Lecture Notes on Dark Bubble Cosmology

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Abstract:

Contents

1 Introduction

TO DO List:

1. Add introduction. Emphasis on why the DB model.
2. Trim Foundations. Specially Swampland to just Non-Susy.
3. Decide about instantons length.
4. Change name of chapters and references to other chapters of the thesis.
5. Add section about propagator and more convincing arguments about gravity in section of gravity and strings.
6. Think about pedagogical approach and how the quantum bubble and decorating waves should be related to the 4D counter parts. Perhaps quick appendices of useful formulae and 4D cosmology stuff?

ADD EXAMPLE BOX FOR WHAT THEY WILL FIND IN EACH CHAPTER. TO EASY NAVIGATE.

In these lecture notes you will learn...

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2 Foundations of the Dark Bubble

In this section you will learn...

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2.1 Non supersymmetric AdS unstability conjecture

REWRITE FOCUSING ON NON-SUSY AND WGC

Chapter **??** was dedicated to providing a (quick) bird's eye view of the Stringy Kingdom's jurisdiction. We visited the prairies of braneworld constructions and learnt that some models lacked gravity, or did not take into account brane backreaction effects in the higher-dimensional picture or that the type of branes needed to induce a four-dimensional cosmological constant had to be supercritical. We then set off to the vast territory of the peaks and valleys of string flux compactifications. And here we learnt that this mechanism faces several fundamental problems. To build models where all the moduli are stabilised, equipped with scale separation and a small and positive cosmological constant Λ_4 seem to be unsolved issues [?, ?]. Even if some flux compactification constructions could avoid the above problems, any attempt to improve the model (i.e. to be able to reproduce some sectors of the standard model of particles) would require finding new solutions free from the problems discussed above.

These difficulties in string compactification have inspired researchers to go beyond the lands of the stringy kingdom, with the aim of **charting** its borders, if there are any. In order to do so, they proposed to take a different approach to the problem at hand; instead of aiming to obtain four-dimensional effective field theories with a positive cosmological constant Λ_4 from string theory, they thought that it could be more illuminating to identify, among all the existing lower-dimensional EFTs, those that can **actually** arise from quantum gravity. In order to avoid examining one by one the incalculable number of existing effective field theories, the idea is to find universal patterns that can be used as practical **criteria** to filter "good" EFTs from "bad" ones based on general characteristics. These screening criteria, which have not been proved yet,[1] are formulated in the form of **conjectures** that represent the very core of the **Swampland** programme [**?**], i.e. to identify the borders between the landscape of good EFTss and the **swamp**, those EFTss that do not meet the criteria to be obtainable from UV-complete theories.

The set of Swampland conjectures has grown since the programme was initiated in [**?**]. Interestingly, it has recently been shown that many of these conjectures are related, suggesting that they are no more than different aspects of some more fundamental principles of quantum gravity [**?**, **?**, **?**, **?**, **?**, **?**]. In this chapter we will not deal with the ever-growing number of swampland conjectures, but we willselect and present the basic concepts of some of them, which are relevant for the problem of finding de Sitter vacua in string theory and which are also the basis of the proposal to be presented in stage **??** of this work. The curious reader about the current state of the art of the Swampland is more than welcome to visit treferences presented above and the subsequent references therein.

This chapter is structured as follows: Section 2.1 discusses the difficulty of obtaining low-dimensional vacua with a positive cosmological constant. This is investigated by means of de Sitter conjecture(s), which set bounds on several features of de Sitter vacua from string theory. Although not related (at first sight) to the issue of positive vacua, we will examine in section 2.1 how gravity must always be the weakest force in any low energy description of quantum gravity with gauge forces. The consequences of the weakness of gravity for the fate of non-supersymmetric Anti-de Sitter spaces will be finally examined in section 2.1.

A watchtower to outlaw de Sitter theories

We have seen in chapter **??** how difficult it can be to construct de Sitter vacua in string theory. Furthermore, the few solutions that can be found are under scrutiny and are the source of debate about the components and/or mechanisms to build them [**?**, **?**, **?**]. One can then say that there is no full-fledged top-down de Sitter construction in a controlable regime of string theory. This suggests two different ways of thinking about this concern; on the one hand, it could be that these are just technical difficulties that will be overcome in the future as our understanding of the mathematical machinery used improves. On the other hand, this absence of de Sitter constructions points to more fundamental obstacles to the construction of de Sitter vacua in string theory. This is the route taken within the swampland programme.

We will now introduce the first swampland conjecture relevant to the present work, motivated by the difficulties of obtaining de Sitter vacua from string theory [**?**, **?**, **?**]:

de Sitter vacua, even if metastable, are excluded from quantum gravity.[2]

This de Sitter Conjecture (dSC) can also be read as: de Sitter belongs to the swampland. This very strong conjecture has been formulated as different bounds on the scalar potential of an effective field theory which is coupled to Einstein gravity. These read:

$$\min\left(\nabla_i \nabla_j V\right) \leq \frac{-c'}{M_{\mathrm{Pl}}^2} V, \qquad \text{and} \qquad |\nabla V| \geq \frac{c}{M_{\mathrm{Pl}}} V, \tag{2.1}$$

with c and c' positive constants of $\mathcal{O}(1)$ in Planck units. The $\min\left(\nabla_i \nabla_j V\right)$ represents the minimum possible eigenvalue of the mass matrix, i.e. the Hessian and $|\nabla V|$ is the norm of the vector derivatives with respect to all scalar fields in the theory. This condition provides a lower bound on the slope of the potential and states that if the potential is positive, it must be step enough not to allow for

[1] See [**?**] for the first steps in proving one of these criteria.
[2] At least in the asymptotic regions of the moduli space. See the next footnote for a definition of such a space.

extrema points [?, ?]. This last bound only needs to be imposed if the Hessian bound is violated, as proposed in the refined version of the de Sitter conjecture [?].

But these are not the only constraints that restrain the shape of the scalar potential. The Transplanckian Censorship Conjecture (TCC) states that any sub-Planckian quantum fluctuation should remain quantum. This implies that the expansion of the universe must slow down before any fluctuation become stretched beyond a Hubble size [?]. This restriction can be expressed in terms of the initial and final scale factors as follows:

$$\frac{a_f}{a_i} \times \ell_{(D)} < \frac{1}{H_f}, \tag{2.2}$$

where $\ell_{(D)}$ is the D-dimensional Planck length and H_f is the final Hubble parameter (i.e. the inverse Hubble radius at which the expansion should end to prevent quantum fluctions from becoming larger than the Hubble radius). Using both the classical Friedmann equations and the equations of motion for a scalar field ϕ coupled to gravity controlled by a potential $V(\phi)$, one can find a bound to the potential given by:

$$|\nabla V| \geq \frac{2}{M_{\rm Pl}\sqrt{(D-1)\,(D-2)}}\, V, \tag{2.3}$$

where the potential $V(\phi)$ is assumed to remain positive along the path connecting ϕ_f and ϕ_i in the moduli space.[3] The bound (2.3) appears at asymptotic regimes, i.e. at the asymptotic boundaries of the moduli space, when $\phi_i, \phi_f \to \infty$.

The similarities between the TCC bound (2.3) with the dSC bound (2.1) are not a coincidence. In fact, it can be shown that the undetermined constant c in the de Sitter bound (2.1) becomes fixed to $2/\sqrt{(D-1)\,(D-2)}$ when one takes the values of the field ϕ to be in the asymptotic regimes of the moduli space. Using the Swampland Distance Conjecture (SDC),[4] one can then show that the asymptotic behaviour of the scalar potential V is the same as that predicted by the TCC.

Memorial to the weakness of gravity

Let us consider a D-dimensional effective field theory with a $U(1)$ gauge symmetry (i.e. a Maxwell theory) coupled to gravity, described by

$$S = \int \mathrm{d}^D X \sqrt{|g_{(D)}|}\left[\frac{1}{2\kappa_D}\, R^{(D)} - \frac{1}{4g^2}\mathfrak{F}_{MN}\mathfrak{F}^{MN}\right], \tag{2.4}$$

where g is the gauge coupling of the theory. The (electric) version[5] states that there must exist a particle in the theory with mass m and charge q that satisfies the inequality [?]:

$$m \leq \sqrt{\frac{D-2}{D-3}}\, g\, q\, \left(M_{\rm Pl}^{(D)}\right)^{\frac{D-2}{2}}, \tag{2.5}$$

where $M_{\rm Pl}^{(D)}$ is the D-dimensional Planck mass (??). This inequality becomes a strict inequality in the absence of supersymmetry.[6] Another interpretation of the inequality (2.5) is as follows:

[3]The moduli space is the space of solutions to the equation of motion of all the fields $\Phi = \{\phi_1, \dots, \phi_n\}$ in the theory. The asymptotic regions of this space contain the weak coupling limit theories. The values ϕ_f and ϕ_i are the coordinates of this space, and represent the final value of the scalar field when the scale factor is a_f and a_i, respectively.

[4]The SDC states that an infinite tower of states of an effective field theory coupled to gravity becomes exponentially light at any infinite field distance limit in the moduli space [?, ?]. As no tower of light states has been observed in our universe, we can conclude that we may not be living in such asymptotic regimes.

[5]As the magnetic version is not relevant for this work, we refer the reader to [?].

[6]According to the Coleman-Mandula theorem, the only symmetry that can relate the mass (related to Poincaré symmetry) and the charge (related to an internal symmetry) is supersymmetry [?].

Gravity is the weakest force of all in a theory of quantum gravity with gauge forces.

This conjecture may seem *ad hoc* at first sight, but it has deep foundations in black hole physics. Consider a four-dimensional charged (i.e. Reisner-Nordström) black hole with line invariant:

$$\mathrm{d}s^2_{\mathrm{RN}} = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2, \tag{2.6}$$

where

$$f(r) = 1 - \frac{2\,M_{\mathrm{ADM}}}{r} + \frac{2\,g^2\,Q^2}{r^2}, \tag{2.7}$$

where M_{ADM} is the mass of the black hole[7] and Q is its charge. The quadratic form of the function $f(r)$ gives two horizons for a Reisner-Nordström black hole as:

$$r_\pm = M_{\mathrm{ADM}} \pm \sqrt{M^2_{\mathrm{ADM}} - 2g^2Q^2}. \tag{2.8}$$

The horizon is **degenerate** when $r_+ = r_-$. Furthermore, the horizon(s) require that the black hole mass satisfies an extremality bound in order to avoid the presence of naked singularities [**?**]. This reads:

$$M_{\mathrm{ADM}} \geq \sqrt{2}\,g\,Q\,M_{\mathrm{Pl}}, \tag{2.9}$$

where M_{Pl} has been restored to match dimensionality. A black hole that saturates the previous bound is called **extremal**. One can then see that the outer horizon shrinks towards the inner horizon as one approaches the extremality bound. The violation of this bound is associated with a naked singularity. Note that when $D = 4$, expression (2.5) and the lower bound (2.9) are the same with **opposite** orientation of the inequality. This coincidence is at the heart of black hole decay [**?**]. This can be understood by studying the evaporation process of a black hole.

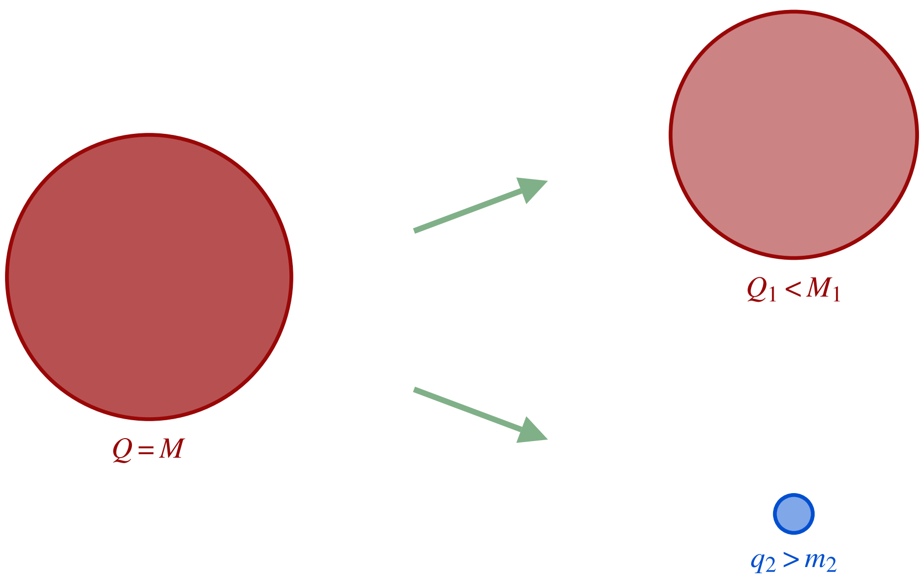

Figure 1: Decay of an extremal black hole. While one of the products can have a charge-to-mass ratio smaller than one to preserve the extremality bound, the other must have a charge-to-mass ratio greater than one. So the latter cannot be a black hole but a particle.

Black holes which are charged under a gauge $U(1)$ symmetry will induce an electric field around their horizon. This field allows for a discharge process analogous to Hawking radiation [**?**, **?**]. While

[7] A definition of the ADM formalism can be found in section B.

there are two different discharge channels, depending on the temperature-to-mass ratio of the black hole,[8] the one we are interested in is that of a cold black hole. This happens when $T_H \ll m$. In this case, the black hole discharges electrically through Schwinger[9] pair production.

Now consider an extremal black hole such that $M_{\rm BH}/Q_{\rm BH} = 1$ as in figure 1. A requirement for the discharge of this black hole is that the theory must contain a charged particle that can be emitted by the Schwinger process. If such particle exists, the black hole can decay into a pair of products such that one satisfies the extremality bound for black holes, i.e. $M_1 \geq Q_1$. By kinematic conservation (i.e. some of the total mass at rest of the initial black hole must be converted into kinetic energy), the second product **cannot** have a charge less than the mass, but greater than! This implies that the second object is not a smaller black hole, but a particle. This kinematic restriction can be derived from the conservation of energy and charge as follows. The mass $M_{\rm BH}$ of the initial black hole state must be greater than the sum of the decay product masses M_i. However, the charge $Q_{\rm BH}$ of the initial black hole should be equal to the sum of the products Q_i. This is

$$M_{\rm BH} \geq \sum_i M_i, \qquad Q_{\rm BH} = \sum_i q_i. \tag{2.10}$$

This entails

$$\frac{M_{\rm BH}}{Q_{\rm BH}} \geq \frac{1}{Q_{\rm BH}} \sum_i M_i = \frac{1}{Q_{\rm BH}} \sum_i \frac{M_i}{q_i} q_i \geq \frac{1}{Q_{\rm BH}} \left(\frac{M}{Q}\right)_{\rm min} \underbrace{\sum_i q_i}_{=Q_{\rm BH}} = \left(\frac{M}{Q}\right)_{\rm min} \tag{2.11}$$

Therefore, the existence of a particle with a charge-to-mass ratio **greater** than that of the black hole is need for the latter to decay.

The previous discussion has been focused on a field theory with a $U(1)$ gauge symmetry. More generally, one can consider antisymmetric tensors or rank p, as the ones that charge objects such as the $(p-1)$-dimensional branes that we saw in the introduction of chapter **??**. It has been proposed that the WGC can be extended to these (black) branes.[10] Hence, given an abelian p-form with gauge coupling g, the WGC requires the existence of a charged $(p-1)$-brane of tension T and integer charge Q such that [**?**, **?**]:

$$\frac{p(D-p-2)}{D-2} T^2 \leq \frac{Q^2}{\kappa_D^{2-D}}. \tag{2.12}$$

We will see that this has interesting consequences for the when applied to the stability of non-supersymmetric Anti-de Sitter vacua in string theory.

The necropolis of the fallen non-susy AdS vacua

In order to understand the implications of the weak gravity conjecture described above, we will consider a D-dimensional Anti-de Sitter vacuum supported by fluxes. As it was discussed in section **??**, the presence of fluxes (let us denote them by the letter f) will generate a potential that stabilises some of the moduli resulting from the compactification. Let us assume that the minimum corresponds to an AdS vacuum. These fluxes are Hodge dual to the top form gauge field strengths

[8]The Hawking temperature of a black hole is proportional to the difference between the outer and inner horizons. It is intrinsically related to the extremality of the black hole [**?**]. At $T_{\rm H} \gg m$, the black hole thermally discharges itself.

[9]A Schwinger effect is a predicted physical phenomenon that states that pairs of matter-antimatter particles can be created as a vacuum decay process in the presence of an electric field [**?**].

[10]The adjective black is added to create an analogy with black holes, as these objects are also surrounded by a horizon.

