

DG-Enhancement of Triangulated Categories

Problems with Triangulated Categories and How to Circumvent Them

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By a **chain complex** we always mean a cochain complex, i.e. the differential increases the degree.

1 Some Problems with Triangulated Categories

1.1 About the Abelianess of Triangulated Categories

Definition 1. An abelian category \mathcal{A} is **semisimple abelian** or simply **semisimple** if every short exact sequence in \mathcal{A} splits.

Lemma 2. In a triangulated category every epimorphism splits.

Proof. Let $f: x \rightarrow y$ be an epimorphism in a triangulated category \mathcal{T} . We may complete f to a distinguished triangle

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma(x).$$

The composition of any two morphisms in a distinguished triangle vanishes, so $gf = 0$. It follows that $g = 0$ since f is an epimorphism. By applying the homological functor $\mathcal{T}(y, -)$ to this distinguished triangle we arrive at the following long exact sequence:

$$\cdots \rightarrow \mathcal{T}(y, x) \xrightarrow{f_*} \mathcal{T}(y, y) \xrightarrow{0} \mathcal{T}(y, z) \rightarrow \cdots$$

We find that for $\text{id}_y \in \mathcal{T}(y, y)$ there exists some $s \in \mathcal{T}(y, x)$ with $\text{id}_y = f_*(s) = fs$. \square

Corollary 3. A triangulated category that is abelian is already semisimple. \square

We see from Corollary 3 that most triangulated categories are not abelian.

Proposition 4. For an abelian category \mathcal{A} the following conditions on \mathcal{A} and its derived category $\mathbf{D}(\mathcal{A})$ are equivalent:

- (1) The derived category $\mathbf{D}(\mathcal{A})$ is abelian.
- (2) The derived category $\mathbf{D}(\mathcal{A})$ is semisimple abelian.
- (3) The abelian category $\mathbf{D}(\mathcal{A})$ is semisimple.

If these equivalent conditions are satisfied then $\mathbf{D}(\mathcal{A}) \simeq \mathcal{A}^{\mathbb{Z}}$ via the homology functor $H_*: \mathbf{D}(\mathcal{A}) \rightarrow \mathcal{A}^{\mathbb{Z}}$. A quasi-inverse $\mathcal{A}^{\mathbb{Z}} \rightarrow \mathbf{D}(\mathcal{A})$ is given by regarding every objects of $\mathcal{A}^{\mathbb{Z}}$ as a chain complex with zero differential.

Proof.

(3) \implies (1): We realize $\mathbf{D}(\mathcal{A})$ by first passing from the category of chain complexes $\mathbf{Ch}(\mathcal{A})$ to its homotopy category $\mathbf{K}(\mathcal{A})$ and then localizing $\mathbf{K}(\mathcal{A})$ at the class of quasi-equivalences.

Every chain complex $X \in \text{Ob}(\mathbf{Ch}(\mathcal{A}))$ splits¹ since \mathcal{A} is semisimple, and can thus be decomposed as $X = X' \oplus X''$ where X' is split acyclic while X'' has zero differential. In the homotopy category $\mathbf{K}(\mathcal{A})$ the chain complex X' becomes zero as it is split acyclic and hence contractible. Every isomorphism class in $\mathbf{K}(\mathcal{A})$ is therefore represented by a chain complex with zero differential, i.e. an object of $\mathcal{A}^{\mathbb{Z}}$. Every nullhomotopy between such chain complexes is trivial (as the differential is zero) so no two morphisms between such chain complexes become identified in $\mathbf{K}(\mathcal{A})$. This shows that the categories $\mathbf{K}(\mathcal{A})$ and $\mathcal{A}^{\mathbb{Z}}$ are equivalent. We note that this equivalence is given by $H_*: \mathbf{K}(\mathcal{A}) \rightarrow \mathcal{A}^{\mathbb{Z}}$.

We also see that a quasi-isomorphism between chain complexes with zero differentials must already be an isomorphism. We therefore see that $\mathbf{D}(\mathcal{A})$ is just $\mathbf{K}(\mathcal{A})$ again. It follows that $\mathbf{D}(\mathcal{A})$ is equivalent to $\mathcal{A}^{\mathbb{Z}}$.

(1) \implies (2): This follows from Corollary 3.

