About Dark Bubbles

CONTENTS

Seems that I need a general introduction here. I do not know what to say, to be honest.

1 HOW TO INDUCE DARK ENERGY ON A BUBBLE SURFACE: A SIMPLE EXAMPLE

ADD REFERENCES!

Let us just start with the most basic set-up: A non-supersymmetric Anti-de Sitter vacuum, with cosmological constant given by:

$$\Lambda_D = -\tfrac{1}{2}(D-1)(D-2)L^{3-D}. \tag{1.1}$$

The parameter L represents the curvature radius of the AdS_D vacuum. For computational convenience, it will be easier to refer to this parameter with its inverse, the vacuum scale $k = \frac{1}{L}$.

As we discussed in section [NON-SUSY], the aforementioned AdS_D vacuum is unstable and will eventually decay to a more metastable solution, via D_p-brane nucleation. This D_p-brane, which is a Coleman-de Luccia/ BT instanton [add ref to CdL and or section], will mediate such decay, separating two different AdS_D vacua. From now on, we will refer to the vacua living inside the hyper-volume encoded by the D_p-brane as the *inside* vacuum, denoted by a minus (-) subscript sign. The whole higher-dimensional space receives the name of *Bulk* and the tandem brane-inside will be colloquially called *Bubble*. On the other hand, the region *outside* the bubble has not yet decayed, and will be denoted by a positive (+) subscript sign.

Finally, as the aim of this thesis is to discuss phenomenological aspects in four dimensions[1], and given the discussion in [ref to GAUSS-CODAZZI] about co-dimension one hyper-surfaces, we will fix $D = 5$. This implies that the boundary ∂B of the nucleated five-dimensional bubble will have $d = D - 1 = 4$ dimensions.

In this dimensionality, the simplest geometry inside and outside the bubble can be described by the following line invariant:

$$\mathrm{ds}_\pm^2 = g_{\mu\nu}^\pm dx^\mu dx^\nu = -f_\pm(r)dt_\pm^2 + f_\pm^{-1}(r)dr^2 + r^2 d\Omega_3^2, \tag{1.2}$$

where

$$f_\pm(r) = 1 + k_\pm^2 r^2 + \chi(r, t, k_\pm, q_1, ..., q_m). \tag{1.3}$$

Here, r will denote the radial coordinate of the AdS_5 vacuum, i.e. the throat direction and the line-invariant $d\Omega_3^2 = \zeta_{ij}dx^i dx^j$ is the metric on S^3, which corresponds to the three-dimensional solid angle for the usual spatial sections. From now on, Greek indices will represent bulk coordinates, while Latin ones will be used to describe quantities associated with the *induced* geometry on the boundary. Finally, it is important to mention that the function $f_\pm(r)$ represents the AdS_5 geometry plus some possible extra features encoded in $\chi(r, t, k_\pm, q_1, ..., q_m)$. These will be relevant in the following sections, but for now on, let us fix $\chi(r, t, k_\pm, q_1, ..., q_m) = 0$.

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[1]Other bubble dimensionalities have been explored in [IVANO STUFF]

Before we start computing the Israel's junction conditions described in [ref to APPENDIX], it is important to realise the parametrical dependence of the expanding bubble. As described in [ref to INSTANTONS], the nucleated bubble can be identified with a Coleman-de Luccia instanton, which is equipped with an $O(4)$-symmetry.[2] The two relevant coordinates for this derivation are the global time t and the radial coordinate r, which also depends on t, as it is expanding, hence evolving in time. Both will be related to the proper time τ an observer living on the boundary ∂B experiments. The coordinate choices to describe the bulk geometry and the induced one are:

$$x^\mu = \{t(\tau), r(\tau), \alpha, \beta, \gamma\}, \qquad y^a = \{\tau, \alpha, \beta, \gamma\}. \tag{1.4}$$

Let us now compute the first Israel junction condition described in [ref to APPENDIX]. This is no more than:

$$ds_{\text{ind}}^2 = h_{ab} dy^a dy^b = -N(\tau)^2 d\tau^2 + a^2(\tau) d\Omega_3^2, \tag{1.5}$$

where

$$N^2 = f(a(\tau))\dot{t}^2 - \frac{\dot{a}(\tau)^2}{f(a(\tau))}. \tag{1.6}$$

The metric h_{ab} represents the four-dimensional induced geometry on the hyper-surface, i.e. the D_3-brane, given by the parametrisation of $\{y^a\}$-coordinates. The lapse function $N(\tau)$ has been introduced to make time reparametrisation invariance manifest. Observe that a dot represents a proper-time derivative (i.e. the time an observer living on the hypersurface will experiment). One can always choose the right parametrical relation between $f(a(\tau))$, $\dot{t}(\tau)$ and $a(\tau)$ such that $N = 1$. In that case, it is always easy to see that Eq. (1.5) represents an expanding four-dimensional cosmology, described by a Friedmann-Robertson-Lemaître-Walker metric with closed spatial sections.

What about the second Israel junction condition? It is important to notice that the presence of the expanding D_3-brane, which divides the bulk space in two: *inside* and *outside*, will generate a discontinuity of the bulk metric across itself. This forces the presence of a non-zero induced energy-stress tensor on the brane, such that the whole configuration remains a solution to the bulk Einstein equation. This is what the second Israel junction condition accounts for. For simplicity in this example, let us assume we have an empty brane. This implies that its energy-momenum tensor only contains information about its tension σ, which can be denoted by:

$$S_{ab} = -\sigma h_{ab}. \tag{1.7}$$

COMMENT ON THE MINUS SIGN? ChECK CRITICAL SIGMA VALUE FOR POINCARE

As discussed in [APPENDIX], the second Israel's condition relates the *jump* in the extrinsic curvature between the inside and the outside geometries that the brane separates. To be as pedagogical as possible in this introductory section, we will compute the most important steps to obtain the induced energy-momentum tensor.[3]

First, we need to identify the normal and tangent vectors $\{n^\mu, e_a^\mu\}$ with respect to the embed-

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[2]Both in Euclidean and Lorentzian time by analytic continuation.
[3]The whole derivation is left as an exercise to the reader.

ding. Using eqs (APPENDIX) and (APPENDIX), these result to be:

$$
e_a^\mu = \begin{pmatrix} \dot{t}(\tau) & 0 & 0 & 0 & 0 \\ \dot{f}(a) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad n^\mu = \left(\frac{-\dot{a}/\dot{t}}{f(a)\sqrt{f(a) - \frac{1}{f(a)}\left(\frac{\dot{a}}{\dot{t}}\right)^2}}, \frac{f(a)}{\sqrt{f(a) - \frac{1}{f(a)}\left(\frac{\dot{a}}{\dot{t}}\right)^2}}, 0, 0, 0 \right). \tag{1.8}
$$

This implies an extrensic curvature K_{ab} of the form:

$$
K_{ab} = \nabla_\beta n_\alpha e_a^\alpha e_b^\beta = \tag{1.9}
$$

ADD THE RIGHT BRIDGE WHEN MATHEMATICA WANTS TO WORK.

