Electrodynamic Solved Problems

Daniel Panizo

Contents

1 Introduction

1.1 About these notes

These notes contain a set of selected problems to discuss during the problem solving session of Classical Electrodynamics subject at Uppsala University (Sweden). The order you see in the table of content correspond to chronological order of the lectures for this course. Each set of problems is related to the correponding lectures where that content was discussed during the course (i.e. **L1** = first lecture). The title of each problem statement is linked to its solution. Try first without looking at... Exercises with an **E** in front of them correspond to old exam ones.

In case you find some typo, mistake or section to improve, please send an email to daniel.panizo@... with indications where the issue is[1].

1.2 Recommended Bibliography

- **Classical Electrodynamics**, John David Jackson. You may not like this book at first glance. Neither second, third... but it contains a formal and serious approach to all the topics that are going to be covered during the lectures. It contains important examples and explanations.

- **Introduction to Electrodyamics**, David J. Griffiths. Excellent book for a first approach to many of the concepts in this course. Its level does not cover the one expected for this course, but after reading once[2] you can jump into Jackson.

- **Electromagnetic Field Theory**, Bo Thidé. It does not contain all the material of the course, but it includes several derivations of formulae and a good final appendix with tons of identities and explanations of the mathematical tools.

- **Space and Geometry: An introduction to General Relativity**, Sean Carroll. This is some extra material to read about tensor notation. The first chapter, and part of the second one, cover the properties of the tensorial language we

[1] Title of the problem, number of equation as reference, etc
[2] Sections, not the whole book.

are going to use. This will be useful for the covariant formalism of electrodynamics and Lagrangian manipulation parts of this course.

- **FMM: Exercise Notes**, S.Giri & G. Kälin. Uploaded to Studium. It contains the most useful mathematical methods and examples that show how to use them. Totally recommended to refresh your mathematical manipulation.

- **Internet**. As you may know, apart from Social Networks and kitten videos, it contains an enormous amount of resources when used in a proper way.

1.3 Tips to enhance your understanding

Here we offer a set of tips in order to enhance your problem-solving capability.

- Read twice/ thrice/ hundredice the statement of a problem until you really understand what is asking you to solve. You can apply the same principle when reading through sections of books, notes, etc.

- "Pachanguera": Although it is a Spanish word to describe dynamic-noisy-low quality music, it can be also used to describe what a drawing sketch is. It is easier to remember what the problem is asking for if you draw a low quality picture of the set up. You can understand a problem in a better way if you translate to a picture the description given in the statement.

- "Explain yourself": It is nice for your future self[3] and for the people who will correct your exercises/exam if you explain with descriptive sentences the process of your calculations. It gives a context to whoever reads through your problems and help you to stay focus on the final target (solution) you are looking for.

- "Tolle, Lege": Take it, read it. Saint Augustine was wise enough to know that if you do not open and read books, you will not learn. It applies from religion to physics. If you do not understand what you are reading, try first point of these recommendations. Also, you are more than encouraged to ask the Teacher or teacher assistant.

[3] Has it not happened to you that you try to do your exercises again to prepare for the exam and you cannot understand why you calculated something in a particular way?

2 Problems

2.1 Electrostatics (L1, L2)

2.1.1 Conducting ball

A conducting ball of radius R and total charge Q sits in a homogeneous electric field $\vec{E} = E_0\hat{z}$. How does the electric field change by the presence of the ball? (Make an Ansatz of the form $\Phi(r,\theta,\phi) = f_0(r) + f_1(r)\cos\theta$ and motivate it.) Tip: $\hat{Z} = \cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}$.

2.1.2 Conducting ball Again

1. A point charge q sits at $\vec{a}$ inside a conducting uncharged sphere that is earthed with radius $R(|\vec{a}| < R)$.Compute the potential and the electric field inside the sphere using the method of mirror charges. Compute also the induced charge density on the surface of the sphere and show that the total charge on the surface is $-q$. What does the Gauss theorem say about the electrical field outside the sphere?

2. Do the same analysis with the change that the sphere is isolated and uncharged. Tip: Determine the electric field outside the sphere with the new b.c.

3. Follow again the same procedure as b for a sphere that is isolated and with charge Q.

2.1.3 The Capacitance of an off-centered Capacitor

A spherical conducting shell centered at the origin has radius R_1 and is maintained at potential V_1. A second spherical conducting shell maintained at potential V_2 has radius $R_2 > R_1$ but is centered at the point $s\hat{\mathbf{z}}$ where $s \ll R_1$.

1. To lowest order in s, show that the charge density induced on the surface of the inner shell is

$$\sigma(\theta) = \epsilon_0 \frac{R_1 R_2 (V_2 - V_1)}{R_2 - R_1}\left[\frac{1}{R_1^2} - \frac{3s}{R_2^3 - R_1^3}\cos\theta\right]. \tag{2.1.1}$$

   Hint: Show first that the boundary of the outer shell is $r_2 \approx R_2 + s\cos\theta$.

2. To lowest order in s, show that the force exerted on the inner shell is:

$$\mathbf{F} = \int dS \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}} = \hat{\mathbf{z}} 2\pi R_1^2 \int^{\pi} d\theta \sin\theta \frac{\sigma^2(\theta)}{2\epsilon_0}\cos\theta = -\frac{Q^2}{4\pi\epsilon_0}\frac{s\hat{\mathbf{z}}}{R_2^3 - R_1^3}. \qquad (2.1.2)$$

2.1.4 Spherical cavity and spherical functions

Consider a sphere of radius a where the surface of the upper hemisphere has a potential $+\Phi_0$ and the surface of the lower hemisphere has a potential $-\Phi_0$. In this case the Green Function is given by:

$$G(r, r') = \frac{1}{\left|\vec{r} - \vec{r'}\right|} - \frac{a}{r'\left|\vec{r} - \frac{a^2}{r'^2}\vec{r'}\right|}, \qquad (2.1.3)$$

where $\vec{r'}$ refers to a unit source outside the sphere and $\vec{r}$ to the point where the potential is evaluated.

1. Using the expression for the expansion of $\frac{1}{\left|\vec{r}-\vec{r'}\right|}$ in the appropriate basis show that the Green's function can be written as

$$G(r, r') = 4\pi \sum_{l,m} \frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right] Y_{l,m}^*(\theta', \phi')\, Y_{l,m}(\theta, \phi), \qquad (2.1.4)$$

2. Using Dirichlet boundary conditions, show that the potential outside the sphere has fol- lowing the expansion.

$$\Phi(r, \theta, \phi) = \sum_{lm} \frac{l+1}{a^2(2l+1)}\left(\frac{a}{r}\right)^{l+1} Y_{l,m}(\theta, \phi) \int \Phi_0(\theta', \phi')\, Y_{l,m}^*(\theta', \phi')\, d\Sigma', \qquad (2.1.5)$$

which tends to 0 as $r \to \infty$.

2.1.5 Green's function between concentric spheres

Consider the green's function for Neumann b.c. in the volume V between two concentric spheres between $r = a$ and $r = b, a < b$. We write the potential as

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x')\, G(x, x')\, d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'} G da', \qquad (2.1.6)$$

where S is the surface of the boundary. This implies that the b.c. for the Green's function is given by:

$$\frac{\partial}{\partial n'} G\left(x, x'\right) = -\frac{4\pi}{S}, \tag{2.1.7}$$

or x' in S. Expanding the Green's function in spherical harmonics we get:

$$G\left(x, x'\right) = \sum_{l=0}^{\infty} g_l\left(r, r'\right) P_l(\cos\gamma), \tag{2.1.8}$$

where $g_l\left(r, r'\right) = \frac{r_l^l}{r_>^{l+1}} + f_l\left(r, r'\right)$, and γ is the angle between the vector x and x'.

Also here one can prove that $P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_m Y_{l,m}^*\left(\theta', \phi'\right) Y_{l,m}(\theta, \phi)$.

1. Show for $l > 0$ that the Green's function takes the symmetric form:

$$g_l\left(r, r'\right) = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1} - a^{2l+1}} \left[\frac{l+1}{l}\left(r r'\right)^l + \frac{l}{l+1} \frac{(ab)^{2l+1}}{\left(r r'\right)^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right] \tag{2.1.9}$$

2. Use the Green's function that you found in the situation that you have a normal electric field $E_r = -E_0 \cos\theta$ at $r = b$ and $E_r = 0$ at $r = a$. Show that the potential inside V is

$$\Phi(x) = E_0 \frac{r\cos\theta}{1-p^3}\left(1 + \frac{a^3}{2r^3}\right), \tag{2.1.10}$$

where $p = \frac{a}{b}$. Find also for the electric field that:

$$E_r(r, \theta) = -E_0 \frac{\cos\theta}{1-p^3}\left(1 + \frac{a^3}{r^3}\right), \quad E_\theta(r, \theta) = E_0 \frac{\sin\theta}{1-p^3}\left(1 + \frac{a^3}{2r^3}\right). \tag{2.1.11}$$

2.2 Multipoles (L3)

2.2.1 Spherical Multiple Moment

Consider the system where you have point charges $+q$ at $(a, 0, 0)$ and $(0, a, 0)$ and charges $-q$ at $(-a, 0, 0)$ and $(0, -a, 0)$. Derive the spherical multiple moment $q_{l,m}$ and write down the first two non vanishing terms. Express the charge density in spherical coordinates and check that the integral over these densities produce the appropriate total charge.

2.2.2 Multiple Moments in Cartesian Coordinates

1. Prove that Q_{ij} is traceless.
2. Assume that $q, \vec{p}, Q_{ij}$ are in a specific coordinate system. Now find the new quantities in a coordinate system which is related to the previous one by an $\vec{R}$ displacement. Assume now that you have charges q at $(0, a, 0)$ and $(0, 0, a)$ and charge $-q$ at $(a, 0, 0)$
3. Find $q, \vec{p}, Q_{ij}$ and check that the later one is traceless.
4. Can you find a coordinate system such that $\vec{p}' = 0$? If yes what is the displacement vector $\vec{R}$?

2.2.3 Exterior Multipoles for a Specified Potential on a Sphere

Let $\Phi(R, \theta, \phi)$ be specified values of the electrostatic potential on the surface of a sphere. Show that the general form of an exterior, spherical multipole expansion implies that,

$$\Phi[\vec{r}] = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{R}{r}\right)^{l+1} Y_{l,m}[\Omega] \int d\Omega' \Phi[R, \Omega'] Y^*_{l',m'}[\Omega'] \tag{2.2.1}$$

For $r > R$. Given the previous potential expression, imagine the eight octants of a spherical shell which are maintained at alternating electrostatic potentials $\pm V$ as shown below in the following picture:

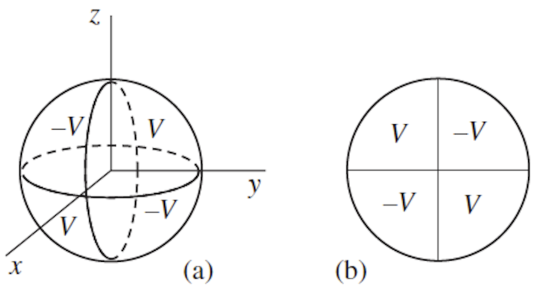


Figure 1: Potential distribution across the octants.

Where view a is in perspective and b is looking down the z axis from above. Use the results from previous section to find the asymptotic ($r \to \infty$) form of the potential produced by this shell configuration.

2.2.4 Radiating Fidget Spinner

Three identical point charges q are at the corners of an imaginary equilateral triangle that lies in the $x-y$ plane. The charges rotate with constant angular velocity ω around the z -axis, which passes through the center of the triangle. Find the angular distribution of electric dipole, magnetic dipole, and electric quadrupole radiation (treated separately) produced by this source.

2.3 Macroscopic Media (L3, L4)

2.3.1 A Conducting Sphere at a Dielectric Boundary

A conducting sphere with radius R and charge Q sits at the origin of coordinates. The space outside the sphere above the $z = 0$ plane has dielectric constant κ_1. The space outside the sphere below the $z = 0$ plane has dielectric constant κ_2.

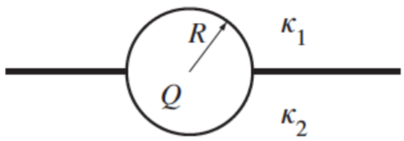


Figure 2: Dielectric distribution around the sphere.

1. Find the potential everywhere outside the conductor.

2. Find the distributions of free charge and polarization charge wherever they may be.

2.3.2 Polarization by Superposition

Two spheres with radius R have uniform but equal and opposite charge densities $\pm\rho$. The centers of the two spheres fail to coincide by an infinitesimal displacement vector δ. Show by direct superposition that the electric field produced by the spheres is identical to the electric field produced by a sphere with a suitably chosen uniform polarization **P**.

2.3.3 The Field at the Center of a Polarized Cube

A cube is polarized uniformly parallel to one of its edges. Show that the electric field at the center of the cube is $\mathbf{E}(0) = -\mathbf{P}/3\epsilon_0$. Compare with $\mathbf{E}(0)$ for a uniformly polarized sphere. Hint: Recall the definition of solid angle.

2.3.4 E and D for an Annular Dielectric

1. The entire volume between two concentric spherical shells is filled with a material with uniform polarization $\mathbf{P}$. Find $\mathbf{E}(\mathbf{r})$ everywhere.

2. The entire volume inside a sphere of radius R is filled with polarized matter. Find $\mathbf{D}(\mathbf{r})$ everywhere if $\mathbf{P} = P\hat{\mathbf{r}}/r^2$.

2.3.5 E: A Charge and A Conducting Sphere

1. A charge q is placed at a distance d away from the center of a conducting sphere of radius $a < d$. Let the potential at infinity and on the surface of the sphere be 0. Using the method of images find the total charge induced on the surface of the sphere.

2. Suppose the conducting sphere and the charge q are as above but the potential on the surface of the sphere is $V \neq 0$ (the potential at infinity is 0). Find the total charge on the surface of the sphere (hint: you need to place a second "image charge" at the center of the sphere).

3. Now consider a different situation. There are two conducting spheres of radius a whose centres are at a distance d that is much greater than a. The potential at infinity is 0. One of the spheres is kept at a potential V and the other at $-V$. Because $a \ll d$ when discussing the fields near one of the spheres you can approximate the other sphere as a single point charge located at its center. Using this approximation find the total charge on the surface of each of the spheres.

4. Finally imagine that the space in between the two spheres is filled with a medium of conductivity σ so that, in the presence of an electric field, there will be a current density $\vec{J} = \sigma\vec{E}$. Using Gauss's law find the total current I flowing between the two spheres. (Note: ignore the effects of any $\vec{B}$ produced by the moving charges). Compute the effective resistance of the circuit $R = \frac{2V}{I}$ as a function of a and d. What happens to R as $d \to \infty$. What happens to R as $a \to 0$?

