
About Dark Bubbles

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Seems that I need a general introduction here. I do not know what to say, to be honest.

1 HOW TO INDUCE DARK ENERGY ON A BUBBLE SURFACE: A SIMPLE EXAMPLE

ADD REFERENCES!

Let us just start with the most basic set-up: A non-supersymmetric Anti-de Sitter vacuum, with cosmological constant given by:

$$\Lambda_D = -\frac{1}{2}(D-1)(D-2)L^{3-D}. \quad (1.1)$$

The parameter L represents the curvature radius of the AdS_D vacuum. For computational convenience, it will be easier to refer to this parameter with its inverse, the vacuum scale $k = \frac{1}{L}$.

As we discussed in section [NON-SUSY], the aforementioned AdS_D vacuum is unstable and will eventually decay to a more metastable solution, via D_p -brane nucleation. This D_p -brane, which is a Coleman-de Luccia/ BT instanton [add ref to CdL and or section], will mediate such decay, separating two different AdS_D vacua. From now on, we will refer to the vacua living inside the hyper-volume encoded by the D_p -brane as the *inside* vacuum, denoted by a minus (-) subscript sign. The whole higher-dimensional space receives the name of *Bulk* and the tandem brane-inside will be colloquially called *Bubble*. On the other hand, the region *outside* the bubble has not yet decayed, and will be denoted by a positive (+) subscript sign.

Finally, as the aim of this thesis is to discuss phenomenological aspects in four dimensions¹, and given the discussion in [ref to GAUSS-CODAZZI] about co-dimension one hyper-surfaces, we will fix $D = 5$. This implies that the boundary ∂B of the nucleated five-dimensional bubble will have $d = D - 1 = 4$ dimensions.

In this dimensionality, the simplest geometry inside and outside the bubble can be described by the following line invariant:

$$ds_{\pm}^2 = g_{\mu\nu}^{\pm} dx^{\mu} dx^{\nu} = -f_{\pm}(r) dt_{\pm}^2 + f_{\pm}^{-1}(r) dr^2 + r^2 d\Omega_3^2, \quad (1.2)$$

where

$$f_{\pm}(r) = 1 + k_{\pm}^2 r^2 + \chi(r, t, k_{\pm}, q_1, \dots, q_m). \quad (1.3)$$

Here, r will denote the radial coordinate of the AdS_5 vacuum, i.e. the throat direction and the line-invariant $d\Omega_3^2 = \zeta_{ij} dx^i dx^j$ is the metric on S^3 , which corresponds to the three-dimensional solid angle for the usual spatial sections. From now on, Greek indices will represent bulk coordinates, while Latin ones will be used to describe quantities associated with the *induced* geometry on the boundary. Finally, it is important to mention that the function $f_{\pm}(r)$ represents the AdS_5 geometry plus some possible extra features encoded in $\chi(r, t, k_{\pm}, q_1, \dots, q_m)$. These will be relevant in the following sections, but for now on, let us fix $\chi(r, t, k_{\pm}, q_1, \dots, q_m) = 0$.

¹Other bubble dimensionalities have been explored in [IVANO STUFF]

Before we start computing the Israel's junction conditions described in [ref to APPENDIX], it is important to realise the parametrical dependence of the expanding bubble. As described in [ref to INSTANTONS], the nucleated bubble can be identified with a Coleman-de Luccia instanton, which is equipped with an $O(4)$ -symmetry.² The two relevant coordinates for this derivation are the global time t and the radial coordinate r , which also depends on t , as it is expanding, hence evolving in time. Both will be related to the proper time τ an observer living on the boundary ∂B experiments. The coordinate choices to describe the bulk geometry and the induced one are:

$$x^\mu = \{t(\tau), r(\tau), \alpha, \beta, \gamma\}, \quad y^a = \{\tau, \alpha, \beta, \gamma\}. \quad (1.4)$$

Let us now compute the first Israel junction condition described in [ref to APPENDIX]. This is no more than:

$$ds_{\text{ind}}^2 = h_{ab} dy^a dy^b = -N(\tau)^2 d\tau^2 + a^2(\tau) d\Omega_3^2, \quad (1.5)$$

where

$$N^2 = f(a(\tau)) \dot{t}^2 - \frac{\dot{a}(\tau)^2}{f(a(\tau))}. \quad (1.6)$$

The metric h_{ab} represents the four-dimensional induced geometry on the hyper-surface, i.e. the D_3 -brane, given by the parametrisation of $\{y^a\}$ -coordinates. The lapse function $N(\tau)$ has been introduced to make time reparametrisation invariance manifest. Observe that a dot represents a proper-time derivative (i.e. the time an observer living on the hypersurface will experiment). One can always choose the right parametrical relation between $f(a(\tau))$, $\dot{t}(\tau)$ and $a(\tau)$ such that $N = 1$. In that case, it is always easy to see that Eq. (1.5) represents an expanding four-dimensional cosmology, described by a Friedmann-Robertson-Lemaître-Walker metric with closed spatial sections.

