UCLouvain

École polytechnique de Louvain

FuxCP: A Constraint Programming Based Tool Formalizing Fux's Musical Theory of Counterpoint

A generalization to three voices

Author: **Anton Lamotte**
Supervisor: **Peter Van Roy**
Readers: **Yves Deville, Karim Haddad, Damien Sprockeels**
Academic year 2023–2024
Master [120] in Computer Science and Engineering

Contents

Chapter 1

Introduction

This thesis is a formalisation of three-part counterpoint based on the writings and rules of Fux. Its aim is to provide a mathematical set of rules and a computer environment capable of translating Fux's teachings into formal logic, and capable of implementing these logical rules in a concrete way to produce Fux-style counterpoint.

This thesis will therefore be divided into several parts: we will first immerse ourselves in *Gradus ad Parnassum*, Fux's central work, from which we will meticulously extract the rules laid down by its author. We will briefly discuss these rules to make them unambiguous, and then translate them into formal logic, so that each rule Fux had in mind when writing his work is mathematically recorded. On this basis, we will create a computer implementation using constraint programming. We will then look at how this implementation finds results, discussing the search algorithm and heuristics used. We then discuss the cost techniques used to obtain the best possible results. Finally, we will analyse the musical compositions produced by the tool created.

It is very important to know that this thesis is based on T. Wafflard's thesis "FuxCP: a constraint programming based tool formalizing Fux's musical theory of counterpoint" [1]. The present work takes up T. Wafflard's concepts and definitions, and could only be understood in its full depth by reading and fully understanding that work as well. A short summary of it is given in section 1.3.

1.1 A brief history of counterpoint: from the writings of Bach to algorithmic generation

And before we get to the heart of the matter (which is the formalisation), let's take a look at Fux's theory of counterpoint, because that's the work that this work aims to formalise. Counterpoint is a compositional technique in which there are several musical lines (or voices) that are independent and distinct from one another but are balanced and sound beautiful [2]. No voice is dominant over the others, and all are main voices, even though some may take a small precedence during part of the composition [3].

Counterpoint has been central to the work of many famous composers from different artistic movements, such as Bach in the Baroque, Mozart in the Classical and Beethoven in the Romantic, and has continued to arouse interest over the centuries, with the development of key texts on the subject, such as Schenker's Counterpoint [4] or Jeppesen's Analysis [5]. And while Bach was probably the master of counterpoint composition in his day [6], the central and founding work in the teaching of counterpoint belongs to another great Baroque composer: the Austrian Fux and his treatise *Gradus ad Parnassum*. In it, this Baroque composer gives a detailed analysis of two-, three- and four-part writing of counterpoint, all told as a discourse between a master and his pupil. *Gradus ad Parnassum* was part of the species counterpoint movement, a way of conceiving counterpoint in five different types that could then be combined. This is the work on which this dissertation is based.

For Fux, but also for many other authors, species counterpoint are governed by many diverse rules, and it is these rules that interest us in the present work. The rules are based on old concepts that go back to older styles [7] and have been discussed by many authors. Those concepts include, for example, the notions of opposite motion and consonance (which in turn can be either perfect or imperfect). These concepts and their application to counterpoint are particularly interesting because they allow us to consider the composition of counterpoint both in a 'vertical' way, in which we consider the harmony of the notes played together, and in a 'horizontal' way, in which we consider the melodic development of each of the parts individually, which provides the independence of the counterpoints from each other and their melodic beauty.

This is what makes it interesting to analyse from a constraint programming point of view. We'll come back to this later, but for now let's concentrate on Fux's music theory.

1.1.1 Fux's theory of counterpoint

As we have just said, Fux uses a set of rules to create 'the right counterpoint'. These rules can be divided into three categories: melodic rules, harmonic rules and rules of motion. We will examine them here, bearing in mind that the formalisation of all the rules for two-voice counterpoint can be found in T. Wafflard's thesis, and that the complete set of rules (in mathematical form) can be found in the appendix A to this thesis.

Melodic Rules

Fux explains that there are rules that apply within parts (the horizontal rules) about the order of the interval between one note and the next: we find, for example, that a melody is more beautiful[1] if the intervals between its successive notes are small, that there is no chromatic succession, if the notes that follow each other are varied, and so on. These 'horizontal' rules are called 'melodic' rules because they concern only the melody. These rules apply within a given counterpoint.

Harmonic Rules

If there is a horizontal perspective to counterpoint, there is also, of course, a vertical perspective. This perspective is expressed in a harmonic relationship between the different voices. At each point in the composition, a series of rules (known as 'harmonic rules' because they concern harmony alone) apply. For example, there is the rule that in the first beats of each measure the interval between the voices must be a consonance [2]; imperfect consonances[3] are preferred to fifths, which are preferred to octaves; and the rule that the different voices must be different at each point in the composition. These rules apply between the counterpoints.

Motion rules

Finally, there is a third type of rule: movement rules. These rules are a hybrid of the two discussed above in that they consider not only vertical interaction, i.e. harmony, but also horizontal interaction, i.e. melody. They can therefore be seen as 'diagonal' rules that relate the unique melody of each counterpoint to its respective harmonies.

[1] Throughout this work we will speak of the "beauty of music". This beauty is highly subjective, and therefore reference will be made to the Fuxian concept of music to define whether a melody is beautiful or not. In other words, music will be considered beautiful if it conforms to the rules of Fux, and vice versa

[2] A third, a fifth, a sixth or an octave.

[3] Thirds and sixths.

These rules include, for example, the fact that we prefer voices that move in opposite directions (i.e. if one voice goes up, we want the other to go down), the rule that there should be no sequence of fifths or octaves between voices, or the rule that we should not arrive at a perfect interval by direct motion.

1.1.2 Species counterpoint

When we talk about species counterpoint, we're talking about five categories of counterpoint. Each species is a separate concept, each with its own peculiarities. We'll go into more detail about the different species below. Let's first concentrate on how species counterpoint works. When composing counterpoint, the starting point is a fixed melodic line, the *cantus firmus*, which is a basic melody composed entirely of whole notes. It is the basis of a composition when writing counterpoint. It is from this voice and in relation to it that the others are composed. It is important to note that once the composition is complete, the *cantus firmus* is no more or less important than the other voices, and has the same melodic independence as the other voices. It is therefore nothing more than the basis from which we begin to write.

Let's take a look at the five species:

1. **First species**: Note against note – the counterpoint is composed entirely of whole notes, and the composition is a sequence of harmonies sounding on the first beat between the counterpoint and the other voices.

2. **Second species**: Half notes against whole notes – the counterpoint is composed entirely of half notes, which introduce dissonant harmonies.

3. **Third species**: Quarters against whole notes – the counterpoint is made up entirely of quarter notes, which allow more different movements and more freedom in the composition.

4. **Fourth species**: The ligature – the counterpoint is delayed by two beats, creating syncopation. The notes are all round (i.e., tied minims, since they span between two measures).

5. **Fifth species**: Florid counterpoint – counterpoint is a mixture of all the other species and is the richest form of counterpoint. It allows great freedom of composition while respecting the rules of the other types.

Although these types could technically be combined to form a three-part composition with two different types of counterpoint, Fux seems to prefer to write a 'special' counterpoint (i.e. second, third, fourth or fifth species) in one voice and only whole notes (i.e. a first species counterpoint) in another voice. He also sometimes says that it is possible and recommended to mix species, but does not do so extensively.

1.1.3 Building a computer tool for writing species counterpoint

We have just seen that counterpoint is an ancient style, and many generations of composers have worked on it (and still do). So it's only natural that, over the years, some computer specialists have thought about automating the writing of counterpoint. The first notorious example of this was Schottstaedt [8], who in 1984 developed an expert system for writing counterpoint based on Fux's rules. Confronted with the technologies and algorithms of his time, he created an algorithm consisting of over 300 if-else clauses, with the drawback that these if-else clauses are far less efficient than what constraints are capable of today.

In 1997, a genetic programming and symbiosis approach to automatic counterpoint generation was developed. A team from Michigan used this approach to optimise counterpoints of the 5th species and make them more attractive [9]. A similar approach was used in 2004 to generate fugues (hence counterpoint), also using genetic algorithms [10].

Many years later, in 2010, a team of researchers from the University of Malaga developed an automated method for the generation of first-order counterpoint using probabilistic logic [11]. Their approach was specifically tailored to compositions in C major, providing a generated counterpoint in response to a given *cantus firmus*. Please note that this application only evaluates the harmonic attributes of the counterpoint, ignoring the melodic aspect.

Two years later, a team from London developed a way to generate high-quality first-species counterpoint using a variable neighbourhood search algorithm [12]. Their research was limited to first-species counterpoint, but they addressed issues such as cost (finding the best counterpoint) and user-friendly interface.

Finally, a research was carried out in 2015 on Fux's counterpoint [13], with the aim of generating the first species counterpoint using dominance relations, has yielded fairly good results. The search demonstrates the use of this paradigm and its applicability, and is a good starting point for composing counterpoints of other species based on the same concept.

If we now focus on applications that have gone as far as the user interface and are now ready to be used, we should mention two namesakes, both called 'Counterpointer', which have the merit of offering a functional tool for composing counterpoint.

The first Counterpointer tool [14], which anyone[4] can use to check the validity (or otherwise) of their counterpoint. It was last released in 2019 as a desktop application, and it works like this: an apprentice composer tries to write a counterpoint, and then submits it to the tool. The tool then decides whether the counterpoint is valid according to the traditional rules of counterpoint[5]. It also provides feedback to help the student composer improve his or her future counterpoint writing. The tool is not able to write counterpoints automatically, nor is it explicit about how it works, as it is completely closed source and has no accessible report. It is therefore impossible to know the paradigm it uses or the exact rules it follows.

Another attempt at automatic counterpoint writing is the Counterpointer project in 2021, created by a team of students at Brown University as part of a software engineering course [15]. The project is less accomplished than the aforementioned application, but it has the merit of being able to generate two-voice counterpoints of the first, second and third species. It is an entirely free and open source project. While the results are encouraging, the project has been discontinued as it was a course project and their method of finding a counterpoint seems much less efficient than the efficiency that a constraint solver can achieve.

This brief overview leads us to conclude that there is no satisfactory tool for composing counterpoint in a user-friendly way, with good quality, quickly and with several voices. It is to fill this gap that this research has been carried out. This was the aim of T. Wafflard's thesis and it is therefore natural that this thesis should follow in his footsteps.

[4] Anyone... or almost, as it is a paid tool.

[5] Not only Fux's rules, but also those of other authors.

1.2 Standing on the shoulders of giants: underlying works and editions of *Gradus ad Parnassum* used

As has just been said, this work is the continuation of T. Wafflard's work. However, it also relies heavily on the work of

- Lapière [16], who presented an interface for using Gecode functions in Lisp called "GiL". This interface was then tested with some rhythm-oriented constraints.
- Sprockeels [17], who explored the use of constraint programming in OpenMusic using GiL. The tool that was produced in this thesis is capable of producing songs with basic harmonic and melodic constraints.
- Chardon, Diels, and Gobbi [18], who created a tool capable of combining the strengths of the first two implementations while continuing to develop support for GiL.

As with T. Wafflard, the musical reference work chosen is Fux's *Gradus ad Parnassum*, because it is a pillar of counterpoint theory and because it is fairly easy to extract rules from it (although Fux is sometimes very vague about his intentions). And as with any book published several centuries ago (1725 in the case of *Gradus ad Parnassum*), there are many versions and translations. This is good news, as Fux can sometimes be really unclear about what he means, and having many versions (some annotated, some not) from many people who also had to interpret Fux to translate it is a great treasure, as it helps to clarify Fux's meanings. This work is therefore based on several different editions and translations of the book, although it is mainly based on Alfred Mann's English translation [19]. French (both Chevalier's [20] and Denis's [21]), German [22] and Latin [23] translations are used when it is necessary to remove an ambiguity or clarify an unclear rule. These translations have been chosen because French is the *lingua franca* of the team; German is the language of Fux and the environment in which he developed; and Latin is the original version, so we can hope that it is the most faithful to what he wanted to convey.

1.3 T. Wafflard's thesis in a nutshell

In 2023, T. Wafflard proposed a complete formalisation of Fux's two-voice counterpoint [1]. This formalisation takes each of the rules given by Fux about two-voice composition and translates them into formal logic. Those formal relations are then translated into constraints and given to a constraint solver. When given an input *cantus firmus*, the solver applies all the constraints it was given and yields an output counterpoint. The following subsection is not intended to be an exhaustive summary of all of T. Wafflard's excellent work, but rather a brief outline of the idea behind it and the procedure followed. This brief overview is given to the reader because many of the concepts in T. Wafflard's work are at the heart of the three-voice generalisation and will be key to understanding the three-voice formalisation.

1.3.1 Variables

In order to formalise Fux's rules, it was necessary to define variables whose purpose is to represent a compositional reality. Thus T. Wafflard created variables to represent various concepts, such as the pitch of notes (written N), the harmonic interval between voices (written H), the melodic interval from one note to the next (written

M), and many others. These variables are then related to each other according to the formalised rules. The constraint solver searches for all possible values of these variables, according to the constraints, and stops when all variables in N (the pitches) are fixed, as this means that a solution has been reached (the notes of the counterpoint are known, and this is the goal of the solver).

Useful constants were also defined and will be reused throughout this work. The most important of these are m, which represents the length of the *cantus firmus* (and thus the number of bars in the composition), n, which represents the number of notes in a given counterpoint, or $\text{Cons}_{\text{all, p, imp}}$, which represents the set of consonances (all, perfect and imperfect).

Of course, there are many different solutions for the same *cantus firmus*, and in order to distinguish between two valid solutions (i.e. all solutions that respect all constraints), some costs have been defined. These costs are intended to convey the preferences expressed by Fux in *Gradus ad Parnassum*. In fact, Fux sometimes expresses a preference that is not an absolute rule: "this should be done if possible, but is not necessary". Thus, the solver considers a valid solution with a low cost to be better than a valid solution with a high cost. The costs scale from 0 (when we don't care) to $64m$ (when something should only happen as a last resort), but most costs scale from 0 to 8. These costs are then added together to form a total cost, which the solver tries to minimise.

1.3.2 Array notation

As we have just seen it is necessary to refer to numerous arrays, that formally represent the musical composition and many of its underlying aspects (pitches, harmony, melody, ...). These arrays always have two indexes: the first index represents the time in question, the second index represents the measure in question. These indices are written in computer notation. For example, $X[3, 7]$ represents the value X on the 3rd beat of the 7th measure.

Let's put it all together. Here is a simple example of how the variables and constants are brought together through the formalisation:

$$H[0, m] \in Cons_{\text{perf}} \tag{1.1}$$

This is a rule which means that the last harmonic interval must be a perfect consonance[6]. All rules from T. Wafflard's thesis can be found in appendix A.

1.3.3 In practice

In practice, to solve this constraint programming problem, the constraints are written in Lisp, and thanks to the Gecode Interface Lisp (GiL) [24], it uses the Gecode constraint solver [25] to find a solution. To make the constraints work, we need a starting point. The starting point is the *cantus firmus*(see 1.1.2). When given a *cantus firmus*, the solver defines a set of variables (those mentioned in the formalisation) to which constraints are then applied (the relations from the formalisation) and produces a counterpoint that obeys all the rules that have been defined and whose quality can be given by the cost. As explained in 1.3.1, in the case of T. Wafflard's implementation, the total cost is the sum of all the costs, and this cost is minimised by a depth first search algorithm that finds the lowest cost, and then gives the corresponding solution.

As for the front-end, all the user sees and interacts with is OpenMusic [26], an object-oriented visual programming environment for musical composition based on

[6] Please refer to section 2.3.4 for more exact details

Common Lisp [27] developed by the Parisian institute IRCAM (Institute for Acoustic/Music Research and Coordination) [28].

The present work has exactly the same basic functioning.

1.4 The contributions of this paper

The aim of this work is to generalise T. Wafflard's formalisation to three-voice counterpoint, still based on Fux's work, and to create the corresponding implementation. It would be too easy to believe (wrongly) that three-voice counterpoint is nothing more than the combination of two two-voice counterpoints. From this point of view, we would then calculate a first counterpoint according to the *cantus firmus*, and then a second counterpoint again according to the *cantus firmus*, and that's it. Obviously, this view is too simplistic and doesn't really capture all the interactions between three voices. It is to this point (the peculiarities brought about by the addition of a third voice) that a whole section of this thesis will be devoted. Another part will be devoted to discussing and analysing the impact of costs. The following is a more detailed summary of the contributions of this thesis.

