

# DG-Enhancement of Triangulated Categories

Problems with  
Triangulated and Derived Categories  
and How to Circumvent Them

Jendrik Stelzner

By a **chain complex** we always mean a cochain complex, i.e. the differential increases the degree.

## 1 Problems with Triangulated Categories

### 1.1 About the Abelianess of Triangulated Categories

**Definition 1.** An abelian category  $\mathcal{A}$  is **semisimple abelian** or simply **semisimple** if every short exact sequence in  $\mathcal{A}$  splits.

**Lemma 2.** In a triangulated category every epimorphism splits.

*Proof.* Let  $f: x \rightarrow y$  be an epimorphism in a triangulated category  $\mathcal{T}$ . We may complete  $f$  to a distinguished triangle

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma(x).$$

The composition of any two morphisms in a distinguished triangle vanishes, so  $gf = 0$ . It follows that  $g = 0$  since  $f$  is an epimorphism. By applying the homological functor  $\mathcal{T}(y, -)$  to this distinguished triangle we arrive at the following long exact sequence:

$$\cdots \rightarrow \mathcal{T}(y, x) \xrightarrow{f_*} \mathcal{T}(y, y) \xrightarrow{0} \mathcal{T}(y, z) \rightarrow \cdots$$

We find that for  $\text{id}_y \in \mathcal{T}(y, y)$  there exists some  $s \in \mathcal{T}(y, x)$  with  $\text{id}_y = f_*(s) = fs$ .  $\square$

**Corollary 3.** A triangulated category that is abelian is already semisimple.  $\square$

We see from Corollary 3 that most triangulated categories are not abelian.

**Proposition 4.** For an abelian category  $\mathcal{A}$  the following conditions on  $\mathcal{A}$  and its derived category  $\mathbf{D}(\mathcal{A})$  are equivalent:

- (1) The derived category  $\mathbf{D}(\mathcal{A})$  is abelian.
- (2) The derived category  $\mathbf{D}(\mathcal{A})$  is semisimple abelian.
- (3) The abelian category  $\mathbf{D}(\mathcal{A})$  is semisimple.

If these equivalent conditions are satisfied then  $\mathbf{D}(\mathcal{A}) \simeq \mathcal{A}^{\mathbb{Z}}$  via the homology functor  $H_*: \mathbf{D}(\mathcal{A}) \rightarrow \mathcal{A}^{\mathbb{Z}}$ . A quasi-inverse  $\mathcal{A}^{\mathbb{Z}} \rightarrow \mathbf{D}(\mathcal{A})$  is given by regarding every objects of  $\mathcal{A}^{\mathbb{Z}}$  as a chain complex with zero differential.

*Proof.*

(3)  $\implies$  (1): We realize  $\mathbf{D}(\mathcal{A})$  by first passing from the category of chain complexes  $\mathbf{Ch}(\mathcal{A})$  to its homotopy category  $\mathbf{K}(\mathcal{A})$  and then localizing  $\mathbf{K}(\mathcal{A})$  at the class of quasi-equivalences.

Every chain complex  $X \in \text{Ob}(\mathbf{Ch}(\mathcal{A}))$  splits<sup>1</sup> since  $\mathcal{A}$  is semisimple, and can thus be decomposed as  $X = X' \oplus X''$  where  $X'$  is split acyclic while  $X''$  has zero differential. In the homotopy category  $\mathbf{K}(\mathcal{A})$  the chain complex  $X'$  becomes zero as it is split acyclic and hence contractible. Every isomorphism class in  $\mathbf{K}(\mathcal{A})$  is therefore represented by a chain complex with zero differential, i.e. an object of  $\mathcal{A}^{\mathbb{Z}}$ . Every nullhomotopy between such chain complexes is trivial (as the differential is zero) so no two morphisms between such chain complexes become identified in  $\mathbf{K}(\mathcal{A})$ . This shows that the categories  $\mathbf{K}(\mathcal{A})$  and  $\mathcal{A}^{\mathbb{Z}}$  are equivalent. We note that this equivalence is given by  $H_*: \mathbf{K}(\mathcal{A}) \rightarrow \mathcal{A}^{\mathbb{Z}}$ .

We also see that a quasi-isomorphism between chain complexes with zero differentials must already be an isomorphism. We therefore see that  $\mathbf{D}(\mathcal{A})$  is just  $\mathbf{K}(\mathcal{A})$  again. It follows that  $\mathbf{D}(\mathcal{A})$  is equivalent to  $\mathcal{A}^{\mathbb{Z}}$ .

(1)  $\implies$  (2): This follows from Corollary 3.