5. (For a bonus point) Can you give a qualitative reason for the behavior of R found above? (Hint: think of resistors in series and parallel).

2.3.6 E: Critical strain

A parallel plate capacitor is made of two identical parallel conducting plates of area A. One plate carries a charge $+q$ and the other a charge $-q$. The capacitor is filled with a dielectric medium with permittivity ϵ. The distance between the two plates d is variable because the dielectric is elastic. The elastic energy stored in the dielectric is:

$$U_{\mathrm{el}} = \frac{1}{2}k\,(d-d_0)^2\,. \tag{2.3.1}$$

where d_0 and k are constants.

1. Find the separation of the plates at equilibrium $d(q)$.

2. Find and plot the potential difference between the plates at equilibrium $V(q)$ as a function of q. Interpret the result.

2.4 Light and Polarisation

2.4.1 Elliptic Polarisation Wave

Assume electromagnetic wave $\vec{E}(x,t)$ and the magnetic part of it that will not contribute in the exercise. The propagation vector is in the z direction $\vec{k} = k\hat{z}$ and the wave has the following form

$$E_x(\vec{x},t) = A\cos(kz-\omega t), \tag{2.4.1a}$$
$$E_y(\vec{x},t) = B\cos(kz-\omega t+\phi). \tag{2.4.1b}$$

1. Show that the vector $\vec{E}(0,t)$ parametrizes an ellipse. Note that this vector describe the polarization. For which values of A, B and ϕ the polarization parametrizes a circle? Tip: The ellipse equation is of the form $ax^2+2bxy+cy^2+f=0$.

2. Show for general A and B that the wave can be written as a superposition of two opposite circular polarized waves

$$\vec{E}(\vec{x},t) = \mathrm{Re}\left(\vec{E}_+(z,t)+\vec{E}_+(z,t)\right) \tag{2.4.2}$$

where $\vec{E}_\pm(z,t) = A_\pm\epsilon_\pm e^{i(kz-\omega t)}$. Here we have that $A_\pm$ are constants that need to be found and $\epsilon_\pm = \frac{1}{\sqrt{2}}(\hat{x}\pm i\hat{y})$.

2.4.2 A Sandwich of Light

Assume two half planes made out of a homogeneous isotropic, non magnetic, loss-free, dielectric medium with refraction index n. The two planes are separated by vacuum and they are d distance away from each-other.

A wave is propagated from the below hitting the first surface of the medium with vacuum with angle α. The wave has frequency ω.

Consider the two cases where the propagation is perpendicular to the plane of incident. Describe the phenomenon and find how much of the wave was transmitted or reflected (energy/time).

2.4.3 Faraday Rotation During Propagation

For propagation along the z -axis, a medium supports left circular polarization with index of refraction n_L and right circular polarization with index of refraction n_R. If a plane wave propagating through this medium has $\mathbf{E}(z=0,t) = \hat{\mathbf{x}}E\exp(-i\omega t)$, find the values of z where the wave is linearly polarized along the y -axis.

2.4.4 Charged Particle Motion in a Circularly Polarized Plane Wave

A particle with charge q and mass m interacts with a circularly polarized plane wave in vacuum. The electric field of the wave is $\mathbf{E}(z,t) = \mathrm{Re}\left\{(\hat{\mathbf{x}} + i\hat{\mathbf{y}})E_0 \exp[i(kz-\omega t)]\right\}$.

1. Let $v_\pm = v_x \pm i v_y$ and $\Omega = 2qE_0/mc$. Show that the equations of motion for the components of the particle's velocity v can be written

$$\frac{dv_z}{dt} = \frac{1}{2}\Omega\left\{v_+ e^{+i(kz-\omega t)} + v_- e^{-i(kz-\omega t)}\right\} \tag{2.4.3a}$$

$$\frac{dv_\pm}{dt} = \Omega\,(c - v_z)\,e^{\mp i(kz-\omega t)} \tag{2.4.3b}$$

2. Let $\ell_\pm = v_\pm e^{\pm i(kz-\omega t)} \pm ic\Omega\omega$ and show that

$$\frac{dv_z}{dt} = \frac{1}{2}\Omega\,(\ell_+ + \ell_-) = i\frac{\Omega}{2\omega}\frac{d}{dt}\,(\ell_+ - \ell_-) \tag{2.4.4}$$

3. Let K be the constant of the motion defined by the two $\dot{v}_z$ equations above. Differentiate the equations in part (a) and establish that

$$\frac{d^2 v_z}{dt^2} + \left[\Omega^2 + \omega^2\right] v_z = \omega^2 K \tag{2.4.5}$$

Use the initial conditions $v(0) = 0$ and $v_z'(0) = 0$ to evaluate K and solve for $v_z(t)$. Describe the nature of the particle acceleration in the z -direction.

2.4.5 E: A Wave and Some Boundary Conditions

Consider an electromagnetic wave propagating in the vacuum in the half-space $x_3 \geq 0$.

$$\vec{E}_i(\vec{x}, t) = \vec{E}_0 e^{\mathrm{i}\vec{k}\cdot\vec{x} - \mathrm{i}\omega t}, \tag{2.4.6a}$$

$$\vec{B}_i(\vec{x}, t) = \frac{\hat{k}}{c} \times \vec{E}, \tag{2.4.6b}$$

where $\vec{E}_i$ satisfies $\vec{k} \cdot \vec{E}_i = 0$ and the components of $\vec{k}$ are real. The frequency satisfies $\omega^2 = c^2 \vec{k} \cdot \vec{k}$.

1. Suppose this wave is incident on a perfectly conducting plane placed at $x_3 = 0$. Let the plane of incidence be formed by $\vec{k}$ and $\hat{x}_3$. Write down an expression for the electric and magnetic fields for the reflected wave $\vec{E}_r$ and $\vec{B}_r$. (Consider separately the case where $\vec{E}_r$ and $\vec{E}_i$ are both perpendicular to the plane of incidence and the case where they are both contained in it.)

2. Now suppose there is a second conducting plane located at $x_3 = d > 0$. Derive what are the conditions on $\vec{k}$, such that in the region $0 < x_3 < d$ the electric and magnetic fields are given by:

$$\vec{E} = \vec{E}_i + \vec{E}_r, \quad \vec{B} = \vec{B}_i + \vec{B}_r, \tag{2.4.7}$$

   where the incident and reflected fields are those found above.

3. Suppose now that the two conducting planes are orthogonal to each other. One is placed at $x_3 = 0$ and the other at $x_1 = 0$. How many plane-waves do you need generically to satisfy the Maxwell equations (with the appropriate boundary conditions) in the region $x_1 > 0, \quad -\infty < x_2 < +\infty, \quad x_3 > 0$? Write down the electric and magnetic fields for one such solution.

2.4.6 E: Waving at the Properties of a Wave

Let $\vec{E} = \hat{y}E_0 e^{i(hz-\omega t)-\kappa x}$ be the electric field of a wave propagating in vacuum. The parameters E_0, h, ω, κ are real.

1. What is the magnetic field of the wave?
2. Use the wave equation for $\vec{E}$ to determine a relation between h, κ and ω.
3. Compute the time averaged Poynting vector.

2.5 Waveguides and Cavities

2.5.1 Electromagnetic Crosswalk

Imagine two electromagnetic beams intersecting at right angles. $(\bar{E_H}, \bar{B_H})$ (moving in the horizontal direction) propagates in the $+x$ axis. $(\bar{E_V}, \bar{B_V})$ (Vertical direction) propagates in the $+y$ direction. For simplicity, each beam is taken as a pure plane wave cut of transversely so its cross section is a perfect square of area λ^2 (Here λ stands for the "space" each beam occupy). The fields are given by:

$$\vec{E}_H = -E_0 e^{i(kx-\omega t)}\hat{z} \tag{2.5.1a}$$
$$c\vec{B}_H = E_0 e^{i(kx-\omega t)}\hat{y} \tag{2.5.1b}$$
$$\vec{E}_V = E_0 e^{i(ky-\omega t)}\hat{y} \tag{2.5.1c}$$
$$c\vec{B}_V = E_0 e^{i(ky-\omega t)}\hat{x} \tag{2.5.1d}$$

Where $|x| = |y| = |z| < \lambda/2$. The beams overlap in a cube centered at the origin where the total fields are given by a linear combination of vertical and horizontal ones.

1. Calculate the time-averaged energy density $\langle u_{EM}(\bar{r})\rangle$ for the horizontal beam, the vertical beam and the total field in the overlap region. Show that the least of these takes its minimum value on the plane $x = y$. Compute $\bar{E}$ and $\bar{B}$ on this plane.
2. Calculate the time-averaged Poynting vector $\langle S(\bar{r})\rangle$ for the H beam, the V beam and the total field as in previous part. Try to make a sketch of $\langle S(x, y)\rangle$ everywhere the fields are defined.

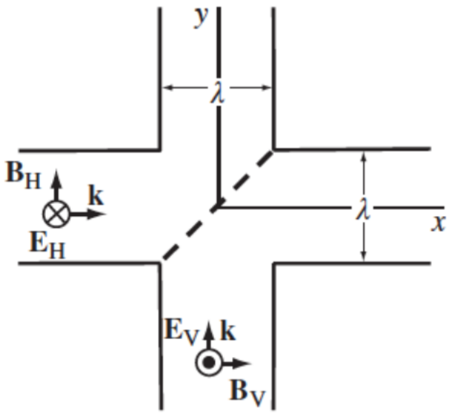


Figure 3: A sketch representation of the crossing beams and their components.

2.5.2 Waveguide Discontinuity

Two rectangular waveguides with different major sides ($a_1 < a_2$) along the x-axis and equal minor sides ($b_1 = b_2$) along the y-axis[4] are joined in the $z = 0$ plane ($x = y = 0$). The first region (a_1) propagates a $TE_{1,0}$ mode in the $+z$-direction towards the second region (a_2). Find the amplitude of some excited modes in the second region. Check also the limit where $a_1 = a_2$[5].

2.5.3 Guess Who? (Wavefilter Edition)

The figure below shows two circular conducting tubes in cross section. Each tube has a thin metal screen inserted at one point along its length. One screen takes the form of metal wires bent into concentric circles. The other takes the form of metal wires arranged like the spokes of a wheel. One of these tubes transmits only a low-frequency TE waveguide mode down the tube. The other transmits only a low-frequency TM waveguide mode down the tube. Explain which tube is which and why, using the fact that the fields of a general waveguide satisfy $\nabla \times \mathbf{E}_\perp = i\omega B_z \hat{z}$.

2.5.4 An Electromagnetic Bat in a Resonant Cavity

The two-dimensional vectors $\mathbf{k}_m$ shown below are inclined at angles $\theta_m = m\pi/3$ with respect to the positive x -axis. The vectors share a common magnitud $|\mathbf{k}_m| = k$. Superpose six waves with alternating amplitudes to form the scalar function

[4] I rotate the axis in my solution, but the result should be the same.

[5] To find the modes and limits, consider that the remaining open space at $x = y = 0$ between waveguides is closed by a perfect conductor, so modes cannot scape from our set up.

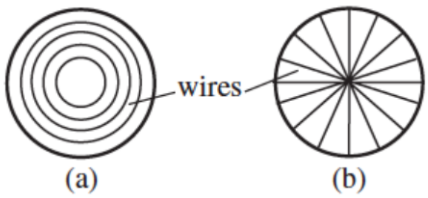


Figure 4: Both described wavefilters.

$$\psi(x, y, t) = \sum_{m=0}^{5} (-1)^k \sin(\mathbf{k}_i \cdot \mathbf{r} - ckt) \tag{2.5.2}$$

Draw the outline of a two-dimensional resonant cavity which supports a TM mode built from $\psi(x, y, t)$.

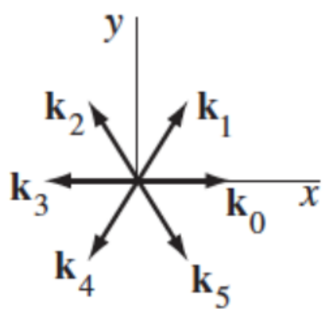


Figure 5: The vectorial distribution of the six waves.

2.5.5 Cutting off the Modes

Transverse electric and magnetic waves are propagated along a hollow, right, circular cylinder with inner radius R and conductivity σ. Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (dominant mode) in terms of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this part, assume that the conductivity of the cylinder is infinite.

2.5.6 E: Rectangular Waveguide and its Modes

Consider a waveguide whose section in the x-y plane is a rectangle with sides of length a and b (see figure). The waveguide walls are perfect conductors. The inside of the waveguide can be considered to be the vacuum.

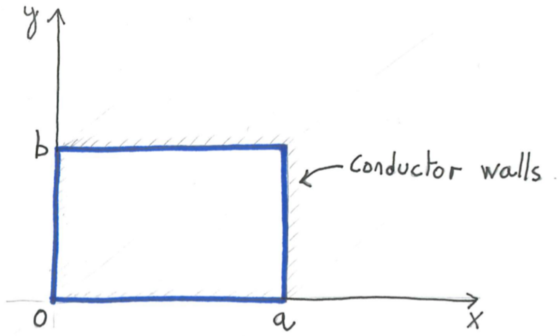


Figure 6: A sketch picture of the waveguide's section.

1. What are the boundary conditions that the electric $\vec{E}$ and magnetic $\vec{B}$ fields need to satisfy at the surface of a perfect conductor?

2. Consider a function $\psi(x, y)$ that satisfies the equation

$$\left(\partial_x^2 + \partial_y^2\right)\psi(x, y) + \gamma^2\psi(x, y) = 0. \tag{2.5.3}$$

in the interior of the rectangle for some $\gamma > 0$. The cutoff frequencies of TE and TM modes for the waveguide are obtained determining the possible values of $\gamma > 0$ in the equation above provided that the function $\psi(x, y)$ satisfies certain boundary conditions at the walls of the waveguide. For TM modes it must be that

$$\psi\big|_{\text{wall}} = 0, \tag{2.5.4}$$

while for TE modes,

$$\left.\frac{\partial\psi}{\partial n}\right|_{\text{wall}} = 0. \tag{2.5.5}$$

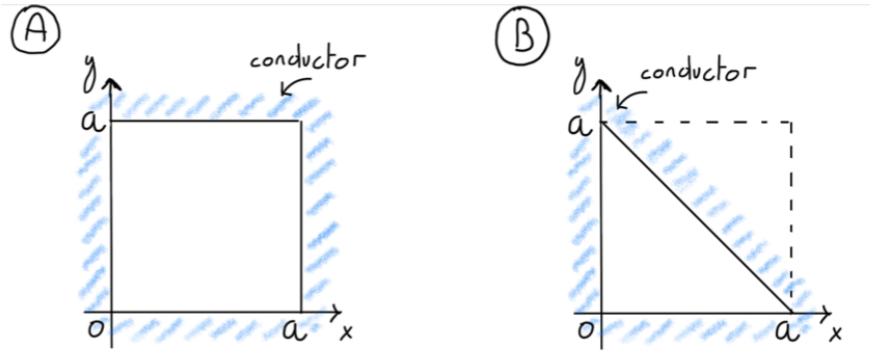
Where $\left.\frac{\partial\psi}{\partial n}\right|_{\text{wall}}$ is the derivative in the direction perpendicular to the wall. The cutoff frequencies are then given by $\omega = c\gamma$.