(2) \implies (3) We first observe that an abelian category \mathcal{B} is semisimple if and only if every morphism $f: x \rightarrow y$ in \mathcal{B} admits a **pseudoinverse** $g: y \rightarrow x$ satisfying $fgf = f$ and $gfg = g$: If every morphism f in \mathcal{B} admits such a pseudoinverse g then it follows for every epimorphism f from $fgf = f$ that $fg = \text{id}$ so that f splits. If on the other hand \mathcal{B} is semisimple and $f: x \rightarrow y$ any morphism in \mathcal{B} then we have decompositions $x = x' \oplus \ker(f)$ and $y = y' \oplus \text{im}(f)$ with f inducing an isomorphism $x' \rightarrow \text{im}(f)$. The inverse $\text{im}(f) \rightarrow x'$ together with the projection $y \rightarrow \text{im}(f)$ and the inclusion $x' \rightarrow x$ then give the desired pseudoinverse $g: y \rightarrow x$.

We find that by assumption every morphism in $\mathbf{D}(\mathcal{A})$ admits a pseudoinverse. Every morphism f in \mathcal{A} hence admits a pseudoinverse g in $\mathbf{D}(\mathcal{A})$ (where we regard \mathcal{A} as chain complexes concentrated in degree 0) which becomes a pseudoinverse $H_0(g)$ to f in \mathcal{A} . This shows that every morphism in \mathcal{A} admits a pseudoinverse, so that \mathcal{A} is semisimple. \square

¹Recall that a chain complex X is said to **split** if it can be (up to isomorphism) degreeewise decomposed as $X^n = B^n \oplus H^n \oplus B_{n+1}$ such that the differential of X is with respect to this decomposition given by $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $Z^n(X) = B^n \oplus H^n$, $B^n(X) \cong B^n$ and $H^n(X) = H^n$. The claimed decomposition $X = X' \oplus X''$ is then given degreeewise by $X'^n = B_{n+1} \oplus B^n$ with differential $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $X''^n = H^n$.

1.2 Non-Functoriality of Cones

Triangulated categories do in general not admit functorial cones. As a consequence of this, we see that for a triangulated category \mathcal{T} and a (small) index category I the diagram category $\mathbf{Fun}(I, \mathcal{T})$ does in general not inherit a triangulated structure from \mathcal{T} :

Given a morphism $f: D \rightarrow D'$ in $\mathbf{Fun}(I, \mathcal{T})$ we would otherwise like to compute its cone for the inherited triangulated structure of $\mathbf{Fun}(I, \mathcal{T})$ pointwise, i.e. via

$$\text{cone}(f)(i) = \text{cone}(f_i)$$

at every position $i \in \text{Ob}(I)$, where $f_i: D(i) \rightarrow D'(i)$.

But $\text{cone}(f)$, which is supposed to be a functor $\text{cone}(f): I \rightarrow \mathcal{T}$, also needs to be defined on morphisms of I . For every such morphism $e: i \rightarrow j$ in I we would hence need an induced morphism $\text{cone}(f)(e): \text{cone}(f)(i) \rightarrow \text{cone}(f)(j)$, i.e. an induced morphism $\text{cone}(f)(e): \text{cone}(f_i) \rightarrow \text{cone}(f_j)$. The good thing is that there exists by the axioms of a triangulated category some morphism $\text{cone}(e): \text{cone}(f_i) \rightarrow \text{cone}(f_j)$ that makes the diagram

$$\begin{array}{ccccc} D(i) & \xrightarrow{f_i} & D'(i) & \longrightarrow & \text{cone}(f_i) \\ D(i) \downarrow & & \downarrow D(j) & & \downarrow \\ D(j) & \xrightarrow{f_j} & D'(j) & \longrightarrow & \text{cone}(f_j) \end{array}$$

commute. But by the missing functoriality of the cone this does in general not define a functor $\text{cone}(f): I \rightarrow \mathcal{T}$ and hence not an object of $\mathbf{Fun}(I, \mathcal{T})$.

The missing functoriality of the cone of \mathcal{T} can in general not be fixed, as the following result asserts:

Proposition 5. Let \mathcal{T} be an idempotent complete triangulated category. If \mathcal{T} admits functorial cones then \mathcal{T} is abelian and semisimple.

1.3 (Non-)Existence of Limits and Colimits

A triangulated category \mathcal{T} is in general neither complete nor cocomplete.

1.4 Difference between $\mathbf{D}(\mathcal{A})^I$ and $\mathbf{D}(\mathcal{A}^I)$

One might suspect that for $\mathcal{T} = \mathbf{D}(\mathcal{A})$, where \mathcal{A} is some abelian category, the above problems can be fixed by using an equivalence $\mathbf{D}(\mathcal{A})^I \simeq \mathbf{D}(\mathcal{A}^I)$. (Note that the category \mathcal{A}^I is again abelian.) The problem is that there is in general no such equivalence.

