Computer Architecture
（计算机体系结构）

Lecture 11
Floating Point II

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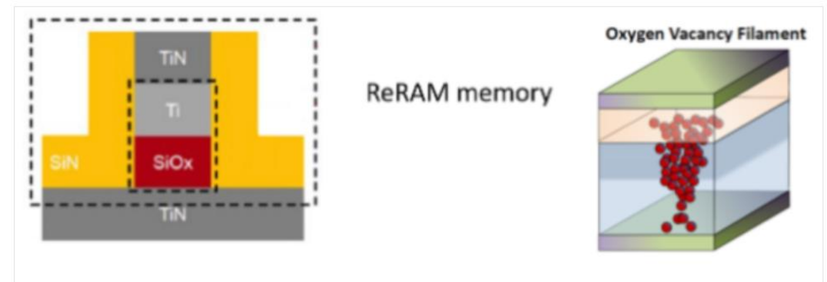
Lecturer Yuanqing Cheng
www.cadetlab.cn/teaching

Emerging Memories May Never Go Beyond Niche Applications



Weebit NAno's ReRAM technology uses two metal layers with a silicon oxide layer between, comprised of materials that can used in existing production lines.

Review

Exponent tells Significand how much (2^i) to count by (…, 1/4, 1/2, 1, 2, …)

- Floating Point lets us:
  - **Represent numbers containing both integer and fractional parts; makes efficient use of available bits.**
  - **Store** approximate **values for very large and very small #s.**
- IEEE 754 Floating Point Standard is most widely accepted attempt to standardize interpretation of such numbers (Every desktop or server computer sold since ~1997 follows these conventions)

Summary (single precision):

31	30 … 23	22 … 0
S	Exponent	Significand
1 bit	8 bits	23 bits

$(-1)^S \times (1 + \text{Significand}) \times 2^{(\text{Exponent}-127)}$

Double precision identical, except with exponent bias of 1023 (half, quad similar)

"Father" of the Floating point standard

IEEE Standard 754 for Binary Floating-Point Arithmetic.

1989 ACM Turing Award Winner!

Prof. Kahan

`www.cs.berkeley.edu/~wkahan/`
`.../ieee754status/754story.html`

Precision and Accuracy

Don't confuse these two terms!

Precision is a count of the number bits in a computer word used to represent a value.

Accuracy is a measure of the difference between the actual value of a number and its computer representation.

High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.

Example: `float pi = 3.14;`

pi will be represented using all 24 bits of the significant (highly precise), but is only an approximation (not accurate).

Representation for ± ∞

- In FP, divide by 0 should produce ± ∞, not overflow.
- Why?
  - OK to do further computations with ∞ E.g., X/0 > Y may be a valid comparison
  - Ask math majors
- IEEE 754 represents ± ∞
  - Most positive exponent reserved for ∞
  - Significands all zeroes

Representation for 0

- Represent 0?
  - exponent all zeroes
  - significand all zeroes
  - What about sign? Both cases valid.

```
+0: 0 00000000 00000000000000000000000
-0: 1 00000000 00000000000000000000000
```

Special Numbers

- What have we defined so far? (Single Precision)

Exponent	Significand	Object
0	0	0
0	nonzero	???
1-254	anything	+/- fl. pt. #
255	0	+/- ∞
255	nonzero	???

- Professor Kahan had clever ideas; "Waste not, want not"
  - We'll talk about Exp=0,255 & Sig!=0 later

Representation for Not a Number

- What do I get if I calculate
  `sqrt(-4.0)` or `0/0`?
  - If ∞ not an error, these shouldn't be either
  - Called Not a Number (NaN)
  - Exponent = 255, Significand nonzero
- Why is this useful?
  - Hope NaNs help with debugging?
  - They contaminate: op(NaN, X) = NaN

Representation for Denorms (1/2)

- Problem: There's a gap among representable FP numbers around 0
  - Smallest representable pos num:

    $a = 1.0\ldots_2 * 2^{-126} = 2^{-126}$
  - Second smallest representable pos num:

    $b = 1.000\ldots\ldots1_2 * 2^{-126}$

    $= (1 + 0.00\ldots1_2) * 2^{-126}$

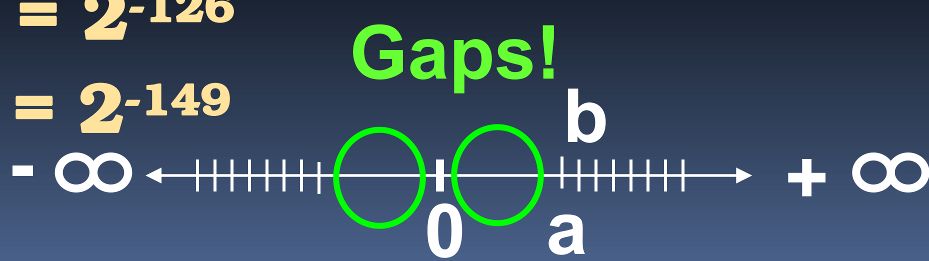
    $= (1 + 2^{-23}) * 2^{-126}$

    $= 2^{-126} + 2^{-149}$

Normalization and implicit 1 is to blame!