(a) For TM modes what is the smallest cut-off frequency?

(b) For TE modes what is the smallest cut-toff frequency?

2.5.7 E: Mirror mirror on the wall...

Consider a waveguide whose section in the x-y plane is a square with sides of length a (see figure A below). The waveguide walls are perfect conductors. The inside of the waveguide can be considered to be the vacuum.



1. What are the boundary conditions that the electric $\vec{E}$ and magnetic $\vec{B}$ fields need to satisfy at the surface of a perfect conductor?

2. Find the TM and TE modes for this wave-guide. For each mode display $\vec{E}\cdot\hat{z}$ for the TM modes and $\vec{B}\cdot\hat{z}$ for the TE modes. Also find the cutoff frequency for every mode.

3. Certain distinct modes have the same cutoff frequency. Why? By taking appropriate linear combinations of the modes sharing the same cutoff frequency construct TM and TE modes for a waveguide whose section is a right isosceles triangle with catheti (short sides) of length a (see figure B above). Show explicitly $\vec{E}\cdot\hat{z}$ for the TM modes and $\vec{B}\cdot\hat{z}$ for the TE modes.

2.6 Radiation and Scattering

2.6.1 Electric Dipole Radiation

Imagine two tiny metal spheres at distance d from each other connected by a wire, where at time t, the one sphere carries a charge $q(t) = q_0\cos(\omega t)$ while the other sphere is given by $-q(t)$.

1. Calculate the electric potential far away from the dipole. Use $d << r$ and $d << \frac{c}{\omega}$

2. Take the limit of $\omega \to 0$. What do you expect?

3. Now look at the case where also $r >> \frac{c}{\omega}$, that is, when we are interested in large distances from the source in comparison to the wavelength. How does the expression for the potential simplify in this case?

4. Obtain an expression for the vector potential in the limit $d << r$ and $d << \frac{c}{\omega}$.

5. Calculate the resulting electric and magnetic fields in the same limit with also $r >> \frac{c}{\omega}$.

2.6.2 Metallic Shells

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. an alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V \cos \omega t$. In the long-wavelength limit, find the radiation field, the angular distribution of radiated power and the total radiated power from the sphere.

2.6.3 Electrostatic Potential from a Dipole

Consider a dipole that has distance $\vec{x}'$ and a point P at distance $\vec{x}$ far away from the dipole. Considering the general expression for the potential without boundary conditions show that at large distances from the charge distribution the potential can be approximated by using the electric dipole moment in first order. Then calculate the potential in the case where the dipole is formed by two charges q^+ and q^- with distance d between them.

2.6.4 Radiation Interference

Let the origin of coordinates be centered on a compact, time-harmonic source of electromagnetic radiation. The time-averaged power radiated into a differential element of solid angle $d\Omega$ centered on an observation point $\mathbf{r}$ has the form

$$\frac{dP}{d\Omega} \propto |\hat{\mathbf{r}} \times \alpha| \tag{2.6.1}$$

The vector $\alpha = \mathbf{p}_0$ if the source has a time-dependent electric dipole moment $\mathbf{p}(t) = \mathbf{p}_0 \cos \omega t$. The vector $\boldsymbol{\alpha} = \mathbf{m}_0 \times \hat{\mathbf{r}}$ if the source has a time-dependent magnetic dipole moment $\mathbf{m}(t) = \mathbf{m}_0 \cos \omega t$. For this problem, consider a source where $\mathbf{p}(t)$ and $\mathbf{m}(t)$ are present simultaneously.

1. Show that the time-averaged angular distribution of power generally exhibits interference between the two types of dipole radiation. Under what conditions is there no interference?

2. Show that the time-averaged total power emitted by the source does not exhibit interference.

2.6.5 Sinusoidal thin Antenna

A thin linear antenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation.

1. Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.

2. Determine the total power radiated and find a numerical value for the radiation resistance.

3. Calculate the multipole moments (electric dipole, magnetic dipole, and electric quadrupole) exactly.

2.6.6 Scattering in Solid Sphere

A solid uniform sphere of radius R and conductivity σ acts as a scatterer of a plane-wave beam of unpolarized radiation of frequency ω, with $\omega R/c << 1$. The conductivity is large enough that the skin depth δ is small compared to R.

1. Justify and use a magnetostatic scalar potential to determine the magnetic field around the sphere, assuming the conductivity is infinite.

2. determine the absorption cross section of the sphere. Tip: The power loss from a waveguide is $\frac{P_{\text{loss}}}{da} = \frac{1}{2\sigma\delta}|\hat{n} \times \vec{H}|^2$.

2.6.7 Aperture (Science)

The aperture or apertures in a perfectly conducting plane screen can be viewed as the location of effective sources that produce radiation (the diffracted fields). An aperture whose dimensions are small compared with a wavelength acts as a source of dipole radiation with the contributions of other multipoles being negligible.

1. Show that the effective electric and magnetic dipole moments can be expressed in terms of integrals of the tangential electric field in the aperture as follows:

$$\vec{p} = \epsilon \hat{n} \int \left(\vec{x} \cdot \vec{E}_{tan}\right) da, \tag{2.6.2a}$$

$$\vec{m} = \frac{2}{i\omega\mu} \int \left(\hat{n} \times \vec{E}_{tan}\right) da. \tag{2.6.2b}$$

where $\vec{E}_{\text{tan}}$ is the exact tangential electric field in the aperture, $\hat{n}$ is the normal to the plane screen, directed into the region of interest, and the integration is over the area of the openings.

2. Show that the expression for the magnetic moment can be transformed into

$$\vec{m} = \frac{2}{\mu} \int \vec{x}(\hat{n} \cdot \vec{B}) da. \tag{2.6.3}$$

2.6.8 Born Scattering from a Dielectric Cube

A plane wave $\mathbf{E}_0 \exp\left[i\left(\mathbf{k}_0 \cdot \mathbf{r} - \omega t\right)\right]$ scatters from a dielectric cube with volume $V = a^3$ and electric susceptibility $\chi \ll 1$. Two cube edges align with $\mathbf{k}_0$ and $\mathbf{E}_0$.

1. Calculate the differential scattering cross section in the Born approximation.

2. Show that $\sigma_{\text{Born}} \approx \frac{1}{4}k^2 a^4 \chi^2$ when $ka \gg 1$. Hint: The near-forward direction dominates the scattering when $ka \gg 1$

3. The weak scattering assumed by the Born approximation implies that

$$|\mathbf{E}_{\text{rad}}| \,/\, |\mathbf{E}_0| \ll 1, \tag{2.6.4}$$

for all $\mathbf{q}$, even when $r \approx a$. Deduce from this that the $ka \gg 1$ result of part (b) is valid only when $\sigma_{\text{Born}} \ll \chi a^2$.

2.6.9 E: Two Antennas Sitting Together

A circular loop of radius a made of conducting wire is centred at the origin and lies in the $x_3 = 0$ plane. Let be the polar angle in the $x_3 = 0$ plane (i.e. figure). The wire carries a current oscillating at frequency

$$\vec{I} = I_0 \hat{\phi} e^{-i\omega t}, \tag{2.6.5}$$

with I_0 real. There is also a small antenna wire of length $2a$ along $\hat{x}_3$ centered at the origin as in figure. An oscillating current is fed into the antenna at its midpoint so that, away from the midpoint, the wire carries a linear charge density

$$\lambda = i\lambda_0 e^{-i\omega t}, \quad \text{for} \quad 0 < x_3 < a, \quad \lambda = -i\lambda_0 e^{-i\omega t} + c.c, \quad \text{for} \quad -a < x_3 < 0. \qquad (2.6.6)$$

Where λ_0 is real.

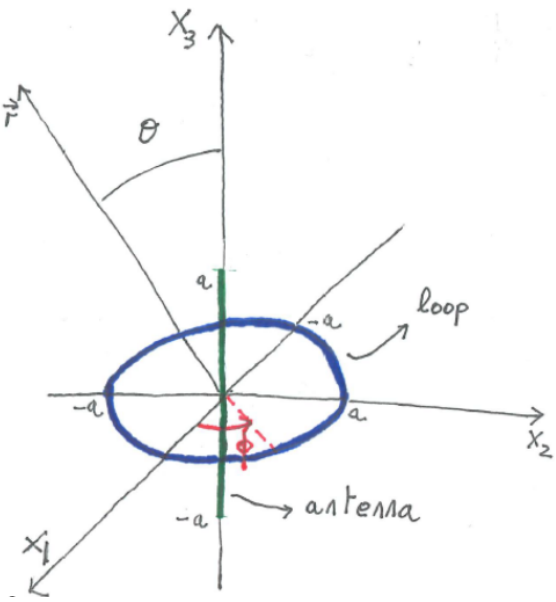


Figure 7: The two described antennas.

1. Find the electric dipole moment $\vec{p}(\omega)$ and the magnetic dipole moment $\vec{m}(\omega)$ at fre- quency ω due to the antenna and to the wire loop.

2. Work in the approximation that $\frac{c}{\omega} \gg a$ so that a multipole expansion is meaningful. Determine the vector potential $\vec{A}(\vec{r},\omega)$ in Lorentz gauge due to the dipole moments above in the radiation zone (that is $|\vec{r}| \gg \frac{c}{\omega}$).

3. In the same approximation write down the electric and magnetic fields $\vec{E}(\vec{r},\omega)$ and $\vec{B}(\vec{r},\omega)$ in the radiation zone.

4. Determine the power emitted per unit solid angle by the antenna and loop in the radiation zone. Write the answer as a function of the angle θ between $\hat{x}_3$ and $\hat{r}$.

2.6.10 E: One... Err, Two Antennas

Consider a small antenna wire of length $2a$ along $\hat{x}_3$. Let the center of the wire be at the origin. A current oscillating at frequency ω is fed into the antenna at its midpoint so that away from the midpoint, the wire carries a linear charge density

$$\lambda = \lambda_0 e^{-i\omega t} \text{ for } 0 < x_3 < a, \quad \lambda = -\lambda_0 e^{-i\omega t} \quad \text{for} \quad -a < x_3 < 0, \tag{2.6.7}$$

Where λ_0 is real.

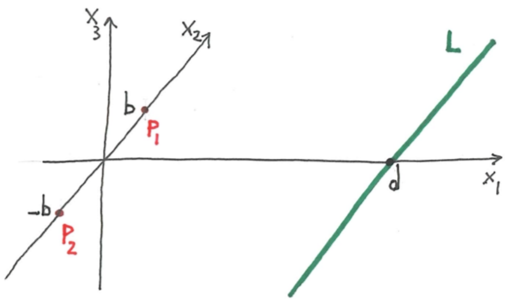


Figure 8: The aforamentioned antennas.

1. Find the electric dipole moment at frequency ω of the antenna $\vec{p}(\omega)$.
2. Determine the current $\vec{I}(x_3)$ flowing along the wire.
3. Work in the approximation that $\frac{c}{\omega} \gg a$ so that a multipole expansion is meaningful. Determine the vector potential $\vec{A}(\vec{r},\omega)$ in Lorentz gauge due to the antenna in the radiation zone (that) is $|\vec{r}| \gg \frac{c}{\omega}$).
4. In the same approximation write down the electric and magnetic fields $\vec{E}(\vec{r},\omega)$ and $\vec{B}(\vec{r},\omega)$ in the radiation zone.
5. Now consider placing two antennas identical to the one above at the two points (see figure)

$$P1 : (x_1 = 0, x_2 = b, x_3 = 0) \text{ and } \mathrm{P2} : (\mathrm{x}_1 = 0, \mathrm{x}_2 = -\mathrm{b}, \mathrm{x}_3 = 0)\,. \tag{2.6.8}$$

   The two antennas are pointing along $\hat{x}_3$ and they are oscillating in phase.

(a) Let $b = \frac{5\pi c}{\omega}$. Determine the electric field $\vec{E}(x_2)$ along the line L (see figure) located at $x_3 = 0, x_1 = d$. Assume that $d \gg b$

(b) Does the electric field you found above vanish somewhere along the line L? If so where? Explain your result.

2.6.11 E : Who bent my Antenna?

An antenna is made of a circular conducting wire loop of radius a centered at the origin. It lies in the $x = 0$ plane. Let $-\pi < \alpha \leq \pi$ be the polar angle in the $x = 0$ plane (see figure at the top of next page). There is a gap in the wire at $\alpha = \pi$ so no current can flow across. The antenna is fed an RF signal at $\alpha = 0$ so that the wire carries a current oscillating at frequency ω

$$\vec{I} = I_0(\pi - |\alpha|)\hat{\alpha}e^{-i\omega t}, \quad -\pi < \alpha < \pi, \tag{2.6.9}$$

with I_0 real.

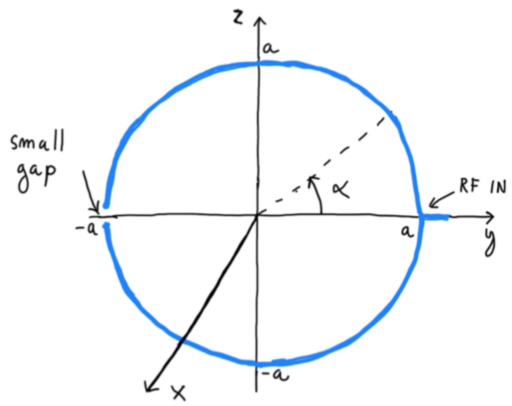


Figure 9: Who bent it?

1. Find the electric dipole moment $\vec{p}(\omega)$ and the magnetic dipole moment $\vec{m}(\omega)$ at frequency ω of the wire loop.

2. Work in the approximation that $\frac{c}{\omega} \gg a$ so that a multipole expansion is meaningful. Determine the vector potential $\vec{A}(\vec{r},\omega)$ in Lorentz gauge in the radiation zone (that is $|\vec{r}| \gg \frac{c}{\omega}$) due to the dipole moments above.

3. In the same approximation write down the electric and magnetic fields $\vec{E}(\vec{r},\omega)$ and $\vec{B}(\vec{r},\omega)$ in the radiation zone.

4. Determine the power emitted per unit solid angle by the loop in the radiation zone.

2.7 Covariant Formalism of Electrodynamics

2.7.1 Getting Familiar with Four-Vectors

In the following exercise, we will learn some basic four-vector manipulations. The greek indices $\mu, \nu, \ldots$ take values $0, 1, \ldots, d$, where d is the dimension of space:

1. Derive the position vector: Let now $x^\mu = \left(x^0, x^1, \ldots, x^d\right)$ and $\partial_\mu = \frac{\partial}{\partial x^\mu}$.What is $\partial_\mu x^\mu$? Can you see that it is indeed a (Lorentz) scalar?

