Hypersurfaces and Junction Conditions

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1 Hypersurfaces

The aim of these notes is to provide the reader with a quick overview of the wonderful world of hypersurfaces, i.e. submanifolds with dimension dim $= D-d$ that are "slices" of D-dimensional manifolds. These notes will follow along the lines of [1] and [2].

The hypersurface's mathematical definition can be cast in the following form:

$$\mathscr{S} = \{x^\alpha \in \mathscr{M} \,|\, \Phi(x^\alpha) = 0\} \subset \mathscr{M}. \tag{1}$$

Perhaps mathematical definitions bring our dear reader out in a rash, so let us ease the previous description into more "peasant" language. In addition, as we also care about our reader's sanity, we will restrict this study to *co-dimension* one hypersurfaces. This is $d = 1$. More valiant mathematical warriors, willing to fight through *co-dimension* d sub-manifolds, are welcome to read the exquisite selected literature on the topic [3–5].

As we said, we define a hypersurface Σ as a "slice" of a higher dimensional space with metric $g_{\mu\nu}$. This is something that it is well known since our good old days in high school. For example, one can define a two dimensional sphere in a three-dimensional flat Euclidean space by

$$\Phi(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0, \tag{2}$$

with R as its radius. The **embedding** map Φ tells us how to "insert" the hypersurface Σ (the sphere) in the manifold $\mathscr{M}$, i.e. the three-dimensional Euclidean space. Observe that this description of the sphere respects the definition given in expression (1). Its coordinates $\{x, y, z\}$ are contained in $\mathscr{M}$, the three dimensional Euclidean space and Eq. (2) corresponds to the restriction in the second part of the definition. Eventuality, one can further choose a new set of coordinates that are **intrinsic** to the sphere itself, as $y^a = \{\phi, \theta\}$, such that we can relate the **extrinsic** coordinates $\{x, y, z\}$ to those of the sphere by the well-known parametrical relations:

$$\begin{aligned} x &= R\cos\phi\sin\theta, \\ y &= R\,\sin\phi\sin\theta, \\ z &= R\cos\theta. \end{aligned} \tag{3}$$

This parametric equation can be written in a more general way as:

$$x^\alpha = x^\alpha(y^a). \tag{4}$$

The following notation is extremely important; please engrave this on your pupils:

- **Extrinsic** coordinates will be denoted by **Greek** letters $\{\alpha, \beta, \gamma, \cdots\}$.
- **Intrinsic** coordinates will be denoted by **Latin** letters $\{a, b, c, \cdots\}$.

To have this notation crystal clear is of extreme importance, as the aim of these notes is to relate extrinsic and intrinsic properties of manifolds one another, so we can get the most information of both coordinate systems. The preceding notation will help us clarify if we are dealing with the total manifold or just the hypersurface Σ.

Returning to our simple spherical example, let us continue with more definitions. As a surface, it can be equipped with vectors. The ones of our interest are two different types: The normal and tangent vectors.

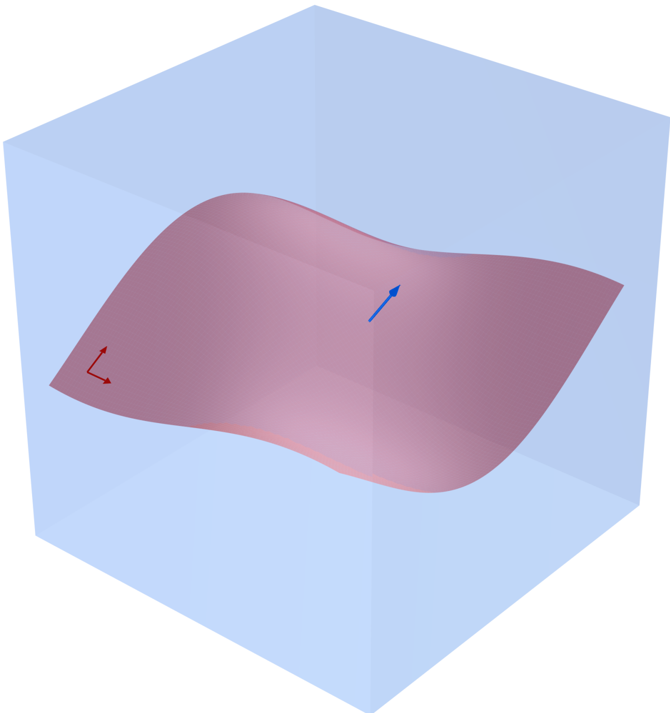

Figure 1: Two of the most basic elements to describe the embedding of a co-dimension one hypersurface Σ in a D-dimensional manifold are the normal vector n^μ and the tangent vectors e^α_a.