(2)  $\implies$  (3) We first observe that an abelian category  $\mathcal{B}$  is semisimple if and only if every morphism  $f: x \rightarrow y$  in  $\mathcal{B}$  admits a **pseudoinverse**  $g: y \rightarrow x$  satisfying  $fgf = f$  and  $gfg = g$ : If every morphism  $f$  in  $\mathcal{B}$  admits such a pseudoinverse  $g$  then it follows for every epimorphism  $f$  from  $fgf = f$  that  $fg = \text{id}$  so that  $f$  splits. If on the other hand  $\mathcal{B}$  is semisimple and  $f: x \rightarrow y$  any morphism in  $\mathcal{B}$  then we have decompositions  $x = x' \oplus \ker(f)$  and  $y = y' \oplus \text{im}(f)$  with  $f$  inducing an isomorphism  $x' \rightarrow \text{im}(f)$ . The inverse  $\text{im}(f) \rightarrow x'$  together with the projection  $y \rightarrow \text{im}(f)$  and the inclusion  $x' \rightarrow x$  then give the desired pseudoinverse  $g: y \rightarrow x$ .

We find that by assumption every morphism in  $\mathbf{D}(\mathcal{A})$  admits a pseudoinverse. Every morphism  $f$  in  $\mathcal{A}$  hence admits a pseudoinverse  $g$  in  $\mathbf{D}(\mathcal{A})$  (where we regard  $\mathcal{A}$  as chain complexes concentrated in degree 0) which becomes a pseudoinverse  $H_0(g)$  to  $f$  in  $\mathcal{A}$ . This shows that every morphism in  $\mathcal{A}$  admits a pseudoinverse, so that  $\mathcal{A}$  is semisimple.  $\square$

<sup>1</sup>Recall that a chain complex  $X$  is said to **split** if it can be (up to isomorphism) degreewise decomposed as  $X^n = B^n \oplus H^n \oplus B_{n+1}$  such that the differential of  $X$  is with respect to this decomposition given by  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . Then  $Z^n(X) = B^n \oplus H^n$ ,  $B^n(X) \cong B^n$  and  $H^n(X) = H^n$ . The claimed decomposition  $X = X' \oplus X''$  is then given degreewise by  $X'^n = B_{n+1} \oplus B^n$  with differential  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $X''^n = H^n$ .

## 1.2 Non-Functoriality of Cones

Triangulated categories do in general not admit functorial cones. As a consequence of this, we see that for a triangulated category  $\mathcal{T}$  and a (small) index category  $I$  the diagram category  $\mathbf{Fun}(I, \mathcal{T})$  does in general not inherit a triangulated structure from  $\mathcal{T}$ :

Given a morphism  $f: D \rightarrow D'$  in  $\mathbf{Fun}(I, \mathcal{T})$  we would otherwise like to compute its cone for the inherited triangulated structure of  $\mathbf{Fun}(I, \mathcal{T})$  pointwise, i.e. via

$$\text{cone}(f)(i) = \text{cone}(f_i)$$

at every position  $i \in \text{Ob}(I)$ , where  $f_i: D(i) \rightarrow D'(i)$ .

But  $\text{cone}(f)$ , which is supposed to be a functor  $\text{cone}(f): I \rightarrow \mathcal{T}$ , also needs to be defined on morphisms of  $I$ . For every such morphism  $e: i \rightarrow j$  in  $I$  we would hence need an induced morphism  $\text{cone}(f)(e): \text{cone}(f)(i) \rightarrow \text{cone}(f)(j)$ , i.e. an induced morphism  $\text{cone}(f)(e): \text{cone}(f_i) \rightarrow \text{cone}(f_j)$ . The good thing is that there exists by the axioms of a triangulated category some morphism  $\text{cone}(e): \text{cone}(f_i) \rightarrow \text{cone}(f_j)$  that makes the diagram

$$\begin{array}{ccccc} D(i) & \xrightarrow{f_i} & D'(i) & \longrightarrow & \text{cone}(f_i) \\ D(i) \downarrow & & \downarrow D(j) & & \downarrow \\ D(j) & \xrightarrow{f_j} & D'(j) & \longrightarrow & \text{cone}(f_j) \end{array}$$

commute. But by the missing functoriality of the cone this does in general not define a functor  $\text{cone}(f): I \rightarrow \mathcal{T}$  and hence not an object of  $\mathbf{Fun}(I, \mathcal{T})$ .

The missing functoriality of the cone can in general not be fixed, as the following result asserts:

**Proposition 5.** Let  $\mathcal{T}$  be an idempotent complete triangulated category. If  $\mathcal{T}$  admits functorial cones then  $\mathcal{T}$  is abelian and semisimple.

## 1.3 Difference between $\mathbf{D}(\mathcal{A})^I$ and $\mathbf{D}(\mathcal{A}^I)$

One might suspect that for  $\mathcal{T} = \mathbf{D}(\mathcal{A})$ , where  $\mathcal{A}$  is some abelian category, the above problems can be fixed by using an equivalence  $\mathbf{D}(\mathcal{A})^I \simeq \mathbf{D}(\mathcal{A}^I)$ . (Note that the category  $\mathcal{A}^I$  is again abelian.) The problem is that there is in general no such equivalence.