2. We can define a general tensor as an object with multiple indices, both up and down, i.e. $A^{\mu\nu\rho}_{\gamma\delta\sigma}$. Its transformation properties follow from those ones of the tensor product of vectors, i.e. $x'^\mu y'^\nu = \Lambda^\mu_\sigma \Lambda^\nu_\gamma x^\sigma y^\gamma$, which implies that $A'^{\mu\nu} = \Lambda^\mu_\sigma \Lambda^\nu_\gamma A^{\sigma\gamma}$.

   Prove however, that not every tensor can be written as a product of vectors. This means that it is not always possible to find a^μ, b^ν such that $\Sigma^{\mu\nu} a^\mu b^\nu$ (even if $S^{\mu\nu}$ is symmetric).

3. In order to distiguish between different tensors, we can tag them depending on their properties. In the following, let $A^{\mu\nu}$ be an antisymmetric tensor, that is $A^{\mu\nu} = -A^{\nu\mu}$ and $S^{\mu\nu}$ to be a symmetric tensor, so $S^{\mu\nu} = S^{\nu\mu}$.

   (a) Show that the (anti)symmetry property of a tensor is preserved by the Lorentz transformations.

   (b) Prove that $S^{\mu\nu} A_{\mu\nu} = 0$.

   (c) Let us now introduce the concept of symmetrization and antisymmetrization of a tensor with two indices. For an arbitrary tensor $C^{\mu\nu}$ we can define that $C^{(\mu\nu)} = \frac{1}{2}\left(C^{\mu\nu} + C^{\nu\mu}\right)$. In the same spirit, its antisymmetrisation goes as $C^{[\mu\nu]} = \frac{1}{2}\left(C^{\mu\nu} - C^{\nu\mu}\right)$.

   Show that a general tensor with two indices can be uniquely decomposed into the symmetric and antisymmetric part $C^{\mu\nu} = C^{(\mu\nu)} + C^{[\mu\nu]}$.

2.7.2 Covariant Formalism of Electrodynamics

1. Given the electromagnetic field tensor $F^{\mu\nu}$ with components

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon^{ijk} B_k, \quad F^{\mu\nu} = -F^{\nu\mu} \tag{2.7.1}$$

where $\epsilon_{123} = 1$, compute in terms of $\vec{E}$ and $\vec{B}$ fields the following tensor objects:

- $-F^{\mu\nu} F_{\mu\nu}$
- $\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$

2. Show that the Maxwell equations,

$$\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0, \tag{2.7.2}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{2.7.3}$$

are equivalent to the Bianchi identity $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$.

3. Given the energy-momentum tensor,

$$T^{\mu\nu} = F^{\mu}_{\rho} F^{\rho\nu} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}, \tag{2.7.4}$$

compute the components of T^{ij} in terms of $\vec{E}$ and $\vec{B}$ fields.

4. Show that the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ is invariant under Lorentz transformations.

2.7.3 Lorentz Transformations for the Electromagnetic Field

1. Prove the general Lorentz transformation of the electric and the magnetic field.
2. Argue what happens to the angle between the electric and the magnetic field under a general boost transformation.

2.7.4 Three Observers. "One Field"

For some event, observer A measures $\mathbf{E} = (\alpha, 0, 0)$ and $\mathbf{B} = (\alpha, 0, 2\alpha,)$ and observer B measures $\mathbf{E}' = \left(E'_x, \alpha, 0\right)$ and $\mathbf{B}' = \left(\alpha, B'_y, \alpha,\right)$. Observer C moves with velocity $v\hat{\mathbf{x}}$ with respect to observer B.

Find:

1. the fields $\mathbf{E}'$ and $\mathbf{B}'$ measured by observer B.

2. the fields $\mathbf{E}''$ and $\mathbf{B}''$ measured by observer C.

2.7.5 Transformation of Force

A cylindrical column of electrons has uniform charge density ρ_0 and radius a.

1. Find the force on an electron at a radius $r < a$.

2. A moving observer sees the column as a beam of electrons, each moving with uniform speed $\mathbf{v}$. What force does this observer report is felt by an electron in the beam at a radius $r < a$?

2.7.6 A Long Wire Moving Fast

An infinitely long straight wire of negligible cross-sectional area is at rest and has a uniform linear charge density q_0 in the inertial frame K'. The frame K' move with a velocity $\vec{v}$ parallel to the direction of the wire with respect to the laboratory frame K.

1. Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory.

2. What are the charge and current densities associated with the wire in its rest frame? In the laboratory?

3. From the laboratory charge and current densities, calculate directly the electric and magnetic fields in the laboratory. Compare with the results of part 1.

2.7.7 Relativistic Ohm's law

In the rest frame of a conducting medium the current density satisfies Ohm's law, $\vec{J}' = \sigma\vec{E}'$ in the rest frame.

1. Taking into account the possibility of convection current as well as conduction current, show that the covariant generalization of Ohm's law is

$$J^\mu - \frac{1}{c^2}\left(U_\nu J^\nu\right)U^\mu = \frac{\sigma}{c}F^{\mu\nu}U_\nu, \tag{2.7.5}$$

where U^μ is the 4 -velocity in the medium.

2. Find the 3 -vector current in a frame where the medium has velocity $\vec{v} = c\vec{\beta}$ with respect to some initial frame.

3. If the medium is uncharged in its rest frame, what is the charge density and the expression of the current density in the above frame.

2.7.8 E: A Loooooong Cylinder and Several Frames

1. An infinitely long cylinder of radius R has a uniform charge density ρ_0 and is at rest in an inertial frame K_0. The frame K_0 moves with a speed $\vec{v}$ parallel to the direction of the cylinder with respect to the laboratory frame K_L.

   (a) Find the electric field $\vec{E}_0$ and the magnetic field $\vec{B}_0$ in the rest frame (inside and outside the cylinder).

   (b) Find the electric field $\vec{E}_L$ and the magnetic field $\vec{B}_L$ in the frame of the laboratory (again both inside and outside the cylinder). Also find the current density $\vec{J}_L$ and the charge density ρ_L in the laboratory.

   (c) Add a second cylinder of radius R parallel to the first. The second cylinder carries a charge density ρ_L and current density $-\vec{J}_L$ in the frame of the laboratory. Let the distance between the axes of the two cylinders in the laboratory be $d > 2R$. Find the electric and magnetic fields outside the cylinders in the rest frame of the first cylinder K_0.

   (d) When there is only one cylinder is there an inertial reference frame where the electric field vanishes? In the situation with the two cylinders is there an inertial reference frame where the magnetic field $\vec{B}$ vanishes? Motivate your answers.

2. Consider the energy momentum tensor $T^{\mu\nu}(x)$ of some theory invariant under transla- tions and Lorentz transformations. The energy momentum is conserved i.e. $\partial_\mu T^{\mu\nu} = 0$.

   (a) Using the energy momentum tensor we can build a new object

   $$cM^{\mu\nu\rho}(x) = x^\rho T^{\mu\nu}(x) - x^\nu T^{\mu\rho}(x). \tag{2.7.6}$$

   Find what condition does $T^{\mu\nu}$ need to satisfy so that $\partial_\mu M^{\mu\nu\rho} = 0$. (that is $M^{\mu\nu\rho}$ is conserved.)

   (b) (For a bonus point) As seen in class the conserved four-momentum is an integral over space at any fixed time $P^\mu = \int_{t=\text{const}} d^3x T^{0\mu}$. Can you give an interpretation to the conserved quantities $N^{\nu\rho} = \int_{t=\text{const}} d^3x M^{0\mu\nu}$? Explain.

2.7.9 E: Planes and Frames

In an inertial frame K_0 there are two planes at $x_3 = 0$ and $x_3 = a$. The plane at $x_3 = 0$ carries a uniform charge surface density σ while the plane at $x_3 = a$ carries a uniform charge surface density $-\sigma$. Both planes are at rest in K_0. The frame K_0 moves with a speed $\vec{v} = v\hat{x}_1$ parallel to the x_1 axis with respect to the laboratory frame K_L.

1. Consider the electric field $\vec{E}_0$ in the inertial frame K_0. Assume that $\vec{E}_0$ vanishes for $x_3 < 0$. What is $\vec{E}_0$ between the two planes (that is for $0 < x_3 < a$) and in the region$x_3 > a$?.

2. Find the electric $\vec{E}_L$ and magnetic $\vec{B}_L$ fields in the frame of the laboratory K_L.

3. Find the charge surface densities on the two planes in the laboratory frame K_L.

4. Find the surface current densities on the two planes in the laboratory frame K_L.

5. Is there an inertial reference frame where the electric field $\vec{E}$ vanishes everywhere?

6. Consider the energy momentum tensor $T^{\mu\nu}(x)$ of some theory invariant under translations and Lorentz transformations. The energy momentum is conserved i.e. $\partial_\mu T^{\mu\nu} = 0$.

   (a) Using the energy momentum tensor we can build a new object

   $$D^\mu(x) = x_\nu T^{\mu\nu}(x). \tag{2.7.7}$$

   Find what condition does $T^{\mu\nu}$ need to satisfy so that $\partial_\mu D^\mu = 0$. (that is D^μ is conserved.)

   (b) Is the condition you found satisfied by the energy momentum tensor of the electromagnetic fields $T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\rho}F^\nu_\rho + \frac{1}{4}g^{\mu\nu}F^{\rho\lambda}F_{\rho\lambda}\right)$?

2.7.10 E: Different Points of View

In an inertial reference frame there is an infinite long wire along the $\hat{z}$ direction. The wire is at rest and carries a nonzero linear charge density λ and a nonzero current $\vec{I} = I\hat{z}$.

1. Boost to a different inertial reference frame moving with speed $\vec{v} = v\hat{z}$ with respect to the rest frame of the wire. What is the linear charge density carried by the wire in the new reference frame? What is the current?

2. Under which condition on the values of λ and $\vec{I}$ in the rest frame of the wire is it possible to boost to a frame where the electric field produced by the wire vanishes? Similarly under which condition on the values of λ and $\vec{I}$ in the rest frame of the wire is it possible to boost to a frame where the magnetic field produced by the wire vanishes?

2.7.11 E: Waves Across Reference Frames

In an inertial reference frame K the electric and magnetic fields of an electromagnetic wave are given by

$$\vec{E} = \hat{z}Ce^{i(k_x x + k_y y - \omega t)}, \quad \vec{B} = \frac{c}{\omega}\left(k_y\hat{x} - k_x\hat{y}\right)Ce^{i(k_x x + k_y y - \omega t)}. \tag{2.7.8}$$

A second reference frame K' moves with speed $\vec{v} = v\hat{x}$ with respect to K. Let the origin of K and K' coincide at $t = t' = 0$.

1. Determine the electric and magnetic fields in the reference frame K' that is $\vec{E}'\left(x', y', z', t'\right)$ and $\vec{B}'\left(x', y', z', t'\right)$.

2. What is the direction of propagation of the wave in K'? what is its frequency?

2.8 Lagrangian Manipulation

2.8.1 A Relativistic Particle Coupled to a Scalar Field

The action for a relativistic point particle coupled by a strength g to a space-time-dependent Lorentz scalar field $\varphi(x)$ is

$$S = -mc^2\int ds - g\int ds\varphi(\mathbf{r}(s)). \tag{2.8.1}$$

Find the equation of motion for the particle. How does the force on the particle differ from the Coulomb force of an electric field?

2.8.2 One-Dimensional Massive Scalar Field

A one-dimensional field theory with scalar potential $\varphi(x, t)$ is characterized by the action

$$S = \frac{1}{2} \iint dt dx \left[\frac{1}{c^2} \left(\frac{\partial \varphi}{\partial t} \right)^2 - \left(\frac{\partial \varphi}{\partial x} \right)^2 - m^2 \varphi^2 \right]. \tag{2.8.2}$$

Find the equation of motion for $\varphi(x,t)$ by both Lagrangian and Hamiltonian methods.

2.8.3 Introduction to Lagrangian Manipulations

An alternative Lagrangian density for the electromagnetic field[6] is,

$$\mathcal{L} = -\frac{1}{8\pi} \partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha. \tag{2.8.3}$$

1. Derive the Euler-Lagrange equations of motion. Are they the Maxwell equations? Under what assumptions?

2. Show explicitly, and with those previous assumptions, that this Lagrangian density differs from the usual one[7] by a four-divergence. Does this divergence affect the action or the equations of motion?

2.8.4 Coupling Extra Fields to

An axionic field [8] $a(x)$ is coupled to a gauge field $A_\mu(\vec{x})$ with an associated field strength $F_{\mu\nu}$. The action describing this system goes as:

$$\begin{aligned} \mathscr{S}[a(\vec{x}), A_\mu(\vec{x})] = -\frac{1}{2} \int d^4\vec{x} \partial_\mu a \partial^\mu a - \frac{1}{4} \int d^4\vec{x} F^{\mu\nu} F_{\mu\nu} \\ - \frac{1}{f} \int d^4\vec{x} \left[a F_{\mu\nu} * F^{\mu\nu} - 2\partial_\mu \left(a A_\nu * F^{\mu\nu} \right) \right]. \end{aligned} \tag{2.8.4}$$

Where $*F$ is dual to F and f is a constant.

1. Under what circumstances is this action Lorentz invariant?

2. Find the Equations of Motion.

[6] The one you have seen during lectures and/or books.

[7] $\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha$.

[8] Can be thought as a scalar. We will see in the solutions that indeed it needs to behave as a pseudoscalar field.

3. Show that $\mathscr{S}$ is invariant under a displacement of the axionic field as $a(\vec{x}) \to a(\vec{x}) + \epsilon$.

4. Calculate the Noether current associated to the previous displacement invariance.

2.8.5 E: Ponderous Light

Consider the following action for the four-potential A^μ and a scalar field ϕ.

$$S = \int d^4x \left(\frac{1}{8\pi} \left(\partial^\mu \phi - mA^\mu\right) \left(\partial_\mu \phi - mA_\mu\right) - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J^\mu A_\mu \right), \tag{2.8.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and J^μ is a conserved current that is $\partial_\mu J^\mu = 0$.

1. Show that the action is invariant under gauge transformations $A_\mu \to A_\mu + \partial_\mu \alpha$ provided that the scalar ϕ also shifts as $\phi \to \phi + m\,\alpha$. Gauge fix by imposing $\phi = 0$. Rewrite the action in this gauge.

2. Using the gauge fixed action write the equations of motion for A^μ.

3. By contracting the equations of motion with ∂_μ obtain an equation for $\partial_\mu A^\mu$. Use this equation to simplify the equations of motion.

4. Find the form of a plane wave solution to the equations of motion with no sources ($J^\mu = 0$). Given a wave-vector $\vec{k}$ what is the frequency of the wave? How many independent polarizations are there?