- **Introducing new concepts and redefining variables**: As we have just mentioned, a three-part composition is much more than a (two+one) part composition. So we had to redefine and define a whole series of concepts to adapt to this reality. The creation of the (lowest, middle and highest) stratum concept is part of this, and is essential for formalising Fux's counterpoint constraints.

- **Mathematical formalisation of three-part counterpoint**: As with the two parts of the formalisation, we rewrote Fux's explanations into unambiguous English and then translated them into logical notation. This formalisation builds on the previous formalisation for two voices, and sometimes (rarely) has to modify it.

- **Implementation of a working constraint solver for a three-voice composition**: Those logical rules were then implemented as constraints and the solver was adapted to allow a search for two counterpoints.

- **Searching for the best cost management techniques**: Three-part composition introduces so many possibilities for the search that it was important to rethink the way we thought about costs. Therefore, some cost management techniques were discussed to find out the best way to use the costs.

- **Musical analysis of the solutions generated by the solver**: Finding the best solution also means being able to assess the quality of current solutions.

- **Adapted user interface to allow them to compose with three voices and set a cost order**: All the new capabilities of the solver and the costing techniques must also be accessible to the user: it is now possible for a user to freely combine any number of species to form a three-part composition, and to set a cost order to indicate their preferences to the solver (in addition to the previous ability to set personalised costs).

Chapter 2

Defining some concepts and redefining the variables

2.1 Voices, parts and strata

Before we start this section, we need to look at some vocabulary to make sure we understand what we are discussing. The most important definition (and distinction) we will introduce is the definition of the terms *part* and *stratum*. The need for these definitions arises from the increasing complexity of the rules of counterpoint when it is generalised to three voices. Indeed, the rules are no longer (as we shall see later) concerned solely with the counterpoints and the *cantus firmus*, but also with new concepts, such as that referred to by Fux as 'the lowest voice'. As the term 'voice' is too generic (it is used in Fux's text to describe notions as different as 'counterpoint', '*cantus firmus*', voice range and the so called 'lowest voice'), we need to create a precise vocabulary that is different from the word 'voice' to talk about these new concepts.

With this in mind, let's explain what 'parts' and 'strata' are, and how they relate to the concept of 'voice'.

Voices

Again, voices are that vague and *general* concept, whereas parts and strata are more precise and *specific* concepts. The concept of 'voice' includes both 'parts' and 'strata'. In other words, each of these two concepts is a type of voice. When we talk about a voice, we could be talking about either a part or a stratum. To make a metaphor out of it, we could say that parts and strata are a type of voice.

Since there are as many parts and layers as there are voices, in a composition with n voices there will also be n parts and n layers.

Parts

Parts are an intuitive and concrete concept because each part corresponds to what a particular person sings or what a particular instrument plays. They correspond to a staff (each staff corresponds to a part). The term 'part' is the same as that used by Fux in his *Gradus ad Parnassum*. The three parts in a three-part composition are: the *cantus firmus*, the first counterpoint and the second counterpoint. Fux distinguishes them by calling them by the name of their range, i.e. "bass", "tenor", "alto" or "soprano" (obviously you cannot have all four in a three-part composition).

Strata

As for the strata, they are defined like this: a stratum delineates discrete layers or levels of pitches at any given moment in the composition. It denotes a vertical alignment of simultaneous notes and organizes them into distinct strata. By definition, the lowest stratum encompasses the lowest sounding notes, the highest stratum comprises the highest sounding notes, and intermediary strata represent pitch levels in between.

This concept is very helpful in identifying and categorising the vertical placement of pitches, creating distinct categories of sound within the overall texture of the counterpoint composition. It provides a way of analysing and understanding the distribution of pitches across different parts, allowing more complex rules to be established: for example, it would now be possible to establish a rule between the notes of the cantus firmus and the highest sounding notes (no matter which part they come from). The full potential of strata lies in harmonic rules, but as we shall see, some melodic rules are also related to it.

The term stratum was chosen in this context for its visual impact. In geology, a stratum "is a rock layer with a lithology (texture, colour, grain size, composition, fossils, etc.) different from the adjacent ones", see figure 2.1.

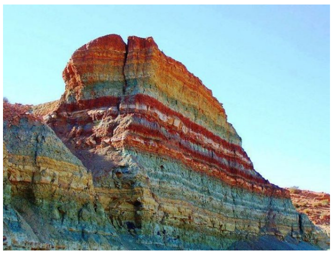


Figure 2.1: Geological strata, for the illustration

When Fux speaks about the lowest stratum, he often uses the word 'bass'. It was deliberately chosen to speak about the 'lowest stratum' instead of the 'bass' (like Fux does), because 'bass' is also the name of a range of voices (on a par with soprano and alto, for example), and there is already enough complexity in all the terminology to add even further ambiguity.

These new terms (parts and strata) are used where the distinction between the concepts is important. Whenever this distinction is not relevant, the more general term 'voice' is used to reduce the complexity of reading. In this case, the 'voice' could refer to both a stratum and a part.

Since a picture is worth a thousand words, Figure 2.2 illustrates the difference between parts (the blue lines) and strata (the red and orange lines). The lowest stratum is shown in its own colour (red) because it is the most meaningful stratum, and it will be particularly important later on.

Here is also the mathematical representation for the notes of the lowest stratum (written $N(a)$, see section 2.3 for the notations):

$$\forall i \in [0,3] \quad \forall j \in [0, m-1) : N(a)[i,j] = \text{min}(N(cf)[i,j], N(cp_1)[i,j], N(cp_2)[i,j]) \tag{2.1}$$

Of the first upper stratum, or medium stratum (written $N(b)$, see section 2.3 for the notations):

$$\forall i \in [0,3] \quad \forall j \in [0, m-1) : N(b)[i,j] = \text{med}^1(N(cf)[i,j], N(cp_1)[i,j], N(cp_2)[i,j]) \tag{2.2}$$

And of the second upper stratum, or uppest stratum (written $N(c)$, see section 2.3 for the notations):

$$\forall i \in [0,3] \quad \forall j \in [0, m-1) : N(c)[i,j] = \text{max}(N(cf)[i,j], N(cp_1)[i,j], N(cp_2)[i,j]) \tag{2.3}$$

One part per stratum and one stratum per part

It is important to note that, for each musical measure, there is a bijection between the individual parts and the corresponding strata. This means that, for any given measure,

[1] Where med(X) means the median value of X.

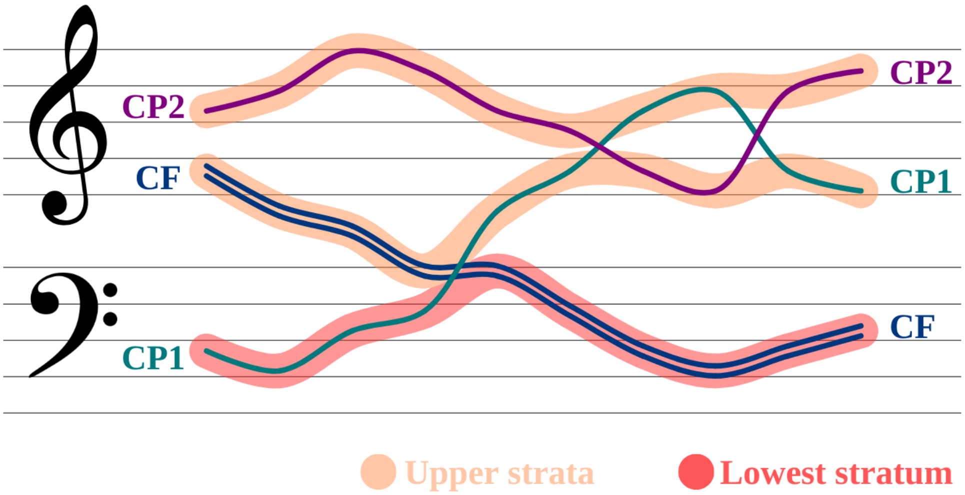


Figure 2.2: Parts and strata in a three voice composition

each stratum uniquely corresponds to a single part, and vice versa. Put differently, if two parts within a measure share the same pitch, they do not constitute the same stratum. Instead, one part corresponds with one stratum, and the other one to a separate stratum.

To illustrate this, consider a scenario in a two-voice composition (see figure 2.3), where part 'cf' and part 'cp1' in measure X both have a pitch value of 67 (representing a G). Despite having identical pitches at the same moment, one part is categorised as the lowest stratum, while the other is designated as the uppermost stratum. This distinction becomes crucial for subsequent analysis, especially when calculating aspects like motions.

To know which part gets to be the lowest stratum in such situations, an arbitrary hierarchical rule is implemented. If the ambivalence is between the *cantus firmus* and another part, the *cantus firmus* is always prioritised and assigned the role of the lowest stratum, over any other part. In the case of a ambivalence between the first counterpointand the second counterpoint, the first counterpoint is given the status of the lowest stratum.

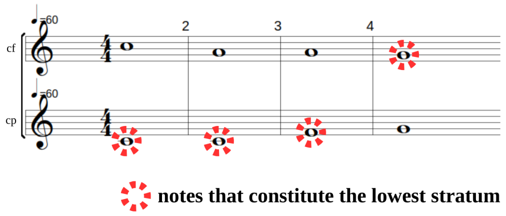


Figure 2.3: Establishing which part corresponds to the lowest stratum

2.2 Exploring the interaction of the parts with the lowest stratum

One of the major differences between the composition of two voices (i.e., one *cantus firmus* and one counterpoint) and the generalisation to three voices (i.e., one *cantus firmus* and two counterpoints) is that the rules no longer necessarily apply between the counterpoints and the *cantus firmus*, but instead of this are mostly applied **between the different parts and the lowest stratum**.

If we go back to the rules for two voices, we see that each of them applied between the single counterpoint and the *cantus firmus*. For example, when it was stated that each interval must be consonant, this referred to the interval between the counterpoint and the *cantus firmus*. On the other hand, in his second part (where he describes the rules for composing in three voices), Fux explains that the rules are not necessarily to be observed between each of the counterpoints and the *cantus firmus*, but rather between "each of the voices and the lowest voice" (i.e. the lowest stratum). Again, if we take the example of the need for consonance between the voices, consonance will be required in the intervals between the notes of any voice and those of the lowest voice (whether or not the latter is the *cantus firmus*). Fux approaches the concept of lowest stratum without ever stating it clearly, mentioning for example that the lowest voice can change (sometimes the bass is the lowest voice, sometimes the tenor, ...), and that at any given moment the lowest voice should be considered. In other words, Fux says that the rules apply between the parts and the lowest stratum.

Generalisation of two-voice counterpoint

One might be tempted to conclude that three-part composition breaks completely with two-part composition, but that would be too hasty a conclusion. Indeed, on closer inspection, the way the rules worked in two-part composition (from counterpoint to *cantus firmus*) is just one particular case of this new vision of things. In two-part composition, too, the rules apply between the parts and the lowest stratum. But of course, since there were only two voices, the lowest stratum was either counterpoint or cantus firmus. This means that when links were established between the upper part and the lowest stratum, links were also established between the counterpoint and the cantus firmus. Considering the rules as being established between the counterpoint and the *cantus firmus* was just a simplification of reality, although it was perfectly correct. We were therefore considering a convenient particular case, and not the general case. Please note that when we talk about "applying constraints from voice A to voice B", it is clear that the constraints are bidirectional and that they also apply from voice B to voice A. What is shown here is rather the philosophy behind the application of these constraints, and the reasons why they were imposed.

The particular case happening when composing with two parts is illustrated in figures 2.4 and 2.5. As we can see on those pseudo-compositions, it does not change anything to apply the constraints between the counterpoints and the *cantus firmus* or between the parts and the lowest stratum.

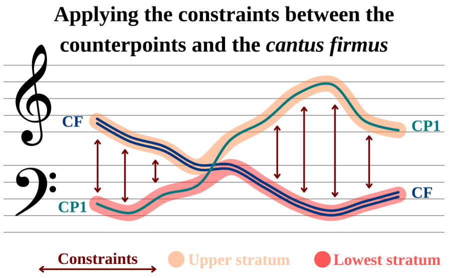

Applying the constraints between the counterpoints and the *cantus firmus*
CF
CP1
CP1
CF
Constraints
Upper stratum
Lowest stratum

Figure 2.4: Applying the constraints from the counterpoint to the *cantus firmus*

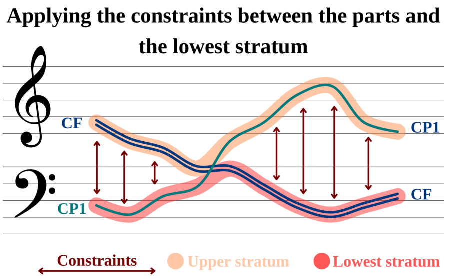

Applying the constraints between the parts and the lowest stratum
CF
CP1
CP1
CF
Constraints
Upper stratum
Lowest stratum

Figure 2.5: Applying the constraints from the parts to the lowest stratum

However, when it comes to generalising the composition of counterpoint for three voices, the same simplification is no longer possible. We are now forced to establish our rules between the parts and the lowest stratum, and no longer between the counterpoints and the *cantus firmus*. In figures 2.6 and 2.7 it becomes clear that establishing the rules between the counterpoints and the *cantus firmus* is really different from applying them between the various parts to the lowest stratum. In these figures, the parts don't intersect and therefore fit perfectly with the strata, so the constraints are always applied to the same counterpoint. This was done for the sake of intelligibility of the graphs, but it is of course possible for the parts to cross and for the "target" of the constraints not always to be the same counterpoint.

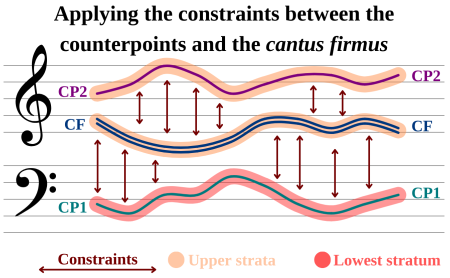

Applying the constraints between the counterpoints and the *cantus firmus*
CP2
CP2
CF
CF
CP1
CP1
Constraints
Upper strata
Lowest stratum

Figure 2.6: Wrong approach: applying the constraints between the counterpoint to the *cantus firmus*.

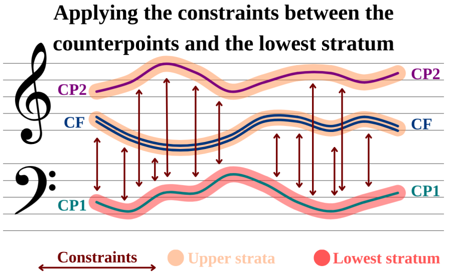

Applying the constraints between the counterpoints and the lowest stratum
CP2
CP2
CF
CF
CP1
CP1
Constraints
Upper strata
Lowest stratum

Figure 2.7: Correct approach: applying the constraints between the parts to the lowest stratum.

It is, of course, possible for the *cantus firmus* to be equal to the lowest stratum all along, in which case nothing changes from the perspective we had when composing for two voices. In this particular case, by applying the rules with respect to the *cantus firmus*, we would find ourselves de facto applying the rules with respect to the lowest stratum (and we would be back to the situation described above, see figures 2.6 and 2.7, only that there is now one more part). It is when the *cantus firmus* pitches are higher up than those of the counterpoints that considering the lowest stratum consideration becomes necessary.

TODO complement the following paragraph to make it clearer

A very important detail, and perhaps the biggest change brought about by this paradigm shift, is since we do not compute all voices with respect to the *cantus firmus* , but with respect to the lowest stratum, it is important that the relationship between the *cantus firmus* and the lowest stratum also get taken into account. This means, when applying the constraints over the parts, we are considering the *cantus firmus* too (as it is a part, like the counterpoints), unless explicitely mentioned otherwise (for example in the variety cost).

A second point to bear in mind, and not the least, is that all this does not mean that *all* the rules are established between the parts and the lowest stratum. Certain rules will continue to apply between the different parts, regardless of whether they are high, low or intermediate.

2.3 Changes induced in the variables

Many changes have been induced as a result of the three part generalisation. In two part composition, it was obvious that the harmonic intervals array was describing the intervals between the *cantus firmus* and the only counterpoint, it was obvious that the motions were those of the only counterpoint, and so it goes for all the variables. When writing a three part composition, we are dealing with many possibilities when speaking about intervals or motions. Intervals between which voices? Motions of which counterpoint? To tackle this, each variable is now related to a voice.

This relationship between a variable and a voice is expressed as function. $X(v)$ represents the variable X of the voice v. The arguments of the function can either be:

- cf - for linking the variable to the *cantus firmus*.
- cp_1 - for linking the variable to the second counterpoint.
- cp_2 - for linking the variable to the third counterpoint.
- a - for linking the variable to the lowest stratum.
- b - for linking the variable to the intermediate stratum.
- c - for linking the variable to the uppermost stratum.