What about the second Israel junction condition? It is important to notice that the presence of the expanding D_3 -brane, which divides the bulk space in two: *inside* and *outside*, will generate a discontinuity of the bulk metric across itself. This forces the presence of a non-zero induced energy-stress tensor on the brane, such that the whole configuration remains a solution to the bulk Einstein equation. This is what the second Israel junction condition accounts for. For simplicity in this example, let us assume we have an empty brane. This implies that its energy-momentum tensor only contains information about its tension σ , which can be denoted by:

$$S_{ab} = -\sigma h_{ab}. \quad (1.7)$$

COMMENT ON THE MINUS SIGN? CHECK CRITICAL SIGMA VALUE FOR POINCARÉ

As discussed in [APPENDIX], the second Israel's condition relates the *jump* in the extrinsic curvature between the inside and the outside geometries that the brane separates. To be as pedagogical as possible in this introductory section, we will compute the most important steps to obtain the induced energy-momentum tensor.³

First, we need to identify the normal and tangent vectors $\{n^\mu, e_a^\mu\}$ with respect to the embed-

²Both in Euclidean and Lorentzian time by analytic continuation.

³The whole derivation is left as an exercise to the reader.

ding. Using eqs (APPENDIX) and (APPENDIX), these result to be:

$$e_a^\mu = \begin{pmatrix} \dot{t}(\tau) & 0 & 0 & 0 & 0 \\ \dot{f}(a) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad n^\mu = \left(\frac{-\dot{a}/\dot{t}}{f(a)\sqrt{f(a) - \frac{1}{f(a)}\left(\frac{\dot{a}}{\dot{t}}\right)^2}}, \frac{f(a)}{\sqrt{f(a) - \frac{1}{f(a)}\left(\frac{\dot{a}}{\dot{t}}\right)^2}}, 0, 0, 0 \right). \quad (1.8)$$

This implies an extrensic curvature K_{ab} of the form:

$$K_{ab} = \nabla_\beta n_\alpha e_a^\alpha e_b^\beta = \quad (1.9)$$

ADD THE RIGHT BRIDGE WHEN MATHEMATICA WANTS TO WORK.

Given the previous results, if one computes the second junction condition Eq. [APPENDIX] assuming a pure constant tension brane, described by Eq. (1.7), it yields:

$$\sigma = \frac{3}{\kappa_5} \left(\sqrt{\frac{f_-(a)}{a^2} + \frac{\dot{a}^2}{N^2 a^2}} - \sqrt{\frac{f_+(a)}{a^2} + \frac{\dot{a}^2}{N^2 a^2}} \right), \quad (1.10)$$

where $\kappa_5 = 8\pi G_5$ and σ corresponds to the tension of the bubble wall. Observe how the tension of the shell is governed by the four-dimensional Hubble parameter $H = \dot{a}/a$ and the difference in the AdS₅ scales of the five-dimensional bulk. This already points to a limit situation; when the five dimensional scales k_\pm are large compared to the four-dimensional H parameter, the tension of the shell approaches the *extremal* tension⁴

$$\sigma_{\text{cr}} = \frac{3}{\kappa_5} (k_- - k_+). \quad (1.11)$$

This critical value of the tension σ_{cr} corresponds to the extremal value of the radius that maximises the bounce action of the instanton described in [APPENDIX]. This also implies that all the energy obtained in the decay process has been invested in generating a shell that will remain static, as $H \ll k_\pm$ in that limit situation.⁵

What now if the tension results to be sub-critical, i.e. a small portion of energy of the decay is invested in accelerating the bubble growth, as described in [APPENDIX]?⁶ In this case, we can assume:

$$\sigma = (1 - \epsilon) \sigma_{\text{cr}}, \quad (1.12)$$

with $\epsilon > 0$. Expanding this sub-critical value σ in Eq. (1.10), results into:

$$H = \left(\frac{\dot{a}}{a} \right)^2 = -\frac{1}{a^2} + \frac{8\pi}{3} \frac{2k_-k_+}{k_+ - k_-} G_5 (\sigma_{\text{cr}} - \sigma) + \mathcal{O}(\epsilon^2). \quad (1.13)$$

This previous equation should be familiar to the reader that does not get distracted by the suppressed corrections $\mathcal{O}(\epsilon^2)$. A little bit of dimensional analysis may help those absent-minded readers.

⁴This can be seen as a *flat* shell, as the curvature parameter a is negligible compared to the AdS₅ scale k_\pm .

⁵We will soon see that this is always the case, as the energy hierarchy of the embedding of this construction into string theory provides $k_\pm \gg H$. These corrections are also independent of $a(\tau)$.

⁶This implies that the bounce [APPENDIX] is not extremised.