Take for example $\mathcal{A} = k\text{-Vect}$ and let $I = (\bullet \rightarrow \bullet)$ be the category consisting of two objects 0 and 1 and one non-identity morphism $0 \rightarrow 1$. The abelian category \mathcal{A} is semisimple whence $\mathbf{D}(\mathcal{A})^I$ is again abelian by Proposition 4. But the functor category $\mathcal{A}^I = \mathbf{Fun}(I, \mathcal{A})$ is equivalent to the category of representations of the quiver $\bullet \rightarrow \bullet$ and therefore not abelian. The derived category $\mathbf{D}(\mathcal{A}^I)$ is thus not again abelian. This shows that $\mathbf{D}(\mathcal{A})^I$ is abelian but $\mathbf{D}(\mathcal{A}^I)$ is not abelian, which entails that these categories are not equivalent.

2 Towards Derivators

3 Towards DG-Enhancement

In the following we denote by k some commutative ring.

A dg enhancement of category \mathcal{T} is, roughly speaking, a dg-category \mathcal{A} together with an equivalence $\mathcal{T} \simeq H^0(\mathcal{A})$. If the category \mathcal{T} carries additional structures which we want to be respected by this equivalence, then we need to make sure that the homotopy category $H^0(\mathcal{A})$ does carry such a structure itself. If \mathcal{T} is a triangulated category then this leads us to the notion of a pretriangulated dg-category.

3.1 Review on DG-Categories

The k -linear category $\mathbf{Ch}(k)$ together with the tensor product of chain complexes becomes a symmetric monoidal category. A **dg-category** is an enriched category over $\mathbf{Ch}(k)$. More precisely, a dg-category \mathcal{A} consists of a class of objects $\text{Ob}(\mathcal{A})$, for any two objects $x, y \in \mathcal{A}$ a chain complex $\mathcal{A}(x, y)$, for any three objects $x, y, z \in \text{Ob}(\mathcal{A})$ a morphism of chain complexes

$$(-) \circ (-): \mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \rightarrow \mathcal{A}(x, z)$$

satisfying the usual associativity diagram, and for every object $x \in \text{Ob}(\mathcal{A})$ an element $1_x \in Z^0(\mathcal{A}(x, y))$ satisfying the usual identity diagrams. As an example we enrich $\mathbf{Ch}(k)$ into a dg-category $\underline{\mathbf{Ch}}(k)$ with $\underline{\mathbf{Ch}}(k)(X, Y) = \underline{\mathbf{Hom}}(X, Y)$ being the usual Hom-chain complex.

To any dg-category \mathcal{A} we can associate the k -linear categories $Z^0(\mathcal{A})$ and $H^0(\mathcal{A})$ that are given by $\text{Ob}(Z^0(\mathcal{A})) = \text{Ob}(H^0(\mathcal{A}))$ and

$$Z^0(\mathcal{A})(x, y) = Z^0(\mathcal{A}(x, y)) \quad \text{and} \quad H^0(\mathcal{A})(x, y) = H^0(\mathcal{A}(x, y)).$$

The composition of morphisms in $Z^0(\mathcal{A})$ and $H^0(\mathcal{A})$ is induced by the compositions in \mathcal{A} . As an example we have $Z^0(\underline{\mathbf{Ch}}(k)) = \mathbf{Ch}(k)$ and $H^0(\underline{\mathbf{Ch}}(k)) = \mathbf{K}(k)$.

This examples motivates that one should think about $Z^0(\mathcal{A})$ as the “underlying k -linear category of \mathcal{A} ”, and as the elements of $Z^0(\mathcal{A}(x, y))$ as the “actual morphisms” from x to y .

For two dg-categories \mathcal{A}, \mathcal{B} a **dg-functor** $F: \mathcal{A} \rightarrow \mathcal{B}$ is given by a map $F: \text{Ob}(\mathcal{A}) \rightarrow \text{Ob}(\mathcal{B})$ together with morphisms of chain complexes $F_{x,y}: \mathcal{A}(x, y) \rightarrow \mathcal{B}(F(x), F(y))$ for any two objects $x, y \in \text{Ob}(\mathcal{A})$, satisfying the usual axioms. Every dg-functor $F: \mathcal{A} \rightarrow \mathcal{B}$ induces k -linear functors $Z^0(\mathcal{A}) \rightarrow Z^0(\mathcal{B})$ and $H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$.