1.1 Normal vector

It is easy to think of a normal vector n^α in the spherical case described above: A stingy arrow pointing orthogonally (outside or inside) respect to the surface Σ. The problem comes when one deals with dimensions greater than three or signatures beyond the Euclidean one. How does one define the normal vector?

We can define a unit normal unit vector n^α imposing unitarity:

$$n^\alpha n_\alpha = \varepsilon = \pm 1, \tag{5}$$

where (+) correspond to a timelike hypersurface and (−) represents spacelike ones.[1] Furhtermore, we demand that n^α points in the direction of increasing Φ. In the case we face a spacelike surface, the normal vector will point in the direction of growing spatial sections, i.e. $n^\alpha \partial_\alpha \Phi > 0$. This implies that the normal vector can be defined as:

$$n_\alpha = \frac{\varepsilon\, \partial_\alpha \Phi}{\sqrt{g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi}}. \tag{6}$$

Observe that for the two-dimensional sphere described above, as the metric $g^\mu_\nu = \mathbb{I}_{3\times 3}$, one recovers its usual euclidean definition.

1.2 Tangent vectors

Contrary to normal vectors, tangent ones live on the hypersurface. They are defined by:

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}, \tag{7}$$

where again, x^α coordinates belong to the manifold $\mathscr{M}$ and y^a are coordinates of Σ, related through the parametric relation (4). Note that e^α_a will be a matrix, i.e. the Jacobian of the parametric transformation, with D rows and $D-1$ columns, as we are dealing with co-dimension one hypersurfaces.

As expected, tangent and normal vectors are orthogonal to each other, which means its scalar product is null,

$$n_\alpha e^\alpha_a = 0. \tag{8}$$

Furthermore, tangents vectors, acting as "projectors" of D-dimensional coordinates x^α onto the hypersurface Σ, can be used to described the **intrinsic** or **induced** line invariant on the surface. This is done by restricting the line element of the D-dimensional space to displacements confined to Σ, as:

$$\begin{aligned} ds_\Sigma &= g_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta \Big|_\Sigma \\ &= g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial y^a} \mathrm{d}y^a \right) \left(\frac{\partial x^\beta}{\partial y^b} \mathrm{d}y^b \right) \\ &= \underbrace{g_{\alpha\beta} \tfrac{\partial x^\alpha}{\partial y^a} \tfrac{\partial x^\beta}{\partial y^b}}_{h_{ab}} \mathrm{d}y^a \mathrm{d}y^b, \end{aligned} \tag{9}$$

where h_{ab} is the **induced** metric or first fundamental form. This will allow us to define the following completeness relation for the metric $g_{\alpha\beta}$ as:

$$g^{\alpha\beta} = \varepsilon\, n^\alpha n^\beta + h^{ab} e^\alpha_a e^\beta_b. \tag{10}$$

This previous expression will be of great use in the following pages, as it relates the tangential and normal parts of the embedding to the line invariant hosting the hypersurface.

Let us now imagine a full tangential tensor $A^{\alpha\beta}$ defined on Σ, with no components in the normal directions (i.e. $A^{\alpha\beta} n_\alpha = 0$). Such tensor admits the decomposition

$$A^{\alpha\beta\cdots} = A^{ab\cdots} e^\alpha_a e^\beta_b \cdots = h^{ai} h^{bj} \cdots A_{ij\ldots} e^\alpha_a e^\beta_b \cdots \tag{11}$$

To understand how these tensors differenciate, we can just apply the usual covariant derivative. However, the resulting information will depend on the chosen set of coordinates. In the case one chooses the hypersurface coordinates y^a, it is easy to prove that:

$$\nabla_b A_a = \nabla_\beta A_\alpha e^\alpha_a e^\beta_b = \cdots = \partial_b A_a - \Gamma^i_{ab} A_i\,, \tag{12}$$

