DG-Enhancement of Triangulated Categories

Problems with Triangulated and Derived Categories and What to do About It

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For this talk chain complexes has differential of degree $+1$.

1. Problems with Triangulated Categories

1.1. The Abelianess of Triangulated Categories

Definition 1. An abelian category $\mathcal{A}$ is **semisimple abelian** or simply **semisimple** if every short exact sequence in $\mathcal{A}$ splits.

Lemma 2. In a triangulated category every epimorphism splits.

Proof. See Appendix A.1. □

Corollary 3. A triangulated category that is abelian is already semisimple. □

We see from Corollary 3 that most triangulated categories are not abelian.

Proposition 4. For an abelian category $\mathcal{A}$ the following conditions on $\mathcal{A}$ and its derived category $\mathbf{D}(\mathcal{A})$ are equivalent:

(1) The derived category $\mathbf{D}(\mathcal{A})$ is abelian.

(2) The derived category $\mathbf{D}(\mathcal{A})$ is semisimple abelian.

(3) The abelian category $\mathcal{A}$ is semisimple.

If these equivalent conditions are satisfied then $\mathbf{D}(\mathcal{A}) \simeq \mathcal{A}^{\mathbb{Z}}$ via the homology functor $\mathrm{H}^* \colon \mathbf{D}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$ with quasi-inverse $\mathcal{A}^{\mathbb{Z}} \to \mathbf{Ch}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$.

Proof. See Appendix A.2. □

1.2. Non-Functoriality of Cones

Triangulated categories do in general not admit functorial cones. As a consequence of this we see that for a triangulated category $\mathcal{T}$ and a (small) index category I the diagram category $\mathcal{T}^I$ does in general not inherit a triangulated structure from $\mathcal{T}$:

Given a morphism $f\colon D \to D'$ in $\mathcal{T}^I$ we would otherwise like to compute its cone for the inherited triangulated structure of $\mathcal{T}^I$ pointwise, i.e. for any morphism $e\colon i \to j$ in I we want the following commutative diagram:

$$\begin{tikzcd}
D(i) \arrow[r, "f_i"] \arrow[d, "D(e)"'] & D'(i) \arrow[r] \arrow[d, "D'(e)"] & \mathrm{cone}_{\mathcal{T}}(f)(i) \arrow[d, dashed, "\mathrm{cone}(f)(e)"] \\
D(j) \arrow[r, "f_j"'] & D'(j) \arrow[r] & \mathrm{cone}_{\mathcal{T}}(f)(j)
\end{tikzcd}$$

The vertical dashed arrow comes from (TR3). But by the missing functoriality of the cone in $\mathcal{T}$ these diagrams do not assemble into a functor $\mathrm{cone}(f)\colon I \to \mathcal{T}$.

The missing functoriality of the cone is in general unfixable, as the following result asserts:

Proposition 5. Let $\mathcal{T}$ be an idempotent complete triangulated category. If $\mathcal{T}$ admits functorial cones then $\mathcal{T}$ is abelian and semisimple.

1.3. Difference between $\mathrm{D}(\mathcal{A})^I$ and $\mathrm{D}(\mathcal{A}^I)$

One might suspect that for $\mathcal{T} = \mathbf{D}(\mathcal{A})$, where $\mathcal{A}$ is some abelian category, the above problems can be fixed by using an equivalence $\mathbf{D}(\mathcal{A})^I \simeq \mathbf{D}(\mathcal{A}^I)$. (Note that the category $\mathcal{A}^I$ is again abelian.) The problem is that there is in general no such equivalence.

Take for example $\mathcal{A} = k\text{-}\mathbf{Vect}$ and let $I = (\bullet \to \bullet)$ be the arrow category. The abelian category $\mathcal{A}$ is semisimple whence $\mathbf{D}(\mathcal{A})$ and then also $\mathbf{D}(\mathcal{A})^I$ is again abelian by Proposition 4. But the functor category $\mathcal{A}^I$ is equivalent to the category of representations of the quiver $\bullet \to \bullet$ and therefore not abelian. (The path algebra of this quiver is isomorphic to the algebra of upper triangular matrices of size 2, which is not semisimple.) The derived category $\mathbf{D}(\mathcal{A}^I)$ is thus not again abelian. This shows that $\mathbf{D}(\mathcal{A})^I$ is abelian but $\mathbf{D}(\mathcal{A}^I)$ is not abelian, which entails that these categories are not equivalent.

1.4. (Non-)Existence of Limits and Colimits

A triangulated category $\mathcal{T}$ is in general neither complete nor cocomplete. See Appendix A.3 for an explicit counterexample.

2. Solution: Working Derived

We give a first approach to dealing with the above problems. We denote for any category $\mathcal{C}$ by $\mathbf{Mor}(\mathcal{C})$ its morphism category. Then $\mathbf{Mor}(\mathcal{C}) \cong \mathcal{C}^I$ for $I = (\bullet \to \bullet)$.