Given the previous results, if one computes the second junction condition Eq. [APPENDIX] assuming a pure constant tension brane, described by Eq. (1.7), it yields:

$$
\sigma = \frac{3}{\kappa_5} \left(\sqrt{\frac{f_-(a)}{a^2} + \frac{\dot{a}^2}{N^2 a^2}} - \sqrt{\frac{f_+(a)}{a^2} + \frac{\dot{a}^2}{N^2 a^2}} \right), \tag{1.10}
$$

where $\kappa_5 = 8\pi G_5$ and σ corresponds to the tension of the bubble wall. Observe how the tension of the shell is governed by the four-dimensional Hubble parameter $H = \dot{a}/a$ and the difference in the AdS$_5$ scales of the five-dimensional bulk. This already points to a limit situation; when the five dimensional scales $k_\pm$ are large compared to the four-dimensional H parameter, the tension of the shell approaches the *extremal* tension[4]

$$
\sigma_{\rm cr} = \frac{3}{\kappa_5} (k_- - k_+). \tag{1.11}
$$

This critical value of the tension $\sigma_{\rm cr}$ corresponds to the extremal value of the radius that maximises the bounce action of the instanton described in [APPENDIX]. This also implies that all the energy obtained in the decay process has been invested in generating a shell that will remain static, as $H \ll k_\pm$ in that limit situation.[5]

What now if the tension results to be sub-critical, i.e. a small portion of energy of the decay is invested in accelerating the bubble growth, as described in [APPENDIX]?[6] In this case, we can assume:

$$
\sigma = (1 - \epsilon)\sigma_{\rm cr}, \tag{1.12}
$$

with $\epsilon > 0$. Expanding this sub-critical value σ in Eq. (1.10), results into:

$$
H = \left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{a^2} + \frac{8\pi}{3}\frac{2\,k_- k_+}{k_+ - k_-} G_5 (\sigma_{\rm cr} - \sigma) + \mathcal{O}\left(\epsilon^2\right). \tag{1.13}
$$

This previous equation should be familiar to the reader that does not get distracted by the suppressed corrections $\mathcal{O}\left(\epsilon^2\right)$. A little bit of dimensional analysis may help those absent-minded readers.

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[4]This can be seen as a *flat* shell, as the curvature parameter a is neglegible compared to the AdS$_5$ scale $k_\pm$.

[5]We will soon see that this is always the case, as the energy hierarchy of the embedding of this construction into string theory provides $k_\pm \gg H$. These corrections are also independent of $a(\tau)$.

[6]This implies that the bounce [APPENDIX] is not extremised.

As observed in Eq. (1.11), $[k_\pm] = [\text{Length}]^{-1}$, while $\kappa_5 = 8\pi G_5 = \ell_5^3$, where ℓ_5 is the five dimensional Planck length. So $[\kappa_5] = [\text{Length}]^3$. Hence, $[\sigma_{\text{cr}}] = [\sigma] = [\text{Length}]^{-4}$. This has the same dimensionality as the dark energy density ρ_Λ in four dimensions as discussed in [SECTION COSMO]. Similarly, we can perform the same dimensional analysis to conclude that $\left[G_5 \frac{k_- k_+}{k_+ - k_-} \right] = [\text{Length}]^2$. This is the dimensionality of the Newton's constant G_4.[7] These do not seem like coincidence, pointing to the following identifications:

$$\kappa_4 = 2\frac{k_- k_+}{k_+ - k_-}\kappa_5, \qquad\qquad \rho_\Lambda = \sigma_{\text{cr}} - \sigma. \qquad\qquad (1.14)$$

These allow us to "unmask" Eq. (1.13). It is no more than a regular Friedmann equation in disguise! From Eq. (1.14), we see that dark energy ρ_Λ is no more than a dynamical property of the expanding D_3-brane in the bulk. The expansion rate, and hence the induced dark energy density, are controlled by the sub-criticallity parameter ϵ.

Let us close this section of the chapter making comments on points that will be relevant and discussed in the following sections.

- de Sitter induced cosmologies can only occur when the tension of the brane is *sub-critical* as seen in Eq. (1.14). The tension of the brane σ will dictate its evolution. As discussed in [APPENDIX to CdL], the brane will always nucleate at rest, i.e. $\dot{a} = 0$. If the tension is critical $\sigma = \sigma_{\text{cr}}$, the repulsion generated by charge of the brane Q, related to the flux difference across the wall (see [SWAMPLAND SECTION] for further details), will be exactly compensated by its tension, preventing its expansion. This results in an induced flat cosmology.

  On the other hand, the brane can nucleate with a sub-critical tension $\sigma < \sigma_{\text{cr}}$, and the whole previous discussion holds, inducing an accelerated expanding cosmology. This is in line with the weak gravity conjecture described in [SWAMPLAND SECTION]. What about supercritical stuff?

- As the simplest example that can be provided, we have omitted the presence of extra dimensions (up to ten or eleven) required from supergravity models. These will be the main discussion in section (2), where we will embed the dark bubble model into supegravity theories. Its connections to the swampland will be also commented in this section.

- So far, we have just discussed how to induced dark energy on the brane, but the observable universe contains more than that. This will be the topic of section (4), where the induced cosmology will be decorated with matter, radiation and waves (plenty of waves)... coming from the extra dimensions. There will be understand the true power of the second Israel condition, its relation to the Gauss-Codazzi equation describe in [APPENDIX] and how to interpret them in the right way to obtain a meaningful induced energy-momentum tensor.

- Where is gravity? Is it localised on the induced geometry or does it propagates across extra dimensions? This will be examined in section (5), where will be study how strings associated to the expanding brane are required to have a good understanding of gravity in this model.

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[7] Recall, as one can see in [SECTION], that $\kappa_D = \ell_D^{D-2}$.

- We have talked about expanding branes in the bulk, but, How did those branes appear there? This will be reviewed, from a quantum cosmology perspective, in section (3)

- Finally, the dark bubble model will be compared against the Randall-Sundrum model. Although both proposals share features, there are key differences at both conceptual and computational levels thay may not be perceived at first glance. Section (6) will be devoted to unravel on these issues.

2 DARK BUBBLES FROM STRING THEORY: AN EXPLICIT CONSTRUCTION

I GUESS THAT HERE I NEED TO WRITE A GOOD SUMMARY OF THE MODEL I WILL USE (OSCAR'S). SPECIALLY WHAT IT WAS USED FOR IN THE PAST AND HOW GOOD IT MAY CONNECT TO THE KIND OF SCENARIO WE HAD IN MIND FOR BUBBLES.

EMPHASISE THAT THE STACK OF BLACK BRANES IS A BLACK HOLE IN HIGHER DIM.

2.1 A black hole in the bulk: Ten-dimensional Kerr or five-dimensional Reissner-Nordström one?

Let us start by defining the parametric coordinates that will be used to describe the 10 dimensional rotating background[8]: OBSERVE THE CHANGE OF COORDINATE DEFINITION WRT THE PAPER. SO THE RESULTING EQUATIONS ARE MORE FAMILIAR WITH THE USUAL THROAT.

$$x^\mu = \{t, \alpha, \beta, \gamma, z, \theta, \psi, \phi_1, \phi_2, \phi_3\}, \tag{2.1}$$

and its line invariant

$$ds_{10}^2 = ds_5^2 + L^2 \sum_{i=1}^{3} \left\{ d\sigma_i^2 + \sigma_i^2 \left(d\phi_i + \tfrac{1}{L} A(z) \right)^2 \right\}, \tag{2.2}$$

where $A(z)$ is a 10 dimensional one-form that will be later discussed and the σ_i-functions are combinations of trigonometric functions of two angles[9] $\{\theta, \psi\}$ of the S^5 as:

$$\sigma_1 = \sin\theta, \qquad \sigma_2 = \cos\theta\sin\psi, \qquad \sigma_3 = \cos\theta\cos\psi. \tag{2.3}$$

As introduced in section 1, L is the AdS_5 radius in Eq. (2.2). It is important to notice that this radius also sets the scale of the extra compact dimensions of S^5. This can already raise some eyebrows, as it points to lack of scale separation, as discussed in [SEC Landscape]. However, this will be a key signature of the dark bubble embedding into supegravity, as this model acquires the aforementioned scale separation via tension-to-charge ratio instead. We will later explore this method in section 2.6.