$\mathfrak{F}_D$, which are related to the gauge fields $\mathfrak{C}_{D-1}$. According to the WGC discussed above, this form will charge a $(D-2)$-dimensional brane, which will play the role of an hypersurface of co-dimension one (see appendix **??** for further details). This object will have a tension T, charge Q and will obey the rules defined in Eq. (2.12).

This hypersurface, located somewhere in the previously discussed D-dimensional AdS space, will interpolate between two different vacua with different values for the fluxes. Due to charge conservation the value of the flux[11] on one side of the brane will be f, while on the other side of the wall it will be equal to $f+Q$. If the vacuum hosting the brane is supersymmetric, then the brane will saturate the inequality (2.12) and remain steady at its position, separating two different supersymmetric configurations. However, if the vacuum is non-supersymmetric, then the weak gravity conjecture requires the strict inequality of equation (2.12). But such a co-dimensional one brane with tension less than the charge will correspond to an instability in the Anti-de Sitter space. Intuitively, we can see this process as simple conservation of energy: As the tension must be less than the charge, the energy cost of expanding the bubble will be less than the energy gain from the electric repulsion between different points on the wall. Therefore the brane will expand, mediating the decay of the vacuum with $f+Q$ to that inside the brane (i.e. the bubble) with vacuum f. We can then say that:

any non-supersymmetric AdS geometry supported by flux is unstable.

In other words, supersymmetry is the only mechanism to protect a vacuum decay from decaying into quantum gravity. This conjecture first appeared in [**?**, **?**], but its foundations can be found in an older observation called *AdS fragmentation* in [**?**]. In this work, it was shown that in a D-dimensional Anti-de Sitter space containing a spacetime filling flux of the same rank, there will be $(D-2)$-branes charged with respect to the flux, which can nucleate and expand towards the boundary of the AdS space, leaving behind an AdS space with one less unit of flux. The viability of this process depends on the charge-to-tension ratio of the brane. In order to understand this process, let us construct the instanton solution[12] that mediates the vacuum decay.

Let us consider the following parametrisation of an AdS_D metric,

$$\mathrm{d}s^2 = L^2\left(\cosh^2(r)\,\mathrm{d}\tau^2 + \mathrm{d}r^2 + \sinh^2(r)\,\mathrm{d}\Omega_{d-2}^2\right), \tag{2.13}$$

where L is the AdS radius. A spherical $(D-2)$-brane with radius $r(\tau)$ wrapping aroud S^{D-2} and evolving in the Euclidean time τ is described by the action [**?**]:

$$S = L^{D-1}\Omega_{D-2}\int \mathrm{d}\tau\left(T\sinh^{D-2}(r)\sqrt{\cosh^2(r)+\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2} - Q\sinh^{D-1}(r)\right), \tag{2.14}$$

where Ω_{D-2} is the volume of the unit $(D-2)$-sphere. One can then solve the Euclidean Einstein's equations, to find the radius for the spherical brane solution that extremises the action to be:

$$r_0 = \tanh^{-1}\left(\frac{T}{Q}\right), \tag{2.15}$$

in Planck units. In the Lorentzian signature, this is associated with the nucleation at rest of a bubble of radius r_0, which begins to expand, and mediates the decay of the vacua. Imposing the strict inequality of the WGC condition, we see that

$$r_0 < \tanh^{-1}(1)\,. \tag{2.16}$$

[11] Understand this value as the integral of F_p over some internal non-trivial p-cycle of the compact space.

[12] The decay via brane nucleation will be a non-perturbative process, governed by the instanton (i.e. the brane) solution of the nucleated brane. More information about these solutions can be found in appendix **??**.

This implies that a finite value of the Euclidean action can only be achieved with the strict inequality version of expression 2.12. On the contrary, if the vacuum is supersymmetric, we could satisfy the WGC by a BPS brane saturating the inequality (2.12), which implies an infinite radius solution, i.e. a non-expanding straight brane interpolating between two different supersymmetric vacua.

With this final conjecture we conclude our brief summary of the swampland program. Perhaps our dear reader is now wondering if a brane like the one mediating the aforementioned decay could be used to realise similar braneworld constructions to those discussed in section **??**. In contrast to de Sitter solutions, Anti-de Sitter ones are quite abundant in supergravity and have been studied in extension. If some of these spaces have broken supersymmetry, they should decay via nucleation of a co-dimension one brane as discussed in 2.1. Inspired by the Randal-Sundrum construction discussed in **??**, one could then find a way to induce a positive cosmological constant on the brane. However, this may not be an easy task, as the Randal-Sundrum construction requires a supercritical brane, i.e. $\sigma > Q$, while the aforementioned decay is mediated by subcritical ones $T < Q$.

At this point we would like to invite the reader to continue reading stage **??** of this work. There, we will jump into one of the nucleated branes discussed above and explore its induced cosmology.

2.2 Instantons: Coleman-de Lucia and Brown-Teitelboim

TO DO: Shall this be rewritten in some way?

We will review two main types of instantons[13] in this appendix: Coleman-de Lucia [**?**] and Brown-Teitelboim [**?**] ones. These types of solutions are constitutional foundations of the dark bubble model proposed in part **??** of this work.

Coleman-De Lucia instantons

Let us consider a theory with a single scalar field $\phi(t, \bar{x})$ which is controlled by a potential $V(\phi)$ in a D-dimensional spacetime. We further assume that this potential has *two* non-degenerate minima $V_\pm$, such that $V_+(\phi) > V_-(\phi)$. The shape of this potential is sketched in figure 2. The action governing the dynamics of such configuration is:

$$S = \int \mathrm{d}^D x \left(\tfrac{1}{2}\partial_M \phi \partial^M \phi - V(\phi)\right), \tag{2.17}$$

where M goes from 0 to $D-1$. From a classical point of view, if the scalar field ϕ is at rest in the local minimum V_+, it will not have enough energy to climb over the potential barrier and reach the global minimum V_-. However, if quantum mechanics is taken into account, there is a probability that the field will "tunnel" through the barrier and end up in the global minimum V_-. In this framework, we will say that the local minimum V_- is *metastable* and a *decay* can take place, so that the field configuration "traverses" from the *false* vacuum V_+ to the *true* vacuum V_-.

The process described above was studied by Sidney Coleman and Curtis Callan in [**?**, **?**]. The underlying physics of these decays is a first order transition[14] and it can be easily understood by a thermodynamic analogy.

Consider a fluid, that is heated homogenously up to the point where it can start boiling. In its initial liquid phase, as much as you can try to have an even distribution of the temperature, there may be thermodynamic fluctuations at some given points where the temperature is slightly different from the surrounding volume. If this variation is favourable towards the phase transition

[13]Instantons are solutions to the classical Euclidean equations of motion which interpolate between real classical motions of the system, and thus provide a semiclassical "path" by which the system tunnels from one classical regime to the other.

[14]i.e. the amount of energy absorbed or released by the system is fixed.

temperature, a *bubble* of vapour phase will appear. This bubble can have two different fates: If its size is smaller that a certain threshold, so that the gain in energy density (i.e. the energy stored inside the bubble volume) is overcompensated by the loss of surface energy, then the bubble will collapse to nothing. On the contrary, a large enough bubble will have a favourable energy balance, which will cause the bubble to expand until all the liquid has undergone the phase transition to vapour.

This is a similar situation to the vacuum decay described some paragraphs above. We are now faced with a field decorating an empty spacetime. The configuration of the field is such that it starts in its *false* vacuum. In this case, not thermodynamical, but quantum fluctuations of the vacuum can occur, triggering the phase transition towards the *true* configuration in some specific region. This will happen through the nucleation of a spherically symmetric bubble of true vacuum.

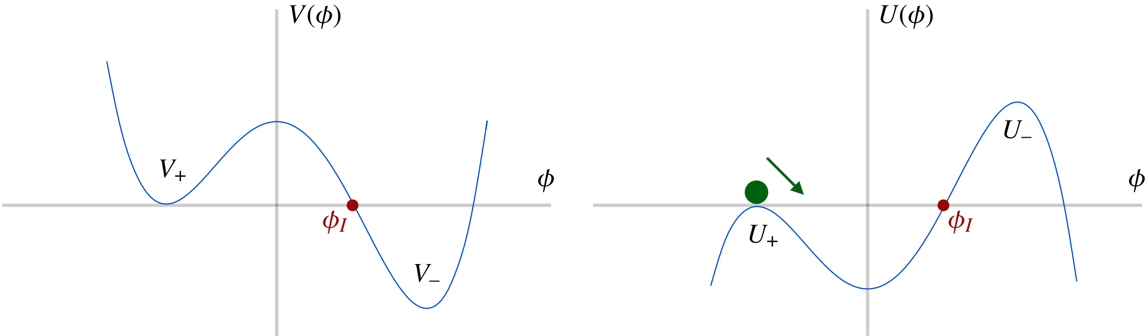

Figure 2: (Left): The potential $V(\phi)$ with its local and global minima. (Right): The inverted potential $U(\phi)$. This allows us to think in terms of the motion of a particle. Starting from ϕ_+, we then see that it can roll up to ϕ_I and bounce back. Hence the name for this type of solutions.

The probability per unit of volume Γ that such an event will occur can be determined in the semi-classical approximation $\hbar \ll 1$ and it is given by:[15]

$$\Gamma \sim e^{-B}\,(1 + \mathcal{O}(\hbar))\,, \tag{2.18}$$

with $B = S_E(\phi_I) - S_E(\phi_-)$. The term $S_E(\phi_-)$ is the Euclidean action of the system evaluated in the false vacuum while $S_E(\phi_I)$ is the Euclidean action for the bounce solution. The bounce is the instanton solution corresponding to the nucleated bubble, i.e. solutions that extremise and give a finite value to the Euclidean version of the action that governs the dynamics of the system. Let us discuss this quantity in more detail.

The bounce

In order to easily understand the bounce, we are going to work on three specific regimes that are the implicit ones in this work:

- The first one, by analytic continuation, we will perform all our computations in the Euclidean realm ($\tau = it$).

[15]This can be deduced from the transition matrix element between the ground state $|0\rangle$ evolving in time. When $\hbar$-corrections are taken into account, the associated eigen-energy of the process is $E = \frac{1}{2}\omega\hbar + \hbar K\,e^{-B/\hbar}$, where $K \in \mathbb{C}$. Identifying the corrections with the non-Hermitian piece of the evolution operator $\mathcal{H}$ of the system, one can then relate the imaginary part of the energy E with the decay channel Γ.

- The second one will affect the shape of the potential $V(\phi)$, as we will be working in the *thin* wall approximation. This is the case when the difference between the energy density of the two vacua is really small, so that one writes $|\Delta V(\phi)| = \varepsilon$ with $|\varepsilon| \ll 1$.
- The last regime will help us to think in terms of particle dynamics. If we invert the potential, i.e. $U(\phi) = -V(\phi)$, such as shown in figure 2, we can then use a motion analogy. In this case, a particle sitting on the "lower" hill could start rolling down, up to the point of the potential where its energy will be equal to that of the starting position. As we know from high school, this is the point that the particle will reach with zero velocity and would then *bounce* back down the valley. Restoring the sign, we then see the meaning of that bouncing point: it would be the final position at which the tunneling process ends. We will call this field position as ϕ_I, which is a solution of the equation of motion and extremise the value of the Euclidean action.

Given the previous assumptions and exploiting the implicit $O(D)$ symmetry of the Euclidean space of study, so that we can rewrite all coordinates $(\tau, \bar{x})$ in terms of a "radial" coordinate $r = \sqrt{\tau^2 + x_i x^i}$, it can be shown that the Euclidean version of action (2.17) is

$$\begin{aligned} S_E^{\text{total}} = \int \mathrm{d}^D x \left(\tfrac{1}{2}\partial_M\phi\partial^M\phi + U(\phi)\right) =& \Omega_{D-1}\int_R^\infty \mathrm{d}r\, r^{D-1}\left[\left(\frac{\mathrm{d}\phi_+}{\mathrm{d}r}\right)^2 + U_+(\phi)\right] \\ &+\Omega_{D-1}\, R^{D-1}\int \mathrm{d}r \underbrace{\left[\left(\frac{\mathrm{d}\phi_I}{\mathrm{d}r}\right)^2 + U_+(\phi_+)\right]}_{S_1} \\ &+\Omega_{D-1}\int_0^R \mathrm{d}r\, r^{D-1}\left[\left(\frac{\mathrm{d}\phi_-}{\mathrm{d}r}\right)^2 + U_-(\phi)\right], \end{aligned} \tag{2.19}$$

where Ω_{D-1} is the area of a unit-radius $(D-1)$-sphere given by:

$$\Omega_{D-1} = \frac{2\,\pi^{D/2}}{\Gamma(D/2)}. \tag{2.20}$$

Note that we have divided the integration regime into three different intervals:

- That corresponding to the *outside* of the bubble (first line in Eq. (2.19)). The phase transition has not yet reach to this region, so the field and associated potential $U(\phi)$ correspond to the *false* configuration.
- The *instanton* solution, which corresponds to the location of the bubble (second line). We will call this piece S_1, and it represents the Euclidean action for the bounce solution.
- The *inside* volume enclosed by it (last line). The decay has already taken place here, so we write ϕ_- and $U(\phi)$.

Let us now specify our computations for a potential such that $U(\phi_+) = 0$ and $U(\phi_-) \simeq -\varepsilon$. Plugging in these considerations into the action (2.19), we will obtain:

$$S_E^{\text{total}} = S_E(\phi_I) - S_E(\phi_-) = \Omega_{D-1}\left(R^{D-1}\,S_1 - \frac{R^D}{D}\varepsilon\right), \tag{2.21}$$

which corresponds to the expression in the argument of Eq. (2.18). If we now derive the action with respect to R, we will find the radius R that extremises the value of the action. This yields:

$$B = \frac{\Omega_{D-1}}{D}\left(\frac{D-1}{\varepsilon}\right)^{D-1} S_1^D. \tag{2.22}$$

In this way we have found the closed-form expression for the coefficient B in the thin-wall approximation. The explicit value of this term will depend on the potential shape integrand in S_1. In any case, this value ensures that the bubble nucleation probability is maximal.

When gravity wants to blow bubbles

The previous discussion was made without considering the role of gravity. When gravity is not present, any false vacuum in quantum field theory can decay. However, this is not the case when the theory is coupled to gravity. This was later studied by Coleman and de Luccia in [?], who showed that gravity can stabilise some false vacua, making them persistant.[16] Let us elaborate on this point:

The bubble, being an instanton solution, is formed at a radius R such that it minimises the Euclidean action. The formation of such a bubble has an energy cost, which is proportional to the bubble tension σ times the area of the bubble. But where does this energy come from? By conservation of energy, it must have come from reducing the vacuum energy inside its interior. This amount of energy is equal to the vacuum energy density times the volume enclosed by the bubble. This relationship between the energy contributions can be seen in Eq. (2.21). If the balance is *precise* so that one term compensates for the other, the bubble will be nucleated with the radius that minimises the action and remain at rest. In the case the bubble gains more energy

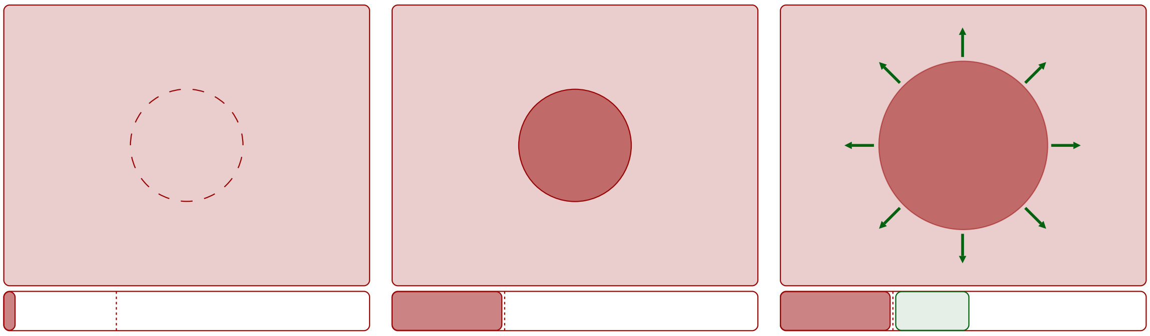

Figure 3: From left to right: Bubble nucleation requires a delicate balance between the surface energy (represented by the dashed line) and the vacuum energy difference (the bar under each rectangle). When the values are *exactly*equal to compensate each other, a bubble will nucleate at rest and remain there. However, if more energy is extracted from the decay than is required to form the bubble, the extra energy will be kinetic and used to expand the bubble.

than is needed to create it by reducing the vacuum energy inside it, the extra energy will provide the kinetics to expand the bubble. The bubble, which will nucleate at rest, is then accelerated outwards, asymptotically approaching the speed of light. This whole process can be understood by the expression

$$\left| V_{\text{after}} - V_{\text{before}} \right| = E_{\text{wall}} + E_{\text{kinetic}}, \tag{2.24}$$

where E_{kinetic} is the kinetic energy of the bubble if more energy was obtained by reducing the energy vacuum density inside. Note that when this term is zero, the tension σ, i.e. the "mass" of the bubble at rest is proportional to difference in vacuum energy discussed above.