For example, $X(cf)$ refers to the variable X of the *cantus firmus*.

When a variable is not explicitly linked to a voice, it is implied that the relation expressed for it is true for all *parts*. In other words, if the variable X is written without any precision, it means $X \equiv \forall v \in \{cf, cp_1, cp_2\} : X(v)$.

This change applies to all variables, namely:

- **N**(v) - the notes (pitches) of voice v (this variable was written cp in T. Wafflard's thesis)
- **H**(v$_1$, v$_2$) - the harmonic intervals between voice v$_1$ and voice v$_2$. This variable is particular, as it needs to arguments to be meaningful.
- **M**(v) - the melodic intervals of voice v,
- **P**(v) - the motions of voice v,
- **IsCfB**(v) - the boolean array representing whether the cantus firmus is lower than a given voice,
- **IsCons**(v) - to the boolean array representing whether voice v is consonant or not.

It also applies to *some* constants, namely:

- **species**(p) - the species of part p,
- **n**(p) - the number of notes in part p,
- **lb**(p) - the lower bound of the range of part p,
- **ub**(p) - the upper bound of the range of part p,
- $\mathcal{R}$(p) - the range of part p,
- **borrow**(p) - the borrowing scale of part p,
- $\mathcal{N}$(p) - the extended domain of part p,
- $\mathcal{B}$(p) - the set of beats[2] in a measure according to the species of part p,
- b(p) - the number of beats[3] in a measure according to the species of part p,
- d(p) - the duration of a note[4] according to the species of part p.

Please note that the constants cannot be linked to a stratum, and only to a part. Indeed, it would have no sense to speak about the species or about the extended domain of a stratum.

The costs are also affected by the change, except for $\mathcal{C}$ (the cost factors) and τ (the total cost). The latter two remain global and are not duplicated.

To make sure that those notations are clear, here are some examples: the notation $N(a)$ corresponds to the variable representing the notes (pitches) of the lowest stratum, whereas $N(cf)$ are the notes of the *cantus firmus*. The species of the second counterpoint is written $species(c)$. If only N is written, then the equation in which N is located holds true for any possible part (not necessary stratum). That is, the relationship $N < 60$ would mean: the pitches *of all parts* must be lower than a middle C (whose representation is 60 is Open Music).

Note regarding the fourth species

Let's recall that the fourth species behaves in a particular way compared to the other species. Its beats are "shifted": its upbeat should be considered as the downbeat, and its downbeat as the upbeat of the previous measure. This means that in the majority of cases, the equations for the fourth species would have to be rewritten, swapping the 0 and 2 indexes (H[2, j] becomes H[0, j] and H[0, j+1] becomes H[2, j]). To avoid duplicating each of the equations (a first equation if it is not of the fourth species and a second equation if it is of the fourth species) and also to avoid equations that are too complex and difficult to read, it was decided that the index swap would be implicit. Except, of course, when a rule concerns only the fourth species: in this case, the indexes are written as they are.

[2] To make it clearer: for the first species, the only beat in a measure is $\{0\}$, as there is only a note on the first beat. For the second species, the set of beats is $\{0, 2\}$. For the third species, it is: $\{0, 1, 2, 3\}$. For the fourth species: $\{0, 2\}$. And for the fifth species: $\{0, 1, 2, 3\}$.

[3] Thus, it is always equal to the size of the set $\mathcal{B}$(p).

[4] For the first species, it is equal to 1, as each note is a whole note. For the second species, it is $\frac{1}{2}$, and so on. It is always equal to $\frac{1}{b(p)}$.

2.3.1 Added constants

Here are described some added constants, that are useful throughout the whole work.

NumberOfParts `*N-PARTS`
This integer describes as many parts there are in a given problem. At this stage, it can either be equal to two (two-part composition) or to three (three-part composition). It is mainly used in the loops of the program as an end condition (`(dotimes (i *N-PARTS))`)

Cons$_{h\_triad}$ `H_TRIAD_CONS`
Set representing all consonances that belong to the harmonic triad

2.3.2 Added variables

A `is-lowest`
A is an array of boolean variables with a size of m, where each variable indicates whether the corresponding part is the lowest stratum. In other words, $A(v)$ is true if v is the lowest stratum. The notation "A" was chosen as the uppercase of "a", which itself represents the lowest stratum. It is also worth to be noted that only one of the parts can be the lowest stratum at the time. This does not mean that two parts cannot equal the lowest stratum at the same time, it *is* indeed possible that two parts blend in unison in the final chord, and that both pitches are the lowest sounding notes. It means that only one of those two is going to be considered to *be* the lowest stratum (and the other one will be the intermediate stratum). This is needed in order for motions to work well. See 2.1 for the details.

Here is the mathematical definition of the A array:

$$\begin{aligned}
&\forall j \in [0, m-1):\\
&\quad A(cf)[j] = \begin{cases} \top & \text{if } N(cf)[0,j] = N(a)[0,j] \\ \bot & \text{else} \end{cases}\\
&\quad A(cp_1)[j] = \begin{cases} \top & \text{if } (N(cp_1)[0,j] = N(a)[0,j]) \land \neg A(cf)[j] \\ \bot & \text{else} \end{cases}\\
&\quad A(cp_2)[j] = \begin{cases} \top & \text{if } \neg A(cf) \land \neg A(cp_1) \\ \bot & \text{else} \end{cases}
\end{aligned} \tag{2.4}$$

As can be seen in these equations, only the downbeat of each measure is taken into account when computing the A array. The reason for this is that it is the downbeat note that determines which chord will be *the* chord of the measure, and the other beats are just fioritures. Another reason for this is also that it is only going to serve in contexts where the first note of the measure is relevant.

In practice, there is only an `is-not-bass` array in the code (which is then equal to $\neg A$), as it is almost always more useful to know if a part is *not* the lowest stratum than knowing if it is the lowest one.

2.3.3 Modified constants

species `species`

The species constant, which represents the species of a given part, can now take also take the value zero. The value zero is specific to the *cantus firmus* (which can be understood as a simplified first species counterpoint). This constant is more useful in the code than in the mathematical notations.

$$species(v) = 0 \iff v = cf \tag{2.5}$$

2.3.4 Redefined variables with respect to the definitions in T. Wafflard's thesis

As most of the rules now apply between parts and the lowest stratum, the meaning of the variables has been modified to reflect this reality. Throughout this section, when reference is made to the past ("this variable used to be", "this variable keeps the same meaning", ...), this means that reference is made to the previous definition of the variable, which was the one defined in T. Wafflard's work.

Nota bene Please take into consideration that all rules from T. Wafflard's thesis are compatible with the new definitions of the variables, as discussed in 2.2.

N(v) `notes`
N is the array corresponding the pitches of each voice. Its size is s_m. It is the same array as the one named `cp` in T. Wafflard's thesis, and it got renamed to `N` (for notes), for the sake of clarity. As we have now three of those arrays (one for the first counterpoint, one for the second counterpoint, and even one for the *cantus firmus*), it needed a less ambiguous name than the one it had before.

$\mathbf{H}_{\text{(abs)}}(\mathrm{v}_1, \mathrm{v}_2)$ `h-intervals h-intervals-abs h-intervals-to-cf ...`
This variable is an array of size s_m and it represents the harmonic interval between one voice and another. The previous definition of this array was that it represented the harmonic intervals between a given voice and the *cantus firmus*. This definition has been extended to include intervals other than that between the counterpoint and the *cantus firmus*. In order to do so, H now accepts two arguments, and it represents the interval between those two arguments. Thus, $H(v1, v2)[i, j]$ represents the intervals between the ith beat of voice v_1 and the *first* beat of voice v_2. v_1 may be a part and v_2 may be a stratum, as you can calculate harmonic intervals between a part and a stratum. When no v_2 is precised, it is equal to a, by default. In other words, $H(v_1)$ represents the intervals between the voice v_1 and the lowest stratum: $H(v_1) \equiv H(v_1, a)$. This default value for v_2 was chosen since it is the most frequently used, and for a good reason: most relevant harmonic intervals are those between the parts and the lowest stratum.

Here is the generalisation explained above, matching to the current definition of the harmonic intervals array:

$$\begin{aligned}
&\forall v_1, v_2 \in \{cf, cp_1, cp_2, a, b, c\}, \quad \forall i \in \mathcal{B}(v_1), \quad \forall j \in [0, m) : \\
&H_{abs}(v_1, v_2)[i, j] = |N(v_1)[i, j] - N(v_2)[0, j]| \\
&H(v_1 - v_2)[i, j] = H_{abs}[i, j] \text{ mod } 12 \\
&\text{where } H_{abs}[i, j] \in [0, 127], H[i, j] \in [0, 11]
\end{aligned} \tag{2.6}$$

$\mathbf{M}_{\text{brut}}(\mathrm{v})$ `m-intervals-brut`
The variable M represents the melodic intervals of a voice. It can either be evaluated on a part or on a stratum, each of those situations leading to different behaviours.

M(p) (i.e. when related to a part) keeps representing the melodic intervals of the voice v and its way of working remains intact as in T. Wafflard's thesis. M(s) (i.e. when related to a stratum) has it own way of working, that is defined in the next paragraph. We are going to focus specifically on M(a), that is, the melodic intervals of the lowest stratum.

Since strata don't have melodic intervals *per se* (they actually do have melodic intervals, but it doesn't really make sense to consider them), we need to redefine what we mean when speaking about the melodic intervals of a stratum. If it is not clear why strata have no inherent melodic intervals, remember that they are an abstract concept that is used only in mathematical relationships (and respective constraints). People who listen to the music hear the different parts (be they different tessitura, different instruments, ...) and the way these parts interact together in melodic movements and harmonic convergences, rendering a beautiful music, or not. Strata are an abstraction of the harmonic interactions between the parts, and because of this, they are a consequence of the parts: they exist because the parts exist, and not the other way round! And since they are defined according to harmonic principles (as was suggested before, they are successions of vertical alignments), speaking about the proper *melodic* intervals of a stratum makes no sense. One could then conclude that melodic intervals do not apply to strata, and go ahead. Nevertheless, Fux *does* speak about computing the motions between a part and the lowest stratum. And to be able to compute motions, one needs to compare two different melodic intervals. So we need to have a definition for the melodic intervals of a stratum.

In order to do so, one must recall that the lowest stratum is just the assemblage of all the lowest sounding notes in the composition. It is thus very logical to consider the melodic intervals of the lowest stratum as the melodic intervals that lead to all those lowest sounding notes. If the lowest stratum is constituted of the notes [C$_{cp_1}$, E$_{cf}$, G$_{cp_2}$], with C$_{cp_1}$ indicating that the C belongs to the first counterpoint, and and that in the *cantus firmus* the interval that lead to the E was a +0 (i.e. staying on the same note) and in the second counterpoint the interval that lead to the G was a -4 (getting down of two tones), the corresponding melodic intervals array of the lowest stratum would be [+0, -2]. This example has been written again in a more visual way in equation 2.7 to make it easier to understand:

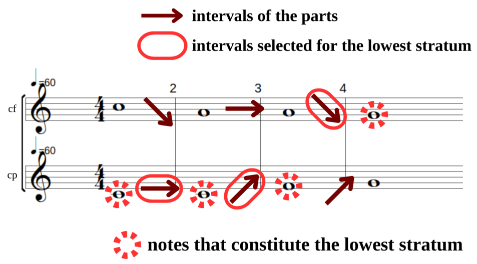
$$\begin{aligned} N(cf) &= [64, \quad \mathbf{64}, \quad 71] & M_{brut}(cf) &= [\mathbf{+0}, \quad +7] \\ N(cp_1) &= [\mathbf{60}, \quad 67, \quad 74] & M_{brut}(cp_2) &= [+7, \quad +7] \\ N(cp_2) &= [72, \quad 71, \quad \mathbf{67}] & M_{brut}(cp_2) &= [-1, \quad \mathbf{-4}] \\ \\ N(a) &= [\mathbf{60}, \quad \mathbf{64}, \quad \mathbf{67}] & M_{brut}(a) &= [\mathbf{+0}, \quad \mathbf{-4}] \end{aligned} \tag{2.7}$$

The formal definition of the melodic intervals of the lowest stratum is hence as follows: the melodic interval in measure j of the lowest stratum is equal to the last melodic interval in measure j of the part that is the lowest stratum in measure $j+1$. Remember that this complex definition is needed in order for the computation of the motions to work fine, and that the motions of the lowest stratum are an abstract notion that serves only in formulas and constraints and *does not intend to represent any concrete motion really happening in the composition, nor does it correspond to the melodic intervals between the pitches of the lowest stratum.*

$$\forall j \in [0, m-2):$$

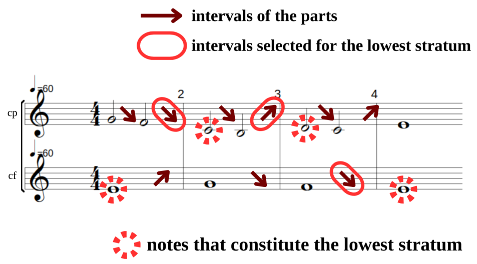
$$M_{brut}(a)[j] = \begin{cases} M_{brut}(cf)[0][j] & \text{if } A(cf)[j+1] \\ M_{brut}(cp_1)[\max(\mathcal{B}(cp_1))][j] & \text{if } A(cp_1)[j+1] \\ M_{brut}(cp_2)[\max(\mathcal{B}(cp_1))][j] & \text{if } A(cp_2)[j+1] \end{cases} \tag{2.8}$$

It might be helpful to have a look at figures 2.8 and 2.9 to understand better how the melodic intervals arrays for the lowest stratum.



intervals of the parts
intervals selected for the lowest stratum
cf
cp
60
60
2
3
4
notes that constitute the lowest stratum

Figure 2.8: Understanding the melodic intervals of the lowest stratum with a first species counterpoint



intervals of the parts
intervals selected for the lowest stratum
cp
cf
60
60
2
3
4
notes that constitute the lowest stratum

Figure 2.9: Understanding the melodic intervals of the lowest stratum with a second species counterpoint

P(p) motions

The motions array was redefined to compute the motions of a voice v with respect to the lowest stratum, instead of computing them with respect to the *cantus firmus*, since Fux considers that the motions should be considered between each voice and the lowest voice. Dealing with the melodic intervals of the lowest stratum is not a problem anymore, since we defined what it exactly means in the section about melodic intervals. However, a problem arises when computing the motions of the part that is also the lowest stratum in some measures. When this happens, we end up calculating motions between a part and itself. Any part is inevitably moving in direct motion with itself, and this situation leads to only direct motions being calculated. This becomes problematic when considering costs (it is bad to have direct motions, but it obviously should not be bad to be the lowest stratum), and when considering some constraints. To tackle this problem, the motions of a part are now equal to -1 when the part is also the lowest stratum (which is denoted A(p), see section 2.3.2).

$$\forall p \in \{cf, cp_1, cp_2\}, \quad \forall x \in \{1,2\}, \quad \forall i \in B, \quad \forall j \in [0, m-1), \quad x := b - i$$

$$motion(p)[i,j] = \begin{cases} 0 & \wedge(M^x_{brut}(p)[i,j] > 0 > M(a)_{brut}[j]) \\ & \qquad\qquad \vee(M^x_{brut}(p)[i,j] < 0 < M(a)_{brut}[j]) \\ 1 & M^x_{brut}(p)[i,j] = 0 \oplus M(a)_{brut}[j] = 0 \\ 2 & (M^x_{brut}(p)[i,j] > 0 \wedge M(a)_{brut}[j] > 0) \\ & \qquad\qquad \vee(M^x_{brut}(p)[i,j] < 0 \wedge M(a)_{brut}[j] < 0) \end{cases}$$

$$P(p)[i,j] = \begin{cases} -1 & \text{if } A(p)[j] \\ motion(p)[i,j] & \text{if } \neg A(p)[j] \end{cases} \tag{2.9}$$

Chapter 3

Formalising Fux rules for three-part composition

This section will be concerned with extracting all the rules that Fux mentions in his work and making sure that they are unambiguous. It will consist of seven subsections: first, a section for the implicit rules (the rules that Fux doesn't mention, but are present in all his examples), then a section for each species, and finally one that considers the interaction between different species. There is no section on rules for all species, yet there *are* rules that apply to all species. In fact, all the rules mentioned in the first species section also apply to the other species. This isn't something that Fux mentions explicitly, but it's obvious when you look at how he teaches these rules and how he composes them. Therefore, the rules for all species are in the first species section. The reason these rules are in the first type section is because Fux himself mentions these rules in his chapter on first type counterpoints.