[1] Null hypersurfaces are trickier. Check [1] for further details.

where $\cdots$ represent intermediate steps of the computation. The expression (12) corresponds to the familiar *intrinsic* covariant differentiation. But this is not the end of the story. We can reproduce the same computation, but splitting components of the metric $g_{\alpha\beta}$ into its normal and tangential pieces. For indices convenience, let us look at the vector $\nabla_\beta A_\alpha e^\beta_b$, whose tangential components are given by Eq. (12). This is:

$$\begin{aligned} \nabla_\beta A_\alpha e^\beta_b &= g_{\alpha\gamma} \nabla_\beta A^\gamma e^\beta_b \\ &= \left(\varepsilon n_\alpha n_\gamma + h_{ij} e^i_\alpha e^j_\gamma\right) \nabla_\beta A^\gamma e^\beta_b \\ &= \varepsilon \left(n_\gamma \nabla_\beta A^\gamma e^\beta_b\right) n_\alpha + h_{ij} \underbrace{\left(\nabla_\beta A^\gamma e^j_\gamma e^\beta_b\right)}_{\nabla_b A^j} e^i_\alpha, \end{aligned} \tag{13}$$

Observe that the first term can be rewritten making use of the fact that $A^\gamma n_\gamma = 0$, as we assume the tensor A_α to be completely tangential. This allows us to rewrite previous expression as:

$$\cdots = \nabla_b A_i e^i_\alpha - \varepsilon A^i \underbrace{\left(\nabla_\beta n_\gamma e^\gamma_i e^\beta_b\right)}_{K_{bi}} n_\alpha, \tag{14}$$

where we have defined the symmetric *extrinsic* curvature of the hypersurface Σ or second fundamental form of the hypersurface as:

$$K_{ab} = \nabla_\beta n_\alpha e^\alpha_a e^\beta_b. \tag{15}$$

with trace computed after contraction against the induced metric h_{ab}

$$K = K^{ab} h_{ab} = \nabla_\alpha n^\alpha. \tag{16}$$

Note that the starting point in Eqs. (12) and (13) is the same; The covariant derivative of the tangent form A^α living on Σ. Eq. (14) shows a pure tangential piece of the vector field (the first term) and its normal component (the second term). This piece carries geometrical information about how the hypersurface Σ is embedded within the hosting space $\mathscr{M}$ and hence, what kind of curvature acquires. This term can only be zero if and only if the extrinsic curvature vanishes.

2 Gauss-Codazzi Equations

The next logical step in this discussion is to explore if the *intrinsic* Riemann tensor of the hypersurface Σ can also be expressed in terms of *extrinsic* information. Let us first recall the definition of a purely intrinsic curvature tensor as:

$$[\nabla_a, \nabla_b]\, A^c = R^c_{dba} A^d. \tag{17}$$

In the same spirit as in previous computations, one can make good use of the identities relating normal and tangent tensors in order to related both *extrinsic* and *intrinsic* curvature tensors. This requires a modest amount of algebra and we refer the curious reader to [1]. Here we will just show the final result of the computation,

$$\begin{aligned} R_{\alpha\beta\gamma\delta} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d &= R_{abcd} + \varepsilon \left(K_{ad} K_{bc} - K_{ac} K_{bd}\right), \\ R_{\mu\alpha\beta\gamma} e^\alpha_a e^\beta_b e^\gamma_c n^\mu &= \nabla_c K_{ab} - \nabla_c K_{ac}. \end{aligned} \tag{18}$$

These are known as the **Gauss-Codazzi** equations. They show that some components of curvature tensor of any geometry in $\mathscr{M}$ can be decomposed in terms of the *intrinsic* and *extrinsic* curvature pieces of the hypersurface it may be hosting.