[16] It can be shown that the bounce solution gets modified to by gravity as:

$$B_{\text{gravity}} = \frac{B}{(1 + (R_0/2\,\Lambda_D))^2}, \tag{2.23}$$

where R_0 is the radius that minimises the action (2.21) and Λ_D is the vacuum energy of the D-dimensional spacetime, i.e. a cosmological constant.

Brown-Teitelboim instantons

The previous discussion only applies if the decay process occurs once. One can then wonder whether a tower of decays can be achieved within the precedent formalism. This was done by Brown and Teitelboim in [**?**], where they proved that one can have a tower of vacuum decays when antisymmetric tensor fields are considered in the bubble action.

Let us now suppose a vacuum supported by a cosmological constant Λ_D which can be lowered in the presence of antisymmetric tensor field strength $F = \mathrm{d}A$, where A is the corresponding tensor field. In this case, the nucleated membrane will be charged under this field A. The Euclidean action to describing this type of instantons is given by:

$$\begin{aligned} S_E = &-\frac{1}{2\kappa_D}\int \mathrm{d}^D x\sqrt{|g|}\left({}^{(D)}R - 2\Lambda_D\right) + \sigma\int \mathrm{d}^{D-1}y\sqrt{|h|} + \frac{1}{\kappa_D}\int \mathrm{d}^{D-1}x\sqrt{|g|}\,K \\ &+ \frac{1}{(D-1)!}\int \mathrm{d}^D x\sqrt{|g|}\nabla_M\left(F^{M\cdots}A_{\dots}\right) - \frac{1}{2D!}\int \mathrm{d}^D x\sqrt{|g|}\,|F|^2 \\ &+ \frac{q}{(D-1)!}\int \mathrm{d}^{D-1}y A_{M_1\cdots M_{D-1}} e^{M_1}_{m_1}\cdots e^{M_{D-1}}_{m_{D-1}}\epsilon^{m_1\cdots m_{D-1}}, \end{aligned} \tag{2.25}$$

with ϵ is the Levi-Civitta symbol. As already discussed in appendix **??**, Eq. (2.58), the first line represents the gravitational contribution of the whole space. In this case, the energy-momentum tensor of all matter living **on** the hypersurface (i.e. the membrane), is its tension σ. The second line (2.25) represents the presence of the field strength in the D-dimensional space and a boundary term for it, while the third line denotes the coupling of the brane to the tensor field A.

As F is a form of top degree, this means that it is proportional to the volume form:

$$F_{M_1\cdots M_D} = E(x)\epsilon_{M_1\cdots M_D}, \tag{2.26}$$

for some scalar function $E(x)$, i.e. the electric field. This can be shown to be constant away from the brane sources by examining the equation of motion for the tensor field A:

$$\partial_L E(x)\,\epsilon^{L\,M_1\cdots M_{D-1}} = -q\int \mathrm{d}^{D-1}y\,\delta\left(x^\alpha - x^\alpha(y)\right)e^{M_1}_{m_1}\cdots e^{M_{D-1}}_{m_{D-1}}\epsilon^{m_1\cdots m_{D-1}}. \tag{2.27}$$

This implies that the electric field $E(x)$ will jump one charge unit q across the brane.[17] This jump will be an additional contribution to lowering the vacuum energy via brane nucleation. Indeed, if one inserts Eq. (2.26) in the action (2.25) one can read off an effective cosmological constant:

$$\Lambda_{\text{eff}} = \Lambda_D + \tfrac{1}{2}\kappa_D\,E^2. \tag{2.28}$$

Thus, when the equation of motion (2.27) holds, we can rewrite the action (2.25) as:

$$\begin{aligned} S_E = &-\frac{1}{2\kappa_D}\int \mathrm{d}^D x\sqrt{|g|}\left({}^{(D)}R - 2\Lambda_D\right) + \frac{1}{2D!}\int \mathrm{d}^D x\sqrt{|g|}\,|F|^2 \\ &+\sigma\int \mathrm{d}^{D-1}y\sqrt{|h|} + \frac{1}{\kappa_D}\int \mathrm{d}^{D-1}x\sqrt{|g|}\,K\,, \end{aligned} \tag{2.29}$$

Let us now discuss the decay channel (2.18) associated with the nucleation of a single bubble of *true* vacuum[18] when the tensor field A permeates the vacuum. Here we will just highlight the most important results of [**?**]. The curious reader interested in the detailed steps of the derivation is referred to that paper.

[17] Due to the conservation of charge, this charge difference will be carried by on the bubble's boundary, i.e. *the (mem)brane.*

[18] As the decay can occur repeatedly, we should speak of a *less false* vacuum. Let us stick to *true* to avoid confussion.

It can be shown that the bounce of the action (2.29) is given by:

$$B=\sigma\,\Omega_{D-1}\,R^{D-1}+\frac{1}{\kappa_D}\Bigg|\Bigg[\frac{2\Lambda_i}{(D-2)}\mathrm{Vol}_D\left(R,\epsilon_i,\Lambda_i\right)\\ +(D-1)\,\epsilon_i\sqrt{\frac{1}{R^2}-\frac{2\Lambda_i}{(D-2)(D-1)}}\Omega_{D-1}\,R^{D-1}\Bigg]\Bigg|_+^-, \tag{2.30}$$

where we have dropped the "eff" superscript of the effective cosmological constant Λ and ϵ_i represents the orientation choice for the normal n_μ of the hypersurface Σ, i.e. the membrane, as discussed in appendix **??**. The area covered by a unit-radius $(D-1)$-dimensional Euclidean membrane is Ω_{D-1} and is given by:

$$\Omega_{D-1}=\int \mathrm{d}^{D-1}\xi\sqrt{^{(D-1)}g}=\frac{2\pi^{D/2}}{\Gamma(D/2)}, \tag{2.31}$$

where $\Gamma(x)$ is the usual gamma function. The volume of the inside and the complement[19] of the outside can be computed by:

$$V_D\left(R,\epsilon_{\mathrm{i}},\Lambda_{\mathrm{i}}\right)=\int_\pm \mathrm{d}^D x\sqrt{g}=\left(\frac{(D-1)(D-2)}{2\,|\Lambda_{\mathrm{i}}|}\right)^{D/2}\Omega_{D-1}\\ \times\left|\int_1^{\sigma_{\mathrm{i}}\left[1-2\Lambda_{\mathrm{i}}R^2/(D-2)(D-1)\right]^{1/2}}\mathrm{d}(\cos x)\sin^{D-2}x\right|. \tag{2.32}$$

In the case that the effective cosmological constant is $\Lambda<0$, the trigonometric functions in Eq. (2.32) should be replaced by hiperbolic trigonometric functions.

In the same spirit as in the case of Coleman-De Lucia instantons, one can find the value of R that extremises the action B by deriving it with respect to R. Substituting the resulting R_0 back into B gives the extremised nucleation probability of a Brown-Teitelboim bubble in D dimensions.

2.3 Hypersurfaces and junction conditions

I THINK THIS SECTION IS GOOD ENOUGH. PERHAPS ADDING SOME INTERMEDIATE STEPS OF COMPUTATIONS?. PUT EMPHASIS ON JUNCTIONS.

The aim of this subsection is to provide the reader with a quick overview of the wonderful world of hypersurfaces, i.e. submanifolds of dimension dim $=D-d$ which are "slices" of D-dimensional manifolds. This subsection will follow along the lines of [**?**] and [**?**].

The mathematical definition of hypersurface can be cast in the following form:

$$\mathcal{S}=\{x^\alpha\in\mathcal{M}\,|\,\Phi(x^\alpha)=0\}\subset\mathcal{M}. \tag{2.33}$$

Perhaps mathematical definitions will cause our dear reader to have a rash, so let us simplify the previous description in more "peasant" language. In addition, as we also care about our reader's sanity, we will restrict this study to hypersurfaces of *co-dimension* one. This is $d=1$. More brave mathematical warriors, willing to fight through *co-dimension* d sub-manifolds, are welcome to read the exquisite selected literature on the topic [**?**, **?**, **?**].

As we said before, we define a hypersurface Σ as a "slice" of a higher-dimensional space with metric $g_{\mu\nu}$. This is something that has been known since the good old high school days. For example, one can define a two-dimensional sphere in a three-dimensional flat Euclidean space by

$$\Phi(x,y,z)=x^2+y^2+z^2-R^2=0, \tag{2.34}$$

[19] This is the volume fraction of the background that is converted into the inside region when the bubble is created, i.e. the dashed volume in figure 3.

with R as its radius. The **embedding** map Φ tells us how to "insert" the hypersurface Σ (the sphere) in the manifold $\mathcal{M}$, i.e. the three-dimensional Euclidean space. Note that this description of the sphere respects the definition given in expression (2.33). Its coordinates $\{x, y, z\}$ are contained in $\mathcal{M}$, the three-dimensional Euclidean space and Eq. (2.34) corresponds to the restriction in the second part of the definition. Eventually, one can also choose a new set of coordinates that are **intrinsic** to the sphere itself, such as $y^a = \{\phi, \theta\}$, so that we can relate the **extrinsic** coordinates $\{x, y, z\}$ to those of the sphere by the well-known parametric relations:

$$\begin{aligned} x &= R \cos\phi \sin\theta, \\ y &= R \ \sin\phi \sin\theta, \\ z &= R \cos\theta. \end{aligned} \tag{2.35}$$

This parametric equation can be written in a more general way as:

$$x^\alpha = x^\alpha(y^a). \tag{2.36}$$

Although it has been already specified in subsection **??**, the significance of the following notation demands to emphasise again that:

- **Extrinsic** coordinates will be denoted by **Greek** letters $\{\alpha, \beta, \gamma, \cdots\}$.
- **Intrinsic** coordinates will be denoted by **Latin** letters $\{a, b, c, \cdots\}$.

It is extremely important that this notation is crystal clear, as the aim of this subsection, and by extension this thesis, is to relate extrinsic and intrinsic properties of manifolds one another, so that we can get as much information as possible from both coordinate systems.

Returning to our simple spherical example, let us continue with more definitions. As a surface, it can be equipped with vectors. The ones of our interest are of two different types: The normal and the tangent vectors.

Normal vector

It is easy to think of a normal vector n^α in the spherical case described above: A stingy arrow pointing orthogonally (outside or inside) with respect to the surface Σ. The problem arises when dealing with dimensions greater than three or signatures beyond the Euclidean one. How do you define the normal vector?

We can define a unit normal unit vector n^α imposing unitarity:

$$n^\alpha n_\alpha = \epsilon = \pm 1, \tag{2.37}$$

where $(+)$ corresponds to a timelike hypersurface and $(-)$ represents spacelike ones.[20] Furhtermore, we require that n^α points in the direction of increasing Φ. In the case we that we are looking at a spacelike surface, the normal vector will point in the direction of growing spatial sections, i.e. $n^\alpha \partial_\alpha \Phi > 0$. This implies that the normal vector can be defined as:

$$n_\alpha = \frac{\epsilon\, \partial_\alpha \Phi}{\sqrt{g^{\mu\nu}\, \partial_\mu \Phi\, \partial_\nu \Phi}}. \tag{2.38}$$

Note that for the two-dimensional sphere described above, as the metric $g^\mu_\nu = \mathbb{I}_{3\times 3}$, one recovers its usual euclidean definition.

[20]Null hypersurfaces are trickier. Check [?] for further details.

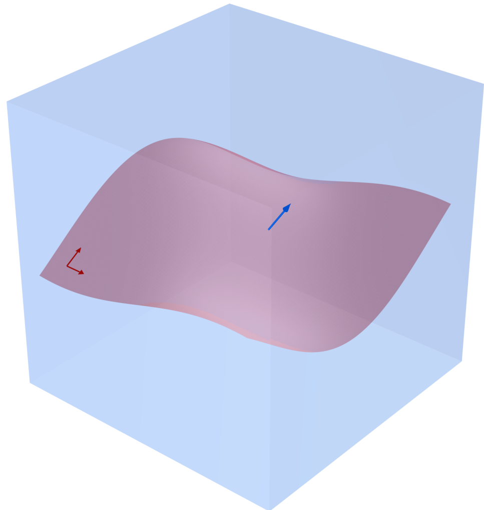

Figure 4: Two of the most basic elements to describe the embedding of a co-dimension one hypersurface Σ in a D-dimensional manifold are the normal vector n^μ and the tangent vectors e^α_a.

Tangent vectors

Unlike normal vectors, tangent vectors live on the hypersurface. They are defined by:

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}, \tag{2.39}$$

where again, x^α coordinates belong to the manifold $\mathcal{M}$ and y^a are coordinates of Σ, related by the parametric relation (2.36). Note that e^α_a will be a matrix, i.e. the Jacobian of the parametric transformation, with D rows and $D-1$ columns, as we are dealing with hypersurfaces of co-dimension one.

As expected, tangent and normal vectors are orthogonal to each other, which means that their scalar product is null,

$$n_\alpha e^\alpha_a = 0. \tag{2.40}$$

Furthermore, tangent vectors, acting as "projectors" of D-dimensional coordinates x^α onto the hypersurface Σ, can be used to describe the **intrinsic** or **induced** line invariant on the surface. This is done by restricting the line element of the D-dimensional space to displacements confined to Σ, as:

$$\begin{aligned} ds_\Sigma &= g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \Big|_\Sigma \\ &= g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a}\mathrm{d}y^a\right)\left(\frac{\partial x^\beta}{\partial y^b}\mathrm{d}y^b\right) \\ &= \underbrace{g_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial y^a}\tfrac{\partial x^\beta}{\partial y^b}}_{h_{ab}}\,\mathrm{d}y^a\mathrm{d}y^b, \end{aligned} \tag{2.41}$$

where h_{ab} is the **induced** metric or first fundamental form. This will allow us to define the following completeness relation for the metric $g_{\alpha\beta}$ as:

$$g^{\alpha\beta} = \epsilon\, n^{\alpha} n^{\beta} + h^{ab} e^{\alpha}_{a} e^{\beta}_{b}. \tag{2.42}$$

This previous expression will be very useful in the following pages, as it relates the tangential and normal parts of the embedding in the line invariant hosting the hypersurface.

Let us now imagine a full tangential tensor $A^{\alpha\beta}$ defined on Σ, with no components in the normal directions (i.e. $A^{\alpha\beta} n_{\alpha} = 0$). Such tensor admits the decomposition

$$A^{\alpha\beta\cdots} = A^{ab\cdots} e^{\alpha}_{a} e^{\beta}_{b} \cdots = h^{ai} h^{bj} \cdots\ A_{ij\cdots} e^{\alpha}_{a} e^{\beta}_{b} \cdots \tag{2.43}$$

To understand how these tensors differenciate, we can simply apply the usual covariant derivative. However, the resulting information will depend on the chosen set of coordinates. If we choose the hypersurface coordinates y^{a}, it is easy to prove that:

$$\nabla_{b} A_{a} = \nabla_{\beta} A_{\alpha} e^{\alpha}_{a} e^{\beta}_{b} = \cdots = \partial_{b} A_{a} - \Gamma^{i}_{ab} A_{i}\,, \tag{2.44}$$

where $\cdots$ are intermediate steps of the computation. The expression (2.44) corresponds to the well-known *intrinsic* covariant differentiation. But this is not the end of the story. We can reproduce the same computation, but splitting the components of the metric $g_{\alpha\beta}$ into its normal and tangential pieces. For indices convenience, let us consider the vector $\nabla_{\beta} A_{\alpha} e^{\beta}_{b}$, whose tangential components are given by Eq. (2.44). This is:

$$\begin{aligned} \nabla_{\beta} A_{\alpha} e^{\beta}_{b} &= g_{\alpha\gamma}\, \nabla_{\beta} A^{\gamma} e^{\beta}_{b} \\ &= \left(\epsilon n_{\alpha} n_{\gamma} + h_{ij} e^{i}_{\alpha} e^{j}_{\gamma}\right)\, \nabla_{\beta} A^{\gamma} e^{\beta}_{b} \\ &= \epsilon \left(n_{\gamma}\, \nabla_{\beta} A^{\gamma} e^{\beta}_{b}\right) n_{\alpha} + h_{ij} \underbrace{\left(\nabla_{\beta} A^{\gamma} e^{j}_{\gamma} e^{\beta}_{b}\right)}_{\nabla_{b} A^{j}} e^{i}_{\alpha}, \end{aligned} \tag{2.45}$$

Note that the first term can be rewritten using the fact that $A^{\gamma} n_{\gamma} = 0$, as we assume that the tensor A_{α} is completely tangential. This allows us to rewrite the previous expression as:

$$\cdots = \nabla_{b} A_{i} e^{i}_{\alpha} - \epsilon A^{i} \underbrace{\left(\nabla_{\beta} n_{\gamma} e^{\gamma}_{i} e^{\beta}_{b}\right)}_{K_{bi}} n_{\alpha}, \tag{2.46}$$

where we have defined the symmetric *extrinsic* curvature of the hypersurface Σ or second fundamental form of the hypersurface as:

$$K_{ab} = \nabla_{\beta} n_{\alpha} e^{\alpha}_{a} e^{\beta}_{b}. \tag{2.47}$$

with trace computed after contraction against the induced metric h_{ab}

$$K = K^{ab}\, h_{ab} = \nabla_{\alpha}\, n^{\alpha}. \tag{2.48}$$

Note that the starting point in Eqs. (2.44) and (2.45) is the same; the covariant derivative of the tangent form A^{α} lives on Σ. Eq. (2.46) shows a pure tangential piece of the vector field (the first term) and its normal component (the second term). This piece carries geometrical information about how the hypersurface Σ is embedded in the hosting space $\mathcal{M}$ and hence, what kind of curvature it acquires. This term can only be zero if and only if the extrinsic curvature vanishes.

