Electrodynamic Solved Problems

Daniel Panizo

Contents

About these notes

These notes contain a set of selected problems to discuss during the problem solving session of Classical Electrodynamics subject at Uppsala University (Sweden). The order you see in the table of content correspond to chronological order of the lectures for this course. The title of each problem statement is linked to its solution. Try first without looking at... Exercises with an **E** in front of them correspond to old exams.

Recommended Bibliography

- **Classical Electrodynamics**, John David Jackson. You may not like this book at first glance. Neither second, third... but it contains a formal and serious approach to all the topics that are going to be covered during the lectures. It contains important examples and explanations.

- **Introduction to Electrodyamics**, David J. Griffiths. Excellent book for a first approach to many of the concepts in this course. Its level does not cover the one expected for this course, but after reading once[1] you can jump into Jackson.

[1] Sections, not the whole book.

- **Electromagnetic Field Theory**, Bo Thidé. It does not contain all the material of the course, but it includes several derivations of formulae and a good final appendix with tons of identities and explanations of the mathematical tools.

- **Space and Geometry: An introduction to General Relativity**, Sean Carroll. This is some extra material to read about tensor notation. The first chapter, and part of the second one, cover the properties of the tensorial language we are going to use. This will be useful for the covariant formalism of electrodynamics and Lagrangian manipulation parts of this course.

- **FMM: Exercise Notes**, S.Giri & G. Kälin. Uploaded to Studium. It contains the most useful mathematical methods and examples that show how to use them. Totally recommended to refresh your mathematical manipulation.

- **Internet**. As you may know, apart from Social Networks and kitten videos, it contains an enormous amount of resources when used in a proper way.

Tips to enhance your understanding

Here we offer a set of tips in order to enhance your problem-solving capability.

- Read twice/ thrice/ hundredice the statement of a problem until you really understand what is asking you to solve. You can apply the same principle when reading through sections of books, notes, etc.

- "Pachanguera": Although it is a Spanish word to describe dynamic-noisy-low quality music, it can be also used to describe what a drawing sketch is. It is easier to remember what the problem is asking for if you draw a low quality picture of the set up. You can understand a problem in a better way if you translate to a picture the description given in the statement.

- "Explain yourself": It is nice for your future self[2] and for the people who will correct your exercises/exam if you explain with descriptive sentences the process of your calculations. It gives a context to whoever reads through your problems and help you to stay focus on the final target (solution) you are looking for.

- "Tolle, Lege": Take it, read it. Saint Augustine was wise enough to know that if you do not open and read books, you will not learn. It applies from religion to physics. If you do not understand what you are reading, try first point of these

[2]Has it not happened to you that you try to do your exercises again to prepare for the exam and you cannot understand why you calculated something in a particular way?

recommendations. Also, you are more than encouraged to ask the Teacher or teacher assistant.

1 Electrostatics

Conducting ball

A conducting ball of radius R and total charge Q sits in a homogeneous electric field $\vec{E} = E_0\hat{z}$. How does the electric field change by the presence of the ball? (Make an Ansatz of the form $\Phi(r,\theta,\phi) = f_0(r) + f_1(r)\cos\theta$ and motivate it.) Tip: $\hat{Z} = \cos\theta\hat{r} + \sin\theta\hat{\theta}$.

Conducting ball Again

1. A point charge q sits at $\vec{a}$ inside a conducting uncharged sphere that is earthed with radius $R(|\vec{a}| < R)$.Compute the potential and the electric field inside the sphere using the method of mirror charges. Compute also the induced charge density on the surface of the sphere and show that the total charge on the surface is $-q$. What does the Gauss theorem say about the electrical field outside the sphere?

2. Do the same analysis with the change that the sphere is isolated and uncharged. Tip: Determine the electric field outside the sphere with the new b.c.

