DG-Enhancement of Triangulated Categories

Problems with Triangulated Categories and How to Circumvent Them

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By a **chain complex** we always mean a cochain complex, i.e. the differenital increases the degree.

1 Some Problems with Triangulated Categories

1.1 About the Abelianess of Triangulated Categories

Definition 1. An abelian category $\mathcal{A}$ is **semisimple abelian** or simply **semisimple** if every short exact sequence in $\mathcal{A}$ splits.

Lemma 2. In a triangulated category every epimorphism splits.

Proof. Let $f\colon x \to y$ be an epimorphism in a triangulated category $\mathcal{T}$. We may complete f to a distinguished triangle

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma(x)\,.$$

The composition of any two morphisms in a distinguished triangle vanishes, so $gf = 0$. It follows that $g = 0$ since f is an epimorphism. By applying the homological functor $\mathcal{T}(y, -)$ to this distinguished triangle we arrive at the followig long exact sequence:

$$\cdots \to \mathcal{T}(y,x) \xrightarrow{f_*} \mathcal{T}(y,y) \xrightarrow{0} \mathcal{T}(y,z) \to \cdots$$

We find that for $\mathrm{id}_y \in \mathcal{T}(y,y)$ there exists some $s \in \mathcal{T}(y,x)$ with $\mathrm{id}_y = f_*(s) = fs$. $\square$

Corollary 3. A triangulated category that is abelian is already semisimple. $\square$

We see from Corollary 3 that most triangulated categories are not abelian.

Proposition 4. For an abelian category $\mathcal{A}$ the following conditions on $\mathcal{A}$ and its derived category $\mathbf{D}(\mathcal{A})$ are equivalent:

(1) The derived category $\mathbf{D}(\mathcal{A})$ is abelian.

(2) The derived category $\mathbf{D}(\mathcal{A})$ is semisimple abelian.

(3) The abelian category $\mathbf{D}(\mathcal{A})$ is semisimple.

If these equivalent conditions are satisfied then $\mathbf{D}(\mathcal{A}) \simeq \mathcal{A}^{\mathbb{Z}}$ via the homology functor $\mathrm{H}_* \colon \mathbf{D}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$. A quasi-inverse $\mathcal{A}^{\mathbb{Z}} \to \mathbf{D}(\mathcal{A})$ is given by regarding every objects of $\mathcal{A}^{\mathbb{Z}}$ as a chain complex with zero differential.

Proof.

(3) $\implies$ (1): We realize $\mathbf{D}(\mathcal{A})$ by first passing from the category of chain complexes $\mathbf{Ch}(\mathcal{A})$ to its homotopy category $\mathbf{K}(\mathcal{A})$ and then localizing $\mathbf{K}(\mathcal{A})$ at the class of quasi-equivalences.

Every chain complex $X \in \mathrm{Ob}(\mathbf{Ch}(\mathcal{A}))$ splits[1] since $\mathcal{A}$ is semisimple, and can thus be decomposed as $X = X' \oplus X''$ where X' is split acyclic while X'' has zero differential. In the homotopy category $\mathbf{K}(\mathcal{A})$ the chain complex X' becomes zero as it is split acyclic and hence contractible. Every isomorphism class in $\mathbf{K}(\mathcal{A})$ is therefore represented by a chain complex with zero differential, i.e. an object of $\mathcal{A}^{\mathbb{Z}}$. Every nullhomotopy between such chain complexes is trivial (as the differential is zero) so no two morphisms between such chain complexes become identified in $\mathbf{K}(\mathcal{A})$. This shows that the categories $\mathbf{K}(\mathcal{A})$ and $\mathcal{A}^{\mathbb{Z}}$ are equivalent. We note that this equivalence is given by $\mathrm{H}_* \colon \mathbf{K}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$.

We also see that a quasi-isomorphism between chain complexes with zero differentials must already be an isomorphsim. We therefore see that $\mathbf{D}(\mathcal{A})$ is just $\mathbf{K}(\mathcal{A})$ again. It follows that $\mathbf{D}(\mathcal{A})$ is equivalent to $\mathcal{A}^{\mathbb{Z}}$.

(1) $\implies$ (2): This follows from Corollary 3.