Gauss-Codazzi Equations

The next logical step in this discussion is to explore if the *intrinsic* Riemann tensor of the hypersurface Σ can also be expressed in terms of *extrinsic* information. Let us first recall the definition of a purely intrinsic curvature tensor as:

$$[\nabla_a, \nabla_b]\, A^c = R^c_{dba}\, A^d. \tag{2.49}$$

In the same spirit as in the previous computations, one can use the identities relating normal and tangent tensors in order to relate both *extrinsic* and *intrinsic* curvature tensors. This requires a modest amount of algebra and we refer the curious reader to [**?**]. Here we will only show the final result of the computation,

$$\begin{aligned} R_{\alpha\beta\gamma\delta} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d &= R_{abcd} + \epsilon \left(K_{ad}K_{bc} - K_{ac}K_{bd}\right), \\ R_{\mu\alpha\beta\gamma} e^\alpha_a e^\beta_b e^\gamma_c n^\mu &= \nabla_c K_{ab} - \nabla_c K_{ac}. \end{aligned} \tag{2.50}$$

These are known as the **Gauss-Codazzi** equations. They show that some components of the curvature tensor of any geometry in $\mathcal{M}$ can be decomposed into terms of the *intrinsic* and *extrinsic* curvature pieces of the hypersurface it may be hosting.

Although the Riemann tensor, in any of its forms, contains valuable information about the geometry it represents, more practical tensorial objects will be found in everyday physic computations. Let us then find expressions for both the Ricci tensor and scalar given the metric decomposition described in Eq. (2.42). For the Ricci tensor we find:

$$\begin{aligned} R_{\alpha\beta} &= g^{\mu\nu} R_{\mu\alpha\nu\beta} \\ &= \left(\epsilon n^\mu n^v + h^{mn} e^\mu_m e^v_n\right) R_{\mu\alpha\nu\beta} \\ &= \epsilon R_{\mu\alpha\nu\beta} n^\mu n^v + h^{mn} R_{\mu\alpha\nu\beta} e^\mu_m e^v_n, \end{aligned} \tag{2.51}$$

while the Ricci scalar gives:

$$\begin{aligned} R &= g^{\alpha\beta} R_{\alpha\beta} \\ &= \left(\epsilon n^\alpha n^\beta + h^{ab} e^\alpha_a e^\beta_b\right) \left(\epsilon R_{\mu\alpha\nu\beta} n^\mu n^\nu + h^{mn} R_{\mu\alpha\nu\beta} e^\mu_m e^\nu_n\right) \\ &= 2\epsilon h^{ab} R_{\mu\alpha\nu\beta} n^\mu e^\alpha_a n^\nu e^\beta_b + h^{ab} h^{mn} R_{\mu\alpha\nu\beta} e^\mu_m e^\alpha_a e^\nu_n e^\beta_b. \end{aligned} \tag{2.52}$$

It can be useful to make good use of relations (2.47) and (2.49) to further simplify the Ricci scalar expression (2.52). Some minutes of patience and algebra yield:

$$R = {}^{(D-1)}R + \epsilon \left(K^2 - K^{ab}K_{ab}\right) + 2\epsilon \nabla_\alpha \left(n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta\right). \tag{2.53}$$

This expression is the evaluation of the D-dimensional Ricci scalar on the $D-1$-dimensional hypersurface Σ. This result is extremely practical in the context of branes and hypersurfaces, especially when the action governing their dynamics requires to be split, as discussed in chapter **??**.

Junction conditions

The previous pages have been devoted to studying how the embedding of a co-dimension one hypersurface Σ can be used to provide a splitting of the hosting manifold $\mathcal{M}$ into its tangential and normal components. However, the physical concepts of such picture have not yet received our attention. For example, one can find the following situation in physics: Suppose that such a hypersurface Σ divides a spacetime geometry into two regions $\mathcal{V}^+$ and $\mathcal{V}^-$. Both regions have different metrics $g^\pm_{\alpha\beta}$. Furthermore, they are both solutions to the Einstein field equations. What conditions should be put on the metrics to ensure that both spaces join smoothly at Σ, so that the whole union of spaces becomes a solution to the Einstein equation? This set of requirements demanded on the geometric features of the spaces $\mathcal{M}^{\pm}$ and Σ are called **Junction conditions**. They were originally discussed in papers like [**?**, **?**, **?**].

Let us first imagine two D-dimensional manifold $\mathcal{M}_{\pm}$, described by different sets of coordinates $\{x^{\alpha}_{\pm}\}$ and equipped with metrics $g^{\pm}_{\alpha\beta}$. Furthermore, both spaces share a boundary $\partial\mathcal{M}$, which is a hypersurface Σ of co-dimension one described by a set of coordinates $\{y^a\}$. On top of all this, we assume that this composition space satisfies the D-dimensional Einstein equation. We can then try

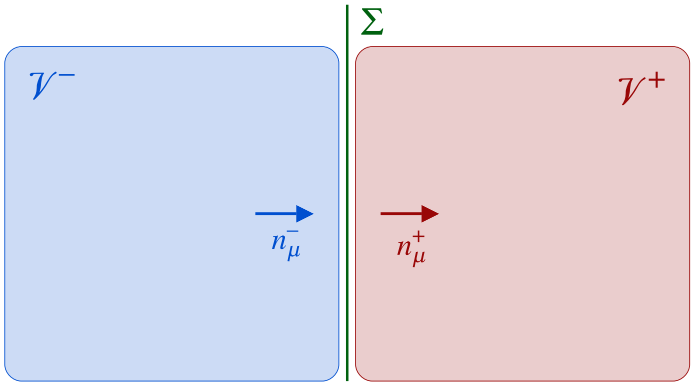

Figure 5: Pictorial representation of the composite space under study: Two different regions $\mathcal{V}^-$ and $\mathcal{V}^+$, with Σ as their common boundary. Note the choice of orientation of the normal vectors $n^{\pm}_{\mu}$. Different choices of normal orientation can be seen in figures 6 and 9. In this work, blue will be associated to the inside/inside construction of Randall-Sundrum model in section **??**, while red will represent the inside/outside construction of the dark bubble model in part **??**.

to define a general metric $g_{\alpha\beta}$ that interpolates between two D-dimensional spaces. This can be written as:

$$g_{\alpha\beta} = \Theta(\lambda) g^{+}_{\alpha\beta} + \Theta(-\lambda) g^{-}_{\alpha\beta}, \tag{2.54}$$

where $\Theta(\pm\lambda)$ is the Heaviside distribution function and λ is an affine parameter describing geodesics connecting both regions, piercing through the hypersurface Σ. In this sense, when $\lambda > 0$, one could say to be placed in $\mathcal{M}^+$. Similarly, the minus sign represents a point of the geodesic in $\mathcal{M}^-$ and its null value is on the hypersurface Σ. Note that more complicated objects like the affine connection Γ or the Riemann tensor depend on derivatives of the metric. One has to be careful then, as we are dealing with distribution functions and combinations of them can give rise to non-distribution terms, which makes it difficult to find a physical interpretation. In fact, one finds oneself in such a situation by simply the deriving expression (2.54) with respect to any coordinate x^{γ}. This yields:

$$\partial_{\gamma} g_{\alpha\beta} = \Theta(\lambda)\, \partial_{\gamma} g^{+}_{\alpha\beta} + \Theta(-\lambda)\, \partial_{\gamma} g^{-}_{\alpha\beta} + \epsilon\delta(\lambda)\, n_{\gamma} \left[g^{+}_{\alpha\beta} - g^{-}_{\alpha\beta} \right], \tag{2.55}$$

where the last term comes from the derivative $\partial_{\lambda}\Theta(\lambda) = \delta(\lambda)$.[21] Note that this term will yield contributions of the form $\Theta(\pm\lambda)\delta(\pm\lambda)$ when computing the Christoffel symbols. But such a combination of distributions is not one![22] It is therefore necessary to get rid of such a term. In order to achieve this task, let us impose continuity of the metric accross the hypersurface Σ,

$$g^{+}_{\alpha\beta}|_{\Sigma} = g^{-}_{\alpha\beta}|_{\Sigma}. \tag{2.56}$$

[21]The required change of variables in the derivative follows from the fact that any displacement away from the hypersurface along one of the geodesics described above is given by $\mathrm{d}x^{\alpha} = n^{\alpha}\mathrm{d}\lambda$.
[22]Note that $\Theta(0) =$ indeterminate, while $\delta(0) = 1$. What is this?

This constraint can be further refined: The completness relation (2.42) allows us to split between tangential and normal components of the continuity relation. Here we note that $[n^\alpha]^+_- = n^+ - n^- = 0$.[23] Furthermore, coordinates $\{y^a\}$ are the same on both sides of the hypersurface Σ. This implies that tangent vectors are uniquely defined on it. With these two facts, one can rewrite Eq. (2.56) as:

$$h^+_{ab} = h^-_{ab}. \tag{2.57}$$

This is called the **first junction condition**, which forces the *induced* metric h_{ab} to be the same on both sides of Σ. This is an essential requirement for a well-defined geometry.

The derivation of the second junction can be done in different ways, with the previous mathematical approach requiring us to elaborate further along the lines discussed above.[24] While this approach can be formal and elegant, we would be need to introduce new concepts and complicated computations. Inspired by [**?**], a more physical approach will be presented in this section of the subsection, in line with the essence of this thesis.

We will first try to understand the geometry of a portion of the composition space presented above; A single manifold $\mathcal{M}$ and its boundary $\partial\mathcal{M}$ equipped with the metrics $g_{\mu\nu}$ and h_{ab}, respectively. The action describing the geometry and content of this space can be given as:

$$S_{\text{Total}} = S_{\mathcal{M}} + S_{\partial\mathcal{M}} + S_{\mathcal{L}_\Sigma}, \tag{2.58}$$

where $S_{\mathcal{M}}$ is the usual Einstein-Hilbert action as:

$$S_{\mathcal{M}} = \int_{\mathcal{M}} \mathrm{d}^D x \sqrt{|g|} \left(\tfrac{1}{2\kappa_D} {}^{(D)}R + \mathcal{L}_{\mathcal{M}} \right), \tag{2.59}$$

where ${}^{(D)}R$ the D-dimensional Ricci scalar and $\mathcal{L}_{\mathcal{M}}$ is the Lagrangian density for any kind of matter content in such space.

The action term $S_{\partial\mathcal{M}}$ in Eq. (2.58) is the Gibbons-Hawking-York term [**?**, **?**] and is necessary for the propert definition of the variational principle since the Ricci scalar R is constructed from second derivatives of the metric. It describes how the submanifold is embedded as a boundary of $\mathcal{M}$. Hence, it can be expressed in terms of the extrinsic geometric pieces as:

$$S_{\partial\mathcal{M}} = \tfrac{\epsilon}{\kappa_D} \int_{\partial\mathcal{M}} \mathrm{d}^{D-1} y \sqrt{|h|} K, \tag{2.60}$$

with the induced metric h_{ab} and the trace of the extrinsic curvature K_{ab} as described in (2.41) and (2.47). Note the presence of the chosen normalisation ϵ of the normal vector n^α. Finally, the term $S_{\mathcal{L}_\Sigma}$ represents any type of matter content living **on** the boundary.

But this discussion is so far only valid for one manifold and its boundary. As described above, we will usually face situations where we find that the hypersurface Σ is the boundary of two different manifolds $\mathcal{M}^\pm$. Hence, we have to "duplicate" the previous action (2.58) and glue them together, along the boundary $\partial\mathcal{M}$ mediating between $\mathcal{M}^-$ and $\mathcal{M}^+$. This is assumed to be as the only boundary present in the construction.

Let us now derive the junction condition by applying the variational principle with respect to $g_{\mu\nu}$. In order to simplify this task and present the results in the tidiest way, we will carry out this computation piece by piece in the action. For each manifold $\mathcal{M}^\pm$ we find:

$$\begin{aligned} \delta S_{\mathcal{M}^\pm} = \tfrac{1}{2\kappa_D} \int_{\mathcal{M}_\pm} \mathrm{d}^D x \sqrt{|g|} \Big[& \left(G^\pm_{\mu\nu} - \kappa_D T^\pm_{\mu\nu} \right) \delta g^{\mu\nu} \\ & + \nabla_\mu \left(g_{\alpha\beta} \nabla^\mu \delta g^{\alpha\beta} - \nabla_\alpha \delta g^{\alpha\mu} \right) \Big], \end{aligned} \tag{2.61}$$

[23] This requirement follows from footnote 2 plus the continuity of λ and x^α accross the hypersurface.
[24] These lines can be found in [**?**].

where $T_{\mu\nu}$ is the energy-momentum tensor corresponding to any matter content **in** the D-dimensional spaces

$$T_{\mu\nu} = \mathcal{L}_{\mathcal{M}}\, g_{\mu\nu} - 2\frac{\delta \mathcal{L}_{\mathcal{M}}}{\delta g^{\mu\nu}}. \tag{2.62}$$

The boundary term yields a variation of the form:

$$\begin{aligned}\delta\partial\mathcal{M}^{\pm} = \tfrac{\epsilon}{2\kappa_D}\int_{\partial\mathcal{M}^{\pm}} \mathrm{d}^{D-1}y\sqrt{|h|}\Big[& \left(K^{\pm}_{\mu\nu} - K^{\pm}g_{\mu\nu}\right)\delta g^{\mu\nu} \\ & + n_\mu\left(\nabla_\alpha \delta g_{\alpha\mu} - g_{\alpha\beta}\nabla^\mu \delta g^{\alpha\beta}\right)\Big].\end{aligned} \tag{2.63}$$

Note that the second line of each expression cancel each other out by the Gauss-Stokes theorem [**?**].