3. Follow again the same procedure as b for a sphere that is isolated and with charge Q.

The Capacitance of an off-centered Capacitor

A spherical conducting shell centered at the origin has radius R_1 and is maintained at potential V_1. A second spherical conducting shell maintained at potential V_2 has radius $R_2 > R_1$ but is centered at the point $s\hat{\mathbf{z}}$ where $s \ll R_1$.

1. To lowest order in s, show that the charge density induced on the surface of the inner shell is

$$\sigma(\theta) = \epsilon_0 \frac{R_1 R_2 (V_2 - V_1)}{R_2 - R_1} \left[\frac{1}{R_1^2} - \frac{3s}{R_2^3 - R_1^3} \cos\theta \right]. \tag{1}$$

   Hint: Show first that the boundary of the outer shell is $r_2 \approx R_2 + s\cos\theta$.

2. To lowest order in s, show that the force exerted on the inner shell is:

$$\mathbf{F} = \int dS \frac{\sigma^2}{2\epsilon_0} \hat{\mathbf{n}} = \hat{\mathbf{z}} 2\pi R_1^2 \int^{\pi} d\theta \sin\theta \frac{\sigma^2(\theta)}{2\epsilon_0} \cos\theta = -\frac{Q^2}{4\pi\epsilon_0} \frac{s\hat{\mathbf{z}}}{R_2^3 - R_1^3}. \tag{2}$$

Spherical cavity and spherical functions

Consider a sphere of radius a where the surface of the upper hemisphere has a potential $+\Phi_0$ and the surface of the lower hemisphere has a potential $-\Phi_0$. In this case the Green Function is given by:

$$G(r, r') = \frac{1}{\left|\vec{r} - \vec{r}'\right|} - \frac{a}{r' \left|\vec{r} - \frac{a^2}{r'^2}\vec{r}'\right|}, \tag{3}$$

where $\vec{r}'$ refers to a unit source outside the sphere and $\vec{r}$ to the point where the potential is evaluated.

1. Using the expression for the expansion of $\frac{1}{\left|\vec{r}-\vec{r}'\right|}$ in the appropriate basis show that the Green's function can be written as

$$G(r, r') = 4\pi \sum_{l,m} \frac{1}{2l+1} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1} \right] Y_{l,m}^*(\theta', \phi')\, Y_{l,m}(\theta, \phi), \tag{4}$$

2. Using Dirichlet boundary conditions, show that the potential outside the sphere has fol- lowing the expansion.

$$\Phi(r, \theta, \phi) = \sum_{lm} \frac{l+1}{a^2(2l+1)} \left(\frac{a}{r}\right)^{l+1} Y_{l,m}(\theta, \phi) \int \Phi_0(\theta', \phi')\, Y_{l,m}^*(\theta', \phi')\, d\Sigma', \tag{5}$$

which tends to 0 as $r \to \infty$.

Green's function between concentric spheres

Consider the green's function for Newnmann b.c. in the volume V between two concentric spheres between $r = a$ and $r = b, a < b$. We write the potential as

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int_V \rho(x')\, G(x, x')\, d^3x' + \frac{1}{4\pi} \oint_S \frac{\partial \Phi}{\partial n'} G da', \tag{6}$$

where S is the surface of the boundary. This implies that the b.c. for the Green's function is given by:

$$\frac{\partial}{\partial n'} G\left(x, x'\right) = -\frac{4\pi}{S}, \tag{7}$$

or x' in S. Expanding the Green's function in spherical harmonics we get:

$$G\left(x, x'\right) = \sum_{l=0}^{\infty} g_l\left(r, r'\right) P_l(\cos\gamma), \tag{8}$$

where $g_l\left(r, r'\right) = \frac{r_l^l}{r_>^{l+1}} + f_l\left(r, r'\right)$, and γ is the angle between the vector x and x'.

Also here one can prove that $P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_m Y^*_{l,m}\left(\theta', \phi'\right) Y_{l,m}(\theta, \phi)$.