2.1. Functorial Cones

We have seen that for a triangulated category $\mathcal{T}$ there exists in general no cone functor $\mathbf{Mor}(\mathcal{T}) \to \mathcal{T}$, i.e. no cone functor $\mathcal{T}^I \to \mathcal{T}$ for $I = (\bullet \to \bullet)$. But if $\mathcal{T} = \mathbf{D}(\mathcal{A})$ for some abelian category $\mathcal{A}$ then we have seen above that the categories $\mathcal{T}^I = \mathbf{D}(\mathcal{A})^I$ and $\mathbf{D}(\mathcal{A}^I)$ are in general not equivalent. We can therefore instead try to construct a cone functor cone: $\mathbf{D}(\mathcal{A}^I) \to \mathbf{D}(\mathcal{A})$.

This is indeed possible: We start with the usual cone functor

$$C\colon \mathbf{Mor}(\mathbf{Ch}(\mathcal{A})) \to \mathbf{Ch}(\mathcal{A})$$

which assigns to each morphism $f\colon X \to Y$ in $\mathbf{Ch}(\mathcal{A})$ the usual mapping cone

$$C(f) = \left(X[1] \oplus Y, \begin{bmatrix} -d_X & 0 \\ f & d_Y \end{bmatrix} \right)$$

and to each morphism $(g, h)\colon f \to f'$ in $\mathbf{Mor}(\mathcal{A})$, i.e. every commmutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "h"] & Y' \end{tikzcd}$$

in $\mathbf{Ch}(\mathcal{A})$ the induced morphism

$$C(g,h)\colon C(f) \xrightarrow{\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}} C(f')\,.$$

This functor is additive, and under the identification

$$\mathbf{Mor}(\mathbf{Ch}(\mathcal{A})) \cong \mathbf{Ch}(\mathcal{A})^I \cong \mathbf{Ch}(\mathcal{A}^I) \cong \mathbf{Ch}(\mathbf{Mor}(\mathcal{A}))$$

this cone functor C respect quasi-isomorphisms: The quasi-isomorphism in $\mathbf{Ch}(\mathcal{A}^I)$ correspond to a pointwise quasi-isomorphism in $\mathbf{Ch}(\mathcal{A})^I$. If $(g, h)\colon f \to f'$ in $\mathbf{Ch}(\mathcal{A})^I$ is such a pointwise quasi-isomorphism, i.e. if both g and h are quasi-isomorphism, then

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}"] \arrow[d, "h"'] & C(f) \arrow[r, "{[\,-1\ 0\,]}"] \arrow[d, "{C(g,h)}"] & X[1] \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & Y' \arrow[r, "{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}"] & C(f') \arrow[r, "{[\,-1\ 0\,]}"] & X'[1] \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram with short exact rows. We get an induced ladder diagram whose rows are the long exact cone sequences and where h and g induce vertical isomorphisms. It follows from the five lemma that $C(g, h)$ also induces isomorphisms in homology, i.e. is a quasi-isomorphism. It follows that the cone functor C descends to an additive functor

$$C\colon \mathbf{D}(\mathbf{Mor}(\mathcal{A})) \to \mathbf{D}(\mathcal{A})\,.$$

2.2. Colimits and Limits

Let I be an index category and let $\Delta\colon \mathcal{C} \to \mathcal{C}^I$ be the constant diagram functor. Recall that the existence of (co)limits of I-shaped diagrams in $\mathcal{C}$ is equivalent to Δ admiting adjoints

$$\operatorname*{colim}_I \dashv \Delta \dashv \lim_I .$$

That a derived category $\mathbf{D}(\mathcal{A})$ does in general not admit (co)limits can be circumvented by considering **homotopy (co)limits** instead:

The constant diagram functor $\Delta\colon \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A})^I \cong \mathbf{Ch}(\mathcal{A}^I)$ is additive and respects quasi-isomorphisms and hence descends to an additive functor

$$\Delta\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A}^I)\,.$$

If $\mathcal{A}$ admits all (co)limits of shape I then it follows that the functor Δ admits adjoints

$$\operatorname*{hocolim}_I \dashv \Delta \dashv \operatorname*{holim}_I .$$

2.3. Description as Total Derived Functors

The above constructions can be understood in terms of total derived functors:

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F\colon \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{B})$ be an additive functor, so that we have the following:

$$\begin{tikzcd}
\mathbf{Ch}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathbf{Ch}(\mathcal{B}) \arrow[d, "\gamma"] \\
\mathbf{D}(\mathcal{A}) & \mathbf{D}(\mathcal{B})
\end{tikzcd}$$

A **total left derived functor** of the functor F is a pair $(\mathrm{L}F, \varepsilon)$ consisting of a functor $\mathrm{L}F\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ together with a natural transformation $\varepsilon\colon \mathrm{L}F \circ \gamma \Rightarrow \gamma \circ F$ which is terminal with this properties (in a suitable sense).

$$\begin{tikzcd}
\mathbf{Ch}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Nearrow\varepsilon"] & \mathbf{Ch}(\mathcal{B}) \arrow[d, "\gamma"] \\
\mathbf{D}(\mathcal{A}) \arrow[r, "\mathrm{L}F"] & \mathbf{D}(\mathcal{B})
\end{tikzcd}$$

By replacing "terminal" with "inital" we arrive at the definition of a **total right derived functor** $\mathrm{R}F$. If F is respects quasi-isomporphisms, i.e. is exact, then $\mathrm{L}F$ (resp. $\mathrm{R}F$) is simply the induced induced functor $\mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ and ε is the identity transformation.