Take for example  $\mathcal{A} = k\text{-Vect}$  and let  $I = (\bullet \rightarrow \bullet)$  be the category consisting of two objects 0 and 1 and one non-identity morphism  $0 \rightarrow 1$ . The abelian category  $\mathcal{A}$  is semisimple whence  $\mathbf{D}(\mathcal{A})^I$  is again abelian by Proposition 4. But the functor category  $\mathcal{A}^I = \mathbf{Fun}(I, \mathcal{A})$  is equivalent to the category of representations of the quiver  $\bullet \rightarrow \bullet$  and therefore not abelian. The derived category  $\mathbf{D}(\mathcal{A}^I)$  is thus not again abelian. This shows that  $\mathbf{D}(\mathcal{A})^I$  is abelian but  $\mathbf{D}(\mathcal{A}^I)$  is not abelian, which entails that these categories are not equivalent.

## 1.4 (Non-)Existence of Limits and Colimits

A triangulated category  $\mathcal{T}$  is in general neither complete nor cocomplete.

We see for example that in the category  $\mathbf{D}(\mathbb{Z}) = \mathbf{D}(\mathbf{Mod}_{\mathbb{Z}})$  the nonzero morphism  $f: \mathbb{Z}/2 \rightarrow \mathbb{Z}/4$  does not admit a cokernel: Suppose otherwise that  $c: \mathbb{Z}/4 \rightarrow C$  is a such a cokernel. Then  $c$  splits in  $\mathbf{D}(\mathbb{Z})$  by Lemma 2. It follows that  $H^0(c): \mathbb{Z}/4 \rightarrow H^0(C)$  splits in  $\mathbf{Mod}_{\mathbb{Z}}$ . But  $\mathbb{Z}/4$  is indecomposable, so it follows that  $H^0(c) = 0$  or  $H^0(c)$  is an isomorphism.

If  $H^0(c) = 0$  then we consider the nonzero morphism  $g: \mathbb{Z}/4 \rightarrow \mathbb{Z}/2$ . It follows from  $g \circ f = 0$  that  $g$  factors through  $C$  in  $\mathbf{D}(\mathbb{Z})$  and therefore factors through  $H^0(C)$  in  $\mathbf{Mod}_{\mathbb{Z}}$ . But this is not possible since  $H^0(C) = 0$  while  $g \neq 0$ .

Suppose that  $H^0(c)$  is an isomorphism. It follows from  $c \circ f = 0$  in  $\mathbf{D}(\mathbf{Mod}_{\mathbb{Z}})$  that  $H^0(c) \circ f = 0$  in  $\mathbf{Mod}_{\mathbb{Z}}$ . It now further follows that  $f = 0$  which is wrong.

## 2 Solution: Working Derived

We picture one approach to circumvent the non-functoriality of the cone: We denote for any category  $\mathcal{C}$  by  $\mathbf{Mor}(\mathcal{C})$  its morphism category. We have  $\mathbf{Mor}(\mathcal{C}) \cong \mathcal{C}^I$  for the arrow category  $I = (\bullet \rightarrow \bullet)$ .

### 2.1 Functorial Cones

We have seen that for a triangulated category  $\mathcal{T}$  there exists in general no cone functor  $\mathbf{Mor}(\mathcal{T}) \rightarrow \mathcal{T}$ , i.e. no cone functor  $\mathcal{T}^I \rightarrow \mathcal{T}$ . But if  $\mathcal{T} = \mathbf{D}(\mathcal{A})$  for some abelian category  $\mathcal{A}$  then we have seen above that the categories  $\mathcal{T}^I = \mathbf{D}(\mathcal{A})^I$  and  $\mathbf{D}(\mathcal{A}^I)$  are in general not equivalent. We can therefore instead try to construct a cone functor cone:  $\mathbf{D}(\mathcal{A}^I) \rightarrow \mathbf{D}(\mathcal{A})$ .

This is indeed possible: We start with the usual cone functor

$$C: \mathbf{Ch}(\mathcal{A})^I \rightarrow \mathbf{Ch}(\mathcal{A})$$

which assigns to each morphism  $f: X \rightarrow Y$  in  $\mathbf{Ch}(\mathcal{A})$  the usual mapping cone

$$C(f) = \left( X[1] \oplus Y, \begin{bmatrix} -d_X & 0 \\ f & d_Y \end{bmatrix} \right).$$

and to each commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{h} & Y' \end{array}$$

the induced morphism

$$C(g, h): C(f) \xrightarrow{\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}} C(f').$$

