Formalizing Fux's Theory of Musical Counterpoint Using Constraint Programming

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Abstract

Acknowledgements

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Introduction

Related Work

Chapter 1

Theoretical Background

1.1 Music Theory

1.1.1 Equivalent American vs British English Terms

- Measure $\equiv$ Bar
- Whole step $\equiv$ Tone
- Half step $\equiv$ Semitone
- Whole note $\equiv$ Semibreve
- Half note $\equiv$ Minim
- Quarter note $\equiv$ Crotchet
- Eighth note $\equiv$ Quaver
- Sixteenth note $\equiv$ Semi-quaver

1.1.2 Concept of Counterpoint

TODO 1

1.1.3 Music Concepts

Note **TODO 2** (temp web def:) A note is a symbol used in sheet music to represent a specific pitch and duration of sound. Notes are written on a staff and can be represented by a variety of symbols, such as a round note head, a diamond-shaped note head, or a rectangular note head.

Beat **TODO 3**

Measure **TODO 4** (temp web def:) A measure is a section of music that is delimited by vertical bar lines in sheet music. Measures are used to organize the rhythm of a piece of music and to indicate when a new section of music begins.

Pitch **TODO 5** (temp web def:) Pitch refers to the highness or lowness of a sound. Pitch is determined by the frequency of the sound wave, and is usually measured in hertz (Hz). Higher pitched sounds have a higher frequency than lower pitched sounds.

MIDI Musical Instrument Digital Interface. A standard protocol for the communication between musical instruments and computers. What is commonly called "MIDI values" refers to the different possible MIDI notes ranging from 0 ($C_{-1} \equiv 8.175799\ Hz$) to 127 ($G_9 \equiv 12543.85\ Hz$). The notes of an 88-key piano are limited to A_0 to C_8. [5]

Tone TODO 6 (temp web def:) A tone is another term for a musical note. It is the sound that is produced by a musical instrument or a human voice.

Semitone TODO 7 (temp web def:) A semitone, also known as a half step, is the smallest interval (the distance between two notes) in Western music. It represents the distance between two adjacent notes on a keyboard or guitar.

Step The melodic interval of one semitone (minor second) one tone (major second). [4]

Stepwise Melodies that move by steps are stepwise.

Whole note TODO 8 (temp web def:) A whole note is a musical note that represents a longer duration of sound. It is represented in sheet music by an open note head and no stem. It lasts for four beats in 4/4 time signature.

Half note TODO 9 (temp web def:) A half note is a musical note that represents a medium duration of sound. It is represented in sheet music by an open note head and a stem. It lasts for two beats in 4/4 time signature.

Quarter note TODO 10 (temp web def:) A quarter note is a musical note that represents a shorter duration of sound. It is represented in sheet music by a filled-in note head and a stem. It lasts for one beat in 4/4 time signature.

Syncopation The displacement of the main beat of a measure. It creates an off-balance rhythm through the accenting of normally unaccented beats.

Dotted half note A dotted half note is a musical note that represents $1.5\times$ the duration of a half note. It is represented in sheet music by an open note head, a stem, and a dot. It lasts for three beats in 4/4 time signature.

Mordent A mordent is a type of ornament referring to a quick alternation between a note and its upper (upper mordent) or lower neighbor (lower/inverted mordent).[3]

Intervals In Western tonal music, the intervals making up an octave are separated into 12 semitones. Table 1.1 shows the MIDI values corresponding to these intervals.

Interval	Unison/Octave	Second		Third		Fourth	Triton	Fifth	Sixth		Seventh	
Type	Perfect	Minor	Major	Minor	Major	Perfect	♯4th / ♭5th	Perfect	Minor	Major	Minor	Major
Value	0	1	2	3	4	5	6	7	8	9	10	11

Table 1.1: MIDI values of the intervals over an octave range.

Tonic TODO 11 (temp web def:) The tonic is the first note of a scale and serves as the foundation or the "home" for the other notes in the scale. It is often used as a reference point for the other notes, and is the note that gives the scale its name.

Scale TODO 12 (temp web def:) A scale is a series of notes arranged in ascending or descending order. The most common scales in Western music are the major and minor scales. Each scale has a unique pattern of whole and half steps between the notes.

Key TODO 13 (temp web def:) A key refers to a specific scale and tonic. For example, a piece of music in the key of C major would use the C major scale and have C as the tonic. The key of a piece of music determines the overall tonality and harmony of the piece.

Mode TODO 14 (temp web def:) A mode is a type of scale that is derived from a specific parent scale. The most common modes in Western music are the Dorian, Phrygian, Lydian, Mixolydian, Aeolian and Locrian modes. Each mode has a unique pattern of whole and half steps between the notes that differs from the parent scale.

Diatonic TODO 15 (temp web def:) A diatonic scale is a scale made up of seven different pitches, where each pitch corresponds to a letter in the musical alphabet (A, B, C, D, E, F, G). The diatonic scale is the foundation of most Western music, and the basic building blocks of melody and harmony.

Chromatic TODO 16 (temp web def:) A chromatic scale is a scale that includes all the notes of the musical alphabet. A chromatic scale contains 12 notes in total, including all the notes in a diatonic scale and additional notes between each of the diatonic scale notes. Chromatic notes are often used to add dissonance or tension to a piece of music.

Borrowed note TODO 17 (temp web def:) A borrowed note is a non-diatonic note borrowed from another key or mode and used temporarily in a piece of music. Borrowed notes can be used to add variety and interest to a melody or harmony. They can also be used to create a sense of tension or dissonance, which can then be resolved back to the original key or mode.

Degree The relative position of a note in a scale to the tonic, also called a scale-step. By default, one degree aside from a note is the closest next note available in the diatonic scale. A degree can be expressed for both melody and harmony (even as chords). The degrees make it possible to understand and convert any tonality through a relative system. By convention, they are written with Roman numerals from I (the tonic) to VII (the sensible). For example, in C major, C (i.e. the tonic) is the I degree while G (i.e. the dominant, the fifth) is the V degree. Transposed to F major, this would give F the I degree and C the V degree. Also, melodies that progress by joint degrees are equivalent to stepwise melodies. [9]

Thesis Aka downbeat. With a common 4/4 time signature, the thesis is the first beat of any measure.

Arsis Aka upbeat. With a common 4/4 time signature, the arsis is the third beat of any measure.

Skip The melodic interval which, unlike the step, is greater than one tone. The term is rather used to refer to the third melodic interval because it is equivalent to *skip* a key on a piano but no convention exists. "Leap" can therefore also be used for the same purpose.

Leap The melodic interval which, unlike the step, is greater than one tone. The term is rather used to refer to melodic intervals larger than a third in constrast with the term "skip". Although, no convention exists so "skip" can also be used for the same purpose.

Diminution An intermediate note that exists between two notes separated by a skip of a third. In other words, a note that fills the space in third skip. This intermediate note is not necessarily below the previous one. Actually, the term refers to the division of a note into several shorter ones (i.e. "passage notes"). [17]

1.2 Constraint Programming Knowledge

Chapter 2

Introduction to the Formalization of Fux's Theory

The formalization of Fux is done in several steps:

1. **Spot the right rules in the Gradus Ad Parnassum.** Fux tended to explain certain rules of music so that they were easy to understand and use for the musicians of the time. This implies that sometimes several rules can be reduced to one. On the other hand, some of the rules of music are not written as such in the book because they are implicit. For example, it goes without saying that counterpoint belongs to a certain key and scale, but this is never explicitly written in the book. In order not to create misunderstanding, it was decided to write them explicitly and separately in the next sections.

2. **Formalize the rules in natural language in a way that is easy to construe as constraints.** Indeed, the Gradus Ad Parnassum is a work dedicated to a 17th century audience. It is necessary to read it with a critical eye and to translate it into modern language. That is, to reduce several rules into one, or at times, some rules are expressed in inclusive terms, whereas it is easier for a mathematician or computer scientist to write them in an equivalent way with exclusive terms or vice versa. Examples will be given in section 3.1.

3. **On the one hand, write the rules in discrete mathematics.** This is a crucial step in order to be able to use these rules precisely in other contexts and with other programming languages. This will also allow us to check whether solutions exist mathematically. Indeed, it is possible that some rules are contradictory and that consequently no solution is possible. It is important to keep in mind that some rules are written in a way that can be easily written with the Gecode tool.

4. **On the other hand, write the rules in constraint programming language.** The final goal of this thesis is to have constraints fixed according to Fux's rules and to find the best possible solutions with Gecode.

2.1 Array Logic and Notation

How the arrays are constructed is particularly important to understand. The rest of the paper relies entirely on the nuances and particularities of this logic and notation. This section is intended for mathematicians and computer scientists.

2.1.1 Logic of the arrays

The majority of the variables are arrays representing "in order" the different constrained values linked to the solution. The solution to the problem is an array of MIDI notes lists representing counterpoint. Before starting, two constants[1] must be defined[2]:

- m as the number of measures of the *cantus firmus* and the counterpoint;
- s_m as the maximum number of notes possible in the counterpoint, i.e. the size of the main arrays used to store Gecode variables. $s_m = m + 3 \times (m-1)$ and by extension, $s_{m-1} = (m-1) + 3 \times (m-2)$.

Intuitively one would separate an array into m lists of each measure with the different notes of a measure inside. Here the reverse applies. With a C-like representation, the access to a variable will be done as $[beat][measure]$ instead of $[measure][beat]$. This is more convenient for applying constraints in Lisp with GiL. Indeed, since the number of beats used by species varies, it is then easier to separate the arrays by lists of beats in order to be able to initialize only those which are treated in the problem. Since these arrays are initialized not with simple integers but with `IntVar` objects from Gecode, these constraint variables would definitely be initialized in the constraint space, which would not be ideal.

All the arrays related directly to the counterpoint are stored in arrays of size s_m (or s_{m-1} for the melody arrays as will be explained later). These arrays are composed of four lists, each representing the corresponding beat all along the measures of the song. The first is of size m while the other three are of size $m-1$ since they do not have a note in the last measure of the counterpoint which is only composed of a single whole note. E.g. `notes[0][9]`[3] would represent the note in the first beat of the tenth measure.

If the chosen species of counterpoint uses only **whole notes**, i.e. the first one, each note in first beat of each measure lasts **four beats**. Consequently, the lists of notes in the second, third and fourth beats are not used because these notes would already be represented by the one in first beat. The same logic applies to the other species: the second and fourth species only use the **first** and **third** "beat lists" because a note lasts **two beats**. While the third and fifth species are the only ones to use the **four available** beat lists because a note (can) last(s) **one beat**. See figure 2.1 (the corresponding midi value is annotated below each note) and table 2.1 for clarity.

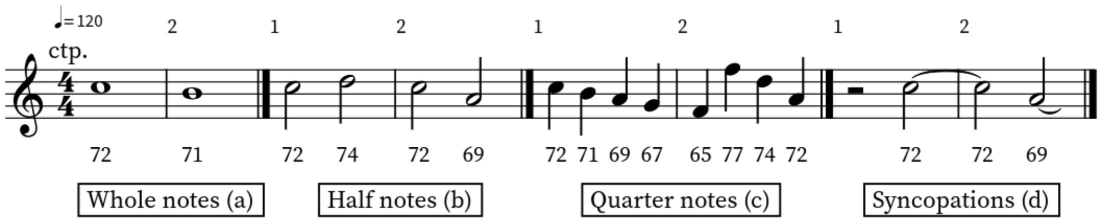


Figure 2.1: The 3 types of notes (N.B.: b ≡ d) over 8 beats for the 4th first species.

Syncopations have been added to illustrate that they work in the same way as half notes. The fifth species repeats the first four ones so it is not shown here. It will be explained in detail in section **TODO-sec 1** .

[1]Careful, these are constants from the point of view of the Gecode solver. They are variables defined once with the input but which are never set as constraint variables in the CSP.

[2]These constants are defined more precisely in subsection 2.2.1.

[3]This array exists only as an example. Here the notation corresponds neither to Lisp notation nor to mathematical notation.

beat, measure	$1^{st}, 1^{st}$	$2^{nd}, 1^{st}$	$3^{rd}, 1^{st}$	$4^{th}, 1^{st}$	$1^{st}, 2^{nd}$	$2^{nd}, 2^{nd}$	$3^{rd}, 2^{nd}$	$4^{th}, 2^{nd}$
Whole notes	72	$\emptyset$	$\emptyset$	$\emptyset$	71	$\emptyset$	$\emptyset$	$\emptyset$
Half notes	72	$\emptyset$	74	$\emptyset$	72	$\emptyset$	69	$\emptyset$
Quarter notes	72	71	69	67	65	77	74	72
Syncopations	$\emptyset$	$\emptyset$	72	$\emptyset$	72	$\emptyset$	69	$\emptyset$

Table 2.1: Relative MIDI values of figure 2.1.

2.1.2 Notations of the arrays

Several notations exist to describe the elements of an array. The one chosen here is close to the computer notation with the indexing starting at zero.

- $A[i, j]$ for element j of list i of array A;
- $L[i]$ for element i of list L;
- $A[i]$ for list i of array A.

Note that another way is also used to represent all the positions of a table. Indeed, as it is shown in the previous subsection, an array representing all measures per beat can be merged as a long list representing all beats one after the other. Therefore, to clarify the notation, $\forall \rho \in positions(m)$ will be used to represent all non-empty positions of an array. For example, for the half notes in the previous table 2.1: $\rho \in \{[0, 0], [2, 0], [0, 1], [2, 1], \ldots\}$. Moreover for notational purposes, $\rho + 1$ will denote the position of the next note such that if $A[\rho] = A[0, 0]$ then $A[\rho + 1] = A[2, 0]$. To explain it properly, the set $\mathcal{B}$ and the constants b and d must be introduced.

$\mathcal{B}$ Set of beats in a measure used by the solver depending on the chosen species. $\mathcal{B}$ can be seen as the location or index of the notes written over a measure on a score.

$$\mathcal{B} = \begin{cases} \{0\} & \text{if species = 1} \\ \{0, 2\} & \text{if species = \{2, 4\}} \\ \{0, 1, 2, 3\} & \text{if species = \{3, 5\}} \end{cases} \tag{2.1}$$

This refers back to the previous table 2.1.

b Number of beat(s) in a measure used by the solver depending on the chosen species. b can be seen as the number of notes written over a measure on a score. b is related to $\mathcal{B}$ since $b = |\mathcal{B}|$.

$$b = \begin{cases} 1 & \text{if species = 1} \\ 2 & \text{if species = \{2, 4\}} \\ 4 & \text{if species = \{3, 5\}} \end{cases} \tag{2.2}$$

d Duration of a note in beat(s) depending on the chosen species. d can be seen as the space between the notes of a measure on a score. d is inversely proportional to b.

$$\begin{gathered} d = 4/b \\ \therefore d = \begin{cases} 4 & \text{if species = 1} \\ 2 & \text{if species = \{2, 4\}} \\ 1 & \text{if species = \{3, 5\}} \end{cases} \end{gathered} \tag{2.3}$$

positions(upto) Function that returns the set of non-empty positions or indexes ordered depending on the species in such a way that all the positions would follow one another to represent all the beats of that species on a score in a single list.

$$\begin{gathered} positions(upto) = \bigcup_{\forall i \in \mathcal{B}, \forall j \in [0, upto)} [i, j] \\ \text{s.t. } \forall x \in [1, 3], \forall y \in [1, upto) \\ [i, j] <_s [i + x, j] <_s [i, j + y] \\ \text{where } <_s \text{ means the sorting order} \end{gathered} \tag{2.4}$$

By extension, $\rho + z >_s \rho$ such that:

$$\begin{gathered} \forall z \in \mathbb{N}^+, \forall \rho = [i, j] \in positions(upto) \\ \rho + z = [i + zd, j + nextm(i + zd)] \\ \text{where } nextm() \text{ is a function that returns the correct number of measure(s) to add.} \end{gathered} \tag{2.5}$$

2.2 Definitions of the Constants, Costs, Variables and Functions

This section is more intended for mathematicians and computer scientists too. Those who don't wish to read the mathematical parts should still broadly understand the variables of harmonic intervals, melodic intervals and motions (**H**, **M** and **P** in section 2.2.3). Subsections 2.2.3 and 2.2.1 describes the various names used in the mathematical parts and in the Lisp code of the solver immediately to their right if they were used (e.g. **n** `*total-cp-len`). These subsections explain also how those constants and variables work. Unless otherwise stated, all domains of constants and variables belong to the domain of integers $\mathbb{N}$.

2.2.1 Constants

Constants are only constant with respect to the Gecode solver, so they are deduced before a solution is sought by the latter.

Cons $_{(all,\ p,\ imp)}$ `ALL_CONS`, `P_CONS`, `IMP_CONS`

Set representing all consonances, perfect consonances and imperfect consonances respectively. By default, the notation $Cons \equiv Cons_{all}$.

$$\begin{gathered} Cons_p := \{0, 7\} \\ Cons_{imp} := \{3, 4, 8, 9\} \\ Cons_{all} := Cons_p \cup Cons_{imp} \equiv \{0, 3, 4, 7, 8, 9\} \end{gathered} \tag{2.6}$$

species `species`

Chosen species of counterpoint. $species \in \{1, 2, 3, 4, 5\}$.

m `*cf-len`

Number of measures which is equivalent to the number of notes in the *cantus firmus.* $m \in [3, 17]$. 3 because the solver needs al least 3 measures to work properly. 17 is arbitrary and comes from $4 \times 4 + 1$, i.e. a commun number of measure $\times$ a number not too large for the computation + one final measure.

n `*total-cp-len`
Number of notes in the counterpoint depending on the chosen species. $n \in [1, b(m-1)+1]$ because the last measure has necessarily a whole note.

s$_m$ Maximum number of notes contained in the counterpoint, all species combined, i.e. if the counterpoint contained only quarter notes, with the exception of the last note being a whole note.

$$s_m = m + 3 \times (m-1) \text{ and } s_{m-1} = (m-1) + 3 \times (m-2) \tag{2.7}$$

Used as the size for an array containing one list of size m (or $m-1$) the notes in thesis and three lists of size $m-1$ (or $m-2$) the other beats. The difference with n is that s does not depend on b.

Cf `*cf`
List of size m representing the MIDI notes of the *cantus firmus*.

$$\begin{gathered} \forall j \in [0, m) \\ Cf[j] \in [0, 127] \end{gathered} \tag{2.8}$$

M$_{cf}$ `*cf-brut-m-intervals`
List of size $m-1$ representing the melodic intervals between the consecutive notes of the *cantus firmus*.

$$\begin{gathered} \forall j \in [0, m-1) \\ M_{cf}[j] = Cf[j+1] - Cf[j] \\ \text{where } M_{cf} \in [-127, 127] \end{gathered} \tag{2.9}$$

lb `RANGE_LB` *DFLT: <62>*
Lower bound of the range of the notes of the counterpoint. $lb \in [0, ub)$.

ub `RANGE_UB` *DFLT: <82>*
Upper bound of the range of the notes of the counterpoint. $ub \in (lb, 127]$.

$\mathcal{R}$ `*cp-range`
Range of the notes of the counterpoint. $\mathcal{R} := [lb, ub]$.

borrow **TODO 18** *DFLT: <major>*
Determines the additional notes that the counterpoint can have in relation to the tonic of the piece. More details will be given on what are the borrowed notes in section 2.3.1.

$$borrow \in \{none, major, minor\} \tag{2.10}$$

$\mathcal{N}^{(\mathcal{R})}_{(all,\ key,\ brw)}$ `*extended-cp-domain` **TODO 19**
Set of values available for the notes of the counterpoint. $\mathcal{N}_{key}$ represents the notes of the key provided by the user's score. $\mathcal{N}_{brw}$ represents the additional borrowed notes that the counterpoint can have in relation to the tonic of the piece. $\mathcal{N}_{all}$ represents the union of the two previous sets. If $borrow = none$ then $\mathcal{N}_{brw} = \emptyset$ and $\mathcal{N}_{all} = \mathcal{N}_{key}$.