(2) $\implies$ (3) We first observe that an abelian category $\mathcal{B}$ is semisimple if and only if every morphism $f \colon x \to y$ in $\mathcal{B}$ admits a **pseudoinverse** $g \colon y \to x$ satisfying $fgf = f$ and $gfg = g$: If every morphism f in $\mathcal{B}$ admits such a pseudoinverse g then it follows for every epimorphism f from $fgf = f$ that $fg = \mathrm{id}$ so that f splits. If on the other hand $\mathcal{B}$ is semisimple and $f \colon x \to y$ any morphism in $\mathcal{B}$ then we have decompositions $x = x' \oplus \ker(f)$ and $y = y' \oplus \mathrm{im}(f)$ with f inducing an isomorphism $x' \to \mathrm{im}(f)$. The inverse $\mathrm{im}(f) \to x'$ together with the projection $y \to \mathrm{im}(f)$ and the inclusion $x' \to x$ then give the desired pseudoinverse $g \colon y \to x$.

We find that by assumption every morphism in $\mathbf{D}(\mathcal{A})$ admits a pseudoinverse. Every morphism f in $\mathcal{A}$ hence admits a pseudoinverse g in $\mathbf{D}(\mathcal{A})$ (where we regard $\mathcal{A}$ as chain complexes concentrated in degree 0) which becomes a pseudoinverse $\mathrm{H}_0(g)$ to f in $\mathcal{A}$. This shows that every morphism in $\mathcal{A}$ admits a pseudoinverse, so that $\mathcal{A}$ is semisimple. $\square$

[1] Recall that a chain complex X is said to **split** if it can be (up to isomorphism) degreewise decomposed as $X^n = B^n \oplus H^n \oplus B_{n+1}$ such that the differential of X is with respect to this decomposition given by $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $\mathrm{Z}^n(X) = B^n \oplus H^n$, $\mathrm{B}^n(X) \cong B^n$ and $\mathrm{H}^n(X) = H^n$. The claimed decompositon $X = X' \oplus X''$ is then given degreewise by $X'^n = B_{n+1} \oplus B^n$ with differential $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $X''^n = H^n$.

1.2 Non-Functoriality of Cones

Triangulated categories do in general not admit functorial cones. As a consequence of this, we see that for a triangulated category $\mathcal{T}$ and a (small) index category I the diagram category $\mathbf{Fun}(I, \mathcal{T})$ does in general not inherit a triangulated structure from $\mathcal{T}$:

Given a morphism $f\colon D \to D'$ in $\mathbf{Fun}(I, \mathcal{T})$ we would otherwise like to compute its cone for the inherited triangulated structure of $\mathbf{Fun}(I, \mathcal{T})$ pointwise, i.e. via

$$\operatorname{cone}(f)(i) = \operatorname{cone}(f_i)$$

at every position $i \in \operatorname{Ob}(I)$, where $f_i\colon D(i) \to D'(i)$.

But $\operatorname{cone}(f)$, which is supposed to be a funcor $\operatorname{cone}(f)\colon I \to \mathcal{T}$, also needs to be defined on morphisms of I. For every such morphism $e\colon i \to j$ in I we would hence need an induced morphism $\operatorname{cone}(f)(e)\colon \operatorname{cone}(f)(i) \to \operatorname{cone}(f)(j)$, i.e. an induced morphismp$\operatorname{cone}(f)(e)\colon \operatorname{cone}(f_i) \to \operatorname{cone}(f_j)$. The good thing is that there exists by the axioms of a triangulated category some morphism $\operatorname{cone}(e)\colon \operatorname{cone}(f_i) \to \operatorname{cone}(f_j)$ that makes the diagram

$$\begin{tikzcd}
D(i) \arrow[r, "f_i"] \arrow[d, "D(i)"'] & D'(i) \arrow[r] \arrow[d, "D(j)"] & \operatorname{cone}(f_i) \arrow[d, dashed] \\
D(j) \arrow[r, "f_j"'] & D'(j) \arrow[r] & \operatorname{cone}(f_j)
\end{tikzcd}$$

commute. But by the missing functoriality of the cone this does in general not define a functor $\operatorname{cone}(f)\colon I \to \mathcal{T}$ and hence not an object of $\mathbf{Fun}(I, \mathcal{T})$.