5. In the electrostatic case we have $\vec{A} = 0$. Find the electrostatic potential $\Phi = A^0$ due to a single electric charge q at rest at the origin. (Hint: you may try a solution of the form $\Phi(\vec{x}) = e^{-\alpha|\vec{x}|} f(\vec{x})$ for some function f and an appropriately chosen constant α)

2.9 Radiation and Relativistic Dynamics

2.9.1 Emission Rates by Lorentz Transformation

An electron enters and exits a capacitor with parallel-plate separation d through two small holes. The electron velocity is given by $v\hat{z}$ and it is parallel to the capacitor electric field $\vec{E}$. The change in the electron velocity is small. Calculate the total energy $\Delta U'_{EM}$ and its linear momentum $\Delta P'_{EM}$ that was radiated by the electron in both rest and laboraty frames (ΔU_{EM} and ΔP_{EM} respectively).

2.9.2 A Merry Go Round of Radiating Particles

N identical, equally spaced[9] point particles, each with a charge q, move in a circle of radius a. All of them have the same constant speed v around the ring. Show that the Lienard-Wiechert electric field is *static* everywhere on the symmetry axis.

2.9.3 The Direction of the Velocity Field

Prove that the "velocity" part of the Lienard-Wiechert electric field points to the observer from the "anticipated position" of the moving point charge. The latter is the position the charge *would* have moved *if* it retained the velocity $\vec{v}_{ret}$ from $t = t_{ret}$ to the present time of observation.

2.9.4 Radiating 14.4 Jackson Problem

Using the Liénard - Wiechert electric field, discuss the time-averaged power radiated per unit solid angle by a charged particle (e^-) in a **non-relativistic** motion in the next two different cases:

1. Along the z axis with position given by $z(t) = acos(\omega t)$,
2. In a circle of radius R in the plane xy with constant angular frecuency w_0.

2.9.5 A Fast Particle in a Constant Electric Field

A relativistic point particle with charge q and mass m moves in response to a uniform electric field $\mathbf{E} = E\hat{\mathbf{z}}$. The initial energy, linear momentum, and velocity are $\mathscr{E}_0$, p_0, and $\mathbf{u}(0) = u_0\hat{\mathbf{y}}$. Find $\mathbf{r}(t)$ and show that eliminating t gives the particle trajectory

$$z = \frac{\mathscr{E}_0}{qE}\cosh\left(\frac{qEy}{cp_0}\right). \tag{2.9.1}$$

Check the non-relativistic limit.

2.9.6 A Ringy Radiating Problem

1. A small current loop moves with constant velocity $\mathbf{v}_0$ as viewed in the laboratory frame. Find the vector potential $\mathbf{A}(\mathbf{r})$ and the scalar potential $\varphi(\mathbf{r})$ in the lab frame. It may be convenient to introduce the vector $\mathbf{R} = \mathbf{r} - \mathbf{v}_0 t$

[9] Is coronavirus still around?

2. Take the limit $v_0 \ll c$ in your formulae and deduce that the moving loop possesses both a magnetic dipole moment and an electric dipole moment.

3 Solutions

3.1 Electrostatics

3.1.1 Conducting ball

We have a conducting ball which is placed in an homogenous electric field $\vec{E} = E_z \hat{z}$. The starting point is to consider the general spherical solution for a "hollow" sphere on a constant field. Let's discuss this in depth.

This sphere has charge that is homogeneously distributed. On top of that, we can infer the following: If the field $\vec{E}$ is constant, the potential of this field should be linear with respect to z, as $-\vec{\nabla}\Phi = \vec{E}$. So $V \sim E_0 z\,\hat{z}$. But this would only be if the sphere was not present there. We also have to account for the field generated by the charge of the sphere and, hence, the potential of it. But here comes the trick. The potential generated by the sphere in negligible from far away. Why? We know that $\vec{E} = 0$ inside conductors, so the potential Φ inside of the sphere is a constant. By Gauss, we also know that the electric field generated by the sphere outside decreases proportionally as $\vec{E} \propto \frac{Q}{r^2}$. When $r \gg R$, with r being the observation position, we have that $\vec{E} \sim 0$.

So we know how the potential Φ (hence $\vec{E}$) looks like at two specific regimes. On the surface of the sphere this is:

$$\Phi_{r=R} = -\frac{Q}{4\pi\epsilon_0 R}, \tag{3.1.1}$$

and far far away from it, which goes as:

$$\Phi_{r\gg R} = \Phi_{\text{field}} + \Phi_{\text{sphere}} = E_0 \underbrace{r\cos\theta}_{z} - \cancel{\frac{Q}{4\pi\epsilon_0 R}}. \tag{3.1.2}$$

How does $\vec{E}$ look like in some mid region? Again, if we know the potential, we know the field. We have seen what the most general solution for the Laplace equation with azimuthal symmetry is given by:

$$\Phi(r,\theta) = \sum_{\ell=0} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta). \tag{3.1.3}$$

Where $P_\ell(\cos\theta)$ are Legendre polynomials... If we expand the first few terms of previous expression we can see that:

$$\Phi(r,\theta) = \underbrace{\left(A_0 + \frac{B_0}{r}\right)}_{f_0(r)} \underbrace{P_0}_{1} + \underbrace{\left(A_1 r + \frac{B_1}{r^2}\right)}_{f_1(r)} \underbrace{P_1}_{\cos\theta} + \cdots \tag{3.1.4}$$

it looks quite similar to the given Ansatz in the statement of the problem! In fact, we can now use this expression to fix the values of the coefficients A_ℓ, B_ℓ with help of boundary conditions $\Phi_{r=R}$ and $\Phi_{r\gg R}$.

$$\begin{aligned} \Phi_{r=R} &= -\frac{Q}{4\pi\epsilon_0 R} = \left(A_0 + \frac{B_0}{R}\right) + \left(A_1 r + \frac{B_1}{R^2}\right)\cos\theta = \\ &= \cdots \text{matching powers of } R \cdots = \\ &\to B_0 = -\frac{Q}{4\pi\epsilon_0}, \\ &\to A_0 + \left(A_1 r + \frac{B_1}{R^2}\right)\cos\theta = 0. \end{aligned} \tag{3.1.5}$$

If we use the boundary condition at ∞, we obtain that:

$$\begin{aligned} \Phi_{r\to\infty} &= -E_0 r\cos\theta = A_0 + \cancel{\frac{B_0}{r}} + \left(A_1 r + \cancel{\frac{B_1}{r^2}}\right)\cos\theta = \\ &= \cdots \text{matching powers of } r \cdots = \\ &\to A_1 = -E_0, \\ &\to A_0 = 0. \end{aligned} \tag{3.1.6}$$

We only need to fix the value of B_1. With all previous values fixed, go back to equation 3.1.5 to see obtain:

$$B_1 = E_0 R^3. \tag{3.1.7}$$

Then, all values together yield the following result:

$$\Phi(r,\theta) = -\frac{Q}{4\pi\epsilon_0 r} - E_0 R\cos\theta\left(\frac{r}{R} - \frac{R^2}{r^2}\right), \tag{3.1.8}$$

which one can easily see that behave as expected when $r \to 0$ and $r = R$. The only remaining thing is to compute the value of the electric field $\vec{E}$ given this potential. As we are dealing with an azimuthal symmetry, the gradient should be used in spherical coordinates. Then:

$$\vec{E} = -\left(\frac{Q}{4\pi\epsilon_0 r} + E_0 R \cos\theta \left(\frac{1}{R} + \frac{2R^2}{r^3}\right)\right)\hat{r} + \sin\theta E_0 R \left(\frac{1}{R} - \frac{R^2}{r^3}\right)\hat{\theta}. \tag{3.1.9}$$

3.1.2 Conducting ball Again

Let start by noting that the method of images consist of creating a fake charge q_f such that we can reproduce the result of the original set up in an easier way. In this specific case, as we have the sphere earthed, we already know that the potential ϕ on its surface is equal to 0 ($\phi(R)$ =0). Several points to consider in the current set up:

1. The charge q_t (t for true) can be located **anywhere** inside the sphere. The important thing is that the sphere is earthed.

2. As we have axial symmetry, this problem can be reduced to a two dimensional problem. We can use polar coordinates.

A rough sketch of this set-up considering a mirror images can be found in fig(10).

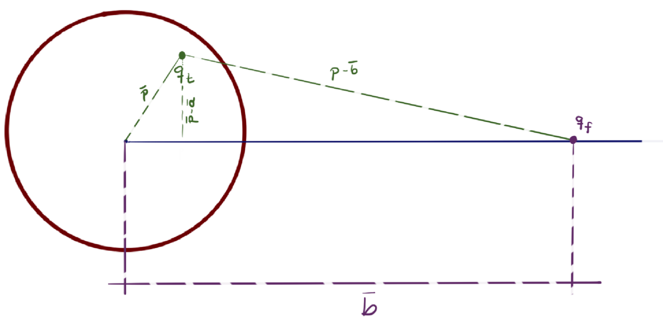


Figure 10: A rough sketch of this system studied by the method of images.

Then, the potential of this two charges, is given by superposition as:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_t}{|\vec{r}_t|} + \frac{q_f}{\left|\vec{r}_f\right|}\right), \tag{3.1.10}$$

where q_f stands for false charge/mirror charge. Imposing the boundary condition on the surface $\phi = 0$, we have:

$$\frac{q_t}{\left|\vec{p} - \vec{a}\right|} = \frac{-q_f}{\left|\vec{p} - \vec{b}\right|}. \tag{3.1.11}$$

Now, we have to realise that we have two potential positions (a, b) to fix, but just one equations... This issue can be easily solved by accounting for two possible set ups:

1. Case where $p_x = R, p_y = 0$. In this case we will arrive to an equation that looks like:

$$\frac{q_t}{|R - a|} = \frac{-q_f}{\left|R - \vec{b}\right|} \to -\frac{q_t}{q_f} = \frac{R - a}{R - b}. \tag{3.1.12}$$

   Which is a good candidate as equation to solve for one of the variables. The other can be obtained by:

2. Case where $p_x = 0, p_y = R$.

   This will give another equation as:

$$-\frac{q_t}{q_f} = \sqrt{\frac{R^2 + a^2}{R^2 + b^2}}, \tag{3.1.13}$$

   which one can square both sides to get rid of the root.

From these two set-ups, one should realise that LHS of equations (3.1.12,3.1.13) are the same. Hence, equate them and manipulate the algebra to arrive to:

$$\left(R^2 + a^2\right) b - a\, b^2 = R^2 a. \tag{3.1.14}$$

This is a well-known second order equation that gives two solutions for b as: $b_1 = a$, which makes no sense, as it tells us to place both charges at the same position, and

$b_2 = \frac{R^2}{a}$, which relates a and b in a non-trivial way. Introduce this result into any of both previous cases (3.1.12, 3.1.13) to find the relation between the charges expressed as:

$$q_f = -\frac{R\, q_t}{a} \tag{3.1.15}$$

With this result the potential ϕ follows as:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_t}{\left|\vec{p}-\vec{a}\right|} - \frac{q_t \frac{R}{a}}{\left|\vec{p}-\left(\frac{R^2}{a},0\right)\right|}\right) = \tag{3.1.16}$$

$$= \frac{q_t}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{1}{\frac{a}{R}\sqrt{r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta}}\right). \tag{3.1.17}$$

The electric field $\vec{E}$ follows from taking the minus gradient of previous expression. To obtain the induced charge density, we just need to recall that it is given the derivative of the potential ϕ respect to the normal of the surface where we want to evaluate such density, i.e.

$$\sigma = -\epsilon\frac{\partial\phi}{\partial\vec{n}}, \tag{3.1.18}$$

In our case, due to spherical sym $\vec{n} = r$ and the surface of the sphere lies on $r = R$, so we just have to evaluate there. If one is careful with all the arithmetic and simplify cautiously, the result is:

$$\sigma = \frac{-q\epsilon}{4\pi\epsilon_0 aR}\left(\frac{1-\frac{R^2}{a^2}}{\sqrt[3]{\frac{R^2}{a^2}+1-2\frac{R}{a}\cos\theta}}\right). \tag{3.1.19}$$

Regarding the Gauss theorem for the electric field outside the sphere, it will the total charge inside the sphere divided by the total area where we evaluate the field.

Solutions b and c for this problems follows exactly the same steps as we have previously done. The main difference can be found in the boundary condition on the surface of the sphere. As this is not any longer connected to earth, the potential on the surface will be given by:

$$\phi(R) = V_0. \tag{3.1.20}$$

In any case, one can repeat both cases for different positions of the inner charge to extract that the new fake charge q'_f is:

$$q'_f = q_f + V_0. \tag{3.1.21}$$

From that point on, the rest of the exercise is straightforward to adapt to this new BC.

3.1.3 The Capacitance of an off-centered Capacitor

Given the description of the statement, the first thing should be to draw something similar to this sketch:

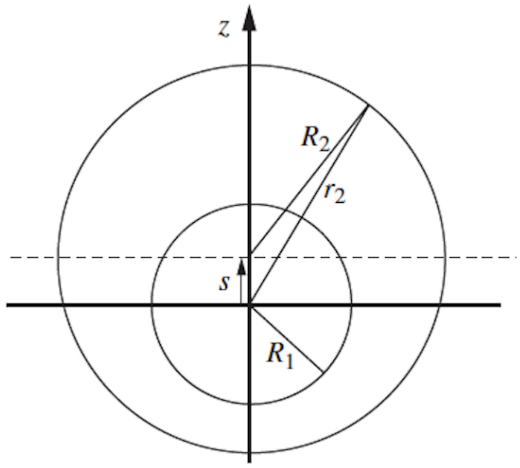


Figure 11: A rough sketch of the system we want to study....