Although the Riemann tensor, in any of its forms, contains valuable information about the geometry it represents, more handable tensorial objects will be found in everyday's physic computations. Let us then find expressions for both the Ricci tensor and scalar given the metric decomposition described in Eq. (10). For the Ricci tensor we find:

$$\begin{aligned} R_{\alpha\beta} &= g^{\mu\nu} R_{\mu\alpha\nu\beta} \\ &= \left(\varepsilon n^{\mu} n^{\nu} + h^{mn} e^{\mu}_{m} e^{\nu}_{n}\right) R_{\mu\alpha\nu\beta} \\ &= \varepsilon R_{\mu\alpha\nu\beta} n^{\mu} n^{\nu} + h^{mn} R_{\mu\alpha\nu\beta} e^{\mu}_{m} e^{\nu}_{n}, \end{aligned} \tag{19}$$

while the Ricci scalar gives:

$$\begin{aligned} R &= g^{\alpha\beta} R_{\alpha\beta} \\ &= \left(\varepsilon n^{\alpha} n^{\beta} + h^{ab} e^{\alpha}_{a} e^{\beta}_{b}\right) \left(\varepsilon R_{\mu\alpha\nu\beta} n^{\mu} n^{\nu} + h^{mn} R_{\mu\alpha\nu\beta} e^{\mu}_{m} e^{\nu}_{n}\right) \\ &= 2\varepsilon h^{ab} R_{\mu\alpha\nu\beta} n^{\mu} e^{\alpha}_{a} n^{\nu} e^{\beta}_{b} + h^{ab} h^{mn} R_{\mu\alpha\nu\beta} e^{\mu}_{m} e^{\alpha}_{a} e^{\nu}_{n} e^{\beta}_{b}. \end{aligned} \tag{20}$$

It can be useful to make good use of relations (15) and (17) to further simplify the Ricci scalar expression (20). Some minutes of patience and algebra yield:

$$R = {}^{(D-1)}R + \varepsilon \left(K^2 - K^{ab} K_{ab}\right) + 2\varepsilon \nabla_{\alpha} \left(n^{\beta} \nabla_{\beta} n^{\alpha} - n^{\alpha} \nabla_{\beta} n^{\beta}\right). \tag{21}$$

This expression can be interpreted as the evaluation of the D-dimensional Ricci scalar on the $D-1$-dimensional hypersurface Σ. This result is exceptionally practical in the context of branes and hypersurfaces, specially when the action governing their dynamics requires to be split.

3 Junction conditions

Previous pages have been devoted to study how the embedding of a co-dimension one hypersurface Σ can be used to provide a splitting of the hosting manifold $\mathscr{M}$ into its tangential and normal components. However, physical concepts of such picture has not yet received our attention. For example, one can find situations in physics as follows: Assume that such hypersurface Σ separates a spacetime geometry in two regions $\mathscr{V}^+$ and $\mathscr{V}^-$. Both regions are equipped with different metrics $g^{\pm}_{\alpha\beta}$. Additionally, they are both solutions to the Einstein field equations. What conditions should be put on the metrics to ensure that both spaces join smoothly at Σ, such that the whole union of spaces becomes a solution to the Einstein equation? This set of requirements demanded on the geometrical features of the spaces $\mathscr{M}^{\pm}$ and Σ are called **Junction conditions**. They were originally discussed in papers like [6–8].

Let us first imagine two D-dimensional manifold $\mathscr{M}_{\pm}$, described by different sets of coordinates $\{x^{\alpha}_{\pm}\}$ and equipped with metrics $g^{\pm}_{\alpha\beta}$. Furthermore, both spaces share a boundary $\partial\mathscr{M}$, which is a co-dimension one hypersurface Σ described with a set of coordinates $\{y^a\}$. On top of all this, we assume that this composition space satifies the D-dimensional Einstein equation. We can then try to define a general metric $g_{\alpha\beta}$ that interpolates between two D-dimensional spaces. This can be written as:

$$g_{\alpha\beta} = \Theta(\lambda) g^{+}_{\alpha\beta} + \Theta(-\lambda) g^{-}_{\alpha\beta}, \tag{22}$$

where $\Theta(\pm\lambda)$ is the Heaviside distribution function and λ is an affine parameter describing geodesics that connect both regions, piercing through the hypersurface Σ. In that sense, when $\lambda > 0$, one could say to be placed in $\mathscr{M}^+$. Similarly, the minus sign represents a point of the geodesic in $\mathscr{M}^-$ and its null value sits on the hypersurface Σ. Observe that more complicated objects like the affine connection Γ or the Riemann tensor will depend on derivatives of the metric. One then has to be careful, as we are dealing with distribution functions and combinations of them may result in non-distribution terms, giving us a hard time finding a physical