With the expressions (2.61) and (2.63) at hand and the action piece $S_{\mathcal{L}_\Sigma}$ in Eq. (2.58) representing matter content **on** the wall, one can then compute the dynamics of the whole composite space. However, one must be careful, as the normal n^μ is chosen to point in the direction of increasing volume in the transverse directions, i.e. from $\mathcal{M}_-$ to $\mathcal{M}_+$. This implies a change of the sign[25] for $n^\mu_\pm$. This will affect the definition of the extrinsic curvature $K_{\mu\nu}$, as it contains the normal vector n^μ inside (see Eq. (2.47) and figure 5). Consequently, we have:

$$n^\mu = -n^\mu_+ = n^\mu_-. \tag{2.64}$$

The whole composition space (2.58) then reads:

$$\begin{aligned}\delta S_{\text{Total}} &= \delta S_{\mathcal{M}_+} + \delta S_{\mathcal{M}_-} - \delta S_{\partial\mathcal{M}_-} + \delta S_{\partial\mathcal{M}_+} + \delta S_{\mathcal{L}_\Sigma} = \\ &= \tfrac{1}{2\kappa_D}\int_{\mathcal{M}_+} \mathrm{d}^D x\sqrt{|g|}\left(G^+_{\mu\nu} - \kappa_D T^+_{\mu\nu}\right)\delta g^{\mu\nu} \\ &+ \tfrac{1}{2\kappa_D}\int_{\mathcal{M}_-} \mathrm{d}^D x\sqrt{|g|}\left(G^-_{\mu\nu} - \kappa_D T^-_{\mu\nu}\right)\delta g^{\mu\nu} \\ &- \tfrac{\epsilon}{2\kappa_D}\int_{\Sigma} \mathrm{d}^{D-1}y\sqrt{|h|}\left(K^+ g_{\mu\nu} - K^+_{\mu\nu}\right)\delta g^{\mu\nu} \\ &+ \tfrac{\epsilon}{2\kappa_D}\int_{\Sigma} \mathrm{d}^{D-1}y\sqrt{|h|}\left(K^- g_{\mu\nu} - K^-_{\mu\nu}\right)\delta g^{\mu\nu} \\ &- \tfrac{1}{2\kappa_D}\int_{\Sigma} \mathrm{d}^{D-1}y\sqrt{|h|}\,\kappa_D\, S_{\mu\nu}\,\delta g^{\mu\nu},\end{aligned} \tag{2.65}$$

where $S_{\mu\nu}$ represents the energy-momentum tensor (2.62) for all matter content living **on** the hypersurface Σ. We can conveniently identify both Einstein equations for each manifold $\mathcal{M}_\pm$ in the first two lines of expression (2.65). Both spaces are independent solutions of the Einstein equation, so they will not contribute to the equation of motion. The last three lines of previous expression correspond to the boundary geometric contributions $\partial M_\pm$ (where the orientation of the normal has already been taken into account) and any possible matter fields living on such hypersurface. For the whole expression (2.65) to be zero, it is required that these last three terms cancel, so

$$\kappa_D\, S_{\mu\nu} = \epsilon\left[\left(K^- g_{\mu\nu} - K^-_{\mu\nu}\right) - \left(K^+ g_{\mu\nu} - K^+_{\mu\nu}\right)\right] \tag{2.66}$$

A small massage and projection down to tangential components, as these are tangential tensors (2.43), we finally find:

$$\kappa_D\, S_{ab} = \epsilon\left([K_{ab}]^+_- - h_{ab}\,[K]^+_-\right), \tag{2.67}$$

[25] but not in the value of the norm ϵ.

with $[A]^+_- = A^+ - A^-$. This is the (second) **junction condition**. Equation (2.67) states that the presence of a localised energy-momentum tensor S_{ab} **on** the hypersurface will source a jump discontinuity in the extrinsic curvature. Alternatively, one can read it backwards: a $D-1$-dimensional hypersurface Σ acting as the boundary of two different D-dimensional manifolds $\mathcal{M}_{\pm}$, which are independent solutions to the Einstein equation, must be equipped with an energy-momentum tensor S_{ab}, proportional to the jump in the extrinsic curvature of the embedding, such that the whole composite space is also solution to the Einstein equation.

2.4 Randall-Sundrum braneworlds

I THINK THIS REQUIRES SOME REWRITING AND WAY MORE EMPHASIS ON ITS FEATURES. TO LATER COMPARE WITH DB.

One of the best known mechanisms for solving the issue of extra dimensions is to consider our universe as a three-dimensional hypersurface (brane) embedded in a higher-dimensional space. Our universe could then move freely along those extra dimensions, but any content of it would be confined to its volume, with no possibility of accessing the higher-dimensional picture.

First proposals of this type of construction date back to [?, ?]. In this toy model, where gravitational effects were not taken into account, matter was confined to a domain wall, the brane, which could move freely along an extra infinite dimension. Higher-dimensional equations of motion admit solutions close enough to the wall. However, gravity was missing, which could be a big problem when one wants to find a consistent theory of gravity which the graviton can propagate freely throught the full spacetime.

The first "braneworld" model which aimed to include gravity was discussed in [?]. This proposal consists of a three-dimensional membrane, in which all matter is confined and a set of extra dimensions which are required to be not infinitely large, but flat, closed (i.e. $y \sim y + 1$) and small.[26] In this sense, matter is still localised on the brane, while now propagating throughout all spacetime. Although this model includes gravity, it does not take into account a subtle but important detail; As we saw in section B, any content in the universe will cause a backreaction in its geometry. In the same way, the presence of the brane carrying the matter in this model should cause changes in the higher-dimensional spacetime that contains it. This is known as *warping* and it describes how the brane deforms as it moves through the extra dimensions.

The first model to take into account both the presence of gravity and any possible deformations of the membrane which is embedded in the higher-dimensional space was that of Randall and Sundrum [?, ?]. This string theory inspired braneworld proposal, not only took into account those subtleties forgotten in the previously discussed models, but also proposed a solution to the gauge hierarchy problem in particle physics (i.e. why the gap between the electroweak scale and that of Planck is so big). This model requires two different branes, with positive and negative tensions respectively, both embedded in a five-dimensional Anti-de Sitter space. Thanks to the backreaction, which translates into curvature, i.e. *warping* in the extra dimensions, any mass trapped on the negative tension brane is much smaller than the Planck scale. This is known as RS-I model. In a later paper [?], a second model was proposed, called RS-II.[27]

Firstly, this second RS model does not require the presence of the negative tension brane. This model involves a single positive tension brane. This brane, as discussed above, will warp the

[26] If extra dimensions were large in this configuration, the force of gravity could leak into them. This would cause modifications in Newton's law and lead to contradiction with experiments.
[27] Again, physicists lack the imagination to nominate.

geometry and it only involves an "infinitely" large extra dimension.[28] Although this may seem counterproductive, generating possible contradictions in Newtonian gravity at low energies on the brane, it is not. Randall and Sundrum managed to avoid this problem by forcing gravity to be localised **on** the brane. Let us review some of the key concepts of this construction.

Following [?], let us work in a five-dimensional effective theory of Einstein gravity with matter as

$$S = \frac{1}{2\kappa_5}\int \mathrm{d}^5x\sqrt{|g_{(5)}|}\left(R_{(5)} - 2\Lambda_5 + \mathcal{L}_{\mathrm{m}}\right) + \frac{1}{\kappa_5}\int \mathrm{d}^4x\sqrt{|h|}K, \tag{2.68}$$

where κ_5 is the Newton's constant (**??**) in five dimensions and the five-dimensional cosmological constant $\Lambda_5 < 0$. The matter Lagrangian is taken to be $\mathcal{L}_m$. As discussed in subsection **??**, the extrinsic curvature K contains information about how the matter, if localised on a hypersurface Σ, is embedded in the higher-dimensional spacetime. If we choose $\mathcal{L}_m = 0$, it can be shown that the metric Ansatz that solves the equation of motion is that of AdS$_5$ space[29]

$$\mathrm{d}s^2_{\mathrm{AdS}_5} = e^{2A(r)}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}r^2, \tag{2.69}$$

where the aforementioned warp is reflected in the factor $A(r) = \pm kr$ and $k = \sqrt{-\Lambda_5/6}$ is the Anti-de Sitter curvature. The AdS$_5$ radial coordinate, i.e. the *holographic* coordinate $r \in (-\infty, \infty)$ increases monotonically from the centre to the boundary of the Anti-de Sitter space.

Let us now imagine a (mem)brane with tension σ placed at $r = 0$.[30] This shell of matter will divide the five-dimensional space into two regions and it will hence generate a jump in the extrinsic curvature $K_{\mu\nu}$ on both sides of the brane (see subsection **??** for further information). This can be denoted in the warp factor as:

$$A(r) = \epsilon_+ k_+\,\Theta(r)\,r + \epsilon_- k_-\,\Theta(-r)\,r, \tag{2.70}$$

where $\epsilon_\pm$ represents the normalisation and orientation choice of the normal to the hypersurface Σ, $\Theta(r)$ is the Heaviside function and $k_\pm$ represent the curvatures of each side of both vacua. One

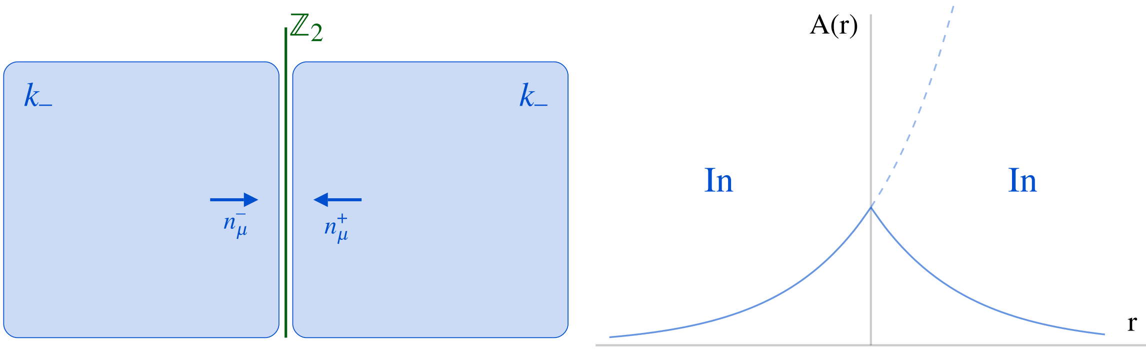

Figure 6: Left: A pictorial representation of the configuration under discussion (inside/inside). The presence of $\mathbb{Z}_2$ and the choice of normal orientations will have important consequences for this construction. Right: The presence of the "mirror" will cause the warp factor $A(r)$ to peak on the brane and to fade to 0 when $r \to \pm\infty$.

of the most important key features of the Randall-Sundrum model is that the brane will act as a

[28] In fact, this is an artefact of the choice of Poincaré coordinates. The presence of the $\mathbb{Z}_2$ and the warp factor indicate a finite volume, as we will see later.

[29] Some of these steps will be discussed in greater detail in chapter **??**.

[30] This object is not a fundamental one coming from string theory, hence the change of notation to σ.

”mirror”, i.e. a $\mathbb{Z}_2$ symmetry is imposed at $r = 0$. This forces us to identify both five-dimensional vacua across the brane;[31] In addition, the Randall-Sundrum model makes a specific choice of normal orientations, which, together with the aforementioned $\mathbb{Z}_2$ translate into:

$$\epsilon_- = -\epsilon_+ = 1 \qquad \text{and} \qquad k = k_+ = k_-. \tag{2.71}$$

This identification is often referred to as *inside-inside* construction. This results in a matter Lagrangian defined as:[32]

$$\mathcal{L}_\mathrm{m} = \frac{6}{\kappa_5}\, k\, \delta(r) = \sigma\delta(r). \tag{2.73}$$

Imposing this matter configuration in the junction condition (2.67), while restricting to the case where $\epsilon_- = -\epsilon_+ = 1$ and $k = k_+ = k_-$ and leaving the warp factor $A(r)$ undetermined,[33] one can then induce an effective four-dimensional cosmological constant **on** the brane which reads:

$$\Lambda_4 = \frac{\kappa_5^2}{12}\sigma^2 - 3k^2. \tag{2.74}$$

The first thing to note is the absence of a cosmological constant when $\sigma = \frac{6}{\kappa_5}k$. This value corresponds *exactly* to that of the matter content defined in Eq. (2.73). This implies that the presence of a positive cosmological constant on the brane requires a brane with tension **greater** than the one discussed above. This is called a *supercritical* brane.

Let us now look at the presence of gravity in the model. Another crucial aspects of Randal-Sundrum models is that gravity still appears as four-dimensional to observers confined to the hypersurface. This can be studied by using first order transverse and traceless gravitational perturbations (B.1) in five dimensions. Following [**?**, **?**] and in a similar way as we did in section B, one can find a differential equation of the form:

$$-\frac{\mathrm{d}^2\zeta}{\mathrm{d}\chi^2} + \underbrace{\left[\frac{3}{4}\left(\frac{A'}{A}\right)^2 + \frac{3}{2}\frac{A''}{A}\right]}_{V(\chi)}\zeta = 0, \tag{2.75}$$

where ζ is the radial piece of the wave and a prime denotes a derivative respect to χ, which is a convenient conformal coordinate of the form:

$$\chi = \int_0^r \frac{\mathrm{d}r'}{A(r')}. \tag{2.76}$$

The second derivative A'' is extremely important for identifying gravity on the braneworld. As the warp factor $A(\chi)$ involves an absolute value, its second derivative will be a delta function $\delta(r)$. Hence, the potential $V(\chi)$ will peak at the location of the brane. Furthermore, when $\Lambda_4 \geq 0$, one can always find a normalisable solution for ζ of the form:

$$\zeta(\chi) \propto A^{3/2}(\chi). \tag{2.77}$$

This implies that the wave peaks around the location of the brane and fades away when $\chi \to \pm\infty$. This solution can be interpreted as the *localisation* of gravity and it means that gravity uniquely

[31] Hence the choice of k_- on either sides of the brane in figure 6.

[32] This can be obtained from the formal Ansatz

$$\mathcal{L}_\mathrm{m} = \frac{3}{\kappa_5}\left(\epsilon_+ k_+ - \epsilon_- k_-\right)\delta(r). \tag{2.72}$$

What happens when we remove the $\mathbb{Z}_2$ symmetry and choose the normal vectors to point in the direction of **always** increasing transverse volume is left to be discussed in chapter **??**.

[33] There will be a backreaction of the brane in the embedding space. The new warping factor is not relevant to the current discussion.

lives **on** the brane [?, ?]. In order to find how strong this four-dimensional gravity is, we can simply perform a dimensional reduction along the infinitely long fifth dimension r, to find that [?]:[34]

$$\kappa_5 = \kappa_4 \int_{-\infty}^{\infty} \mathrm{d}r e^{2[\epsilon_+ k_+ \,\theta(r)\, r + \epsilon_- k_- \,\theta(-r)\, r]} = \frac{\kappa_4}{k}, \tag{2.78}$$

where we have imposed the choices (2.71). Although the extra dimension r is not compact, it will still give a finite value for the dimensional reduction of the Newton's constant, as the volume along this direction is finite, due to the warping and the presence of the $\mathbb{Z}_2$ symmetry on the bubble wall.

Although the Randall-Sundrum construction uses elements that can easily be provided from string theory, it is not yet clear whether a phenomenological model can be obtained at all. The main point is that the type of brane required in order to induce a positive cosmological constant Λ_4, i.e. supercritical branes, are not those that can be found in string theory [?]. Moreover, the proposal discussed above seems to be in contradiction with the Maldacena-Nuñez no-go theorem [?].[35] Similarly, it was found in [?, ?] that the spacetime transverse to a domain wall interpolating between two different vacuum values always approaches the boundary of AdS. This implies that Randall-Sundrum four-dimensional de Sitter braneworld constructions are forbidden in string theory.

In any case, several iterations and variations of this model with interesting cosmological consequences have recently been studied in [?, ?]. The curious reader is welcome to get a better grasp of the proposals by reading these papers. In addition, we will extensively revisit braneworld constructions in stage **??** of this work. For now, we will take a quick look to the most famous mechanism for hiding extra dimensions: Compactifications.

3 Inducing Dark Energy on a Bubble

I WOULD SAY THIS IS FINE. REMOVE PERHAPS LAST SECTION.

In this section you will learn...

- what the dark bubble model is and its stringy/swampy foundations
- how to induce four-dimensional positive dark energy on an expanding D_3-brane mediating a vacua decay

We have modestly explored the state of art of (string) cosmology in previous chapters. Chapters **??** and **??** were devoted to explaining our understanding of the universe and its dynamics from four-dimensional classical and quantum perspectives, respectively. These dynamics are strongly influenced by the presence of an unknown type of energy called *dark energy*.

We also discussed the physics community's hope of gaining an understanding of dark energy by deriving four-dimensional cosmologies from the higher-dimensional theories with ultraviolet completion, such as string theory. Chapter **??** narrates how, infused with determination, they departured to the plains of the **Braneworlds** and mountains and valleys of the **Landscape**, promised lands of effective low-dimensional vacua derived from string theory.

Focusing only on the Landscape, this plethora of vacua had the potential to host solutions equipped with a small, yet positive four dimensional cosmological constant. But some sort of

[34]From action (2.68), perform a dimensional reduction along r and identify the Planck mass in five and four dimensions.

[35]The Maldacena-Nuñez no-go theorem states that de Sitter solutions are not allowed in regular warped compactifications at tree level, i.e. de Sitter solutions would require from perturbative and/or non-perturbative corrections and/or the presence of O_3 planes, non-dynamical objects similar to branes, with mirror properties and negative tension.

stringy complot seemed to make such achievement almost impossible. This fact motivated the community to explore the **Swampland** surrounding the Stringy Kingdom. Perhaps here, we could gain some understanding of why their expectations regarding the behaviour of UV-complete theories at low energies may have been wrong. Some of the highlights of this yet unfinished journey were elaborated in chapter **??**.