Each species subsection is divided into two parts: the first part is about setting up Fux's rules and discussing them. The second part is about translating the rules into formal logic.

Please note that all the rules from T. Wafflard's thesis still apply to three-part compositions. The numbering of the rules in this work is the same as in T. Wafflard's work. If a rule is defined with the same number as an existing rule for two-part compositions, this means that the corresponding rule from T. Wafflard's work does not apply to three-part compositions, and that the new rule should be used instead. To make this clearer, there is a green dot (●) next to these rules.

3.1 Implicit rules

3.1.1 Formalisation in English

1.H2 and 1.H3 ● *First and last notes have not to be perfect consonances anymore.*

Fux doesn't state this in his text, but in many of his examples, we see that when he composes with three voices (and more), the first and last harmonic intervals between the parts and the lowest stratum are not necessary a perfect consonance anymore.

G8 *The last chord must be composed only of the notes of the harmonic triad.*

Again, this isn't stated explicitly, but we see that all of his examples end with a chord containing exclusively the notes of the harmonic triad.

G9 *The last chord must have the same fundamental as the one of the scale used throughout the composition.*

This rule emanates from an observation of Fux's examples throughout the chapter. The last chord of all his compositions always have the same fundamental as the fundamental of the scale used throughout the composition. When the *cantus firmus* is the lowest stratum, this is not a problem, as the *cantus firmi* always end with the fundamental note of the scale. But when not, it has to be imposed by a constraint, or we may end up with surprising results.

3.1.2 Formalisation into constraints

1.H2 and 1.H3 ● *First and last notes have not to be perfect consonances anymore.*

There is no constraint associated with this rule, as it is a relaxation of a rule from the two-part composition rule set.

G8 *The last chord must be composed only of the notes of the harmonic triad.*

$$\forall s \in \{b, c\}\colon H(s)[0, m-1] \in Cons_{h\_triad} \tag{3.1}$$

G9 *The last chord must have the same fundamental as the one of the scale used throughout the composition.*

Since the fundamental of the scale is defined by being the first note of the *cantus firmus*, we impose that the last note of the lowest stratum must be equal to the first one of the *cantus firmus* (taking the modulos into account).

$$N(a)[0, m-1] \mod 12 = N(cf)[0, 0] \mod 12 \tag{3.2}$$

3.2 First species

Concerning the rules applying to all species (those marked with a red dot): when a constraint described in this section applies to all species, it means that it applies to the first beat of every species, unless mentioned otherwise, and excepted for the fourth species, where it applies to the third beat.

3.2.1 Formalisation into English

Structural constraints

1.S1 *All notes are whole notes.*

> "This species consists of three whole notes in each instance." Mann [19, p.71]

This pretty straightforward rule is the very definition of the first species. It adds nothing in comparison with the rules for the two part comparison. It is hence already implemented by the first species for two voices and does not need any consideration.

Harmonic constraints

1.H1 ● *All notes on the downbeat are consonant with the notes of the lowest stratum.*

> "This species consists of three notes, the upper two being consonant with the lowest." Mann [19, p.71]

This rule is an update of previous **1.H1** (that previously was saying that *all* intervals must be consonants). Fux states that the upper voices and the lowest one are consonant, and not all voices together.

1.H8 *The harmonic triad should be used as much as possible.*

> "The harmonic triad should be employed in every measure if there is no special reason against it." Mann [19, p.71]

As a footnote states it [19, footnote, p.71], Fux refers to the "harmonic triad" as being a chord in this position: 1-3-5 (contrary to what is today understood as a harmonic triad). The rule says it is not obligated, but it is preferred, to use the 1-3-5 chord, considering that 1 is the lowest voice.

1.H9 *One might use sixths or octaves.*

> "Occasionally, one uses a consonance not properly belonging to the triad, namely, a sixth or an octave." Mann [19, p.72]

Here, Fux explains that when it is not possible to have a harmonic triad, you can use sixths or octaves instead. Remember that the sixths or the octaves are calculated from the lowest stratum. Since the rule **1.H1** obligates the use of a perfect consonance (i.e. a third, a fifth, a sixth or an octave), when the harmonic triad cannot be used, it is already naturally replaced by a third or a sixth, because no other intervals are allowed. It is thus not a new rule but a restatement of rule **1.H1**.

1.H10 *Tenths are prohibited in the last chord.*

> "One feels that the degree of perfection and repose which is required of the final chord does not become sufficiently positive with this imperfect consonance [(speaking about a tenth)]." Mann [19, p.77]

When Fux says this, he takes a tenth as an example, but it here understood that the final chord cannot include a tenth (third + octave), nor an eight-teenth (third + two octaves), etc. Nevertheless, "simple" third are considered completely valid.

1.H11 *Octaves should be used over unisons.*

> "Unison is less harmonious than the octave." Mann [19, p.79]

This rule is not bringing anything new, as there is already a rule stating that two parts cannot blend in unison. If no unison is possible, then the octaves will always be preferred over the unison.

1.H12 *Last chord cannot include a minor third.*

> "The minor third is not capable of giving a sense of conclusion." Mann [19, p.80]

Fux later states that minor modes should not include a third altogether, but that sometimes it is impossible to do without it, so the major third *is* allowed in minor modes.

Melodic constraints

1.M3 *Steps are preferred to skips.*

> "[Each part] follows the natural order closely." Mann [19, p.73]

After having said this, Fux complements his explication by saying the counterpoints should be "moving gracefully, stepwise without any skip". This is clearly a preference, and has already been covered when implementing the first species for two voices. It can thus be ignored in the scope of this thesis.

1.M4 *The notes of each part should be as diverse as possible.*

> "[Each part] follows the principle of variety." Mann [19, p.73]

Fux never defines clearly what he means by "principle of variety". Nevertheless, the examples he provides are of a great help as he corrects his student for not following the principle by augmenting the variety of different notes in a single voice. This means, the principle of variety can be understood as having as many different notes in a single voice.

1.M5 *Each part should stay in its voice range.*

> "One should not exceed the limits of the five lines without grave necessity." Mann [19, p.79]

Fux says here that each part should stay on the musical staff (Fux's "five lines"). Since every staff can be represented differently according to the clef that is used, this rule could be always true. Obviously, Fux meant the staff corresponding to the voice range (treble clef for a soprano, bass clef for a bass, ...).

This is actually something that is already handled when declaring the $N(p)$ arrays, as they are declared with an upper and lower bound ($ub(p)$ and $lb(p)$), corresponding to their voice range.

1.M6 *Melodic intervals cannot be greater or equal to a sixth.*

> "The skip of a major sixth is prohibited." Mann [19, p.79]

This rule is only a restatement of rule 1.M2, saying that melodic intervals cannot exceed a minor sixth interval.

Motion constraints

1.P1 ● *Reaching a perfect consonance by direct motion should be avoided.*

> "[Reaching] perfect consonance by direct motion [is allowed if] there is no other possibility." Mann [19, p.77]

This is a new rule as it only applies to three-part composition, but it cancels an already existing rule that used to be applied in two-part composition. The same rule in two-part composition states that it is prohibited to reach a perfect consonance using a direct motion. In three-part composition, this is not prohibited anymore, as not doing it is sometimes impossible, and you may thus derogate from this rule.

1.P4 *Successive perfect consonances are prohibited.*

> "The necessity of avoiding the succession of two perfect consonances [...]." Mann [19, p.72]

Fux here implies that there should be no two successive perfect consonances. He does not specify whether this rule applies to all three parts at once (i.e. if there was a consonance at bar X between part 1 and part 2, there cannot be one between part 2 and 3 at bar X+1), or whether it applies to each pair of parts separately. That said, in his example (Fig. 91 of the English version), we can clearly see that there is perfect consonance in every bar (parts 1-3, then 1-2, then 1-3, then 2-3, then 1-2). From this we can deduce that *for each pair of parts* it is forbidden for two perfect consonances to follow each other.

1.P5 *Each part starts distant from the lowest stratum.*

> "To allow enough space for the voices to move toward each other by contrary motion, the upper voices begin distant from the bass." Mann [19, p.75]

This preference cannot be made clearer: the voices start distant from the lowest stratum.

1.P6 *It is prohibited that all parts move in the same direction.*

> "All voices ascend[ing] [is] a progression which can hardly be managed without awkwardness resulting." Mann [19, p.76]

What Fux says here is just that the three parts cannot be moving in the same direction.

1.P7 *It is prohibited to use successive ascending sixths on a direct upwards motion.*

> "Ascending sixths on the downbeat sound harsh." Mann [19, p.77]

This rule is pretty straightforward and states that if an interval is a sixth, the next one cannot be one.

3.2.2 Formalisation into constraints

Structural constraints

1.S1 *All notes are whole notes.*
This rule needs no special constraint, since it is the very definition of the first species to consist only of whole notes.

Harmonic constraints

1.H1 ● *All notes on the downbeat are consonant with the notes of the lowest stratum.*
The new definition of variable H already captures the change in the rule. This means that the equation of rule **1.H1** stays the same.

1.H8 *The harmonic triad should be used as much as possible.*
As this rule is actually a preference and not a mandatory rule, it has been implemented as a cost. If the harmonic triad is used, then the cost is 0. Else, it is 1.

$$\begin{aligned}&\forall j \in [0, m-1):\\&((H(b)[0,j] \neq 3) \wedge (H(b)[0,j] \neq 4)) \vee (H(c)[0,j] \neq 7))\\&\iff cost_{prefer-harmonic-triad}[j] = 1\end{aligned} \tag{3.3}$$

1.H10 *Tenths are prohibited in the last chord.*

$$H_{brut}[0, m-1] > 12 \implies H[0, m-1] \notin \{3, 4\} \tag{3.4}$$

If the interval is bigger than an octave, then you cannot use thirds anymore.

1.H12 *Last chord cannot include a minor third.*

$$H[0, m-1] \neq 3 \tag{3.5}$$

Melodic constraints

1.M3 *Steps are preferred to skips.*
As discussed in **1.M3**, there is no constraint to add as it already exists.

1.M4 *The notes of each part should be as diverse as possible.*
As it is not explained either if this has to be true for the whole partition or only for two following notes, it has been chosen as an arbitrary seven successive notes to apply the rule on. This means that the solution is penalized if a note in measure X was already present in measures [X-3, X+3]. This amount was chosen because it represents the number of flat notes that exist, pushing for the solver to find a solution that contain all of them.

$$\begin{aligned}
&\forall p \in \{cp_1, cp_2\}, \quad \forall j \in [0, m-1), \quad \forall k \in [j+1, \text{min}(j+3, m-1)] : \\
&N(p)[0, j] = N(p)[0, j+k] \iff cost_{variety}[j + m * k] = 1
\end{aligned} \tag{3.6}$$

Motion constraints

1.P1 ● *Reaching a perfect consonance by direct motion should be avoided.*
Since there is no way in constraint programming to implement a rule that must be obeyed only if possible other than by using a cost, the initial constraint was rewritten to a new one.

$$\begin{aligned}
&\forall j \in [0, m-2) : \\
&P[0, j] = 2 \land H[0, j+1] \in Cons_p \\
&\iff cost_{\text{direct\_move\_to\_p\_cons}}[j] = 8
\end{aligned} \tag{3.7}$$

Remember: $P[0, j] = 2$ means that the motion is direct.

1.P4 *Successive perfect consonances are prohibited.*

$$\begin{aligned}
&\forall v_1, v_2 \in \{cf, cp_1, cp_2\}, \quad v_1 \neq v_2, \\
&\forall j \in [0, m-2) \colon H(v_1, v_2)[0, j] \in Cons_p \\
&\implies H(v_1, v_2)[0, j+1] \notin Cons_p
\end{aligned} \tag{3.8}$$

The meaning of this equation is that a harmonic interval and the following one cannot be consonant at the same time.

1.P5 *Each part starts distant from the lowest stratum.*
This is not a strict rule but an indication to make easier for the composer to have contrary motions. Given that is not an obligation, nor a preference, it was simply added as a heuristic for the solver. This is discussed in section 4.1.1.

1.P6 *It is prohibited that all parts move in the same direction.*

To prevent this, we need only look at the motions between the parts and the lowest stratum. If one of their motions is contrary, then it is guaranteed that the three voices will not go in the same direction (because at least one is contrary). The same applies if one of the motions is oblique. The problem arises when all the movements are direct, because this would mean that the three voices are going in the same direction. So it was forbidden to have all motions direct at the same time.

$$\begin{aligned}&\forall j \in [0, m-2):\\ &\bigvee_{p \in \{cf, cp_1, cp_2\}} M(p)[0, j] \neq 2\end{aligned} \tag{3.9}$$

1.P7 *It is prohibited to use successive ascending sixths on a direct upwards motion.* Either the harmonic interval is not a sixth in any of both positions, or one of them is not moving up.

$$\begin{aligned}&\forall j \in [1, m-1), \quad \forall v_1, v_2 \in \{cf, cp_1, cp_2\} \text{ where } v_1 \neq v_2, \quad \text{sixth} := \{8, 9\}:\\ &(H(v_1, v_2)[0, j-1] \notin \text{sixth}) \vee (H(v_1, v_2)[0, j] \notin \text{sixth})\\ &\vee M(v_1)[0, j] > 0 \vee M(v_2)[0, j] > 0\end{aligned} \tag{3.10}$$

3.3 Second species

3.3.1 Formalisation in English

Harmonic constraints

2.H4 *Major thirds are now allowed in the last chord.*

> "A major third [may] appear in the last chord." Mann [19, p.87]

This is a consequence of now using three voices instead of two. Fux makes explicit two implicit rules we had already defined (**1.H2 and 1.H3** and **G8**). It has thus already been implemented in the first species for two voices.

2.H5 *The half notes must be coherent with respect to the whole notes.*

> "The half notes are always concordant with the two whole notes." Mann [19, p.88]

One might ask what Fux meant when he wrote "concordant". Did he mean to say "consonant"? Our take on the question is that he meant that the half notes are written whilst taking the whole notes into account. This interpretation is aligned with the French translation, and even with the Latin original. In other words, Fux just says "there are constraints on the half notes". It is thus not a rule *per se*.

Melodic constraints

2.M2 ● *It is allowed to ligature the fourth-to-last with the third-to-last or to ligature the third-to-last with the second-to-last.*

> "Ligatures have no place in this species [except] in the final cadence." Mann [19, p.87]

Fux explains that in some cases, you have no other option than ligaturing the fourth-to-last and the third-to-last notes. The reasons he gives for this are all part of the previous mentioned rules (no successive perfect consonances, no unison, ...).

Later on, he also says that the third-to-last and the second-to-last notes can be ligatured (hence producing a whole note).

> "A whole note may occasionally be used in the next to last measure." Mann [19, p.93]

He says that in the chapter about third species, but it seems that this applies even in cases where the second species is not used in combination with the third (see figures 134, 173 and 174 of the English version).

He doesn't state clearly if the three of them can get ligatured, but it seems quite obvious that this is not allowed, as it would introduce a lot of redundancy in the composition. It is hence decided that the rule is: a ligature may happen in one case or in the other, but not in both.

We thus have to relax the already existing constraint from two-part composition stating that no two consecutive notes can be the same, to accept it in some cases.

Motion constraints

2.P3 *Successive fifths on the downbeat are only allowed when they are separated by a third on the upbeat.*

> "A half note may, for the sake of the harmonic triad, occasionally make a succession of two parallel fifth acceptable - which can be effected by the skip of a third." Mann [19, p.86]

Fux didn't speak about prohibiting two parallel (i.e. consecutive) fifths in the second species for two voices. That being said, it is indeed prohibited in three parts composition as you cannot have two successive perfect consonances (see constraint **1.P4**). We thus have to relax constraint **1.P4** in order to accept two successive consonances, when the two successive fifths flank a third.

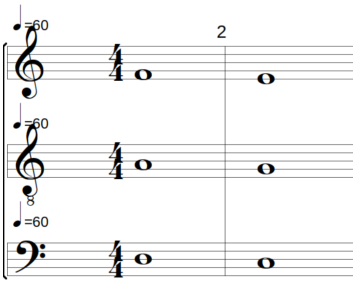


Figure 3.1: Successive fifths - prohibited

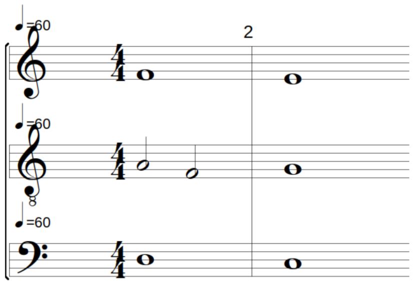


Figure 3.2: Successive fifths separated by a third - valid

3.3.2 Formalisation into constraints

Harmonic constraints

2.H4 *Major thirds are now allowed in the last chord.*

No need to add a new constraint as this rule is already covered by rules **1.H2 and 1.H3** and **1.H8**. As discussed in 3.3.1, no new harmonic rules were defined by Fux for the second species.