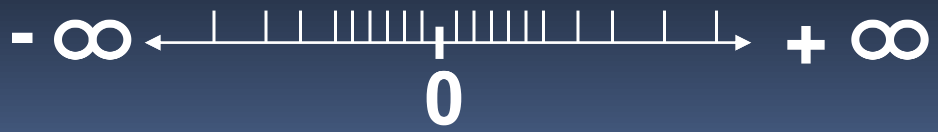
$a - 0 = 2^{-126}$

$b - a = 2^{-149}$



Representation for Denorm (2/2)

- Solution:
  - We still haven't used Exponent=0, Significand nonzero
  - Denormalized number: no (implied) leading 1, implicit exponent = -126
  - Smallest representable pos num:
    - $A = 2^{-149}$
  - Second smallest representable pos num:
    - $b = 2^{-148}$



Special Numbers Summary

- Reserve exponents, significands:

Exponent	Significand	Object
0	0	0
0	nonzero	Denorm
1-254	Anything	+/- fl. Pt #
255	0	+/- ∞
255	nonzero	NaN

Rounding

- When we perform math on real numbers, we have to worry about rounding to fit the result in the significant field.
- The FP hardware carries two extra bits of precision, and then round to get the proper value
- Rounding also occurs when converting:
  double to a single precision value, or
  floating point number to an integer

IEEE FP Rounding Modes

- Halfway between two floating point values (rounding bits read 10)? Choose from the following:
- Round towards + ∞
  - **Round "up": 1.01 10 → 1.10 , -1.01 10 → -1.01**
- Round towards - ∞
  - **Round "down": 1.01 10 → 1.01, -1.01 10 → -1.10**
- Truncate
  - **Just drop the extra bits (round towards 0)**
- Unbiased (default mode). Round to nearest EVEN number
  - **Half the time we round up on tie, the other half time we round down. Tends to balance out inaccuracies.**
  - **In binary, even means least significant bit is 0.**
- Otherwise, not halfway (00, 01, 11)! Just round to the nearest float.

Peer Instruction

1. Converting `float` -> `int` -> `float` produces same `float` number
2. Converting `int` -> `float` -> `int` produces same `int` number
3. FP add is associative: `(x+y)+z = x+(y+z)`

	ABC
1:	FFF
2:	FFT
3:	FTF
4:	FTT
5:	TFF

Peer Instruction Answer

1. Converting `float` -> `int` -> `float` produces same `float` number
   FALSE
2. Converting `int` -> `float` -> `int` produces same `int` number
   FALSE
3. FP add is associative:
   `(x+y)+z = x+(y+z)`
   FALSE

1. 3.14 -> 3 -> 3
2. 32 bits for signed int, but 24 for FP mantissa?
3. x = biggest pos #, y = -x, z = 1 (x != inf)

	ABC
1:	FFF
2:	FFT
3:	FTF
4:	FTT
5:	TFF

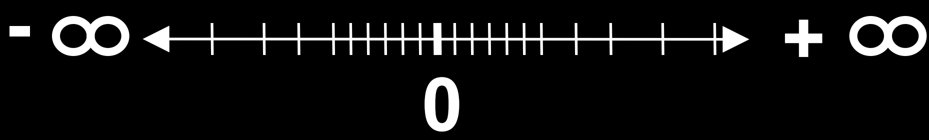
Peer Instruction

- Let `f(1,2)` = # of floats between 1 and 2
- Let `f(2,3)` = # of floats between 2 and 3

```
1: f(1,2) < f(2,3)
2: f(1,2) = f(2,3)
3: f(1,2) > f(2,3)
```

Peer Instruction Answer

- Let `f(1,2)` = # of floats between 1 and 2
- Let `f(2,3)` = # of floats between 2 and 3



`1: f(1,2) < f(2,3)`
`2: f(1,2) = f(2,3)`
`3: f(1,2) > f(2,3)`

"And in conclusion…"

- Reserve exponents, significands:

Exponent	Significand	Object
0	0	0
0	nonzero	Denorm
1-254	Anything	+/- fl. Pt #
255	0	+/- ∞
255	nonzero	NaN

- 4 Rounding modes (default: unbiased)
- MIPS Fl ops complicated, expensive

Bonus slides

- These are extra slides that used to be included in lecture notes, but have been moved to this, the “bonus” area to serve as a supplement.
- The slides will appear in the order they would have in the normal presentation

Bonus

FP Addition

- More difficult than with integers
- Can’t just add significands
- How do we do it?
  - De-normalize to match exponents
  - Add significands to get resulting one
  - Keep the same exponent
  - Normalize (possibly changing exponent)
- Note: If signs differ, just perform a subtract instead.