$\mathcal{N}^{\mathcal{R}}_{(all,\ key,\ brw)}$ represents the set of notes bounded to the range, i.e. the intersection of $\mathcal{N}_{(all,\ key,\ brw)}$ and $\mathcal{R}$. By default, $\mathcal{N}$ refers to $\mathcal{N}_{all}$ not bounded to the range.

$$\begin{gathered}
\mathcal{N}_{key} := buildScale(key, scale) \\
\mathcal{N}_{brw} := \begin{cases} \emptyset & \text{if } borrow = none \\ buildScale(Cf[0] \bmod 12, "borrowed") & \text{if } borrow = major \\ buildScale([Cf[0]+3] \bmod 12, "borrowed") & \text{if } borrow = minor \end{cases} \\
\mathcal{N}_{all} := \mathcal{N}_{key} \cup \mathcal{N}_{brw} \\
\mathcal{N}^{\mathcal{R}}_{(all,\ key,\ brw)} := \mathcal{N}_{(all,\ key,\ brw)} \cap \mathcal{R}
\end{gathered} \tag{2.11}$$

Where $buildScale(key, scale)$ (see function 2.24) is a function that returns the set of notes in the key on the basis of the $scale$ used. Also more details on the borrowed notes will be given in section 2.3.1.

2.2.2 Costs

The costs are constants chosen by the user that have default values supposed to represent Fux's preferences.

pref and cost **TODO 20**

A preference can have 7 levels of intensity ranging from "no cost" to "forbidden". For any cost $cost$ and any preference $pref$, it can be defined that:

$$cost = \begin{cases} 0 & \text{if } pref = \text{no cost} \\ 1 & \text{if } pref = \text{low cost} \\ 2 & \text{if } pref = \text{medium cost} \\ 4 & \text{if } pref = \text{high cost} \\ 8 & \text{if } pref = \text{last resort} \\ 2m & \text{if } pref = \text{cost prop. to length} \\ 64m & \text{if } pref = \text{forbidden} \end{cases} \tag{2.12}$$

Cond$_{costs}$ and cost$_{Cond}$ All costs work the same way: a list of repeated conditions determines whether it is true that a certain cost should be established. Only for the explanation, the list of assigned costs for a certain condition is noted $Cond_{costs}$. The elements of $Cond_{costs}$ are thus equivalent to any cost $cost$. The different costs of the different conditions each have their own identifier noted $cost_{Cond}$ and it is this value that is determined by the user. To sum up:

$$\begin{gathered}
\forall c \in Cond_{costs}, \forall cond \in Cond \\
c = \begin{cases} cost_{Cond} & \text{if } cond \text{ is true} \\ 0 & \text{otherwise} \end{cases} \\
\text{where } cost_{Cond} \in dom(cost)
\end{gathered} \tag{2.13}$$

$64m$ is a ridiculously huge value that will never be reached by all the other costs combined even if they were all *high*.

$\mathcal{C}$ **and** τ `*cost-factors`, `*total-cost`.

The final goal of the solver is to find a solution while minimizing the total cost. The latter is represented by τ while $\mathcal{C}$ is a set pf integers representing all the sums of the different lists of costs. τ is thus the sum of all the elements of $\mathcal{C}$. If $Costs$ is the set of all the different $Cond_{costs}$ lists then:

$$\begin{gathered}\mathcal{C} = \bigcup_{\forall \chi \in Costs} \sum_{\forall c \in \chi} c \\ \tau := \sum_{\forall \sigma \in \mathcal{C}} \sigma \\ \min \tau\end{gathered} \tag{2.14}$$

By definition, for any forbidden $pref$ to be indeed *forbidden*, the following constraint must be added:

$$\sum_{\forall \sigma \in \mathcal{C}} \sigma < 64m \tag{2.15}$$

2.2.3 Variables

Variables are fully deduced by the Gecode solver and their values can be evaluated only after a solution has been found.

Many variables have a general definition so that they can be used in all equations, this does not mean that all possible combinations have been defined in the Lisp code but only those that are actually used. For example, there is no need to have access to all possible melodic intervals in the solver, however the mathematical notation would allow it.

If some letters are not defined, it means that they have already been defined in the constants or in the previous variables.

Cp   `*cp`

Array of size s_m representing the MIDI notes of the counterpoint. This array is thus composed of four lists, each representing a beat on all the measures of the song. As explained above, this is how all the other arrays related to the countrepoint (i.e. the Cp array) are constructed.

For example, for a whole notes counterpoint: the relevant Cp would be only the list $Cp[0]$. For a half notes counterpoint: it would be the merge of $Cp[0]$ and $Cp[2]$. For a quarter notes counterpoint: it would be the merge of $Cp[0]$, $Cp[1]$, $Cp[2]$ and $Cp[3]$.

$$\forall i \in \mathcal{B}, \forall j \in [0, m) : Cp[i, j] \in \mathcal{N}^{\mathcal{R}} \tag{2.16}$$

$\mathbf{H}_{(abs)}$ `*h-intervals, *h-intervals-abs.`

Array of size s_m representing each harmonic interval between the counterpoint and the *cantus firmus.* There are four lists of harmonic intervals, each representing a beat along the whole counterpoint. The harmonic intervals are calculated so that they represent the absolute difference between the pitch of the counterpoint and the pitch of the *cantus firmus.* Since the values are absolute, it does not matter if the *cantus firmus* is lower or upper, the intervals will always be calculated according to the lowest note. Any harmonic interval is calculated according to the notes played at the same time in the *cantus firmus* and the counterpoint. Therefore, up to four notes in the counterpoint can be calculated with respect to the same note in the *cantus firmus.*

Two versions of that array-variable exist: the main one H which is modulo 12 and H_{abs} which is not. It is always true that $H = H_{abs}\ mod\ 12$. Unless mentioned, when talking about "harmonic intervals" or "harmonies", it refers to the variables of the array H.

$$\begin{gathered}\forall i \in \mathcal{B}, \forall j \in [0, m) \\ H_{abs}[i,j] = |Cp[i,j] - Cf[j]| \\ H[i,j] = H_{abs}[i,j]\ mod\ 12 \\ \text{where } H_{abs}[i,j] \in [0,127], H[i,j] \in [0,11]\end{gathered} \tag{2.17}$$

12 representing the number of semitones in an octave. This allows the interval between a note and any note higher at different octaves to always be the same. This implies that $H \in$ table 1.1 values. For example for the gap between C_4 (60) and G_4 (67) and the gap between C_4 (60) and G_5 (79), the H_{abs} values will be 7 and 19 while the H values will be 7 and 7. See figure **TODO 21** and table **TODO 22** for clarity.

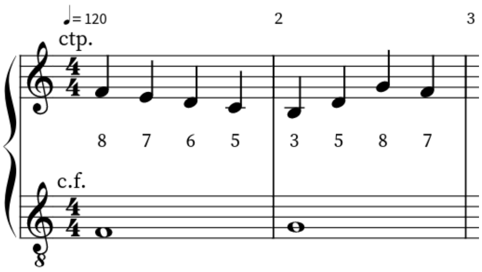


Figure 2.2: Harmonic intervals between the counterpoint and the *cantus firmus*.

Beware that the numbers noted are those used on scores. They refer to the names of the intervals and not to the relative MIDI values. See section **TODO-sec 2** for more information. By contrast, table 2.2 below shows the MIDI values of the intervals for this figure.

measure j	$H[0,j]$	$H[1,j]$	$H[2,j]$	$H[3,j]$
0	12	11	9	7
1	4	7	12	10

Table 2.2: Relative MIDI values of figure 2.2.

$\mathbf{M}^{(x)}_{(brut)}$ `*m-intervals, *m-intervals-brut, *m2-intervals, *m2-intervals-brut, *m-succ-intervals, *m-succ-intervals-brut.`

Arrays of size s_{m-x} representing each melodic interval between a note of the counterpoint at a specific beat and another further note of the counterpoint at another specific beat. The melodic intervals are calculated so that they represent the difference between the two notes involved.

The array is noted M^x where x is the number of d[4] beat(s) that separates the initial note to the further one. x represents the desired number of notes between the current note and the one of interest to calculate the melodic interval. In other words, $M^x[i,j]$ represents the melodic interval between the note at beat i in measure j and the note at beat $[(i+xd)\ mod\ 4]$ in measure $[j+nextm(i+xd)]$. If x is not present then its default is

[4]Duration of a note in beat(s) depending on the chosen species (see d in above section 2.2.1).

1. For example, with whole notes (i.e. $d = 4$): $M[0,5]$ represents the melodic interval between the note in the sixth measure ($j = 5$) and the note in the seventh measure ($j = 6$).

There are two versions of that array-variable: the main one M^x which is absolute and M^x_{brut} which is not. It is always true that $M^x = |M^x_{brut}|$. Unless mentioned, when talking about "melodic intervals" or "melodies", it refers to the variables of the array M^1. See figure 2.3 (the corresponding midi value is annotated below each note) and table 2.3 for clarity.

$$\begin{gathered} \forall x \in \{1,2\}, \forall i \in \mathcal{B}, \forall j \in [0, m-x) \\ M^x_{brut}[i,j] = Cp[(i+xd) \; mod \; 4, j + nextm(i+xd)] - Cp[i,j] \\ M^x[i,j] = |M^x_{brut}[i,j]| \\ \text{where } M^x_{brut}[i,j] \in [-12, 12], M^x[i,j] \in [0, 12] \end{gathered} \tag{2.18}$$

The intervals are limited to 12 because the octave leap is the maximum that can be reached.

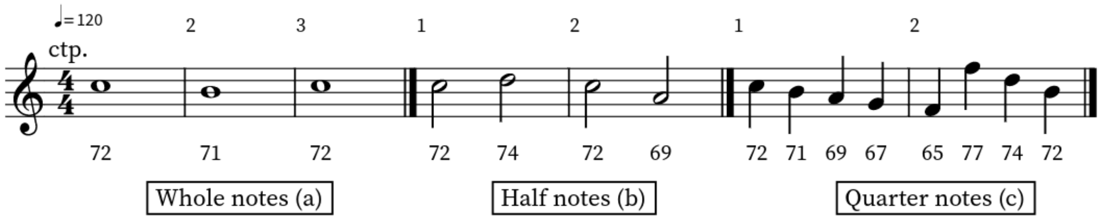


Figure 2.3: The 3 types of notes that can be used in the counterpoint.

$M^x_{(brut)}$	Whole notes (a)	Half notes (b)	Quarter notes (c)
$M[0,0]$	1 (-1)	2 (2)	1 (-1)
$M[1,0]$	$\emptyset$	$\emptyset$	2 (-2)
$M[2,0]$	$\emptyset$	2 (-2)	2 (-2)
$M[3,0]$	$\emptyset$	$\emptyset$	2 (-2)
$M[0,1]$	1 (1)	3 (-3)	12 (12)
$M^2[0,0]$	(0)	0 (0)	3 (-3)
$M^2[2,0]$	$\emptyset$	5 (-5)	4 (-4)

Table 2.3: Some relative MIDI values of figure 2.3 with $x = \{1, 2\}$.

In the solver, melodic intervals used are stored in several lists by beat pair, e.g. one list for all the intervals between the first and second beats of all measures. The constraints to represent these calculations are done separately from one table to another with the same function. From example, all the melodic intervals between the fourth beat note and the next first beat note in the thrid species are computed like in equation 2.19:

$$\begin{gathered} \forall j \in [0, m-1) \\ M_{brut}[3,j] = Cp[0, j+1] - Cp[3,j] \\ M[3,j] = |M_{brut}[3,j]| \end{gathered} \tag{2.19}$$

P *motions

Array of size $4 \times (m-1)$ representing each motion between two consecutive measures. The letter P is for *passage* since M is already taken. Contrary, oblique and direct motions are represented by 0, 1 and 2 respectively.

$$\forall x \in \{1,2\}, \forall i \in \mathcal{B}, \forall j \in [0, m-1), x := b - i$$
$$P[i,j] = \begin{cases} 0 & \text{if } (M^x_{brut}[i,j] > 0 > M_{cf}[j]) \vee (M^x_{brut}[i,j] < 0 < M_{cf}[j]) \\ 1 & \text{if } M^x_{brut}[i,j] = 0 \vee M_{cf}[j] = 0 \\ 2 & \text{if } (M^x_{brut}[i,j] > 0 \wedge M_{cf}[j] > 0) \vee (M^x_{brut}[i,j] < 0 \wedge M_{cf}[j] < 0) \end{cases} \tag{2.20}$$

$x := b - i$ represents the fact that the motion is obtained between the current note and the first note of the next measure. For example, with quarter notes, the gap between the third note and the first note of the next measure is defined as: $b = 4$, $i = 2$ and $x = 4 - 2 = 2$. The first note of the next measure is therefore 2 notes away.

The motions require relatively many constraints to be computed. Indeed, a boolean variable is needed for each type of direction of the counterpoint melody (3) as well as that of the *cantus firmus* (3). This gives 3*3 different possibilities to be divided into 3 categories of motions for each measure. This is not a problem in itself but with GiL, any boolean operation must be computed via a constrained boolean variable. Ideally one should use argument variables provided by Gecode that are intended to be temporary variables. Implementing this in GiL would probably improve performance.

IsCfB `*is-cf-bass-arr`

Boolean array of size $m + 3 \times (m-1)$ representing if the *cantus firmus* is below. Each list of this array represents a beat along the whole counterpoint and is calculated by comparing the pitch of the counterpoint with the pitch of the *cantus firmus* at the same time.

$$\forall i \in \mathcal{B}, \forall j \in [0, m) : IsCfB[i,j] = \begin{cases} \top & \text{if } Cp[i,j] \geq Cf[j] \\ \bot & \text{otherwise} \end{cases} \tag{2.21}$$

By default, if both notes are the same then the *cantus firmus* is considered as the bass.

IsCons$_{(all,\ p,\ imp)}$ `*is-cons-arr`

Boolean array of size $m + 3 \times (m-1)$ representing if harmonic intervals are consonances, perfect consonantes or imperfect consonances. Each list of this array represents a beat along the whole counterpoint and is calculated by checking that harmonies belong to the corresponding set of consonances. By default, $IsCons \equiv IsCons_{all}$.

$$\forall i \in \mathcal{B}, \forall j \in [0, m) : IsCons[i,j]_{(all,\ p,\ imp)} = \begin{cases} \top & \text{if } H[i,j] \in Cons_{(all,\ p,\ imp)} \\ \bot & \text{otherwise} \end{cases} \tag{2.22}$$

2.2.4 Fonctions

Functions are a way to improve the readability of some more complex mathematical notations. The majority remain relatively simple.

nextm(x) Returns the number of measure(s) to add in 4/4 time signature depending on the number of beat x.

$$nextm(x) = \begin{cases} 1 + nextm(x-4) & \text{if } x \geq 4 \\ 0 & \text{otherwise} \end{cases} \tag{2.23}$$

buildScale(key, scale) Returns the set of notes in the key on the basis of the $scale$ used. key is a value between 0 and 11 such that $0 \equiv C$ and $11 \equiv B$.

$$\forall x \in [-11, 127], \forall \delta := key + x \in [0, 127]$$
$$buildScale(key, scale) = \begin{cases} \bigcup_{\delta \bmod 12 \in key + \{0,2,4,5,7,9,11\}} \delta & \text{if } scale = \text{major} \\ \bigcup_{\delta \bmod 12 \in key + \{0,2,3,5,7,8,10\}} \delta & \text{if } scale = \text{minor} \\ \bigcup_{\delta \bmod 12 \in key + \{0,5,9,11\}} \delta & \text{if } scale = \text{borrowed} \end{cases} \tag{2.24}$$
$$\text{where } key \in [0, 11], scale \in \{"major", "minor", "borrowed"\}$$

N.B.: $buildScale(key, "minor") \equiv buildScale([key + 3] \bmod 12, "major")$.

Membership function $e \in E$ Returns a boolean value that is true if e is in the set E. In mathematical expressions, this notation is often used to represent a multitude of constraints in one.

$$E := \{e_0, \ldots, e_n\} : e \in E = \begin{cases} \top & \text{if } (e = e_0) \vee \cdots \vee (e = e_n) \\ \bot & \text{otherwise} \end{cases} \tag{2.25}$$

In the code the constraints are often expressed separately for each element. For example for a constraint cst which is applied if $e \in x, y, z$, it would state:

$$(e = x) \implies cst; \quad (e = y) \implies cst; \quad (e = z) \implies cst$$

By extension *in the context of this paper* when an expression uses only $\in$, it implies that this expression is true, i.e the element must belong to the set: $e \in E \equiv (e \in E \iff \top)$. This refers directly to the way Gecode allows this constraint.

2.3 Implicit General Rules of Counterpoint

In this section, all the following rules are implicit, sometimes taken from Fux's examples, sometimes from music theory in general.

2.3.1 Formalization in English

G1 *Harmonic intervals are always calculated from the lower note.*

Indeed, any harmonic interval is a calculation of the absolute difference between two notes. This implies that they adapt to where the counterpoint is in relation to the *cantus firmus.* .

G2 *The number of measure of the counterpoint must be the same as the number of measure of the cantus firmus.*

The goal is to compose complete counterpoints which lasts the same time as the *cantus firmus.*

G3 *The counterpoint must have the same time signature and the same tempo as the cantus firmus.*

The notes must be played in sync.

G4 *The counterpoint must be in the same key as the cantus firmus.*

This is a fundamental rule of music in general. Since music of the baroque period does not follow the same standards as today's music, this rule is a bit more complicated than it seems. Indeed, it often happens that Fux gives examples with accidentals, i.e. notes that do not belong to the diatonic scale. There are therefore notes "borrowed" from other scales which do not appear as a basis for the key signature.

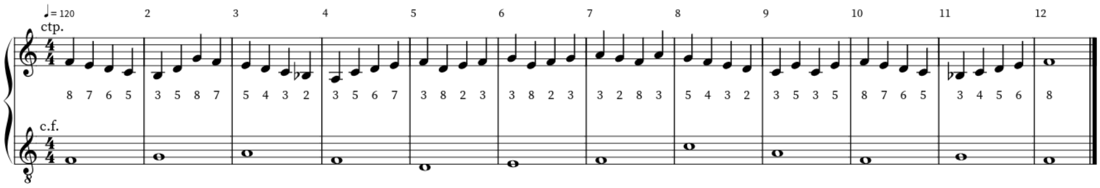


Figure 2.4: Example of a C major key signature starting on F with $B\flat$'s [18, p.54].

This makes it somewhat difficult to determine the precise domain of notes available for counterpoint. It is possible to determine a logic behind these borrowed notes. One way of looking at it is as follows: Fux composes with several different modes throughout his work: the F (lydian) mode, the D (dorian) mode and others (see section **TODO-sec 3**). In the rules of the first species (see section 3.1 at **1.H4**), it will be seen that Fux determines the use of a mode according to the first note of the *cantus firmus* in relation to the key of the musical work. Since a mode can be either major or minor, some notes can be borrowed from the major or minor diatonic scale of the first note of the *cantus firmus* respectively.

In the figure 2.4, the key is C major, i.e. $[C, D, E, F, G, A, B]$. These notes can therefore be used in counterpoint, but that is not all. Since the first note is an F, this implies that the tonic of this work is F, although it uses the major scale of C, so it is an example of the use of the F mode, the lydian mode. The lydian mode being a major mode, some notes of the diatonic major scale of F can be used sparingly by counterpoint. Looking at several examples given by Fux, the notes borrowed are I ($[F]$[5] necessarily included since it determines the tonic of the work), IV ($[B\flat]$ the fourth), VI ($[D]$ note of the relative minor) and VII ($[E]$ the sensible which is most often used in the penultimate measure). These notes are probably not arbitrary, but for the purposes of this work, it is simply the examples provided by Fux that allow to say that these notes can be used sparingly if necessary.