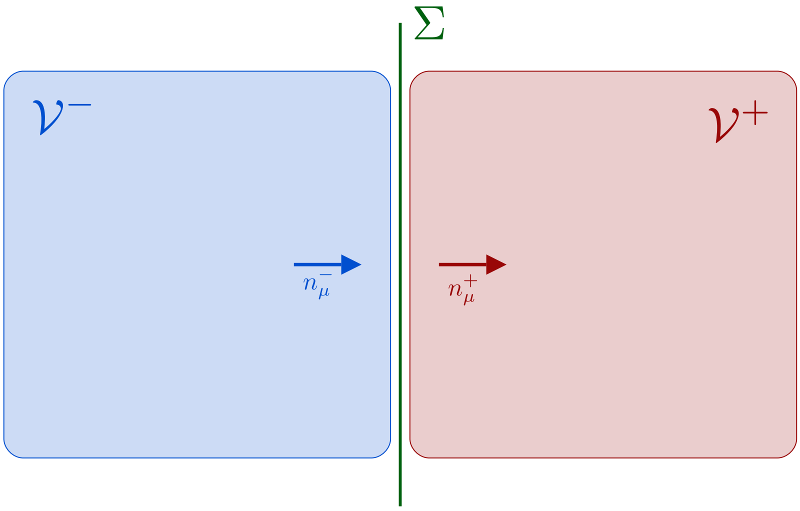

Figure 2: Pictorical representation of the composite space of study: Two different regions $\mathscr{V}^-$ and $\mathscr{V}^+$, with Σ as a common boundary to them. Note the choice of orientation of the normal vectors $n_\mu^\pm$. This will be relevant in further computations.

interpretation. In fact, one finds themself in such situation by just deriving expression (22) respect to any coordinate x^γ. This yields:

$$\partial_\gamma g_{\alpha\beta} = \Theta(\lambda)\,\partial_\gamma g^+_{\alpha\beta} + \Theta(-\lambda)\,\partial_\gamma g^-_{\alpha\beta} + \varepsilon\delta(\lambda)\,n_\gamma\left[g^+_{\alpha\beta} - g^-_{\alpha\beta}\right], \tag{23}$$

where the last term comes from the derivative $\partial_\lambda\Theta(\lambda) = \delta(\lambda)$.[2] Observe that this term will yield contributions of the form $\Theta(\pm\lambda)\delta(\pm\lambda)$ when computing the Christoffel symbols. But such combination of distributions is not one![3] Therefore, this requires to get rid of such term. In order to be succesful in this task, let us impose continuity of the metric accross the hypersurface Σ,

$$g^+_{\alpha\beta} = g^-_{\alpha\beta}. \tag{24}$$

This restriction can be polished further: Completness relation (10) allows us to split between tangential and normal components of the continuity relation. Here we note that $[n^\alpha]^+_- = n^+ - n^- = 0$.[4] Furthermore, coordinates $\{y^a\}$ are the same on both sides of the hypersurface Σ. This implies that tangent vectors are uniquely defined on it. With these two facts, one can rewrite Eq. (24) as:

$$h^+_{ab} = h^-_{ab}. \tag{25}$$

This is called the **first junction condition**, which constrains the *induced* metric h_{ab} to be the same on both sides of Σ. This is an essential requirement to have a well-defined geometry.

The derivation of the second junction can be done in different fashions, with the precedent mathematical approach requiring us to continue elaborating along the lines discussed above.[5] While this approach can be formal and elegant, we would be in need of introducing new concepts and convoluted computations. Inspired by [2], a more physical approach will be presented in this section of the notes.

We will first try to understand the geometry of one portion of the composition space presented above; A single manifold $\mathscr{M}$ and its boundary $\partial\mathscr{M}$ equipped with metrics $g_{\mu\nu}$ and h_{ab}, respectively. The action describing the geometry and contents of this space can be given as:

$$S_{\text{Total}} = S_{\mathscr{M}} + S_{\partial\mathscr{M}} + S_{\mathscr{L}_\Sigma}, \tag{26}$$

[2] The required change of variables in the derivative follows the fact that any displacement away from the hypersurface along one of the geodesics described above is given as $\mathrm{d}x^\alpha = n^\alpha \mathrm{d}\lambda$.