This functor is additive, and under the identification of  $\mathbf{Ch}(\mathcal{A})^I \cong \mathbf{Ch}(\mathcal{A}^I)$  this cone functor  $C$  respect quasi-isomorphisms: The quasi-isomorphism in  $\mathbf{Ch}(\mathcal{A}^I)$  correspond to a pointwise quasi-isomorphism in  $\mathbf{Ch}(\mathcal{A})^I$ . If  $(g, h): f \rightarrow f'$  in  $\mathbf{Ch}(\mathcal{A})^I$  is such a pointwise quasi-isomorphism, i.e. if both  $g$  and  $h$  are quasi-isomorphism, then

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} & C(f) & \xrightarrow{[-1 \ 0]} & X[1] \longrightarrow 0 \\ & & \downarrow h & & \downarrow C(g, h) & & \downarrow g \\ 0 & \longrightarrow & Y' & \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} & C(f') & \xrightarrow{[-1 \ 0]} & X'[1] \longrightarrow 0 \end{array}$$

is a commutative diagram with short exact rows. We get an induced ladder diagram from the functoriality of the long exact cone sequence, with  $h$  and  $g$  inducing vertical isomorphisms. It follows from the five lemma that  $C(g, h)$  also induces isomorphisms in homology, i.e. is a quasi-isomorphism. It follows that the cone functor  $C$  descends to an additive functor

$$C: \mathbf{D}(\mathcal{A}^I) \rightarrow \mathbf{D}(\mathcal{A})$$

where  $\mathbf{D}(\mathcal{A}^I) = \mathbf{D}(\mathbf{Mor}(\mathcal{A}))$ .

## 2.2 Colimits and Limits

Let  $I$  be an index category and let  $\Delta: \mathcal{C} \rightarrow \mathcal{C}^I$  be the constant diagram functor. Recall that the existence of (co)limits of  $I$ -shaped diagrams in  $\mathcal{C}$  is equivalent to  $\Delta$  admitting adjoints

$$\operatorname{colim}_I \dashv \Delta \dashv \lim_I .$$

That a derived category  $\mathbf{D}(\mathcal{A})$  does in general not admit (co)limits can be circumvented by considering **homotopy (co)limits** instead:

The constant diagram functor  $\Delta: \mathbf{Ch}(\mathcal{A}) \rightarrow \mathbf{Ch}(\mathcal{A})^I \cong \mathbf{Ch}(\mathcal{A}^I)$  is additive and respects quasi-isomorphisms and hence descends to an additive functor

$$\Delta: \mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A}^I) .$$

If  $\mathcal{A}$  admits all (co)limits of shape  $\mathcal{A}^I$  then it follows This functor admits both a left and a right adjoint:

$$\operatorname{hocolim}_I \dashv \Delta \dashv \operatorname{holim}_I .$$

## 2.3 Description as Total Derived Functors

## 3 Solution: DG-Enhancement

In the following we denote by  $k$  some commutative ring.

A dg enhancement of category  $\mathcal{T}$  is, roughly speaking, a dg-category  $\mathcal{A}$  together with an equivalence  $\mathcal{T} \simeq \mathbf{H}^0(\mathcal{A})$ . If the category  $\mathcal{T}$  carries additional structures which we

want to be respected by this equivalence, then we need to make sure than the homotopy category  $H^0(\mathcal{A})$  does carry such a structure itself. If  $\mathcal{T}$  is a triangulated category then this leads us to the notion of a pretriangulated dg-category.

### 3.1 Review on DG-Categories

The  $k$ -linear category  $\mathbf{Ch}(k)$  together with the tensor product of chain complexes becomes a symmetric monoidal category. A **dg-category** is an enriched category over  $\mathbf{Ch}(k)$ . More precisely, a dg-category  $\mathcal{A}$  consists of a class of objects  $\text{Ob}(\mathcal{A})$ , for any two objects  $x, y \in \mathcal{A}$  a chain complex  $\mathcal{A}(x, y)$ , for any three objects  $x, y, z \in \text{Ob}(\mathcal{A})$  a morphism of chain complexes

$$(-) \circ (-): \mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \rightarrow \mathcal{A}(x, z)$$

satisfying the usual associativity diagram, and for every object  $x \in \text{Ob}(\mathcal{A})$  an element  $1_x \in Z^0(\mathcal{A}(x, y))$  satisfying the usual identity diagrams. As an example we enrich  $\mathbf{Ch}(k)$  into a dg-category  $\underline{\mathbf{Ch}}(k)$  with  $\underline{\mathbf{Ch}}(k)(X, Y) = \underline{\text{Hom}}(X, Y)$  being the usual Hom-chain complex.

To any dg-category  $\mathcal{A}$  we can associated the  $k$ -linear categories  $Z^0(\mathcal{A})$  and  $H^0(\mathcal{A})$  that are given by  $\text{Ob}(Z^0(\mathcal{A})) = \text{Ob}(H^0(\mathcal{A}))$  and

$$Z^0(\mathcal{A})(x, y) = Z^0(\mathcal{A}(x, y)) \quad \text{and} \quad H^0(\mathcal{A})(x, y) = H^0(\mathcal{A}(x, y)).$$

The composition of morphisms in  $Z^0(\mathcal{A})$  and  $H^0(\mathcal{A})$  is induced by the compositions in  $\mathcal{A}$ . As an example we have  $Z^0(\underline{\mathbf{Ch}}(k)) = \mathbf{Ch}(k)$  and  $H^0(\underline{\mathbf{Ch}}(k)) = \mathbf{K}(k)$ .