1):

Let (r_2, θ) denote a point on the outer shell with respect to the origin of the inner shell (The small one). By the law of cosines, the difference between R_2^2 (big capacitor centre) and r^2 (small centre) is given by: $R_2^2 = r_2^2 + s^2 - 2r_2 s\cos\theta$. Therefore, expanding the square root of r_2 to first order in s, the boundary of the outer shell is:

$$r_2 = R_2 + s\cos\theta. \tag{3.1.22}$$

If the shells were exactly concentric, the potential between them would have the form $\varphi(r) = a + b/r$. Therefore, in light of the expansion to first order and the general solution of Laplace's equation in polar coordinates, we expect the potential in the space between the displaced shells to take the form[10]:

$$\varphi(r,\theta) = a + \frac{b}{r} + s\left(cr + \frac{d}{r^2}\right)\cos\theta + O\left(s^2\right) \tag{3.1.23}$$

To order s, fixing the boundary conditions at the shell surfaces we get

$$\begin{aligned} V_1 &= \varphi\left(R_1,\theta\right) = a + \frac{b}{R_1} + s\left(cR_1 + \frac{d}{R_1^2}\right)\cos\theta, \\ V_2 &= \varphi\left(r_2,\theta\right) = a + \frac{b}{R_2 + s\cos\theta} + s\left(c\left[R_2 + s\cos\theta\right] + \frac{d}{\left[R_2 + s\cos\theta\right]^2}\right)\cos\theta = \\ &= a + \frac{b}{R_2} + s\left(cR_2 + \frac{d}{R_2^2} - \frac{b}{R_2^2}\right)\cos\theta. \end{aligned} \tag{3.1.24}$$

We know that the potential on the BC V_1 and V_2 are constants, so the coefficients of $\cos\theta$ must vanish in (3.1.24). This fixes $d = -cR_1^3$ and $b = c\left(R_2^3 - R_1^3\right)$. Moreover, subtracting both conditions in (3.1.24) we get an extra equation as:

$$b = \left(V_1 - V_2\right) R_1 R_2 / \left(R_2 - R_1\right). \tag{3.1.25}$$

so c and d written in terms of R_i are:

$$c = \left(V_1 - V_2\right)\frac{R_1 R_2}{\left(R_2^3 - R_1^3\right)\left(R_2 - R_1\right)}, \qquad d = -\left(V_1 - V_2\right)\frac{R_1^4 R_2}{\left(R_2^3 - R_1^3\right)\left(R_2 - R_1\right)}. \tag{3.1.26}$$

Using (3.1.23), we can determine that the charge density on the surface of the inner shell is:

$$\sigma(\theta) = -\epsilon_0 \left.\frac{\partial\varphi}{\partial r}\right|_{r=R_1} = \epsilon_0 \frac{R_1 R_2\left(V_2 - V_1\right)}{R_2 - R_1}\left[\frac{1}{R_1^2} - \frac{3s}{R_2^3 - R_1^3}\cos\theta\right]. \tag{3.1.27}$$

[10]This is an Ansatz of the Laplace equation. Observe that the first term is the zeroth order in the expansion in terms of Legendre polynomials for the most general solution, and the second term, is the subleading order, but, there is an "s" in front of everything. This is considered as a perturbation, as the inner sphere is slightly out of the centre.

The angular term in $\sigma(\theta)$ integrates to zero. Therefore, the total charge on the inner shell and the capacitance (to first order in s) are identical to the zero-order case of a concentric capacitor:

$$C_0 = \frac{Q}{V_1 - V_2} = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1} \tag{3.1.28}$$

2):

By symmetry, there is only a z-component to the force on inner shell. Explicitly,

$$\mathbf{F} = \int dS \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}} = \hat{\mathbf{z}} 2\pi R_1^2 \int^{\pi} d\theta \sin\theta \frac{\sigma^2(\theta)}{2\epsilon_0}\cos\theta = -\frac{Q^2}{4\pi\epsilon_0}\frac{s\hat{\mathbf{z}}}{R_2^3 - R_1^3} \tag{3.1.29}$$

3.1.4 Spherical cavity and spherical functions

1): First thing we should do in order to understand this problem, is to sketch how our geometrical distribution of potentials look like. The following sketch shows that:

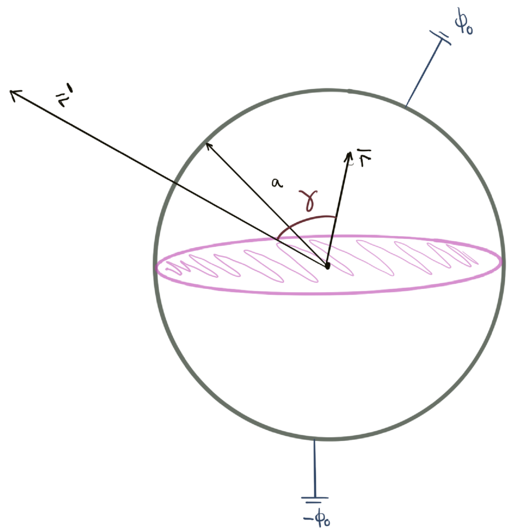


Figure 12: A rough sketch of the system we want to study....

We also know that the final appearance of the Green's function:

$$G\left(r,r'\right) = \underbrace{\frac{1}{\left|\vec{r}-\vec{r'}\right|}}_{G_1} - \underbrace{\frac{a}{r'\left|\vec{r}-\frac{a^2}{r'^2}\vec{r'}\right|}}_{G_2}. \tag{3.1.30}$$

So we have to basically massage two previous terms G_1 and G_2 to arrive to the desired result. Let's start by studying G_1. We know that, by addition theorem for spherical harmonics, G_1 can be expressed as:

$$G\left(r,r'\right) = \frac{1}{\left|\vec{r}-\vec{r'}\right|} = 4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{l,m}\left(\theta',\phi'\right)Y_{l,m}(\theta,\phi). \tag{3.1.31}$$

So this part of the Green's function does not need more explanation. On the other hand, G_2 requires some changes before we can apply previous expression. Starting from:

$$G_2(r,r') = -\vec{r'} - \frac{a}{r'\left|\vec{r}-\frac{a^2}{r'^2}\vec{r'}\right|}, \tag{3.1.32}$$

We can expand the norm in the denominator, put r' inside the square root and extract an overall a from the square root to see:

$$G_2(r,r') = \frac{-\cancel{a}}{\sqrt{\cancel{a^2}\left(\frac{(r'r)^2}{a^2}+a^2-2r'r\cos\gamma\right)}}, \tag{3.1.33}$$

where γ is the angle between both vectors $\vec{r}$ and $\vec{r'}$. We can see rr' as a general vector and $\vec{a}$ as the vector position on the surface. If we undo square root to move back to an expression in terms of the norm, we immediately see that:

$$G_2(r,r') = \frac{-1}{\left|\frac{r\vec{r'}}{a}-\vec{a}\right|}. \tag{3.1.34}$$

From here, is easy to connect this expression with that of (3.1.31), as they have the same form. Introducing this new term in eq (3.1.30) we arrive to the conclusion that (massaging powers along the way):

$$G(r,r') = 4\pi \sum_{l,m} \frac{1}{2l+1} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1} \right] Y_{l,m}^*(\theta',\phi')\, Y_{l,m}(\theta,\phi), \tag{3.1.35}$$

As we wanted to show.

2):

We would like to arrive to an expression of the form:

$$\Phi(r,\theta,\phi) = \sum_{lm} \frac{1}{a^2}\left(\frac{a}{r}\right)^{l+1} Y_{l,m}(\theta,\phi) \int \Phi_0(\theta',\phi')\, Y_{l,m}^*(\theta',\phi')\, d\Sigma', \tag{3.1.36}$$

This implies that we should start from the most general expression for a potential in terms of the Green's function, i.e:

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x')\, G(x,x')\, d^3x - \frac{1}{4\pi}\oint_S \Phi(x') \frac{\partial G}{\partial n'} dA'. \tag{3.1.37}$$

As we do not have any charge distribution in this given problem, this means $\rho(x') = 0$. So the first term will not contribute. The next step, is to evaluate the interior of the remaining integral on the boundary where we can fix some parameters. In this case, the boundary condition is such that:

$$\left.\frac{\partial G}{\partial n'}\right|_{n'=a} \qquad\qquad \Phi(a,\theta,\phi) = \pm\phi_0. \tag{3.1.38}$$

Then, we have to evaluate the derivative of the Green's function G with respect to the normal $n' = r'$ on the surface of the sphere (i.e $r' = a$ after derivation). After some algebra and powers manipulation, the result is:

$$\left.\frac{\partial G}{\partial n'}\right|_{n'=a} = 4\pi \sum_{l,m} \frac{l+1}{2l+1}\left(\frac{a^{l-1}}{r^{l+1}}\right) Y_{l,m}^*(\theta',\phi')\, Y_{l,m}(\theta,\phi). \tag{3.1.39}$$

So taking this result and introducing inside expression (3.1.37), together with the boundary condition on the surface that $\Phi(a,\theta,\phi) = \pm\phi_0$, we obtained the final desired formula as:

$$\Phi(r,\theta,\phi) = \sum_{lm} \frac{l+1}{a^2(2l+1)} \left(\frac{a}{r}\right)^{l+1} Y_{l,m}(\theta,\phi) \int \Phi_0\left(\theta',\phi'\right) Y^*_{l,m}\left(\theta',\phi'\right) d\Sigma', \tag{3.1.40}$$

which obviously fades out when $r \to \infty$.

3.1.5 Green's function between concentric spheres

1):

First of all, we should proceed as always; To draw a sketch of the system we want to study:

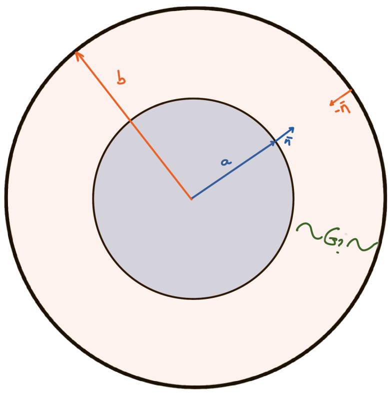


Figure 13: Some fancy sketch of the cavity we want to study.

This first part of the problem is asking us to show that, for a given concentric spherical geometry, the radial part of the Green's function looks like:

$$g_l\left(r,r'\right) = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1} - a^{2l+1}} \left[\frac{l+1}{l} \left(r r'\right)^l + \frac{l}{l+1} \frac{(ab)^{2l+1}}{\left(r r'\right)^{l+1}} + a^{2l+1} \left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}} \right) \right]. \tag{3.1.41}$$

With a hint stating that any Green's function can be decomposed in its radial part and its spherical part, as a linear combination of spherical harmonics of the form:

$$G\left(x,x'\right)=\sum_{l=0}^{\infty}g_l\left(r,r'\right)P_l(\cos\gamma), \tag{3.1.42}$$

Furthermore, we know that the Neumann's boundary condition states that:

$$\frac{\partial}{\partial n'}G\left(x,x'\right)=-\frac{4\pi}{S}, \tag{3.1.43}$$

So we basically have all ingredients to solve this part of the problem. If we know the boundary condition appearance for the general Green's function, we also know it for its radial part. We just have to fix the coefficients A_l and B_l inside g_l making use of the double boundary condition. Double, because we have two surface where we can evaluate.
When evaluating on the inner sphere with radius a, we get:

$$\left.\frac{\partial G}{\partial n'}\right|_a=\left.\sum_l\partial_r g_l(r,r')P_l(\cos\gamma)\right|_a=\frac{1}{a^2+b^2}. \tag{3.1.44}$$

Where we have used that the normal is pointing outwards, so it has positive signature. It is also remarkable to realise the following; Our previous expression (3.1.44) has neither θ nor ϕ dependence, although there is a Legendre polynomial involved in its LHS. So this is already stating that the only set of spherical harmonics that will contribute in this problem to fix the value of the coefficients are those ones with $l=0$ (i.e $Y_{00}=\frac{1}{\sqrt{4\pi}}$.) This implies that one can express the radial part of the Green's function as:

$$\left.\partial_{r'}g_l\right|_{r'=a}=\frac{-1}{a^2+b^2}\delta_{l,0}, \tag{3.1.45}$$

Now, we can do exactly the same for the outer sphere. Here there is a crucial difference; The normal to this surface, as we want to evaluate the Green's function in the region between both spheres, is pointing *inwards*, so it will cary a negative sign. The same arguments we used for the inner one applies in this case too, so the result looks like:

$$\left.\partial_{r'}g_l\right|_{r'=b}=\frac{1}{a^2+b^2}\delta_{l,0}. \tag{3.1.46}$$

Equipped with this knowledge, let's the value of the coefficients. We know that:

$$g_l\left(r,r'\right) = \frac{r_<^l}{r_>^{l+1}} + f_l\left(r,r'\right) = \frac{r_l^l}{r_>^{l+1}} + A_l r'^l + B_l r'^{-(l+1)}. \tag{3.1.47}$$

In this specific geometry we cannot drop any of the coefficients as we have done in previous exercises. This is due to the fact that we are now dealing with things happening inside and outside two different spheres that generate the given geometry. In any case, We have two boundary conditions values two fix two different equations, so we can solve for two different variables, A_l and B_l. Introducing evaluated green's radial function (3.1.45) and (3.1.46) as RHS of derivative with respect r of (3.1.47), we will get two expressions as:

$$\frac{l\,a^{l-1}}{r^{l+1}} + l\,a^{l-1} A_l + B_l \frac{-(l+1)\,a^l}{a^{2(l+1)}} = \frac{1}{a^2+b^2}\delta_{l,0}, \tag{3.1.48}$$

$$\frac{-(l+1)r^l}{b^{l+2}} + l\,b^{l-1} A_l + B_l \frac{-(l+1)\,b^l}{b^{2(l+1)}} = \frac{-1}{a^2+b^2}\delta_{l,0}. \tag{3.1.49}$$

Next step we have to perform, is just to solve for A_l and B_l in this coupled system of linear equations. Without loss of generality, we can set $l \neq 0$, so we get LHS of both expressions for the most general coefficients. The good part is that this simplifies RHS to 0. Solving then, one arrives to:

$$A_l = \frac{1}{a^{2l+1}+b^{2l+1}}\left(\frac{(l+1)\,r^l}{l} - \frac{a^{2l+1}}{r^{l+1}}\right), \tag{3.1.50}$$

$$B_l = \frac{-1}{b^{2l+1}-a^{2l+1}}\left(\frac{l\,(ab)^{2l+1}}{(l+1)r^{l+1}} + r^l a^{2l+1}\right). \tag{3.1.51}$$

The last step, after all despair and suffer we have gone through, it is just to introduce these values inside expression (3.1.47), massaging it a little bit to obtain:

$$g_l\left(r,r'\right) = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1}-a^{2l+1}}\left[\frac{l+1}{l}\left(r\,r'\right)^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{\left(r\,r'\right)^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right], \tag{3.1.52}$$

As we wanted to show.

2):

From our previous result, we can then continue in order to obtain a close expression for the potential $\Phi(r,\theta,\phi)$ in all the concentric region. Furthermore, with the potential, we can compute the value of the electric field $\vec{E}$ in that geometry. Hence, our starting point will be the expression given Neumann boundary conditions is:

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x')\, G(x,x')\, d^3x + \frac{1}{4\pi}\oint_S G(r,r')\frac{\partial\Phi}{\partial n'}dA'. \tag{3.1.53}$$

As always, the first question we have to raise when this expression pops up is: *Is there any charge in our system?* As we do not have, this implies that $\rho = 0$, so the first term will not contribute and it is the second one that does. As in the previous part, the normal n' is given by the radius r'. Then, what is $\partial_{r'}\Phi$ inside previous expression? As we know, the gradient of the potential is minus the electric field $\vec{E}$, so in this case, we get that the derivative of Φ respect to $-r'$ is just the radial component of the electric field $\vec{E}$.