One can rephase the above constructions in the language of total derived functors:

(1) The cone functor $C\colon \mathbf{D}(\mathbf{Mor}(\mathcal{A})) \to \mathbf{D}(\mathcal{A})$ constructed above is the total left derived of the cokernel functor

$$\operatorname{coker}\colon \mathbf{Ch}(\mathbf{Mor}(\mathcal{A})) \cong \mathbf{Mor}(\mathbf{Ch}(\mathcal{A})) \to \mathbf{Ch}(\mathcal{A})\,.$$

(2) Let I be an index category and suppose that $\mathcal{A}$ admits all (co)limits of shape I. Then the constant diagram functor $\Delta\colon \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A})^I$ admits adjoints

$$\operatorname*{colim}_I \dashv \Delta \dashv \lim_I\,.$$

The functors

$$\operatorname*{colim}_I, \lim_I\colon \mathbf{Ch}(\mathcal{A}^I) \cong \mathbf{Ch}(\mathcal{A})^I \to \mathbf{Ch}(\mathcal{A})$$

admit total derived functors

$$\operatorname*{Lcolim}_I, \operatorname*{Rlim}_I\colon \mathbf{D}(\mathcal{A}^I) \to \mathbf{D}(\mathcal{A})\,.$$

These are precisely the functors $\operatorname{hocolim}_I$ and holim_I introduced above. Hence the adjunction

$$\operatorname*{colim}_I \dashv \Delta \dashv \lim_I\,.$$

descends to a derived adjunction

$$\operatorname*{Lcolim}_I = \operatorname*{hocolim}_I \dashv \Delta \dashv \operatorname*{holim}_I = \operatorname*{Rlim}_I\,.$$

Remark 6. Let I be an index set and let $\mathcal{A}$ be an abelian category. The localization functor $\mathbf{Ch}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ induces functor of diagram categories

$$\mathbf{Ch}(\mathcal{A}^I) \cong \mathbf{Ch}(\mathcal{A})^I \to \mathbf{D}(\mathcal{A})^I$$

This functor maps quasi-isomorphisms to isomorphisms and hence induces a functor

$$\mathbf{D}(\mathcal{A}^I) \to \mathbf{D}(\mathcal{A})^I\,,$$

which we will call the **forgetful functor**. One may think about $\mathbf{D}(\mathcal{A}^I)$ as consisting of diagrams in $\mathcal{A}$ of shape I which strictly commute, whereas $\mathbf{D}(\mathcal{A})^I$ then consists of diagram of shape I which commute only "up to homotopy".

We have seen above that we have functorial constructions $\mathbf{D}(\mathcal{A}^I) \to \mathbf{D}(\mathcal{A})$ for some of our problems. We can now see why these does not lead to solutions to the original problems, which require functors $\mathbf{D}(\mathcal{A})^I \to \mathbf{D}(\mathcal{A})$: This would require the functors $\mathbf{D}(\mathcal{A}^I) \to \mathbf{D}(\mathcal{A})$ to extend along the forgetful functor $\mathbf{D}(\mathcal{A}^I) \to \mathbf{D}(\mathcal{A})^I$, which they have no reason to do.

3. Solution: DG-Enhancement

In the following we denote by k some commutative ring.

A dg enhancement of category $\mathcal{T}$ is, roughly speaking, a dg-category $\mathcal{A}$ together with an equivalence $\mathcal{T} \simeq \mathrm{H}^0(\mathcal{A})$. If the category $\mathcal{T}$ carries additional structures which we want to be respected by this equivalence, then we need to make sure that the homotopy category $\mathrm{H}^0(\mathcal{A})$ does carry such a structure itself. If $\mathcal{T}$ is a triangulated category then this leads us to the notion of a pretriangulated dg-category.

3.1. Notations on DG-Categories

We denote by $\underline{\mathbf{Ch}}(k)$ the dg-category of chain complexes over k. For any dg-category $\mathcal{A}$ we denote by

$$\mathbf{dgMod}_{\mathcal{A}} = \mathbf{dgFun}(\mathcal{A}^{\mathrm{op}}, \underline{\mathbf{Ch}}(k))$$

the k-linear dg-category of right dg-$\mathcal{A}$-modules, and by

$$\underline{\mathbf{dgMod}}_{\mathcal{A}} = \underline{\mathbf{dgFun}}(\mathcal{A}^{\mathrm{op}}, \underline{\mathbf{Ch}}(k))$$

its dg-enrichement. The category $\mathbf{dgMod}_{\mathcal{A}}$ is abelian and (co)complete; all (co)limits are computed pointwise. Recall that

$$\mathrm{Z}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}}) = \mathbf{dgMod}_{\mathcal{A}}\,.$$

(See Appendix A.4 for a more detailed review on dg-categories.)

