Computer Architecture
（计算机体系结构）

Lecture #2 – Number Representation

Lecturer Yuanqing Cheng （成元庆）

`www.cadetlab.cn`

News & Analysis
Qualcomm Prepares to Take Nvidia for a Ride

Review

- Continued rapid improvement in computing
  - 2X every 2.0 years in memory size;
    every 1.5 years in processor speed;
    every 1.0 year in disk capacity;
  - Moore's Law enables processor (2X transistors/chip every 2 yrs)
- 5 classic components of all computers

a	b	c	d	e
Control	Datapath	Memory	Input	Output

$\underbrace{\text{Control} \quad \text{Datapath}}_{\text{Processor}}$

What'll be the most important part of a computer in the future?

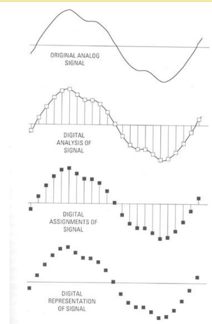
Putting it all in perspective…

"If the automobile had followed the same development cycle as the computer,

– Robert X. Cringely

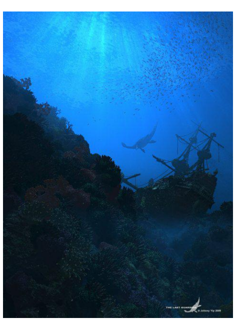
Data input: Analog ➔ Digital

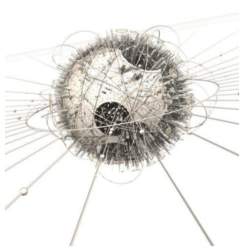
- Real world is analog!
- To import analog information, we must do two things
  - Sample
    - E.g., for a CD, every 44,100ths of a second, we ask a music signal how loud it is.
  - Quantize
    - For every one of these samples, we figure out where, on a 16-bit (65,536 tic-mark) "yardstick", it lies.



www.joshuadysart.com/journal/archives/digital_sampling.gif

Digital data not nec born Analog…





`hof.povray.org`

BIG IDEA: Bits can represent anything!!

- Characters?
  - 26 letters ⇒ 5 bits ($2^5 = 32$)
  - upper/lower case + punctuation ⇒ 7 bits (in 8) ("ASCII")
  - standard code to cover all the world's languages ⇒ 8,16,32 bits ("Unicode") `www.unicode.com`
- Logical values?
  - 0 ⇒ False, 1 ⇒ True
- colors ? Ex: *Red (00)* *Green (01)* *Blue (11)*
- locations / addresses? commands?
- MEMORIZE: N bits ⇔ at most 2^N things

How many bits to represent π ?

a) 1

b) 9 (π = 3.14, so that's 011 "." 001 100)

c) 64 (Since Macs are 64-bit machines)

d) Every bit the machine has!

e) ∞

What to do with representations of numbers?

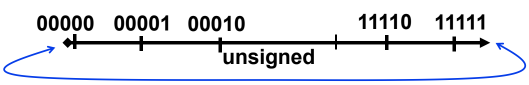
- Just what we do with numbers!
  - Add them
  - Subtract them
  - Multiply them
  - Divide them
  - Compare them

```
      1   1
      1   0   1   0
+ 0   1   1   1
-------------------------
1   0   0   0   1
```

- Example: 10 + 7 = 17
  - …so simple to add in binary that we can build circuits to do it!
  - subtraction just as you would in decimal
  - Comparison: How do you tell if X > Y ?

What if too big?

- Binary bit patterns above are simply representatives of numbers. Strictly speaking they are called "numerals".
- Numbers really have an ∞ number of digits
  - with almost all being same (00…0 or 11…1) except for a few of the rightmost digits
  - Just don't normally show leading digits
- If result of add (or -, *, /) cannot be represented by these rightmost HW bits, overflow is said to have occurred.



How to Represent Negative Numbers?

