Classical Cosmology

What do we know about the Universe?

Several satellites that we have sent there, outside[1], to the empty and cold darkness of space, have provided enough data to prove that any galaxy far away (and not so far away) from us is in a process of getting further away. On top of that, increasing their distance today even faster than yesterday. It could perhaps be due to the fact that the Universe itself sees human beings as a potential plague, and want to avoid us. Or could be because some sort of colossal multidimensional being decided to streach the fabric of spacetime itself, just for fun.

Nonsense apart, nowadays, the most accepted reason why the Universe is pushed apart in an accelerated manner is *Dark Energy*. Ah, wonderful. And what is *Dark Energy*? To explain this is still an open problem[2][3]. But let us first step back more than a century ago, to understand the synthesis of our current understanding of Cosmology.

In 1915, good old Einstein published his theory of *General Relativity* (GR) []. The world did not become a better place due to this, but at least we were provided with a ridiculous powerful tool to describe low-energy gravitational events. In the following years after the publication, Friedmann, Hubble, Lemaître [] (among many others) used GR technology to describe the Universe as a whole. Their work set the foundations of what we call today *The Standard Model of Cosmology* []. This model is good enough to describe our current observations of the Universe. And not only that. If we reversed the observed expansion back to very close the beginning of everything, we could still have a really good description of the events happening in the almost newborn Universe. It has 3 really simple foundations:

1. *Copernican*: Our planet occupies not special position in the Universe.
2. *GR + Expansion*: Einstein equation describe gravitational dynamics with accuracy at low-energy physics and Hubble's discovery (The expansion of the Universe) in 1929 is correct.
3. *Perfect fluidity*: We can assume that all contents in the Universe behave as a perfect fluid.

Of course this model has its flaws, but we leave these downsides for future lines.

At really big scales, Copernican principle holds. No point in space occupies a special position. Wherever you sit at and look at, everything will be more or less the same. In technical words, this implies *homogeneity* and *isotropy*. Good, seems simple. Next step is to find a reliable way to measure distance, hence to be able to describe the geometry of spacetime. This is given by the *line invariant*, which takes the famous *FRLW* form, adequate to describe a Lorentzian signature spacetime with a high degree of symmetry as the one we seem to live in. This can be written as:

[1] Plus all evidence collected also from earth surface.

[2] A really big one. Or a small one?

[3] As one could say in Spanish, *"Un problema nimio"*. Nimio is an adjective that can be used for both small and big.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2(t)dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin[\theta]^2 d\phi^2\right)\right), \tag{1}$$

Where $N(t)$ is a lapse function and $a(t)$ is the scale factor that describes the expansion (or contraction) of $3D$ spacial slices. Before we start talking about the spatial properties of previous line invariant (1), let us first discuss about the lapse function $N(t)$.

This function is in charge of time reparametrization invariance. As we do not want to overcomplicate our computation, the most useful and convienent choices for $N(t)$ are:

- $N(t) = 1$: This is the choice of *global time*. Add some explanation
- $N(t) = a(t)$: This is the so-called *conformal time* gauge choice. This choice allows the observer to co-move with the expansion of the universe, even in the time coordinate, accounting for any funky effect... Add some explanation

The extra parameter we have not talked about yet is k. This has the power to change curvature of space. It comes in three different flavours:

- $k = 0$: A quite boring case. No curvature, where spatial sections of the geometry are *flat*, like $\mathbb{R}^3$.
- $k = 1$: With this value, spatial sections are *closed*, like in $\mathbb{S}^3$.
- $k = -1$: Spatial sections are *open*, as in the hyperboloid $\mathbb{H}^3$.

So the line invariant (1) allows us to describe a dynamical universe with different types of spatial curvature. But, what do we do with this tool? How do we get a specific equation(s) that explictly describe the evolution of the universe? It is at this point where Einstein equation have something to say.

Einstein equation is an *Equation of Motion* (EOM) that captures the universe's dynamics and it is obtained by extremising the Einstein-Hilbert action with respect to variations of the metric $\delta g_{\mu\nu}$, which reads:

$$S[g_{\mu\nu}, \phi_i] = \int d^d x\sqrt{-g}\left(\frac{R}{2\kappa_d^2} + \mathscr{L}_{mat}(\phi_i, \partial\phi_i)\right), \tag{2}$$

where $\mathscr{L}_{mat}$ is the matter lagrangian of some fields ϕ_i, coupled to gravity and R is the Ricci scalar, which carries geometrical information. κ_d^2 encodes information about d dimensional Newton's gravitational constant as:

$$\kappa_d^2 = 8\pi G_d = M_{pl}^{2-d}, \tag{3}$$

where M_{pl} is the Planck mass for d dimensions. As we are so far working out the classical cosmology scenario, we will stick to $d = 4$. To obtain Einstein equation, we just have to vary the action (2) with respect to $g_{\mu\nu}$ to obtain:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \kappa_4^2\left(\mathscr{L}_{mat}\, g_{\mu\nu} - 2\frac{\delta\mathscr{L}_{mat}}{\delta g^{\mu\nu}}\right) = \kappa_4^2 T_{\mu\nu}. \tag{4}$$

The left hand side of (4) condense pure geometry information, while the right hand side stands for matter field contribution. This is packaged inside the *energy-momentum tensor* $T_{\mu\nu}$. As we are under the assumption that the universe is homogenous, isotropic and its content can be described as perfect fluid, the energy-momentum (EM) tensor becomes of the simple form:

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p\, g_{\mu\nu}, \tag{5}$$

where $u_\mu = (-N, 0, 0, 0)$ is the fluid four-velocity, and (ρ, p) its energy density and pressure. These can be used to describe presureless matter (dust) or relativistic one (radiation), among others. As energy is a conserved quantity, so must be the EM tensor, i.e. $\nabla_\mu T^{\mu\nu} = 0$.