If the key notes and the borrowed notes are merged, then the following set of notes is get: $[C, D, E, F, G, A, B\flat, B]$. Since the modes are variations of the diatonic scale, only a few notes are added in the end (one in this case). It is more complicated to understand when exactly these borrowed notes are used. Fux explains that these notes can be used to avoid certain intervals at certain times, which otherwise the melody would harshly imply the relationship of *mi* against *fa* [18, p.35]. Again, his approach to music is probably stricter than the current one, especially when his music was intended to be religious songs. That is why this setting is user-definable.

G5 *The range of the counterpoint must be consistent with the instrument used.*

[5]Notes corresponding to the example are put in square brackets.

This rule is relatively arbitrary and should be managed by the software user. Fux's treatise is mainly concerned with sung counterpoint, although it is applicable to any instrument. Most of the time, counterpoint is composed either in a higher register or in a lower register and more rarely both simultaneously. By default, the software has been designed for a range of two octaves, but it is possible to enlarge this range according to the user's needs.

G6 *Chromatic melodies are forbidden.*

In this work, a melody is considered chromatic when three notes in a row are separated by semitones in the same direction. For example, $C \rightarrow C\sharp \rightarrow D$ or $C \rightarrow B \rightarrow B\flat$ are chromatic melodies. As a rule, this should never happen because the diatonic scale does not have those intervals. However, it might be possible to compose chromatic melodies by using borrowed notes in the use of certain modes. Fux refrains from doing so but it still deactivatable.

G7 *Melodic intervals should be small.*

The purpose of a melody is to be melodious, but how to define that? This question is several centuries old and still does not have an answer that suits everyone. In his treatise, Fux argues that one should never neglect the beauty of singing. As a result according to his examples, most melodies consist of stepwise[6] motions with occasional leaps. One solution to represent this is to give higher cost to larger melodic intervals. The appropriate cost function will be discussed in each chapter of species.

G8 *Penultimate notes of the counterpoint tend to rise.*

While at the end the *cantus firmus* tends to fall, the counterpoint tends to rise. It makes sense because the last motion must always be contrary (or oblique as will be seen in the next chapter). In Fux's examples, most of them tend to confirm this trend for the last two or even three notes of the counterpoint depending on the species. The examples given in this thesis are therefore strongly influenced by this idea which is omnipresent in the *Gradus ad Parnassum*.

2.3.2 Formalization into Constraints Language

G1 *Harmonic intervals are always calculated from the lower note.*

Already handled by making the difference value absolute as seen in section 2.2.3 for the **H** variable.

G2, G3 *Same number of measures and same time signature.*

Only 4/4 time signatures are currently considered. The array Cp is therefore composed of four lists as explained in section 2.2.3 at **Cp**.

Listing 2.1: Definition of Cp in the first species.

```
(defvar *cp (list nil nil nil nil))
; ...
;; FIRST SPECIES ;;
; setting the first list of *cp with
;   integer *cf-len as size
;   set *extended-cp-domain as available notes
(setf (first *cp)
    (gil::add-int-var-array-dom *sp* *cf-len *extended-cp-domain))
```

[6]Which moves by scale steps (i.e. one tone or one semitone)[4].

G4 *The counterpoint must be in the same key as the cantus firmus.*

This rule is already handled by the creation of the set $\mathcal{N}$ as shown in section 2.2.3. The example of the actual rule given above will clarify the explanations. Let k be the value of the key determined by the key signature, i.e. 60 for C; and t the tonic of the piece, i.e. $Cf[0] = 65$ (**TODO 23** check the value). Then:

$$\begin{aligned}\mathcal{N}_{key} &= buildScale(k \; mod \; 12, "major") = \{0, 2, 4, 5, 7, 9, 11, 12, \ldots, 127\} \\ \mathcal{N}_{brw} &= buildScale(t \; mod \; 12, "borrowed") = \{2, 4, 5, \mathbf{10}, 14, \ldots, 125\} \\ \therefore \mathcal{N}_{all} &= \{0, 2, 4, 5, 7, 9, \mathbf{10}, 11, 12, \ldots, 127\}\end{aligned}$$

To ensure that borrowed notes are used sparingly, they must be given a cost to use. Let $OffKey$ be the set of notes outside the key and $OffKey_{costs}$ the list of costs associated with each note. The cost for a note will be *<no cost>* or $cost_{OffKey}$ (*DFLT: <high cost>*).

$$\begin{gathered} OffKey = [0, 1, 2, \ldots, 127] \setminus \mathcal{N}_{key} \\ \forall \rho \in positions(m) \\ OffKey_{costs}[\rho] = \begin{cases} cost_{OffKey} & \text{if } Cp[\rho] \in OffKey \\ 0 & \text{otherwise} \end{cases} \\ \text{moreover } \mathcal{C} = \mathcal{C} \cup \sum_{c \in OffKey_{costs}} c \end{gathered} \tag{2.26}$$

This equation is trivial but requires several adjustments in the program. Indeed, there is no boolean constraint in Gecode that assign the value *true* to a variable if an element belongs to a set[7]. This can be solved by creating the following constraints (see code sample 2.2). The idea is to add a 1 each time the candidate element $\equiv$ a member of the set. If the sum of this list ≥ 1 then the candidate appears at least once in the set.

Listing 2.2: Function that constrains b-member to be true if candidate is in member-list.

```
(defun add-is-member-cst (candidate member-list b-member)
    (let (
        (results (gil::add-int-var-array *sp* (length member-list) 0 1)) ; where candidate == m
        (sum (gil::add-int-var *sp* 0 (length member-list))) ; sum(results)
    )
        (loop for m in member-list for r in results do
            (let (
                (b1 (gil::add-bool-var *sp* 0 1)) ; b1 = (candidate == m)
            )
                (gil::g-rel-reify *sp* candidate gil::IRT_EQ m b1) ; b1 = (candidate == m)
                (gil::g-ite *sp* b1 ONE ZERO r) ; r = (b1 ? 1 : 0)
            )
        )
        (gil::g-sum *sp* sum results) ; sum = sum(results)
        (gil::g-rel-reify *sp* sum gil::IRT_GR 0 b-member) ; b-member = (sum >= 1)
)   )
```

G5 *The range of the counterpoint must be consistent with the instrument used.*

This rule is already handled by the creation of the set $\mathcal{N}^{\mathcal{R}} = \mathcal{N} \cap \mathcal{R}$ as shown in section 2.2.3. When Cp is created its domain is set to $\mathcal{N}^{\mathcal{R}}_{all}$ as seen in the code sample 2.1: `*extended-cp-domain` refers to the set $\mathcal{N}^{\mathcal{R}}_{all}$.

[7]To our knowledge, Gecode provides only a constraint such that an element must be a member of a certain set. Ideally, we would need a reified version of this constraint to allow a boolean associated with the result.

G6 *Chromatic melodies are forbidden.*

A three-note melody is chromatic if the interval between the first, second and third notes is one semitone in the same direction each time. This can be translated into the two following constraints.

$$\begin{gathered}\forall \rho \in positions(m-2) \\ (M_{brut}[\rho] = 1 \wedge M_{brut}[\rho+1] = 1) \iff \perp \\ (M_{brut}[\rho] = -1 \wedge M_{brut}[\rho+1] = -1) \iff \perp\end{gathered} \tag{2.27}$$

Listing 2.3: Function that prevents chromatic melodies.

```
; add melodic interval constraints such that there is no chromatic interval:
;   - no m1 == 1 and m2 == 1 OR
;   - no m1 == -1 and m2 == -1
; @m-intervals-brut: list of all the melodic intervals
(defun add-no-chromatic-m-cst (m-intervals-brut)
    (loop
        for m1 in m-intervals-brut
        for m2 in (rest m-intervals-brut) do
        (let (
            (b1 (gil::add-bool-var *sp* 0 1)) ; s.f. (m1 == 1)
            (b2 (gil::add-bool-var *sp* 0 1)) ; s.f. (m2 == 1)
            (b3 (gil::add-bool-var *sp* 0 1)) ; s.f. (m1 == -1)
            (b4 (gil::add-bool-var *sp* 0 1)) ; s.f. (m2 == -1)
        )
            (gil::g-rel-reify *sp* m1 gil::IRT_EQ 1 b1) ; b1 = (m1 == 1)
            (gil::g-rel-reify *sp* m2 gil::IRT_EQ 1 b2) ; b2 = (m2 == 1)
            (gil::g-op *sp* b1 gil::BOT_AND b2 0) ; not(b1 and b2)
            (gil::g-rel-reify *sp* m1 gil::IRT_EQ -1 b3) ; b3 = (m1 == -1)
            (gil::g-rel-reify *sp* m2 gil::IRT_EQ -1 b4) ; b4 = (m2 == -1)
            (gil::g-op *sp* b3 gil::BOT_AND b4 0) ; not(b3 and b4)
)   )   )
```

The previous function takes care of setting this constraint using GiL. This is a classical example that shows how constraints on all notes of the counterpoint are set when there is no distinction to be made between beats. In this case, `m-intervals-brut` always represent all the melodic intervals of the counterpoint and not the melodic intervals of a single beat as will often be the case later on. Indeed, one must always adapt to the rule to make it as simple as possible.

The functions often all look the same, a `let` block declaring the local variables, which are often all the booleans required to determine a situation. Then comes the execution block where the constraints determining the booleans (`g-rel-reify`) and the restrictive constraints (`g-op` states that `b1` and `b2` must not happen) are set. In the end, putting several constraints one after the other is the same thing as having these same constraints gathered in one separated by $\vee$.

G7 *Melodic intervals should be small.*

Just a global minimization of the melodic intervals could be asked to Gecode during the search for solutions but this would not be fully consistent with the stepwise principle. Having a stepwise melody considers that an interval of a semitone is worth the same as having one of a whole tone. Giving a specific cost to each degree of interval could be done but would be too exhaustive and less meaningful. To have a modular but non-exhaustive set of constraints, it was decided to give different costs to:

- the one degree interval wit no cost;

- the two degree/octave[8] interval with *DFLT: <low cost>*;
- the other intervals with *DFLT: <medium cost>*.

$$\forall \rho \in positions(m-1)$$
$$Mdeg_{costs}[\rho] = \begin{cases} 0 & \text{if } M[\rho] \in \{0,1,2\} \\ cost_{closeMdeg} & \text{if } M[\rho] \in \{3,4,12\} \\ cost_{farMdeg} & \text{if } M[\rho] \in \{5,6,7,8,9,10,11\} \end{cases} \tag{2.28}$$
$$\text{moreover } \mathcal{C} = \mathcal{C} \cup \sum_{c \in Mdeg_{costs}} c$$

TODO 24 paste code sample

Listing 2.4: Function that gives a cost to non-stepwise melodies.

```
; TODO
; TODO
; TODO
; TODO
; TODO
; TODO
; TODO
; TODO
; TODO
; TODO
```

G8 *Penultimate notes of the counterpoint tend to rise.*

It is mostly a self-created trend due to other constraints. So there is no constraint to put on it because it is only a consequence of the next ones. Nevertheless, it is necessary to have it in mind in order to understand certain subtleties that are coming.

2.4 Types of rules

Three types of rules are distinguished in the next chapters:

- **Harmonic rules**: harmonic rules concern the harmonic intervals between the different voices, i.e. the harmony created by the *cantus firmus* and the counterpoint of the same measure. They are often the most important and the most numerous. These rules are noted by the letter **H**.
- **Melodic rules**: melodic rules refer to the melodic intervals of counterpoint or *cantus firmus*, which usually correspond to the gap between two consecutive notes of the same voice. These rules are noted by the letter **M**.
- **Motion or Harmonic and Melodic rules**: these rules use both of the above types of intervals. They are more complex and often relate to specific motions. These rules are noted by the letter **P** for *passage* since *M* is already taken[9].

The notation of the rules is: **S.TX** where **S** is the species, **T** is the type of rule (H, M or P), and **X** is the number of the rule. For example, the sixth harmonic rule of the first species is written **1.H6**.

[8]The melodic octave interval is important to be able to quickly return to a comfortable pitch.

[9]This way of classifying is only intended to clarify the idea behind a rule. It remains quite abstract and subjective because some rules are classified as melodic while they also use harmonic constraints. In no case does Fux make a delimitation between his explanations in this way.

Chapter 3

First Species of Counterpoint

> "With God's help, then let us begin composition for two voices. We take as a basis for this given melody or cantus formus, which we invent ourselves or select from a book of chorales. To each of these notes, now, should be set a suitable consonance in a voice above [...]." Mann [18, p.27]

The first species of counterpoint consists of one note by measure, note against note. In other words, only whole notes.

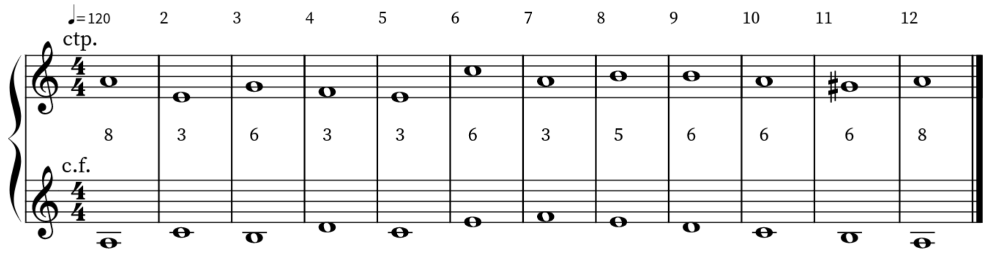


Figure 3.1: Example of a first species counterpoint. Listen here [24].

As a reminder, *unless mentioned*, harmonic and melodic intervals are considered in absolute values. Moreover, harmonic intervals are modulo 12, so an octave interval is equivalent to a unison interval (see section 2.2.3).

3.1 Formalization in English

3.1.1 Harmonic Rules of the First Species

1.H1 *All harmonic intervals must be consonances*[1]. Chevalier [1, p.53]

> "[The master addressing his pupil] I shall explain to you. It is the simplest composition of two voices [...] which, having notes of equal length, consists only of consonances." Mann [18, p.27]

1.H2 *The first harmonic interval must be a perfect consonance*[2]. [1, p.54]

Perfect consonances are not those that bring the most harmony but those that give the most sense of stability and rest. They clarify the key and provide a strong foundation for the entire musical work. This rule applies to all species.

[1]This excludes dissonances which are seconds, fourths and sevenths.
[2]Perfect consonances are fifths and octaves (or unisons).

1.H3 *The last harmonic interval must be a perfect consonance.* [1, p.54]

Same logic as the previous rule. This one also applies to all species.

1.H4 *The key tone is tuned according to the first note of the cantus firmus.* [1, p.56]

As seen in section **TODO-sec 4** , Fux sees the modes as variations of a single scale with different tonics. While the key signature gives the usable diatonic notes, the first note of the *cantus firmus* gives the tonic of the piece. This implies that some notes, the borrowed ones, will be available accidentally (e.g. $\sharp$ and $\flat$ in the key of C major) in relation to the tonic of the piece as explained in rule **G4**.

This rule also implies that the bass at the first and last note must be the tonic. To explain it another way, this means that if the counterpoint is in the lower part, only octave or unison harmonic intervals are available for the first and last note because of rules **1.H2** and **1.H3**. A wrong example would be the figure 3.2.

Figure 3.2: Ctp. not keeping the key tone set by the *cantus firmus.*

G is used as a bass note to make a fifth instead of the D note required to keep the key of the *cantus firmus* [3]. This rule applies to all species.

1.H5 *The counterpoint and the cantus firmus cannot play the same note at the same time except in the first and last measure.* [1, p.62]

It does not mean that the harmonic interval cannot be equal to zero because an octave can occur. But unison in the strict sense of the term cannot be used in this case. This rule applies to all species for all thesis[4] notes.

1.H6 *Imperfect consonances[5] are preferred to perfect consonances.* [1, p.54]

Preferred means that all consonances are allowed but some cost, or "punishment", will be associated with the use of perfect consonances. This rule applies to all species for all thesis notes.

1.H7 *If the cantus firmus is in the lower part, then the harmonic interval of the penultimate note must be a major sixth.* [1, p.54]

This rule seems a bit strange at first, but there is a rational explanation for this. Indeed, traditional *cantus firmus* almost always end with a descending melody of one degree, for example $E \to D$ or $F \to E$ (figure 3.3).

From this example, the rule makes sense because the major sixths of E and F are $C\sharp$[6] and D respectively. These notes are only one degree away from the tonic and lead perfectly by contrary motion to the tonics E and F. However, this implies several things.

[3] As it is, the work would be in G mixolydian instead of D dorian.
[4] Thesis means the note on the downbeat.
[5] Imperfect consonances are thirds and sixths.
[6] $C\sharp$ is a leading-tone to D. Leading-tone is a note that resolves to the next one semitone higher (or lower) note. It begins to be used from the late Middle Ages [2].

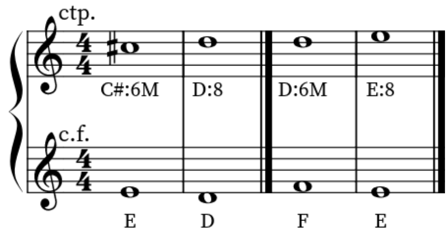


Figure 3.3: *Cantus firmus* ending with descending melodic intervals.

First, if big leaps are to be avoided in general, the last consonance will necessarily be an octave or unison because, as explained above, the closest note is necessarily the tonic.

Secondly, if a composer wants to use the tool to compose from a *cantus firmus* which does not have the particularity of ending on a melody descending by one degree, then the solutions will not be very coherent on the penultimate measure. This point will be explained in more detail in section **TODO-sec 5** . This rule applies to all species.

1.H8 *If the cantus firmus is in the upper part, then the harmonic interval of the penultimate note must be a minor third.* [1, p.54]

This rule goes hand in hand with the previous one. Indeed, a minor third is an inverted major sixth[7]. With the previous example, the notes of the counterpoint used would be exactly the same but this time would be below the *cantus firmus* (see figure 3.4). This rule applies to all species.

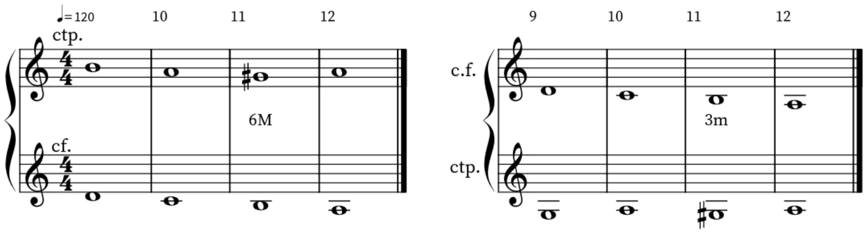


Figure 3.4: Equivalence between 6^{th}M and 3^{rd}m in penultimate measures, 1^{st} species.

3.1.2 Melodic Rules of the First Species

1.M1 *Tritone*[8] *melodic intervals are forbidden.* [1, p.59]

The tritone, sometimes called the devil's interval by some [18, p.35], is a three-tone interval just below the perfect fifth [12]. It brings a lot of dissonance that is best avoided, even melodically. This is an common rule of classical music in the broad sense but it is more used in today's music, so it can be deactivated. This rule applies to all species.

1.M2 *Melodic intervals cannot exceed a minor sixth interval.* [1, p.61]

> "[The master addressing his pupil] You shouldn't be so impatient, though I am most glad about your care not to depart from the rules. But how should you avoid those small errors for which you have yet had no rules? [...] you

[7] If the octave interval is defined by 12 semitones, then the minor third is 3 and the major sixth is 9. The same note is found because $(Cf - 3) \ mod \ 12 = (Cf + 9) \ mod \ 12$. In other words, any note is the minor third of its major sixth.

[8] If you want to hear what is a tritone, you can check the video *What is a Tritone? Tritone Explained in 2 Minutes (Music Theory)* at `https://youtu.be/JJIO-Jr0E8o` [28].

used a skip of a major sixth, which is prohibited in strict counterpoint where everything should be as singable as possible." Mann [18, p.37]

As Fux explains, this rule applies especially for singers. As explained in the rule **G7**, it is not very melodious to make big leaps in the melody anyway. This rule applies to all species with some exceptions.