The missing functoriality of the cone of $\mathcal{T}$ can in general not be fixed, as the following result asserts:

Proposition 5. Let $\mathcal{T}$ be an idempotent complete triangulated category. If $\mathcal{T}$ admits functorial cones then $\mathcal{T}$ is abelian and semisimple.

1.3 (Non-)Existence of Limits and Colimits

A triangulated category $\mathcal{T}$ is in general neither complete nor cocomplete.

1.4 Difference between $\mathbf{D}(\mathcal{A})^I$ and $\mathbf{D}(\mathcal{A}^I)$

One might suspect that for $\mathcal{T} = \mathbf{D}(\mathcal{A})$, where $\mathcal{A}$ is some abelian category, the above problems can be fixed by using an equivalence $\mathbf{D}(\mathcal{A})^I \simeq \mathbf{D}(\mathcal{A}^I)$. (Note that the category $\mathcal{A}^I$ is again abelian.) The problem is that there is in general no such equivalence.

Take for example $\mathcal{A} = k$-$\mathbf{Vect}$ and let $I = (\bullet \to \bullet)$ be the category consisting of two objects 0 and 1 and one non-identity morphism $0 \to 1$. The abelian category $\mathcal{A}$ is semisimple whence $\mathbf{D}(\mathcal{A})^I$ is again abelian by Proposition 4. But the functor category $\mathcal{A}^I = \mathbf{Fun}(I, \mathcal{A})$ is equivalent to the category of representations of the quiver $\bullet \to \bullet$ and therefore not abelian. The derived category $\mathbf{D}(\mathcal{A}^I)$ is thus not again abelian. This shows that $\mathbf{D}(\mathcal{A})^I$ is abelian but $\mathbf{D}(\mathcal{A}^I)$ is not abelian, which entails that these categories are not equivalent.

2 Towards Derivators

3 Towards DG-Enhancement

In the following we denote by k some commutative ring.

A dg enhancement of category $\mathcal{T}$ is, roughly speaking, a dg-category $\mathcal{A}$ together with an equivalence $\mathcal{T} \simeq \mathrm{H}^0(\mathcal{A})$. If the category $\mathcal{T}$ carries additional structures which we want to be respected by this equivalence, then we need to make sure than the homotopy category $\mathrm{H}^0(\mathcal{A})$ does carry such a structure itself. If $\mathcal{T}$ is a triangulated category then this leads us to the notion of a pretriangulated dg-category.

3.1 Review on DG-Categories

The k-linear category $\mathbf{Ch}(k)$ together with the tensor product of chain complexes becomes a symmetric monoidal category. A **dg-category** is an enriched category oever $\mathbf{Ch}(k)$. More precisely, a dg-category $\mathcal{A}$ consists of a class of objects $\mathrm{Ob}(\mathcal{A})$, for any two objects $x, y \in \mathcal{A}$ a chain complex $\mathcal{A}(x, y)$, for any three objects $x, y, z \in \mathrm{Ob}(\mathcal{A})$ a morphism of chain complexes

$$(-) \circ (-) \colon \mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \to \mathcal{A}(x, z)$$

satisfying the usual associativity diagram, and for every object $x \in \mathrm{Ob}(\mathcal{A})$ an element $1_x \in \mathrm{Z}^0(\mathcal{A}(x, y))$ satifying the usual identity diagrams. As an example we enrich $\mathbf{Ch}(k)$ into a dg-category $\underline{\mathbf{Ch}}(k)$ with $\underline{\mathbf{Ch}}(k)(X, Y) = \underline{\mathrm{Hom}}(X, Y)$ being the usual Hom-chain complex.

To any dg-category $\mathcal{A}$ we can associated the k-linear categories $\mathrm{Z}^0(\mathcal{A})$ and $\mathrm{H}^0(\mathcal{A})$ that are given by $\mathrm{Ob}(\mathrm{Z}^0(\mathcal{A})) = \mathrm{Ob}(\mathrm{H}^0(\mathcal{A}))$ and

$$\mathrm{Z}^0(\mathcal{A})(x, y) = \mathrm{Z}^0(\mathcal{A}(x, y)) \quad \text{and} \quad \mathrm{H}^0(\mathcal{A})(x, y) = \mathrm{H}^0(\mathcal{A}(x, y)) .$$

The composition of morphisms in $\mathrm{Z}^0(\mathcal{A})$ and $\mathrm{H}^0(\mathcal{A})$ is induced by the compositions in $\mathcal{A}$. As an example we have $\mathrm{Z}^0(\underline{\mathbf{Ch}}(k)) = \mathbf{Ch}(k)$ and $\mathrm{H}^0(\underline{\mathbf{Ch}}(k)) = \mathbf{K}(k)$.