1. Show for $l > 0$ that the Green's function takes the symmetric form:

$$g_l\left(r, r'\right) = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1} - a^{2l+1}} \left[\frac{l+1}{l} \left(r r'\right)^l + \frac{l}{l+1} \frac{(ab)^{2l+1}}{\left(r r'\right)^{l+1}} + a^{2l+1} \left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}} \right) \right] \tag{9}$$

2. Use the Green's function that you found in the situation that you have a normal electric field $E_r = -E_0 \cos\theta$ at $r = b$ and $E_r = 0$ at $r = a$. Show that the potential inside V is

$$\Phi(x) = E_0 \frac{r\cos\theta}{1 - p^3} \left(1 + \frac{a^3}{2r^3}\right), \tag{10}$$

where $p = \frac{a}{b}$. Find also for the electric field that:

$$E_r(r, \theta) = -E_0 \frac{\cos\theta}{1 - p^3} \left(1 + \frac{a^3}{r^3}\right), \quad E_\theta(r, \theta) = E_0 \frac{\sin\theta}{1 - p^3} \left(1 + \frac{a^3}{2r^3}\right). \tag{11}$$

2 Multipoles

2.1 Spherical Multiple Moment

Consider the system where you have point charges $+q$ at $(a,0,0)$ and $(0,a,0)$ and charges $-q$ at $(-a,0,0)$ and $(0,-a,0)$. Derive the spherical multiple moment $q_{l,m}$ and write down the first two non vanishing terms. Express the charge density in spherical coordinates and check that the integral over these densities produce the appropriate total charge.

2.2 Multiple Moments in Cartesian Coordinates

1. Prove that Q_{ij} is traceless.
2. Assume that $q, \vec{p}, Q_{ij}$ are in a specific coordinate system. Now find the new quantities in a coordinate system which is related to the previous one by an $\vec{R}$ displacement. Assume now that you have charges q at $(0,a,0)$ and $(0,0,a)$ and charge $-q$ at $(a,0,0)$
3. Find $q, \vec{p}, Q_{ij}$ and check that the later one is traceless.
4. Can you find a coordinate system such that $\vec{p}' = 0$? If yes what is the displacement vector $\vec{R}$?

2.3 Exterior Multipoles for a Specified Potential on a Sphere

Let $\Phi(R,\theta,\phi)$ be specified values of the electrostatic potential on the surface of a sphere. Show that the general form of an exterior, spherical multipole expansion implies that,

$$\Phi[\vec{r}] = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{R}{r}\right)^{l+1} Y_{l,m}[\Omega] \int d\Omega' \Phi[R,\Omega'] Y^*_{l',m'}[\Omega'] \tag{12}$$

For $r > R$. Given the previous potential expression, imagine the eight octants of a spherical shell which are maintained at alternating electrostatic potentials $\pm V$ as shown below in the following picture:

Where view a is in perspective and b is looking down the z axis from above. Use the results from previous section to find the asymptotic $(r \to \infty)$ form of the potential produced by this shell configuration.

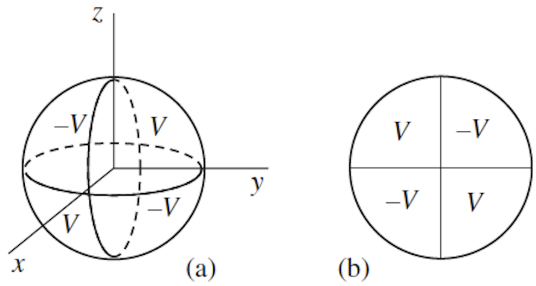

Figure 1: Potential distribution across the octants.

2.4 Radiating Fidget Spinner

Three identical point charges q are at the corners of an imaginary equilateral triangle that lies in the $x-y$ plane. The charges rotate with constant angular velocity ω around the z -axis, which passes through the center of the triangle. Find the angular distribution of electric dipole, magnetic dipole, and electric quadrupole radiation (treated separately) produced by this source.

