{array}$$

where $C(f)$ denotes the usual mapping cone.

Proof. We calculate the pushout as

$$\begin{aligned} & \left(X \oplus X[1] \oplus Y, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_Y \end{bmatrix} \right) / \left\langle \begin{bmatrix} x \\ d(x) \\ -f(x) \end{bmatrix} \mid x \in X \right\rangle \\ & \cong \left(X[1] \oplus Y, \begin{bmatrix} -d_X & 0 \\ f & d_Y \end{bmatrix} \right) \\ & = C(f) \end{aligned}$$

as claimed. \square

Remark 11. The above result is (a posteriori) not surprising: It is a standard statement from homological algebra that for a morphism of chain complexes $f: X \rightarrow Y$ the data of a morphism $h: C(f) \rightarrow Z$ is the same as that of a morphism $g: Y \rightarrow Z$ together with a null homotopy of the composition $g \circ f: X \rightarrow Z$. Whence $C(f)$ corepresents morphisms going out of Y whose composition with f are zero “up to homotopy”.

We hence find that the pushout

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ i_M \downarrow & & \downarrow \\ IM & \longrightarrow & C(f) \end{array}$$

is given by $(C(f))_x = C(f_x)$ at every $x \in \text{Ob}(\mathcal{A})$, i.e. the dg- \mathcal{A} -module $C(f)$ is pointwise given by the usual mapping cone of chain complexes.

We see that $C(f)$ is actually the cone of f in the triangulated category $H^0(\mathbf{dgMod}_{\mathcal{A}})$: We have the following commutative diagram in which the left hand side is a pushout, and where the upper row is exact and contained in the Frobenius exact structure $\mathcal{S}_{\mathcal{A}}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{i_M} & IM & \xrightarrow{p_M} & M[1] \longrightarrow 0 \\ & & f \downarrow & & \downarrow & & \\ & & N & \longrightarrow & C(f) & & \end{array}$$

It follows from the standard lemma of homological algebra that we can extend the above diagram to a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & M & \xrightarrow{i_M} & IM & \xrightarrow{p_M} & M[1] \longrightarrow 0 \\
& & \downarrow f & & \downarrow & & \downarrow \text{\tiny $\scriptstyle \circ$} \\
0 & \dashrightarrow & N & \longrightarrow & C(f) & \dashrightarrow & M[1] \dashrightarrow 0
\end{array}$$

whose row are exact. It follows from the axioms of a Frobenius exact structure that the lower row is again contained in $\mathcal{S}_{\mathcal{A}}$. We have seen in the previous review on Frobenius exact structures that

$$M \xrightarrow{f} N \rightarrow C(f) \rightarrow M[1]$$

is therefore a distinguished triangle in $H^0(\mathbf{dgMod}_{\mathcal{A}})$. This entails that $C(f)$ is a cone of f in $H^0(\mathbf{dgMod}_{\mathcal{A}})$.

We have thus seen that one can think about the cone in $H^0(\mathbf{dgMod}_{\mathcal{A}})$ as a kind of “derived cokernel” of f , coming from a pushout in $\mathbf{dgMod}_{\mathcal{A}}$ that was chosen as a “cokernel up to homotopy”.

3.5 Extending to $\mathbf{dgMod}_{\mathcal{A}}$

The object $C(f)$ of the dg-category $\mathcal{B} := \mathbf{dgMod}_{\mathcal{A}}$ gives a corepresentable dg-functor, i.e. left dg-module

$$\widehat{C} := \mathcal{B}(C(f), -) : \mathcal{B} \rightarrow \mathbf{Ch}(k).$$

At every object $P \in \text{Ob}(\mathcal{B})$ this left dg-module is given by the chain complex $\widehat{C}(P)$ which is given as a graded module by

$$\widehat{C}(P) = \mathcal{B}(N, P)[1] \oplus \mathcal{B}(M, P)$$

and whose differential is

$$d_{\widehat{C}(P)} = \begin{bmatrix} d_{\mathcal{B}(N, P)} & 0 \\ (-) \circ f & -d_{\mathcal{B}(M, P)} \end{bmatrix}.$$

For any two objects $P, Q \in \text{Ob}(\mathcal{B})$ the morphism of chain complexes

$$\widehat{C}_{P, Q} : \mathcal{B}(P, Q) \rightarrow \mathcal{B}(\widehat{C}(P), \widehat{C}(Q))$$

is given by

$$g \mapsto ((a, b) \mapsto (g \circ a, (-1)^{|g|} g \circ b)).$$

We note that these formulae make sense in any dg-category. We can therefore define for every dg-category \mathcal{B} and every $f \in Z^0(\mathcal{B}(x, y))$ a left dg- \mathcal{B} -module $C : \mathcal{B} \rightarrow \mathbf{Ch}(k)$ by the same expressions as above. One can then define a **cone** of f in \mathcal{B} as a representing object for this dg-functor. This allows us to talk about cones of morphisms in dg-categories.