3.1.3 Motion Rules of the First Species

1.P1 *Perfect consonances cannot be reached by direct motion.* [1, p.51, 57]

This rule is a good example of Fux overloading the explanations for perhaps a better understanding of the yesteryear audience.

> "First rule: From one perfect consonance to another perfect consonance one must proceed in contrary or oblique motion.
> Second rule: From a perfect consonance to an imperfect consonance one may proceed in any of the three motions.
> Third rule: From an imperfect consonance to a perfect consonance one must proceed in contrary or oblique motion.
> Fourth rule: From one imperfect consonance to another imperfect consonance one may proceed in any of the three motions." Mann [18, p.22]

As Martini [19, p.23] explains, these rules can be reduced to one such that the direct motion into perfect consonances is the only forbidden progression. Figure 3.5 violates the rule. This rule applies to all species.

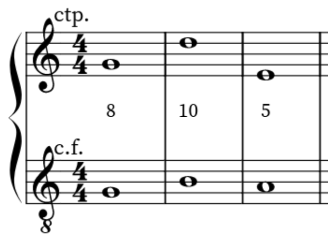


Figure 3.5: Perfect consonance reached by direct motion.

1.P2 *Contrary motions are preferred to oblique motions which are preferred to direct motions.* [1, p.53]

According to Fux, this would avoid making mistakes. Since the purpose of counterpoint is to have different melodies, it is understandable that contrary motion is preferable as the melodies will naturally differ. He is nevertheless criticized for the use of oblique motions which are, by some authors, forbidden.

Sachs and Dahlhaus [21] say that "The repetition of a note, causing oblique motion, is sometimes permitted only in the cantus, but may be used in either part (or even in both simultaneously, as a repeated note); it is not however the recommended 'next step'." Fabre [13][9] explains that in the treatises from Marcel Bitsch, Marcel Dupré or treatises of the 19th century, they proscribes the repetition of a note.

Since the preference of the motion is different according to the musical context, this parameter is manageable by the user. This rule applies to all species.

[9] Jean-Louis Fabre has a long experience of teaching and practicing music. He has taught piano, music writing and analysis at the conservatory and more [20].

1.P3 *In the start of any measure, an octave cannot be reached by the lower voice going up and the upper voice going down more than a third skip.* [1, p.61-62]

This rule may seem arbitrary because it is. The original rule forbids this *battuta* octave[10] no matter how far the upper voice travels. Fux explains that "it is of little importance"[18, p.39] because he has found no particular reason for this rule, which is respected by authoritative composers. However, he thinks that the octave reached by a leap in the same context should be avoided.

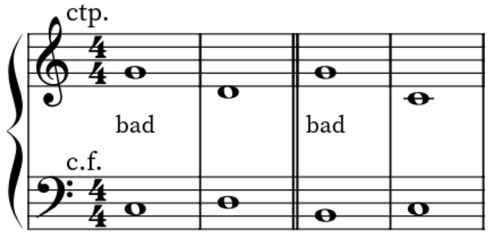


Figure 3.6: Example of battuta octaves.

On the right of the figure 3.6, the octave is reached by a skip which is not good. While the example on the left is admitted by Fux. This rule applies to all species with some exceptions.

3.2 Formalization into Constraints Language

3.2.1 Harmonic Constraints of the First Species

1.H1 *All harmonic intervals must be consonances.*

$$\forall j \in [0, m) \quad H[0, j] \in Cons \tag{3.1}$$

This can be expressed with the constraint `(gil::g-member *sp* ALL_CONS_VAR h-intervals)` (see original code for more details).

1.H2, 1.H3 *The first and last harmonic intervals must be a perfect consonances.*

$$\begin{aligned} H[0, 0] &\in Cons_p \\ H[0, m-1] &\in Cons_p \end{aligned} \tag{3.2}$$

1.H4 *The key tone is tuned according to the first note of the cantus firmus.*

Rule **G4** already handles the set of available additional notes. The only rule to add is that the first and last bass notes of the piece must have the same letter as the first note of the *cantus firmus* (i.e. unison or octaves).

$$\begin{aligned} \neg IsCfB[0, 0] &\implies H[0, 0] = 0 \\ \neg IsCfB[0, m-1] &\implies H[0, m-1] = 0 \end{aligned} \tag{3.3}$$

This is a good example of how implication works. `RM_IMP` on code sample 3.1 means that the boolean to its left implies the relation again to its left.

Listing 3.1: Function that constrains the first and last harmonies to be unisons or octaves.

[10] Literally translated from Italian to "beaten". It refers to the downbeat.

```
1  ; @h-interval: the harmonic interval array
2  ; @is-cf-bass-arr: boolean variables indicating if cf is at the bass
3  (defun add-tonic-tuned-cst (h-interval is-cf-bass-arr)
4      (let (
5          (bf-not (gil::add-bool-var *sp* 0 1)) ; s.f. !(first is-cf-bass-arr)
6          (bl-not (gil::add-bool-var *sp* 0 1)) ; s.f. !(lastone is-cf-bass-arr)
7      )
8          ; bf-not = !(first is-cf-bass-arr)
9          (gil::g-op *sp* (first is-cf-bass-arr) gil::BOT_EQV FALSE bf-not)
10         ; bl-not = !(lastone is-cf-bass-arr)
11         (gil::g-op *sp* (lastone is-cf-bass-arr) gil::BOT_EQV FALSE bl-not)
12         ; bf-not => h-interval[0, 0] = 0
13         (gil::g-rel-reify *sp* (first h-interval) gil::IRT_EQ 0 bf-not gil::RM_IMP)
14         ; bl-not => h-interval[-1, -1] = 0
15         (gil::g-rel-reify *sp* (lastone h-interval) gil::IRT_EQ 0 bl-not gil::RM_IMP)
16 )   )
```

Since the negation of IsCfBass is required and Gecode does not offer a $\neg$ operation, it must be written in the form: $!p \equiv (p \iff \bot)$ where p is any predicate (see lines 9 and 11).

1.H5 *The counterpoint and the cantus firmus cannot play the same note at the same time except in the first and last measure.*

$$\forall j \in [1, m-1) \quad Cp[0, j] \neq Cf[j] \tag{3.4}$$

1.H6 *Imperfect consonances are preferred to perfect consonances.*

Only the cost for perfect consonance is definable (*DFLT: <low cost>*) which leaves a null cost for the imperfect consonances.

$$\begin{gathered} \forall j \in [0, m) \\ Pcons_{costs}[j] = \begin{cases} cost_{Pcons} & \text{if } H[0, j] \in Cons_p \\ 0 & \text{otherwise} \end{cases} \\ \text{moreover } \mathcal{C} = \mathcal{C} \cup \sum_{c \in Pcons_{costs}} c \end{gathered} \tag{3.5}$$

1.H7, 1.H8 *The harmonic interval of the penultimate note must be a major sixth or a minor third depending on the cantus firmus pitch.*

These two rules can be expressed with a single *if-then-else* constraint like this: `(gil::g-ite *sp* (penult *is-cf-bass-arr) NINE THREE (penult *h-intervals))`.

$$\begin{gathered} \rho := \max(positions(m)) - 1 \\ H[\rho] = \begin{cases} 9 & \text{if } IsCfB[\rho] \\ 3 & \text{otherwise} \end{cases} \\ \text{where } \rho \text{ represents the penultimate index of any counterpoint.} \end{gathered} \tag{3.6}$$

3.2.2 Melodic Constraints of the First Species

1.M1 *Tritone melodic intervals are forbidden.*

Instead of prohibiting this type of melodic interval, a cost is assigned (*DFLT: <forbidden>*) because it is a popular dissonant interval in today's music. Any major chord with a minor seventh has a tritone and this chord is the very basis of the blues [7]. In addition, some less conventional *cantus firmus* than those of Fux might require a tritone on the last motion because of the number of constraints on the penultimate measure.

$$\forall \rho \in positions(m-1)$$
$$Mtritone_{costs}[\rho] = \begin{cases} cost_{Mtritone} & \text{if } M[\rho] = 6 \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$
$$\text{moreover } \mathcal{C} = \mathcal{C} \cup \sum_{c \in Mtritone_{costs}} c$$

1.M2 *Melodic intervals cannot exceed a minor sixth interval.*

$$\forall j \in [0, m-1) \quad M[0,j] \leq 8 \tag{3.8}$$

For simple rules that apply to the whole list, it is possible to add a single line constraint like this: `(gil::g-rel *sp* m-intervals gil::IRT_LQ 8)`.

3.2.3 Motion Constraints of the First Species

1.P1 *Perfect consonances cannot be reached by direct motion.*

$$\forall j \in [0, m-1) \quad H[0, j+1] \in Cons_p \implies P[0,j] \neq 2 \tag{3.9}$$

This can be read as *if a harmony belongs to the perfect consonances then the motion to reach it is not direct* ($2 \equiv direct$, see **P** in section 2.2.3).

1.P2 *Contrary motions are preferred to oblique motions which are preferred to direct motions.*
By default, motions have the same cost as their identifiers:

- $cost_{contrary}$ *DFLT: <0>*
- $cost_{oblique}$ *DFLT: <1>*
- $cost_{direct}$ *DFLT: <2>*

$$\forall j \in [0, m-1)$$
$$P_{costs}[j] = \begin{cases} cost_{contrary} & \text{if } P[0,j] = 0 \\ cost_{oblique} & \text{if } P[0,j] = 1 \\ cost_{direct} & \text{if } P[0,j] = 2 \end{cases} \tag{3.10}$$
$$\text{moreover } \mathcal{C} = \mathcal{C} \cup \sum_{c \in P_{costs}} c$$

1.P3 *In the start of any measure, an octave cannot be reached by the lower voice going up and the upper voice going down more than a third skip.*

This rule can be represented by two sets of constraints. The first line of equation 3.11 represents the case where the counterpoint is on top while the second represents the case where the *cantus firmus* is on top.

$$i := \max(\mathcal{B}), \forall j \in [0, m-1)$$
$$H[0, j+1] = 0 \land P[i,j] = 0 \land \begin{cases} M_{brut}[i,j] < -4 \land IsCfB[i,j] \iff \bot \\ M_{cf}[i,j] < -4 \land \neg IsCfB[i,j] \iff \bot \end{cases} \tag{3.11}$$

where i stands for the last beat index in a measure.

Chapter 4

Second Species of Counterpoint

The second species of counterpoint consists of two notes by measure, two notes against one note. In other words, only half notes.

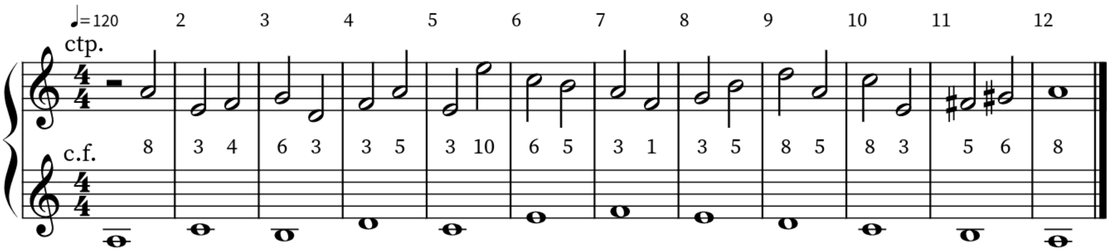


Figure 4.1: Example of a second species counterpoint. Listen here [26].

Since the second species is distinguished by a strong beat followed by a weak beat, the first species must be seen as a counterpoint composed of strong beats only. Therefore, all the rules of the first species that only apply per measure apply in thesis (e.g. rule **2.H1**). However, rule **1.M2** applies generally with the exception **2.M1**. Although the rules concerning the motions still hold, motions themselves are determined in a different way (see rule **2.P1**).

To sum up, first species harmonic rules are applied in thesis, while first species melodic rules are applied for all notes, and first species motions rules are adapted to the species.

4.1 Formalization in English

4.1.1 Harmonic Rules of the Second Species

2.H1 *Thesis*[1] *notes cannot be dissonant.* Chevalier [1, p.64]

As explained above, this rule is consistent with the **1.H1** one. Actually, it is written only to illustrate the associated logic because in terms of constraints, the same are applied.

2.H2 *Arsis*[2] *harmonies cannot be dissonant except if there is a diminution*[3]*.* [1, p.64]

This might sound like a very restrictive rule but in reality it is a common rule that applies itself in tonal music. In fact, any dissonance is surrounded by a consonance on each side.

[1]Thesis means the note on the down beat.
[2]Arsis means the note on the upbeat.
[3]Diminution means an intermediate note that exists between two notes separated by a skip of a third.

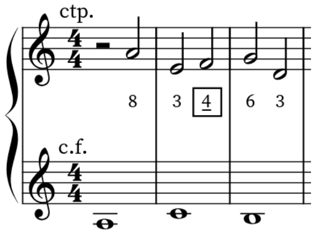


Figure 4.2: Diminution in arsis, 2^{nd} species.

Since rule **G7** insinuates that the melodic intervals are small, it makes perfect sense to go from one thesis consonance to the next thesis consonance through an arsis dissonance.

2.H3 *In addition to rules **1.H7** and **1.H8**, in the penultimate measure the harmonic interval of perfect fifth* (*unless exception **2.H4***) *must be used for the thesis note.* [1, p.64-65]

The rules of the penultimate measure, although too strict for today's music (see rule **1.H7**), are still consistent with the other rules of the species. Since the penultimate note is a major sixth or a major third, the closest consonance in thesis is a fifth[4] (see figure 4.3).

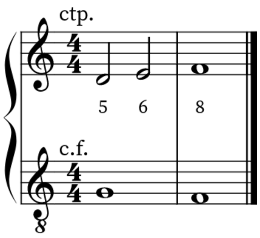


Figure 4.3: Basic penultimate measure, 2^{nd} species.

2.H4 *In the penultimate measure, if the harmonic interval of fifth in thesis is not avaible, then a sixth interval must be used.* [1, p.69]

When Fux makes exceptions, it can get tricky so it is highly recommended to understand rules **G4**, **G8** and **1.H4** and read section **TODO-sec 6** on modes.

Every musician knows that the seventh of the diatonic major scale does not have a perfect fifth in its key. That's why this rule exists. In the figure 4.4a, the mode of E (i.e. the phrygian mode) is used and the *cantus firmus* plays an F above. To have a perfect fifth, a $B\flat$ would have to be played, which is not available and is therefore replaced by an A to form a sixth.

Where it gets tricky is when Fux shows this example (figure 4.4b) using the A mode (i.e. the aeolian mode, the relative minor). Why does Fux allow himself to use $F\sharp$ which gives a perfect fifth to B? As always the key used is C major (no $\sharp$ or $\flat$), but since the tonic is A the scale used will be extended to notes of the A major scale (i.e. $F\sharp$ and $G\sharp$)[5].

One might ask: why not a sixth as in the first example (figure 4.4a)? There are two reasons for this choice. First, the implicit rule **G6** that says chromaticism is forbidden prevents a minor sixth because the melody would then be: $G \rightarrow G\sharp \rightarrow A$. Secondly, it could suggest that Fux prefers to go outside the diatonic scale to get a perfect fifth if

[4] With respect to the trend **G7** that says that the melody is stepwise.

[5] For more experienced musicians, this penultimate measure is immediately reminiscent of the melodic minor scale [10], which is common in classical music.

(a) 6th in thesis. (b) ♯5th in thesis.

Figure 4.4: Different penultimate measures, 2nd species.

the mode allows it rather than breaking the ground rule **2.H3**. More details regarding the costs will be given in the next mathematical section.

4.1.2 Melodic Rules of the Second Species

2.M1 *If the two voices are getting so close that there is no contrary motion possible without crossing each other, then the melodic interval of the counterpoint can be an octave leap*[6]. [1, p.67-68]

> "[...] if the parts have been led so close together that one does not know where to take them; and if there is no possibility of using contrary motion, this motion can be brought about by using the skip of [...] an octave [...]." Mann [18, p.45]

More explicitly, this case occurs when:

- the brut harmonic gap is a third or less;
- the *cantus firmus* is both below (/above) and rising (/falling).

Why a third? Because there is no more closed consonance than the latter.

According to Fux's examples, this rule applies only to $thesis \rightarrow arsis$ melodic intervals. Octave leaps seem to be unconditional in the case of $arsis \rightarrow thesis$ intervals. Moreover, it goes hand in hand with rule **G7** which says that melodic intervals should be small. Indeed, the octave skip allows to reset the pitch of the melody to go down (or up) again stepwisely (see figure 4.5).

Figure 4.5: Octave leap, 2nd species.

Implicit melodic rule

2.M2 *Two consecutive notes cannot be the same.*
In Fux's examples, none of them have oblique motions. This makes sense with the criticisms made for rule **1.P2**. This rule applies to the third species.

[6]The octave leap is quite natural and easy to sing because it is the first harmonic of the sound [6].

4.1.3 Motion Rules of the Second Species

2.P1 *If the melodic interval of the counterpoint between the thesis and the arsis is larger than an third, then the motion is perceived on the basis of the arsis note.* [1, p.65-67]

Fux explains that the melodic interval between the note in thesis and the note in arsis determines which note will be kept in our mind. A third skip does not deviate enough from the thesis note to forget the latter. This implies that a perfect consonance to a perfect consonance cannot be saved by a third skip (see figure 4.6a) because the motion will be considered as direct, which is not in accordance with rule **1.P1**. However, this rule allows the following situation in figure 4.6b.

(a) Bad direct motion with a 3rd skip.

(b) Good contrary motion with a 4th leap.

Figure 4.6: Different motions based on different leaps, 2nd species.

Implicit adapted motion rule

2.P2 *Rule **1.P3** on the battuta octave is adapted such that it focuses on the motion from the note in arsis.*

Fux does not mention it in the second species. Instead of not applying the rule, it is adapted to prevent the same situation but considering only the note in arsis. Given the limited information and interest in this rule, it is entirely user-deactivatable.

4.2 Formalization into Constraints

4.2.1 Harmonic Constraints of the Second Species

2.H1 *Thesis harmonies cannot be dissonant.*

As explained above, there is no constraint to add because it would be a duplicate of rule **1.H1**.

2.H2 *Arsis harmonies cannot be dissonant except if there is a diminution.*

Let $IsDim$ be a list of booleans of size $m-1$ representing if an arsis note is a diminution. A diminution can be described as follows: the interval between the notes in thesis is a third and the two intervals that compose it are seconds (one or two semi-tones).

$$\forall j \in [0, m-1)$$
$$IsDim[j] = \begin{cases} \top & \text{if } M^2[0,j] \in \{3,4\} \wedge M^1[0,j] \in \{1,2\} \wedge M^1[2,j] \in \{1,2\} \\ \bot & \text{otherwise} \end{cases} \tag{4.1}$$

There is no need to use the brut melodic intervals to check if the melody always goes in the same direction[7]. This is because the constraint of third ensures the conditions to be met: $M^2[0,j] = \left|M^1_{brut}[0,j] + M^1_{brut}[2,j]\right|$. Besides, the constraint $<= 2$ can be used to represent $\in \{1,2\}$ because the melodic intervals are never zero as will be seen later.

[7] The note would be a mere ornament like a suspended or added note instead of a diminution.