As observed in Eq. (1.11), $[k_{\pm}] = [\text{Length}]^{-1}$, while $\kappa_5 = 8\pi G_5 = \ell_5^3$, where ℓ_5 is the five dimensional Planck length. So $[\kappa_5] = [\text{Length}]^3$. Hence, $[\sigma_{\text{cr}}] = [\sigma] = [\text{Length}]^{-4}$. This has the same dimensionality as the dark energy density ρ_{Λ} in four dimensions as discussed in [SECTION COSMO]. Similarly, we can perform the same dimensional analysis to conclude that $\left[G_5 \frac{k_- k_+}{k_+ - k_-}\right] = [\text{Length}]^2$. This is the dimensionality of the Newton's constant G_4 .⁷ These do not seem like coincidence, pointing to the following identifications:

$$\kappa_4 = 2 \frac{k_- k_+}{k_+ - k_-} \kappa_5, \quad \rho_{\Lambda} = \sigma_{\text{cr}} - \sigma. \quad (1.14)$$

These allow us to "unmask" Eq. (1.13). It is no more than a regular Friedmann equation in disguise! From Eq. (1.14), we see that dark energy ρ_{Λ} is no more than a dynamical property of the expanding D_3 -brane in the bulk. The expansion rate, and hence the induced dark energy density, are controlled by the sub-criticality parameter ϵ .

Let us close this section of the chapter making comments on points that will be relevant and discussed in the following sections.

- de Sitter induced cosmologies can only occur when the tension of the brane is *sub-critical* as seen in Eq. (1.14). The tension of the brane σ will dictate its evolution. As discussed in [APPENDIX to CdL], the brane will always nucleate at rest, i.e. $\dot{a} = 0$. If the tension is critical $\sigma = \sigma_{\text{cr}}$, the repulsion generated by charge of the brane Q , related to the flux difference across the wall (see [SWAMPLAND SECTION] for further details), will be exactly compensated by its tension, preventing its expansion. This results in an induced flat cosmology.

On the other hand, the brane can nucleate with a sub-critical tension $\sigma < \sigma_{\text{cr}}$, and the whole previous discussion holds, inducing an accelerated expanding cosmology. This is in line with the weak gravity conjecture described in [SWAMPLAND SECTION]. **What about supercritical stuff?**

- As the simplest example that can be provided, we have omitted the presence of extra dimensions (up to ten or eleven) required from supergravity models. These will be the main discussion in section (2), where we will embed the dark bubble model into supegravity theories. Its connections to the swampland will be also commented in this section.
- So far, we have just discussed how to induced dark energy on the brane, but the observable universe contains more than that. This will be the topic of section (4), where the induced cosmology will be decorated with matter, radiation and waves (plenty of waves)... coming from the extra dimensions. There will be understand the true power of the second Israel condition, its relation to the Gauss-Codazzi equation describe in [APPENDIX] and how to interpret them in the right way to obtain a meaningful induced energy-momentum tensor.
- Where is gravity? Is it localised on the induced geometry or does it propagates across extra dimensions? This will be examined in section (5), where will be study how strings associated to the expanding brane are required to have a good understanding of gravity in this model.

⁷Recall, as one can see in [SECTION], that $\kappa_D = \ell_D^{D-2}$.

- We have talked about expanding branes in the bulk, but, How did those branes appear there? This will be reviewed, from a quantum cosmology perspective, in section (3)
- Finally, the dark bubble model will be compared against the Randall-Sundrum model. Although both proposals share features, there are key differences at both conceptual and computational levels that may not be perceived at first glance. Section (6) will be devoted to unravel on these issues.

2 DARK BUBBLES FROM STRING THEORY: AN EXPLICIT CONSTRUCTION

I GUESS THAT HERE I NEED TO WRITE A GOOD SUMMARY OF THE MODEL I WILL USE (OSCAR'S). SPECIALLY WHAT IT WAS USED FOR IN THE PAST AND HOW GOOD IT MAY CONNECT TO THE KIND OF SCENARIO WE HAD IN MIND FOR BUBBLES.

EMPHASISE THAT THE STACK OF BLACK BRANES IS A BLACK HOLE IN HIGHER DIM.