(C's `unsigned int`, C99's `uintN_t`)

- So far, unsigned numbers

Binary odometer

00000 00001 ... 01111 10000 ... 11111

- Obvious solution: define leftmost bit to be sign!
  - 0 ➔ + 1 ➔ –
  - Rest of bits can be numerical value of number
- Representation called sign and magnitude

Binary odometer

00000 00001 ... 01111

11111 ... 10001 10000

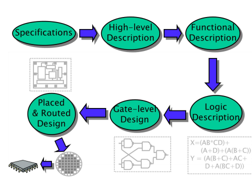
META: Ain't no free lunch

Shortcomings of sign and magnitude?

- Arithmetic circuit complicated
  - Special steps depending whether signs are the same or not
- Also, two zeros
  - 0x00000000 = +0 $_{ten}$
  - 0x80000000 = –0 $_{ten}$
  - What would two 0s mean for programming?
- Also, incrementing "binary odometer", sometimes increases values, and sometimes decreases!
- Therefore sign and magnitude abandoned

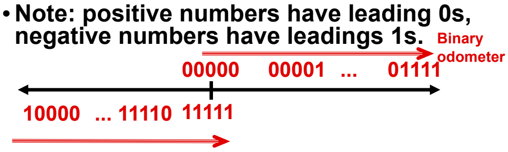
Great EDA course I supervise

- Introduction to VLSI Design Automation
  - The first EDA course in Beihang University
  - Learn physical design or design automation of ICs
  - Prereqs (data structures, programming language, algorithms, VLSI design)
  - `http://www.cadetlab.cn/courses`



Another try: complement the bits

- Example: $7_{10} = 00111_2$ $-7_{10} = 11000_2$
- Called One's Complement
- Note: positive numbers have leading 0s, negative numbers have leadings 1s.



- What is -00000 ? Answer: 11111
- How many positive numbers in N bits?
- How many negative numbers?

Shortcomings of One's complement ?

- Arithmetic still a somewhat complicated.
- Still two zeros
  - `0x00000000` = $+0_{ten}$
  - `0xFFFFFFFF` = -0_{ten}
- Although used for a while on some computer products, one's complement was eventually abandoned because another solution was better.

Standard Negative # Representation

- Problem is the negative mappings "overlap" with the positive ones (the two 0s). Want to shift the negative mappings left by one.
  - Solution! For negative numbers, complement, then add 1 to the result
- As with sign and magnitude, & one's compl. leading 0s is positive, leading 1s is negative
  - 000000...xxx is ≥ 0, 111111...xxx is < 0
  - except 1…1111 is -1, not -0 (as in sign & mag.)
- This representation is Two's Complement
  - This makes the hardware simple!

(C's `int`, aka a "signed integer")
(Also C's `short`, `long long`, …, C99's `intN_t`)

Two's Complement Formula

- Can represent positive and negative numbers in terms of the bit value times a power of 2:

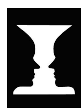
$d_{31} \times -(2^{31}) + d_{30} \times 2^{30} + \ldots + d_2 \times 2^2 + d_1 \times 2^1 + d_0 \times 2^0$

- Example: 1101_{two} in a nibble?

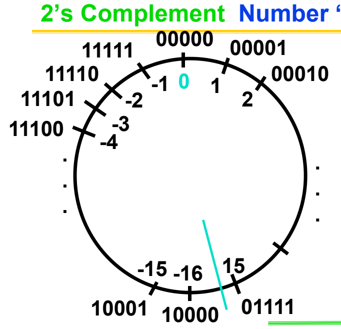
$= 1 \times -(2^3) + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
$= -2^3 + 2^2 + 0 + 2^0$
$= -8 + 4 + 0 + 1$
$= -8 + 5$
$= -3_{ten}$

Example: -3 to +3 to -3 (again, in a nibble):

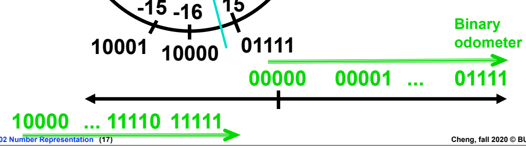
x : 1101_{two}
x': 0010_{two}
+1: 0011_{two}
()': 1100_{two}
+1: 1101_{two}