Listing 4.1: Function that constrains $IsDim$ to reprensent diminutions.

```
; @m-intervals-ta: the melodic interval between each thesis and its following arsis
; @m-intervals: the melodic interval between each thesis and its following thesis
; @m-intervals-arsis: the melodic interval between each arsis and its following thesis
; @is-dim-arr: the array of BoolVar to fill
(defun create-is-dim-arr (m-intervals-ta m-intervals m-intervals-arsis is-dim-arr)
    (loop
    for mta in m-intervals-ta ; inter(thesis, arsis)
    for mtt in m-intervals ; inter(thesis, thesis + 1)
    for mat in m-intervals-arsis ; inter(arsis, thesis + 1)
    for b in is-dim-arr ; the BoolVar to constrain
    do (let (
        (btt3 (gil::add-bool-var *sp* 0 1)) ; s.f. mtt == 3
        (btt4 (gil::add-bool-var *sp* 0 1)) ; s.f. mtt == 4
        (bta-2nd (gil::add-bool-var *sp* 0 1)) ; s.f. mat <= 2
        (btt-3rd (gil::add-bool-var *sp* 0 1)) ; s.f. mtt == 3 or 4
        (bat-2nd (gil::add-bool-var *sp* 0 1)) ; s.f. mta <= 2
        (b-and (gil::add-bool-var *sp* 0 1)) ; temporary BoolVar
    )
        (gil::g-rel-reify *sp* mtt gil::IRT_EQ 3 btt3) ; btt3 = (mtt == 3)
        (gil::g-rel-reify *sp* mtt gil::IRT_EQ 4 btt4) ; btt4 = (mtt == 4)
        (gil::g-rel-reify *sp* mta gil::IRT_LQ 2 bta-2nd) ; bta-2nd = (mta <= 2)
        (gil::g-rel-reify *sp* mat gil::IRT_LQ 2 bat-2nd) ; bat-2nd = (mat <= 2)
        (gil::g-op *sp* btt3 gil::BOT_OR btt4 btt-3rd) ; btt-3rd = btt3 || btt4
        (gil::g-op *sp* bta-2nd gil::BOT_AND btt-3rd b-and) ; temporay operation
        (gil::g-op *sp* b-and gil::BOT_AND bat-2nd b) ; b = bta-2nd && btt-3rd && bat-2nd
)   ))
```

To represent an action that produces only in one situation, this action must imply that situation. So it can be established that a dissonance in arsis implies a diminution like this:

$$\forall j \in [0, m-1) \quad \neg IsCons[2, j] \implies IsDim[j] \tag{4.2}$$

2.H3, 2.H4 *In the penultimate measure the harmonic interval of perfect fifth must be used for the thesis note if possible. Otherwise, a sixth interval should be used instead.*

If one wants to follow Fux's rules, it is important that the cost of leaving the diatonic scale is less than the cost of not having a fifth. For this, $cost_{penulthesis}$ is set to *<last resort>* which is greater than $cost_{OffKey}$ (*<high cost>*).

$$\begin{gathered} H[0, m-2] \in \{7, 8, 9\} \\ \therefore penulthesis_{cost} = \begin{cases} cost_{penulthesis} & \text{if } H[0, m-2] \neq 7 \\ 0 & \text{otherwise} \end{cases} \\ \text{moreover } \mathcal{C} = \mathcal{C} \cup penulthesis_{cost} \end{gathered} \tag{4.3}$$

4.2.2 Melodic Constraints of the Second Species

2.M1 *If the two voices are getting so close that there is no contrary motion possible without crossing each other, then the melodic interval of the counterpoint can be an octave leap.*

$$\begin{gathered} \forall j \in [0, m-1), \forall M_{cf}[j] \neq 0 \\ M[0, j] = 12 \implies (H_{abs}[0, j] \leq 4) \land (IsCfB[j] \iff M_{cf}[j] > 0) \end{gathered} \tag{4.4}$$

Where $H_{abs}[0, j] \leq 4$ states that there is no smaller consonance and $IsCfB[j] \equiv M_{cf}[j] > 0$ that the *cantus firmus* is getting closer to the counterpoint. As a reminder, M_{cf} is not absolute so $M_{cf} > 0$ states that the *cantus firmus* is necessarily rising.

2.M2 *Two consecutive notes cannot be the same.*

$$\forall \rho \in positions(m) \quad Cp[\rho] \neq Cp[\rho + 1] \tag{4.5}$$

4.2.3 Motion Constraints of the Second Species

2.P1 *If the melodic interval of the counterpoint between the thesis and the arsis is larger than a third, then the motion is perceived on the basis of the arsis note.*

Let P_{real} be a list of size $m - 1$, with the same domain as a list of P, representing which motion is perceived between that coming from the thesis note and that coming from the arsis note. This implies that the costs of the motions and the first species constraints on the motions are deducted from P_{real}.

$$\forall j \in [0, m-1) \quad P_{real}[j] = \begin{cases} P[2,j] & \text{if } M[0,j] > 4 \\ P[0,j] & \text{otherwise} \end{cases} \tag{4.6}$$

Listing 4.2: Function that constrains P_{real} to represent the real motions.

```
; @m-intervals-ta: melodic intervals between the thesis and the arsis note
; @motions: motions perceived from the thesis note
; @motions-arsis: motions perceived from the arsis note
; @real-motions: motions perceived by the human ear
(defun create-real-motions (m-intervals-ta motions motions-arsis real-motions)
    (loop
    for tai in m-intervals-ta
    for t-move in motions
    for a-move in motions-arsis
    for r-move in real-motions
    do (let (
        (b (gil::add-bool-var *sp* 0 1)) ; s.f. (tai > 4)
    )
        (gil::g-rel-reify *sp* tai gil::IRT_GR 4 b) ; b = (tai > 4)
        (gil::g-ite *sp* b a-move t-move r-move) ; r-move = (b ? a-move : t-move)
)   ))
```

2.P2 *Rule **1.P3** on the battuta octave is adapted such that it focuses on the motion from the note in arsis.*

This constraint already had an adapted mathematical notation in the chapter of the first species. Note that this constraint would indeed use $P[2]$ and not P_{real}.

Chapter 5

Third Species of Counterpoint

The third species of counterpoint consists of four notes by measure, four notes against one note. In other words, only quarter notes.

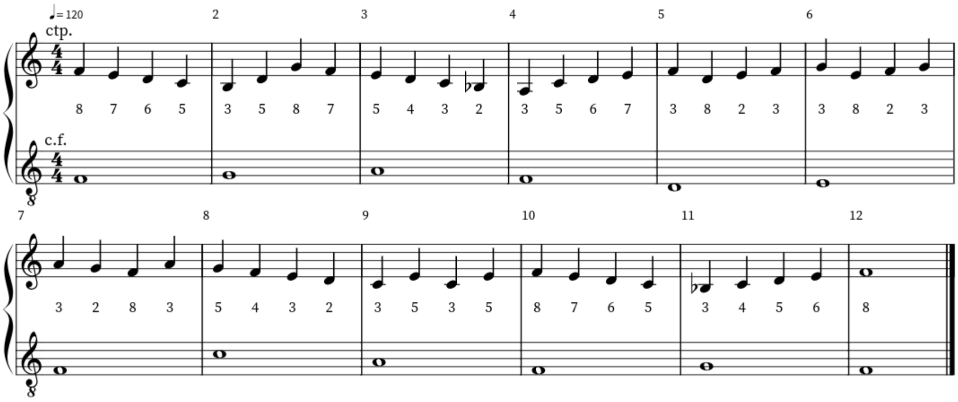


Figure 5.1: Example of a third species counterpoint. Listen here [27].

As in the previous chapter, the rules of the first species are applied to the thesis note, i.e. the first note of the group of four quarter notes. The first note of a measure is always the most important[1], it is the one that establishes the main harmony perceived by the human ear. To sum up, first species harmonic rules are applied in thesis, while first species melodic rules are applied for all notes, and first species motions rules are adapted to the species.

The third species is the one that starts to be vague in the explanations given by Fux. Admittedly, he probably didn't expect his work to be formalized through constraint programming. But even for musicians, there's no denying that some rules lack illustrative examples and are a bit skimmed over. In addition, the original treatise is in Latin and, despite access to several translations in French and English, the explanations do not always mean exactly the same thing, and everyone knows that the devil's in the details. This is reflected, for example, in the formalization of the first two harmonic rules, which are both created from fuzzy explanations and different translations.

[1]Unless there is syncopation as it will be explained in the next chapter.

5.1 Formalization in English

5.1.1 Harmonic rules of the third species

3.H1 *If five notes follow each other by joint degrees in the same direction, then the harmonic interval of the third note must be consonant.* Chevalier [1, p.73]

The following analysis is more the work of a historian than a computer scientist (**TODO 25** move to a dedicated part ?). The resulting formalization is therefore not the only way to go. As explained above, not all translations are equivalent. Chevalier's French translation, which is the most recent and used as the main source in this thesis, says (see the original text in the appendix at A.3):

> If it happens that five quarter notes follow each other **by joint degrees**, either ascending or descending, the first one must be consonant, the second one may be dissonant, the third one again necessarily consonant, the fourth one may be dissonant **if** the fifth one is a consonance.

In contrast, Mann's English translation says:

> "[...] if fives quarters follow each other either ascending or descending, the first one [...]. The fourth one may be dissonant **if** the fifth is consonant [...]." Mann [18, p.50]

Alternatively, other older references as [8, p.51] and [22, p.4] from the XVIII century basically say:

> When five quarters follow one another **gradually** either rising or falling, the first, third **and** fifth note **must** be consonant. While the second and fourth may be dissonant.

Several issues arise from these previous sentences. First, Mann's English version does not say "gradually" or "by joint degree" which changes the rule itself. These terms make the constraint much more precise and therefore less restrictive. It can be said without too much hesitation that the rule must be applied only in the case of joint degrees because most translations propose a "gradually"[2]. Moreover, Fux's examples confirm this hypothesis.

Second problem: "**if** the fifth note is consonant". Why "if"? Actually it's more complicated than that. For this rule, Fux does not explain if he is talking about:

(a) the four quarter notes of a measure plus the first one of the next measure;

(b) any five-note tuple;

(c) any independent five-note tuple that don't overlap with the previous one.

In Fux's examples, more than five notes follow each other several times, up to nine notes in a row in some. If the second assumption were true, then the following figure 5.2 from the book would not be correct.

The third hypothesis (c) that states that Fux talks of *any five-note tuple as long as it is not itself in a previous five-note tuple* does not work either. Otherwise this figure 5.3 would not be right.

It is clear that the third note is dissonant whereas with assumption (a), the rule would be maintained. As a result, it was decided that the first hypothesis was the right

[2] In the original Latin text, Fux [15, p.63] states "continuò gradatim", which can be translated by "step by step".

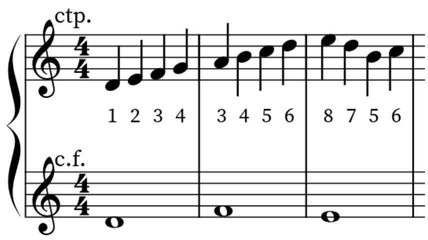


Figure 5.2: Nine quarters that follow each other gradually, 3rd species.

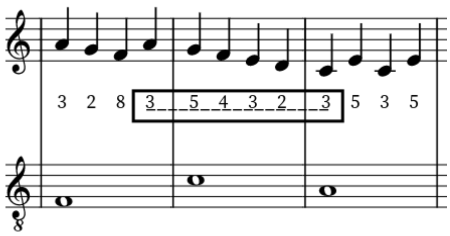


Figure 5.3: Six quarters that follow each other gradually where the 3rd one is dissonant, 3rd species.

one. But it does not explain why it is said "if the fifth note is consonant". With this hypothesis, the fifth note is a thesis note and is therefore necessarily consonant thanks to rule **2.H1**. In the end, since saying that a note "may be dissonant" actually means that no constraint is added, the only additional constraint is the one on the third note.

3.H2 *If the third harmonic interval of a measure is dissonant then the second and the fourth interval must be consonant and the third note must be a diminution*[3]. [1, p.73-74]

Stepping back, this rule can be *partly* written in another more meaningful way: *any dissonance implies that it is surrounded by consonances*. Which makes sense in music because in a melody, dissonances are often used to link the consonant notes of an explicit or implicit chord. The logical proof is given in the mathematical section 5.2 that follows.

3.H3 *It is best to avoid the second and third harmonies of a measure to be consonant with a one degree melodic interval between them.* [1, p.74-75]

Fux calls this rule the *cambiata* note[4]. This rule is followed by composers of authority who stimulate the use of dissonances. As shown in figure 5.4, the seventh interval of the second note should be played rather than the sixth.

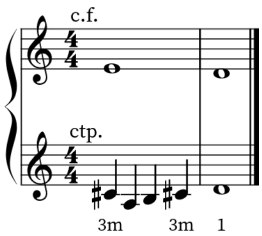
Figure 5.4: Use of the *cambiata* note in the second quarter.

3.H4 *In addition to rule **1.H8**, in the penultimate measure, if the cantus firmus is in the upper part, then the harmonic interval of the first note should be a minor third.* [1, p.75]

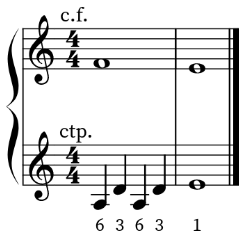
[3] An intermediate note that fills a skip of third.

[4] Literally translated from Italian to the "*exchanged* note". [18, p.51]

Fux, for some reason, does not always follow this rule, which he gives in a very crude way with a single example (figure 5.5a) to follow without further explanation. The only particularity of this measure is in the first and last note which are minor thirds, which is consistent.



(a) Standard penultimate measure.



(b) Fux's deviation.

Figure 5.5: Different penultimate measures, 3rd species.

However, Fux gives this example (figure 5.5b) which is not detailed. Luckily, Mann has footnoted that:

> "The forming of sequences (the so-called monotonia) ought to be avoided as far as possible. In the original [a] correction for the next to the last measure was added in manuscript". Mann [18, p.54]

This correction is yet another way of writing the penultimate measure. There is nothing wrong with Fux allowing deviations, that is what music is about in a way. But it makes systematic formalization more difficult. It was chosen to ignore this example and leave this rule optional because of its inconsistency with the rest.

5.1.2 Melodic rules of the third species

The melodic rule **2.M2** of the second species is applied to all notes.

Implicit melodic rule

3.M1 *Each note and its two beats further peer are preferred to be different.*

This implicit rule is already generally present. It is kind of complementary to rule **2.M2** but in a softer way. It happens several times in Fux's work that the pupil prefers to put himself in difficulty to avoid monotony in the melody. An important aspect of this monotony can be found in the repetition of notes. In this species, it becomes important because not taking that into account could lead to having only two different notes per measure (see figure 5.6), which could be considered "boring". The cost of this parameter is still adjustable by the user.

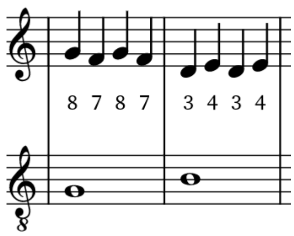


Figure 5.6: "Boring" example with only two different notes per measure, 3rd species.

5.1.3 Motion rules of the third species

Implicit motion rule

3.P1 *The motion is perceived on the basis of the fourth note.*

Fux stops talking about motions explicitly from the chapter on the third species. But the legacy of the first species, the idea of reaching perfect consonances by contrary motion, remains present in all his examples.

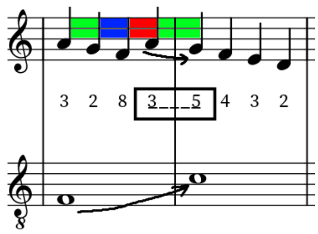


Figure 5.7: Contrary motion based on the fourth note.

The motion is here (figure 5.7) perceived from the note of the *cantus firmus* with the fourth note of the counterpoint of the corresponding measure towards the note of the next *cantus firmus* with the first note of the counterpoint of the corresponding measure. In fact, the third species allows more flexibility in the motions because with more notes it is possible to go up during the first three notes to come down (or vice versa) just before the start of the next measure to obtain the desired motion as seen in figure 5.7.

5.2 Formalization into Constraints

5.2.1 Harmonic Constraints of the Third Species

3.H1 *If five notes follow each other by joint degrees in the same direction, then the harmonic interval of the third note must be consonant.*

$$\begin{gathered}\forall j \in [0, m-1) \\ \left(\bigwedge_{i=0}^{3} M[i,j] \leq 2\right) \wedge \left(\bigwedge_{i=0}^{3} M_{brut}[i,j] > 0 \vee \bigwedge_{i=0}^{3} M_{brut}[i,j] < 0\right) \\ \implies IsCons[2,j]\end{gathered} \tag{5.1}$$

On the one hand, the M is used for the "joint degrees" property while the M_{brut} for the "same direction" one.

3.H2 *If the third harmonic interval of a measure is dissonant then the second and the fourth interval must be consonant and the third note must be a diminution.*

To avoid negation in the code, which would require an additional step, the implication has been transformed into a logical or. The following constraints are set to be true.

$$\begin{gathered}\forall j \in [0, m-1) \\ IsCons[2,j] \vee (IsCons[1,j] \wedge IsCons[3,j] \wedge IsDim[j])\end{gathered} \tag{5.2}$$

where $IsDim[j] = \top$ when the 3rd note of the measure j is a diminution.

3.H3 *It is best to avoid the second and third harmonies of a measure to be consonant with a one degree melodic interval between them.*

The default value of $cost_{Cambiata}$ is <*last resort*> because Fux almost seems to forbid it but without a real musical reason to justify this convention.

$$\forall j \in [0, m-1)$$
$$Cambiata_{costs}[j] = \begin{cases} cost_{Cambiata} & \text{if } IsCons[1,j] \wedge IsCons[2,j] \wedge M[1,j] \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

3.H4 *In the penultimate measure, if the cantus firmus is in the upper part, then the harmonic interval of the first note should be a minor third.*

$$\neg IsCfB[m-2] \implies H[0, m-2] = 3 \tag{5.4}$$

5.2.2 Melodic Constraints of the Third Species

3.M1 *Each note and its two beats further peer are preferred to be different.*

This rule is implicit so the default value of $cost_{MtwobSame}$ is <*low cost*>.

$$\forall \rho \in positions(m-2)$$
$$MtwoSame_{costs}[i,j] = \begin{cases} cost_{MtwobSame} & \text{if } M^2[\rho] = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

5.2.3 Motion Constraints of the Third Species

3.P1 *The motion is perceived on the basis of the fourth note.*

This implies that the costs of the motions and the first species constraints on the motions are deducted from $P[3]$.

Chapter 6

Fourth Species of Counterpoint

The fourth species of counterpoint consists of syncopations[1], one note shifted half a measure late against one note. In other words, only pairs of half notes[2].

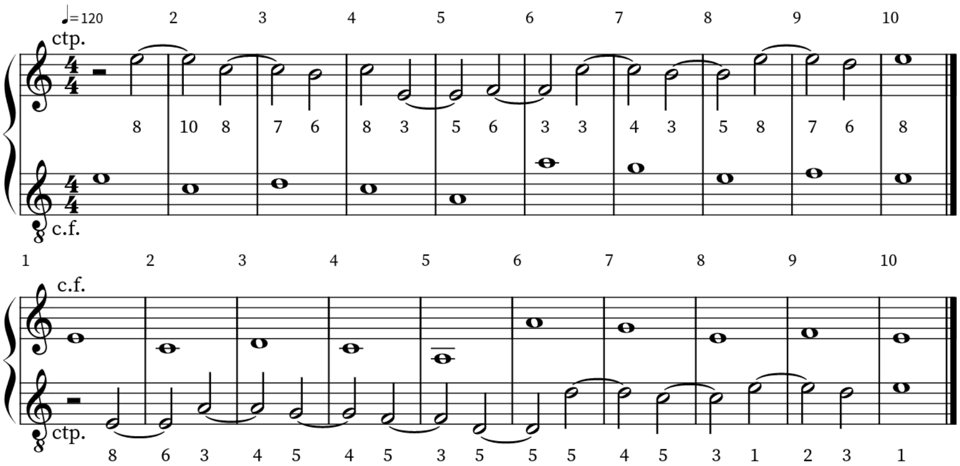


Figure 6.1: Two examples of a fourth species counterpoint. Listen here [25].

The fourth species is particular because it does not have more notes than the preceding species, it even has less. Indeed, this species is more like the first one. Here the syncopations are delays, which is roughly equivalent to using the first species with the whole note in thesis shifted in arsis which then lasts until the next arsis beat. While in the first species, all notes were consonant, here the syncopation requires more flexibility because the same whole note (here represented by a pair of half notes) is confronted with two different notes of the *cantus firmus.* First the second half of the first and then the first half of the second. If the syncopation is a delay of the note in thesis, then it is logical that the harmony it creates in arsis must be consonant (see rule **4.H1**). The specificity of the fourth species comes from the fact that dissonances can appear in thesis.