As we want to find a close expression for the potential, let us evaluate our previous expression for $r' = b$ (You can also do it at $r' = a$, but you will get no information, as $E_{r=a} = 0$). Recall the sign of the norm when computing the derivative.

$$\begin{aligned}\Phi(r' = b,\theta,\phi) &= \frac{1}{4\pi}\oint_S G(r,b) \underbrace{\frac{\partial\Phi}{\partial n'}}_{-E_r = E_0\cos\theta'} \left(\underbrace{b^2\sin\theta'}_{\text{jacobian}}\right) d\Omega', \\ &= \frac{1}{4\pi}E_0 b^2 \oint G(r,b)\sin\theta'\cos\theta' d\Omega'.\end{aligned} \tag{3.1.54}$$

Inside of previous equation, we see our Green's function evaluated at b. Recall that this function can be decomposed in its radial part, as we shown in previous section (see eq(3.1.52)) and spherical harmonics. Introducing expression (3.1.42) in previous formula, and using eq(3.1.52) we obtain:

$$\Phi(b,\theta,\phi) = \frac{E_0 b^2}{4\pi}\sum_l \frac{4\pi}{2l+1}\, g_l(r,b) \oint_S \sum_m Y_{l,m}(\Omega)\, Y^*_{l',m'}(\Omega')\cos\theta'\sin\theta' d\Omega'. \tag{3.1.55}$$

Now we have a crucial point. A happy idea. Maria virgin, visiting us[11], to give us a hint. We have a cos in the game. We can exploit its presence and transform it into a spherical harmonic that can help us fix the previous expression. As you may know:

[11] This a rough translation of a say we have in Spanish.

$$Y_{1,0}(\Omega') = \sqrt{\frac{3}{4\pi}}\cos\theta'. \tag{3.1.56}$$

Introducing this in expression (3.1.55), and making use of spherical harmonics orthonormality, we obtain:

$$\Phi(b,\theta,\phi) = \sum_l \frac{E_0 b^2}{2l+1}\, g_l(r,b)\, \sqrt{\frac{4\pi}{3}}\, Y_{l,m}(\Omega)\, \delta_{m,0}\delta_{l,1}. \tag{3.1.57}$$

Simplifying annoying factors and evaluating $g_l(r,b)$ for $l = 1$, we arrive to the well deserved expression as:

$$\Phi(r,\theta) = E_0 \frac{r\cos\theta}{1-p^3}\left(1+\frac{a^3}{2r^3}\right), \tag{3.1.58}$$

with notation $p = \frac{a}{b}$. We are just there. Now take this potential and derive it respect to θ and ϕ to compute those electric field components as:

$$E_r(r,\theta) = -E_0\frac{\cos\theta}{1-p^3}\left(1+\frac{a^3}{r^3}\right), \quad E_\theta(r,\theta) = E_0\frac{\sin\theta}{1-p^3}\left(1+\frac{a^3}{2r^3}\right), \quad E_\phi(r,\theta) = 0. \tag{3.1.59}$$

Finally, we are done. Congratulations to us. Take a rest, you deserve it. And some chocolate and/or fancy beverage. You deserve it even more!

4 Multipoles

4.1 Spherical Multiple Moment

1):

The first thing one should do is to sketch how the system looks like:

Given this charge distribution, we already know that the total charge of the system is $Q = 0$. Let's first prove that the charge distribution integrated over the whole space yields also a zero. We also know that the total charge is:

$$Q_T = \int \rho(\vec{x}')d^3x' = \int \rho(r',\theta',\phi')\, r'^2 \sin\theta' dr'\, d\phi'\, d\theta'. \tag{4.1.1}$$

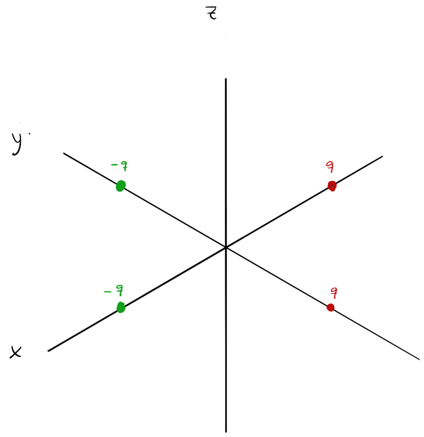


Figure 14: The distribution of the charges.

Recall that δ changes between Cartesian and spherical coordinates as:

$$\delta(\vec{x}' - \vec{x}) \rightarrow \frac{1}{r'^2 \sin\phi'} \delta(r' - r)\delta(\theta' - \theta)\delta(\phi' - \phi). \tag{4.1.2}$$

So the charge density for the four point charged particles is:

$$\begin{aligned} \rho(r') = \frac{q}{r'^2 \sin\phi'} (\delta(r' - a)\delta(\theta' - 0)\delta(\phi' - \tfrac{\pi}{2}) + \delta(r' - a)\delta(\theta' - \tfrac{\pi}{2})\delta(\phi' - \tfrac{\pi}{2}) - \\ \delta(r' - a)\delta(\theta' - \pi)\delta(\phi' - \tfrac{\pi}{2}) - \delta(r' - a)\delta(\theta' - \tfrac{3\pi}{2})\delta(\phi' - \tfrac{\pi}{2})). \end{aligned} \tag{4.1.3}$$

Which integrated over the whole space $r\epsilon[0,\infty], \theta\epsilon[0,2\pi], \phi\epsilon[0,\pi]$ yields a total charge $Q_T = 0$ as expected. What about the quadrupoles?

2):

They also want us to calculate the multiple moments (recall: The charge density $\times$ the harmonics). This is given by the following formula as:

$$
\begin{aligned}
q_{lm} &= \int Y^*_{lm}(\theta',\phi') r'^l \rho(\vec{x}') d^3x' = \\
&= \int Y^*_{lm} r'^l \frac{q\,\delta(r'-a)\delta(\phi'-\frac{\pi}{2})}{r'^2 \sin\phi} \left(\delta(\theta'-0)+\delta(\theta'-\tfrac{\pi}{2})-\delta(\theta'-\pi)-\delta(\theta'-\tfrac{3\pi}{2})\right) r'^2 \sin\phi\, d^3x' = \\
&= q\,a^l \left(Y^*_{lm}(0,\tfrac{\pi}{2})+Y^*_{lm}(\tfrac{\pi}{2},\tfrac{\pi}{2})-Y^*_{lm}(\pi,\tfrac{\pi}{2})-Y^*_{lm}(\tfrac{3\pi}{2},\tfrac{\pi}{2})\right).
\end{aligned}
\tag{4.1.4}
$$

This can be further simplified, simply using the definition of spherical harmonics given by:

$$
Y_{lm}(\theta,\phi) = \sqrt{\tfrac{2l+1}{4\pi}\tfrac{(l-m)!}{(l+m)!}} P_l^m(\cos\phi) e^{im\theta}, \qquad P_l^m(x) = (-1)^m (1-x^2)^{m/2} \tfrac{d^m}{dx^m} P_l(x).
\tag{4.1.5}
$$

So a more specific expression for q_{lm} is:

$$
q_{lm} = q\,a^l \sqrt{\tfrac{2l+1}{4\pi}\tfrac{(l-m)!}{(l+m)!}} P_l^m(\cos\tfrac{\pi}{2}) \left(1+(-i)^m-(-1)^m-i^m\right)
\tag{4.1.6}
$$

From here, one just have to compute the first non-zero entries. It follows that:

$$
q_{0,0}=q_{1,0}=q_{0,1}=0, \qquad q_{1,1} = a\sqrt{\tfrac{3}{2\pi}}(1-i), \quad q_{1,-1} = -a\sqrt{\tfrac{3}{2\pi}}(1+i).
\tag{4.1.7}
$$

4.2 Multiple Moments in Cartesian Coordinates

UNDER CONSTRUCTION

4.3 Exterior Multipoles for a Specified Potential on a Sphere

1):

The general form of a spherical multipole expansion is given by:

$$
\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}\right) Y_{\ell m}(\Omega).
\tag{4.3.1}
$$

As we are asked to show the general form of an **exterior** multipole expansion, we have to get rid of $A_{\ell m}$ as this coefficient is only for $r < R$ cases. For $B_{\ell m}$ we have the basic description:

$$B_{\ell m} = \frac{4\pi}{2\ell+1} \int \rho\left(\mathrm{r}'\right) r'^{\ell}\, Y^{\star}_{\ell m}\left(\Omega'\right) dV'. \tag{4.3.2}$$

As we have to express eq(4.3.1) without the B coefficient, we need to find its explicit value for $\forall r > R$. Here we can abuse from spherical harmonic properties as:

$$\begin{aligned} \Phi(R,\Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} \frac{Y_{\ell m}(\Omega)}{R^{\ell+1}} = \\ = \text{Multiply both sides times } Y^{\star}_{\ell' m'} \rightarrow \\ \int \Phi\left(R,\Omega'\right) Y^{\star}_{\ell' m'}\left(\Omega'\right) d\Omega' = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{B_{\ell m}}{R^{\ell+1}} \int Y_{\ell m}\left(\Omega'\right) Y^{\star}_{\ell' m'}\left(\Omega'\right) d\Omega'. \end{aligned} \tag{4.3.3}$$

The orthonormality of the spherical harmonics gives the expansion coefficients as:

$$\begin{aligned} \int d\Omega\, Y^{\star}_{\ell',m'} Y_{\ell m} = \delta_{\ell'\ell}\, \delta_{m'm} \rightarrow \\ B_{\ell m} = R^{\ell+1} \int \Phi\left(R,\Omega'\right) Y^{\star}_{\ell' m'}\left(\Omega'\right) d\Omega'. \end{aligned} \tag{4.3.4}$$

Then, all together looks like:

$$\Phi(\mathrm{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{R}{r}\right)^{\ell+1} Y_{\ell m}(\Omega) \int \varphi\left(R,\Omega'\right) Y^{\star}_{\ell' m'}\left(\Omega'\right) d\Omega', \quad r > R. \tag{4.3.5}$$

2):

We need to solve how the potential Φ looks like in asymptotic limit ($r \rightarrow \infty$) for the given distribution. In order to craft this, we have to find a linear combination of harmonics that can reproduce the behaviour of the combination of the octants. This can be done by brute force, integrating each of the different 8 regions for given solid angle or to think a little bit to reduce the required computation.

We can see that V changes to $\pm$ every $\pi/2$ in ϕ angle. We know that the associated number in the harmonics to this angle is m. Checking the values of $Y_{\ell m}$, we see that the change will occur when $m = \pm 2$, because:

$$Y_{\ell\pm2} \propto e^{\pm 2i\phi} \to e^{\pm i\pi} = \pm 1. \tag{4.3.6}$$

So, if $|m| = 2$, $\ell \geq 2$. We find a minimum value for ℓ. What about 2? Then, the associated spherical harmonic value reads:

$$Y_{22} \propto \sin^2\theta e^{2i\phi}. \tag{4.3.7}$$

This cannot be,as we want the associated value of θ to vary as $\pm$ when moving around the octants. But if we check the following harmonic:

$$Y_{3\pm2} = \tfrac{1}{4\pi}\sqrt{\tfrac{105}{2\pi}}\sin^2\theta\cos\theta e^{\pm 2i\phi}. \tag{4.3.8}$$

Where cos will allow that variation. Then $\Phi = \text{L.C}\left(Y_{3,\pm2}\right)$. We are close to be able to offer an expression for the potential when $r \to \infty$. Recall also that $\Phi|_{r=R} = \pm V$ and when $r \to \infty$, only the smallest allowed value of ℓ will contribute in the leading order of the expression. With all this, we can state that:

$$\Phi(\mathrm{r}) = V\left(\frac{R}{r}\right)^4 2\sqrt{\frac{2\pi}{105}}\left(Y_{32} + Y_{3-2}\right) = V\left(\frac{R}{r}\right)^4 \sin^2\theta\cos\theta\cos 2\phi, \quad r \to \infty. \tag{4.3.9}$$

4.4 Radiating Fidget Spinner

So we want to study this system and get its values of $\mathbf{p}, \mathbf{m}$ and Q_{ij}. Let the distance from the axis of rotation to the charges be R, as displayed in the following sketch:

The electric dipole moment, as we know, is given by:

$$\mathbf{p} = \int d^3r\, \mathrm{r}\, \rho(\mathbf{r}, t). \tag{4.4.1}$$

Hence, we need to write down the position of the charges in this spinning device. We know that each of them are at $2\pi/3$ angular distance, so:

$$\begin{aligned} \mathbf{x}_1 &= \left(R\cos(\omega t + 0), R\sin(\omega t + 0), 0\right), \\ \mathbf{x}_2 &= \left(R\cos\left(\omega t + \tfrac{2\pi}{3}\right), R\sin\left(\omega t + \tfrac{2\pi}{3}\right), 0\right), \\ \mathbf{x}_3 &= \left(R\cos\left(\omega t + \tfrac{4\pi}{3}\right), R\sin\left(\omega t + \tfrac{4\pi}{3}\right), 0\right). \end{aligned} \tag{4.4.2}$$

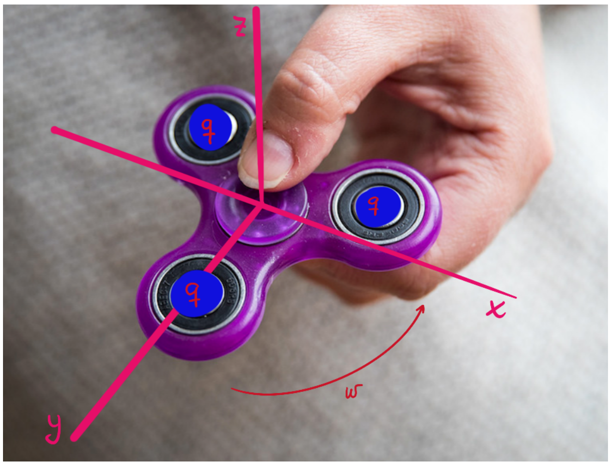


Figure 15: We can imagine we placed a charge in each of the inner holes.

We therefore have:

$$p_x = q\sum_{i=1}^{3} x_i = Rq(\cos(\omega t) + \cos(\omega t + 2\pi/3) + \cos(\omega t - 2\pi/3)) = 0 \tag{4.4.3}$$

and, similarly, $p_y = p_z = 0$. So, there is no electric dipole radiation.

The current density is given by $\mathbf{j}(\mathbf{r}, t) = \mathbf{v}\rho(\mathbf{r}, t)$ where the velocity $\mathbf{v}$ points (locally) in the direction of the particle motion with magnitude $v = R\omega$. The magnetic dipole moment can be calculated as:

$$\begin{aligned} \mathbf{m} = \int d^3r\, \mathbf{r}\times\mathbf{j} = \int d^3r \underbrace{\mathbf{r}\times\mathbf{v}}_{(x,y,0)\times(x,y,0)=\hat{z}} \rho(\mathbf{r}, t) = \\ = R^2\omega\,\hat{\mathbf{z}} \int d^3r\,\rho(\mathbf{r}, t) = 3\omega q R^2\,\hat{\mathbf{z}}. \end{aligned} \tag{4.4.4}$$

Observe that the magnetic dipole moment is time-independent! Therefore, there is no magnetic dipole radiation.