This examples motivates that one should think about  $Z^0(\mathcal{A})$  as the “underlying  $k$ -linear category of  $\mathcal{A}$ ”, and as the elements of  $Z^0(\mathcal{A}(x, y))$  as the “actual morphisms” from  $x$  to  $y$ .

A **dg-functor**  $F: \mathcal{A} \rightarrow \mathcal{B}$  is given by a map  $F: \text{Ob}(\mathcal{A}) \rightarrow \text{Ob}(\mathcal{B})$  and morphisms of chain complexes  $F_{x,y}: \mathcal{A}(x, y) \rightarrow \mathcal{B}(F(x), F(y))$  for any two objects  $x, y \in \text{Ob}(\mathcal{A})$ , satisfying the usual axioms. Every dg-functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  induces  $k$ -linear functors  $Z^0(\mathcal{A}) \rightarrow Z^0(\mathcal{B})$  and  $H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$ .

A dg-natural transformation  $\alpha: F \Rightarrow G$  between dg-functors  $F, G: \mathcal{A} \rightarrow \mathcal{B}$  assigns to is a family  $\alpha = (\alpha_x)_{x \in \text{Ob}(\mathcal{A})}$  of “actual morphisms”  $\alpha_x \in Z^0(\mathcal{B}(F(x), G(x)))$  with  $\alpha_y \circ F(f) = G(f) \circ \alpha_x$  for every  $f \in \mathcal{A}(x, y)$ . We get a  $k$ -linear abelian category  $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$  whose objects are dg-functors  $\mathcal{A} \rightarrow \mathcal{B}$  and whose morphisms are dg-natural transformations. This category can be enriched into a dg-category  $\underline{\mathbf{dgFun}}(\mathcal{A}, \mathcal{B})$  with

$$\underline{\mathbf{dgFun}}(\mathcal{A}, \mathcal{B})(F, G)^i = \left\{ (\alpha_x)_{x \in \text{Ob}(\mathcal{A})} \left| \begin{array}{l} \alpha_x \in \mathcal{B}(F(x), G(x))^i \text{ with} \\ \alpha_y \circ F(f) = (-1)^{ij} G(f) \circ \alpha_x \\ \text{for every } f \in \mathcal{A}(x, y)^j \end{array} \right. \right\}$$

being a subcomplex of  $\prod_{x \in \text{Ob}(\mathcal{A})} \mathcal{B}(F(x), G(x))$ . Then in particular

$$Z^0(\underline{\mathbf{dgFun}}(\mathcal{A}, \mathcal{B})) = \mathbf{dgFun}(\mathcal{A}, \mathcal{B}),$$

so that  $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$  is the underlying  $k$ -linear category of  $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$ .

For a dg-category  $\mathcal{A}$  a **(right) dg- $\mathcal{A}$ -module** is a dg-functor  $M: \mathcal{A}^{\text{op}} \rightarrow \mathbf{Ch}(k)$ . This means that at every object  $x \in \text{Ob}(\mathcal{A})$  we have a chain complex  $M_x$ , and for every  $f \in \mathcal{A}(x, y)^i$  we have associated a map  $M(f): M(y) \rightarrow M(x)$  of degree  $i$  such that  $M(g \circ f) = (-1)^{ij} M(f) \circ M(g)$  for all  $f \in \mathcal{A}(x, y)^i$  and  $g \in \mathcal{A}(y, z)^j$ . The category of dg- $\mathcal{A}$ -modules is given by  $\mathbf{dgMod}_{\mathcal{A}} = \mathbf{dgFun}(\mathcal{A}^{\text{op}}, \mathbf{Ch}(k))$ , and its dg-enrichment by  $\mathbf{dgMod}_{\mathcal{A}} = \mathbf{dgFun}(\mathcal{A}^{\text{op}}, \mathbf{Ch}(k))$ . The category  $\mathbf{dgMod}_{\mathcal{A}}$  is abelian, complete and cocomplete, and all (co)limits are computed pointwise.