6.1 Formalization in English

For a better reading experience, the subsection on motion rules has been placed first as it is fundamental to understanding the other types of rules.

[1]Syncopation creates an off-balance rhythm through the accenting of normally unaccented beats.

[2]Except that the penultimate measure never has syncopation and it happens in certain measures that no syncopation is available.

6.1.1 Motion Rules of the Fourth Species

For this species, no rule concerning the motions is given by Fux. Moreover, no invariant, which could have served as a basis for creating an implicit rule, has been found in these examples. From another point of view, it could be seen that the motion created by a syncopation is nothing else than the oblique motion because one note stays in place while the other changes. This is of little importance because the rules concerning motions are somewhat adapted by rule **4.P2**.

4.P1 *Dissonant harmonies must be followed by the next lower consonant harmony.* Chevalier [1, p.78-81]

Any dissonant syncopation[3] should be resolved by moving downwards. This implies that if the *cantus firmus* is below, a second will resolve into a unison, narrowing the harmonic gap. Whereas if the *cantus firmus* is above, a second will resolve into a third, widening the harmonic gap. Figure 6.2 shows some examples of this rule.

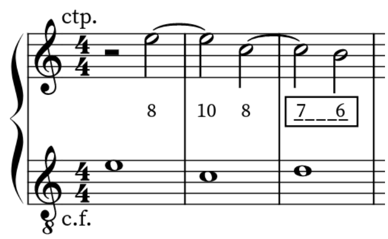


Figure 6.2: Dissonant syncopations resolved, 4^{th} species.

4.P2 *If the cantus firmus is in the lower part then no second harmony can be preceded by a unison/octave harmony.* [1, p.79-80]

The idea behind this rule is that *no octave/unison harmony in arsis can be followed by an octave/unison harmony in the next arsis with a dissonant harmony in between*.

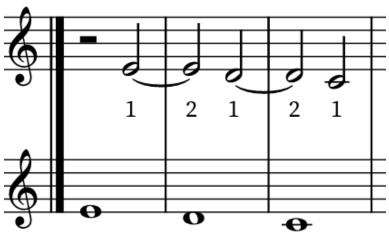


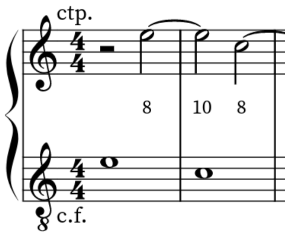
Figure 6.3: Seconds preceded by a unisons, 4^{th} species.

It is a kind of adaptation of rule **1.P1** which says that perfect consonances cannot be reached by direct motion. Indeed, according to rule **4.P1**, a second that is dissonant must resolve into a unison. This would result in a unison sequence (see figure 6.3) if the retardation is removed, i.e. the second, which would violate rule **1.P1**.

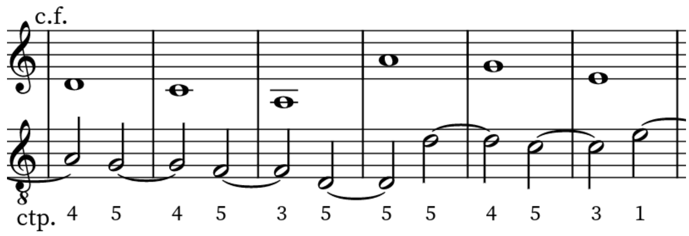
[3] A dissonant syncopation is a syncopation that becomes dissonant at the changing note of the *cantus firmus*. It differs from a consonant syncopation which is strictly always consonant with the *cantus firmus*.

In-depth review TODO 26 move all this reasoning to an appropriate section ?

Although Fux's explanation is logical and the rule is applied in his examples, the logic itself is not applied to other similar problems later on. An example will speak for itself:



(a) Two consecutive arsis octaves.



(b) Two consecutive arsis fifths.

Figure 6.4: Consecutive perfect consonances in arsis, 4$^{\text{th}}$ species.

In figure 6.4a, a consonant syncopation consisting of an octave and a third[4] is then followed by an octave again. No problem, the rule is respected since no second has appeared, but why put an octave whereas if the delay is removed, one falls back into the same issue that originated this rule, (i.e. two consecutive arsis octaves)? Mann [18, p.95] suggests that "[...] in measures containing dissonant syncopations the essential part is the upbeat, the second, consonant, half." This can be paraphrased to say that the human ear is only interested about the first consonance of a measure. This explains why the succession of octaves in the previous figure 6.4a is not one. Because the consonant third cuts off this impression.

What about fifths, which are also perfect consonances? In figure 6.4b, a consonant fifth $(G - D)$ turns into a dissonant fourth $(G - C)$ which is, as rule **4.P1** requires, resolved into a fifth again $(F - C)$. There is clearly a succession of fifths. But for a reason that Fux does not detail but that Mann [18, p.57] points out: "In the case of fifths, however, the retardation can mitigate the effect of parralel motion. Successions of fifths may therefore be used with syncopations." Probably because the fifth brings a little harmony where the octave does not really[5]. It is therefore only the current rule **4.P2** specific to octaves that is admitted.

All this thinking is explained for a reason: the purpose of the final software is to assist a composer and that he can choose thanks to an obvious logic that some rules are obsolete in his own case. It is therefore preferable to have logical rules such as "no two perfect consonances in a row without another imperfect consonance in between". This rule would be more contextual, more global and would speak more to a composer. Here, the rule is adapted only for octaves so that it keeps the associated logic instead of explaining it in the form of forbidding a second after a unison.

6.1.2 Harmonic Rules of the Fourth Species

4.H1 *Arsis harmonies must be consonant.* [1, p.78]

Although explicitly described by Fux, this rule is only an adaptation of fundamental rule **1.H1** as explained above.

4.H2 *If the cantus firmus is in the upper part, then no harmonic seventh interval can occur.*

[4]Here the third is actually a tenth.

[5]The octave is the simplest harmonic of its basic note with a frequency ratio of 2:1. Since it is the same note in a higher register it is not really about "harmony" as such. [11]

The origin of this rule is the same as rule **4.P2**. It is just less specific and therefore more restrictive because it does not depend on the previous or next harmony. Fux explains that this rule has no logical reason to exist. Nevertheless, the authoritative composers respected it, as did Fux as a result. It is optional for the previous reason.

4.H3 *For rule **1.H7** to be satisfied in the penultimate measure, if the cantus firmus is in the lower part, then the harmonic interval of the thesis note must be a seventh.*

The penultimate note cannot be a syncopation because the last note necessarily ends at the same time as the last note of the *cantus firmus* (see figure 6.5).

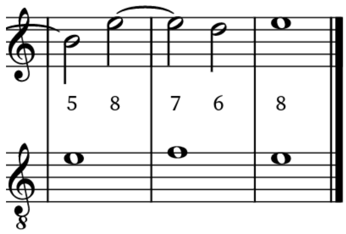


Figure 6.5: Penultimate measure, 4$^{\text{th}}$ species.

As usual in this case, the penultimate note is always a major sixth. The syncopation ending on the penultimate measure must be a dissonant seventh [6]. Following rule **4.P1**, the dissonance is resolved to the nearest consonance below.

4.H4 *For rule **1.H8** to be satisfied in the penultimate measure, if the cantus firmus is in the upper part, then the harmonic interval of the thesis note must be a second.*

The logic of the previous rule also applies to this one.

6.1.3 Melodic Rules of the Fourth Species

4.M1 *Arsis half notes should be the same as their next halves in thesis.*

In other words, *syncopations should occur if possible*. In theory, they are mandatory except in the penultimate measure. However, it happens that Fux breaks this rule to avoid monotony which is reflected by a repetition of a pattern in the musical work. This means that the cost of not putting a syncopation is lower than the cost of repeating the same syncopations. The difficulty is to know which cost best represents the monotony, which is quite subjective. Although all costs in the program have functional defaults, it's up to the composer to test various combinations to make the software shine. This will be discussed in section **TODO-sec 7** .

4.M2 *Each arsis note and its two measures further peer are preferred to be different.*

This is a more or less implicit consequence of the previous rule and is also an adaptation of rule **3.M1**. For the same reason as the latter, it is better to avoid alternating only between two different syncopations. But this remains totally subjective, because one could look for this very repetition in the syncopations. This is why the associated cost is customizable by the user.

6.2 Formalization into Constraints

Note that the arrays in index $[0, 0]$ are empty because the syncope arrives two beats late and leaves a silence in first thesis.

[6] Because of the structure of the *cantus firmus*, the seventh is often the tonic. This is a classic melodic progression at the end of a piece in tonal music that makes I - VII - I in degree (see *degree* at section 1.1.3).

6.2.1 Motion Constraints of the Fourth Species

4.P1 *Dissonant harmonies must be followed by the next lower consonant harmony.*

There is no need to add the constraint $IsCons[2, j] = \top$ because it is already included by rule **4.H1** (see equation 6.3).

$$\begin{gathered} \forall j \in [1, m-1) \\ \neg IsCons[0, j] \implies M_{brut}[0, j] \in \{-1, -2\} \end{gathered} \tag{6.1}$$

Listing 6.1: Function that constrains a dissonance to be followed by a consonance.

```
; @m-succ-intervals-brut: list of IntVar, s.f. brut melodic intervals (thesis ->
    arsis)
; @is-cons-arr: list of BoolVar, s.f. 1 -> the note is consonant
(defun add-h-dis-imp-cons-below-cst (m-succ-intervals-brut is-cons-arr)
    (loop for m in m-succ-intervals-brut for b in is-cons-arr do
    (let (
        (b-not (gil::add-bool-var *sp* 0 1)) ; s.f. !b (dissonance)
    )
        (gil::g-op *sp* b gil::BOT_EQV FALSE b-not) ; b-not = !b (dissonance)
        (gil::g-rel-reify *sp* m gil::IRT_LE 0 b-not gil::RM_IMP) ; b-not => m < 0
        (gil::g-rel-reify *sp* m gil::IRT_GQ -2 b-not gil::RM_IMP) ; b-not => m >= -2
)   ))
```

4.P2 *If the cantus firmus is in the lower part then no second harmony can be preceded by a unison/octave harmony.*

$$\begin{gathered} \forall j \in [1, m-1) \\ IsCfB[j+1] \implies H[2, j] \neq 0 \land H[0, j+1] \notin \{1, 2\} \end{gathered} \tag{6.2}$$

6.2.2 Harmonic Constraints of the Fourth Species

4.H1 *Arsis harmonies must be consonant.*

$$\forall j \in [0, m-1) \quad H[2, j] \in Cons \tag{6.3}$$

4.H2 *If the cantus firmus is in the upper part, then no harmonic seventh interval can occur.*

$$\forall j \in [1, m-1) \quad \neg IsCfB[j] \implies H[0, j] \notin \{10, 11\} \tag{6.4}$$

4.H3, 4.H4 *In the penultimate measure, the harmonic interval of the thesis note must be a major sixth or a minor third depending on the cantus firmus pitch.*

$$H[0, m-2] = \begin{cases} 9 & \text{if } IsCfB[m-2] \\ 3 & \text{otherwise} \end{cases} \tag{6.5}$$

6.2.3 Melodic Constraints of the Fourth Species

4.M1 *Arsis half notes should be the same as their next halves in thesis.*

The cost of not having syncope is by default <*last resort*>. It is because of costs like this that it is not really possible to compare the quality of two works of the same length just with the raw cost. Indeed, some *cantus firmus* may not have possibilities with syncopations only, which will artificially increase the total cost. It is therefore important to keep in mind that the costs are only relative to the *cantus firmus* used.

$$\forall j \in [0, m-1) \quad NoSync_{costs} = \begin{cases} cost_{NoSync} & \text{if } M[2,j] \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.6}$$

4.M2 *Each arsis note and its two measures further peer are preferred to be different.*

The default cost is <*high cost*> because monotony is very much avoided by Fux. It is unclear whether this cost should be higher than the cost of not having syncope.

$$\forall j \in [0, m-1)$$
$$MtwomSame_{costs} = \begin{cases} cost_{MtwomSame} & \text{if } Cp[2,j] = Cp[2,j+2] \\ 0 & \text{otherwise} \end{cases} \tag{6.7}$$

Chapter 7

Fifth Species of Counterpoint

The fifth species of counterpoint, also called *florid counterpoint*, consists of a combination of the four preceding species but mainly of the third and fourth. Indeed, a florid counterpoint in Fux's work looks like an alternation between quarter notes and syncopations, with a few dotted half notes (part of the syncopations) and eighth notes (binding quarter notes). It is more uncommon to find half notes which it is difficult to determine whether they come from the second or the fourth species.

Figure 7.1: Two examples of a florid counterpoint. Listen here [23].

The florid counterpoint is much free than its predecessors because the number of possibilities increases drastically with the possibility of adapting the species and thus the rhythm and the rules to obtain certain notes more easily than with the previous species. Here, more flexibility means easier to find a solution but more possibilities to explore for the solver. It is partly for these reasons that this chapter will be completely different from the previous ones. Where the others were formalizations of rules, this one is concerned with another problem: the relations between the different constraints of the previous species and the notion of rhythm that comes from them.

This chapter is mainly intended for computer scientists and mathematicians. This implies that the reader is aware of the different notions established in chapter 2 and that he understands the role of the variables previously introduced. Where the previous chapters were divided into two parts (natural language and constraints in the form of mathematics), the present chapter develops logic as a whole.

7.1 Problem Differences from Previous Species

Several points differentiate this species from others and influence the approach to be adopted:

- Fux does not describe new rules specific to this species, there are only new constraints linking the third and fourth species together. This is undoubtedly the lesson with the least information about its functioning.

- Fux shows variants at syncopations such that whole notes are replaced by dotted half notes; and variations on quarter notes by replacing them with eighth notes to fill in third skips or add mordent[1].

- So far, the solver has no notion of rhythm, its only goal was to find a list of note pitches. Now it must also be able to calculate which species are used where so that a rhythm can be deduced.

- Since notes must be constrained differently depending on the species they are part of, all species constraints cannot simply be applied to all notes. Furthermore, it is impossible with Gecode to dynamically remove constraints after they have been applied[2]. Therefore, another way must be found to have the constraints applied fully dynamically.

7.2 Representation of Species as Constraints

As explained before, the only values that had to be calculated and explicitly provided by the solver were the list of MIDI notes that form the generated counterpoint. Since each species only has notes of equal duration (apart from the last note which is necessarily a whole note), there is no constraint determining whether a note must exist or not at a certain position. Moreover, in the lesson of the fifth species Fux gives only too little information on any rhythm to follow to extract hard constraints.

7.2.1 Naive Solution

A naive solution would be to individually generate solutions from the previous species and somehow merge them. The problem with this approach is that the flexibility offered by the fifth species would be lost. Indeed, it is possible that certain notes of a certain species are only accessible from the use of a note of another species. Therefore there would be no interaction between the species and the main asset of the solver would be lost, i.e. being able to find a better solution according to preferences and associated costs.

7.2.2 Species Array System

The only approach that seems correct is to create an array of integer variables the same size as the counterpoint array Cp. Each variable would then represent which species the note belongs to at the same location in Cp. In this case, all the variables will be used, i.e. as many as the number of notes in a counterpoint of the third species containing only quarter notes. If this array determines to which species the corresponding note

[1] A mordent is a type of ornament referring to a quick alternation between a note and its upper or lower neighbor.[3]

[2] It is possible to *add* constraints dynamically after the CSP has been created, but nothing has been found to perform the operation the other way around, which seems much more complex.

belongs, it also determines if a note does *not* belong to any species. That is, whether a note at a certain beat of a certain measure exists or not. This is how the notion of rhythm appears. Be careful, declaring that a note does not exist implies that it is not in the final result of the counterpoint that the user see in the interface. In reality, for the solver the note does indeed exist in the space of constraints. There is an important distinction between the notes displayed to the user and the notes calculated by the solver. All this will be explained in more detail later.

A mathematical formalization is necessary. Let S be an array of same size and structure as Cp[3] representing to which species belongs the note at the same index in the array Cp.

$$\forall \rho \in positions(m)$$
$$S[\rho] = \begin{cases} 0 & \text{if } Cp[\rho] \text{ is not constrained by any species} \\ 1 & \text{if } Cp[\rho] \text{ is constrained by the first species} \\ 2 & \text{if } Cp[\rho] \text{ is constrained by the second species} \\ 3 & \text{if } Cp[\rho] \text{ is constrained by the third species} \\ 4 & \text{if } Cp[\rho] \text{ is constrained by the fourth species} \end{cases} \tag{7.1}$$

Without going into details for the moment, the solver never generates solutions with $S[\rho] \in \{1, 2\}$ which gives in the current state a domain equal to $\{0, 3, 4\}$.

Figure 7.2: Representation of the species array S along a counterpoint, 5th species.

By analyzing figure 7.2, one may notice that some patterns are emerging: all syncopations are distinguished by $4 - 0 - 4$ while quarter notes are never followed by 0. These patterns are rhythm constraints imposed in the solver but for now let's leave that and assume that S has coherent values, i.e. syncopations and quarter notes where it is possible to have them.

Now let IsS_x be another array of same size and structure as Cp representing whether a note belongs to species x where x is the number assigned to the species in S just above.

$$\forall x \in \{0, 1, 2, 3, 4\}, \forall \rho \in positions(m)$$
$$IsS_x[\rho] = \begin{cases} \top & \text{if } S[\rho] = x \\ \bot & \text{otherwise} \end{cases} \tag{7.2}$$

For example, $IsS_0[i, j] = \top$ means that the note at the beat i from the measure j is not contrained by any species. This does not mean that no constraint is placed on this note, only that the constraints of the species placed on this note are in this case necessarily respected. When an $\vee\top$ is added to a constraint, it renders the original constraint useless because the whole thing then becomes a tautology which is equivalent to remove the original constraint.

[3]Size of s_m, composed of four lists each representing a beat over the entirety of the measures, as always.

7.3 Formalization of the Species Rhythm into Constraints

In order for the array S and IsS to have relevant values, that is to say values which respect a format making it possible to produce a coherent rhythm, there must be constraints imposing that certain species may or may not exist at certain positions. These constraints come from common sense and have been created from the examples of the *Gradus ad Parnassum*. The context is no longer Fux's music theory but computer logic. The first rules are mandatory for the proper functioning of the system while additional rules have been added to match the style of the Fux.

5.R1 *There must always be a note in thesis and in arsis, except the very first thesis and the very last arsis.*

No species would allow not to have a note in thesis and only the first species does not have a note in arsis, a species which is not used in florid counterpoint (the last whole note of the counterpoint is the same in all species and is therefore not considered a particularity of any species).

$$\begin{gathered} \forall j \in [0, m) \\ \neg IsS_0[0, j] \quad \text{where } j \neq 0 \\ \neg IsS_0[2, j] \quad \text{where } j \neq m-1 \end{gathered} \tag{7.3}$$

5.R2 *The 4^{th} species can only exist in first and third beat.*

Indeed, the notes beginning or ending a syncopation in this species are always located in these beats.

$$\forall i \in \{1, 3\}, \forall j \in [0, m) \quad \neg IsS_4[i, j] \tag{7.4}$$

5.R3 *A 4^{th} species in the third beat necessarily implies a 4^{th} species in the first beat of the following measure and vice versa. The fourth beat should then have no note.*

This simply describes the usual syncopation which consists of the mandatory $4-0-4$ sequence (see figure 7.3).