A dg-natural transformation $\alpha: F \Rightarrow G$ between dg-functors $F, G: \mathcal{A} \rightarrow \mathcal{B}$ assigns to is a family $\alpha = (\alpha_x)_{x \in \text{Ob}(\mathcal{A})}$ of “actual morphisms” $\alpha_x \in Z^0(\mathcal{B}(F(x), G(x)))$ with $\alpha_y \circ F(f) = G(f) \circ \alpha_x$ for every $f \in \mathcal{A}(x, y)$. We get a k -linear abelian category $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$ whose objects are dg-functors $\mathcal{A} \rightarrow \mathcal{B}$ and whose morphisms are dg-natural transformations. This category can be enriched into a dg-category $\underline{\mathbf{dgFun}}(\mathcal{A}, \mathcal{B})$

with

$$\underline{\mathbf{dgFun}}(\mathcal{A}, \mathcal{B})(F, G)^i = \left\{ (\alpha_x)_{x \in \text{Ob}(\mathcal{A})} \left| \begin{array}{l} \alpha_x \in \mathcal{B}(F(x), G(x))^i \text{ with} \\ \alpha_y \circ F(f) = (-1)^{ij} G(f) \circ \alpha_x \\ \text{for every } f \in \mathcal{A}(x, y)^j \end{array} \right. \right\}$$

being a subcomplex of $\prod_{x \in \text{Ob}(\mathcal{A})} \mathcal{B}(F(x), G(x))$. Then in particular

$$Z^0(\underline{\mathbf{dgFun}}(\mathcal{A}, \mathcal{B})) = \mathbf{dgFun}(\mathcal{A}, \mathcal{B}),$$

so that $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$ is the underlying k -linear category of $\underline{\mathbf{dgFun}}(\mathcal{A}, \mathcal{B})$.

For a dg-category \mathcal{A} a **(right) dg- \mathcal{A} -module** is a dg-functor $M: \mathcal{A}^{\text{op}} \rightarrow \mathbf{Ch}(k)$. This means that at every object $x \in \text{Ob}(\mathcal{A})$ we have a chain complex M_x , and for every $f \in \mathcal{A}(x, y)^i$ we have associated a map $M(f): M(y) \rightarrow M(x)$ of degree i such that $M(g \circ f) = (-1)^{ij} M(f) \circ M(g)$ for all $f \in \mathcal{A}(x, y)^i$ and $g \in \mathcal{A}(y, z)^j$. The category of dg- \mathcal{A} -modules is given by $\mathbf{dgMod}_{\mathcal{A}} = \mathbf{dgFun}(\mathcal{A}^{\text{op}}, \mathbf{Ch}(k))$, and its dg-enrichment by $\underline{\mathbf{dgMod}}_{\mathcal{A}} = \underline{\mathbf{dgFun}}(\mathcal{A}^{\text{op}}, \mathbf{Ch}(k))$. The category $\mathbf{dgMod}_{\mathcal{A}}$ is abelian, complete and cocomplete, and all (co)limits are computed pointwise.

3.2 Review on Frobenius Exact Structures

The abelian category $\mathbf{Ch}(k)$ admits a Frobenius exact structure \mathcal{S} whose associated \mathcal{S} -stable triangulated category is precisely $\mathbf{K}(k)$. For every chain complex X its **\mathcal{S} -injective envelope** is given by the chain complex

$$IX = \left(X \oplus X[1], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

which fits into the short exact sequence

$$0 \rightarrow X \xrightarrow{i_X} IX \xrightarrow{p_X} X[1] \rightarrow 0 \quad (1)$$

belonging to \mathcal{S} , where

$$i_X = \begin{bmatrix} 1 \\ d_X \end{bmatrix} \quad \text{and} \quad p_X = \begin{bmatrix} -d_X & 1 \end{bmatrix}.$$

The k -linear functor $I: \mathbf{Ch}(k) \rightarrow \mathbf{Ch}(k)$ and natural transformations $i: 1 \Rightarrow I$ and $p: I \Rightarrow [1]$ extends to a dg-functor $I: \underline{\mathbf{Ch}}(k) \rightarrow \underline{\mathbf{Ch}}(k)$ and dg-natural transformations $i: 1 \Rightarrow I$ and $p: I \Rightarrow [1]$. By applying $\mathbf{dgFun}(\mathcal{A}^{\text{op}}, -)$ these in turn induce a k -linear functor

$$I: \mathbf{dgMod}_{\mathcal{A}} \rightarrow \mathbf{dgMod}_{\mathcal{A}}$$

together with natural transformations $i: 1 \Rightarrow I$ and $p: I \Rightarrow [1]$. We get for every dg- \mathcal{A} -module M a short exact sequence of dg- \mathcal{A} -modules