This examples motivates that one should think about $\mathrm{Z}^0(\mathcal{A})$ as the "underlying k-linear category of $\mathcal{A}$", and as the elements of $\mathrm{Z}^0(\mathcal{A}(x, y))$ as the "actual morphisms" from x to y.

For two dg-categories $\mathcal{A}, \mathcal{B}$ a **dg-functor** $F \colon \mathcal{A} \to \mathcal{B}$ is given by a map $F \colon \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$ together with morphisms of chain complexes $F_{x,y} \colon \mathcal{A}(x, y) \to \mathcal{B}(F(x), F(y))$ for any two objects $x, y \in \mathrm{Ob}(\mathcal{A})$, satisfying the usual axioms. Every dg-functor $F \colon \mathcal{A} \to \mathcal{B}$ induces k-linear functors $\mathrm{Z}^0(\mathcal{A}) \to \mathrm{Z}^0(\mathcal{B})$ and $\mathrm{H}^0(\mathcal{A}) \to \mathrm{H}^0(\mathcal{B})$.

A dg-natural transformation $\alpha \colon F \Rightarrow G$ between dg-functors $F, G \colon \mathcal{A} \to \mathcal{B}$ assigns to is a family $\alpha = (\alpha_x)_{x \in \mathrm{Ob}(\mathcal{A})}$ of "actual morphisms" $\alpha_x \in \mathrm{Z}^0(\mathcal{B}(F(x), G(x)))$ with $\alpha_y \circ F(f) = G(f) \circ \alpha_x$ for every $f \in \mathcal{A}(x, y)$. We get a k-linear abelian category $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$ whose objects are dg-functors $\mathcal{A} \to \mathcal{B}$ and whose morphisms are dg-natural transformations. This category can be enriched into a dg-category $\underline{\mathbf{dgFun}}(\mathcal{A}, \mathcal{B})$

with

$$\underline{\mathbf{dgFun}}(\mathcal{A},\mathcal{B})(F,G)^i = \left\{ (\alpha_x)_{x\in\mathrm{Ob}(\mathcal{A})} \,\middle|\, \begin{array}{c} \alpha_x \in \mathcal{B}(F(x),G(x))^i \text{ with} \\ \alpha_y \circ F(f) = (-1)^{ij} G(f)\circ \alpha_x \\ \text{for every } f \in \mathcal{A}(x,y)^j \end{array} \right\}$$

being a subcomplex of $\prod_{x\in\mathrm{Ob}(\mathcal{A})} \mathcal{B}(F(x),G(x))$. Then in particular

$$\mathrm{Z}^0(\underline{\mathbf{dgFun}}(\mathcal{A},\mathcal{B})) = \mathbf{dgFun}(\mathcal{A},\mathcal{B})\,,$$

so that $\mathbf{dgFun}(\mathcal{A},\mathcal{B})$ is the underlying k-linear category of $\underline{\mathbf{dgFun}}(\mathcal{A},\mathcal{B})$.

For a dg-category $\mathcal{A}$ a **(right) dg-$\mathcal{A}$-module** is a dg-functor $M\colon \mathcal{A}^{\mathrm{op}} \to \underline{\mathbf{Ch}}(k)$. This means that at every object $x \in \mathrm{Ob}(\mathcal{A})$ we have a chain complex M_x, and for every $f \in \mathcal{A}(x,y)^i$ we have associated a map $M(f)\colon M(y) \to M(x)$ of degree i such that $M(g\circ f) = (-1)^{ij} M(f)\circ M(g)$ for all $f \in \mathcal{A}(x,y)^i$ and $g \in \mathcal{A}(y,z)^j$. The category of dg-$\mathcal{A}$-modules is given by $\mathbf{dgMod}_{\mathcal{A}} = \mathbf{dgFun}(\mathcal{A}^{\mathrm{op}}, \underline{\mathbf{Ch}}(k))$, and its dg-enrichement by $\underline{\mathbf{dgMod}}_{\mathcal{A}} = \underline{\mathbf{dgFun}}(\mathcal{A}^{\mathrm{op}}, \underline{\mathbf{Ch}}(k))$. The category $\mathbf{dgMod}_{\mathcal{A}}$ is abelian, complete and cocomplete, and all (co)limits are computed pointwise.