2.1 A black hole in the bulk: Ten-dimensional Kerr or five-dimensional Reissner-Nordström one?

Let us start by defining the parametric coordinates that will be used to describe the 10 dimensional rotating background⁸: **OBSERVE THE CHANGE OF COORDINATE DEFINITION WRT THE PAPER. SO THE RESULTING EQUATIONS ARE MORE FAMILIAR WITH THE USUAL THROAT.**

$$x^\mu = \{t, \alpha, \beta, \gamma, z, \theta, \psi, \phi_1, \phi_2, \phi_3\}, \quad (2.1)$$

and its line invariant

$$ds_{10}^2 = ds_5^2 + L^2 \sum_{i=1}^3 \left\{ d\sigma_i^2 + \sigma_i^2 \left(d\phi_i + \frac{1}{L} A(z) \right)^2 \right\}, \quad (2.2)$$

where $A(z)$ is a 10 dimensional one-form that will be later discussed and the σ_i -functions are combinations of trigonometric functions of two angles⁹ $\{\theta, \psi\}$ of the S^5 as:

$$\sigma_1 = \sin \theta, \quad \sigma_2 = \cos \theta \sin \psi, \quad \sigma_3 = \cos \theta \cos \psi. \quad (2.3)$$

As introduced in section 1, L is the AdS_5 radius in Eq. (2.2). It is important to notice that this radius also sets the scale of the extra compact dimensions of S^5 . This can already raise some eyebrows, as it points to lack of scale separation, as discussed in [SEC Landscape]. However, this will be a key signature of the dark bubble embedding into supegravity, as this model acquires the aforementioned scale separation via tension-to-charge ratio instead. We will later explore this method in section 2.6.

Conducting our attention back to the line invariant (2.2), we can define the five dimensional asymptotically-AdS metric ds_5 as:

$$ds_5^2 = -h(z)^{-2} f(z) dt^2 + h(z) [f(z)^{-1} dz^2 + z^2 d\Omega_3^2], \quad (2.4)$$

⁸It is important to notice the choice of z to describe the AdS_5 throat direction. Spoiler alert: There will be a convenient change of coordinates soon to adequate r -direction. Stay alert.

⁹**ADD REF AND COMMENTS ABOUT THE PROCEDURE OF THIS ANGLES (from where to where do they run) AND PROBABLY, COMMENTS ABOUT THE SPHERE?**

where $d\Omega_3^2$ is the usual unit metric of the 3-sphere [ref to COSMO SECTION]. The radial functions $h(z)$ and $f(z)$ are expressed as

$$\begin{aligned} h(z) &= 1 + \frac{q^2}{z^2}, \\ f(r) &= 1 - \frac{m}{z^2} + \frac{z^2}{L^2} h(z)^3, \end{aligned} \quad (2.5)$$

where m and q are respectively the mass and charge of the five dimensional black. Wait? What five-dimensional black hole? Let us then perform a "massage" to Eqs. (2.4, 2.5) to provide a more familiar black hole-ish line invariant. One can start this task by performing a simple change of coordinates, defining a new radial coordinate r as:

$$r^2 = z^2 h(z) = q^2 + z^2. \quad (2.6)$$

This change of coordinates will substantially transform Eq. (2.4), yielding:

$$ds_5^2 = -g(r)dt^2 + g(r)^{-1}dr^2 + r^2 d\Omega_3^2, \quad g(r) = 1 + k^2 r^2 - \frac{2\kappa_5 M}{r^2} + \frac{\kappa_5^2 Q^2}{r^4}, \quad (2.7)$$

with

$$M = \frac{1}{2\kappa_5} (m + 2q^2), \quad Q^2 = \frac{1}{\kappa_5^2} q^2 (m + q^2), \quad (2.8)$$

with $\kappa_5 = 8\pi G_5$. It is now easy to see that the change of variables described in Eq. (2.6) casts the line invariant (2.4) in the familiar form¹⁰ of a five dimensional Reissner-Nordström black hole living in an AdS vacuum [REF]. Observe that, for a small black hole, with $r_H \ll L$, we recover a flat space description and it is required $Q < M$ to get a horizon and no naked singularity. On the other hand, if we want to study a horizon larger than the AdS₅ radius, this immediately implies $Q \ll M$.

Let us recap now the interpretation of both ten and five dimensional geometries. In the ten dimensional approach, we will observe a Kerr black hole which rotates in three of the compact directions of S^5 (i.e. $\{\phi_i\}$). When we zoom out and not have resolution of the five dimensional sphere, the black hole no longer rotates. It becomes "static" from the point of view of a five dimensional observer, but it acquires a charge q due to its motion in the compact directions (This will be further explained later). This charge becomes effective (i.e. charge Q) when the change of coordinates (2.6) is performed to recover the more familiar description of a Reissner-Nordström black hole living in AdS.