3.2. Review on Frobenius Exact Structures

The abelian category $\mathbf{Ch}(k)$ admits a Frobenius exact structure $\mathcal{S}_k$ whose associated $\mathcal{S}_k$-stable triangulated category is precisely $\mathbf{K}(k)$. For every chain complex X its **$\mathcal{S}_k$-injective envelope** is given by the chain complex

$$IX = \left(X \oplus X[1], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

which fits into the short exact sequence

$$0 \to X \xrightarrow{i_X} IX \xrightarrow{p_X} X[1] \to 0 \tag{1}$$

belonging to $\mathcal{S}_k$, where

$$i_X = \begin{bmatrix} 1 \\ d_X \end{bmatrix} \quad \text{and} \quad p_X = \begin{bmatrix} -d_X & 1 \end{bmatrix}\,.$$

The k-linear functors and natural transformations

$$I, [1] \colon \mathbf{Ch}(k) \to \mathbf{Ch}(k)\,, \qquad i \colon \mathrm{id} \Rightarrow I,\, p \colon I \Rightarrow [1]$$

extends to dg-functors and dg-natural transformations

$$I, [1] \colon \underline{\mathbf{Ch}}(k) \to \underline{\mathbf{Ch}}(k)\,, \qquad i \colon \mathrm{id} \Rightarrow I,\, p \colon I \Rightarrow [1]$$

and by applying $\mathbf{dgFun}(\mathcal{A}^{\mathrm{op}}, -)$ we arrive at k-linear functors and natural transformation

$$I, [1] \colon \mathbf{dgMod}_{\mathcal{A}} \to \mathbf{dgMod}_{\mathcal{A}}\,, \quad i \colon \mathrm{id} \Rightarrow I,\, p \colon I \Rightarrow [1]\,.$$

We get for every dg-$\mathcal{A}$-module M a short exact sequence of dg-$\mathcal{A}$-modules

$$0 \to M \xrightarrow{i_M} IM \xrightarrow{p_M} M[1] \to 0 \tag{2}$$

that gives at every point $x \in \mathrm{Ob}(\mathcal{A})$ the short exact sequence of chain complexes

$$0 \to M_x \xrightarrow{i_{M_x}} IM_x \xrightarrow{p_{M_x}} M_x[1] \to 0$$

from (1) with $X = M_x$. The short exact sequence (2) belongs to a Frobenus exact structure $\mathcal{S}_{\mathcal{A}}$ on $\mathbf{dgMod}_{\mathcal{A}}$ whose stable triangulated category is precisely the homotopy category $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$, with $i_M \colon M \to IM$ being an $\mathcal{S}_{\mathcal{A}}$-injective envelope of M for every dg-$\mathcal{A}$-module M. This description of $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ and its triangulated structure has two consequences we will need:

(1) A morphism $f \colon M \to N$ in $\mathbf{dgMod}_{\mathcal{A}}$ vanishes in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ if and only it factors through some $\mathcal{S}_{\mathcal{A}}$-injective object I of $\mathbf{dgMod}_{\mathcal{A}}$, i.e. if there exists in $\mathbf{dgMod}_{\mathcal{A}}$ a commutative diagram of the following form:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr] & & N \\ & I \arrow[ur] & \end{tikzcd}$$

The morphism $M \to I$ factors through the morphism $i_M \colon M \to IM$ since we have by the definition of an $\mathcal{S}_{\mathcal{A}}$-injective object the following diagram:

$$\begin{tikzcd} & I & & & \\ 0 \arrow[r] & M \arrow[u] \arrow[r, "i_M"] & IM \arrow[ul, dashed, bend right] \arrow[r, "p_M"] & M[1] \arrow[r] & 0 \end{tikzcd}$$

It follows that f already factors through $i_M \colon M \to IM$.

(2) We get a description of the distinguished triangles in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ from Happel's theorem: Let

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i_M"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & IM \arrow[r, "p_M"] \arrow[d] & M[1] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r, "g"'] & P \arrow[r, "h"'] & M[1] \arrow[r] & 0 \end{tikzcd}$$

be be a commutative diagram in $\mathbf{dgMod}_{\mathcal{A}}$ whose rows are contained in $\mathcal{S}_{\mathcal{A}}$ and where the left hand square is a pushout square. Then the resulting sequence

$$M \xrightarrow{f} N \xrightarrow{g} P \xrightarrow{h} M[1]$$

in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ is a distinguished triangle. (And every distinguished triangle is up to isomorphism of this form.)

3.3. Pretriangulated DG-Categories

Definition 7. A dg-functor $F\colon \mathcal{A} \to \mathcal{B}$ is **dg-fully faithful** if for every two objects $x, y \in \mathcal{A}$ the morphism of chain complexes $F_{x,y}\colon \mathcal{A}(x,y) \to \mathcal{B}(F(x), F(y))$ is an isomorphism.

Proposition 8 (dg-Yoneda embedding)**.** Let $\mathcal{A}$ be a dg-category. Then the mapping

$$\mathcal{A} \to \underline{\mathbf{dgMod}}_{\mathcal{A}}\,, \quad x \mapsto \mathcal{A}(-,x) = \mathcal{A}^{\mathrm{op}}(x,-)$$

extends (in the usual way) to a dg-fully faithful dg-functor.

Definition 9. A dg-category $\mathcal{A}$ is **pretriangulated** if the fully faithful k-linear functor

$$\mathrm{H}^0(\mathcal{A}) \to \mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$$

that is induced by the Yoneda embedding $\mathcal{A} \to \underline{\mathbf{dgMod}}_{\mathcal{A}}$ identifies $\mathcal{A}$ with a triangulated subcategory of $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$.