[3] Note that $\Theta(0)$ = indeterminate, while $\delta(0) = 1$. What is that?

[4] This requirement follows from footnote 16 plus the continuity of λ and x^α accross the hypersurface.

[5] These lines can be found in [1].

where $S_{\mathscr{M}}$ is the usual Einstein-Hilbert action as:

$$S_{\mathscr{M}} = \int_{\mathscr{M}} \mathrm{d}^D x \sqrt{|g|} \left(\tfrac{1}{2\kappa_D} {}^{(D)}R + \mathscr{L}_{\mathscr{M}} \right), \tag{27}$$

with ${}^{(D)}R$ the D-dimensional Ricci scalar and $\mathscr{L}_{\mathscr{M}}$ the Lagrangian density for any type of matter content in such space.

The action term $S_{\partial\mathscr{M}}$ in Eq. (26) is the Gibbons-Hawking-York term [REF] and is required for the variational principle to be properly defined since the Ricci scalar R is constructed from second derivatives of the metric. It describes how the submanifold is embedded as a boundary of $\mathscr{M}$. Hence, it can be expressed in terms of the extrinsic geometrical pieces as:

$$S_{\partial\mathscr{M}} = \tfrac{\varepsilon}{\kappa_D} \int_{\partial\mathscr{M}} \mathrm{d}^{D-1} y \sqrt{|h|} K, \tag{28}$$

with the induced metric h_{ab} and the trace of the extrinsic curvature K_{ab} as described in (9) and (15). Note the presence of the chosen normalisation ε of the normal vector n^α. Finally, the term $S_{\mathscr{L}_\Sigma}$ represents any type of matter content living **on** the boundary.

But this discussion is so far only valid for one manifold with its boundary. As described above, we are going to usually face situations where we find that the hypersurface Σ is the boundary of two different manifolds $\mathscr{M}^\pm$. Hence, we have to "duplicate" previous action (26) and glue them together, along the boundary $\partial\mathscr{M}$ mediating between $\mathscr{M}^-$ and $\mathscr{M}^+$. This will be assumed as the only boundary present in the construction.

Let us now derive the junctions condition by applying the variational principle respect to $g_{\mu\nu}$. In order to simplify this task and present results in the tidiest way, we will perform this computation piece by piece in the action. For each manifold $\mathscr{M}^\pm$ we find:

$$\begin{aligned} \delta S_{\mathscr{M}^\pm} = \tfrac{1}{2\kappa_D} \int_{\mathscr{M}_\pm} \mathrm{d}^D x \sqrt{|g|} \Big[& \left(G^\pm_{\mu\nu} - \kappa_D T^\pm_{\mu\nu} \right) \delta g^{\mu\nu} \\ & + \nabla_\mu \left(g_{\alpha\beta} \nabla^\mu \delta g^{\alpha\beta} - \nabla_\alpha \delta g^{\alpha\mu} \right) \Big], \end{aligned} \tag{29}$$

where $T_{\mu\nu}$ is the energy-momentum tensor corresponding to any matter content **in** the D-dimensional spaces

$$T_{\mu\nu} = \mathscr{L}_{\mathscr{M}} \, g_{\mu\nu} - 2 \frac{\delta \mathscr{L}_{\mathscr{M}}}{\delta g^{\mu\nu}}. \tag{30}$$

The boundary term yields a variation of the form:

$$\begin{aligned} \delta\partial\mathscr{M}^\pm = \tfrac{\varepsilon}{2\kappa_D} \int_{\partial\mathscr{M}^\pm} \mathrm{d}^{D-1} y \sqrt{|h|} \Big[& \left(K^\pm_{\mu\nu} - K^\pm g_{\mu\nu} \right) \delta g^{\mu\nu} \\ & + n_\mu \left(\nabla_\alpha \delta g_{\alpha\mu} - g_{\alpha\beta} \nabla^\mu \delta g^{\alpha\beta} \right) \Big]. \end{aligned} \tag{31}$$

Observe that the second line of each expression cancel against each other by Gauss-Stokes theorem [1].