3 Macroscopic Media

3.1 A Conducting Sphere at a Dielectric Boundary

A conducting sphere with radius R and charge Q sits at the origin of coordinates. The space outside the sphere above the $z = 0$ plane has dielectric constant κ_1. The space outside the sphere below the $z = 0$ plane has dielectric constant κ_2.

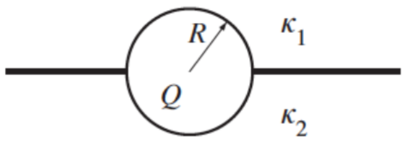

Figure 2: Dielectric distribution around the sphere.

1. Find the potential everywhere outside the conductor.

2. Find the distributions of free charge and polarization charge wherever they may be.

3.2 Polarization by Superposition

Two spheres with radius R have uniform but equal and opposite charge densities $\pm\rho$. The centers of the two spheres fail to coincide by an infinitesimal displacement vector δ. Show by direct superposition that the electric field produced by the spheres is identical to the electric field produced by a sphere with a suitably chosen uniform polarization $\mathbf{P}$.

3.3 The Field at the Center of a Polarized Cube

A cube is polarized uniformly parallel to one of its edges. Show that the electric field at the center of the cube is $\mathbf{E}(0) = -\mathbf{P}/3\epsilon_0$. Compare with $\mathbf{E}(0)$ for a uniformly polarized sphere. Hint: Recall the definition of solid angle.

3.4 E and D for an Annular Dielectric

1. The entire volume between two concentric spherical shells is filled with a material with uniform polarization $\mathbf{P}$. Find $\mathbf{E}(\mathbf{r})$ everywhere.

2. The entire volume inside a sphere of radius R is filled with polarized matter. Find $\mathbf{D}(\mathbf{r})$ everywhere if $\mathbf{P} = P\hat{\mathbf{r}}/r^2$.

3.5 E: A Charge and A Conducting Sphere

1. A charge q is placed at a distance d away from the center of a conducting sphere of radius $a < d$. Let the potential at infinity and on the surface of the sphere be 0. Using the method of images find the total charge induced on the surface of the sphere.

2. Suppose the conducting sphere and the charge q are as above but the potential on the surface of the sphere is $V \neq 0$ (the potential at infinity is 0). Find the total charge on the surface of the sphere (hint: you need to place a second "image charge" at the center of the sphere).

3. Now consider a different situation. There are two conducting spheres of radius a whose centres are at a distance d that is much greater than a. The potential at infinity is 0. One of the spheres is kept at a potential V and the other at $-V$. Because $a \ll d$ when discussing the fields near one of the spheres you can approximate the other sphere as a single point charge located at its center. Using this approximation find the total charge on the surface of each of the spheres.

4. Finally imagine that the space in between the two spheres is filled with a medium

of conductivity σ so that, in the presence of an electric field, there will be a current density $\vec{J} = \sigma\vec{E}$. Using Gauss's law find the total current I flowing between the two spheres. (Note: ignore the effects of any $\vec{B}$ produced by the moving charges). Compute the effective resistance of the circuit $R = \frac{2V}{I}$ as a function of a and d. What happens to R as $d \to \infty$. What happens to R as $a \to 0$?

5. (For a bonus point) Can you give a qualitative reason for the behavior of R found above? (Hint: think of resistors in series and parallel).

3.6 E: Critical strain

A parallel plate capacitor is made of two identical parallel conducting plates of area A. One plate carries a charge $+q$ and the other a charge $-q$. The capacitor is filled with a dielectric medium with permittivity ϵ. The distance between the two plates d is variable because the dielectric is elastic. The elastic energy stored in the dielectric is:

$$U_{\rm el} = \frac{1}{2}k\,(d - d_0)^2 . \tag{13}$$

where d_0 and k are constants.

1. Find the separation of the plates at equilibrium $d(q)$.

2. Find and plot the potential difference between the plates at equilibrium $V(q)$ as a function of q. Interpret the result.

4 Solutions: Electrostatics