3.2 Review on Frobenius Exact Structures

The abelian category $\mathbf{Ch}(k)$ admits a Frobenius exact structure $\mathcal{S}$ whose associated $\mathcal{S}$-stable triangulated category is precisely $\mathbf{K}(k)$. For every chain complex X its **$\mathcal{S}$-injective envelope** is given by the chain complex

$$IX = \left(X \oplus X[1], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

which fits into the short exact sequence

$$0 \to X \xrightarrow{i_X} IX \xrightarrow{p_X} X[1] \to 0 \tag{1}$$

belonging to $\mathcal{S}$, where

$$i_X = \begin{bmatrix} 1 \\ d_X \end{bmatrix} \quad \text{and} \quad p_X = \begin{bmatrix} -d_X & 1 \end{bmatrix}\,.$$

The k-linear functors $I, [1]\colon \mathbf{Ch}(k) \to \mathbf{Ch}(k)$ and natural transformations $i\colon 1 \Rightarrow I$ and $p\colon I \Rightarrow [1]$ extends to a dg-functors $I,[1]\colon \underline{\mathbf{Ch}}(k) \to \underline{\mathbf{Ch}}(k)$ and dg-natural transformations $i\colon 1 \Rightarrow I$ and $p\colon I \to [1]$. By applying $\mathbf{dgFun}(\mathcal{A}^{\mathrm{op}}, -)$ these in turn induce a k-linear functors

$$I, [1]\colon \mathbf{dgMod}_{\mathcal{A}} \to \mathbf{dgMod}_{\mathcal{A}}$$

together with natural transformations $i\colon 1 \Rightarrow I$ and $p\colon I \Rightarrow [1]$. We get for every dg-$\mathcal{A}$-module M a short exact sequence of dg-$\mathcal{A}$-modules

$$0 \to M \xrightarrow{i_M} IM \xrightarrow{p_M} M[1] \to 0 \tag{2}$$

that gives at every point $x \in \mathrm{Ob}(\mathcal{A})$ the short exact sequence of chain complexes

$$0 \to M_x \xrightarrow{i_{M_x}} IM_x \xrightarrow{p_{M_x}} M_x[1] \to 0$$

from (1) with $X = M_x$. The short exact sequence (2) belongs to a Frobenus exact structure $\mathcal{S}$ on $\mathbf{dgMod}_{\mathcal{A}}$ whose stable triangulated category is precisely the homotopy category $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$, with $i_M \colon M \to IM$ being an $\mathcal{S}$-injective envelope of M for every dg-$\mathcal{A}$-module M. This description of $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ and its triangulated structure has two consequences we will need:

(1) A morphsim $f \colon M \to N$ in $\mathbf{dgMod}_{\mathcal{A}}$ vanishes in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ if and only it factors through some $\mathcal{S}$-injective object I of $\mathbf{dgMod}_{\mathcal{A}}$, i.e. if there exists in $\mathbf{dgMod}_{\mathcal{A}}$ a commutative diagram of the following form:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr] & & N \\ & I \arrow[ur] & \end{tikzcd}$$

The morphism $M \to I$ factors through the morphism $i_M \colon M \to IM$ since we have by the definition of an $\mathcal{S}$-injective object the following diagram:

$$\begin{tikzcd} & I & & & \\ 0 \arrow[r] & M \arrow[u] \arrow[r, "i_M"] & IM \arrow[ul, dashed, bend right] \arrow[r, "p_M"] & M[1] \arrow[r] & 0 \end{tikzcd}$$

It follows that f already factors through $i_M \colon M \to IM$.