The above definition ensures that for a pretriangulated dg-category $\mathcal{A}$ its homotopy category $\mathrm{H}^0(\mathcal{A})$ does carry in a canonical way the structure of a triangulated category.

Definition 10. A **dg-enhancement** of a triangulated category $\mathcal{T}$ is a pretriangulated category $\mathcal{A}$ together with an equivalence of triangulated categories $\mathcal{T} \simeq \mathrm{H}^0(\mathcal{A})$.

A dg-enhancement of a triangulated category $\mathcal{T}$ allows us to identity $\mathcal{T}$ with a full triangulated subcategory of $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ for some dg-category $\mathcal{A}$. We can then try to understand the original triangulated category $\mathcal{T}$ through the "higher structure" of the dg-category $\mathcal{A}$.

3.4. Cones as Derived Cokernels (DG Version)

Recall that if $f\colon x \to y$ is a morphism in an (pre)additive category $\mathcal{A}$ then its cokernel (which does not need to exists) can be thought of in two equivalent ways:

- A morphism $\operatorname{coker}(f) \to z$ into any other object is "the same" as a morphism $g\colon y \to z$ with $gf = 0$.
- The cokernel $\operatorname{coker}(f)$ is the pushout of the following diagram:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \\ 0 & \end{tikzcd}$$

In a triangulated category $\mathcal{T}$ we do in general have neither cokernels nor pushouts. But given a dg-enhancement $\mathcal{T} \simeq \mathrm{H}^0(\mathcal{A})$ for some pretriangulated dg-category $\mathcal{A}$ we can identify $\mathcal{T}$ with a full triangulated subcategory of $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$. We can then do the desired calculations in the abelian category $\mathbf{dgMod}_{\mathcal{A}} = \mathrm{Z}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ which is (co)complete.

More precisely, given a fixed morphism $f\colon M \to N$ in $\mathbf{dgMod}_{\mathcal{A}}$ we try to understand all morphisms $g\colon N \to P$ in $\mathbf{dgMod}_{\mathcal{A}}$ for which the composition $g \circ f$ vanishes in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$. We have previously seen that $g \circ f$ vanishes if and only if $g \circ f$ factors through $i_M\colon M \to IM$. We are hence interested in the pushout of the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i_M"'] & N \\ IM & \end{tikzcd}$$

in $\mathbf{dgMod}_{\mathcal{A}}$. This pushout is computed pointwise, hence we want to compute for every $x \in \mathrm{Ob}(\mathcal{A})$ the pushout

$$\begin{tikzcd} M_x \arrow[r, "f_x"] \arrow[d, "i_{M_x}"'] & N_x \\ IM_x & \end{tikzcd}$$

in the category $\mathbf{Ch}(k)$ where

$$IM_x = \left(M_x \oplus M_x[1], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \quad \text{and} \quad i_{M_x} = \begin{bmatrix} 1 \\ d_{M_x} \end{bmatrix}.$$

Lemma 11. Any morphism of chain complexes $f\colon X \to Y$ gives a pushout diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"] \\ IX \arrow[r, "\begin{bmatrix} -d_X & 1 \\ f & 0 \end{bmatrix}"'] & C(f) \end{tikzcd}$$

where $C(f)$ denotes the usual mapping cone.

Proof. See Appendix A.5. $\square$

We hence find that the pushout

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i_M"'] \arrow[dr, phantom, "\ulcorner"] & N \arrow[d] \\ IM \arrow[r] & C(f) \end{tikzcd}$$

is given by $(C(f))_x = C(f_x)$ at every $x \in \mathrm{Ob}(\mathcal{A})$, i.e. the dg-$\mathcal{A}$-module $C(f)$ is pointwise given by the usual mapping cone of chain complexes.

We see that $C(f)$ is actually the cone of f in the triangulated category $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$: We have the following commutative diagram in which the left hand side is a pushout, and where the upper row is exact and contained in the Frobenius exact structure $\mathcal{S}_{\mathcal{A}}$:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i_M"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & IM \arrow[r, "p_M"] \arrow[d] & M[1] \arrow[r] & 0 \\
& N \arrow[r] & C(f) & &
\end{tikzcd}$$

A standard lemma from homological algebra asserts that we can extend the above diagram to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i_M"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & IM \arrow[r, "p_M"] \arrow[d] & M[1] \arrow[r] \arrow[d, dashed, equal] & 0 \\
0 \arrow[r, dashed] & N \arrow[r] & C(f) \arrow[r, dashed] & M[1] \arrow[r, dashed] & 0
\end{tikzcd}$$

whose row are exact. It follows from the axioms of a Frobenius exact structure that the lower row is again contained in $\mathcal{S}_{\mathcal{A}}$. We have seen in the previous review on Frobenius exact structures that

$$M \xrightarrow{f} N \to C(f) \to M[1]$$

is therefore a distinguished triangle in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$. This entails that $C(f)$ is a cone of f in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$.

We have thus seen that one can think about the cone in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ as a kind of "derived cokernel" of f, coming from a pushout in $\mathbf{dgMod}_{\mathcal{A}}$ that was choosen as a "cokernel up to homotopy".