Conducting our attention back to the line invariant (2.2), we can define the five dimensional asymptotically-AdS metric ds_5 as:

$$ds_5^2 = -h(z)^{-2} f(z) dt^2 + h(z) \left[f(z)^{-1} dz^2 + z^2 d\Omega_3^2 \right], \tag{2.4}$$

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[8]It is important to notice the choice of z to describe the AdS_5 throat direction. Spoiler alert: There will be a convenient change of coordinates soon to adequate r-direction. Stay alert.

[9]ADD REF AND COMMENTS ABOUT THE PROCEDENCE OF THIS ANGLES (from where to where do they run) AND PROBABLY, COMMENTS ABOUT THE SPHERE?

where $d\Omega_3^2$ is the usual unit metric of the 3-sphere [ref to COSMO SECTION]. The radial functions $h(z)$ and $f(z)$ are expressed as

$$h(z) = 1 + \frac{q^2}{z^2},$$

$$f(r) = 1 - \frac{m}{z^2} + \frac{z^2}{L^2} h(z)^3,$$

(2.5)

where m and q are respectively the mass and charge of the five dimensional black. Wait? What five-dimensional black hole? Let us then perform a "massage" to Eqs. (2.4, 2.5) to provide a more familiar black hole-ish line invariant. One can start this task by performing a simple change of coordinates, defining a new radial coordinate r as:

$$r^2 = z^2 h(z) = q^2 + z^2.$$

(2.6)

This change of coordinates will substantialy transform Eq. (2.4), yielding:

$$ds_5^2 = -g(r)dt^2 + g(r)^{-1}dr^2 + r^2 d\Omega_3^2, \qquad g(r) = 1 + k^2 r^2 - \frac{2\kappa_5 M}{r^2} + \frac{\kappa_5^2 Q^2}{r^4},$$

(2.7)

with

$$M = \frac{1}{2\kappa_5}\left(m + 2q^2\right), \qquad\qquad Q^2 = \frac{1}{\kappa_5^2} q^2(m + q^2),$$

(2.8)

with $\kappa_5 = 8\pi G_5$. It is now easy to see that the change of variables described in Eq. (2.6) casts the line invariant (2.4) in the familiar form[10] of a five dimensional Reissner-Nordström black hole living in an AdS vacuum [REF]. Observe that, for a small black hole, with $r_H \ll L$, we recover a flat space description and it is required $Q < M$ to get a horizon and no naked singularity. On the other hand, if we want to study a horizon larger that the AdS$_5$ radius, this immediately implies $Q \ll M$.

Let us recap now the interpretation of both ten and five dimensional geometries. In the ten dimensional approach, we will observe a Kerr black hole which rotates in three of the compact directions of S^5 (i.e. $\{\phi_i\}$). When we zoom out and not have resolution of the five dimensional sphere, the black hole no longer rotates. It becames "static" from the point of view of a five dimensional observer, but it acquires a charge q due to its motion in the compact directions (This will be further explained later). This charge becomes effective (i.e. charge Q) when the change of coordinates (2.6) is performed to recover the more familiar description of a Reissner-Nordström black hole living in AdS.

As it is well known, this type of black holes enjoy two different horizons [REFS to book]: The *outer* horizon (i.e. event horizon) and the *inner* one (i.e. the Cauchy horizon). When these two horizon become degenerate (i.e. $r_H = r_h$), this corresponds to an extremal black hole description. We will not further discuss charged black holes and their connection to the weak gravity conjecture here (see [SWAMP] for more details), but, for computational purposes in the incoming sections, we find more convenient to express (2.7) in terms of the two aforementioned horizons: $\{r_h, r_H\}$ as the inner and outer horizons. In this way, we can express $g(r)$ in Eq. (2.7) with the following Ansatz:

$$g(r) = \frac{k^2}{r^4}\left(r^2 + c\right)\left(r^2 - r_h^2\right)\left(r^2 - r_H^2\right),$$

(2.9)

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[10]The patch we are describing with the metric (2.4) corresponds to the requirement $r \geqslant q$. This is actually not an issue since the horizon r_H will be covered by the patch.

with $c \in \mathbb{R}$. A simple match between Eqs. (2.7, 2.9) shows that:

$$M = \frac{1}{2\kappa_5}\left[r_h^2 + r_H^2 + k^2\left(r_h^4 + r_h^2 r_H^2 + r_H^4\right)\right],$$

$$Q^2 = \frac{r_h^2 r_H^2}{\kappa_5^2}\left[1 + k^2\left(r_h^2 + r_H^2\right)\right], \tag{2.10}$$

$$c = r_h^2 + r_H^2 + \frac{1}{k^2}.$$

which simplifies when the black hole is extremal. Let us keep in mind these identifications for future section and conduct our attention back to the one-form potential $A(z)$ in Eq. (2.2). This form acts as a gauge field from the 5D point of view WHY?. In term of the old coordinates (i.e. $\{t, \alpha, \beta, \gamma, z\}$) it can be written as:

$$A(z) = \frac{q}{z_H^2 + q^2}\sqrt{\left(q^2 + z_H^2\right) + \frac{z_H^4}{L^2}h(z_H)^3}\left(1 - \frac{z_H^2 + q^2}{z^2 + q^2}\right)dt. \tag{2.11}$$

Observe that gauge freedom has been summoned to add a constant that sets $A(z_H) = 0$. This is required as a $A(z)$ is temporal gauge potential, and must vanish at the horizon. This potential can be expressed in a more handable way in two steps:

1. Part of the argument of the root in Eq. (2.11) is the mass m of the black hole[11]. This is:

$$m = z_H^2\left(1 + \frac{z_H^2}{L^2}h(z_H)^3\right). \tag{2.12}$$

2. Furthermore, performing the change of variables (2.6) and making use of the identifications (2.8) to further simplify in the new coodinate system, one obtains:

$$A(r) = \kappa_5 Q\left(\frac{1}{r_H^2} - \frac{1}{r^2}\right). \tag{2.13}$$

The stack of rotating D_3-branes in the ten dimensional background will source the self dual F_5 Ramond-Ramond (RR) field strength, which can be written in terms of the radial function $h(z)$ and the one-form $A(z)$. However, what we will need for future computations is the corresponding four-form potential C_4, with $dC_4 = F_5$. This is:

$$C_4(z, z_H, q) = \frac{1}{L}\left[(z^2 + q^2)^2 - (z_H^2 + q^2)^2\right]dt \wedge \epsilon_3 + L^2 q\sqrt{z_H^2 + q^2 + \frac{z_H^4}{L^2}h(z_H)^3}\sum_i \sigma_i^2\,d\phi_i \wedge \epsilon_3\,, \tag{2.14}$$

where ϵ_3 corresponds to the volume-form associated with $d\Omega_3$ in Eq. (2.4). Following the same recipe describe above, one cast the four-form C_4 in terms of the new variables as:

$$C_4(r, r_H, Q) = \underbrace{\frac{1}{L}\left[r^4 - r_H^4\right]dt \wedge \epsilon_3}_{(C_4)_t} + \underbrace{L^2\frac{Q}{\kappa_5}\sum_i \sigma_i^2\,d\phi_i \wedge \epsilon_3}_{(C_4)_\phi = \sum_i (C_4)_{\phi_i}}. \tag{2.15}$$

---

[11]This can be easily obtained by computing the outer horizon in Eq. (2.4)

Observe that we have split the four-form C_4 into its time-dependent and ϕ_i-dependent pieces. This notation will further simplify computations to come.

With this we finish the black hole description, both from ten and five dimensional perspectives. Let us now explore the explicit embedding of the D_3-brane in the ten dimensional ambient space.