(2) We get a description of the distinguished triangles in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ from Happel's theorem: Let

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i_M"] \arrow[d, "f"'] & IM \arrow[r, "p_M"] \arrow[d] & M[1] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r, "g"'] & P \arrow[r, "h"'] & M[1] \arrow[r] & 0 \end{tikzcd}$$

be be a commutative diagram in $\mathbf{dgMod}_{\mathcal{A}}$ whose rows are contained in $\mathcal{S}$ and where the left hand square is a pushout square. Then the resulting sequence

$$M \xrightarrow{f} N \xrightarrow{g} P \xrightarrow{h} M[1]$$

in $H^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ is a distinguished triangle. (And every distinguished triangle is up to isomorphism of this form.)

3.3 Pretriangulated DG-Categories

Definition 6. A dg-functor $F\colon \mathcal{A} \to \mathcal{B}$ is **dg-fully faithful** if for every two objects $x, y \in \mathcal{A}$ the morphism of chain complexes $F_{x,y}\colon \mathcal{A}(x,y) \to \mathcal{B}(F(x), F(y))$ is an isomorphism.

Proposition 7 (dg-Yoneda embedding)**.** Let $\mathcal{A}$ be a dg-category. Then the mapping

$$\mathcal{A} \to \underline{\mathbf{dgMod}}_{\mathcal{A}}, \quad x \mapsto \mathcal{A}(-,x) = \mathcal{A}^{\mathrm{op}}(x,-)$$

extends (in the usual way) to a dg-fully faithful dg-functor.

Definition 8. A dg-category $\mathcal{A}$ is **pretriangulated** if the fully faithful k-linear functor

$$\mathrm{H}^0(\mathcal{A}) \to \mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$$

induces by the Yoneda embedding $\mathcal{A} \to \underline{\mathbf{dgMod}}_{\mathcal{A}}$ identifies $\mathcal{A}$ with a triangulated subcategory of $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$.

The above definition ensures that for a pretriangulated dg-category $\mathcal{A}$ its homotopy category $\mathrm{H}^0(\mathcal{A})$ does carry in a canonical way the structure of a triangulated category.

Definition 9. A **dg-enhancement** of a triangulated category $\mathcal{T}$ is a pretriangulated category $\mathcal{A}$ together with an equivalence of triangulated categories $\mathcal{T} \simeq \mathrm{H}^0(\mathcal{A})$.

A dg-enhancement of a triangulated category $\mathcal{T}$ allows us to identity $\mathcal{T}$ with a full triangulated subcategory of $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ for some dg-category $\mathcal{A}$. We can then try to understand the original triangulated category $\mathcal{T}$ through the "higher structure" of the dg-category $\mathcal{A}$.

3.4 Cones as Derived Cokernels

Recall that if $f\colon x \to y$ is a morphism in an (pre)additive category $\mathcal{A}$ then its cokernel (which does not need to exists) can be thought of in two equivalent ways:

- A morphism $\operatorname{coker}(f) \to z$ into any other object is "the same" as a morphism $g\colon y \to z$ with $gf = 0$.
- The cokernel $\operatorname{coker}(f)$ is the pushout of the following diagram:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \\ 0 & \end{tikzcd}$$

In a triangulated category $\mathcal{T}$ we do in general have neither cokernels nor pushouts. But given a dg-enhancement $\mathcal{T} \simeq \mathrm{H}^0(\mathcal{A})$ for some pretriangulated dg-category $\mathcal{A}$ we can identify $\mathcal{T}$ with a full triangulated subcategory of $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$. We can then do the desired calculations in the abelian category $\mathbf{dgMod}_{\mathcal{A}} = \mathrm{Z}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ wich admits all (small) colimits and limits.

More precisely, given a fixed morphism $f\colon M \to N$ in $\mathbf{dgMod}_{\mathcal{A}}$ try to understand all morphisms $g\colon N \to P$ in $\mathbf{dgMod}_{\mathcal{A}}$ for which the composition $g \circ f$ vanishes in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$. We have previously seen that $g \circ f$ vanishes if and only if $g \circ f$ factors through $i_M\colon M \to IM$. We are hence interested in the pushout of the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i_M"'] & N \\ IM & \end{tikzcd}$$

in $\mathbf{dgMod}_{\mathcal{A}}$. This pushout is computed pointwise, hence we want to compute for every $x \in \mathrm{Ob}(\mathcal{A})$ the pushout

$$\begin{tikzcd} M_x \arrow[r, "f_x"] \arrow[d, "i_{M_x}"'] & N_x \\ IM_x & \end{tikzcd}$$

in the category $\mathbf{Ch}(k)$ where

$$IM_x = \left(M_x \oplus M_x[1], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \quad \text{and} \quad i_{M_x} = \begin{bmatrix} 1 \\ d_{M_x} \end{bmatrix}.$$