At this point, one fact has to be addressed: Almost all our attention has been devoted to the mechanism of compactifications throughout stage **??** in this thesis. However, we have also presented an alternative to string compactifications, **Braneworld** scenarios, in section **??**. Can a tweaked version of this mechanism be used to realise four-dimensional de Sitter cosmologies from string theory and at the same time avoid some of the swampland conjectures discussed in chapter **??**? Yes, and we would like to raise the stakes: There is a braneworld scenario which can realise four-dimensional accelerated expanding cosmos making use of some of the aforementioned conjectures in chapter **??**. This is model is known as the **Dark Bubble** model. The inspirational bedrock of this proposal has already been presented in sections 2.1 and 2.1. In summary, the clues are:

- Unlike to de Sitter vacuum solutions, Anti-de Sitter ones are more common in string theory. For example, $\mathrm{AdS}_5 \times S^5$ is one of the most studied and well–understood backgrounds for any flux value [**?**]. However, AdS vacua generally do not exhibit scale separation [**?**].
- If the AdS vacuum is non-supersymmetric and supported by fluxes, the conjecture described in section 2.1 states that such AdS_5 space is unstable and will decay non-perturbately, via brane nucleation [**?**, **?**]. This brane is a spherical co-dimension one hypersurface that mediates the decay between the initial configuration of the vacuum and that enclosed by the brane, with one unit of flux less.
- The nucleated brane, which can be described as a Brown-Teitelboim instanton [**?**], will obey the rules of the weak gravity conjecture 2.1, i.e. its electric (gauge) repulsion will be greater than its gravitational attraction. This is known as a *subcritical* tension brane. This will be the fundamental feature driving its expansion within the vacua.

The Dark Bubble model makes use of the previous facts, starting from a non-supersymmetric five-dimensional Anti-de Sitter time independent flux compactification in string theory.[36] This vacuum solution is supported, by at least, one $\mathfrak{F}_5$ flux. The presence of this field strength will be responsible of fixing the compact dimension moduli potential to a minimum value Λ_5. According to the non-supersymmetric AdS conjecture, this configuration is unstable and will eventually decay. The decay channel will be the nucleation of a D_3-brane, mediating between the new vacuum with "deeper" minimum Λ_5 (which we will call *True* vacuum) and the old vacuum (*False* vacuum) that has not yet undergone the decay. The brane, which is a non-saturated object ($Q > \sigma$) will experience a gauge repulsion greater than its gravitational attraction.[37] This repulsion will force it to expand, "eating" up the old vacuum and "replacing" it with the one enclosed in its everg-rowing inside.

The setup discussed above grants us the playground to develop the Dark Bubble proposal. This model can be summarised as:

The Dark Bubble model proposes that a four-dimensional accelerated expanding cosmology can be realised on the co-dimensional one boundary of a five-dimensional true vacuum bubble, mediating the decay of a false AdS one.

In the remainder of this work, we will present and elaborate on this novel proposal suggested by Banerjee, Danielsson et al. in [**?**] and developed in subsequent papers [**?**, **?**, **?**]. This first chapter of

[36] Chapter **??** describes an explicit embedding of this model in supergravity.

[37] We will denote the tension with the letter σ. We will explain its relation to the tension of a fundamental brane in chapter **??**.

stage ?? will introduce how to induce a small and positive dark energy density ρ_Λ on the boundary of the aforementioned expanding bubble. The procedure will be extremely easy: First, we will dispose and prepare the ingredients to use; The clues commented on above are part of the constituents, but the star of the course will be the AdS$_5$ geometry hosting the bubble. This delicate procedure will be discuss in section 3.1. With our first ingredients ready, section 3.2 will guide us through the creation of the desired four-dimensional de Sitter cosmology riding the bubble of true vacua. Section 3.3 will tease out the details on the Dark Bubble recipe to be discussed, leading the hungry minds towards the different culinary aspects of such creation.

3.1 Chopping the geometry

Let us start with the usual Einstein-Hilbert action for a D-dimensional geometry,

$$S[g_{MN}] = \frac{1}{2\kappa_D} \int \mathrm{d}^D X \sqrt{|g_{(D)}|}\,(R - 2\Lambda_D)\,. \tag{3.1}$$

Note that we are not (yet) accounting for any matter content in this geometry, hence the absence of Lagrangian representing it.

Anti-de Sitter spaces are maximally symmetric Lorentzian spaces characterised by a negative Ricci scalar. For convenience, we will express the D-dimensional cosmological constant as:[38]

$$\Lambda_D = -\tfrac{1}{2}(D-1)(D-2)L_{\mathrm{AdS}}^{3-D}\,. \tag{3.2}$$

The L_{AdS} parameter has dimensions of length and sets the overall scale of the space-time geometry. From now on, we will drop the subscript AdS in order not to oversaturate our notation. A general Ansatz for the metric of such a space can be given as:[39]

$$\mathrm{d}s_D^2 = -e^{2A(r)}\mathrm{d}t^2 + e^{-2A(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{D-2}^2, \tag{3.3}$$

where $\mathrm{d}\Omega_{D-2}^2$ represents the line-invariant of a $(D-2)$-sphere. As usual, varying Eq. (3.1) with respect to g_{MN} will provide the already well-known Einstein equation (??). Introducing the previous Ansatz (3.3) will result in a set of differential equations with solution

$$e^{2A(r)} = 1 + \frac{r^2}{L^2}. \tag{3.4}$$

This solution represents the **global** set of coordinates for a D-dimensional anti-de Sitter space. Note that the whole space is covered, as the radial coordinate r varies in the range of $0 \leq r < \infty$. The angular variables cover the $(D-2)$-sphere[40] and time ranges between $-\infty < t < \infty$.

Although the previous set of global coordinates covers the whole anti-de Sitter space, one can think of approximations for specific regimes. A very useful regime is that of large radius r. This is when $(r/L)^2 \gg 1$. In this sense, one can think of really large spatial sections of S_{D-2} along the radial direction r, such that the curvature of the sphere is negligible. The "unhygenic" approximation $\{\alpha, \beta, \gamma, \cdots\} \simeq \frac{1}{L}\{x, y, z, \cdots\}$ would allow us to rewrite the line invariant as:[41]

$$\mathrm{d}s_D^2 = k^2 r^2\, \eta_{MN}\mathrm{d}X^M \mathrm{d}X^N + \frac{1}{k^2 r^2}\mathrm{d}r^2, \tag{3.5}$$

[38] This is a generalisation of Eq. (??) for D dimensions and a negatively curved space-time. Note that one can easily see the dimensionality of the cosmological constant in this way.

[39] Actually, a more general geometry would be with $A(r)$ and $B(r)$ as the argument of each exponent, but these can be fixed to $A(r) = -B(r)$ by imposing maximal symmetry in the space-time geometry.

[40] From a five-dimensional point of view, this would be the S^3, i.e. our usual spatial directions with a closed geometry.

[41] Such dirty tricks should not be used when cooking. Dear reader, please keep some cleanliness when preparing food!

where η_{MN} is the usual Minkowski metric for representing patches of a really large S_{D-2}, so that they look flat. Note the presence of the new variable k we have written; This is the AdS$_D$ vacuum scale, i.e. the inverse of the AdS-length $k = L^{-1}$. This new coordinate system is named after Henri **Poincaré**. Poincaré coordinates are simpler than global ones but they do not cover all AdS$_D$ geometry.

We are just one step away from having our geometry ready; the "gluing" process. Recall the presence of the nucleated brane mediating the decay of the old vacuum into the new one. This D_p-brane, which is an $O(D-1)$ Coleman-de Luccia instanton [**?**], will expand radially. The position of this hypersurface with co-dimension one along the throat direction can be then parametrised by $r = a(t)$, generating the two aforementioned regions: the *inside*, located at $r \in (0, a(t))$ and the *outside*, located at $r \in (a(t), \infty)$.

It may be convenient to introduce new notation at this point: From now on, we will refer to the vacuum living inside the hyper-volume encoded by the D_p-brane, i.e. the *inside* vacuum, with a minus (-) subscript sign. The whole higher-dimensional space receives the name of *Bulk* and the tandem brane-inside will be colloquially called *Bubble*. On the other hand, the *outside* of the bubble that has not yet decayed will be denoted by a positive (+) subscript sign.

The notation presented above allows us to write an overall bulk geometry taking into account both regions ($\pm$). The complete metric is given by:

$$\mathrm{d}s^2 = g_{MN}\,\mathrm{d}X^M \mathrm{d}X^N = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2_{D-2}, \tag{3.6}$$

where

$$f(r) = 1 + k_-^2 r^2 + \Theta\left(r - a(t)\right)\left(k_+^2 - k_-^2\right) r^2. \tag{3.7}$$

Here $\Theta(x)$ is the Heaviside **step** function. It is important to emphasise on the jump accross the bubble wall. The change in the AdS$_D$ vacuum scale $k_\pm$ will cause a jump in the extrinsic curvature K_{ab}.[42] We will see in the next section that this jump is responsible for the geometry and contents that will be induced on the hypersurface, helping us identify an expanding cosmology which decorates the $D-1$-boundary of the bubble.

3.2 Recipe to induce dark energy on a bubble surface

The aim of this thesis is to discuss cosmological and phenomenological aspects in four dimensions, and in view of the discussion in appendix **??** about co-dimension one hypersurfaces, we will fix $D = 5$. This implies that the boundary ∂B of the nucleated five-dimensional bubble will have $d = D - 1 =$ four dimensions.

From now on, **Greek** indices will represent the five-dimensional bulk coordinates, while **Latin** ones will be used to describe quantities associated with the *induced* geometry on the four-dimensional boundary.[43] In this dimensionality, the line invariant (3.6) can be cast in the ($\pm$)-form:

$$\mathrm{d}s^2_\pm = g^\pm_{\mu\nu} dx^\mu dx^\nu = -f_\pm(r)dt^2_\pm + f^{-1}_\pm(r)dr^2 + r^2 d\Omega^2_3, \tag{3.8}$$

where

$$f_\pm(r) = e^{2A_\pm(r)} = 1 + k^2_\pm r^2 + \chi(r, t, k_\pm, q_1, ..., q_m). \tag{3.9}$$

The extra piece $\chi(r, t, k_\pm, q_1, ..., q_m)$ in the function $f_\pm(r)$ has been added to encode additional properties of the bulk, i.e. a mass M, any charge q_i or any other parametric dependence. However, these will only be relevant in the following chapters. For now on, let us fix $\chi(r, t, k_\pm, q_1, ..., q_m) = 0$.

[42] This will contain explicit information on how the brane is embedded from both points of view; inside and outside. See appendix **??** for further details.

[43] Furthermore, first letters of the latin alphabet $\{a, b, c\}$ will be used to describe all four dimensions, while $\{i, j, k\}$ will represent just the spatial sections. See appendices **??** and **??** for further information.

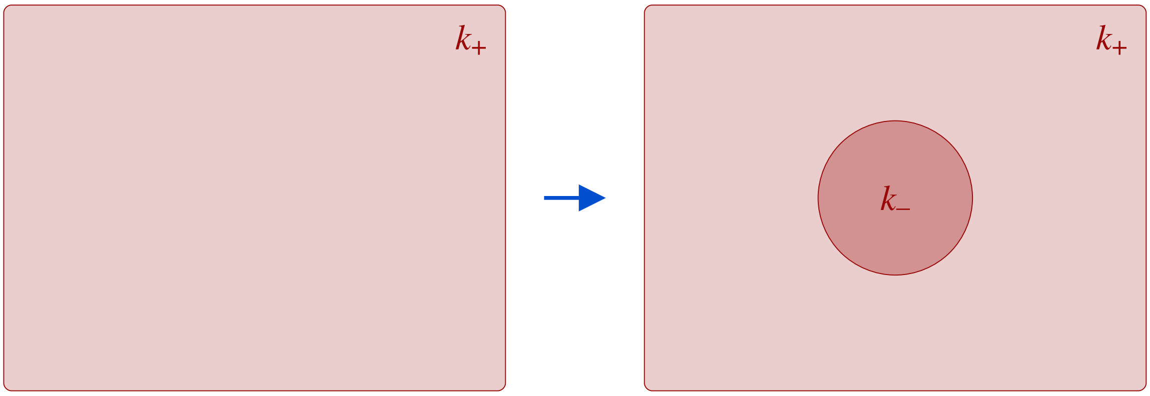

Figure 7: Pictorical representation of the distribution of AdS-vacua before and after the nucleation of the bubble mediating between them. Note that a strong saturation of the inside of the bubble corresponds to a lower (more negative) five-dimensional cosmological constant Λ_5^-, contrary to the outside, which preserves the initial Λ_5^+ value.

Before we start computing the Israel's junction conditions described in appendix **??**, it is important to realise the parametrical dependence of the expanding bubble. As described in appendix **??**, the nucleated bubble is a Coleman-de Luccia instanton [**?**], which has an $O(4)$-symmetry. The two relevant coordinates for this derivation are the global time t and the radial coordinate r, which also depends on t, as the bubble expands, hence evolves in time. Both will be related to the proper time τ experienced by any observer living on the boundary ∂B. The choice of coordinates to describe the bulk geometry and the induced one are:

$$x^\mu = \{t(\tau), r(\tau), \alpha, \beta, \gamma\}, \qquad\qquad y^a = \{\tau, \alpha, \beta, \gamma\}. \tag{3.10}$$

Let us now calculate the first junction condition described in Eq. (2.41). This is no more than:

$$\mathrm{d}s^2_{\text{ind}} = h_{ab}\mathrm{d}y^a\mathrm{d}y^b = -N(\tau)^2 d\tau^2 + a^2(\tau)\mathrm{d}\Omega_3^2, \tag{3.11}$$

where

$$N(\tau)^2 = f(a(\tau))\dot{t}^2 - \frac{\dot{a}(\tau)^2}{f(a(\tau))}. \tag{3.12}$$

The metric h_{ab} represents the four-dimensional induced geometry on the hypersurface, i.e. the D_3-brane, given by the parametrisation of $\{y^a\}$, the set of coordinates on the brane. The lapse function $N(\tau)$ has been introduced to manifest the invariance of the reparametrisation of time. One can always choose the right parametrical relation between $f(a(\tau)), \dot{t}(\tau)$ and $a(\tau)$ such that $N(\tau) = 1$. In that case, a dot will represent a proper-time derivative (i.e. the time an observer living on the hypersurface will experience). Thus, Eq. (3.11) will describe an expanding four-dimensional cosmology, characterised by a Friedmann-Robertson-Lemaître-Walker metric with closed spatial sections.

Let us now compute the second Israel junction condition. As discussed above, the presence of the expanding D_3-brane divides the bulk space into an *inside* and an *outside*. This forces the presence of a non-zero induced energy-stress tensor on the brane, so that the whole configuration remains a solution to the bulk Einstein equation. This is what the second Israel junction condition (2.67) accounts for. To simplify this example, let us assume we have an empty brane. This implies that its energy-momenum tensor S_{ab} only contains information about its tension σ. [44] This can be

[44]Recall that the tension of a D_3-brane is given by $T_{D_3} \propto \frac{1}{\ell_s^4}$, with ℓ_s the string length. So its dimensionality is $[\text{Length}]^{-4}$.

expressed as:

$$S_{ab} = -\sigma h_{ab}. \tag{3.13}$$

Israel's second condition refers to the discontinuity in the extrinsic curvature between the inside and the outside geometries separated by the brane. To be as pedagogical as possible in this introductory chapter, we will also compute the most relevant steps to obtain the induced energy-momentum tensor.[45]

First, we need to identify the normal and tangent vectors $\{n^\mu, e^\mu_a\}$ in relation to the embedding. These vectors can be found using Eqs. (2.38) and (2.39):

$$n^\mu = \left(\frac{-\dot{a}/\dot{t}}{f(a)\sqrt{f(a) - \frac{1}{f(a)}\left(\frac{\dot{a}}{\dot{t}}\right)^2}}, \frac{f(a)}{\sqrt{f(a) - \frac{1}{f(a)}\left(\frac{\dot{a}}{\dot{t}}\right)^2}}, 0, 0, 0 \right) \tag{3.14}$$

and

$$e^\mu_a = \begin{pmatrix} \dot{t}(\tau) & 0 & 0 & 0 & 0 \\ \dot{f}(a) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.15}$$

Note that the parametric dependence on the proper time τ is not explicitly stated, but is still implicit in $a(\tau)$ and $t(\tau)$. Following the guidelines of appendix **??**, the next computation is that of the extrinsic curvature K_{ab} (2.47). It is explicitly stated by

$$\begin{aligned} K_{\tau\tau} &= \frac{N(\tau)^2}{\dot{a}^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{N(\tau)}\sqrt{f(a)N(\tau)^2 + \dot{a}^2}\right], \\ K_{ij} &= -\frac{a}{N(\tau)}\gamma_{ij}\sqrt{f(a)N(\tau)^2 + \dot{a}^2}, \end{aligned} \tag{3.16}$$

with γ_{ij} as the metric on the sphere S^3. It is then straightforward to compute the second junction condition (2.67). Assuming a pure constant tension brane (3.13), the following expression can be derived:

$$\sigma = \frac{3}{\kappa_5}\left(\sqrt{\frac{f_-(a)}{a^2} + \frac{\dot{a}^2}{N(\tau)^2 a^2}} - \sqrt{\frac{f_+(a)}{a^2} + \frac{\dot{a}^2}{N(\tau)^2 a^2}}\right), \tag{3.17}$$

where $\kappa_5 = 8\pi G_5$ and σ corresponds to the tension of the bubble wall. Note how the tension of the shell is governed by the four-dimensional Hubble parameter $H = \dot{a}/\left(N(\tau)a\right)$ and the difference in the AdS_5 scales of the five-dimensional bulk.