2.H5 *The half notes must be coherent with respect to the whole notes.*
No need to add a new constraint as this is not an actual rule.

Melodic constraints

2.M2 ● *It is allowed to ligature the fourth-to-last with the third-to-last or to ligature the third-to-last with the second-to-last.*

This is a relaxation of the two-voice rule **2.M2** "two consecutive notes cannot be the same".

The reason why this rule has been implemented a constraint relaxation instead than as a cost is because Fux does not say that ligaturing is bad, he just presents it as a new option offered by the three-part composition.

$$\begin{aligned}&\forall j \in [1, m-1), \quad j \neq m-2 : \\ &((N[2, j-1] \neq N[0, j]) \wedge (N[0, j] \neq \wedge N[2, j])) \\ &\wedge \\ &((N[2, m-3] \neq N[0, m-2]) \vee (N[0, m-2] \neq N[2, m-2]))\end{aligned} \tag{3.11}$$

The first line prohibits the ligatures except where they are allowed, and the second lines states that only one ligature can occur.

Motion constraints

2.P3 *Successive fifths on the downbeat are only allowed when they are separated by a third on the upbeat.*

In the following equation, only p_1 needs to be a second species counterpoint, p_2 can be of any species.

$$\begin{aligned}&\forall p_1, p_2 \in \{cf, cp_1, cp_2\} \text{ where } p_1 \neq p_2, \quad \forall j \in [0, m-2) : \\ &(H(p_1, p_2)[0, j] = 7) \wedge (H(p_1, p_2)[0, j+1] = 7) \\ &\iff (H(p_1, p_2)[2, j] = 3) \vee (H(p_1, p_2)[2, j] = 4)\end{aligned} \tag{3.12}$$

3.4 Third species

3.4.1 Formalisation in English

Harmonic constraints

3.H5 *The quarter notes must be coherent with respect to the whole notes.*

> "The quarters have to concur with the whole notes of the other voices."
> Mann [19, p.91]

When using the word "concur", Fux more than probably means "are put in relation", and not "are consonant". As for rule **2.H5** in the second species, where the half notes had to be *concordant* with the *cantus firmus*. It is not a rule by itself, Fux is only annunciating that some rules should be followed (i.e. the other constraints).

3.H6 *If the harmonic triad could not be used on the downbeat, it should be used on the second or third beat.*

> "Take care whenever you cannot use the harmonic triad on the first quarter occurring on the upbeat, to use it on the second or third quarters." Mann [19, p.91]

This rule is quite clear and speaks for itself.

Melodic constraints

Fux introduces no new melodic constraints for the third species.

Motion constraints

Fux introduces no new motion constraints for the third species.

3.4.2 Formalisation into constraints

Harmonic constraints

3.H5 *The quarter notes must be coherent with respect to the whole notes.* As has been discussed in the English section, there is no constraint to add for this rule, which isn't really a rule.

3.H6 *If the harmonic triad could not be used on the downbeat, it should be used on the second or third beat.*

This rule is quite clear and speaks for itself. Since this is not a strict rule but an advice, it was treated as a cost.

$$\begin{aligned} &\forall j \in [0, m-1): \\ &(H[1,j] \notin Cons_{h\_triad}) \land (H[2,j] \notin Cons_{h\_triad}) \\ &\iff cost_{harmonic-triad-3rd-species}[j] = 1 \end{aligned} \tag{3.13}$$

3.5 Fourth species

3.5.1 Formalisation in English

Structural constraints

4.S1 *The fourth species is staggered by two beats.*

> "The ligature is nothing but a delaying of the note following." Mann [19, p.95]

Fux here insists on a fact that we have already discussed in 2.3. The fourth species behaves as if its upbeat were the downbeat and its downbeat were the upbeat of the previous measure.

4.S2 *All parts can become the lowest stratum somewhere in the composition.*

> "The tenor takes the place of the bass in the first measure - a thing that not only the tenor may do, but also the alto and even the soprano." Mann [19, p.100]

Fux speaks here about our concept of strata. The tenor can become the lowest stratum, just like the alto and the soprano may do. This is a fundamental concept of the generalization of Fux counterpoint to three voices, and has already been extensively discussed before (see section 2.1).

Harmonic constraints

4.H5 *Imperfect consonances are preferred over fifth intervals, which in turn are preferred over octaves.*

> "The fifth is a perfect consonance, the octave a more perfect one, and the unison the most perfect of all; and the more perfect a consonance, the less harmony it has." Mann [19, p.97]

This rule is as clear as it gets.

Melodic constraints

Fux introduces no new melodic constraints for the third species.

Motion constraints

4.P3 *Successive fifths are allowed when using ligatures.*

> "[It would be impossible to remove] the ligatures because of another consideration, the immediate succession of several fifths." Mann [19, p.95]

By saying that it exists a rule that prohibits the succession of fifths (which is actually just a particular case of rule **1.P4**, stating that you cannot have two successive perfect consonances) when there is no ligature, Fux is telling us in an indirect way that this rule is not applicable when there are ligatures. He further complements by saying "there is great power in ligatures - the ability to avoid or improve incorrect passages".

4.P4 *Resolving to a fifth is preferred over resolving to an octave.*

> "A dissonance that resolves to a fifth is more acceptable than a dissonance that resolves to an octave." Mann [19, p.98]

This rule could not be clearer.

4.P5 *Stationary movement in the bass implies dissonance in the fourth species part.*

> "If I said that the first note of the ligature must always be consonant, that applies only to the instances in which the lower voice moves from bar to bar, but not the instances in which the bass remains on a pedal point, that is, in the same position. In such a case a ligature involving only dissonances is not only correct but even very beautiful." Mann [19, p.98]

The rule evoked here cancels the previous rule **4.H1W** that stated that all notes should be consonant. From now on, if the lowest stratum has a stationary movement, the corresponding delayed note in the fourth species must be a dissonance, instead of a consonance.

4.P6 *A note provoking a hidden fifth gets replaced by a rest.*

> "Here a hidden succession of fifths occurs, which is easily perceptible to the ear and should be avoided in three part composition. This may be managed by using a rest." Mann [19, p.98]

Fux's uses the term 'hidden fifth' without any prior definition. It is therefore difficult to be sure of what he meant, since the traditional terms for such progressions are as vague and variable as the traditional rules that govern them. Nevertheless, most people seem to agree on the following definition of a 'hidden interval': a hidden fifth or hidden octave is when you approach a perfect fifth or perfect octave by direct motion. [29, p.31]. Looking closely at figures 137, 151 and 152 of the English version of *Gradus ad Parnassum*, this definition is consistent with Fux's interpretation.

The point of the rule then is: if a solution leads to a hidden fifth, then the note that provokes the fifth is replaced by a rest. This rule is an *a posteriori* rule: it applies after the solution has been found. The current rule thus complements the rule **4.P3** (about successive fifths in fourth species) and the rule **1.P1** (about direct moves to perfect consonances) without changing them.

See figures 3.3 and 3.4:

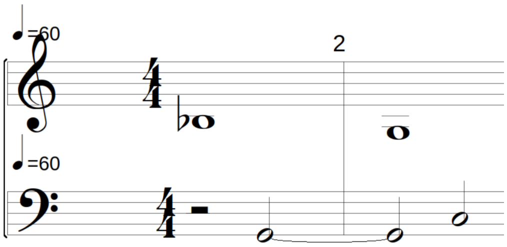


Figure 3.3: Invalid solution featuring hidden fifths

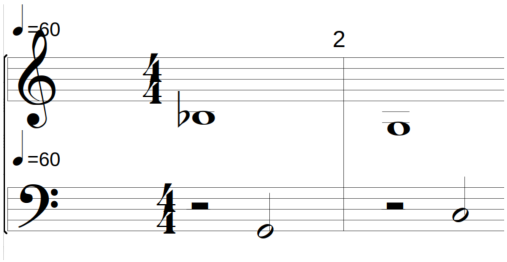


Figure 3.4: Valid solution replacing the hidden fifth by a rest.

3.5.2 Formalisation into constraints

Structural constraints

4.S1 *The fourth species is staggered by two beats.* There is no constraint to add since this rule is the very definition of the fourth species.

4.S2 *All parts can become the lowest stratum somewhere in the composition.* Again, there is no constraint to add since this rule is already covered by the concept of lowest stratum.

Harmonic constraints

4.H5 *Imperfect consonances are preferred over fifth intervals, which in turn are preferred over octaves.*

This rule is almost covered by the existing costs, as a perfect consonance has a cost of 1, where an imperfect consonance has a cost of 0. But Fux says not only that imperfect consonance should be preferred over perfect ones, he says that fifths should be preferred over octaves. This precision in the rule (fifth is better than octave) could be solved by either putting a cost of 2 to octaves and a cost of 1 to fifths, or to put the cost for fifth before the cost for octaves in the lexicographical array of costs, but this is discussed in the parts about costs (see 4.2).

Motion constraints

4.P3 *Successive fifths are allowed when using ligatures.*

The point of this rule is that Fux introduces an exception to **1.P4**: successive fifths are allowed in the fourth species, thanks to the delaying it induces.

We shall then amend rule **1.P4** to only prohibit successive octaves, and rewrite it as:

$$\begin{aligned}
&\forall p_1, p_2 \in \{cf, cp_1, cp_2\} \text{where} p_1 \neq p_2, \quad \forall n \in \{0, 1\} \\
&i_n := \begin{cases} 0 & \text{if } species(p_n) \neq 4 \\ 2 & \text{if } species(p_n) = 4 \end{cases} \\
&\forall j \in [0, m-2) \colon H(p_1, p_2)[i_n, j] = 0 \implies H(p_1, p_2)[i_n, j+1] \neq 0
\end{aligned} \tag{3.14}$$

4.P4 *Resolving to a fifth is preferred over resolving to an octave.*

This is already covered by the rule **4.H5** (prefer fifths over octaves), since preferring fifths over octaves in *all* cases implies preferring to resolve to a fifth rather than to an octave.

4.P5 *Stationary movement in the bass implies dissonance in the fourth species part.*

$$\begin{aligned}
&\forall j \in [0, m-1) : \\
&M(a)[0, j] \neq 0 \iff H[2, j] \in Cons \\
&M(a)[0, j] = 0 \iff H[2, j] \in Dis
\end{aligned} \tag{3.15}$$

4.P6 *A note provoking a hidden fifth gets replaced by a rest.*

$$\begin{aligned}
&\forall j \in [1, m-1) : \\
&H[0, j] = 7 \land P[0, j] = 2 \iff N(0, j-1) = \emptyset
\end{aligned} \tag{3.16}$$

3.6 Fifth species

No additional rules (be they harmonic rules, melodic rules or motion rules) were observed by Fux for the fifth species. However, in order to have some melodic and rhythmic variety when composing with two fifth counterpoints, it was decided to impose a rule that Fux never mentioned: knowing that the fifth is just all the others combined, not more than 50% of the composition of the parts can be made by using the same species for the same measure.

To better understand this rule, please remember that the fifth species works by combining all the other species, and that it assigns a species to each beat (i.e. each of the beats of a fifth species counterpoint actually belongs to a given species). If we want to force the solver to find different solutions when searching for a solution of two fifth species counterpoints, a good trick is to enforce that as many notes as possible should belong to different species.

The following equation holds true if both counterpoints are fifth species counterpoints.

$$\sum_{i=0}^{3} \sum_{j=0}^{m-1} (S(cp_1)[i, j] = S(cp_2)[i, j]) < \frac{s_m}{2} \tag{3.17}$$

3.7 Writing a three-part composition using various species

In *Gradus ad Parnassum*, Fux almost always writes his counterpoints using a combination of the first species and another species, i.e. either a counterpoint of the first species and another of the first species, a counterpoint of the first species and another of the second species, and so on, up to a counterpoint of the first species and another of the fifth species. However, he sometimes creates other combinations, either with two of the same species (for example, two of the fifth species) or with two different species (for example, the second and the third).

When creating combinations of different species, Fux doesn't give any specific rules for combining them. It seems that the different species interact with each other as they would do with the first species, taking into account only the first beat of the measure. For example, if we compute a first counterpoint belonging to the third species and a second counterpoint belonging to the second species, the rules that apply between the two counterpoints will be set between all the beats of the first counterpoint and the first beat of the second counterpoint and between all the beats of the second counterpoint and the first beat of the first counterpoint. This corresponds to what we would have done if we had applied the rules between a counterpoint and the *cantus firmus*, where the rules apply between all beats of the *cantus firmus*and the first beat of the counterpoint.

Chapter 4

Searching for the best existing solution

Three parts composition brings in way more possibilities than two parts composition. But more possibilities also mean an increased computation complexity. The search space has been extended a lot by adding a whole new set of variables, and the time taken for a solution to be found might to be too elevated if one does not think about optimizing the search. In addition to that, adding a third voice to a composition is not bringing many new constraints (which would help discarding some potential solutions faster), but instead comes with many preferences, which in constraint programming, are translated to costs.

In addition, we need to find a way of arranging the costs in a way that comes as close as possible to what Fux was trying to express in his book. There are several ways to do this, which we will go through and discuss.

This chapter will begin by explaining the search algorithm that is used to find a solution, continue by discussing the different ways of considering costs, and end by analysing the results that these different perspectives produce.

4.1 Using Branch-And-Bound as a search algorithm

To cope with the increased complexity brought about by the three-part composition, it was decided to switch from the Depth First Search algorithm (used in T. Wafflard's thesis) to a more efficient Branch and Bound (BAB). This allows us to handle costs properly and to find faster solutions. Moreover, the BAB algorithm can also produce non-optimal results, which is very valuable since finding the best overall solution can be time-consuming. When starting the search for a solution, it is now possible to ask for the next solution (i.e. a better solution than the one found previously, and if none was found previously, then just any valid solution), or for the best solution. In the latter case, the solver will continue to search until it finds the best solution or until it is stopped, returning a better solution each time it finds one.

4.1.1 Heuristics

When it comes to finding a solution, we obviously need some heuristics to guide the search, as there are so many different possibilities for a three-part composition. To know which heuristics to use, simply think about the most important variable to fix first. In case it is not clear enough, the key to writing counterpoint for many voices is to know what the bass is doing. This is true whether the composer is a human or a constraint solver. So the first heuristic follows naturally, and it is: branch on the lowest stratum array, take the highest constrained variable yet, and try with its lowest possible value.

Rule **1.P5** states that the voices should start distant, and as was proposed in the section about the rules, this should translate into a heuristic. However, when we tried to implement this

4.1.2 Time to find a solution

Many factors come into play in determining how long it will take the solver to find a valid solution. These factors are mainly: the species of counterpoints, the spacing between the voice ranges of the counterpoints, and the key of the *cantus firmus*.

In general, the solver is able to find a valid solution fairly quickly (in the order of a second). In some cases, however, it cannot find a valid solution (and so searches endlessly). These cases are generally related to the three factors mentioned above, and we will discuss them a little. Each of these factors makes the search a little more difficult. A single factor may have no effect on the search time, and it is sometimes at the intersection of the factors that an effect is discovered.

- **Species of the counterpoint** – The more complex the species of the counterpoints (think of the 3rd and 5th species, for example), the longer it will probably take the solver to find a solution. The reason is quite obvious: the more complex the species, the more variables the solver has to play with and the greater the range of possibilities.

- **Distance between the voice ranges** – The closer the ranges of the voices, the greater the risk that the solver will take a long time to find a solution. For example, finding a solution by giving the two counterpoints the same voice range as the *cantus firmus* is more time-consuming than selecting distant voice ranges. This is because the voices cannot form a unison, and the possibilities for each voice are therefore smaller when their range is close together.

- **Key of the *cantus firmus*** – The solver finds it much easier to find a counterpoint when the key is C, probably because counterpoints are played with as many natural notes as possible (no flats or sharps), and in the case of C this corresponds to the Ionian mode (also known as the C mode). When using the key of X, the further away the X mode is from the C mode, the more difficult it will be for the counterpoint. For example, take a composition in E: the E (Phrygian) mode is far from the C (Ionian) mode, whereas the G (Myxolydian) mode is quite close. The solver will therefore find it slightly more difficult to find a solution in E than in G, which in turn will be more complicated than in C.

There are also cases where the exact combination of two given vocal ranges for two given species with a given mode does not give a solution, but by changing the vocal range a little the solver finds a solution immediately, quite surprisingly. It is still unclear why some given combinations do not produce solutions. Our best guess is that there are some combinations of parameters for which the solver has difficulty finding a solution, given the very large number of constraints that apply to the voices. However, this doesn't happen very often.