So far, no interesting properties for this system. What about the quadrupole momentum? The components of the electric quadrupole tensor are:

$$Q_{ij} = \int d^3x \left(3x^i x^j - r^2\delta^{ij}\right)\rho(\mathbf{x}, t). \tag{4.4.5}$$

The symmetry of ρ with respect to the x -axis dictates that $Q_{xy} = 0$ (Compute it yourself to check that everything nicely cancels). Since the charges all lie in the plane $z = 0$, we find that $Q_{xz} = Q_{yz} = Q_{zz} = 0$. So the only non-zero entries of Q_{ij} are:

$$\begin{aligned} Q_{xx} &= q\sum_{i=1}^{3} x_i^2 = R^2 q\left(\cos^2(\omega t) + \cos^2(\omega t + 2\pi/3) + \cos^2(\omega t - 2\pi/3) - 1\right) = \\ &= 3R^2 q(\cos(2\omega t) + \cos(2\omega t - 2\pi/3) + \cos(2\omega t + 2\pi/3)) = \frac{3}{2}R^2 q. \end{aligned} \tag{4.4.6}$$

and, recall that $Tr(Q) = 0$, so $Q_{yy} = -Q_{xx} = -\frac{3}{2}R^2 q$. Since Q is time-independent, there is no electric quadrupole radiation either.

5 Macroscopic Media

5.1 A Conducting Sphere at a Dielectric Boundary

We know that the general solution of Laplace's equation is given by:

$$\Phi(\mathbf{r},\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}}\right) Y_{\ell m}(\Omega). \tag{5.1.1}$$

1):

Let the polar z -axis pass through the center of the sphere perpendicular to the dielectric interface. Then, the solution of Laplace's equation outside the sphere is:

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta). \tag{5.1.2}$$

At the sphere boundary, we must have $\Phi(R,\theta) = V =$ const. This tells us that $B_\ell = 0$ for all $\ell \neq 0$ (as in the previous exercise, higher orders of r will not contribute) so:

$$\Phi(r,\theta) = \frac{B_0}{r} \Rightarrow \mathbf{E} = \frac{B_0}{r^2}\hat{\mathbf{r}}. \tag{5.1.3}$$

Therefore, wherever the dielectric constant is $\kappa_i (i = 1,2)$:

$$\mathbf{D}_i(r) = \epsilon_0 \kappa_i \frac{B_0}{r^2}\hat{\mathbf{r}}. \tag{5.1.4}$$

The constant B_0 can be obtained using one of Maxwell equations, $\nabla \cdot \mathbf{D} = \rho_c$. Using a spherical Gaussian surface,

$$\int_S d\mathbf{S}\cdot\mathbf{D} = \epsilon_0 B_0 2\pi \left[\kappa_1 \int_0^{\pi/2} d\theta \sin\theta + \kappa_2 \int_{\pi/2}^{\pi} d\theta \sin\theta\right] = 2\pi\epsilon_0 B_0 (\kappa_1 + \kappa_2) = Q. \tag{5.1.5}$$

Then we have:

$$\Phi(r) = \frac{Q}{2\pi\epsilon_0(\kappa_1+\kappa_2)}\frac{1}{r} \tag{5.1.6}$$

2):

The free charge on the surface of the sphere follows from Gauss' law as:

$$\sigma_c = \mathbf{D}(R)\cdot\hat{\mathbf{r}} = \begin{cases} \frac{\kappa_1}{\kappa_1+\kappa_2}\frac{Q}{2\pi R^2} & \text{in region } \kappa_1 \\ \frac{\kappa_2}{\kappa_1+\kappa_2}\frac{Q}{2\pi R^2} & \text{in region } \kappa_2 \end{cases} \tag{5.1.7}$$

There is polarization charge at the sphere boundary, as we have such a free charge distribution on the surface. Its value is $\sigma_P = (1-\kappa)\sigma_c/\kappa$. This charge is compensated by polarization charge at infinity. There is no polarization charge at the κ_1/κ_2 interface because $\mathbf{E}$ and hence $\mathbf{P}$ are everywhere radial. This means that $\mathbf{P}\cdot\hat{\mathbf{n}} = 0$ at the interface.

5.2 Polarization by Superposition

The Gauss' law electric field produced by a sphere with uniform charge density ρ centred at the origin is:

$$\mathbf{E}(r) = \begin{cases} \frac{\rho}{3\epsilon_0}\mathbf{r}, & r < R \\ \frac{\rho}{3\epsilon_0}\frac{R^3}{r^3}\mathbf{r} & r > R \end{cases} \tag{5.2.1}$$

An identical sphere, but with charge density $-\rho$ displaced from the origin by δ, produces the negative version of the previous field except that $\mathbf{r} \to \mathbf{r} - \delta$. With this in mind, the following can be approximated, such that:

$$
\begin{aligned}
|\mathbf{r}-\boldsymbol{\delta}|^{-3} = [(\mathbf{r}-\boldsymbol{\delta})\cdot(\mathbf{r}-\boldsymbol{\delta})]^{-3/2} &= \\
&= \frac{1}{r^3}\left[1-\frac{2\mathbf{r}\cdot\boldsymbol{\delta}}{r^2}+\frac{\delta^2}{r^2}\right]^{-3/2} = \\
&\approx \frac{1}{r^3}\left[1+\frac{3\mathbf{r}\cdot\boldsymbol{\delta}}{r^2}\right].
\end{aligned}
\tag{5.2.2}
$$

Hence, the total field produced by the superposition of the two spheres is:

$$
\mathbf{E}(r) = \begin{cases} \frac{\rho}{3\epsilon_0}[\mathbf{r}-(\mathbf{r}-\boldsymbol{\delta})] = \frac{\rho\boldsymbol{\delta}}{3\epsilon_0} & r < R \\ \frac{\rho R^3}{3\epsilon_0}\left\{\frac{\mathbf{r}}{r^3}-\frac{\mathrm{r}-\delta}{r^3}\left[1+\frac{3\mathbf{r}\cdot\boldsymbol{\delta}}{r^2}\right]\right\} = \frac{\rho R^3}{3\epsilon_0}\left[\frac{\delta-3(\hat{\mathbf{r}}\cdot\boldsymbol{\delta})\hat{\mathbf{r}}}{r^3}\right] & r > R \end{cases}
\tag{5.2.3}
$$

Comparing these previous results with the field produced by a sphere with volume V and polarization $\mathbf{P}$:

$$
\mathbf{E}(r) = \begin{cases} -\frac{\mathbf{P}}{3\epsilon_0} & r < R \\ \frac{V}{4\pi\epsilon_0}\left[\frac{3(\hat{\mathbf{r}}\cdot\mathbf{P})\hat{\mathbf{r}}}{r^3}-\frac{\mathbf{P}}{r^3}\right] & r > R \end{cases}
\tag{5.2.4}
$$

We find that the two of them are identical if we identify $\mathbf{P} = -\rho\boldsymbol{\delta}$.

5.3 The Field at the Center of a Polarized Cube

Our starting point will be the field produced by a polarised object, which reads:

$$
\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_V -\nabla'\mathbf{P}'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}dV' - \int_S dS'\mathbf{P}'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right]
\tag{5.3.1}
$$

For this specific cubic case, we know that ρ_P is 0, as our cube is homogeneously polarised, so $\nabla\mathbf{P} = 0$. The surface polarization $\sigma_{\mathbf{P}} = \mathbf{P}\cdot\hat{\mathbf{n}}$ is P on the right (R) face of the cube and $-P$ on the left (L) face of the cube. Since we only have surface charge,

$$
\mathbf{E}(\mathbf{r}) = \frac{P}{4\pi\epsilon_0}\left[\int_R dS'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} - \int_L dS'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right].
\tag{5.3.2}
$$

To simplify our lives, better to consider the value of all this when $\mathbf{r} = 0$. Then, at the origin:

$$
\mathbf{E}(0) = -\frac{P}{4\pi\epsilon_0}\left[\int_R dS'\frac{\mathbf{r}'}{r'^3} - \int_L dS'\frac{\mathbf{r}'}{r'^3}\right].
\tag{5.3.3}
$$

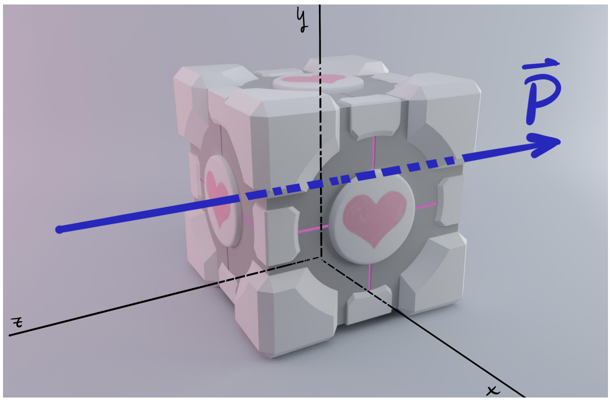


Figure 16: But the cake is still a lie...

By symmetry, the x and y components of these integrals are zero, as the polarisation only happens along the z-axis. Therefore, if the origin of the primed system is at the centre of the cube, we have:

$$\begin{aligned} E_z(0) &= -\frac{P}{4\pi\epsilon_0}\left[\int_R dS'\frac{z'}{r'^3} - \int_L dS'\frac{z'}{r'^3}\right] = -\frac{2P}{4\pi\epsilon_0}\int_R dS'\frac{z'}{r'^3} = \\ &= -\frac{2P}{4\pi\epsilon_0}\int_R d\mathbf{S}'\cdot\frac{\mathbf{r}'}{r'^3} = \frac{2P}{4\pi\epsilon_0}\int_R d\Omega'. \end{aligned} \tag{5.3.4}$$

The integral is the solid angle subtended by the right face at the centre of the cube. By symmetry, this number must be $4\pi/6$. Therefore, the electric field at the centre of the cube is:

$$\mathrm{E}(0) = -\frac{\mathrm{P}}{3\epsilon_0} \tag{5.3.5}$$

This is exactly the same as the electric field at the centre of a uniformly polarized sphere.

5.4 E and D for an Annular Dielectric

1):

We are going to treat the geometry shown below as the superposition of a ball with radius b and uniform polarization P and a concentric ball with radius a and uniform polarization −P.

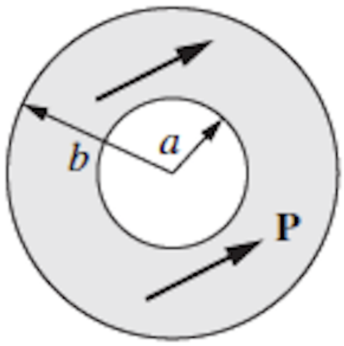


Figure 17: The two concentric spheres and the polarisation **P**.

From the text, the field produced by an origin-centered polarized ball with volume V is:

$$\mathbf{E}(\mathbf{r}) = \begin{cases} -\frac{\mathbf{P}}{3\epsilon_0} & r < R \\ \frac{V}{4\pi\epsilon_0}\left\{\frac{3(\mathbf{r}\cdot\mathbf{P})\mathbf{r}}{r^5} - \frac{\mathbf{P}}{r^3}\right\} & r > R \end{cases} \tag{5.4.1}$$

Therefore, the field we want to study is given by:

$$\mathbf{E}(\mathbf{r}) = \begin{cases} 0 & r < a, \\ -\frac{\mathbf{P}}{3\epsilon_0} - \frac{a^3}{3\epsilon_0}\left\{\frac{3(\mathrm{r}\cdot\mathbf{P})\mathbf{r}}{r^5} - \frac{\mathbf{P}}{r^3}\right\} & a < r < b, \\ \frac{b^3 - a^3}{3\epsilon_0}\left\{\frac{3(\mathbf{r}\cdot\mathbf{P})\mathbf{r}}{r^5} - \frac{\mathbf{P}}{r^3}\right\} & r > b \end{cases} \tag{5.4.2}$$

2):

By symmetry, we should have $\mathbf{D}(\mathbf{r}) = D(r)\hat{\mathbf{r}}$. Therefore, the choice of a spherical Gaussian surface of radius r gives:

$$\int_S d\mathbf{S}\cdot\mathbf{D} = D(r)4\pi r^2 = Q_{c,\mathrm{encl}} = 0. \tag{5.4.3}$$

Therefore, $\mathbf{D} = 0$ everywhere.

5.5 E: A Charge and A Conducting Sphere

UNDER CONSTRUCTION

5.6 E: Critical strain

1):

The capacitor system with two plates can vary its width, as the dielectric filling the space between the plates is elastic. If we want to find an equilibrium point, we need to compute the minima of stability of the potential controlling this system. In this case we will have two different potential energies: The electric one, stored inside the capacitor itself and the mechanical one, given by the elastic properties of the "spring" between the plates. The electromagnetic energy stored inside a capacitor is given by:

$$U_{EM} = \frac{1}{2} A\, d \vec{E} \cdot \vec{D} = \frac{d\, q^2}{2\, A \epsilon}. \tag{5.6.1}$$

As the charge q is constant at equilibrium, the equilibrium position $d(q)$ will be given by just the derivative of the total energy U of the system derived respect to d, i.e.:

$$U_T = U_{EM} + U_{\text{elas}} \to \partial_d U_T = 0, \to d = d_0 - \frac{q^2}{2A\epsilon k}. \tag{5.6.2}$$

2):

The potential inside a capacitor is given the norm of the electric field times the separation of plates, as:

$$\Delta V = \|\vec{E}\| d = \frac{q\, d(q)}{A\epsilon} = \frac{q}{A\epsilon}\left(d_0 - \frac{q^2}{2A\epsilon k}\right). \tag{5.6.3}$$

Observe that this expression has two roots: $q = 0$ and $q = \sqrt{2A\epsilon k d_0}$. It also has a maximum for q lying at $q_{max} = \frac{\sqrt{3}\, q_0}{3}$. If we plot these results, we see that:

At the critical point q_0, the whole capacitor collapses, pointing to the instability of this system.

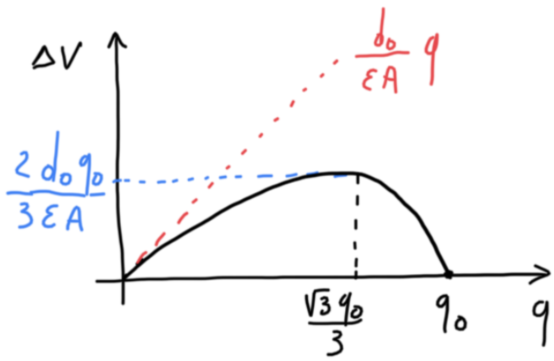


Figure 18: The potential for this elastic capacitor.