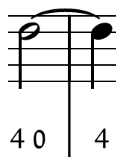


Figure 7.3: Syncopation implication in the S array, 5^{th} species.

$$\begin{gathered} \forall j \in [0, m-1) \\ IsS_4[2, j] \iff IsS_4[0, j+1] \\ IsS_4[2, j] \implies IsS_0[3, j] \end{gathered} \tag{7.5}$$

5.R4 *A 3^{rd} species cannot be followed by no note.*

If a quarter note is followed by no note then there would be at least one beat of silence, which is not intraseccally bad in music but is undesirable in counterpoint.

$$\forall \rho \in positions(m-1) \quad IsS_3[\rho] \implies \neg IsS_0[\rho+1] \tag{7.6}$$

5.R5 *Only 3^{rd} species and 4^{th} species are used.*

It has already been mentioned but as it stands, florid counterpoint is only composed of the third and fourth species in the solver. The formulation that Fux say that the fifth species is a mixture of the previous ones is confusing. Although species are based on common rules, Fux's examples clearly show a mixture of quarter notes and syncopations. Moreover, the half notes in a florid counterpoint can be generated by the second species as well as by the fourth (if the cost of not having syncopations is low).

In S the species are in the original domain in case future developments lead to adding the first and second species.

$$\forall \rho \in positions(m) \quad \neg IsS_1[\rho] \wedge \neg IsS_2[\rho] \tag{7.7}$$

5.R6 *The first and penultimate measures are linked to the 4th species.*

Fux begins all of these counterpoints with an oblique motion created by syncopation and always ends them with a syncopation resolution before the last note. This can result in a first measure and a penultimate measure comprising the sequences $0-0-4-0$ and $4-0-4-0$ respectively (see figure 42). Rule **5.R3** placed above ensures that the syncopations are completed correctly.

$$\begin{gathered} IsS_0[0,0] \wedge IsS_0[1,0] \wedge IsS_4[2,0] \\ IsS_4[0,m-2] \wedge IsS_0[1,m-2] \wedge IsS_4[2,m-2] \end{gathered} \tag{7.8}$$

Figure 7.4: First and penultimate measures in the S array, 5th species.

It is worth noting that the only silence occurs at the beginning of the counterpoint and is defined by the sequence $0-0$. This is the only time this sequence occurs. Another point, with the addition of this constraint, the last note of the counterpoint is necessarily linked to the fourth species, which has no particular impact because this note has the same role in all species, i.e. to be in perfect consonance with the *cantus firmus*.

7.4 Logic Implication of the Species Constraints

Now that the solver knows when a note must be constrained by the rules of a species, it is necessary to represent this concept in the form of constraints.

7.4.1 Generalization of the Species Implications

For this, it is necessary that the previously established rules have the possibility of being activated only if the variables concerned by a rule are variables linked to the species to which the present rule belongs. In other words, a constraint of x^{th} species on a set of variables V must be true only and only if the variables V are bound to notes belonging to this x^{th} species. Unfortunately, this concept cannot be generalized to all the rules because some still apply when only part of the notes concerned is linked to the corresponding species. But a try to generalize this idea can be written as such:

$$
\begin{gathered}
\forall x \in \{3,4\}, \forall cst_x \in Constraints(x), \forall V \in Variables(cst_x) \\
\left(\bigwedge_{\forall v \in V} IsS_x[v_{pos}] \right) \implies cst_x(V) \\
\text{where } Constraints(x) \text{ is the set of constraints of the species } x, \\
\text{and } Variables(cst_x) \text{ is the set of set of variables concerned by the constraint } cst_x, \\
\text{and } v_{pos} \text{ is the position of the } v \text{ related note in the array } Cp.
\end{gathered} \tag{7.9}
$$

It will be seen in equation 7.13 in the next section that all the variables concerned by a constraint do not necessarily have to belong to the species in question. From the point of view of programming, each rule had to be re-examined according to its basic operation. This part of the work revealed some architectural concerns that the software was not well enough adapted to handle this new logic, but this will be discussed in section 9.

Let's continue, in the current state of the program, florid counterpoint is considered to use either the third species or the fourth species. This means that a note has only three possible states: 0, 3 or 4. For example, rule **1.H1** states by extension that notes in *thesis* for the third species must be consonant but rule **4.H1** states that they are the notes in *arsis* for the fourth species which must be consonant. The two rules, hitherto distinct in two different species, result now in the fifth species in parallel.

Following the generalization:

$$
\begin{aligned}
\forall V \in Variables(1.H1_3) \quad & \left(\bigwedge_{\forall v \in V} IsS_3[v_{pos}] \right) \implies 1.H1_3(V) \\
\forall V \in Variables(4.H1_4) \quad & \left(\bigwedge_{\forall v \in V} IsS_4[v_{pos}] \right) \implies 4.H1_4(V)
\end{aligned} \tag{7.10}
$$

And concretely:

$$
\begin{aligned}
\forall j \in [0, m) \quad IsS_3[0, j] \implies (H[0, j] \in Cons) \\
\forall j \in [0, m-1) \quad IsS_4[2, j] \implies (H[2, j] \in Cons)
\end{aligned} \tag{7.11}
$$

It may seem simple but applying this logic to all the constraints of species 3 and 4 is not an easy task with the use of GiL which does not simply allow the addition of an implication on top of a constraint already written. The example above is one of the only cases where this is possible in this way but it must be understood that with GiL, which is only a precarious interface of Gecode, any intermediate step requires a new basic equation with only one operator. Mathematically, the equations would all follow the same notation which would basically just be a copy paste from the previous chapters. The rest of this chapter will therefore focus on the sometimes very specific relationships between species for certain rules that lead to slightly more complex constraints.

7.4.2 Avoiding Multiple Same Final Solutions

One might ask the question: what about notes where $S = 0$? These notes will not show up in the end user interface but the solver still calculates values for these notes. Does this mean that for a single solution on the user side there are a multitude of solutions on the solver side?

No, this is not the case because there is a constraint on the non-displayed notes, aka the non-constrained notes: they must be of the same value as the note of the next beat.

In fact, it's the same as putting a fixed note on all the notes that don't appear, but for a branching issue, it's a little more efficient to work like that. The formulation is written as such:

$$\forall \rho \in positions(m-1) \quad IsS_0[\rho] \implies (Cp[\rho] = Cp[\rho+1]) \tag{7.12}$$

7.5 Formalization of Inter-species Rules into Constraints

Fux, before beginning the lesson of the fifth species, describes variations in syncopations and the introduction of eighth notes, without going into too much detail. Chevalier [1, p.85]

Figure 7.5: Variation of a syncopation with quarter and eighth notes, 5th species.

This kind of variation is used a lot to get a rhythm and a melody more interesting. This can be considered as an inter-species rule and requires more attention than the simple example given above (equation 7.11). The figure 7.5 shows two things:

1. In relation to rule **4.P1**[4], the addition of quarter notes between the thesis and the arsis does not change the requirement to have an arsis consonance.

2. If the second eighth note is omitted, the melody does not move, which then implies that eighth notes can be used as mordents when the melodic interval between two beats is zero.

How to formalize these concepts with the new species array system? For the observation 1, it must first be understood what is the role of the first quarter note in thesis. Since this is a quarter, must not it be constrained by the third species? No, because this quarter note is part of the syncopation and is actually a 1/3 of the dotted half note[5] played in arsis in the previous measure. This quarter note has no difference with the half note found in the original version of the syncope apart from its duration. So this note must be constrained by the fourth species. In fact, whether the duration of the note in thesis is one beat (quarter note) or two beats (half note) is only determined by whether or not a quarter note takes place in the second beat of the measure. To summarize the constraint that must be imposed: *an arsis note, regardless of its species, must be the consonance just below the thesis note if the latter belongs to the fourth species*. This can be mathematically described by:

$$\begin{gathered} \forall j \in [1, m-1) \\ \neg IsCons[0,j] \wedge IsS_4[0,j] \implies M^2_{brut}[0,j] \in \{-1,-2\} \wedge IsCons[2,j] \end{gathered} \tag{7.13}$$

[4] A dissonant harmony in thesis must resolve in arsis with the next lower consonant harmony.

[5] If a whole note is 1 unit long, then a half note lasts 1/2 unit and a quarter note lasts 1/4 unit. A dotted half note then lasts 3/4 unit which is equivalent to three quarter notes.

There is indeed a constraint which is applied to the notes in $[2, j]$ whereas the latter do not necessarily belong to the fourth species according to the S array.

For the observation 2, Fux adds that:

> "Furthermore, two eighths may occasionally be used in the next species; that is, on the second and fourth beats of the measure but never on the first and third." Mann [18, p.63]

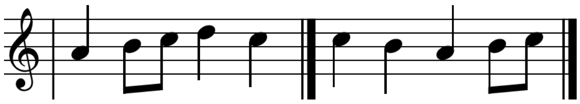


Figure 7.6: Addition of eighth notes in second and fourth beat, 5$^{\text{th}}$ species.

One might be disappointed to learn that these rules are not added as constraints in the CSP but several points led to this.

First, even though the solver does not know anything about the second eighth note (the first one being considered as the original quarter), the algorithm that generates the rhythm (see next section) after the solver has run still creates eighth notes. The end user therefore obtains counterpoints with eighth notes.

Second, the second eighth note of the eighths-pair is not bound by any rule. This means that no new solution with eighth notes can be found by the solver except the original solution with a quarter note instead. The eighth note only completes an already existing leap of third or adds a mordent.

Third, the architecture of the program was not designed to handle a whole new note subdivision, especially compared to the almost non-existent interest.

However, the only constraint which changes, or rather which withdraws with this system of eighths is that the melodic interval is not obliged to be zero between the second and third beat and between the fourth and first beat of the next measure. Therefore, rule **2.M2** which stated that two consecutive notes cannot be the same no longer applies at these positions.

7.6 Parsing of the Species Array in Rhythm

Rhythm species parsing occurs after the solver finds a solution. The parser therefore does not deal with Gecode variables but with values. Figure 7.8 is a simplified diagram of the parser. It represents a recursive function that takes as input the entire ordered lists Cp and S. This function outputs the final solution which will be shown to the user. The parser checks what is the next sequence of species and notes to find the corresponding note and associated duration. On the diagram, the notes are kept in the list `N` and the durations of the notes are kept in the list `R`. Once a sequence is found, it is removed from the lists Cp and S to be able to repeat the function again, this until the Cp and S lists are empty. This looks like a classic recursion pattern where the operation is performed on the head of the list and only the tail is kept for the next step. Here it is not necessarily a single element that is processed at a time but one to four elements.

The duration of notes in OpenMusic is represented as a fraction such that one unit represents an entire measure. Therefore, 1 represents the duration of one whole note; $1/2$ that of a half note; $1/4$ that of a quarter note; etc. If the value is negative, then a silence is played instead of a note. Also, in the diagram, the notation `L=[x:]` means that the list `L` is stripped of its first `x` elements. This means that the previously checked sequence occupied the space of x beats in total. Finally, for the parser to work correctly, the last value of S is replaced by 1 to signify that it is a whole note.

For example, if the values of the lists Cp and S are the following:

Cp	*72*	*72*	**72**	*72*	**72**	**71**	**71**	**69**	**67**	**69**	**69**	*69*	**69**	*68*	**68**	*69*	**69**
S	*0*	*0*	**4**	*0*	**4**	**3**	**3**	**3**	**3**	**3**	**4**	*0*	**4**	*0*	**4**	*0*	**1**

Table 7.1: Example of Cp and S, 5th species. Only the values in **bold** will be kept in the final solution.

Then the parsed output will be the following:

N		**72**	**71**	69	**71**	**69**	**67**	**69**	71	**69**	**68**	**69**
R	-1/2	**3/4**	**1/8**	1/8	**1/4**	**1/4**	**1/4**	**1/8**	1/8	**1**	**1/2**	**1**

Table 7.2: Parsed output of table 7.1, 5th species.

Note that the sum of the absolute values of R will always be equal to m, the number of measures. On the user side, this would appear:

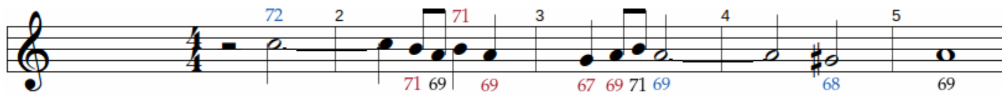


Figure 7.7: Final outcome from table 7.2, 5th species.

TODO 27 Ensure diagram is good.

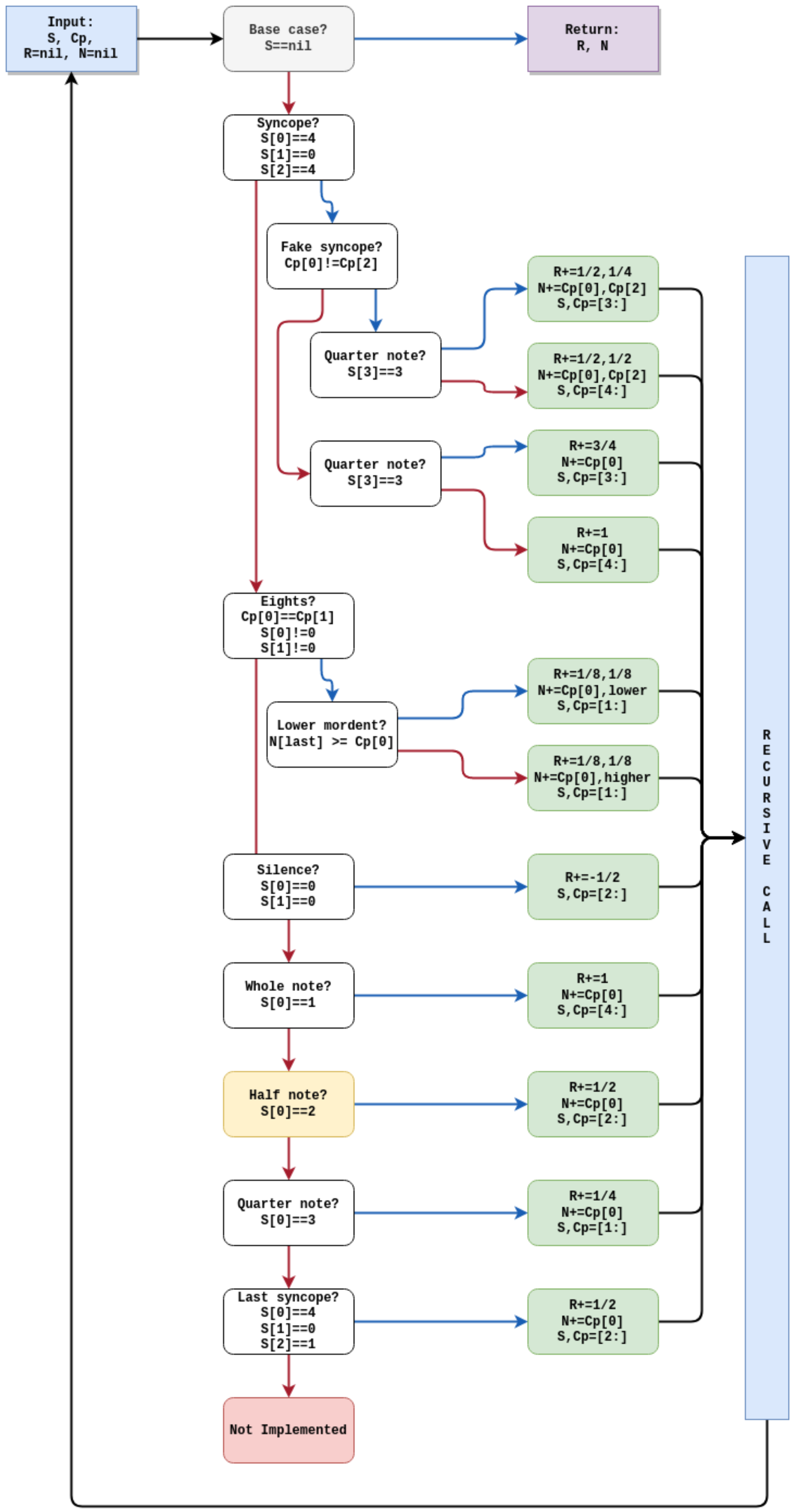


Figure 7.8: Rhythm species parser algorithm diagram, 5th species. A red arrow means the test failed while a blue one means it passed.

Chapter 8

Evaluation of the Solver Solutions Compared to Fux's Examples

This chapter can be seen as the culmination of this dissertation. All the constraints have been described but what about the results? Do the counterpoints found by the solver equal those of Johann Joseph Fux? This is what this chapter will try to answer by comparing the species one by one. The evaluations of the first four species will be simple analyzes of the differences and common points between the first counterpoint produced by the solver with the default values and the Fux counterpoint presented at the beginning of each species chapter. The analysis of the fifth species will be more advanced by tweaking the solver parameters to obtain more interesting counterpoints.

Determining what good counterpoint is is subjective and cultural. The following criticisms are therefore also subjective and cultural. It will be tried to make sociological objectivity and axiological neutrality[1]. It is thus good to note that these last are given by a man of Belgian culture appreciating Western music. Most people would say that Fux's counterpoints "look very baroque". It is therefore hoped that the counterpoints of the solver, presented below, will also be baroque. Moreover, the first four species are complicated to judge because with the absence of rhythm, an interesting melody will remain monotonous. Let's not forget that the main goal is to observe if constraint programming can be useful in the field of music. Finally, these tests are performed with a version of the solver still under development (dated May 17, 2023). It is possible that some default values have changed in the meantime during updates.

8.1 Evaluation of the First Species

The two counterpoints (see figure 8.1[2]) are globally very similar. A few differences are notable: the solver uses a fifth in 1st mesure and does not use a sixth leap from the 5th to the 6th mesure. This makes sense because the sixth leap has an cost of 2. Moreover this leap is surprising on the part of Fux because it is not melodically very interesting. Between the 9th and 10th mesure, a fifth and an oblique motion are used. They both have a cost of 1. It would be the same to not have a fifth, but a sixth and therefore have a direct motion between the 9th and 10th mesure. It would make the end of the song more moving and interesting. Another point is that Fux uses five direct motions (motions supposed to be avoided) while the solver only uses one. This first example shows what

[1]Axiological neutrality is a methodological posture proposed by the sociologist Max Weber. This consists of the researcher becoming aware of his own values during his scientific work, in order to reduce as much as possible the biases that his own value judgments could cause. [14]

[2]The solver solutions come from OpenMusic and have been stretched to better see the score lines.

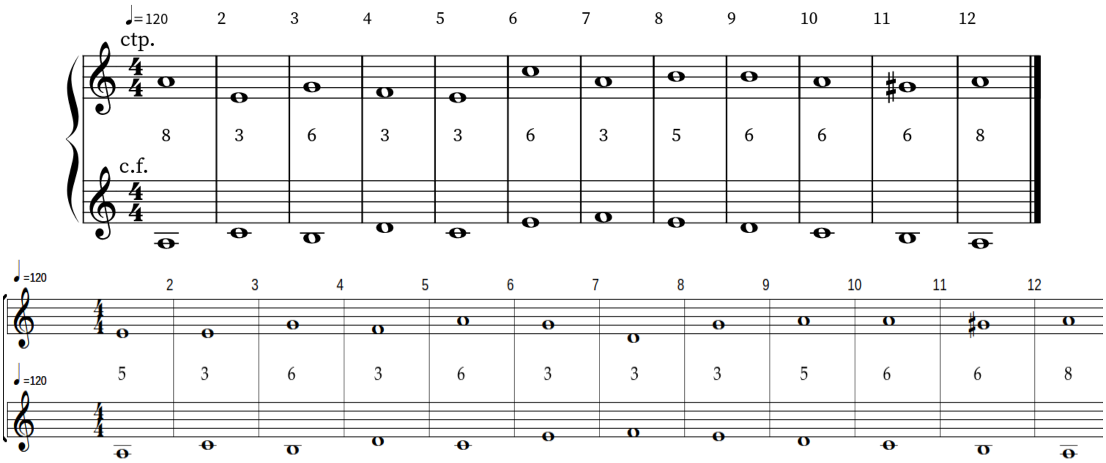


Figure 8.1: 1st species ctp. of Fux (above) vs. ctp. of the solver [0.132 s] (below).

the solver is capable of. It respects the rules well and never surprises because it is not aware of it. This point will be discussed further later.