It is worth noting that the solver's greatest difficulty (in all cases) is finding a valid solution. Once a valid solution has been found, the solver quickly finds a whole series of solutions, each one better than the previous one (until, of course, it is difficult to find the best solution).

4.2 Designing the costs to be as faithful as possible to *Gradus ad Parnassum*

Knowing that we are looking for the solution whose cost must be as low as possible, the question arises: how can we calculate the cost in order to best reflect the preferences expressed in *Gradus ad Parnassum*?

The way to translate each preference into a corresponding cost has of course been formalised in the previous sections, but that's not the crux of the matter. The question we face here is: what is the best way to combine all these individual costs to get the most accurate result in terms of what Fux is trying to convey?

Three main ways of doing this have been identified: a linear combination between costs, a search that minimises costs by lexicographic order, and a cost ordering that involves the calculation of minima. We will first describe each of these techniques and their respective advantages, and then compare them (and the results they produce).

4.2.1 Linear Combination

The first method of calculating our costs is a linear combination. This is the technique used in T. Wafflard's thesis. More precisely, it uses a linear combination in which all the weights are equal to one.

To be more precise about the method used to calculate the total cost in T. Wafflard's thesis, here is a more detailed explanation: there exists a total cost, τ, which is equal to the sum of all individual costs, $\mathcal{C}$. The next step is to minimise τ. Each $\mathcal{C}_i$ is usually itself a sum of sub-costs. Take, for example, the cost of motions, $\mathcal{C}_{motions} = \sum_j P_{costs}[j]$. This cost is the sum of all sub-costs of the motions (one per motion): by default, a contrary motion has a sub-cost of 0, an oblique motion has a sub-cost of 1 and a direct motion has a sub-cost of 2. These default values can be changed by the user to be set somewhere on a scale that ranges from 0 to $64m$. For example, the user could set the oblique motion cost to be equal to 0, and the cost for direct and contrary motion to be equal to $64m$, in order to get a composition filled with as many oblique motions as possible (always in accordance with the basic rules from *Gradus ad Parnassum*, i.e. all voices are never going to go in the same direction, see **1.P6**).

As mentioned at the beginning of this subsection, this procedure can be understood as a linear combination with weights of one only. However, since the cost factors are given different values according to the user's choices, this method is actually more like a regular linear combination, except that the weights are not multiplied by the costs once the latter have been set, but the costs are themselves made larger or smaller before the linear combination is calculated.

The linear combination has two major advantages: ease of implementation and high comprehensibility.

However, it has a major drawback: since the total cost τ we are minimising in a linear combination is the sum of all costs $\mathcal{C}$, the best solution might be a solution where one cost is absolutely huge and all the others are small. This might not be a problem if the outstanding cost is not really relevant, but if it is the cost of not using a harmonic triad, it goes completely against the preferences that Fux conveys in his work, making the solution inappropriate. A representation of this situation can be found in the figure 4.1.

Another drawback of linear combination is that the result is pretty and unpredictable: changing the value of the cost may or may not make a difference, and you may need to set huge values to see a real effect. For example, if a composer really wants oblique motion, they may be forced to set the cost of the other types of motion to a huge value, or they may not see the difference between the default solution and their personalised solution. This is due to the fact that all the costs are mixed together and form an indistinguishable soup that the solver considers as a whole, and a small increase in the cost of the direct and contrary motions is very likely to be absorbed into this soup without any change being noticed.

These two drawbacks make the linear combination solution for the costs hardly acceptable when it comes to representing the preferences. We will therefore examine

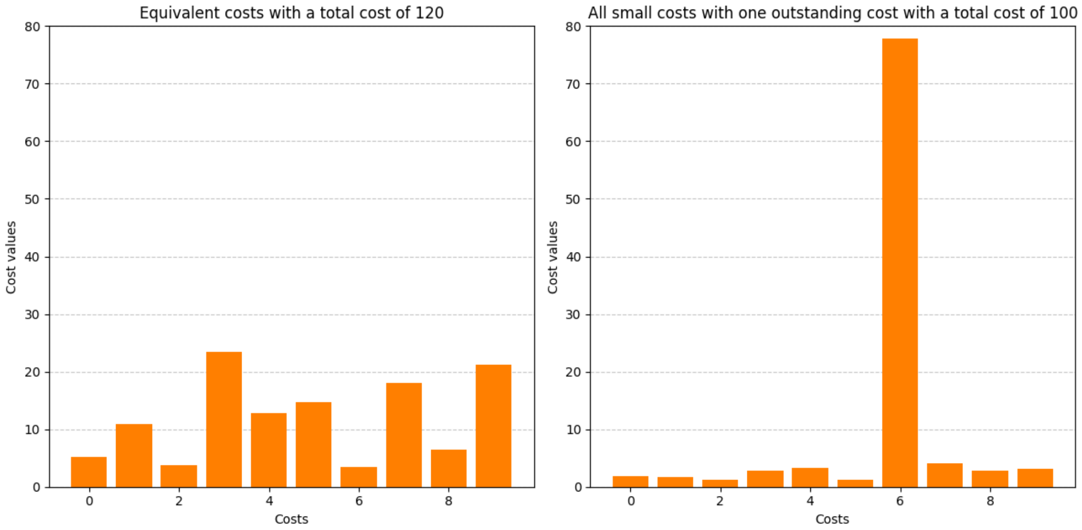


Figure 4.1: Example of a situation where a solution with an outstanding cost is preferred to a solution with equivalent low costs when using a linear combination

the other two options for adjusting costs.

4.2.2 Minimising the maxima

In order to overcome the problem of outstanding costs that we encountered when considering the linear combination solution, one might consider using some minimums when calculating τ, the total cost. For example, τ could be the maximum of all costs. By doing this, the solver would try to find a solution where the focus is on the worst cost and try to reduce it before trying to reduce the other costs.

The problem with this method arises when one cost is significantly higher than the others because it has been defined that way. Let's go back to our example of the composer wanting as many oblique motions as possible. You will set the cost for direct motion and contrary motion to the highest possible cost and start the search. As we've already discussed, it is not possible to have only oblique motions, since this would contradict the rule that not all voices can move in the same direction (**1.P6**). As a result, there will always be contrary motions, and since the cost for them has been set very high, it would be impossible for the solver to converge to a good solution. This creates a bottleneck effect, where once the solver has reached the best potential value of the worst cost, it cannot continue to find better solutions.

Furthermore, even when considering a less extreme case (e.g. the default setting), this method requires a normalisation of the costs: there are $3 \times (m - 2)$ sub-costs for the variety cost, $3 \times (m - 1)$ sub-costs for the motion cost, but only m sub-costs for the octave cost. This means that without normalisation, the motion cost will be on average three times larger than the octave cost, which means that the solver will put three times more effort into minimising the motion cost than the octave cost, which is unfair and unpractical.

4.2.3 Lexicographical Order

The second way of dealing with the costs is to arrange them in an array and then perform a lexicographic minimisation. In other words, the costs would be arranged in order of importance: from most important to least important. The most important cost to minimise would be placed first in this array, and the solver would only try to

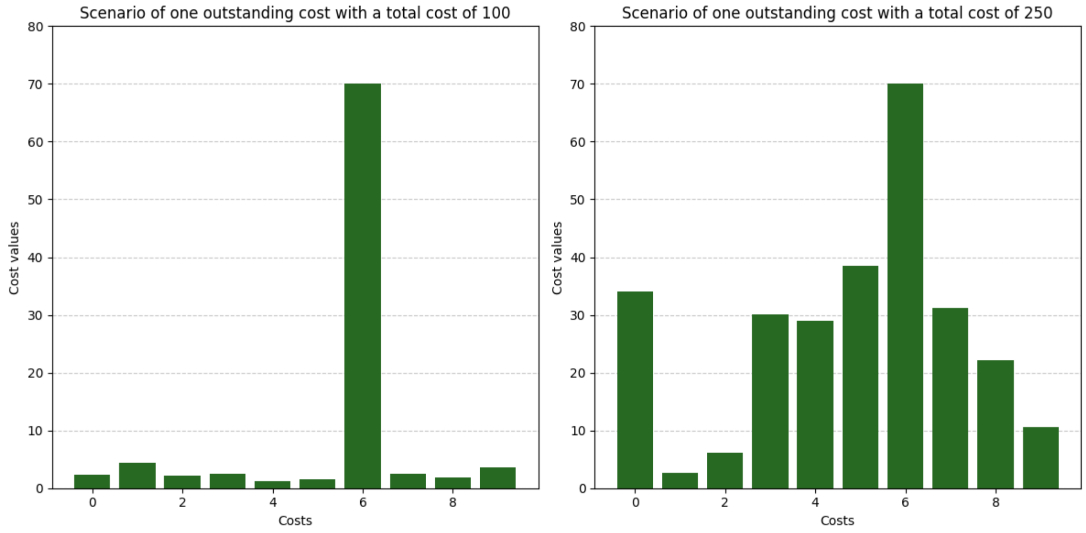


Figure 4.2: Example of two situations where a cost causes a bottleneck in the search because the solver cannot distinguish between left and right situations. The solver will blindly choose one of the two solutions, even if the solution on the left is obviously better.

minimise the other costs if the first cost remained the same or decreased. This method makes a lot of sense when you think about the rules that emanate of *Gradus ad Parnassum*. For example, Fux says that perfect consonance can be achieved by direct motion if there is no other possibility. This means that, all other things being equal, we would prefer to achieve perfect consonance by oblique or contrary motion, but that between a bad solution (respecting almost no preferences) in which perfect consonance is not achieved by direct motion, and a good solution (respecting almost all preferences) in which perfect consonance is achieved by direct motion, we would choose the good solution.

Some costs are also more important than others in absolute terms. For example, when Fux says that an imperfect consonance is preferred to a fifth, which is preferred to an octave. This amounts to lexicographically ranking the cost of using an octave first (because we really don't want octaves), and then the cost of using a fifth (and there is no cost of using an imperfect consonance, since Fux indicates that this is preferable).

$$\tau = [\underbrace{\mathcal{C}_{\text{octaves}}}_{\text{minimise this first}}, \mathcal{C}_{\text{fifths}}] \tag{4.1}$$

Figure 4.3: Array of costs demonstrating the practicality of a lexicographical order solving.

A second example, which ties in particularly well with the first, is that Fux tells us that the harmonic triad must be used in every measure unless a rule forbids it. In saying this, he places the preference for the harmonic triad above all other preferences, because the only reason that can prevent the use of a harmonic triad is a fixed constraint (and not a preference). You'll notice that the harmonic triad consists of a fifth (which is a perfect consonant), so Fux is telling us that we'd rather use a fifth in a harmonic triad than an imperfect consonant outside a harmonic triad. The lexicographic order search is the only one that allows this kind of concept to be taken into account, because in a linear combination these two preferences would be mutually "exclusive"[1]: the first preference would add a cost where the second preference would not, and the second preference would add a cost where the first would not.

$$\tau = [\underbrace{\mathcal{C}_{\text{harmonic\_triad}}}_{\text{minimize this first}}, \underbrace{\mathcal{C}_{\text{octaves}}}_{\substack{\text{and start minimizing this only if it is} \\ \text{not possible anymore to minimize the} \\ \text{harmonic triad cost}}}, \mathcal{C}_{\text{fifths}}] \tag{4.2}$$

Figure 4.4: Array of costs demonstrating the practicality of a lexicographical order solving.

And in this way we can keep integrating the different costs until we get a full array τ with all the costs ordered in a lexicographical way.

Of course, it is not always as simple as in the examples above, because it is not always easy to determine which cost has priority over which other. Sometimes Fux is very clear about it (e.g. for the harmonic triad cost, which Fux says has priority over everything else), and sometimes he isn't (do we prefer no off-key notes, or as much variety as possible?) This is a disadvantage of this method, because we have to hierarchise the costs, even if the choice is difficult. What's more, once the costs are ranked, their order becomes absolute and the solver loses some of its flexibility.

Knowing this, we made a suggested order that should be as close as possible to Fux's preferred order (or at least what we understood him to convey as his preferred order in *Gradus ad Parnassum*). This order should of course be changeable at the composer's discretion. The default order we have agreed upon is as follows:

1. $\mathcal{C}_{\text{no\_syncope}}$[2]
2. $\mathcal{C}_{\text{harmonic\_triad}}$
3. $\mathcal{C}_{\text{harmonic\_triad\_3rd\_species}}$
4. $\mathcal{C}_{\text{octaves}}$
5. $\mathcal{C}_{\text{penult\_thesis\_is\_fifth}}$[3]
6. $\mathcal{C}_{\text{fifths}}$
7. $\mathcal{C}_{\text{off\_key}}$[4]
8. $\mathcal{C}_{\text{variety}}$
9. $\mathcal{C}_{\text{m2\_eq\_zero}}$[5]
10. $\mathcal{C}_{\text{not\_cambiata}}$[6]
11. $\mathcal{C}_{\text{motions}}$
12. $\mathcal{C}_{\text{m\_degrees}}$[7]
13. $\mathcal{C}_{\text{direct\_move\_to\_p\_cons}}$

Some notes on the proposed order:

- One cost comes even before the harmonic triad cost, namely: the $\mathcal{C}_{\text{no\_syncope}}$ cost. This cost is at the heart of the fourth species, and a fourth species counterpoint without syncopations is not really a fourth species counterpoint. This is why syncope is considered even more important than the harmonic triad.

[1] In the sense that their effects would work against each other.

[2] The cost of not using a syncope in the fourth species.

[3] A specific cost for the second species, which applies when a penultimate thesis note does not make a fifth interval with the lowest stratum.

[4] The cost of using sharps or flats.

[5] The cost of having the same note in the downbeat and the upbeat.

[6] The cost of not using a *cambiata* if it is possible. The *cambiata* can be characterised by the following scheme: `consonance - dissonance - consonance`.

[7] The cost of using big or small melodic intervals.

- The cost of $\mathcal{C}_{\text{penult\_thesis\_is\_fifth}}$ comes before the cost of $\mathcal{C}_{\text{fifths}}$, as it is an exception to the latter (similar to the interaction explained above in the section between $\mathcal{C}_{\text{harmonic\_triad}}$ and $\mathcal{C}_{\text{fifths}}$).

- $\mathcal{C}_{\text{off\_key}}$ was added to its ranking because it is actually an absolute rule not to use off-key notes, but Fux does use some, and so it was decided to put this cost after the very important costs to allow off-key notes to happen.

- The costs $\mathcal{C}_{\text{variety}}$, $\mathcal{C}_{\text{motions}}$ and $\mathcal{C}_{\text{m\_degrees}}$ were ranked in order from least to most restrictive. First we say that we would prefer the note to change as much as possible (with the variety cost), then we indicate our preference for the direction (with the motion cost), and finally we indicate our preference for the size of the motion (with the melodic interval cost). This gives the solver as much flexibility as possible. The other way round would have been more restrictive, since the solver would have minimised the melodic intervals first, setting them all to one, which doesn't leave much room for the motion cost to have an effect, and forcing the variety cost to be high in any case, as with intervals the melody tends to vary only a little.

- $\mathcal{C}_{\text{m2\_eq\_zero}}$ and $\mathcal{C}_{\text{m\_degrees}}$ were classified right after the variety cost as they are an in-measure variation of the variety preference.

NB Please note that using the lexicographic order does not *not* mean that the last costs are not taken into account, they will be *too* minimised by the solver. It just means that if the solver has to choose between minimising one cost or another, it will minimise the first one in the lexicographic order.

4.2.4 Comparison between the three types of costs.

We have now discussed the advantages and disadvantages of each of the three methods. These are all listed in 4.1. As you can see, no one method is definitively better than another, and the only way to know which method is better in practice is to *test* them in practice to find out which of the methods gives the best results.

Table 4.1: Comparison of Three Methods According to Criteria

Criteria	**Linear Combination**	**Minimising the maximum**	**Lexicographic Search**
Outsanding costs	Yes	No	Only for minor costs
Predictability[8]	No	Some	Yes
One cost might be a bottleneck	No	Yes	No
Need to normalise costs	No	Yes	No
Possibility to ensure a preference of one cost over another	No	No	Yes
Need for hierarchisation of costs	No	No	Yes
Flexibility	Medium	High	Low

[8] Changing a cost has a big impact on the result.

Even more, one could think about a combination of all the methods to get rid of their disadvantages. In fact, we could enjoy the advantages of all the methods by combining them and cleverly designing a lexicographical order search in which the cost is a linear combination of a maximum minimisation.

4.2.5 Experimenting with the three types

To experiment which method gives the best results, we will follow this plan: start with a purely lexicographical order, using the default preference order as defined in 4.2.3, then solve the problem of which preference is more important than the other by grouping them in the same cell of the lexicographical order (either by linear combination or by maximum minimisation).