As it is well known, this type of black holes enjoy two different horizons [REFS to book]: The *outer* horizon (i.e. event horizon) and the *inner* one (i.e. the Cauchy horizon). When these two horizon become degenerate (i.e. $r_H = r_h$), this corresponds to an extremal black hole description. We will not further discuss charged black holes and their connection to the weak gravity conjecture here (see [SWAMP] for more details), but, for computational purposes in the incoming sections, we find more convenient to express (2.7) in terms of the two aforementioned horizons: $\{r_h, r_H\}$ as the inner and outer horizons. In this way, we can express $g(r)$ in Eq. (2.7) with the following Ansatz:

$$g(r) = \frac{k^2}{r^4} (r^2 + c) (r^2 - r_h^2) (r^2 - r_H^2), \quad (2.9)$$

¹⁰The patch we are describing with the metric (2.4) corresponds to the requirement $r \geq q$. This is actually not an issue since the horizon r_H will be covered by the patch.

with $c \in \mathbb{R}$. A simple match between Eqs. (2.7, 2.9) shows that:

$$\begin{aligned} M &= \frac{1}{2\kappa_5} [r_h^2 + r_H^2 + k^2 (r_h^4 + r_h^2 r_H^2 + r_H^4)], \\ Q^2 &= \frac{r_h^2 r_H^2}{\kappa_5^2} [1 + k^2 (r_h^2 + r_H^2)], \\ c &= r_h^2 + r_H^2 + \frac{1}{k^2}. \end{aligned} \quad (2.10)$$

which simplifies when the black hole is extremal. Let us keep in mind these identifications for future section and conduct our attention back to the one-form potential $A(z)$ in Eq. (2.2). This form acts as a gauge field from the 5D point of view **WHY?** In term of the old coordinates (i.e. $\{t, \alpha, \beta, \gamma, z\}$) it can be written as:

$$A(z) = \frac{q}{z_H^2 + q^2} \sqrt{(q^2 + z_H^2) + \frac{z_H^4}{L^2} h(z_H)^3} \left(1 - \frac{z_H^2 + q^2}{z^2 + q^2} \right) dt. \quad (2.11)$$

Observe that gauge freedom has been summoned to add a constant that sets $A(z_H) = 0$. This is required as a $A(z)$ is temporal gauge potential, and must vanish at the horizon. This potential can be expressed in a more handable way in two steps:

1. Part of the argument of the root in Eq. (2.11) is the mass m of the black hole¹¹. This is:

$$m = z_H^2 \left(1 + \frac{z_H^2}{L^2} h(z_H)^3 \right). \quad (2.12)$$

2. Furthermore, performing the change of variables (2.6) and making use of the identifications (2.8) to further simplify in the new coordinate system, one obtains:

$$A(r) = \kappa_5 Q \left(\frac{1}{r_H^2} - \frac{1}{r^2} \right). \quad (2.13)$$

The stack of rotating D_3 -branes in the ten dimensional background will source the self dual F_5 Ramond-Ramond (RR) field strength, which can be written in terms of the radial function $h(z)$ and the one-form $A(z)$. However, what we will need for future computations is the corresponding four-form potential C_4 , with $dC_4 = F_5$. This is:

$$C_4(z, z_H, q) = \frac{1}{L} [(z^2 + q^2)^2 - (z_H^2 + q^2)^2] dt \wedge \epsilon_3 + L^2 q \sqrt{z_H^2 + q^2 + \frac{z_H^4}{L^2} h(z_H)^3} \sum_i \sigma_i^2 d\phi_i \wedge \epsilon_3, \quad (2.14)$$

where ϵ_3 corresponds to the volume-form associated with $d\Omega_3$ in Eq. (2.4). Following the same recipe describe above, one cast the four-form C_4 in terms of the new variables as:

$$C_4(r, r_H, Q) = \underbrace{\frac{1}{L} [r^4 - r_H^4] dt \wedge \epsilon_3}_{(C_4)_t} + \underbrace{L^2 \frac{Q}{\kappa_5} \sum_i \sigma_i^2 d\phi_i \wedge \epsilon_3}_{(C_4)_\phi = \sum_i (C_4)_{\phi_i}}. \quad (2.15)$$

¹¹This can be easily obtained by computing the outer horizon in Eq. (2.4)

Observe that we have split the four-form C_4 into its time-dependent and ϕ_i -dependent pieces. This notation will further simplify computations to come.

With this we finish the black hole description, both from ten and five dimensional perspectives. Let us now explore the explicit embedding of the D_3 -brane in the ten dimensional ambient space.

2.2 The D_3 -brane dynamics

Let us now move on and study the motion of a nucleated D_3 -brane. The starting point of such enterprise is the action governing its dynamics:

$$S_{D_3} = S_{\text{DBI}} + S_{\text{WZ}} = -T_3 \int d^4\xi \sqrt{-\det P[G]} + T_3 \int P[C_4]. \quad (2.16)$$

The Dirac-Born-Infeld (DBI) term will be in charge of describing the geometrical features of the embedding, while the Wess-Zumino (WZ) term with C_4 captures the non-gravitational forces on the brane due to fields external to it. $P[\dots]$ denotes the pullback of spacetime fields to the world volume of the brane. This is no more than the projection of ten-dimensional coordinates onto the three-dimensional geometry of the D_3 -brane, with tension T_3 . These, the brane's embedding functions will be denoted by capital versions of the spacetime coordinates: $\mathcal{X}^\mu(\xi)$. In this set up, we will set the dilaton ϕ constant **WHY?**