From the clear relation between geometry and matter content in Einstein equation (4), Wheeler once stated: *"Spacetime tells matter how to move; matter tells spacetime how to curve"*. For the specific FRLW metric (1) with $N(t) = 1$ (i.e. global time coordinate), the Einstein equation yields two equations as:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G_4}{3}\sum_i \rho_i - \frac{k}{a^2}, \tag{6}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_4}{3}\left(\sum_i \rho_i + 3p_i\right), \tag{7}$$

where $\frac{\dot{a}^2}{a^2}$ is the *Hubble rate* H, which measures the expansion of the universe (static when $H = 0$). These are the *Friedmann equations.*

The first Friedmann equation (6) describes how the rate of expansion of the universe is governed by its content and topology, while the second equation (7) accounts for its accelaration. It is easy to observe that if the universe is expanding and faster everyday, this implies $\left(\sum_i \rho_i + 3p_i\right) < 0$ in eq (7). In order to obtain information about who can be responsible for this behaviour, we need to characterise different types of matter. The *equation of state* is in charge of that, as relates pressure to energy density as $\omega = \frac{p}{\rho}$. Any type of matter with $\omega > -1/3$, will be responsible of any decelerated expansion of the universe. On the other hand, any content with $\omega < -1/3$ will accelerate the expansion[4]. But there is nothing that we have detected (yet) that can be identify with that equation of state. As the universe is a dark place, and this unknown energy density seems to be the source of accelerated expansion, it has received the famous name of *dark energy.*

But, how can be determine the value of the state parameter for each energy density? Let us derive their values from basic principles:

- In the case we consider non-relativistic matter, the energy density ρ_m will be dominated by the rest mass energy $E = mc^2$, as the momentum, and hence, the exerted pressure are neglegible compare to ρ_m. Thus, to a good approximation, $\omega \simeq 0$.
- For radiation, we are now dealing with relativistic effects. Following the same reasoning as before, the pressure, which is proportional to the velocity $v \simeq c$ will be relevant. Assuming an isotropic distribution of the pressure accross the three spatial dimensions, this can be described by $\omega = 1/3$.

[4]Due to the null energy condition [], $-1 \leq \omega \leq 1$.

- Finally, vacuum energy seems to apply a *repulsive* force, making the universe to expand. This is described by a state parameter $\omega = -1$. This value can be observationally constrained by data extracted from *Barionic Accustic Oscillations* (BAO), perturbations of the hot plasma of the early universe, imprinted in the *Cosmic Microwave Background* (CMB). The latest measured value (2018) is $\omega_\Lambda = -1.028 \pm 0.031$ [1].

Equipped with this information, we are one step closer to understand the true power of Friedmann equations (6, 7). But one step at a time, cosmic hitchhiker! If the universe is evolving, also the energy density of its content. This is governed by energy conservation. The explicit expression for the covariant derivative of the EM tensor (5) in a FRLW geometry (1) yields:

$$\rho \propto a^{-3(1+\omega)}, \tag{8}$$

ADD TABLE!

One of the most interesting features of relation (8) is that ρ_Λ is the same yesterday, today and tomorrow. It is not diluted with the spacetime expansion[5]. This can be interpreted as an intrisic property of spacetime itself, which can be captured by an extra EM tensor of the form:

$$T_\Lambda = -\rho_\Lambda g_{\mu\nu} = -\frac{\Lambda}{\kappa_4^2} g_{\mu\nu}, \tag{9}$$

With Λ being a constant. This is the famous *cosmological constant* (CC)[6], which encodes the information of a non-varying energy density across the expanding cosmos. As this can be interpreted as an intrisic geometrical property, it is perhaps more appropiated to move it to the LHS of Einstein equation (4) as:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa_4^2 T_{\mu\nu} \tag{10}$$

Which is the usual Einstein equation for cosmology.

Now that we have all characters of this piece completely identified, let us come back to the Friedmann equation (6). Accounting for all possible types of energy densities ρ_i, one obtains the following expression:

$$\dot{a}^2 = \frac{8\pi G_4}{3}\left(\frac{\rho_{mat}}{a} + \frac{\rho_\gamma}{a^2} + \frac{\rho_\Lambda}{a^2} - \frac{3k}{8\pi G_4}\right) \tag{11}$$

CONTINUE HERE. CAREFUL WITH PROPER TIME GAUGE STUFF. TALK ABOUT POTENTIAL, DRAW AND EXPLAIN ANALOGY TO CLASSICAL MECHANICS THING.

Join to the CC after discussing who energy densities evolve with scale factor In order to capture dark energy in our description, we can just add an extra

[2]

[5] I wish also my economical capital to behave in this way. No matter what my expenses are, it remains the same.

[6] Yeah, the one that *Albertito* introduced in his equation to obtain a static universe. When Hubble discovered the expansion of the universe in 1929, Einstein remarked that introducing that constant was his bigger mistake. Well, it was a good one, Albert. Thank you!

References

[1] N. Aghanim, Y. Akrami, M. Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini et al., *Planck 2018 results-vi. cosmological parameters*, *Astronomy & Astrophysics* **641** (2020) A6.

[2] U.H. Danielsson and T. Van Riet, *What if string theory has no de Sitter vacua?*, *Int. J. Mod. Phys. D* **27** (2018) 1830007 [1804.01120].