8.2 Evaluation of the Second Species

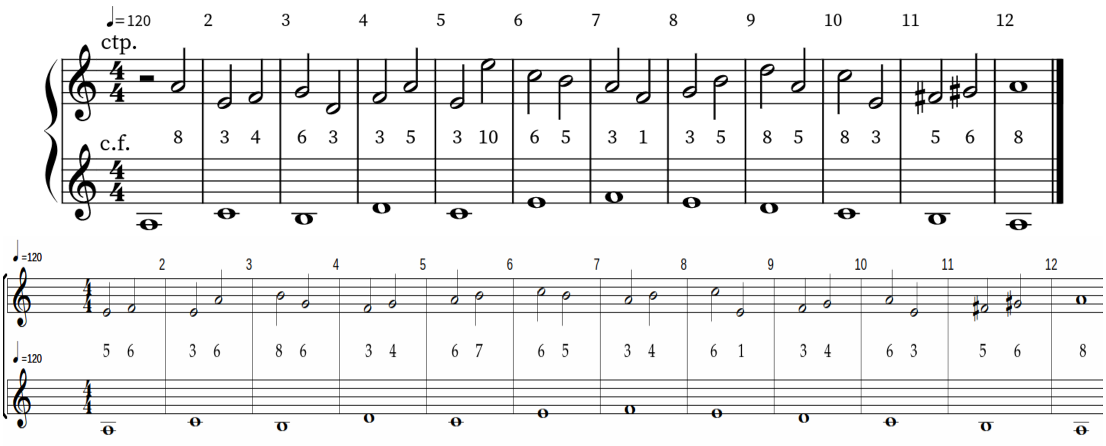


Figure 8.2: 2nd species ctp. of Fux (above) vs. ctp. of the solver [26.849 s] (below).

For the general feeling, the counterpoint of the solver is relatively of the same quality as that of Fux. Besides that, the solver's solution has a four-note motif from the arsis in the 8th mesure to the thesis in the 10th mesure ($E \to F \to G \to A$). This motif is repeated immediately raising the F and the G by a semitone. It sounds both strange and interesting but one can doubt that Fux would appreciate this melody.

A surprising point is the use that Fux makes of the big leaps between the notes in thesis and those in arsis. For example, he makes a fourth leap in the 3rd mesure. According to rule **2.P1**[3], the resulting motion is perceived from the note in arsis, i.e. the motion from the 3rd to the 4th measure is considered direct (cost of 2) instead of contrary (no cost). This is typically the kind of behavior that does not occur with the solver.

[3] Reminder: If the melodic interval of the counterpoint between the thesis and the arsis is larger than a third, then the motion is perceived on the basis of the arsis note.

Finally, one can notice that the search time for the answer is much higher than the previous one. This is a problem that particularly affects the second species and sometimes the fifth. This seems to come from rule **2.P1** discussed just before. Indeed, the best solutions of the solver (in terms of costs) often use large leaps to have more contrary motions. It goes against stepwise melodies and therefore takes more time. As proof, if the cost of the motions is not taken into account, the solution is found in 0.2 seconds. Alternatively, a trade-off can be made by first adding branching from small values to the motions costs. Therefore, small costs for motions are calculated before other costs. The first solution found then deviates from the lowest possible cost but is found in 8 seconds. This is a fairly common optimization problem when the overall cost minimization is composed of inversely proportional costs.

8.3 Evaluation of the Third Species

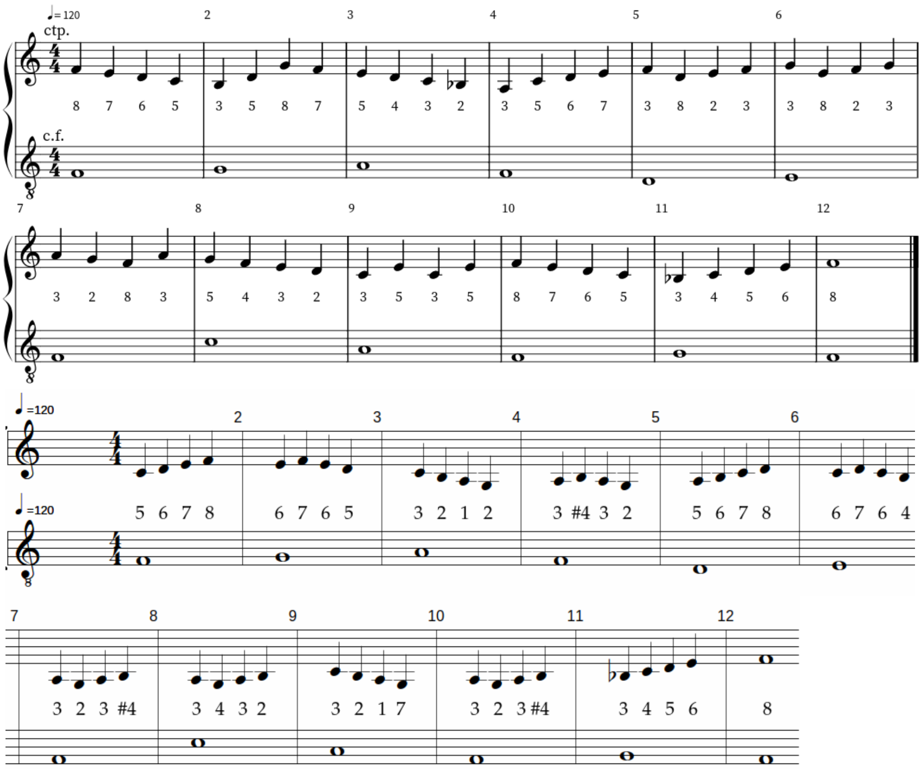


Figure 8.3: 3$^{\text{rd}}$ species ctp. of Fux (above) vs. ctp. of the solver [1.789 s] (below).

The solver's counterpoint is musically quite poor. Generally speaking, it is monotonous and rambling. Compared to that of Fux, it does not sound really baroque. The big negative point that emerges is the permanent use of stepwise melodic intervals. It is true that Fux is quite mysterious about the rules that make up a good melody[4] and that he uses a lot of one-step intervals. However, "a lot" does not mean "all". This is a very important notion that will be developed later: adding to the solver this notion of compromise, of surprise, of "a little bit of that, a little bit of this", etc. Typically, a way to force a minimum of melodic skips has been added to the solver to counter this problem. Another way to solve that is to put no cost to the melodic intervals of third for example.

[4]Which is understandable because the quality of a melody strongly depends on its context.

Also, the solver's counterpoint contains a redundant melody $(A \rightarrow G \rightarrow A \rightarrow B)$ which isn't bad in itself but seems to be randomly repeated and unsatisfactory. Obviously, the solver has no notion concerning the repetition of a pattern. This is also a major point to improve so that the solver can generate more human melodies. This solver really lacks an adjustable notion of monotony.

8.4 Evaluation of the Fourth Species

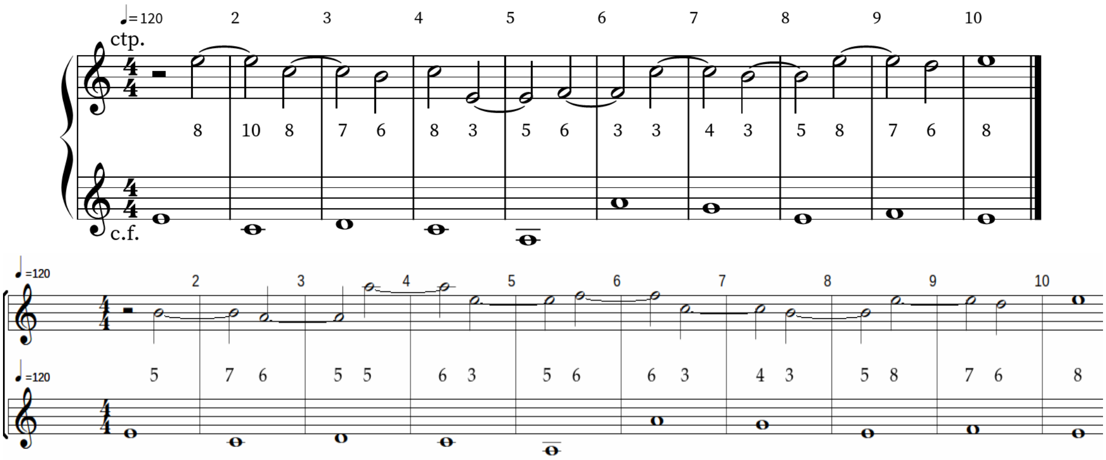


Figure 8.4: 4th species ctp. of Fux (above) vs. ctp. of the solver [0.012 s] (below).

This example strongly highlights a defect of the solver: a poor melody. Indeed, the counterpoint is supposed to be composed of several melodies which, *independently*, sound melodious and which, *together*, sound harmonious. This horizontal vision of music is transcribed only through the fact that the counterpoint is generated from a counterpoint. But with Fux, no rule defines what the counterpoint should be as a consistent whole. In fact, his rules could be considered the "micro rules" of counterpoint. It would therefore be necessary for the solver to have "macro rules" defining the very structure of the counterpoint in its entirety.

In the 5th and 6th mesure of Fux's counterpoint, the crossing between the two voices creates, for the time of three half notes $(F \rightarrow A \rightarrow C)$, a rising melody by skips of third. This intertwining brings out an F major chord giving that nostalgic feeling to the song. On the side of the solver, this opportunity is missed. But actually, the notes are "identical"[5] from the second half of the 4th measure. The only two real differences between those counterpoints are that the solver starts on a fifth and that it prefers an octave leap to the interruption of syncopations. This last point also shows that Fux exaggerates when he explains that syncope should be used "wherever possible"[18, p.89].

Although the generated counterpoint is average, it can allow a more or less experienced composer to find a good counterpoint by shifting a few notes by one octave. It's not perfect, but for a musician who likes to experiment, the solver gives him a good basis instantly that he can then exploit.

[5] In terms of the diatonic scale.

8.5 Evaluation of the Fifth Species

For this species, the analysis will be more advanced. First, the counterpoints will only be compared and secondly, a more compositional approach will be put forward. In the section 8.5.1, the solver counterpoints are the first results obtained with the default values. In the section 8.5.2, the solver will be used in a more intelligent way in order to obtain a more interesting solution.

8.5.1 Comparison

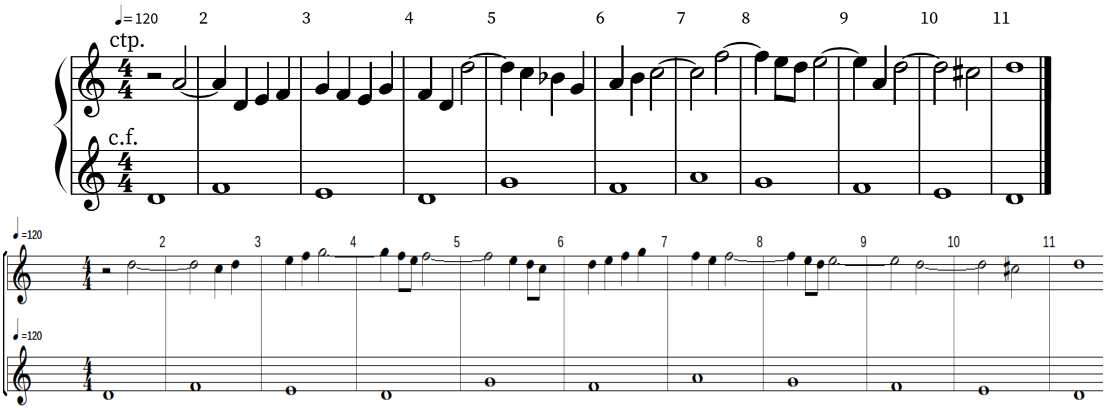


Figure 8.5: 5th species upper ctp. of Fux (above) vs. upper ctp. of the solver [0.174 s] (below).

Figure 8.6: 5th species upper ctp. of Fux (above) vs. upper ctp. of the solver [2.690 s] (below).

Whether it is the lower or upper counterpoint, those of Fux are clearly more baroque and are more melodious in general. Solvers' counterpoints aren't bad, but they're far from interesting. In figure 8.5, what Fux does in the 4th and 5th measure is the strong point of the work. The 4th measure has a *D* three times, which provides a pleasant rest before the repeat. He can afford this repetition because the *D* is the tonic of the piece. The solver does not have this notion of rest and tension related to the underlying chord.

Also, the $B\flat$ in the 5th measure adds a more nostalgic touch by suggesting a *G* minor chord. This $B\flat$ is not repeated in the next measure, which is rather original. Again, it's these kinds of little details that make Fux's counterpoints sound better than solver ones. The problem is the same as with the third species, i.e. the melodies are too "stepwise".

The counterpoint in the lower part has been added for information but the criticisms are generally the same.

One topic that hasn't been covered so far is cost comparisons. Indeed, if we force the solver with the same notes as the Fux counterpoint, it is possible to know what its total cost is. This can give a good idea of how well Fux applies its own rules and whether the costs assigned by the solver are consistent in determining what is or is not good counterpoint.

In this case, the solver's solution costs 14 while that of Fux costs 29. It makes sense that the solver finds solutions with a lower cost since that is the goal of its heuristic, unlike Fux. The cost discrepancy comes mainly from the common use of skips and leaps at Fux. This already represents 9 cost where the solver has none. This represents almost a third of the total cost. In fact, this way of optimizing costs is not entirely consistent with Fux's music. On the other hand, it is important that the melody is mainly stepwise. Maybe there is an alternative?

8.5.2 Refinement

A point which was not specified in the previous section but which is important is the branching of the species array S. One might have noticed that the rhythm of the species was the same for figure 8.5(below) and figure 8.6(below). It's not really a problem but the solver first randomly[6] determines which species is going to be used before it starts determining the associated notes. Indeed, it is much harder for the solver to first find an inexpensive solution and then determine if a rhythm can be associated with it. However, it is expensive but not impossible. This further minimizes the cost.

Another point discussed above was the possibility of having more diversified solutions at the level of melodic intervals. Three options have therefore been added to the user interface.

- *Irreverence* allows to artificially increase the minimum cost of the solution. This has two purposes: to prevent over-respecting solutions and/or to reduce the search time because the solver starts cost minimization with a higher lower bound.
- *Minimum percentage of skips* allows to force the solver to use larger melodic intervals.
- *Force contrary melody after skip* allows to activate a rule (coming from a work of Gallon and Bitsch [16]) obliging the melody to change direction if a skip occurs.

By using these options (see figure 8.7) and lowering the costs associated with the melodic intervals of thirds, fourths and fifths by one notch, a more interesting solution a can be generated.

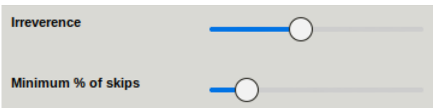


Figure 8.7: *Irreverence* and *Minimum percentage of skips* used for the solution 8.8.

This solution took nearly 3 minutes to be found, which is not huge but not negligible for a composer. Note that the notes boxed in red were changed to $B\flat$[7] to try the solver in a more realistic context[8].

[6]The randomization is controlled and is done from a seed. Currently, the same seed is always used and there is no way to change it from the UI.

[7]These notes have only been transposed by one or two semitones to stay close to the original solution.

[8]Where a composer allows himself to change the few notes that bother him.

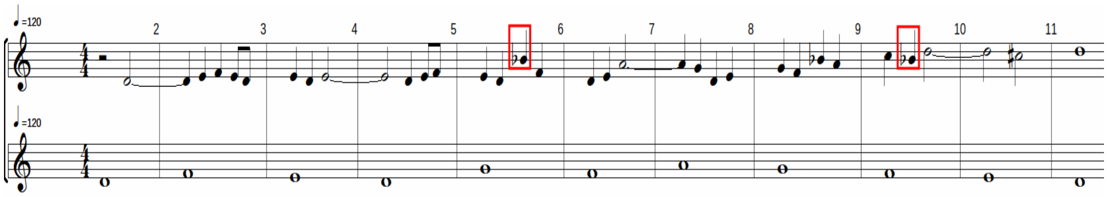


Figure 8.8: 5th species refined counterpoint of the solver found in 2 min 58 s.

With a few manipulations and tweaking, a good counterpoint is obtained in a few clicks and minutes. The solver itself may not be convincing, but it shows that using it as a support tool can be very inspiring. On our side, as a "*cantus firmus*", a more contemporary bass of 17 measures including chromatisms was tested and the result was stunning. This example is presented at the end of the presentation video (**TODO 28** make the video) and shows the ability of the solver to adapt to "*cantus firmus*" which are not at all classic.

8.6 Conclusion of the Evaluation

Chapter 9

Future Improvements

9.1 Software Architecture

9.2 Solver Performances

9.3 Solution Quality

Conclusion

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Appendix A

Transcriptions

> "Ça ce n'est pas bien, j'ai trois fois sol, même deux fois je m'en prive. Alors bon, exceptionnellement je peux permettre de temps en temps d'avoir deux fois la même note mais c'est vrai que dans les traités tels qu'on les utilise, ceux de par exemple: Marcel Bitsch, Marcel Dupré, les traités du XIXème siècle, on évite, enfin on proscrit même la répétition de la note. Bon et bien ça c'est une règle de bon sens en fait. Ce n'est pas une règle imposée comme ça de manière arbitraire. C'est que le contrepoint doit être une ligne en perpétuel mouvement [...]. Attention, chez Fux il le fait, donc c'est intéressant de voir que lui se permet ce genre de choses." Fabre [Jean-Louis Fabre's opinion on the repetition of the same note in counterpoint. 13, 1min 11]

Transcription A.1: French transcription of the video *Le contrepoint, les règles mélodiques et les règles harmoniques* for rule **1.P2**.

Which can be translated as:

> This is not good, I have three times G, even twice I do not use it. So, exceptionally, I can allow from time to time to have the same note twice, but it is true that in the treatises as we use them, those of for example: Marcel Bitsch, Marcel Dupré, the treatises of the XIXth century, we avoid, well we even proscribe the repetition of the note. Well, this is a rule of common sense in fact. It is not a rule imposed arbitrarily. It is that the counterpoint must be a line in perpetual movement [...]. Mind you, Fux does this, so it's interesting to see that he allows himself this kind of thing.

Transcription A.2: English translation of the above quotation A.1.

> "[...] s'il arrive que cinq noires se suivent par degrés conjoints, soit en montant soit en descendant, la première doit être consonante, la deuxième peut être dissonante, la troisième à nouveau nécessairement consonante, la quatrième pourra être dissonante si la cinquième est une consonance;"

Transcription A.3: Original text from Chevalier [1, p.73] for rule **3.H1**.

> "Tertia Contrapuncti Species est quatuor semiminimarum contra unam semibrevem Compositio. Ubi principiò animadvertendum est, quòd, si quinque semiminimas vei ascendendo, vel descendendo **continuò gradatim** se sequi contingat, prima Consonans esse debeat, secunda dissonans esse possit. Tertia denuo Consonans sit, necesse est. Quarta dissonans esse poterit, **si** quinta Consonantia fuerit;"

Transcription A.4: Original text from Fux [15, p.63-64] for rule **3.H1**.

Appendix B

Additional Material

Range	C	$C\sharp$ / $D\flat$	D	$D\sharp$ / $E\flat$	E	F	$F\sharp$ / $G\flat$	G	$G\sharp$ / $A\flat$	A	$A\sharp$ / $B\flat$	B
-1	0	1	2	3	4	5	6	7	8	9	10	11
0	12	13	14	15	16	17	18	19	20	**21**	22	23
1	24	25	26	27	28	29	30	31	32	33	34	35
2	36	37	38	39	40	41	42	43	44	45	46	47
3	48	49	50	51	52	53	54	55	56	57	58	59
4	60	61	62	63	64	65	66	67	68	69	70	71
5	72	73	74	75	76	77	78	79	80	81	82	83
6	84	85	86	87	88	89	90	91	92	93	94	95
7	96	97	98	99	100	101	102	103	104	105	106	107
8	**108**	109	110	111	112	113	114	115	116	117	118	119
9	120	121	122	123	124	125	126	127	-	-	-	-

Table B.1: MIDI note values.

Appendix C

User Guide

Appendix D

Source Code