Lemma 10. For every morphism of chain complexes $f\colon X \to Y$ the pushout of

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] & Y \\ IX & \end{tikzcd}$$

in $\mathbf{Ch}(k)$ is given by

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] & Y \arrow[d, "\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]"] \\ IX \arrow[r, "\left[\begin{smallmatrix} -d_X & 1 \\ f & 0 \end{smallmatrix}\right]"'] & C^f \end{tikzcd}$$

where C^f denotes the usual mapping cone

$$C^f = \left(X[1] \oplus Y, \begin{bmatrix} -d_X & 0 \\ f & d_Y \end{bmatrix} \right).$$

Proof. We calculate the pushout as

$$\begin{aligned}
&\left(X \oplus X[1] \oplus Y, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_Y \end{bmatrix} \right) \Big/ \left\langle \begin{bmatrix} x \\ d(x) \\ -f(x) \end{bmatrix} \,\middle|\, x \in X \right\rangle \\
&\cong \left(X[1] \oplus Y, \begin{bmatrix} -d_X & 0 \\ f & d_Y \end{bmatrix} \right) \\
&= C^f
\end{aligned}$$

as claimed. □

Remark 11. The above results is (a posteri) not surprising: It is a standard statement from homological algebra that for a morphism of chain complexes $f\colon X \to Y$ the data of a morphism $h\colon C^f \to Z$ is the same as that of a morphism $g\colon Y \to Z$ together with a nullhomotopy of the composition $g \circ f\colon X \to Z$. Whence C^f corepresents morphisms going out of Y whose composition with f are zero "up to homotopy".

We hence find that the pushout

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "i_M"'] & N \arrow[d] \\
IM \arrow[r] & C^f
\end{tikzcd}$$

is given by $(C^f)_x = C^{f_x}$ at every $x \in \mathrm{Ob}(\mathcal{A})$, i.e. the dg-$\mathcal{A}$-module C^f is pointwise the usual mapping cone of chain complexes.

We can see that C^f is actually the cone of f in the triangulated category $\mathrm{H}^0(\underline{\mathbf{dgMod}_{\mathcal{A}}})$: We have the following commutative diagram in which the left hand side is a pushout, and where the upper row is exact and contained in the Frobenius exact structure of $\mathbf{dgMod}_{\mathcal{A}}$:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i_M"] \arrow[d, "f"'] & IM \arrow[r, "p_M"] \arrow[d] & M[1] \arrow[r] & 0 \\
& N \arrow[r] & C^f & &
\end{tikzcd}$$

It follows from the standard lemma of homological algebra that we can extend the above diagram to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i_M"] \arrow[d, "f"'] & IM \arrow[r, "p_M"] \arrow[d] & M[1] \arrow[r] \arrow[d, equal, dashed] & 0 \\
0 \arrow[r, dashed] & N \arrow[r] & C^f \arrow[r, dashed] & M[1] \arrow[r, dashed] & 0
\end{tikzcd}$$

whose row are exact. It follows from the axioms of a Frobenius exact structure that the lower row in this diagram is again contained in the Frobenius exact structure of $\mathbf{dgMod}_{\mathcal{A}}$. We have seen in the previous review on Frobenius exact structures that

$$M \xrightarrow{f} N \to C^f \to M[1]$$

is therefore a distinguished triangle in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$. This entails that C^f is a cone of f in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$.

We have thus seen that one can think about the cone in $\mathrm{H}^0(\underline{\mathbf{dgMod}}_{\mathcal{A}})$ as a kind of "derived cokernel" of f, coming from a pushout in $\mathbf{dgMod}_{\mathcal{A}}$ that was choosen as a "cokernel up to homotopy".

3.5 Extending to $\underline{\mathrm{dgMod}}_{\mathcal{A}}$