A little bit of algebra and patience reveals a Friedmann-like equation in the form:

$$H^2 = -\frac{1}{a^2} + \underbrace{\frac{1}{3}\left(\frac{\kappa_5^2\sigma^2}{12} - \frac{3}{2}\left(k_+^2 + k_-^2\right) + \frac{27}{4}\left(\frac{k_-^2 - k_+^2}{\kappa_5\sigma}\right)^2\right)}_{\Lambda_4}, \tag{3.18}$$

where we have identified the second term with a four-dimensional cosmological constant Λ_4. Dimensional analysis can easily demonstrate that: $[k_\pm] = [\text{Length}]^{-1}$, while $\kappa_5 = 8\pi G_5 = \ell_5^3$, where ℓ_5 is the five-dimensional Planck length. So $[\kappa_5] = [\text{Length}]^3$. From Eq. (**??**), we also know that $[\sigma] = [\text{Length}]^{-4}$. It is then easy to conclude that the dimensionality of that piece is that of a four-dimensional cosmological constant. What values can it take?

[45] The whole derivation is left as an exercise for the reader.

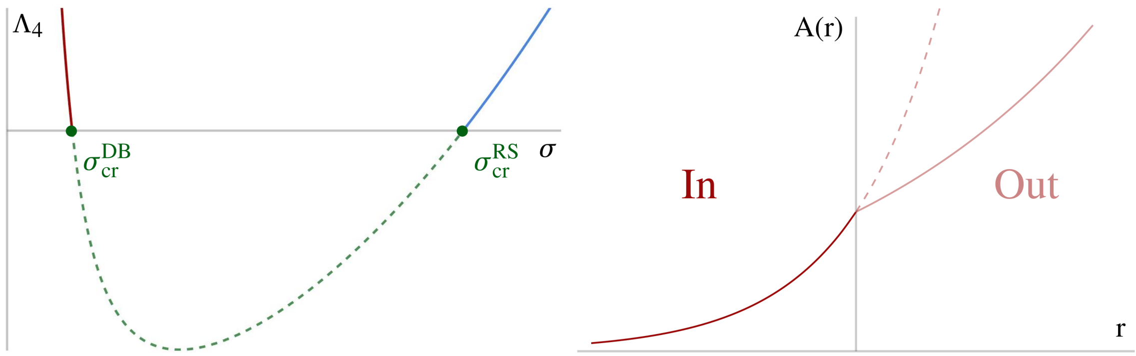

Figure 8: Left: Parametric dependence of Λ_4 on the tension σ. Since only subcritical branes can nucleate, only the solid lines are valid. The red solid line are the values for the Dark bubble construction, while the blue solid line are the supercritical values required by the Randall-Sundrum model discussed in section **??**. Right: A change of coordinates $r \to \frac{1}{k}\log(kr)$ in the Poincaré patch (3.5) reveals the change in the transverse direction warping, which will be reflected in the extrinsic curvature K_{ab}. The brane is lcoated at $r = 0$, and the spatial sections grow monotonically throughout the AdS throat.

Note that the non-linear expression for Λ_4 in Eq. (3.18) has two zeros[46] at:

$$\sigma_{\rm cr} = \frac{3}{\kappa_5}\left(k_- \pm k_+\right). \tag{3.19}$$

These correspond to the value of the *critical* tension.[47] Based only in observational evidence, our objective is to find solutions with a small and positive Λ_4, then the lower half of the plot can be omitted. In addition, recall that any nucleated brane according to the *non-susy AdS* conjecture is a non-saturated object, so its tension must be subcritical. This restricts the set of valid solutions with $\Lambda_4 > 0$ to be in the regime:

$$\sigma < \sigma_{\rm cr} = \frac{3}{\kappa_5}\left(k_- - k_+\right). \tag{3.20}$$

The Friedmann Eq. (3.18) can be further simplified, such that the cosmological constant Λ_4 becomes a linear expression of $k_\pm$ and σ. In fact, our main phenomenological target in this chapter is to provide late-time cosmological descriptions, where a **small** dark energy density ρ_Λ dominates the dynamics of the induced cosmos. This phenomenological regime can be reproduced when:

$$k_\pm\, a \gg 1, \qquad\qquad \sigma = (1-\varepsilon)\,\sigma_{\rm cr}. \tag{3.21}$$

The first requirement implies that the *scaled* scale factor a is large enough, so that observers living on the hypersurface do not infer any spatial curvature. The second condition adjusts the brane to a tension which is nearly subcritical. Expanding Eq. (3.18) for small ε, we can find:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = -\frac{1}{a^2} + \frac{8\pi}{3}\,\frac{2\,k_- k_+}{k_+ - k_-} G_5\,\left(\sigma_{\rm cr} - \sigma\right) + \mathcal{O}\left(\epsilon^2\right). \tag{3.22}$$

We can then perform a similar dimensional analysis as above to conclude that $\left[G_5\frac{k_-k_+}{k_+-k_-}\right] =$ [Length]^2. This is matches four-dimensional Newton's constant G_4 dimensionality.[48] This will

[46]Negative values of the tension σ have no physical meaning for this construction.
[47]Note that the choice of the solution with the plus sign corresponds to that of the Randall-Sundrum construction discussed in section **??**.
[48]Recall, as one can see in Eq. (**??**), that $\kappa_D = \ell_D^{D-2}$.

allow us to make the following identifications:

$$\kappa_4 = 2\frac{k_- k_+}{k_- - k_+}\kappa_5, \qquad \rho_\Lambda = \sigma_{\rm cr} - \sigma. \tag{3.23}$$

We can see that this regime allows us to relate the induced dark energy ρ_Λ to the "subcriticality" of the nucleated bubble. This requires a tuning of $\varepsilon \sim 10^{-120}$ (in Planck units) to make contact with its observed value. We will see that this somewhat *ad hoc* parameter ε can find a stringy interpretation in section 7.7. It is also important to note the unique relationship between four and five-dimensional Newton's constant in this model. The difference between the AdS vacuum scales in the denominator of Eq. (3.23) already points to an inversion of the common scheme of scales discussed in chapter **??**. In fact, when $k_+ \sim k_- \sim k$, one finds that gravity will be a stronger force in four dimensions than five. Although concerns about divergence may make our legs tremble, we will see in section 7.7 how this hierarchy is under control when the bubble is embedded in a ten-dimensional description.

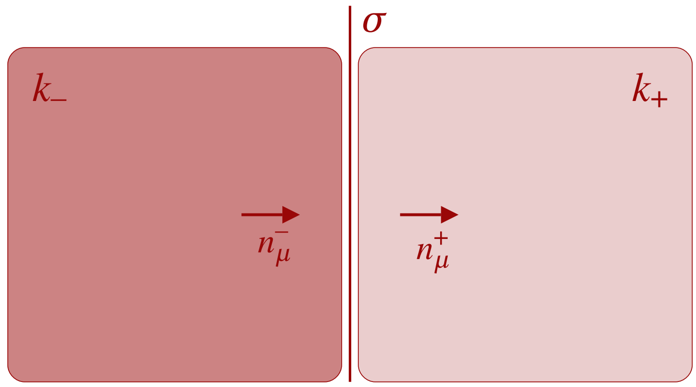

Figure 9: Pictorical representation of the inside/outside distribution of AdS-vacua and the bubble with tension σ mediating between them. Note the orientation of the normals $n_\mu^\pm$, which will always point in the direction of increasing volume of spatial sections, contrary to the Randall-Sundrum case in figure 6.

It could be convenient to point out some of the main differences between the Dark Bubble model [?] and the Randall-Sundrum model [?] discussed in section **??**:

- The first notable absence is that of the $\mathbb{Z}_2$ symmetry, which identifies the vacua scales k to both sides of the brane. Even more important is the natural choice of the normal orientation for the dark bubble, where both point in the direction of growing spatial sections. This can be seen from the behaviour of the warp factors for both dark bubbles (8) and the Randall-Sundrum model (6).
- This brings us to the next consequence, that of the criticality of the brane tension σ. While a subcritical tension is required to have a dark bubble nucleate and induce a positive cosmological constant at its boundary, the opposite, i.e. a supercritical tension is necessary to achieve the same result for the Randall-Sundrum scenario. However, it is clear from the swampland conjectures discussed in chapter **??**, that only branes with $\sigma < Q$ are accepted.
- Finally, it is important to notice the difference between the Newton's constant in Randall-Sundrum and the dark bubble models. While the former scenario has a four-dimensional constant controlled by the vacuum scale k of the mirrored AdS$_5$, the latter depends on the difference. This is crucial, as if this difference is extremely small, it can yield a situation where

gravity results stronger in four dimensions than five, i.e. $G_4 \gg G_5$, as we will see in section 7.7.

In summary: The **Dark** Bubble model proposes the realisation of a four-dimensional accelerated expanding cosmology on the boundary of a nucleated five-dimensional bubble of a decaying AdS_5 vacuum. In this model, the low-dimensional **dark** energy density ρ_Λ finds a dynamical interpretation in the higher-dimensional picture, where the difference between the *critical* and *subcritical* tension of the nucleate brane controls the motion of this in the extra dimensions.

3.3 Towards a gourmet cosmology

We would like to conclude this first chapter of stage **??** whit a discussion on points that will be relevant and hence discussed in the following chapters. In this way, our dear reader can easily find those topics that align with their sophisticated palate.

- This chapters ommits the presence of extra dimensions required by supergravity models. This will be the main discussion in chapter **??**, where we will embed the dark bubble model in string theory. Its connections to the swampland will be also discussed in that chapter.
- We have talked about expanding branes in the bulk, but, how probable is such bubble nucleation? Can this event in higher dimensions be related to the standard cosmology in four dimensions? This will be reviewed and connected to the quantum cosmology discussion of chapter **??** in chapter **??**.
- How can we decorate the four-dimensional universe in this proposal? Where is gravity? Does it remain localised on the induced geometry or does it propagate across extra dimensions? This will be examined in chapter **??**, where we will be study how strings associated with the expanding brane are required to gain a good understanding of gravity in this model.
- We have only discussed how to induce dark energy ρ_Λ on the brane, but the observable universe encompasses much more than that. This will be the topic of , where the induced cosmology will be decorated with matter, radiation and waves coming from the extra dimensions. There we will explore the true power of the second Israel condition, its relation to the Gauss-Codazzi equation as described in appendix **??** and also how to interpret these concepts correctly to obtain a meaningful induced energy-momentum tensor.

4 Quantum Bubbles

TO THINK IF THIS IS THE ORGANISATION I WANT. TO THINK HOW TO CONNECT WITH ITS 4D COUNTERPART

In this section you will learn...

- to quantise the action of an expading bubble
- how likely the bubble nucleation event is
- the relationship between a higher-dimensional nucleation event and the choice of boundary conditions in four-dimensional quantum cosmology

We have discussed in detail the technicalities of how to induce dark energy ρ_Λ on the bubble wall in chapter **??** and how that simple model can be easily embedded in string theory through chapter **??**. In that same chapter, we made some comments on how likely this process is to occur, but we did not provide an explicit and detailed calculation of the nucleation probability P. In this chapter we will present the results of Danielsson, Panizo et al. in [?] on the bubble nucleation process and any possible implications for the low-dimensional cosmology.

As we have seen in chapter ??, there is still an open discussion about the right choice of boundary conditions for the wave function describing our universe. An out-of-the-box perspective to address this issue can be to analyse this problem from a higher-dimensional picture. In other words, we ask ultraviolet-complete models of cosmology within string theory what the right choice of boundary conditions should be in lower-dimensional quantum cosmology. Given the previously presented dark bubble scenario, capable of reproducing an expanding four-dimensional cosmology equipped with a positive cosmological constant Λ_4, one question to raise could be: Can a quantised description of the dark bubble provide insight into the four-dimensional quantum cosmological realm? Let us explore this further.

This is a short and insightful chapter divided into two sections. Section 4.1 will be devoted to presenting the classical bubble action to quantise and how we can extract information from its components. This will allow us to identify similarities and differences between the bubble motion and that of those of the four-dimensional cosmos discussed in chapter ??. Following the aforementioned similarities, we will be able to identify the bubble nucleation probability for the right choice of boundary conditions for the four-dimensional quantum cosmological description in section 4.2.

4.1 Scene reconstruction of the quantum bubble

We will now study the quantum nucleation of such a bubble by closely following the treatment of [?] for bubble nucleation in five dimensions.[49] The five-dimensional action describing the dynamics of the subcritical tension bubble is given by:

$$S_{\text{bubble}} = S_{\text{bulk}} + S_{\text{matter}} + S_{\text{GHY}}. \tag{4.1}$$

Let us discuss each contribution individually:

- The bulk contribution, written as

$$S_{\text{bulk}} = \frac{1}{2\kappa_5}\int \mathrm{d}^5x\sqrt{|g_{(5)}|}R_{(5)}, \tag{4.2}$$

  has the role of describing the five-dimensional curvature of the ambient space where the bubble will live in.

- We should also consider any matter content in the bulk and on the brane. This is the duty of the second term in expression (4.1), described as:

$$S_{\text{matter}} = \int \mathrm{d}^5x\sqrt{|g_{(5)}|}\mathcal{L}_{\text{matter}}, \tag{4.3}$$

  where

$$\mathcal{L}_{\text{matter}} = \begin{cases} \Lambda_-^{(5)}/\kappa_5 & \text{in} \quad \mathcal{M}_-, \\ \Lambda_+^{(5)}/\kappa_5 & \text{in} \quad \mathcal{M}_+, \\ -\rho\,\delta(\Sigma) & \text{on} \quad \Sigma. \end{cases} \tag{4.4}$$

  Here $\mathcal{M}_i$ represents each piece of the bulk geometry on the outside and the inside of the mediating brane, while ρ encodes its content. The position of the brane in the bulk is represented by $\delta(\Sigma)$.

[49] Note that we will not include in our description the compact directions of S^5. We assume a small compact dimension with respect to the size of the five-dimensional bubble.

- Finally, one should also take into account any possible information about how the brane is embedded in the five-dimensional bulk. This is done by the contribution of the extrinsic curvature K_{ab} of the bubble into the five-dimensional space. This a higher-dimensional extension of the Gibbons-Hawking-York term presented in chapter **??**, which looks like:

$$S_{\mathrm{GHY}} = \frac{1}{\kappa_5} \int \mathrm{d}^4 y \sqrt{|h|} K \,. \tag{4.5}$$

It should be clear by now that the whole system is made up of: An outside bulk, a brane mediating the decay and an inside bulk geometries. This implies that one should be careful when integrating along the radial direction r, as the integral will be split into two integration regimes.

As the aim of this section is to provide a clear and simple conceptual picture of the bubble nucleation problem, for the sake of simplicity in our computations, we will choose an empty[50] AdS-bulk described by Eq. (3.8), with

$$f_{\pm}(r) = 1 - \frac{\Lambda_{\pm}^{(5)}}{6} r^2. \tag{4.6}$$

The induced geometry on the bubble is again an FRLW expanding cosmology (3.11). For computational convenience, let us define a handy notation of the form:

$$\beta_{\pm}(\tau) = f_{\pm}(r) \frac{\mathrm{d}t_{\pm}(\tau)}{\mathrm{d}\tau}\bigg|_{r=a(\tau)} = \sqrt{f\left(a(\tau)\right) N(\tau)^2 + \dot{a}^2}, \tag{4.7}$$

where again, $N(\tau)$ is the lapse function (3.12) and a dot represent a derivative with respect to τ.