2.2 The D_3-brane dynamics

Let us now move on and study the motion of a nucleated D_3-brane. The starting point of such enterprise is the action governing its dynamics:

$$S_{D_3} = S_{\text{DBI}} + S_{\text{WZ}} = -T_3 \int \mathrm{d}^4\xi \sqrt{-\det P[G]} + T_3 \int P[C_4]. \tag{2.16}$$

The Dirac-Born-Infeld (DBI) term will be in charge of describing the geometrical features of the embedding, while the Wess-Zumino (WZ) term with C_4 captures the non-gravitational forces on the brane due to fields external to it. $P[\dots]$ denotes the pullback of spacetime fields to the world volume of the brane. This is no more than the projection of ten-dimensional coordinates onto the three-dimensional geometry of the D_3-brane, with tension T_3. These, the brane's embedding functions will be denoted by capital versions of the spacetime coordinates: $\mathscr{X}^\mu(\xi)$. In this set up, we will set the dilaton ϕ constant WHY?.

The dynamical description of the branes is as follows: The stack of D_3-branes, and hence the nucleated one, wrap the three-sphere $\mathrm{d}\Omega_3$ inside the AdS$_5$ piece of the metric, with the radial r-coordinate (hence $\mathscr{R}$) depending only on the time component. The ten dimensional black hole in the background rotates in the azimuthal ϕ_i-directions on S_5. This implies that a D_3-brane escaping from the stack will preserve a portion of the angular momentum of this. Therefore, it will also move in those directions. Let us, for the sake of simplicity in the construction of this model, set the three angular momenta associated to each ϕ_i-directions equal in the background. This implies that the brane will rotate with the same angular velocity in all ϕ_i directions. Finally, we will assume a constant position in the remaining coordinates of the compact S_5, i.e. $\{\Psi = \psi_0, \Theta = \theta_0\}$. The brane's embedding functions are then given by:

$$\mathscr{T} = \mathscr{T}(\tau), \qquad \mathscr{R} = \mathscr{R}(\tau), \qquad \Theta = \theta_0, \qquad \Psi = \psi_0, \qquad \Phi_1 = \Phi_2 = \Phi_3 = \Phi(\tau). \tag{2.17}$$

Here, although we write τ as a time parameter on the induced geometry, we do not yet specify if it denotes its proper time. We will later see that this is case. From the point of view of an observer living in AdS$_5$, the rotation translates into an effective charge Q under the gauge field A. When a brane nucleates, carrying a portion of the charge, it will also effectively lower the charge of the remaining black hole.

Before we start to analyse the action (2.16), we need some more practical notation. Derivatives with respect to τ will be denoted by a dot. The components of the ten dimensional metric (2.1) will be denoted by $G_{\mu\nu}$. It will also be useful to define the mixing of the azimuthal-time and azimuthal-azimuthal components in the compact piece of Eq. (2.1) as $G_{t\phi} = \sum_i G_{t\phi_i}$ and $G_{\phi\phi} = \sum_i G_{\phi_i\phi_i}$ respectively.

Let us start the analysis of Eq. (2.16). The pullback $P[G]$ is no more than the induced $g_{\tau\tau}$-metric entry of the expanding D_3-brane. This is:

$$P[G] = g_{\tau\tau} = G_{tt}\dot{\mathscr{T}}^2 + G_{rr}\dot{\mathscr{R}}^2 + 2G_{t\phi}\dot{\Phi}\dot{\mathscr{T}} + G_{\phi\phi}\dot{\Phi}^2. \tag{2.18}$$

This is also the square of the velocity that a comoving observer living on the brane will experiment,

$$\dot{X}^2 = \dot{X}_\mu \dot{X}^\mu = G_{tt}\dot{\mathscr{T}}^2 + G_{rr}\dot{\mathscr{R}}^2 + 2G_{t\phi}\dot{\Phi}\dot{\mathscr{T}} + G_{\phi\phi}\dot{\Phi}^2 \, , \tag{2.19}$$

with

$$\dot{X} \equiv \frac{\partial X^\mu}{\partial \tau}\partial_\mu = \dot{\mathscr{T}}\partial_t + \dot{\mathscr{R}}\partial_r + \dot{\Phi}\sum_{i=1}^{3}\partial_{\phi_i} \, . \tag{2.20}$$

The induced line element on the brane worldvolume is:

$$\mathrm{d}s_4^2 = \dot{X}^2\mathrm{d}\tau^2 + \mathscr{R}(\tau)^2\mathrm{d}\Omega_3^2 \, . \tag{2.21}$$

Now one can see that picking τ to be the proper time of an observer living on the brane would set $\dot{X}^2 = -1$, and the induced metric takes the usual FLRW-form, with $\mathscr{R}(\tau)$ acting as the scale factor.

Similarly, the pullback of the four-form C_4 is:

$$P[C_4] = \left((C_4)_t \frac{\partial \mathscr{T}}{\partial \tau} + \sum_i (C_4)_\phi \frac{\partial \Phi_i}{\partial \tau} \right)\mathrm{d}\tau \wedge \epsilon_3. \tag{2.22}$$

Finally, the action (2.16) can be rewritten as:

$$S_{D_3} \equiv \int d\tau \, \mathscr{L}_{D_3} = -2\pi^2 T_3 \int d\tau \left\{ \mathscr{R}^3\sqrt{-\dot{X}^2} - (C_4)_t\dot{\mathscr{T}} - (C_4)_\phi\dot{\Phi} \right\}, \tag{2.23}$$

where $\mathscr{L}_{D_3}$ represents the Lagrangian density of the D_3-brane. Observe that integration over the S^3 has been already taken into account. As one can notice, the Lagrangian density does not depend on the time component $\mathscr{T}$ or the angular coordinates Φ_i, just their derivatives. This points to conserved quantities in the system. For our obscure purposes,[12] let us define the conserved angular momentum

$$J \equiv \frac{1}{2\pi^2 T_3}\frac{\mathrm{d}\mathscr{L}_{D_3}}{\mathrm{d}\dot{\Phi}} = \frac{\mathscr{R}^3}{\sqrt{-\dot{X}^2}}\left(G_{t\phi}\dot{\mathscr{T}} + G_{\phi\phi}\dot{\Phi} \right) + (C_4)_\phi. \tag{2.24}$$

With this, let us Legendre transform to substitute $\dot{\Phi}$ for J, automatically satisfying Φ's EoM. Additionally, let's define $J_C = J - (C_4)_\phi$, which we will give a physical interpretation later. We then obtain why do we choose positive root?:

$$\dot{\Phi} = -\frac{G_{t\phi}}{G_{\phi\phi}}\dot{\mathscr{T}} \pm J_C \sqrt{\frac{\left(\frac{G_{t\phi}^2}{G_{\phi\phi}^2} - \frac{G_{tt}}{G_{\phi\phi}} \right)\dot{\mathscr{T}}^2 - \frac{G_{rr}}{G_{\phi\phi}}\dot{\mathscr{R}}^2}{\left(\mathscr{R}^6 G_{\phi\phi} + J_C^2 \right)}}. \tag{2.25}$$

and the Legendre transformation becomes:

$$\mathscr{L}_{D_3}^J = \mathscr{L}_{D3} - \dot{\Phi}J$$

$$= -2\pi^2 T_3\dot{\mathscr{T}}\left\{ -(C_4)_t - \frac{G_{t\phi}}{G_{\phi\phi}}J_C + \sqrt{-\left(\mathscr{R}^6 + \frac{J_C^2}{G_{\phi\phi}} \right)\left(\underbrace{G_{tt} - \frac{G_{t\phi}^2}{G_{\phi\phi}}}_{B} + G_{rr}\mathscr{R}'^2 \right)} \right\}, \tag{2.26}$$

---

[12]i.e. to make our computations as simple as possible.