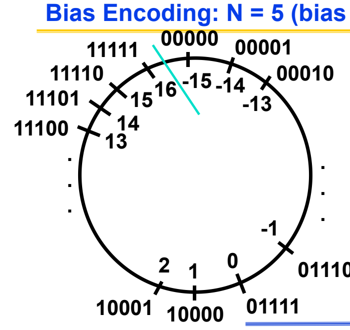
2's Complement Number "line": N = 5



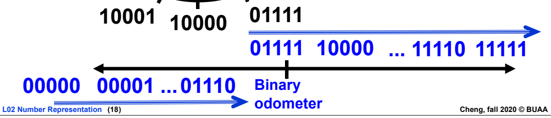
- 2^{N-1} non-negatives
- 2^{N-1} negatives
- one zero
- how many positives?



Bias Encoding: N = 5 (bias = -15)



- # = unsigned + bias
- Bias for N bits chosen as $-(2^{N-1}-1)$
- one zero
- how many positives?



How best to represent -12.75?

a) 2s Complement (but shift binary pt)

b) Bias (but shift binary pt)

c) Combination of 2 encodings

d) Combination of 3 encodings

e) We can't

Shifting binary point means "divide number by some power of 2. E.g., $11_{10} = 1011.0_2$ ➔ $10.110_2 = (11/4)_{10} = 2.75_{10}$

And in summary...

META: We often make design decisions to make HW simple

- We represent "things" in computers as particular bit patterns: N bits ⇒ 2^N things
- These 5 integer encodings have different benefits; 1s complement and sign/mag have most problems.
- unsigned (C99's `uintN_t`) :

00000 00001 ... 01111 10000 ... 11111

- 2's complement (C99's `intN_t`) universal, learn!

00000 00001 ... 01111

10000 ... 11110 11111

- Overflow: numbers ∞ ; computers finite, errors!

META: Ain't no free lunch

REFERENCE: Which base do we use?

- Decimal: great for humans, especially when doing arithmetic
- Hex: if human looking at long strings of binary numbers, its much easier to convert to hex and look 4 bits/symbol
  - Terrible for arithmetic on paper
- Binary: what computers use; you will learn how computers do +, -, *, /
  - To a computer, numbers always binary
  - Regardless of how number is written:
  - 32_{ten} == 32_{10} == 0x20 == 100000_2 == 0b100000
  - Use subscripts "ten", "hex", "two" in book, slides when might be confusing

Two's Complement for N=32

0000 ... 0000 0000 0000 0000 $_{two}$ =	0_{ten}
0000 ... 0000 0000 0000 0001 $_{two}$ =	1_{ten}
0000 ... 0000 0000 0000 0010 $_{two}$ =	2_{ten}
. . .	
0111 ... 1111 1111 1111 1101 $_{two}$ =	2,147,483,645 $_{ten}$
0111 ... 1111 1111 1111 1110 $_{two}$ =	2,147,483,646 $_{ten}$
0111 ... 1111 1111 1111 1111 $_{two}$ =	2,147,483,647 $_{ten}$
1000 ... 0000 0000 0000 0000 $_{two}$ =	–2,147,483,648 $_{ten}$
1000 ... 0000 0000 0000 0001 $_{two}$ =	–2,147,483,647 $_{ten}$
1000 ... 0000 0000 0000 0010 $_{two}$ =	–2,147,483,646 $_{ten}$
. . .	
1111 ... 1111 1111 1111 1101 $_{two}$ =	-3_{ten}
1111 ... 1111 1111 1111 1110 $_{two}$ =	-2_{ten}
1111 ... 1111 1111 1111 1111 $_{two}$ =	-1_{ten}

- One zero; 1st bit called sign bit
- 1 "extra" negative:no positive 2,147,483,648 $_{ten}$

Two's comp. shortcut: Sign extension

- Convert 2's complement number rep. using n bits to more than n bits
- Simply replicate the most significant bit (sign bit) of smaller to fill new bits
  - 2's comp. positive number has infinite 0s
  - 2's comp. negative number has infinite 1s
  - Binary representation hides leading bits; sign extension restores some of them
  - 16-bit -4_{ten} to 32-bit:

1111 1111 1111 1100 $_{two}$

1111 1111 1111 1111 1111 1111 1111 1100 $_{two}$