$$0 \rightarrow M \xrightarrow{i_M} IM \xrightarrow{p_M} M[1] \rightarrow 0 \quad (2)$$

that gives at every point $x \in \text{Ob}(\mathcal{A})$ the short exact sequence of chain complexes

$$0 \rightarrow M_x \xrightarrow{i_{M_x}} IM_x \xrightarrow{p_{M_x}} M_x[1] \rightarrow 0$$

from (1) with $X = M_x$. The short exact sequence (2) belongs to a Frobenius exact structure \mathcal{S} on $\mathbf{dgMod}_{\mathcal{A}}$ whose stable triangulated category is precisely the homotopy category $H^0(\mathbf{dgMod}_{\mathcal{A}})$, with $i_M: M \rightarrow IM$ being an \mathcal{S} -injective envelope of M for every dg- \mathcal{A} -module M . This description of $H^0(\mathbf{dgMod}_{\mathcal{A}})$ and its triangulated structure has two consequences we will need:

- (1) A morphism $f: M \rightarrow N$ in $\mathbf{dgMod}_{\mathcal{A}}$ vanishes in $H^0(\mathbf{dgMod}_{\mathcal{A}})$ if and only if it factors through some \mathcal{S} -injective object I of $\mathbf{dgMod}_{\mathcal{A}}$, i.e. if there exists in $\mathbf{dgMod}_{\mathcal{A}}$ a commutative diagram of the following form:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow & \nearrow \\ & I & \end{array}$$

The morphism $M \rightarrow I$ factors through the morphism $i_M: M \rightarrow IM$ since we have by the definition of an \mathcal{S} -injective object the following diagram:

$$\begin{array}{ccccccc} & & I & & & & \\ & & \uparrow & \swarrow \text{---} & & & \\ 0 & \longrightarrow & M & \xrightarrow{i_M} & IM & \xrightarrow{p_M} & M[1] \longrightarrow 0 \end{array}$$

It follows that f already factors through $i_M: M \rightarrow IM$.

- (2) We get a description of the distinguished triangles in $H^0(\mathbf{dgMod}_{\mathcal{A}})$ from Happel's theorem: Let

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{i_M} & IM & \xrightarrow{p_M} & M[1] \longrightarrow 0 \\ & & \downarrow f & & \downarrow & & \parallel \\ 0 & \longrightarrow & N & \xrightarrow{g} & P & \xrightarrow{h} & M[1] \longrightarrow 0 \end{array}$$

be a commutative diagram in $\mathbf{dgMod}_{\mathcal{A}}$ whose rows are contained in \mathcal{S} and where the left hand square is a pushout square. Then the resulting sequence

$$M \xrightarrow{f} N \xrightarrow{g} P \xrightarrow{h} M[1]$$

in $H^0(\mathbf{dgMod}_{\mathcal{A}})$ is a distinguished triangle. (And every distinguished triangle is up to isomorphism of this form.)

3.3 Review on Cokernels

3.4 Pretriangulated DG-Categories

Definition 6. A dg-functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is dg-fully faithful if for every two objects $x, y \in \mathcal{A}$ the morphism of chain complexes $F_{x,y}: \mathcal{A}(x, y) \rightarrow \mathcal{B}(F(x), F(y))$ is an isomorphism.

Proposition 7 (dg-Yoneda embedding). Let \mathcal{A} be a dg-category. Then the mapping

$$\mathcal{A} \rightarrow \underline{\mathbf{dgMod}}_{\mathcal{A}}, \quad x \mapsto \mathcal{A}(-, x) = \mathcal{A}^{\mathrm{op}}(x, -)$$

extends (in the usual way) to a dg-fully faithful dg-functor.

Definition 8. A dg-category \mathcal{A} is **pretriangulated** if the fully faithful k -linear functor

$$\mathrm{H}^0(\mathcal{A}) \rightarrow \mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$$

induces by the Yoneda embedding $\mathcal{A} \rightarrow \underline{\mathbf{dgMod}}_{\mathcal{A}}$ identifies \mathcal{A} with a triangulated subcategory of $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$.

The above definition ensures that for a pretriangulated dg-category \mathcal{A} its homotopy category $\mathrm{H}^0(\mathcal{A})$ does carry in a canonical way the structure of a triangulated category.

Definition 9. A **dg enhancement** of a triangulated category \mathcal{T} is a pretriangulated category \mathcal{A} together with an equivalence of triangulated category $\mathcal{T} \simeq \mathrm{H}^0(\mathcal{A})$.