### 3.2 Review on Frobenius Exact Structures

The abelian category  $\mathbf{Ch}(k)$  admits a Frobenius exact structure  $\mathcal{S}_k$  whose associated  $\mathcal{S}_k$ -stable triangulated category is precisely  $\mathbf{K}(k)$ . For every chain complex  $X$  its  $\mathcal{S}_k$ -**injective envelope** is given by the chain complex

$$IX = \left( X \oplus X[1], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

which fits into the short exact sequence

$$0 \rightarrow X \xrightarrow{i_X} IX \xrightarrow{p_X} X[1] \rightarrow 0 \quad (1)$$

belonging to  $\mathcal{S}_k$ , where

$$i_X = \begin{bmatrix} 1 \\ d_X \end{bmatrix} \quad \text{and} \quad p_X = \begin{bmatrix} -d_X & 1 \end{bmatrix}.$$

The  $k$ -linear functors  $I, [1]: \mathbf{Ch}(k) \rightarrow \mathbf{Ch}(k)$  and natural transformations  $i: 1 \Rightarrow I$  and  $p: I \Rightarrow [1]$  extends to a dg-functors  $I, [1]: \mathbf{Ch}(k) \rightarrow \mathbf{Ch}(k)$  and dg-natural transformations  $i: 1 \Rightarrow I$  and  $p: I \Rightarrow [1]$ . By applying  $\mathbf{dgFun}(\mathcal{A}^{\text{op}}, -)$  these in turn induce a  $k$ -linear functors

$$I, [1]: \mathbf{dgMod}_{\mathcal{A}} \rightarrow \mathbf{dgMod}_{\mathcal{A}}$$

together with natural transformations  $i: 1 \Rightarrow I$  and  $p: I \Rightarrow [1]$ . We get for every dg- $\mathcal{A}$ -module  $M$  a short exact sequence of dg- $\mathcal{A}$ -modules

$$0 \rightarrow M \xrightarrow{i_M} IM \xrightarrow{p_M} M[1] \rightarrow 0 \quad (2)$$

that gives at every point  $x \in \text{Ob}(\mathcal{A})$  the short exact sequence of chain complexes

$$0 \rightarrow M_x \xrightarrow{i_{M_x}} IM_x \xrightarrow{p_{M_x}} M_x[1] \rightarrow 0$$

from (1) with  $X = M_x$ . The short exact sequence (2) belongs to a Frobenius exact structure  $\mathcal{S}_{\mathcal{A}}$  on  $\mathbf{dgMod}_{\mathcal{A}}$  whose stable triangulated category is precisely the homotopy category  $\mathbf{H}^0(\mathbf{dgMod}_{\mathcal{A}})$ , with  $i_M: M \rightarrow IM$  being an  $\mathcal{S}_{\mathcal{A}}$ -injective envelope of  $M$  for every dg- $\mathcal{A}$ -module  $M$ . This description of  $\mathbf{H}^0(\mathbf{dgMod}_{\mathcal{A}})$  and its triangulated structure has two consequences we will need:

- (1) A morphism  $f: M \rightarrow N$  in  $\mathbf{dgMod}_{\mathcal{A}}$  vanishes in  $H^0(\mathbf{dgMod}_{\mathcal{A}})$  if and only if it factors through some  $\mathcal{S}_{\mathcal{A}}$ -injective object  $I$  of  $\mathbf{dgMod}_{\mathcal{A}}$ , i.e. if there exists in  $\mathbf{dgMod}_{\mathcal{A}}$  a commutative diagram of the following form:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow & \nearrow \\ & I & \end{array}$$

The morphism  $M \rightarrow I$  factors through the morphism  $i_M: M \rightarrow IM$  since we have by the definition of an  $\mathcal{S}_{\mathcal{A}}$ -injective object the following diagram:

$$\begin{array}{ccccccc} & & I & & & & \\ & & \uparrow & \swarrow & & & \\ 0 & \longrightarrow & M & \xrightarrow{i_M} & IM & \xrightarrow{p_M} & M[1] \longrightarrow 0 \end{array}$$

It follows that  $f$  already factors through  $i_M: M \rightarrow IM$ .

- (2) We get a description of the distinguished triangles in  $H^0(\mathbf{dgMod}_{\mathcal{A}})$  from Happel's theorem: Let

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{i_M} & IM & \xrightarrow{p_M} & M[1] \longrightarrow 0 \\ & & \downarrow f & & \downarrow & & \parallel \\ 0 & \longrightarrow & N & \xrightarrow{g} & P & \xrightarrow{h} & M[1] \longrightarrow 0 \end{array}$$

be a commutative diagram in  $\mathbf{dgMod}_{\mathcal{A}}$  whose rows are contained in  $\mathcal{S}_{\mathcal{A}}$  and where the left hand square is a pushout square. Then the resulting sequence

$$M \xrightarrow{f} N \xrightarrow{g} P \xrightarrow{h} M[1]$$

in  $H^0(\mathbf{dgMod}_{\mathcal{A}})$  is a distinguished triangle. (And every distinguished triangle is up to isomorphism of this form.)

### 3.3 Pretriangulated DG-Categories

**Definition 6.** A dg-functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  is **dg-fully faithful** if for every two objects  $x, y \in \mathcal{A}$  the morphism of chain complexes  $F_{x,y}: \mathcal{A}(x, y) \rightarrow \mathcal{B}(F(x), F(y))$  is an isomorphism.