Notice that we do not longer have a dot, but a prime as a derivative. For computational convenience, one can opt to write $\frac{\partial R}{\partial \tau} = \dot{\mathscr{T}} \frac{\partial R}{\partial \mathscr{T}}$, which implies that prime is a derivative with respect to the spacetime time-coordinate. Introducing the components of the ten-dimensional line invariant (2.2), we can write (2.26) as

$$\mathscr{L}_{D_3}^J = -2\pi^2 T_3 \dot{\mathscr{T}} \left\{ -(C_4)_t - \frac{J_C}{L} A_t + \sqrt{-\left(\mathscr{R}^6 + \frac{J_C^2}{L^2} \right) \left(\underbrace{g_{tt}}_{B} + g_{rr} \mathscr{R}'^2 \right)} \right\}, \qquad (2.27)$$

where $B = g_{tt}$ results after simplification of the metric entries. The linear term J_C in (2.27) shows that the brane carries charge and couples to the bulk gauge field. Note that it is J_C rather than J that measures the charge. This relative shift can be understood by frame dragging due to the rotation in the extra dimensions.[13]

Let us elaborate on the conceptual picture that the square root term in Eq. (2.27) represents: In the absence of J_C, it is no more than an effective five-dimensional DBI-term. The presence of J_C can be understood as an extra flux representing an uniform density of charged particles over the D3-brane. This dissolved particle density spread over the brane can be interpreted as the charge Q from a five-dimensional point of view.[14]

Lastly, the $(C_4)_t$-term can be studied as a five-dimensional WZ-term; One can always define an effective five-dimensional flux $F_5^{\text{eff}} = d\left[(C_4)_t dt \wedge \epsilon_3\right]$ with F_5^{eff} proportional to the volume form of the line invariant ds_5^2.

Our D_3-dynamics equation (2.27) can be further massaged to a more simple expression in terms of the Hamiltonian

$$H = 2\pi^2 T_3 \dot{\mathscr{T}} \left(\frac{-g_{tt} \sqrt{\mathscr{R}^6 + \frac{J_C^2}{L^2}}}{\sqrt{-\left(g_{tt} + g_{rr} \mathscr{R}'^2\right)}} - (C_4)_t - \frac{J_C}{L} A_t \right), \qquad (2.29)$$

given in terms of the proper time of the brane. Careful here! It is of extreme importance to realize that this is **not** the time variable relevant for a four-dimensional observer like our dear reader. Such a witness comoves with the nucleated brane in AdS$_5$, but does not move together with the brane in its compact dimensions dance. This is a similar argument to that Kaluza Klein (KK) compactifications to four dimensions. Any observer, made out of KK particles, is governed by a time component which does **not** involve the internal momenta or velocities of their particles. We will refer to the relevant four dimensional proper time as $\tilde{\tau}$, which is computed as

$$d\tilde{\tau} = dt \sqrt{-\left(g_{tt} + g_{rr} \mathscr{R}'^2\right)}. \qquad (2.30)$$

---

[13]The right angular momentum of the expading bubble it is obtained when its rotation in the compact direction is measured with respect a "static" reference system of the background. Hence the relative shift.

[14]One can formally collapse the brane to a point (i.e. set $\mathscr{R} = 0$), to see Eq. (2.27) become:

$$-\frac{2\pi^2 T_3 J_C}{L} \dot{\mathscr{T}} \sqrt{-\left(g_{tt} + g_{rr} \mathscr{R}'^2\right)}. \qquad (2.28)$$

In the absence of the dissolved particles, the total mass of the D_3 approaches zero in this limit. In this case the mass converges to $\frac{2\pi^2 T_3 J_C}{L}$. This can be interpreted as the total mass of the dissolved particles.

This allows us to rewrite the Hamiltonian (2.29) in terms of the proper time of the four-dimensional observer as:

$$H = 2\pi^2 T_3 \frac{d\mathcal{T}}{d\tilde{\tau}} \left(\sqrt{\left(\mathcal{R}^6 + \frac{J_C^2}{L^2} \right)} \sqrt{g(\mathcal{R}) + \dot{\mathcal{R}}^2} - \frac{1}{L} \left(\mathcal{R}^4 - r_H^4 \right) - \frac{\kappa_5 J_C Q}{L} \left(\frac{1}{r_H^2} - \frac{1}{\mathcal{R}^2} \right) \right), \quad (2.31)$$

where Eq. (2.30) has been used to rewrite the first term in the Hamiltonian (2.29), while its second and third pieces correspond to expressions (2.15) and (2.13) respectively. Recall that $g(\mathcal{R})$ is the function to describe the AdS$_5$-Reissner-Nordström geometry (2.7).

This Hamiltonian should be put equal to a conserved energy E,[15] while the constant in $(C_4)_t$ can be thought of a gauge choice that also shifts E. In order to fix such conserved energy, one can recall that the bubble, as a Coleman-de Lucia instanton, will nucleate at *rest*. This translates to zero total energy in the approximately extremal background.[16] Imposing $E = 0$, one can then solve Eq. (2.31) to find the equation of motion controlling the dynamics of the brane as:

$$\dot{\mathcal{R}}^2 = -f(\mathcal{R}) - k^2 \mathcal{R}^2 + \frac{k^2 \left(\mathcal{R}^4 - r_H^4 + \frac{Q\kappa_5 J_c}{r_H^2} \left(1 - \frac{r_H^2}{\mathcal{R}^2} \right) \right)^2}{k^2 J_c^2 + \mathcal{R}^6}, \quad (2.32)$$

where $f(\mathcal{R}) = g(\mathcal{R}) - k^2 \mathcal{R}^2$ and terms in violet and teal will be discussed in the next section. Eq. (2.32) still carries both geometrical, through its first DBI term (i.e. $g(\mathcal{R})$) and WZ information (i.e the last term). This equation of motion is the furthest we can go simplifying the action of the D_3-brane (2.16) into the ten-dimensional background. In order to understand why the embedding of the dark bubble into string **works**, we will explicitly compare Eq. (2.32) to the equation governing a four dimensional expanding cosmology derived from the junction conditions, as was already done in Eq. (1.13). This will be the aim of the next section.

2.3 Comparing junctions to EOM

As stated above, the five dimensional picture involves a AdS$_5$-Reissner-Nordström geometry. The black hole (i.e. the stack of D_3-branes), is unstable due to the presence of the large chemical potential μ_c and one or several can escape (i.e. nucleate) from the outer horizon r_H. Let's beginning our comparison by considering one single nucleated brane. It has tunneled from the outer horizon r_H at radius $r > r_H$ and starts to expand. Similarly to the simplest example described in section 1, this shell will mediate the decay between two different vacua: The *outside*, which has no idea yet that neither vacuum scale k, nor the black hole charge Q and M have changed and the *inside*, where this information has already accounted by the geometry and spacetime has changed accordingly. Hence, the inner and outer metric argument $g(r)$ can be written as:

$$g_\pm(r) = k_\pm^2 r^2 + \underbrace{1 - \frac{2\kappa_5 M_\pm}{r^2} + \frac{\kappa_5^2 Q_\pm^2}{r^4}}_{f_\pm(r)}, \quad (2.33)$$

---

[15]Although we did not derive this conserved quantity, we saw that the Lagrangian density (2.23) does not depend on the coordinate $\mathcal{T}$. This implies E to be conserved.

[16]As described in the introduction of this chapter, the ten-dimensional ambient space has broken supersymmetry, due to the presence of close-to-zero temperature T.