With expressions (29) and (31) at hand and the action piece $S_{\mathscr{L}_\Sigma}$ in Eq. (26) representing matter content **on** the wall, one can then compute the dynamics of the whole composite space. However, one has to be careful, as the normal n^μ is chosen to point in the direction of increasing volume in the transverse directions, i.e. from $\mathscr{M}_-$ to $\mathscr{M}_+$. This implies a change of the sign[6] for $n^\mu_\pm$. This will affect the definition of the extrinsic curvature $K_{\mu\nu}$, as it contains the normal vector n^μ inside (see Eq. (15) and figure 2). Consequently, we have:

$$n^\mu = -n^\mu_+ = n^\mu_-. \tag{32}$$

[6] but not in the value of the norm ε.

The whole composition space (26) reads then:

$$
\begin{aligned}
\delta S_{\text{Total}} = & \delta S_{\mathscr{M}_+} + \delta S_{\mathscr{M}_-} - \delta S_{\partial\mathscr{M}_-} + \delta S_{\partial\mathscr{M}_+} + \delta S_{\mathscr{L}_\Sigma} = \\
& = \tfrac{1}{2\kappa_D} \int_{\mathscr{M}_+} \mathrm{d}^D x \sqrt{|g|} \left(G^+_{\mu\nu} - \kappa_D T^+_{\mu\nu} \right) \delta g^{\mu\nu} \\
& + \tfrac{1}{2\kappa_D} \int_{\mathscr{M}_-} \mathrm{d}^D x \sqrt{|g|} \left(G^-_{\mu\nu} - \kappa_D T^-_{\mu\nu} \right) \delta g^{\mu\nu} \\
& - \tfrac{\varepsilon}{2\kappa_D} \int_{\Sigma} \mathrm{d}^{D-1} y \sqrt{|h|} \left(K^+ g_{\mu\nu} - K^+_{\mu\nu} \right) \delta g^{\mu\nu} \\
& + \tfrac{\varepsilon}{2\kappa_D} \int_{\Sigma} \mathrm{d}^{D-1} y \sqrt{|h|} \left(K^- g_{\mu\nu} - K^-_{\mu\nu} \right) \delta g^{\mu\nu} \\
& - \tfrac{1}{2\kappa_D} \int_{\Sigma} \mathrm{d}^{D-1} y \sqrt{|h|}\, \kappa_D S_{\mu\nu} \delta g^{\mu\nu},
\end{aligned}
\tag{33}
$$

where $S_{\mu\nu}$ represents the energy-momentum tensor (30) for all matter content living **on** the hypersurface Σ. We can comfortably identify both Einstein equation for each manifold $\mathscr{M}_\pm$ in the first two lines of expression (33). Both spaces are independent solutions to the Einstein equation, so they will not contribute to the equation of motion. Last three lines of preceding expression correspond to the boundary geometrical contributions $\partial M_\pm$ (where the orientation of the normal has already been taken into account) and any possible matter fields living on such hypersurface. For the whole expression (33) to be zero, it is required that these last three term cancel, so

$$
\kappa_D S_{\mu\nu} = \varepsilon \left[\left(K^- g_{\mu\nu} - K^-_{\mu\nu} \right) - \left(K^+ g_{\mu\nu} - K^+_{\mu\nu} \right) \right] \tag{34}
$$

A small massage and projecting down to tangent components, as these are tangential tensors (11), we finally find:

$$
\kappa_5 S_{ab} = \varepsilon \left([K_{ab}]^+_- - h_{ab} [K]^+_- \right) \tag{35}
$$

This is the (second) **junction condition**. Equation (35) states that the presence of a localised energy-momentum tensor S_{ab} **on** the hypersurface will source a jump discontinuity in the extrinsic curvature. Alternatively, one can read it backwards: a $D-1$-dimensional hypersurface Σ acting as the boundary of two different D-dimensional manifolds $\mathscr{M}_\pm$, which are independent solutions to the Einstein equation, is required to be equipped with an energy-momentum tensor S_{ab}, proportional to the jump in the extrinsic curvature of the embedding, such that the whole composite space is also solution to the Einstein equation.

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