These experiments are carried out using two different counterpoint combination setups. The first setup is a combination of two first-species counterpoints, since this is the basic case of three-part composition. The second setup is a combination of a second species counterpoint and a fourth species counterpoint. This setup was chosen because it's rich enough to capture the complexity of counterpoint writing, but not too rich, like the third and fifth species, which makes the way it works more opaque and difficult to analyse in terms of the effects of cost. In addition, the search for an optimal solution is quite time-consuming, and this is even more the case when we are talking about complex species such as the third and the fifth, and when they are combined. This means that it is more difficult to grasp the impact of the cost method when studying its impact on these species, as the solver won't have the time to find two significantly different solutions by the time we stop the search. What's more, the vast majority of the costs are related to the interaction of the voices in the first beat of the measures: the behaviour we want to observe, i.e. the interaction between the cost method and the resulting composition, will be just as observable with complex species as with simple ones.

The analyses in this section are superficial and do not deal with in-depth music theory. They will consist of general surface and impression remarks. They are highly subjective and should not be taken at face value. The aim of this analysis is to provide an initial critical view of the results offered by the solver.

The selected *cantus firmi* were chosen from *Gradus ad Parnassum*. If the search time exceeds 30 seconds, the search is stopped and the current solution is analysed.

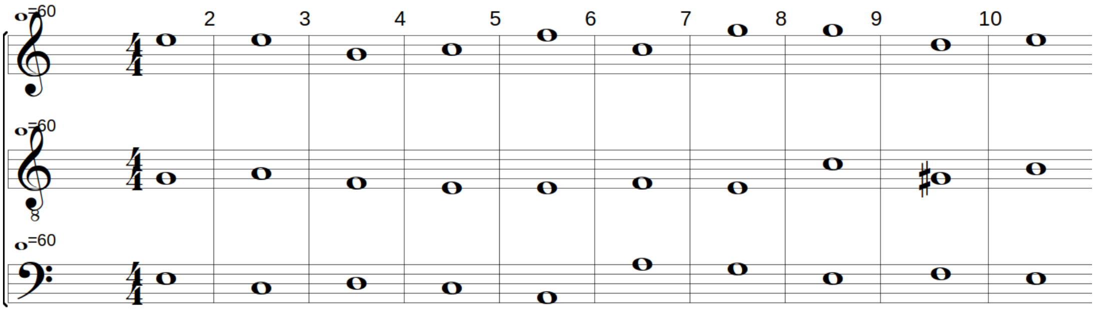


Figure 4.5: Result of the linear combination method with default costs.

Here are the first two results: 4.5 and 4.6. To the average ear, there's not much difference between these two solutions, except that the second is perhaps a tad more vibrant.

In more technical terms, the two solutions are fairly similar, and both feature the same number of harmonic triads, i.e. three. The lexicographic solution offers five di-

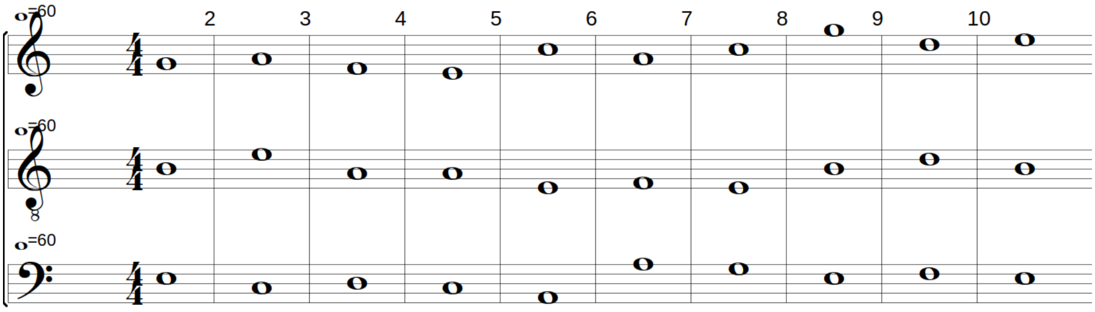


Figure 4.6: Result of the lexicographic search method with default costs.

rect motions, whereas the linear combination solution only offers three. However, the solution does not suffer from any particular redundancy to the ear, as you do not get the impression of hearing the same melody three times. The lexicographic composition gives a more melodic impression than the composition by linear combination, which gives more of an impression of a chord progression.

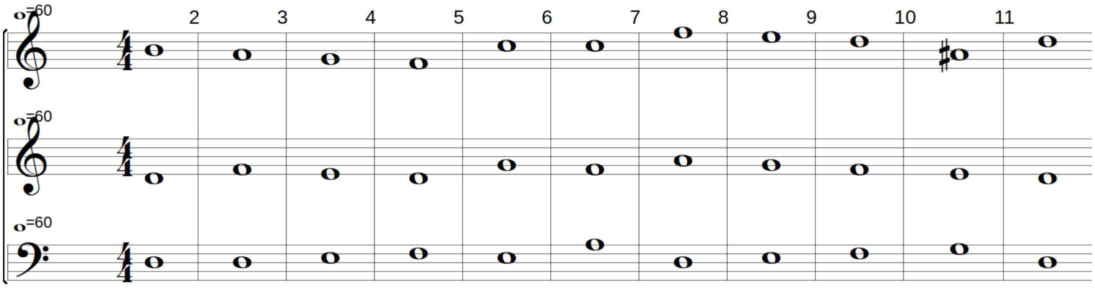


Figure 4.7: Result of the linear combination method with default costs.

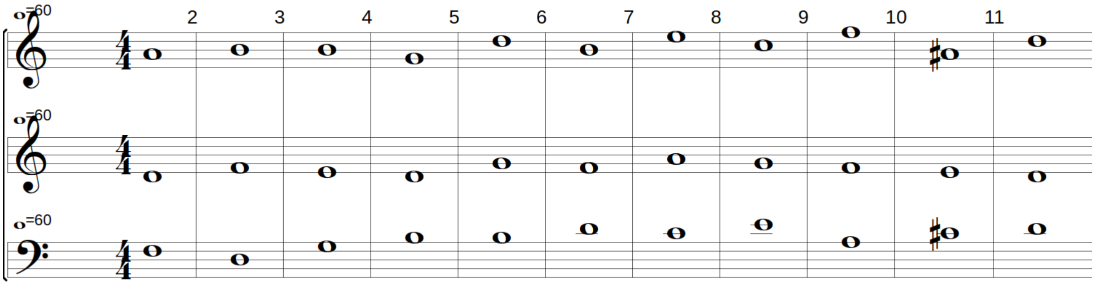


Figure 4.8: Result of the lexicographic search method with default costs.

Let's look at the second result (obtained with a different *cantus firmus*): 4.7 and 4.8. Once again, there's no major difference between the two solutions, except that this time it's perhaps the linear combination search that narrowly wins out over the melodic line question.

Once again, the technical results are fairly similar: two harmonic triads for the two compositions (which is surprisingly few, given that there are eleven bars). The lexicographic solution proposes two direct motions, whereas the linear combination solution proposes only one.

The first result, combining the species, is as follows: 4.9 and 4.10.

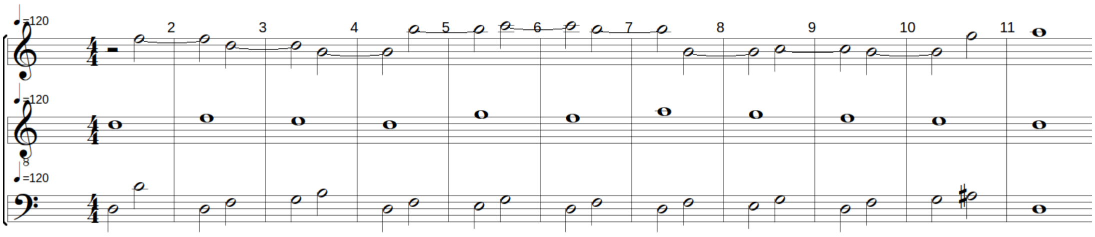


Figure 4.9: Result of the linear combination method with default costs.

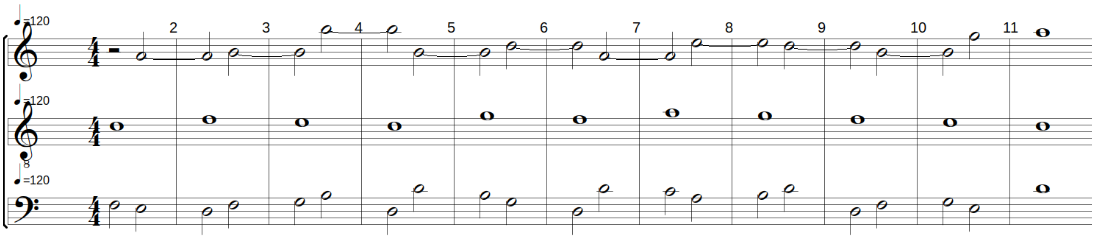


Figure 4.10: Result of the lexicographic search method with default costs.

In both cases, the melody is a little dull and lacks dynamism, and if we had to differentiate the two, we would perhaps observe a slightly more pronounced length construction in the case of the lexicographic order.

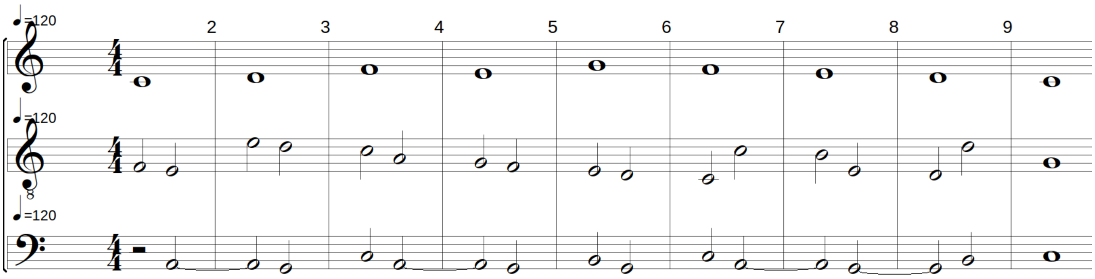


Figure 4.11: Result of the linear combination method with default costs.

And finally, here is a result combining the two species and putting the 4th species at the bottom: 4.11 and 4.12.

We immediately notice something with the 4th species on the bass, which is not related to the costs: there are a few dissonances on the downbeat, as the solver doesn't really take into account the harmonic interaction between the notes of the downbeat of the fourth species and the notes of the downbeat of the other species, but rather the harmonic interaction between the upbeat of the fourth species and the downbeat of the others: which leads to a few surprises, as we can see here.

As far as the costs are concerned, one thing is clear: not all the notes of the 4th species obtained with a linear combination are linked, whereas they all are in the solution of the lexicographic order. This is obviously due to the order of priority, and it shows that the cost of ligating notes is essential, unless you no longer write in 4th species.

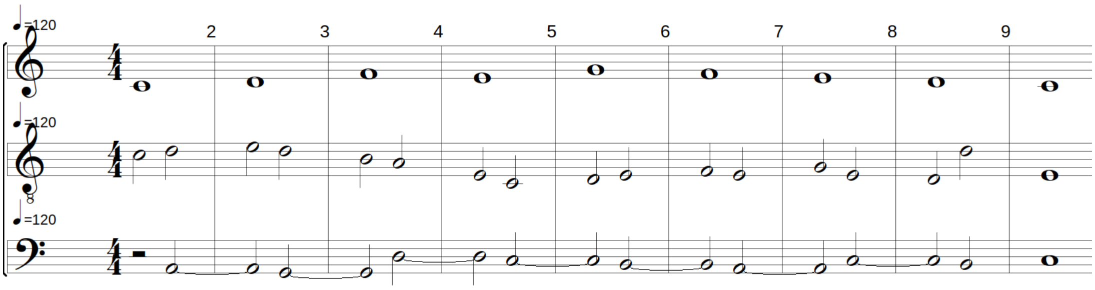


Figure 4.12: Result of the lexicographic search method with default costs.

Chapter 5

Conclusion

It is time to look back at the work that has been done, to highlight the progress that has been made, but also the shortcomings and gaps that need to be filled by future improvements. We will therefore use this chapter to discuss some of the key points that emerge from this thesis.

5.1 Intended use of the FuxCP tool

Throughout this work, it is clear that all the examples provided are fairly short (fourteen bars at most). This is primarily due to Fux himself, as the examples he gives are always of the same shortness, probably for pedagogical purposes. He does not mention this explicitly, but there may also be a practical reason for considering only such short compositions. Indeed, these small compositions can be considered as ’blocks’, which can then be arranged to form a whole. The great advantage of this approach is that the countrepoints can be given different species between the blocks, allowing the composition to be constantly renewed.

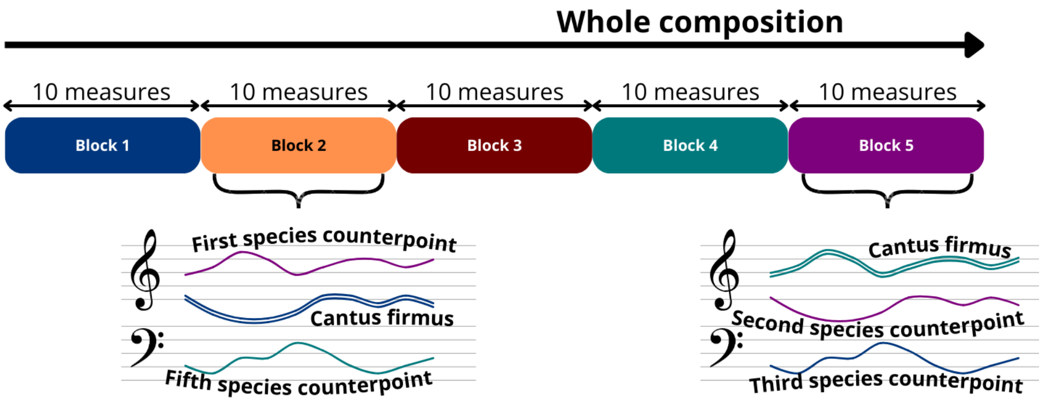


Figure 5.1: Example of what of a composition in blocks could look like

Bibliography

[1] Thibault Wafflard. "FuxCP: a constraint programming based tool formalizing Fux's musical theory of counterpoint". MA thesis. École polytechnique de Louvain, Université catholique de Louvain, 2023. URL: http://hdl.handle.net/2078.1/thesis:40739.

[2] Klaus-Jürgen Sachs and Carl Dahlhaus. *Counterpoint*. Oxford University Press. 2001. DOI: 10.1093/gmo/9781561592630.article.06690. URL: https://www.oxfordmusiconline.com/grovemusic/view/10.1093/gmo/9781561592630.001.0001/omo-9781561592630-e-0000006690.

[3] Juliet Hess. "Balancing the counterpoint: Exploring musical contexts and relations". In: *Action, Criticism & Theory for Music Education* 15.2 (2016), p. 50.

[4] Heinrich Schenker. *Kontrapunkt*. German. Vol. 2: iss.1. Vol. 2 of Neue musikalische Theorien und Phantasien. Stuttgart: J.G. Cotta'sche Buchhandlung Nachfolger, 1906.

[5] Knud Jeppesen and Glen Haydon. *Counterpoint: The Polyphonic Vocal Style of the Sixteenth Century*. English. Englewood Cliffs, N. J.: Prentice-Hall, 1960.

[6] David Gaynor Yearsley. *Bach and the Meanings of Counterpoint*. English. Cambridge ; New York: Cambridge University Press, 2002.

[7] Richard L. Crocker. "Discant, Counterpoint, and Harmony". In: *Journal of the American Musicological Society* 15.1 (1962), pp. 1–21. ISSN: 00030139, 15473848. URL: http://www.jstor.org/stable/830051.

[8] Bill Schottstaedt. *Automatic Species Counterpoint*. Research Report STAN-M-19. Departement of Music, Stanford University, 1984. URL: https://ccrma.stanford.edu/files/papers/stanm19.pdf.

[9] John Polito, Jason M Daida, and Tommaso F Bersano-Begey. "Musica ex machina: Composing 16th-century counterpoint with genetic programming and symbiosis". In: *Evolutionary Programming VI: 6th International Conference, EP97 Indianapolis, Indiana, USA, April 13–16, 1997 Proceedings 6*. Springer. 1997, pp. 113–123.

[10] Andres Garay Acevedo. "Fugue composition with counterpoint melody generation using genetic algorithms". In: *International symposium on computer music modeling and retrieval*. Springer. 2004, pp. 96–106.