The dynamical description of the branes is as follows: The stack of D_3 -branes, and hence the nucleated one, wrap the three-sphere $d\Omega_3$ inside the AdS_5 piece of the metric, with the radial r -coordinate (hence \mathcal{R}) depending only on the time component. The ten dimensional black hole in the background rotates in the azimuthal ϕ_i -directions on S_5 . This implies that a D_3 -brane escaping from the stack will preserve a portion of the angular momentum of this. Therefore, it will also move in those directions. Let us, for the sake of simplicity in the construction of this model, set the three angular momenta associated to each ϕ_i -directions equal in the background. This implies that the brane will rotate with the same angular velocity in all ϕ_i directions. Finally, we will assume a constant position in the remaining coordinates of the compact S_5 , i.e. $\{\Psi = \psi_0, \Theta = \theta_0\}$. The brane's embedding functions are then given by:

$$\mathcal{T} = \mathcal{T}(\tau), \quad \mathcal{R} = \mathcal{R}(\tau), \quad \Theta = \theta_0, \quad \Psi = \psi_0, \quad \Phi_1 = \Phi_2 = \Phi_3 = \Phi(\tau). \quad (2.17)$$

Here, although we write τ as a time parameter on the induced geometry, we do not yet specify if it denotes its proper time. We will later see that this is case. From the point of view of an observer living in AdS_5 , the rotation translates into an effective charge Q under the gauge field A . When a brane nucleates, carrying a portion of the charge, it will also effectively lower the charge of the remaining black hole.

Before we start to analyse the action (2.16), we need some more practical notation. Derivatives with respect to τ will be denoted by a dot. The components of the ten dimensional metric (2.1) will be denoted by $G_{\mu\nu}$. It will also be useful to define the mixing of the azimuthal-time and azimuthal-azimuthal components in the compact piece of Eq. (2.1) as $G_{t\phi} = \sum_i G_{t\phi_i}$ and $G_{\phi\phi} = \sum_i G_{\phi_i\phi_i}$ respectively.

Let us start the analysis of Eq. (2.16). The pullback $P[G]$ is no more than the induced $g_{\tau\tau}$ -metric entry of the expanding D_3 -brane. This is:

$$P[G] = g_{\tau\tau} = G_{tt}\dot{\mathcal{T}}^2 + G_{rr}\dot{\mathcal{R}}^2 + 2G_{t\phi}\dot{\mathcal{T}}\dot{\Phi} + G_{\phi\phi}\dot{\Phi}^2. \quad (2.18)$$

This is also the square of the velocity that a comoving observer living on the brane will experiment,

$$\dot{X}^2 = \dot{X}_\mu \dot{X}^\mu = G_{tt} \dot{\mathcal{T}}^2 + G_{rr} \dot{\mathcal{R}}^2 + 2G_{t\phi} \dot{\Phi} \dot{\mathcal{T}} + G_{\phi\phi} \dot{\Phi}^2, \quad (2.19)$$

with

$$\dot{X} \equiv \frac{\partial X^\mu}{\partial \tau} \partial_\mu = \dot{\mathcal{T}} \partial_t + \dot{\mathcal{R}} \partial_r + \dot{\Phi} \sum_{i=1}^3 \partial_{\phi_i}. \quad (2.20)$$

The induced line element on the brane worldvolume is:

$$ds_4^2 = \dot{X}^2 d\tau^2 + \mathcal{R}(\tau)^2 d\Omega_3^2. \quad (2.21)$$

Now one can see that picking τ to be the proper time of an observer living on the brane would set $\dot{X}^2 = -1$, and the induced metric takes the usual FLRW-form, with $\mathcal{R}(\tau)$ acting as the scale factor.

Similarly, the pullback of the four-form C_4 is:

$$P[C_4] = \left((C_4)_t \frac{\partial \mathcal{T}}{\partial \tau} + \sum_i (C_4)_\phi \frac{\partial \Phi_i}{\partial \tau} \right) d\tau \wedge \epsilon_3. \quad (2.22)$$

Finally, the action (2.16) can be rewritten as:

$$S_{D_3} \equiv \int d\tau \mathcal{L}_{D_3} = -2\pi^2 T_3 \int d\tau \left\{ \mathcal{R}^3 \sqrt{-\dot{X}^2} - (C_4)_t \dot{\mathcal{T}} - (C_4)_\phi \dot{\Phi} \right\}, \quad (2.23)$$

where \mathcal{L}_{D_3} represents the Lagrangian density of the D_3 -brane. Observe that integration over the S^3 has been already taken into account. As one can notice, the Lagrangian density does not depend on the time component \mathcal{T} or the angular coordinates Φ_i , just their derivatives. This points to conserved quantities in the system. For our obscure purposes,¹² let us define the conserved angular momentum

$$J \equiv \frac{1}{2\pi^2 T_3} \frac{d\mathcal{L}_{D_3}}{d\dot{\Phi}} = \frac{\mathcal{R}^3}{\sqrt{-\dot{X}^2}} (G_{t\phi} \dot{\mathcal{T}} + G_{\phi\phi} \dot{\Phi}) + (C_4)_\phi. \quad (2.24)$$