**Proposition 7** (dg-Yoneda embedding). Let  $\mathcal{A}$  be a dg-category. Then the mapping

$$\mathcal{A} \rightarrow \mathbf{dgMod}_{\mathcal{A}}, \quad x \mapsto \mathcal{A}(-, x) = \mathcal{A}^{\text{op}}(x, -)$$

extends (in the usual way) to a dg-fully faithful dg-functor.



**Definition 8.** A dg-category  $\mathcal{A}$  is **pretriangulated** if the fully faithful  $k$ -linear functor

$$H^0(\mathcal{A}) \rightarrow H^0(\mathbf{dgMod}_{\mathcal{A}})$$

induces by the Yoneda embedding  $\mathcal{A} \rightarrow \mathbf{dgMod}_{\mathcal{A}}$  identifies  $\mathcal{A}$  with a triangulated subcategory of  $H^0(\mathbf{dgMod}_{\mathcal{A}})$ .

The above definition ensures that for a pretriangulated dg-category  $\mathcal{A}$  its homotopy category  $H^0(\mathcal{A})$  does carry in a canonical way the structure of a triangulated category.

**Definition 9.** A **dg-enhancement** of a triangulated category  $\mathcal{T}$  is a pretriangulated category  $\mathcal{A}$  together with an equivalence of triangulated categories  $\mathcal{T} \simeq H^0(\mathcal{A})$ .

A dg-enhancement of a triangulated category  $\mathcal{T}$  allows us to identify  $\mathcal{T}$  with a full triangulated subcategory of  $H^0(\mathbf{dgMod}_{\mathcal{A}})$  for some dg-category  $\mathcal{A}$ . We can then try to understand the original triangulated category  $\mathcal{T}$  through the “higher structure” of the dg-category  $\mathcal{A}$ .

### 3.4 Cones as Derived Cokernels

Recall that if  $f: x \rightarrow y$  is a morphism in an (pre)additive category  $\mathcal{A}$  then its cokernel (which does not need to exist) can be thought of in two equivalent ways:

- A morphism  $\text{coker}(f) \rightarrow z$  into any other object is “the same” as a morphism  $g: y \rightarrow z$  with  $gf = 0$ .
- The cokernel  $\text{coker}(f)$  is the pushout of the following diagram:

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \\ 0 & & \end{array}$$

In a triangulated category  $\mathcal{T}$  we do in general have neither cokernels nor pushouts. But given a dg-enhancement  $\mathcal{T} \simeq H^0(\mathcal{A})$  for some pretriangulated dg-category  $\mathcal{A}$  we can identify  $\mathcal{T}$  with a full triangulated subcategory of  $H^0(\mathbf{dgMod}_{\mathcal{A}})$ . We can then do the desired calculations in the abelian category  $\mathbf{dgMod}_{\mathcal{A}} = Z^0(\mathbf{dgMod}_{\mathcal{A}})$  which admits all (small) colimits and limits.

More precisely, given a fixed morphism  $f: M \rightarrow N$  in  $\mathbf{dgMod}_{\mathcal{A}}$  try to understand all morphisms  $g: N \rightarrow P$  in  $\mathbf{dgMod}_{\mathcal{A}}$  for which the composition  $g \circ f$  vanishes in  $H^0(\mathbf{dgMod}_{\mathcal{A}})$ . We have previously seen that  $g \circ f$  vanishes if and only if  $g \circ f$  factors through  $i_M: M \rightarrow IM$ . We are hence interested in the pushout of the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ i_M \downarrow & & \\ IM & & \end{array}$$

in  $\mathbf{dgMod}_{\mathcal{A}}$ . This pushout is computed pointwise, hence we want to compute for every  $x \in \text{Ob}(\mathcal{A})$  the pushout

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & N_x \\ i_{M_x} \downarrow & & \\ IM_x & & \end{array}$$

in the category  $\mathbf{Ch}(k)$  where

$$IM_x = \left( M_x \oplus M_x[1], \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \quad \text{and} \quad i_{M_x} = \begin{bmatrix} 1 \\ d_{M_x} \end{bmatrix}.$$

**Lemma 10.** For every morphism of chain complexes  $f: X \rightarrow Y$  the pushout of

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_X \downarrow & & \\ IX & & \end{array}$$

in  $\mathbf{Ch}(k)$  is given by

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_X \downarrow & & \downarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ IX & \xrightarrow{\begin{bmatrix} -d_X & 1 \\ f & 0 \end{bmatrix}} & C(f) \end{array}$$

where  $C(f)$  denotes the usual mapping cone

$$C(f) = \left( X[1] \oplus Y, \begin{bmatrix} -d_X & 0 \\ f & d_Y \end{bmatrix} \right).$$