3.5. Generalizing Cones

The object $C(f)$ of the dg-category $\mathcal{B} := \underline{\mathbf{dgMod}}_{\mathcal{A}}$ gives a corepresentable dg-functor, i.e. a left dg-module

$$\widehat{C} := \mathcal{B}(C(f), -) \colon \mathcal{B} \to \underline{\mathbf{Ch}}(k)\,.$$

At every object $P \in \mathrm{Ob}(\mathcal{B})$ this left dg-module is given by the chain complex $\widehat{C}(P)$ which is given as a graded module by

$$\widehat{C}(P) = \mathcal{B}(N, P)[1] \oplus \mathcal{B}(M, P)$$

and whose differential is

$$d_{\widehat{C}(P)} = \begin{bmatrix} d_{\mathcal{B}(N,P)} & 0 \\ (-) \circ f & -d_{\mathcal{B}(M,P)} \end{bmatrix}.$$

For any two objects $P, Q \in \mathrm{Ob}(\mathcal{B})$ the morphism of chain complexes

$$\widehat{C}_{P,Q} \colon \mathcal{B}(P, Q) \to \mathcal{B}\big(\widehat{C}(P), \widehat{C}(Q)\big)$$

is given by

$$g \mapsto ((a,b) \mapsto (g \circ a, (-1)^{|g|} g \circ b) .$$

We note that these formulae make sense in any dg-category. We can therefore define for every dg-category $\mathcal{B}$ and every $f \in \mathrm{Z}^0(\mathcal{B}(x,y))$ a left dg-$\mathcal{B}$-module $C\colon \mathcal{B} \to \mathbf{Ch}(k)$ by the same expressions as above. One can then define a **cone** of f in $\mathcal{B}$ as a representing object for this dg-functor. This allows us to talk about cones of morphisms in dg-categories.

A. Appendix

A.1. Proof of Lemma 2

Let $f\colon x \to y$ be an epimorphism in a triangulated category $\mathcal{T}$. We may complete f to a distinguished triangle

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma(x) .$$

The composition of any two morphisms in a distinguished triangle vanishes, so $gf = 0$. It follows that $g = 0$ since f is an epimorphism. By applying the homological functor $\mathcal{T}(y,-)$ to this distinguished triangle we arrive at the followig long exact sequence:

$$\cdots \to \mathcal{T}(y,x) \xrightarrow{f_*} \mathcal{T}(y,y) \xrightarrow{0} \mathcal{T}(y,z) \to \cdots$$

We find that for $\mathrm{id}_y \in \mathcal{T}(y,y)$ there exists some $s \in \mathcal{T}(y,x)$ with $\mathrm{id}_y = f_*(s) = fs$.

A.2. Proof of Proposition 4

(3) $\implies$ (1): We realize $\mathbf{D}(\mathcal{A})$ by first passing from the category of chain complexes $\mathbf{Ch}(\mathcal{A})$ to its homotopy category $\mathbf{K}(\mathcal{A})$ and then localizing at the class of quasi-isomorphisms.

Every chain complex $X \in \mathrm{Ob}(\mathbf{Ch}(\mathcal{A}))$ splits[1] since $\mathcal{A}$ is semisimple and can thus be decomposed as $X = X' \oplus X''$ where X' is split acyclic and X'' has zero differential. In the homotopy category $\mathbf{K}(\mathcal{A})$ the chain complex X' becomes zero as it is split acyclic and hence contractible. Every isomorphism class in $\mathbf{K}(\mathcal{A})$ is therefore represented by a chain complex with zero differential, i.e. an object of $\mathcal{A}^{\mathbb{Z}}$. No two morphisms between such chain complexes become identified in $\mathbf{K}(\mathcal{A})$ so that the categories $\mathbf{K}(\mathcal{A})$ and $\mathcal{A}^{\mathbb{Z}}$ are equivalent. We note that this equivalence is indeed given by $\mathrm{H}^*\colon \mathbf{K}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$.

We also see that a quasi-isomorphism between chain complexes with zero differentials must already be an isomorphism. We therefore see that $\mathbf{D}(\mathcal{A})$ is just $\mathbf{K}(\mathcal{A})$ again.

[1] Recall that a chain complex X is **split** if it can be (up to isomorphism) degreewise decomposed as $X^n = B^n \oplus H^n \oplus B^{n+1}$ such that the differential of X is with respect to this decomposition given by $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $\mathrm{Z}^n(X) = B^n \oplus H^n$, $\mathrm{B}^n(X) = B^n$ and $\mathrm{H}^n(X) \cong H^n$. The claimed decompositon $X = X' \oplus X''$ is then given degreewise by $(X')^n = B^n \oplus B^{n+1}$ with differential $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $(X'')^n = H^n$.

(1) $\implies$ (2): This follows from Corollary 3.

(2) $\implies$ (3) We first observe that an abelian category $\mathcal{B}$ is semisimple if and only if every morphism $f\colon x \to y$ in $\mathcal{B}$ admits a **pseudoinverse** $g\colon y \to x$ satisfying $fgf = f$ and $gfg = g$: If every morphism f in $\mathcal{B}$ admits such a pseudoinverse g then it follows for every epimorphism f in $\mathcal{B}$ from $fgf = f$ that $fg = \mathrm{id}$ so that f splits. If on the other hand $\mathcal{B}$ is semisimple and $f\colon x \to y$ is any morphism in $\mathcal{B}$ then we have decompositions $x = x' \oplus \ker(f)$ and $y = y' \oplus \mathrm{im}(f)$ with f inducing an isomorphism $x' \to \mathrm{im}(f)$. The inverse $\mathrm{im}(f) \to x'$ composed with the projection $y \to \mathrm{im}(f)$ and the inclusion $x' \to x$ then give the desired pseudoinverse $g\colon y \to x$.