With this, let us Legendre transform to substitute $\dot{\Phi}$ for J , automatically satisfying Φ 's EoM. Additionally, let's define $J_C = J - (C_4)_\phi$, which we will give a physical interpretation later. We then obtain **why do we choose positive root?**:

$$\dot{\Phi} = -\frac{G_{t\phi}}{G_{\phi\phi}} \dot{\mathcal{T}} \pm J_C \sqrt{\frac{\left(\frac{G_{t\phi}^2}{G_{\phi\phi}^2} - \frac{G_{tt}}{G_{\phi\phi}} \right) \dot{\mathcal{T}}^2 - \frac{G_{rr}}{G_{\phi\phi}} \dot{\mathcal{R}}^2}{(\mathcal{R}^6 G_{\phi\phi} + J_C^2)}}. \quad (2.25)$$

and the Legendre transformation becomes:

$$\begin{aligned} \mathcal{L}_{D_3}^J &= \mathcal{L}_{D_3} - \dot{\Phi} J \\ &= -2\pi^2 T_3 \dot{\mathcal{T}} \left\{ -(C_4)_t - \frac{G_{t\phi}}{G_{\phi\phi}} J_C + \sqrt{-\left(\mathcal{R}^6 + \frac{J_C^2}{G_{\phi\phi}} \right) \underbrace{\left(G_{tt} - \frac{G_{t\phi}^2}{G_{\phi\phi}} + G_{rr} \mathcal{R}^2 \right)}_B} \right\}, \end{aligned} \quad (2.26)$$

¹²i.e. to make our computations as simple as possible.

Notice that we do not longer have a dot, but a prime as a derivative. For computational convenience, one can opt to write $\frac{\partial R}{\partial \tau} = \dot{\mathcal{T}} \frac{\partial R}{\partial \mathcal{T}}$, which implies that prime is a derivative with respect to the spacetime time-coordinate. Introducing the components of the ten-dimensional line invariant (2.2), we can write (2.26) as

$$\mathcal{L}_{D_3}^J = -2\pi^2 T_3 \dot{\mathcal{T}} \left\{ -(C_4)_t - \frac{J_C}{L} A_t + \sqrt{-\left(\mathcal{R}^6 + \frac{J_C^2}{L^2}\right) \left(\underbrace{g_{tt}}_B + g_{rr} \mathcal{R}'^2\right)} \right\}, \quad (2.27)$$

where $B = g_{tt}$ results after simplification of the metric entries. The linear term J_C in (2.27) shows that the brane carries charge and couples to the bulk gauge field. Note that it is J_C rather than J that measures the charge. This relative shift can be understood by frame dragging due to the rotation in the extra dimensions.¹³

Let us elaborate on the conceptual picture that the square root term in Eq. (2.27) represents: In the absence of J_C , it is no more than an effective five-dimensional DBI-term. The presence of J_C can be understood as an extra flux representing an uniform density of charged particles over the D3-brane. This dissolved particle density spread over the brane can be interpreted as the charge Q from a five-dimensional point of view.¹⁴

Lastly, the $(C_4)_t$ -term can be studied as a five-dimensional WZ-term; One can always define an effective five-dimensional flux $F_5^{\text{eff}} = d[(C_4)_t dt \wedge \epsilon_3]$ with F_5^{eff} proportional to the volume form of the line invariant ds_5^2 .

Our D_3 -dynamics equation (2.27) can be further massage to a more simple expression in terms of the Hamiltonian

$$H = 2\pi^2 T_3 \dot{\mathcal{T}} \left(\frac{-g_{tt} \sqrt{\mathcal{R}^6 + \frac{J_C^2}{L^2}}}{\sqrt{-(g_{tt} + g_{rr} \mathcal{R}'^2)}} - (C_4)_t - \frac{J_C}{L} A_t \right), \quad (2.29)$$

given in terms of the proper time of the brane. Careful here! It is of extreme importance to realize that this is **not** the time variable relevant for a four-dimensional observer like our dear reader. Such a witness comoves with the nucleated brane in AdS_5 , but does not move together with the brane in its compact dimensions dance. This is a similar argument to that Kaluza Klein (KK) compactifications to four dimensions. Any observer, made out of KK particles, is governed by a time component which does **not** involve the internal momenta or velocities of their particles. We will refer to the relevant four dimensional proper time as $\tilde{\tau}$, which is computed as

$$d\tilde{\tau} = dt \sqrt{-(g_{tt} + g_{rr} \mathcal{R}'^2)}. \quad (2.30)$$

¹³The right angular momentum of the expading bubble it is obtained when its rotation in the compact direction is measured with respect a "static" reference system of the background. Hence the relative shift.