In the same spirit as in section 1, the junction condition (1.10) becomes:

$$T_3 = \frac{3}{\kappa_5} \left(\sqrt{\frac{g_-(a)}{a^2} + \frac{\dot{a}^2}{a^2}} - \sqrt{\frac{g_+(a)}{a^2} + \frac{\dot{a}^2}{a^2}} \right). \tag{2.34}$$

Here T_3 represents the tension of the brane (we will later see why T_3 and not σ). The brane's location along the AdS$_5$-throat is $r = a(\tau)$.[17]

In order to obtain a similar equation to (2.32) to compare to, one can solve exactly for $\dot{a}^2$ to obtain a Friedmann-like equation:

$$\dot{a}^2 = \frac{1}{4\sigma^2 a^2} \left[\left(k_-^2 - k_+^2 \right) a^2 + \left(f_-(a) - f_+(a) \right) \right]^2 + \frac{\sigma^2}{4} a^2 - \frac{1}{2} \left[\left(f_-(a) + f_+(a) \right) + \left(k_-^2 + k_+^2 \right) a^2 \right], \tag{2.35}$$

As a handy notation, let us refer to the *critical* tension of this brane (1.11) as $\sigma = T_3 \frac{\kappa_5}{3} = (k_- - k_+) = \Delta k$. In order to compare with (2.32), we must make sure to be in the right regime, i.e. that of the probe approximation. This implies that we must work at linear order in the perturbations when making expansions of the form this has to be changed/improved:

$$k_\pm = k \mp \tfrac{1}{2}\Delta k, \qquad Q_\pm = Q \mp \tfrac{1}{2}\Delta Q, \qquad M_\pm = M \mp \tfrac{1}{2}\Delta M. \tag{2.36}$$

Observe that $\Delta Q = Q_- - Q_+ < 0$. Similarly to the mass M, the a brane leaves the stack, it carries mass (i.e. tension) and a little bit of charge (the angular momentum J_c from the ten-dimensional perspective) with it. In this way, one ensures that the physical picture is respected by the approximation.

With approximations (2.36) at hand, let us substitute them in Eq. (2.35) to read:

$$\dot{a}^2 = -f(a) - k^2 a^2 + \frac{k^2}{a^6} \left(a^4 + \frac{a^2}{2k\Delta k} \left(f_-(a) - f_+(a) \right) \right)^2, \tag{2.37}$$

where only linear order terms have been considered. Here $f(a)$ condenses zeroth order information about the mass M and charge Q, while $f_-(a) - f_+(a)$ encodes linear terms in ΔQ and ΔM. The structure of Eq. (2.37) starts to resemble that of the embedding potential (2.32)... But this is not enough. We aim for a closer match. Notice that this expression can be simplified even further. If one evaluates the junction condition [REF appendix] and $g(r)$ at the horizon r_H, accounting that $g_\pm(r)$ musth vanish at the horizon r_H, it results in the following constraints at the zeroth and first order:

$$\begin{aligned} 0^{\text{th}}: &\quad k^2 a^2 + g(a) = 0, \\ 1^{\text{st}}: &\quad k\Delta k\, a_H^2 - \frac{\kappa_5 \Delta M}{a_H^2} + \frac{\kappa_5 Q \Delta Q}{a_H^4} = 0. \end{aligned} \tag{2.38}$$

Substitute (2.38) in Eq. (2.37) results in:

$$\dot{a}^2 = -f(a) - k^2 a^2 + \frac{k^2}{\textcolor{teal}{a^6}} \left(a^4 - a_H^4 - \frac{\kappa_5^2 Q \Delta Q}{a_H^2\, k\Delta k} \left(1 - \frac{a_H^2}{a^2} \right) \right)^2. \tag{2.39}$$

Overall, the whole structure of Eq. (2.39) is identical to that of expression (2.32). Recall that, while we had chosen capital letters $\mathcal{X}$ to denote the embedding functions of the brane, the five dimensional picture uses lowercase letters. In any case, both represents the same coordinates. Let us discuss the coloured terms:

---

[17]This τ is the **real** proper time discussed in Eq. (2.30). We dropped the tilde to obtain a cleaner notation.

- **fraction**: The comparison of both terms easily yields:

$$J_C = -\frac{\kappa_5 \Delta Q}{k \Delta k}. \tag{2.40}$$

  This has been already discussed; It is the realisation that the brane's effective angular momentum in the compact directions translates to an effective charge ΔQ carried by the brane when observing the system from a five-dimensional perspective.

- **denominator**: At first glance, one can observe the absence of a J_C^2 term in Eq. (2.39). This is because we forgot to account for one important character that is present in the string theory brane action, but not in the junction conditions; The flux related to the dissolved charge density discussed above. Opposite to the DBI + WZ action, the junction condition (2.34) did not account for the presence of this dissolved flux. In fact, this can be used to enhance the junction conditions. Replacing T_3 in expression (2.34) by $T_3\sqrt{1 + k^2 J_C^2 a^{-6}}$ yields exactly the same result as in Eq. (2.32).

After having related the two problematic terms, the match is complete. One can easily see that, the equation of motion for the D_3-brane given by Eq. (2.16) and the junction condtions accross the brane (2.34) are the **same** expression. This just tells us that the construction from the DBI+WZ action (2.16) from the ten dimensional stringy ambient space description yields the same dynamics as the junction condition for the co-dimension 1 shell in the five dimensional AdS$_5$ bulk. This can just point to one simple fact: The dark bubble model has been embedded into string theory. But there is one problem. We have not induced a small and positive dark energy density yet...

2.4 Collapsing cosmology: Evacuation required

There is one little detail we have not commented on yet: This embedding of the dark bubble model within string theory is not able to reproduce a late time cosmology dominated by a cosmological constant. This implies that the curvature of the universe will eventually dominate, reaching a maximum value for the scale factor $a(\tau)$ and collapsing back, in the form of a bouncing cosmology. To realise this, one just need to check the behaviour of the Friedmann-like expression (2.32) for late-time cosmologies (i.e. $\mathscr{R} \to \infty$). In this regime, the $k^2 \mathscr{R}(\tau)^2$ cancels, pointing to a brane's tension that equals the critical one, as discussed in section 1. But let us explore this issue from the brane tension T_3 perspective.

Junction conditions are formulated from a general relativity perspective; If both vacua (inside and outside) are standing alone solutions to Einstein equation, What induced energy momentum tensor (1.7) should the brane be equipped with such that the whole spacetime is a solution to the Einstein equation? This implies that Newton's constant G_5 is **constant** from this perspective. This does not imply that any other combination of variables to describe this constant cannot change, as long as the resulting Newton's variable G_5 remains constant.

Work out a better introduction to the dictionary and relevant relations. Given the AdS/CFT dictionary, one finds that:

$$G_5 = \frac{\pi L^3}{2N^2} \tag{2.41}$$

where again L is the curvature radius of the compact direction and the AdS-throat, while N is number of branes sitting in the stack of D_3-branes sourcing the geometry of the ambient

space. Imposing the Newton's constant to be constant accross the brane (i.e. $\Delta G_5 = 0$), it follows that:

$$\frac{\Delta L}{L} = \frac{2}{3}\frac{\Delta N}{N},\tag{2.42}$$

where ΔL represent the difference in the curvature radius accross the brane and ΔN encodes how many D_3-branes have nucleated together, mediating the decay. This relation (2.42) can be used to rewrite the critical tension of the brane as:

$$T_3 = \frac{3\Delta k}{8\pi G_5} \rightarrow -\frac{3}{8\pi G_5}\frac{\Delta L}{L^2} = -\frac{1}{4\pi G_5 L}\frac{\Delta N}{N} = -\frac{1}{2\pi^2 L^4}N\,\Delta N$$
$$= -\frac{\Delta N}{(2\pi)^3 g_s \ell_s^4} = -\Delta N\, T_{D_3}.\tag{2.43}$$

Observe that relation (2.42) has been used to reach the third step. The fourth equality is obtained using Eq. (2.41), and the final expression just requires us to use another AdS/CFT dictionary key: $L^4 = 4\pi g_s N\ell_s^4$. When $\Delta N = -1$ (i.e. just one nucleated brane), Eq. (2.43) states that the critical tension is exactly that of a D_3-brane, and hence the cosmological constant will be zero. Before we decide to delve down in the intricacies of inducing an effective positive, yet small four-dimensional dark energy density in the model, let us first analyse how the induced four dimensional cosmology, driven to the failure due to the absence of a late-time cosmological constant domination, behaves under the parameters controlling its dynamics.