By assumption every morphism in $\mathbf{D}(\mathcal{A})$ admits a pseudoinverse. Every morphism f in $\mathcal{A}$ hence admits a pseudoinverse g in $\mathbf{D}(\mathcal{A})$ (where we regard $\mathcal{A}$ as chain complexes concentrated in degree 0) which becomes the pseudoinverse $\mathrm{H}_0(g)$ to f in $\mathcal{A}$. This shows that every morphism in $\mathcal{A}$ admits a pseudoinverse, so that $\mathcal{A}$ is semisimple.

A.3. (Counter)example to Section 1.4

In the category $\mathbf{D}(\mathbb{Z}) = \mathbf{D}(\mathbf{Mod}_{\mathbb{Z}})$ the nonzero morphism $f\colon \mathbb{Z}/2 \to \mathbb{Z}/4$ does not admit a cokernel: Suppose otherwise that $c\colon \mathbb{Z}/4 \to C$ is a such a cokernel. Then c is a split epimorphism in $\mathbf{D}(\mathbb{Z})$ by Lemma 2. It follows that $\mathrm{H}^0(c)\colon \mathbb{Z}/4 \to \mathrm{H}^0(C)$ is a split epimorphism in $\mathbf{Mod}_{\mathbb{Z}}$. But $\mathbb{Z}/4$ is indecomposable, so it follows that $\mathrm{H}^0(c) = 0$ or $\mathrm{H}^0(c)$ is an isomorphism.

If $\mathrm{H}^0(c) = 0$ then we consider the nonzero morphism $g\colon \mathbb{Z}/4 \to \mathbb{Z}/2$. It follows from $g \circ f = 0$ that g factors trough c in $\mathbf{D}(\mathbb{Z})$ and therefore factors through $\mathrm{H}^0(c)$ in $\mathbf{Mod}_{\mathbb{Z}}$. But this is not possible since $\mathrm{H}^0(c) = 0$ while $g \neq 0$.

Suppose now that $\mathrm{H}^0(c)$ is an isomorphism. It follows from $c \circ f = 0$ in $\mathbf{D}(\mathbb{Z})$ that $\mathrm{H}^0(c) \circ f = 0$ in $\mathbf{Mod}_{\mathbb{Z}}$. It now further follows that $f = 0$ which is wrong.

A.4. More Detailed Review on DG-Categories

The k-linear category $\mathbf{Ch}(k)$ of chain complexes becomes a symmetric monoidal category with respect to the tensor product of chain complexes. A dg-category $\mathcal{A}$ is a category enriched over $(\mathbf{Ch}(k), \otimes)$. More explicitely, a dg-category $\mathcal{A}$ consists of a class of objects $\mathrm{Ob}(\mathcal{A})$, for any two objects $x, y \in \mathcal{A}$ a chain complex $\mathcal{A}(x, y)$, for any three objects $x, y, z \in \mathrm{Ob}(\mathcal{A})$ a morphism of chain complexes

$$(-) \circ (-)\colon \mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \to \mathcal{A}(x, z)$$

satisfying the usual associativity diagram, and for every object $x \in \mathrm{Ob}(\mathcal{A})$ an element $1_x \in \mathrm{Z}^0(\mathcal{A}(x, y))$ satifying the usual identity diagrams. As an example we enrich $\mathbf{Ch}(k)$ into a dg-category $\underline{\mathbf{Ch}}(k)$ with $\underline{\mathbf{Ch}}(k)(X, Y) = \underline{\mathrm{Hom}}(X, Y)$ being the usual Hom-chain complex.

To any dg-category $\mathcal{A}$ we can associated the k-linear categories $\mathrm{Z}^0(\mathcal{A})$ and $\mathrm{H}^0(\mathcal{A})$ that are given by $\mathrm{Ob}(\mathrm{Z}^0(\mathcal{A})) = \mathrm{Ob}(\mathrm{H}^0(\mathcal{A})) = \mathrm{Ob}(\mathcal{A})$ and

$$\mathrm{Z}^0(\mathcal{A})(x, y) = \mathrm{Z}^0(\mathcal{A}(x, y)) \qquad \text{and} \qquad \mathrm{H}^0(\mathcal{A})(x, y) = \mathrm{H}^0(\mathcal{A}(x, y))\,.$$

As an example we have $\mathrm{Z}^0(\underline{\mathbf{Ch}}(k)) = \mathbf{Ch}(k)$ and $\mathrm{H}^0(\underline{\mathbf{Ch}}(k)) = \mathbf{K}(k)$.

This examples motivates that one should think about $\mathrm{Z}^0(\mathcal{A})$ as the **underlying k-linear category** of $\mathcal{A}$, and as the elements of $\mathrm{Z}^0(\mathcal{A}(x,y))$ as the "actual morphisms" from x to y. The category $\mathrm{H}^0(\mathcal{A})$ is the **homotopy category** of $\mathcal{A}$.