¹⁴One can formally collapse the brane to a point (i.e. set $\mathcal{R} = 0$), to see Eq. (2.27) become:

$$-\frac{2\pi^2 T_3 J_C}{L} \dot{\mathcal{T}} \sqrt{-(g_{tt} + g_{rr} \mathcal{R}'^2)}. \quad (2.28)$$

In the absence of the dissolved particles, the total mass of the D_3 approaches zero in this limit. In this case the mass converges to $\frac{2\pi^2 T_3 J_C}{L}$. This can be interpreted as the total mass of the dissolved particles.

This allows us to rewrite the Hamiltonian (2.29) in terms of the proper time of the four-dimensional observer as:

$$H = 2\pi^2 T_3 \frac{d\mathcal{T}}{d\tilde{\tau}} \left(\sqrt{\left(\mathcal{R}^6 + \frac{J_C^2}{L^2} \right)} \sqrt{g(\mathcal{R}) + \dot{\mathcal{R}}^2} - \frac{1}{L} (\mathcal{R}^4 - r_H^4) - \frac{\kappa_5 J_C Q}{L} \left(\frac{1}{r_H^2} - \frac{1}{\mathcal{R}^2} \right) \right), \quad (2.31)$$

where Eq. (2.30) has been used to rewrite the first term in the Hamiltonian (2.29), while its second and third pieces correspond to expressions (2.15) and (2.13) respectively. Recall that $g(\mathcal{R})$ is the function to describe the AdS₅-Reissner-Nordström geometry (2.7).

This Hamiltonian should be put equal to a conserved energy E ,¹⁵ while the constant in $(C_4)_t$ can be thought of a gauge choice that also shifts E . In order to fix such conserved energy, one can recall that the bubble, as a Coleman-de Lucia instanton, will nucleate at *rest*. This translates to zero total energy in the approximately extremal background.¹⁶ Imposing $E = 0$, one can then solve Eq. (2.31) to find the equation of motion controlling the dynamics of the brane as:

$$\dot{\mathcal{R}}^2 = -f(\mathcal{R}) - k^2 \mathcal{R}^2 + \frac{k^2 \left(\mathcal{R}^4 - r_H^4 + \frac{Q\kappa_5 J_C}{r_H^2} \left(1 - \frac{r_H^2}{\mathcal{R}^2} \right) \right)^2}{k^2 J_C^2 + \mathcal{R}^6}, \quad (2.32)$$

where $f(\mathcal{R}) = g(\mathcal{R}) - k^2 \mathcal{R}^2$. Eq. (2.32) still carries both geometrical, through its first DBI term (i.e. $g(\mathcal{R})$) and WZ information (i.e the last term). This equation of motion is the furthest we can go simplifying the action of the D_3 -brane (2.16) into the ten-dimensional background. In order to understand why the embedding of the dark bubble into string **works**, we will explicitly compare Eq. (2.32) to the equation governing a four dimensional expanding cosmology derived from the junction conditions, as was already done in Eq. (1.13). This will be the aim of the next section.

2.3 Comparing junctions to EOM

Basically, compare as we did in the paper. Just to prove, that at some specific limit EOM = JC.

2.4 Collapsing cosmology: Evacuation required

2.5 Higher curvature corrections to the rescue

Point to the fact that previous EOM has no CC. Discuss about how to obtain it and elaborate on all pieces in the game. End with the famous $1/N$ corrections that transform in CC. (First part of 5th paper)

2.6 Energy scales from Dark Bubble embedding: A new hope

Compare to observed CC and obtain N to fix all scales. Discuss and defend the size of extra dimensions and the stringy scales.

¹⁵Although we did not derive this conserved quantity, we saw that the Lagrangian density (2.23) does not depend on the coordinate \mathcal{T} . This implies E to be conserved.

¹⁶As described in the introduction of this chapter, the ten-dimensional ambient space has broken supersymmetry, due to the presence of close-to-zero temperature T .

3 QUANTUM BUBBLES: HIGHER DIMENSIONS TO SOLVE BOUNDARY CHOICES

4 DECORATING THE COSMOS: GAUSS-CODAZZI EQUATION AT WORK

Argue somehow that we are going to look at things from a 5D perspective and will start to induce stuff in the 4D cosmos.

4.1 Matter

4.2 Radiation

4.3 Gravitational waves

4.4 The standard model of particle physics

4.4.1 Electromagnetism

4.4.2 Weak force: Neutrinos

5 HOLOGRAPHIC BUBBLES AND HANGING STRINGS

6 NO, THIS IS NOT A RANDALL-SUNDRUM MODEL ON STEROIDS

REFERENCES

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