For starters, let us express Eq. (2.39) in terms of the horizons $\{a_h, a_H\}$ (recall that from a four dimensional perspective $\mathcal{R}(\tau) = r(\tau) = a(\tau)$) by substitution of Eq. (2.10). This yields:

$$\dot{a}^2 = -\frac{k^2(a_H^2 - a^2)(a_h^2 - a^2)\left(a_h^2 + a_H^2 + \frac{1}{k^2} + a^2\right)}{a^4} + \frac{k^2\left(a^4 - a_H^4 + \frac{a_h J_c\sqrt{1 + \left(k^2 a_H^2 + a_h^2\right)}}{a_H}\left(1 - \frac{a_H^2}{a^2}\right)\right)^2}{k^2 J_c^2 + a^6}.\tag{2.44}$$

The first characteristic one should notice is that $\dot{a} = 0$ when $a = a_H$. As stated above, the brane nucleates at rest, which not only the junction condition, but the DBI action from string theory remarks in this construction. When both horizons coincide (i.e. $r_h = r_H$), one recovers the extremal description of the Reissner-Nordström black hole living in the AdS$_5$ bulk. Figure (1) will allow us to further explore other critical points of this construction. TRANSFORM TO SVG !!

There are three (four) special points to consider in figure (1):

- Both horizons $\{a_h, a_H\}$ (first two roots in the non-extremal case): They respectively correspond to the inner and outer horizons of the Reisser-Nordström black hole. In the extremal case, both horizons coincide, which corresponds to the first root of the blue plot.

- The nucleation point $\{a_{\text{nuc}}\}$ (2nd or 3rd roots, respectively): This is the value of a at which the D_3 brane (i.e. the four-dimensional universe) will nucleate. The region between a_H and a_{nuc} is the classical forbidden region that the brane will have to tunnel through. More to be discussed in section (3).

- The maximal size $\{a_b\}$ of the collapsing universe: This is given by the last root of the potential $V(a)$. At this point, the size of the bubble is maximal, and the four dimensional
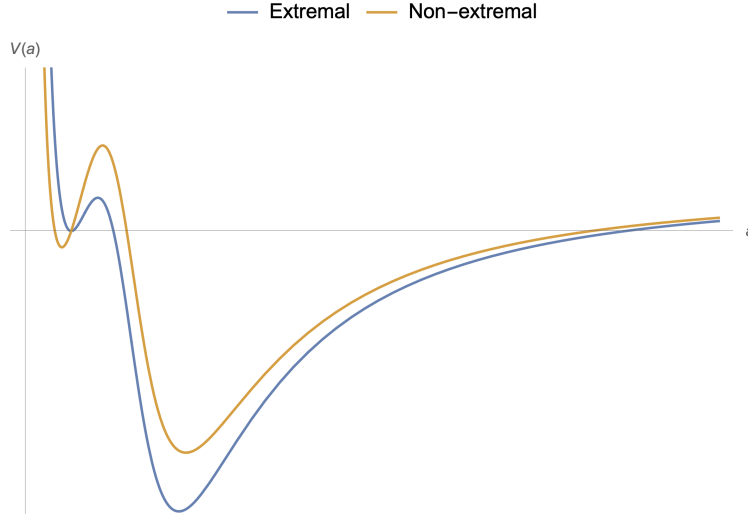
Figure 1: The potential controlling the dynamics of the embedded four dimensional universe.

universe, without positive dark energy density, but equipped with positive curvature, will reach its maximum size and start contracting, in the form of a bouncing cosmology.

But, What different features are controlled by which parameters? By construction, the potential (2.44) goes to 1 when the scale factor $a \to \infty$. The AdS scale k^{-1} is in charge of controlling the overall value of the potential (i.e. basically scaling the y-axis). The angular momentum of the brane J_C will regulate the barrier of the potential.

The potential described in Eq. (2.44) lacks a sustained cosmological constant and experiences only a fraction of an e-folding just after creation. The expansion rate vanishes at nucleation, the universe then accelerates for a short time after which it enters a decelerated phase. Since the curvature is positive, and there is no dark energy in the game, the universe will eventually stop expanding and start to contract, entering a bouncing phase.

Of course, this is not the desired fate for the dark bubble model in its stringy embedding. It is our aim to find a low-dimensional model capable of describing dark energy derived from string theory. Does previous computations indicate that the dark bubble model embedding should be ruled out? Absolutely not. All stringy computation so far have been performed assuming the brane to be BPS, although the ten-dimensional ambient space hosting it has broken supersymmetry. This will be taken into account in the incoming sections.

2.5 Higher curvature corrections to Λ_4 rescue

If one scrutinise Eqs. (2.43) and (2.44), it is easy to notice the absence of an induced dark energy density ρ_Λ, at least, at leading order in the brane action. At the beginning of this section, we argued that the angular momentum of the stack of branes[18] breaks supersymmetry. We also discussed the implications that the WGC would have in non-supersymmetric backgrounds in section [ref SWAMP]. Broken supersymmetry results in a non-saturated inequality (i.e. $Q_{D_p} < T_{D_p}$) for extended objects in string theory. This would require the presence

---

[18]The high chemical potential μ and non-zero temperature T in dual field theory.

of corrections to the D_p-brane action, going beyond the rough sketch that the leading order provides, specially in the description of the brane's tension, making it sub-critical. In the dual field theory description of this embedding, the nucleation of a D_3-brane can be understood as if some of the $\mathcal{N} = 4$ SYM scalar fields obtained a non-vanishing expectation value. This results in a Higgsing of the gauge group as $SU(N) \rightarrow SU(N-1) \times U(1)$. Coming back to the gravity side, this process can be described as corrections of the order $1/N$, where N is the number of D_3-branes in the stack.[19]

If this is the case, What candidates to correct the action do we have? There are two types of corrections (up to a power of the string coupling g_s) that can roughly contribute at the same order of magnitude $\frac{1}{N}$.

- **"Stringy" corrections**: These corrections at $\mathcal{O}(\alpha'^2)$ can be obtained for both relevant pieces of the D_p-brane action. On the one hand, curvature corrections to the DBI action were found in [1]. On simple grounds, each term in the effective action is required to be consistent to those terms of order $\mathcal{O}(\alpha'^2)$ of the corresponding disk-level scattering amplitude [2]. On the other hand, the WZ curvature corrections can be found by imposing that the chiral anomaly on the world volume of intersecting D-branes cancels with the anomalous variation of the WZ action [3]. These extra terms would schematically a correction to the tension $\delta\sigma$ of the form:

$$\delta\sigma \sim T_{D_3} \frac{\alpha'^2}{L^4} \sim T_{D_3} \frac{1}{g_s N} \sim \frac{1}{g_s L G_5 N^2} \sim \frac{1}{g_s L^4}, \quad (2.45)$$

  where we have used same set of relations as in Eq. (2.43). Observe that, since the length scale $\ell_s^2 = \alpha'$ is a classical scale set by the tension of the string, these contributions can be interpreted as a tree level effect.

- **Quantum loop corrections**: Quantum loop corrections can be written as:

$$\frac{1}{N} \sim \frac{\ell_{10}^4}{L^4}, \quad (2.46)$$

  with ℓ_{10} as the ten dimensional Planck length. The Planck length is obtained from the classical Newton's constant by introducing Planck's constant, showing that this correction comes from quantum loops. A renormalized one-loop calculation for a massive field of mass m, leads to a contribution to the vacuum energy of order m^4. Hence, we see that the loops above would be generated by fields with $m \sim 1/L$, which are present in the theory. [20]

So we have two possible candidates. It is not hard to see that the most prominet correction is that of higher curvature corrections (2.45), as want to obtain results for vacua that are weakly coupled (i.e. $g_s \ll 1$). Let us then dissect each contribution piece to the D_p-brane action.