MIPS Floating Point Architecture (1/4)

- MIPS has special instructions for floating point operations:
  - **Single Precision:**
    `add.s, sub.s, mul.s, div.s`
  - **Double Precision:**
    `add.d, sub.d, mul.d, div.d`
- These instructions are far more complicated than their integer counterparts. They require special hardware and usually they can take much longer to compute.

MIPS Floating Point Architecture (2/4)

- Problems:
  - It's inefficient to have different instructions take vastly differing amounts of time.
  - Generally, a particular piece of data will not change from FP to int, or vice versa, within a program. So only one type of instruction will be used on it.
  - Some programs do no floating point calculations
  - It takes lots of hardware relative to integers to do Floating Point fast

MIPS Floating Point Architecture (3/4)

- 1990 Solution: Make a completely separate chip that handles only FP.
- Coprocessor 1: FP chip
  - contains 32 32-bit registers: `$f0`, `$f1`, …
  - most registers specified in `.s` and `.d` instruction refer to this set
  - separate load and store: `lwc1` and `swc1` ("load word coprocessor 1", "store …")
  - Double Precision: by convention, even/odd pair contain one DP FP number: `$f0/$f1`, `$f2/$f3`, … , `$f30/$f31`

MIPS Floating Point Architecture (4/4)

- 1990 Computer actually contains multiple separate chips:
  - **Processor: handles all the normal stuff**
  - **Coprocessor 1: handles FP and only FP;**
  - **more coprocessors?… Yes, later**
  - **Today, cheap chips may leave out FP HW**
- Instructions to move data between main processor and coprocessors:
  - `mfc0`, `mtc0`, `mfc1`, `mtc1`, **etc.**
- Appendix pages A-70 to A-74 contain many, many more FP operations.

Example: Representing 1/3 in MIPS

- 1/3
  - $= 0.33333..._{10}$
  - $= 0.25 + 0.0625 + 0.015625 + 0.00390625 + ...$
  - $= 1/4 + 1/16 + 1/64 + 1/256 + ...$
  - $= 2^{-2} + 2^{-4} + 2^{-6} + 2^{-8} + ...$
  - $= 0.0101010101..._2 * 2^0$
  - $= 1.0101010101..._2 * 2^{-2}$
    - Sign: 0
    - Exponent = -2 + 127 = 125 = 01111101
    - Significand = 0101010101...

0	0111 1101	0101 0101 0101 0101 0101 010

Casting floats to ints and vice versa

```
(int) floating_point_expression
```

Coerces and converts it to the nearest integer (C uses truncation)

```
i = (int) (3.14159 * f);
```

```
(float) integer_expression
```

converts integer to nearest floating point

```
f = f + (float) i;
```

int → float → int

```
if (i == (int)((float) i)) {
 printf("true");
}
```

- **Will not always print "true"**
- **Most large values of integers don't have exact floating point representations!**
- **What about `double`?**

float → int → float

```
if (f == (float)((int) f)) {
 printf("true");
}
```

- **Will not** always print "true"
- Small floating point numbers (<1) don't have integer representations
- For other numbers, rounding errors

Floating Point Fallacy

- FP add associative: FALSE!
  - $x = -1.5 \times 10^{38}$, $y = 1.5 \times 10^{38}$, and $z = 1.0$
  - $x + (y + z) = -1.5\text{x}10^{38} + (1.5\text{x}10^{38} + 1.0)$
    $= -1.5\text{x}10^{38} + (1.5\text{x}10^{38}) = \underline{0.0}$
  - $(x + y) + z = (-1.5\text{x}10^{38} + 1.5\text{x}10^{38}) + 1.0$
    $= (0.0) + 1.0 = \underline{1.0}$
- <u>Therefore, Floating Point add is not associative!</u>
  - Why? FP result <u>approximates</u> real result!
  - This example: 1.5×10^{38} is so much larger than 1.0 that $1.5 \times 10^{38} + 1.0$ in floating point representation is still 1.5×10^{38}