Chain Rule

Given $f(z) = \log_e (1+z)$ where $z = X^T X$, $X \in \mathbb{R}^d$

Sol^n:

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$

Then, $X^T = [x_1 \quad x_2 \quad \cdots \quad x_d]$

$X^T X = [x_1^2 + x_2^2 + \cdots \; x_d^2]$

Applying chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$= \frac{\partial}{\partial z} (\log (1+z)) \cdot \frac{\partial}{\partial x} (X^T . X)$$

$$= \frac{1}{1+z} \cdot \frac{\partial}{\partial z} (z) \cdot \frac{\partial}{\partial x} (x_1^2 + x_2^2 + \cdots + x_d^2)$$

$$= \frac{1}{1+z} (2x_1 + 2x_2 + \cdots \; 2x_d)$$

$$= \frac{1}{1+z} \cdot 2 \left(x_1 + x_2 + \cdots x_d\right)$$

$$= \frac{2}{1+z} \sum_{i=1}^{d} x_i$$

(Ans)

$f(z) = e^{-z/2}$, where $z = g(z)$, $g(y) = y^T S^{-1} y$,

$y = h(x)$, $h(x) = x - \mu$

Sol^n:

Applying chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

Now,

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(e^{-z/2}\right)$$

$$= -\frac{e^{-z/2}}{2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(y^T s^{-1} y\right)$$

$$= \lim_{h\to 0} \frac{(y^T + h)\, s^{-1} (y+h) - y^T s^{-1} y}{h}$$

$$= \lim_{h\to 0} \frac{(y^T s^{-1} + h s^{-1})\,(y+h) - y^T s^{-1} y}{h}$$

$$= \lim_{h\to 0} \frac{y^T s^{-1} y + y^T s^{-1} h + h s^{-1} y + h^2 s^{-1} - y^T s^{-1} y}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{h\,(y^T s^{-1} + s^{-1} y + h s^{-1})}{h}$$

$$\Rightarrow \lim_{h\to 0} \left(y^T s^{-1} + s^{-1} y + h s^{-1}\right)$$

$$= y^T s^{-1} + s^{-1} y$$

$$\frac{\partial y}{\partial x} = \frac{\partial (x-\mu)}{\partial x}$$

$$= 1$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$= -\frac{e^{-z/2}}{2} \left(y^T S^{-1} + S^{-1} y \right) \cdot 1$$

$$= -\frac{e^{-z/2}}{2} \cdot \frac{1}{1} \left(y^T + y \right)$$

(A)