[11] Gabriel Aguilera et al. "Automated generation of contrapuntal musical compositions using probabilistic logic in Derive". In: *Mathematics and Computers in Simulation* 80.6 (2010). Fifth IMACS Seminar on Monte Carlo Methods Applications of Computer Algebra 2007 (ACA 2007) special session on Nonstandard Applications of Computer Algebra Computational Biomechanics and Biology, a collection of papers presented at the 1st IMACS International Conference on the Computational Biomechanics and Biology ICCBB 2007, pp. 1200–1211. ISSN: 0378-4754. DOI: https://doi.org/10.1016/j.matcom.2009.04.012. URL: https://www.sciencedirect.com/science/article/pii/S0378475409001165.

[12] Dorien Herremans and Kenneth Sörensen. "Composing first species counterpoint with a variable neighbourhood search algorithm". In: *Journal of Mathematics and the Arts* 6.4 (2012), pp. 169–189. DOI: 10.1080/17513472.2012.738554.

[13] Maciej Komosinski and Piotr Szachewicz. "Automatic species counterpoint composition by means of the dominance relation". In: *Journal of Mathematics and Music* 9.1 (2015), pp. 75–94.

[14] Ars Nova Software. *Counterpointer*. Developed and produced by Ars Nova Software. Address: PO Box 3333, Kirkland, WA 98083-3333, USA. Available on Microsoft Store and Mac App Store. 2019. URL: https://www.ars-nova.com/counterpointer3.html.

[15] Alex Ding et al. *Counterpointer*. Project for Brown University's CSCI 0320 course. 2021. URL: https://github.com/counter-pointer/counterpointer.

[16] Baptiste Lapière. "Computer-aided musical composition Constraint programming and music". Prom. by Peter Van Roy. MA thesis. Ecole polytechnique de Louvain, Université catholique de Louvain, 2020.

[17] Damien Sprockeels. "Melodizer: A Constraint Programming Tool For Computer-aided Musical Composition". Prom. by Peter Van Roy. MA thesis. Ecole polytechnique de Louvain, Université catholique de Louvain, 2021.

[18] Clément Chardon, Amaury Diels, and Federico Gobbi. "Melodizer 2.0: A Constraint Programming Tool For Computer-aided Musical Composition". Prom. by Peter Van Roy. MA thesis. Ecole polytechnique de Louvain, Université catholique de Louvain, 2022.

[19] Alfred Mann. *The Study of Counterpoint. From Johann Joseph Fux's Gradus ad Parnassum*. English. Ed. and trans. Latin by Alfred Mann. Revised Edition. 500 Fifth Avenue, New York, N.Y. 10110: W.W. Norton & Company, 1971. ISBN: 0-393-00277-2. URL: http://www.opus28.co.uk/Fux_Gradus.pdf.

[20] Simonne Chevalier. *Gradus ad Parnassum. Johann Joseph Fux*. French. Ed. by Gabriel Foucou. Trans. Latin by Simmone Chevalier. 2019. ISBN: 978-2-9556093-6-1.

[21] Johann Joseph Fux. *Gradus ad Parnassum*. French. Trans. by Pierre Denis. Paris: Diod, Bijoutier, Garnier & Cadet, 1773. URL: https://s9.imslp.org/files/imglnks/usimg/b/b2/IMSLP231222-PMLP187246-fux_traite_de_composition_1773.pdf.

[22] Johann Joseph Fux. *Gradus ad Parnassum*. German. Ed. and trans. by Lorenz Christoph Mizler. Leipzig: Mizler, 1742. URL: https://s9.imslp.org/files/imglnks/usimg/6/67/IMSLP273867-PMLP187246-gradusadparnassu00fuxj.pdf.

[23] Johann Joseph Fux. *Gradus ad Parnassum*. Reprinted in 1966 by Broude Bros., New York. Vienna: Johann Peter van Ghelen, 1725. URL: https://s9.imslp.org/files/imglnks/usimg/f/fd/IMSLP91138-PMLP187246-Fux_-_Gradus_ad_Parnassum.pdf.

[24] Damien Sprockeels. *Gecode interface Lisp*. GitHub. May 3, 2023. URL: https://github.com/sprockeelsd/GiL.

[25] Guido Tack and Mikael Zayenz Lagerkvist. *Generic Constraint Development Environment*. Version 6.2.0. Gecode. URL: https://www.gecode.org/index.html.

[26] IRCAM STMS Lab. *OpenMusic*. Version 7.2. URL: https://openmusic-project.github.io.

[27] *Common Lisp*. Available at: `https://common-lisp.net/`. Common Lisp Foundation.

[28] IRCAM. *IRCAM - Institut de recherche et coordination acoustique/musique*. IRCAM official website. `https://www.ircam.fr/`.

[29] Walter Piston. *Harmony: Fifth Edition*. English. Ed. by Mark DeVoto. 5th. W. W. Norton & Company, 1987, p. 31.

Appendix A

Complete set of rules for two and three part compositions

Constraints of the First Species

Harmonic Constraints of the First Species

1.H1 *All harmonic intervals must be consonances.*

$$\forall j \in [0, m) \quad H[0, j] \in Cons \tag{A.1}$$

This can be expressed with the constraint `(gil::g-member *sp* ALL_CONS_VAR h-intervals)` (see original code for more details).

1.H2 *The first harmonic interval must be a perfect consonance.* When dealing with two-part composition:

$$H[0, 0] \in Cons_p \tag{A.2}$$

1.H3 *The last harmonic intervals must be a perfect consonance.* When dealing with three-part composition:

$$H[0, m-1] \in Cons_p \tag{A.3}$$

1.H4 *The key tone is tuned according to the first note of the cantus firmus.*

$$\begin{aligned} \neg IsCfB[0, 0] &\implies H[0, 0] = 0 \\ \neg IsCfB[0, m-1] &\implies H[0, m-1] = 0 \end{aligned} \tag{A.4}$$

1.H5 *The counterpoint and the cantus firmuscannot play the same note at the same time except in the first and last measure.*

$$\forall j \in [1, m-1) \quad Cp[0, j] \neq Cf[j] \tag{A.5}$$

1.H6 *Imperfect consonances are preferred to perfect consonances.*

$$\begin{gathered} \forall j \in [0, m) \\ Pcons_{costs}[j] = \begin{cases} cost_{Pcons} & \text{if } H[0, j] \in Cons_p \\ 0 & \text{otherwise} \end{cases} \\ \text{moreover } \mathcal{C} = \mathcal{C} \cup \sum_{c \in Pcons_{costs}} c \end{gathered} \tag{A.6}$$

1.H7 and **1.H8** *The harmonic interval of the penultimate note must be a major sixth or a minor third depending on the cantus firmuspitch.*

$$\rho := \max(positions(m)) - 1$$
$$H[\rho] = \begin{cases} 9 & \text{if } IsCfB[\rho] \\ 3 & \text{otherwise} \end{cases} \tag{A.7}$$
where ρ represents the penultimate index of any counterpoint.

Melodic Constraints of the First Species

1.M1 *Tritone melodic intervals are forbidden.*

$$\begin{gathered} \forall \rho \in positions(m-1) \\ M[\rho] = 6 \implies Mdeg_{costs}[\rho] = cost_{tritoneMdeg} \end{gathered} \tag{A.8}$$

1.M2 *Melodic intervals cannot exceed a minor sixth interval.*

$$\forall j \in [0, m-1) \quad M[0,j] \leq 8 \tag{A.9}$$

Motion Constraints of the First Species

1.P1 *Perfect consonances cannot be reached by direct motion.*

When dealing with two-part composition:

$$\forall j \in [0, m-1) \quad H[0, j+1] \in Cons_p \implies P[0,j] \neq 2 \tag{A.10}$$

When dealing with three-part composition:

$$\begin{aligned} &\forall j \in [0, m-2) : \\ &P[0,j] = 2 \land H[0, j+1] \in Cons_p \\ &\iff cost_{\text{direct\_move\_to\_p\_cons}}[j] = 8 \end{aligned} \tag{A.11}$$

1.P2 *Contrary motions are preferred to oblique motions which are preferred to direct motions.*

- $cost_{con}$ *DFLT: <no cost>*
- $cost_{obl}$ *DFLT: <low cost>*
- $cost_{dir}$ *DFLT: <medium cost>*

$$\forall j \in [0, m-1)$$
$$P_{costs}[j] = \begin{cases} cost_{con} & \text{if } P[0,j] = 0 \\ cost_{obl} & \text{if } P[0,j] = 1 \\ cost_{dir} & \text{if } P[0,j] = 2 \end{cases} \tag{A.12}$$
$$\text{moreover } \mathcal{C} = \mathcal{C} \cup \sum_{c \in P_{costs}} c$$

1.P3 *At the start of any measure, an octave cannot be reached by the lower voice going up and the upper voice going down more than a third skip.*

$$i := \max(\mathcal{B}), \forall j \in [0, m-1)$$

$$H[0, j+1] = 0 \land P[i, j] = 0 \land \begin{cases} M_{brut}[i, j] < -4 \land IsCfB[i, j] \iff \bot \\ M_{cf}[i, j] < -4 \land \neg IsCfB[i, j] \iff \bot \end{cases} \tag{A.13}$$

where i stands for the last beat index in a measure.

Constraints of the Second Species

Harmonic Constraints of the Second Species

2.H1 *Thesis harmonies cannot be dissonant.*

As explained above, there is no constraint to add because it would be a duplicate of rule **1.H1**.

2.H2 *Arsis harmonies cannot be dissonant except if there is a diminution.*

$$\forall j \in [0, m-1)$$

$$IsDim[j] = \begin{cases} \top & \text{if } M^2[0, j] \in \{3, 4\} \land M^1[0, j] \in \{1, 2\} \land M^1[2, j] \in \{1, 2\} \\ \bot & \text{otherwise} \end{cases} \tag{A.14}$$

$$\forall j \in [0, m-1) \quad \neg IsCons[2, j] \implies IsDim[j] \tag{A.15}$$

2.H3 and **2.H4** *In the penultimate measure the harmonic interval of perfect fifth must be used for the thesis note if possible. Otherwise, a sixth interval should be used instead.*

$$H[0, m-2] \in \{7, 8, 9\}$$

$$\therefore penulthesis_{cost} = \begin{cases} cost_{penulthesis} & \text{if } H[0, m-2] \neq 7 \\ 0 & \text{otherwise} \end{cases} \tag{A.16}$$

$$\text{moreover } \mathcal{C} = \mathcal{C} \cup penulthesis_{cost}$$

Melodic Constraints of the Second Species

2.M1 *If the two voices are getting so close that there is no contrary motion possible without crossing each other, then the melodic interval of the counterpoint can be an octave leap.*

$$\forall j \in [0, m-1), \forall M_{cf}[j] \neq 0$$

$$M[0, j] = 12 \implies (H_{abs}[0, j] \leq 4) \land (IsCfB[j] \iff M_{cf}[j] > 0) \tag{A.17}$$

2.M2 *Two consecutive notes cannot be the same.* When dealing with two-part composition:

$$\forall \rho \in positions(m) \quad Cp[\rho] \neq Cp[\rho + 1] \tag{A.18}$$

When dealing with three-part composition:

$$\begin{aligned}&\forall j \in [1, m-1), \quad j \neq m-2 : \\ &((N[2, j-1] \neq N[0, j]) \wedge (N[0, j] \neq \wedge N[2, j])) \\ &\wedge \\ &((N[2, m-3] \neq N[0, m-2]) \vee (N[0, m-2] \neq N[2, m-2]))\end{aligned} \tag{A.19}$$

Motion Constraints of the Second Species

2.P1 *If the melodic interval of the counterpoint between the thesis and the arsis is larger than a third, then the motion is perceived based on the arsis note.*

$$\forall j \in [0, m-1) \quad P_{real}[j] = \begin{cases} P[2, j] & \text{if } M[0, j] > 4 \\ P[0, j] & \text{otherwise} \end{cases} \tag{A.20}$$

2.P2 *Rule **1.P3** on the battuta octave is adapted such that it focuses on the motion from the note in arsis.*

2.P3 This constraint already had an adapted mathematical notation in the chapter of the first species. Note that this constraint would indeed use P[2] and not P$_{real}$.

Constraints of the Third Species

Harmonic Constraints of the Third Species

3.H1 *If five notes follow each other by joint degrees in the same direction, then the harmonic interval of the third note must be consonant.*

$$\begin{gathered}\forall j \in [0, m-1) \\ \left(\bigwedge_{i=0}^{3} M[i, j] \leq 2\right) \wedge \left(\bigwedge_{i=0}^{3} M_{brut}[i, j] > 0 \vee \bigwedge_{i=0}^{3} M_{brut}[i, j] < 0\right) \\ \implies IsCons[2, j]\end{gathered} \tag{A.21}$$

3.H2 *If the third harmonic interval of a measure is dissonant then the second and the fourth interval must be consonant and the third note must be a diminution.*

$$\begin{gathered}\forall j \in [0, m-1) \\ IsCons[2, j] \vee (IsCons[1, j] \wedge IsCons[3, j] \wedge IsDim[j])\end{gathered} \tag{A.22}$$

where $IsDim[j] = \top$ when the 3$^{\text{rd}}$ note of the measure j is a diminution.

3.H3 *It is best to avoid the second and third harmonies of a measure to be consonant with a one-degree melodic interval between them.*

$$\begin{gathered}\forall j \in [0, m-1) \\ Cambiata_{costs}[j] = \begin{cases} cost_{Cambiata} & \text{if } IsCons[1, j] \wedge IsCons[2, j] \wedge M[1, j] \leq 2 \\ 0 & \text{otherwise} \end{cases}\end{gathered} \tag{A.23}$$

3.H4 *In the penultimate measure, if the cantus firmusis in the upper part, then the harmonic interval of the first note should be a minor third.*

$$\neg IsCfB[m-2] \implies H[0,m-2] = 3 \tag{A.24}$$

Melodic Constraints of the Third Species

3.M1 *Each note and its two beats further peer are preferred to be different.*

$$\begin{gathered}\forall \rho \in positions(m-2) \\ MtwoSame_{costs}[i,j] = \begin{cases} cost_{MtwobSame} & \text{if } M^2[\rho] = 0 \\ 0 & \text{otherwise} \end{cases}\end{gathered} \tag{A.25}$$

Motion Constraints of the Third Species

3.P1 *The motion is perceived based on the fourth note.*

This implies that the costs of the motions and the first species constraints on the motions are deducted from $P[3]$.

Constraints of the Fourth Species

Motion Constraints of the Fourth Species

4.P1 *Dissonant harmonies must be followed by the next lower consonant harmony.*

$$\forall j \in [1, m-1) \quad \neg IsCons[0,j] \implies M_{brut}[0,j] \in \{-1,-2\} \tag{A.26}$$

4.P2 *If the cantus firmusis in the lower part then no second harmony can be preceded by a unison/octave harmony.*

$$\begin{gathered}\forall j \in [1, m-1) \\ IsCfB[j+1] \implies H[2,j] \neq 0 \wedge H[0,j+1] \notin \{1,2\}\end{gathered} \tag{A.27}$$

Harmonic Constraints of the Fourth Species

4.H1 *Arsis harmonies must be consonant.*

$$\forall j \in [0, m-1) \quad H[2,j] \in Cons \tag{A.28}$$

4.H2 *If the cantus firmusis in the upper part, then no harmonic seventh interval can occur.*

$$\forall j \in [1, m-1) \quad \neg IsCfB[j] \implies H[0,j] \notin \{10,11\} \tag{A.29}$$

4.H3 and **4.H4** *In the penultimate measure, the harmonic interval of the thesis note must be a major sixth or a minor third depending on the cantus firmuspitch.*

$$H[0,m-2] = \begin{cases} 9 & \text{if } IsCfB[m-2] \\ 3 & \text{otherwise} \end{cases} \tag{A.30}$$

Melodic Constraints of the Fourth Species

4.M1 *Arsis half notes should be the same as their next halves in thesis.*

$$\forall j \in [0, m-1) \quad NoSync_{costs} = \begin{cases} cost_{NoSync} & \text{if } M[2,j] \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{A.31}$$

4.M2 *Each arsis note and its two measures further peer are preferred to be different.*

$$\forall j \in [0, m-1)$$
$$MtwomSame_{costs} = \begin{cases} cost_{MtwomSame} & \text{if } Cp[2,j] = Cp[2,j+2] \\ 0 & \text{otherwise} \end{cases} \tag{A.32}$$

UNIVERSITÉ CATHOLIQUE DE LOUVAIN
École polytechnique de Louvain
Rue Archimède, 1 bte L6.11.01, 1348 Louvain-la-Neuve, Belgique | **www.uclouvain.be/epl**