*Proof.* We calculate the pushout as

$$\begin{aligned} & \left( X \oplus X[1] \oplus Y, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_Y \end{bmatrix} \right) / \left\langle \begin{bmatrix} x \\ d(x) \\ -f(x) \end{bmatrix} \mid x \in X \right\rangle \\ & \cong \left( X[1] \oplus Y, \begin{bmatrix} -d_X & 0 \\ f & d_Y \end{bmatrix} \right) \\ & = C(f) \end{aligned}$$

as claimed.  $\square$

**Remark 11.** The above results is (a posteriori) not surprising: It is a standard statement from homological algebra that for a morphism of chain complexes  $f: X \rightarrow Y$  the data of a morphism  $h: C(f) \rightarrow Z$  is the same as that of a morphism  $g: Y \rightarrow Z$  together with a nullhomotopy of the composition  $g \circ f: X \rightarrow Z$ . Whence  $C(f)$  corepresents morphisms going out of  $Y$  whose composition with  $f$  are zero “up to homotopy”.

We hence find that the pushout

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ i_M \downarrow & & \downarrow \\ IM & \longrightarrow & C(f) \end{array}$$

is given by  $(C(f))_x = C(f_x)$  at every  $x \in \text{Ob}(\mathcal{A})$ , i.e. the dg- $\mathcal{A}$ -module  $C(f)$  is pointwise the usual mapping cone of chain complexes.

We see that  $C(f)$  is actually the cone of  $f$  in the triangulated category  $H^0(\mathbf{dgMod}_{\mathcal{A}})$ : We have the following commutative diagram in which the left hand side is a pushout, and where the upper row is exact and contained in the Frobenius exact structure of  $\mathbf{dgMod}_{\mathcal{A}}$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{i_M} & IM & \xrightarrow{p_M} & M[1] \longrightarrow 0 \\ & & f \downarrow & & \downarrow & & \\ & & N & \longrightarrow & C(f) & & \end{array}$$

It follows from the standard lemma of homological algebra that we can extend the above diagram to a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{i_M} & IM & \xrightarrow{p_M} & M[1] \longrightarrow 0 \\ & & f \downarrow & & \downarrow & & \parallel \\ & & & & & & 0 \\ 0 & \dashrightarrow & N & \longrightarrow & C(f) & \dashrightarrow & M[1] \dashrightarrow 0 \end{array}$$

whose row are exact. It follows from the axioms of a Frobenius exact structure that the lower row in this diagram is again contained in the Frobenius exact structure of  $\mathbf{dgMod}_{\mathcal{A}}$ . We have seen in the previous review on Frobenius exact structures that

$$M \xrightarrow{f} N \rightarrow C(f) \rightarrow M[1]$$

is therefore a distinguished triangle in  $H^0(\mathbf{dgMod}_{\mathcal{A}})$ . This entails that  $C(f)$  is a cone of  $f$  in  $H^0(\mathbf{dgMod}_{\mathcal{A}})$ .

We have thus seen that one can think about the cone in  $H^0(\mathbf{dgMod}_{\mathcal{A}})$  as a kind of “derived cokernel” of  $f$ , coming from a pushout in  $\mathbf{dgMod}_{\mathcal{A}}$  that was chosen as a “cokernel up to homotopy”.

### 3.5 Extending to $\mathbf{dgMod}_{\mathcal{A}}$

The object  $C(f)$  of the dg-category  $\mathcal{B} := \mathbf{dgMod}_{\mathcal{A}}$  gives a corepresentable dg-functor, i.e. left dg-module

$$\widehat{C} := \mathcal{B}(C(f), -) : \mathcal{B} \rightarrow \mathbf{Ch}(k).$$

At every object  $P \in \text{Ob}(\mathcal{B})$  this left dg-module is given by the chain complex  $\widehat{C}(P)$  which is given as a graded module by

$$\widehat{C}(P) = \mathcal{B}(N, P)[1] \oplus \mathcal{B}(M, P)$$

differential

$$d_{\widehat{C}(P)} = \begin{bmatrix} d_{\mathcal{B}(N,P)} & 0 \\ (-) \circ f & -d_{\mathcal{B}(M,P)} \end{bmatrix}.$$

For any two objects  $P, Q \in \text{Ob}(\mathcal{B})$  the morphism of chain complexes

$$\widehat{C}_{P,Q}: \mathcal{B}(P, Q) \rightarrow \mathcal{B}(\widehat{C}(P), \widehat{C}(Q))$$

is given by

$$g \mapsto ((a, b) \mapsto (g \circ a, (-1)^{|g|} g \circ b)).$$

We note that these formulae make sense in any dg-category. We can therefore define for every dg-category  $\mathcal{B}$  and every  $f \in Z^0(\mathcal{B}(x, y))$  a left dg- $\mathcal{B}$ -module  $C: \mathcal{B} \rightarrow \mathbf{Ch}(k)$  by the same expressions as above. One can then define a **cone** of  $f$  in  $\mathcal{B}$  as a representing object for this dg-functor. This allows us to talk about cones of morphisms in dg-categories.