---

[19]One could argue that expected corrections should be of order $1/N^2$, because all fields would belong to the adjoint representation. However, the relevant description of the group is **after** nucleation, hence the fundamental representation of $SU(N-1)$, with expected correction translation to $1/N$.

[20]Note that this also is the mass-scale of the neutrino.

Corrections to the DBI term

As discussed before, string theory corrections to the DBI-action at order α'^2 due to curvature were found in [1]:

$$
\begin{aligned}
S_{\text{DBI}}^{\alpha'^2} = -\, T_{D_3} \frac{\pi^2 \alpha'^2}{48} \int d^4 x\, e^{-\Phi} \sqrt{-g} \big[(R_T)_{abcd}(R_T)^{abcd} - \\
- 2(R_T)_{ab}(R_T)^{ab} - (R_N)_{abij}(R_N)^{abij} + 2(\bar{R})_{ij}(\bar{R})^{ij} \big].
\end{aligned}
\tag{2.47}
$$

Observe that $(R_T)_{abcd} = R^{(4)}_{abcd}$ is the induced curvature on the brane. $(R_N)_{abij}$ is the curvature of the normal bundle. Both are related to the bulk and extrinsic curvatures through the Gauss-Codazzi equation as described in [Appendix JC]. However, there is one term that cannot be found in the appendix. This is $(\bar{R})_{ij}$, which can be defined as:

$$
\bar{R}^{ij} = h^{ab} R^{(4+d)\,ij}_{ab} + h^{ab} h^{cd} \left(K^i_{ac} K^j_{bd} - \chi K^i_{ab} K^j_{cd} \right).
\tag{2.48}
$$

Contrary to other corrections to the action, Eq. (2.48) has no clear geometrical interpretation. Let us focus our attention on its two terms containing extrinsic curvature pieces. There is an important story here.

The first one, $K^i_{ac} K^{ac|j}$, was derived for the first time in [1]. The second term, $K^{a\,i}_a K^{b|j}_b$, carries an undetermined coefficient χ, which could not be determined through the scattering amplitudes in [1]. How do we then fix its value? In fact, this term seems to be the responsible one to contribute in a non-trivial way to correct the DBI piece of the dark bubble's action (2.16). Long story short: T and S-dualities of string theory can be used in order to fix $\chi = 1$, as was done in [4]. One has to be careful with terms that may be mixed by the action of S-duality, but luckily, none result mixed in the computation.[21] Hence the result $\chi = 1$ from [4] can be trusted.

Let us now use this non-trivial correction to shift the tension of the brane in the DBI piece (2.47). As the cosmological constant we would like to induce will only be relevant at late-time descriptions of the expanding cosmology, it will be enough to study a pure AdS_5-background. Any matter component induced by the Reissner-Nordström piece of the metric (2.2) can be ignored. Furthermore, as the cosmological constant is small compared to any other scales in the model, we can ignore the expansion of the dark bubble itself when calculating its value.[22] WHY?. In this spirit, let us assume that the brane sits at a fixed position along the AdS_5-throat, (i.e. $r = r_0$) and on the S^5. WHY 2? With $AdS_5 \times S^5$ as a direct product, and the brane positioned within the AdS_5, all contributions to the extrinsic curvature $K^{|i}_{ab}$ with $i \neq r$ vanish. In this sense, we have recover the co-dimension 1 shell extensively discussed in [APPENDIX]. With such single extra dimension, one can start to simplify terms. By definition, $(R_N)_{ab}{}^{rr} = 0$, as it is anti-symmetric. R_T will also vanish, as the brane is flat in the late-time cosmology approximation that has been taken. Hence, the only surviving term in (2.47) is $\bar{R}^{ij}$. Trusting $\chi = 1$ in Eq. (2.48), as derived in [4], it yields:

$$
\bar{R}^{rr} = -16 k^2 g^{rr}.
\tag{2.49}
$$

Substituting this result back into Eq. (2.47), and simplifying factors, it is easy to see:

$$
T_{D_3} \to T_{D_3} \left(1 - \frac{512}{48} \pi^2 \alpha'^2 k^4 \right).
\tag{2.50}
$$

---

[21] We refer the reader to [5], in order to discover the subtle details of such derivation.

[22] Correction coming from induced curvature itself will be sub-sub-leading and irrelevant for the computations to be shown.

As expected, there has been a shift in the tension, making it *smaller* than the original leading contribution. This is in line with WGC implications, as discussed above. Let us now return our attention to the Wess-Zumino piece of the action.

Corrections to the WZ term

To discuss the possible correction contributions to the Wess-Zumino piece of the D_3-action (2.16), let us first examine its most general expression. It describes the coupling of the brane to Ramond-Ramond fields of various dimensions (represented by C). It is given by:

$$S_{\text{WZ}} = T_{D_p} \int d^{p+1}x \, C \wedge \sqrt{\frac{\mathscr{A}(4\pi^2\alpha'^2 R_T)}{\mathscr{A}(4\pi^2\alpha'^2 R_N)}}. \tag{2.51}$$

Here $\mathscr{A}$ represents the amplitude computation that contains tangential or normal terms. In our specific case, since R_T and R_N vanish, there are no relevant corrections from this expression. But, there is a potential contribution related to the aforementioned, as it was shown in [6]. This is:

$$S_{WZ} \supset -\frac{\pi^2\alpha'^2}{12} \int d^4x e^{-\Phi} \sqrt{-g} \epsilon^{a_0 a_1 a_2 a_3} \frac{1}{(p+1)!} \partial_r \mathscr{F}^{(5)}_{r a_0 a_1 a_2 a_3} \bar{R}^{rr}, \tag{2.52}$$

where $\mathscr{F}^{(5)}$ is the flux sourced by the presence of the D_3-brane along the non-compact directions and ϵ the Levi-Civita symbol. This is the only relevant term of all derived ones in [6] for the studied case. Observe the presence of $\bar{R}^{rr}$, which contains the trace of the extrinsic curvature discussed above. Unfortunately, one cannot argue in the same manner as before to completely trust results derived in [6].

However, one can argue the following: Eq. (2.52) is equipped with a term of the form

$$\partial_r \mathscr{F}^{(5)}_{r a_0 a_1 a_2 a_3}. \tag{2.53}$$

In the analysis of [6], assuming the trace of the extrinsic curvature to vanish, there is no difference between ∂ and the full covariant derivative $\nabla = \partial + \dots$. One could therefore argue that the correct expression should be (2.52) with a *c*ovariant derivative. Then the correction to the WZ will simply vanish as a result of the equations of motion for $\mathscr{F}$!

Surprisingly, previous argument leaves the correction to be just the simple term derived in Eq. (2.50). What consequences can this have to dark bubble embedding? Let us explore them. I need a stronger argument/explanation here.

2.6 Energy scales from Dark Bubble embedding: A new hope

Compare to observed CC and obtain N to fix all scales. Discuss and defend the size of extra dimensions and the stringy scales.

3 Quantum bubbles: Higher dimensions to solve boundary choices

4 Decorating the cosmos: Gauss-Codazzi equation at work

Argue somehow that we are going to look at things from a 5D perspective and will start to induce stuff in the 4D cosmos.

4.1 Matter

4.2 Radiation

4.3 Gravitational waves

4.4 The standard model of particle physics

4.4.1 Electromagnetism

4.4.2 Weak force: Neutrinos

5 Holographic bubbles and hanging strings

6 No, this is not a Randall-Sundrum model on steroids

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