A **dg-functor** $F\colon \mathcal{A} \to \mathcal{B}$ is given by a map $F\colon \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$ and morphisms of chain complexes $F_{x,y}\colon \mathcal{A}(x,y) \to \mathcal{B}(F(x),F(y))$ for any two objects $x,y \in \mathrm{Ob}(\mathcal{A})$, satisfying the usual axioms. Every dg-functor $F\colon \mathcal{A} \to \mathcal{B}$ induces k-linear functors $\mathrm{Z}^0(\mathcal{A}) \to \mathrm{Z}^0(\mathcal{B})$ and $\mathrm{H}^0(\mathcal{A}) \to \mathrm{H}^0(\mathcal{B})$.

A dg-natural transformation $\alpha\colon F \Rightarrow G$ between dg-functors $F,G\colon \mathcal{A} \to \mathcal{B}$ assigns to is a family $\alpha = (\alpha_x)_{x\in\mathrm{Ob}(\mathcal{A})}$ of "actual morphisms" $\alpha_x \in \mathrm{Z}^0(\mathcal{B}(F(x),G(x)))$ with $\alpha_y \circ F(f) = G(f) \circ \alpha_x$ for every $f \in \mathcal{A}(x,y)$. We get a k-linear abelian category $\mathbf{dgFun}(\mathcal{A},\mathcal{B})$ whose objects are dg-functors $\mathcal{A} \to \mathcal{B}$ and whose morphisms are dg-natural transformations. This category can be enriched into a dg-category $\underline{\mathbf{dgFun}}(\mathcal{A},\mathcal{B})$ with

$$\underline{\mathbf{dgFun}}(\mathcal{A},\mathcal{B})(F,G)^i = \left\{ (\alpha_x)_{x\in\mathrm{Ob}(\mathcal{A})} \;\middle|\; \begin{array}{c} \alpha_x \in \mathcal{B}(F(x),G(x))^i \text{ with} \\ \alpha_y \circ F(f) = (-1)^{ij} G(f) \circ \alpha_x \\ \text{for every } f \in \mathcal{A}(x,y)^j \end{array} \right\}$$

being a subcomplex of $\prod_{x\in\mathrm{Ob}(\mathcal{A})} \mathcal{B}(F(x),G(x))$. This k-linear category can be enriched in a dg-category $\underline{\mathbf{dgFun}}(\mathcal{A},\mathcal{B})$. Then in particular

$$\mathrm{Z}^0(\underline{\mathbf{dgFun}}(\mathcal{A},\mathcal{B})) = \mathbf{dgFun}(\mathcal{A},\mathcal{B}),$$

so that $\mathbf{dgFun}(\mathcal{A},\mathcal{B})$ is the underlying k-linear category of $\underline{\mathbf{dgFun}}(\mathcal{A},\mathcal{B})$.

For a dg-category $\mathcal{A}$ a **(right) dg-$\mathcal{A}$-module** is a dg-functor $M\colon \mathcal{A}^{\mathrm{op}} \to \underline{\mathbf{Ch}}(k)$. This means that at every object $x \in \mathrm{Ob}(\mathcal{A})$ we have a chain complex M_x, and for every $f \in \mathcal{A}(x,y)^i$ we have associated a map $M(f)\colon M(y) \to M(x)$ of degree i such that $M(g\circ f) = (-1)^{ij} M(f) \circ M(g)$ for all $f \in \mathcal{A}(x,y)^i$ and $g \in \mathcal{A}(y,z)^j$. The category of dg-$\mathcal{A}$-modules is given by

$$\mathbf{dgMod}_{\mathcal{A}} = \mathbf{dgFun}(\mathcal{A}^{\mathrm{op}}, \underline{\mathbf{Ch}}(k))$$

and its dg-enrichement is given by

$$\underline{\mathbf{dgMod}}_{\mathcal{A}} = \underline{\mathbf{dgFun}}(\mathcal{A}^{\mathrm{op}}, \underline{\mathbf{Ch}}(k)).$$

The category $\mathbf{dgMod}_{\mathcal{A}}$ is abelian, complete and cocomplete, and all (co)limits are computed pointwise.

A.5. Proof of Lemma 11

We calculate the pushout as

$$
\begin{aligned}
&\left(X \oplus X[1] \oplus Y, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_Y \end{bmatrix}\right) \Big/ \left\langle \begin{bmatrix} x \\ d(x) \\ -f(x) \end{bmatrix} \middle| \, x \in X \right\rangle \\
\cong\ & \left(X[1] \oplus Y, \begin{bmatrix} -d_X & 0 \\ f & d_Y \end{bmatrix}\right) \\
=\ & C(f)
\end{aligned}
$$

as claimed.

Remark 12. The claim of Lemma 11 is (a posteriori) not surprising:c It is a standard statement from homological algebra that for a morphism of chain complexes $f\colon X \to Y$ the data of a morphism $h\colon C(f) \to Z$ is the same as that of a morphism $g\colon Y \to Z$ together with a null homotopy of the composition $g \circ f\colon X \to Z$. Whence $C(f)$ corepresents morphisms going out of Y whose composition with f are zero "up to homotopy".