Our aim now is to rewrite Eq. (4.1) to obtain a Lagrangian $L^{(5)}_{\mathrm{total}}$. We will then extract a Hamiltonian $\mathcal{H}$, which we will later quantise, etc. Basically, we will follow the discussion in chapter **??** applied to this bubble cosmology. In this sense, we will move the cosmological constant $\Lambda^{(5)}$ to the bulk contribution (4.2) and leave the brane content alone. Furthermore, we will use Eq. (4.7) to write the action in terms of the proper time τ. Finally, each term contribution is given by:

$$\begin{aligned}
S_{\mathrm{bulk}} &= \frac{2\pi^2}{\kappa_5} \int \mathrm{d}\tau \left[3\beta(\tau) a^2 + \Lambda^{(5)} a^4 + \frac{a^3}{\beta(\tau)} \left(\ddot{a} + \frac{N(\tau)^2}{2} \frac{df(\tau)}{da} - \frac{\dot{a}\dot{N}(\tau)}{N(\tau)} \right) \right]_+^- , \\
S_{\mathrm{GHY}} &= -\frac{2\pi^2}{\kappa_5} \int \mathrm{d}\tau \left[3a^2 \dot{a} \tanh^{-1}\left(\frac{\dot{a}}{\beta(\tau)} \right) + \frac{a^3}{\beta(\tau)} \left(\ddot{a} - \frac{\dot{a}^2}{2a} \frac{df(\tau)}{da} - \frac{\dot{a}\dot{N}(\tau)}{N(\tau)} \right) \right]_+^- , \\
S_{\mathrm{brane}} &= -2\pi^2 \sigma \int \mathrm{d}\tau a^3 \, N(\tau), \,.
\end{aligned} \tag{4.8}$$

were we have defined $[A]_+^- = A_- - A_+$. Summing up all terms, and ignoring terms that do not affect the dynamics of the shell,[51] the mini-superspace Lagrangian looks like:

$$L^{(5)}_{\mathrm{total}} = \frac{6\pi^2}{\kappa_5} \left[-a^2 \dot{a} \, \tanh^{-1}\left(\frac{\dot{a}}{\beta(\tau)} \right) + a^2 \beta(\tau) \right]_+^- - 2\pi^2 a^3 \sigma N(\tau) \,. \tag{4.9}$$

Note that in the phenomenological "bubble" regime, i.e. when $ka \gg 1$ and $H/k \gg 1$, Eq. (4.9) becomes the simple four-dimensional Lagrangian described in Eq. B.46. This already points out to

[50]The presence of mass in the bulk will be dynamically irrelevant for the computations of the equation of motion. Moreover, the charge Q associated with the five-dimensional Reisser-Nordström black hole in chapter **??** will also be irrelevant. Hence, these terms will not affect the resulting Hartle-Hawking equation. However, this need not be the case when computing the nucleation probability of the bubble. As our aim is to recapitulate the results of Danielsson, Panizo et al. [**?**], we will stick to an empty AdS computation and leave further modifications of the bulk space to future work.

[51]This only affects the $\Lambda^{(5)}$-term, which results to be dynamically irrelevant, i.e. it will not appear in the equations of motion. This can be seen from the first junction condition (2.57) applied to the bulk geometry.

a connection between the four-dimensional description of quantum cosmology discussed in chapter **??** and that of the higher-dimensional bubble. This shows that our five-dimensional ultraviolet completion gives full control over an infinite tower of corrections (coming from the argument of tanh) to the four-dimensional cosmological model. But this is not the only connection to be found.

Let us now compute the Hamiltonian. Following the discussion in chapter **??**, the conjugate momenta are then given by:

$$\pi_{N(\tau)} = \frac{\partial L}{\partial \dot{N}(\tau)} = 0, \qquad \pi_a = \frac{\partial L}{\partial \dot{a}} = -\frac{6\pi^2 a^2}{\kappa_5} \tanh^{-1}\left(\dot{a}/\beta(\tau)\right)\Big|_{+}^{-}. \tag{4.10}$$

The conjugate momentum $\pi_{N(\tau)}$ is already pointing to an important feature of the model: The lapse-function $N(\tau)$ is not dynamical and will act as a Lagrange multiplier, enforcing the constraint $H = 0$. In fact, this allows us to write the Hamiltonian as:

$$\mathcal{H} = 2\pi^2 N(\tau) a^3 \left(\sigma - \frac{3(\beta_-(\tau) - \beta_+(\tau))}{a\,\kappa_5}\right). \tag{4.11}$$

Substituting the β-expression (4.7) and enforcing the constraint $H = 0$, it is easy to see that the Hamiltonian (4.11) is identical to the junction condition (3.17). This is nothing new to us, as we have already shown it in section 7.4. In any case, it is reassuring to find this exact relation from a different construction.

Although Eq. (4.11) shows the true power of the junction condition,[52] it is not well prepared to be quantised and become the Wheeler-de-Witt equation discussed in Eq. (B.49). For this, the conjugate momentum π_a should be given explicitly in the expression. By rewriting it as:

$$\cosh\left(\frac{\kappa_5 \pi_a}{6\pi^2 a^2}\right) = \frac{\beta_-(\tau)\beta_+(\tau) - \dot{a}^2}{N(\tau)^2\sqrt{f_-(\tau) f_+(\tau)}}, \tag{4.12}$$

one can substitute in the Hamiltonian H, to yield:

$$\begin{aligned} \mathcal{H} = -\frac{6\pi^2}{\kappa_5} &\left(f_-(\tau) + f_+(\tau) - 2\sqrt{f_-(\tau) f_+(\tau)} \cosh\left(\frac{\kappa_5 \pi_a}{6\pi^2 a^2}\right)\right)^{1/2} \\ &+ 2\pi^2 N(\tau) a^3 \sigma = 0. \end{aligned} \tag{4.13}$$

The presence of the momentum π_a in the argument of cosh implies that the Wheeler-de-Witt equation becomes one of infinite order in π_a. Therefore, we will focus on the limit for small π_a, up to quadratic order.[53] If we now quantise the system by replacing the operator

$$\pi_a \to -\frac{i}{a^{3/2}} \frac{\mathrm{d}}{\mathrm{d}a} a^{3/2}, \tag{4.14}$$

one obtains,

$$\mathcal{H}\psi_{5D} = \left(-\frac{1}{24\pi^2\, a^{3/2}} \frac{d^2}{da^2}\left(a^{3/2}\right) + 6\pi^2 V(a)\right)\psi_{5D} = 0. \tag{4.15}$$

Note that this Wheeler-de-Witt equation is *exactly* Eq. (B.49). Here the wavefunction ψ is supported in four spatial dimensions and it is related to that of the four-dimensional cosmology approach (B.51) by the identification $\psi_{4D} = a^{3/2}\psi_{5D}$. All in all, it is remarkable to see that the dynamics used to describe the four-dimensional cosmology discussed in chapter **??** can be encoded by those of an expanding bubble in a higher-dimensional environment [**?**].

[52] i.e. the junction condition is the Hamiltonian as described in section 7.4.
[53] Recall that the bubble nucleates at rest $\dot{a} = 0$ and will expand from there.

4.2 Whatever remains, however improbable, must be the truth

Although the results obtained in Eq. (4.15) are fine on their own, that was not the aim of this chapter. We wanted to use the higher-dimensional approach to determine the choice of boundary conditions in the description of four-dimensional quantum cosmology. A relevant question for this purpose might be: How likely is the nucleation of a dark bubble?

In the four-dimensional description discussed in chapter **??**, we saw that the nucleation probability (B.55) differed by a sign in the exponential argument, depending on the choice of boundary conditions. If we now turn our attention to the five-dimensional bubble nucleation probability, we should recall that the dark bubble is a Brown-Teitelboim instanton [**?**]. Details of this type of solution can be found in appendix **??**. According to the Brown-Teitelboim computations presented there, the probability for such bubble nucleation to happen is given by:

$$P = e^{-B}, \tag{4.16}$$

where B is the instanton bounce, as discussed in appendix **??**. The Euclidean instanton can be obtained by integrating (4.1) over an $O(5)$ sphere of radius $r = a$ (i.e. the radius at which the bubble nucleates). Evaluating Eq. (2.30) for this case gives:

$$B = \sigma\, \Omega_4\, a^4 + \frac{1}{\kappa_5} \left[4k^2\, \mathrm{Vol}_5(a,k) - \frac{4}{R}\beta\, \Omega_4 \right]_+^-, \tag{4.17}$$

with $\Omega_4 = 8\pi^2/3$ and $\mathrm{d}\,\mathrm{Vol}_5/\mathrm{d}a = \Omega_4/\beta$. Extremising using $\mathrm{d}B/\mathrm{d}a = 0$ implies the junction condition (4.11) and fixes a to the critical value a_*. Expanding in the late big bubble regime, i.e. $ka_* \gg 1$, we have

$$\mathrm{Vol}_5 = \frac{\Omega_4}{4k} \left(1 - \frac{1}{k^2 a_*^2} \right). \tag{4.18}$$

This expression can be substituted back into Eq. (4.17). Furthermore, if we want to connect to four-dimensional quantities, i.e. κ_5 to κ_4, we must use of Eq. (3.23). Finally, this yields:[54]

$$B = \frac{24\pi^2}{\kappa_4^2 \rho_{\Lambda_4}}. \tag{4.19}$$

Similarly, if one decides to use the WKB wall penetration probability given by:

$$P \sim e^{-B}, \qquad \text{with} \qquad B = 2 \int \pi_a\, \mathrm{d}\tau, \tag{4.20}$$

one then recovers the same result as in (4.19).

Either way, the results are clear: this higher-dimensional approach to the quantum cosmology boundary choice problem leads us to select unambigously the Vilenkin amplitude (B.55) from the four-dimensional point of view. The bubble nucleation probability in five dimensions is equivalent to a "tunneling from nothing" probability in the four-dimensional realm. Furthermore, there is no *Big Bang* singularity to worry about. It does correspond to the fiducial zero size of the bubble (i.e.

[54] It could be tempting to replace the dark energy description (7.65) from chapter **??**. This would return the nucleation probability as $P \propto e^{-N^2}$, which is more highly suppresed than the one discussed in chapter **??**, for a D_3-brane to escape from the N stack. It is important to recall that we have not used the five-dimensional Reisser-Nordström black hole in this chapter, as we followed the results obtained by Danielsson, Panizo et al. in [**?**]. This suppressed difference can be caused by the absence of angular momentum in the description, which translates into a charge Q in the five-dimensional AdS. The absence of extra fields in the background could be the reason for this difference in the arguments for both constructions. In any case, it might be interesting to revisit this quantum construction equipped with a more populated mini-superspace.

its absence before nucleation). Moreover, we have shown that the physics at all the length scales involved in the process of nucleation and expansion are essentially understood (see Eq. (4.9)): it is semi-classical gravity in five dimensions, that provides our ultraviolet completion of four-dimensional cosmology.

Finally, it is reassuring to see that this choice of boundary conditions follows naturally from such construction. This low-dimensional "tunneling into existence" as a higher-dimensional false vacuum decay interpretation has been suggested as the only way to realise bubble nucleation in string theory [?].

5 Gravity and Strings of the Dark Bubble

I THINK THIS HAS TO GO BEFORE THE DECORATION. PERHAPS IT MAKES SENSE THAT IT GOES BEFORE THE EMBEDDING, TO IMPROVE THE ARGUMENTS FOR THE BEHAVIOUR OF GRAVITY IN HIGHER DIMENSIONS. OBS!!! IT REQUIRES A SECTION TALKING ABOUT THE PROPAGATOR. IT COULD ALSO FIT SOME OTHER SECTION COMPARING TO RS MODEL?

In this section you will learn...

- how to induce massive point particles on the four-dimensional cosmos by using strings in the bulk
- the bending reaction of the brane to any changes in the bulk
- how gravity, which extends along the bulk, dictates the brane dynamics and how it behaves on it

In chapter **??**, we focused on obtaining a positive and small cosmological constant Λ_4 for the induced cosmology living on the D_3-brane mediating the decay of two AdS$_5$-vacua. Furthermore, we showed how to embed this "bubbly" five-dimensional construction into string theory in chapter **??**. We have also discussed the likelihood of the nucleation of such a bubble and how this higher-dimensional "pop" sheds light on the long standing problem of boundary choice in four-dimensional quantum cosmology reviewed in chapter **??**. But nothing has been said so far about any other energy densities, such as matter or radiation, on the induced cosmology, other than that of dark energy ρ_Λ. In order to match the observational evidence of our cosmos, we need to extend this phenomenological bubbly-construction, so that it can predict the existence and host familiar energy densities ρ_i on top of the brane.

Any impatient reader willing to decorate the induced cosmology as fast as possible is gently invited to take a shortcut and visit chapter **??**. However, learning to run without having developed a steady and solid walk may not be the best idea. This chapter, inspired (in part) by [?, ?], is devoted to start the decoration of such an empty universe with the most basic element associated with branes: strings.

This chapter is organised as follows: In section 5.1, we will explore the simplest stringy configuration; A radially stretching string with one of its end points **on** the brane. This will induce a massive point particle on the bubble wall. Any deformations caused on the brane by the presence of the string and/or matter on the brane will be studied in section 5.2. We will learn there that the brane and the bulk geometries depend on each other's changes. At first sight, some of the results obtained might appear to be problematic. Section 5.3 will be devoted to remove any concerns about these results, as it will be explicitly shown that gravity behaves in the familiar four-dimensional way when projected onto the induced cosmos.

5.1 Diddley bow string[55]

Let us imagine just **one** string with one of its end points on our familiar D_3-brane.[56] Furthermore, for simplicity in further computations and concept development along this section, we will assume the following points:

- In the same spirit as in chapter **??**, we will leave the bulk "empty". In addition to this, we are interested in presenting results for late-time cosmologies, in such a way that the bubble's patch of interest looks flat along the throat direction. This can be easily seen by changing the variables to Poincaré coordinates with conformally flat slices along the radial direction, giving:[57]
$$\mathrm{d}s_5^2 = \mathrm{d}r^2 + a(r)\eta_{ab}\mathrm{d}x^a x^b, \tag{5.1}$$
with the scale factor $a(r) = e^{2kr}$.
- Let us freeze the expansion of the bubble and assume that it is located at $r = r_0$. Note that in this set of coordinates, the Poincaré horizon sits at $r \to -\infty$, while the boundary to AdS$_5$ is at $r \to \infty$.
- The string, which is assumed to stretch along the radial direction r, will have one of its end points on the brane, while the other one will extend all the way up to the AdS$_5$-horizon.[58]

Given the previous points, we are then ready to study how the presence of such string decorates the five-dimensional geometry. Its dynamics are controlled by the Polyakov action [?],

$$S_{\text{string}} = -\frac{1}{2}T_s \int d^2\xi \sqrt{|\omega_{\alpha\beta}|}\,\omega^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}. \tag{5.2}$$

Note that $\omega_{\alpha\beta}$ represents the world-sheet's metric, with the indices $\{\alpha, \beta\}$ representing its coordinates, while the usual indices $\{\mu, \nu\}$ refer to the coordinates of the target space. T_s is the fundamental tension of the string, given by:

$$T_s = \frac{1}{2\pi\alpha'}. \tag{5.3}$$

The associated energy-momentum tensor of the string can be computed by varying the action (5.2) with respect to $g_{\mu\nu}$. This gives:

$$T^{\mu\nu} = -T_s \int \mathrm{d}^2\xi \frac{1}{\sqrt{|g_{\mu\nu}|}} \sqrt{|\omega_{\alpha\beta}|}\,\omega^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \delta^5 (X - x)\,, \tag{5.4}$$

where the δ-function represents the source divergences due to the presence of the string, i.e. the position at which it is localised. In order to simplify our computations, we will use the *static*

[55] A diddley bow is a single-stringed American instrument that influenced the development of the blues sound. It consists of a single string of baling wire stretched between two nails on a board. It was traditionally considered a beginner's or children's instrument in the Deep South of the USA, especially in the African-American community. It may have been influenced to some extent by West African instruments.

[56] We will not consider extra dimensions in this chapter as our focus is on large length scales and small energy scales.

[57] Starting from a Poincaré line invariant (3.5), the required change of variables is $r = \frac{1}{k}\log(k\zeta)$. Notice that the throat direction is always represented by the letter r, although the warping associated with this direction may vary from chapter to chapter.

[58] In case the title of this chapter is not clear, the brane represents the saddle of the guitar while the horizon is the nut.

gauge of the string. This is $\xi^0 = t$ and $\xi^1 = r$. It is then easy to see that the only non-vanishing components of the energy-momentum tensor are given by [?]:

$$T^t{}_t = T^r{}_r = \frac{T_s}{a(r)^3}\,\delta\left(x^a - x_0^a\right), \tag{5.5}$$

where x^a describes the three spatial directions tangent to the wall of the bubble and x_0^a the exact position of the string on it. However, there is a catch here. This energy-momentum tensor corresponds to an equation of state $p = \rho$ along the throat direction r and $p = 0$ in the transverse directions. This is consistent with what to expect from a string stretched *infinitely* in the radial direction. But this is not what we assumed in the third point stated above. In this sense, we need to "cut" the string to end at the brane. This implies that the pressure component of the energy-momentum tensor should also be cut in the radial direction. This can be done by a Heaviside function $\Theta(r)$, such as that:

$$T^r{}_r = \frac{T_s}{a(r)^3}\,\delta\left(x^a - x_0^a\right)\,\Theta(r - r_0). \tag{5.6}$$

One can then obtain the new associated energy density ρ of a string ending on the brane by imposing the covariant conservation[59] $\nabla_\alpha T^\alpha_\beta = 0$, which requires T^t_t to be:

$$T^t{}_t = \frac{T_s}{a(r)^3}\,\delta\left(x^a - x_0^a\right)\,\Theta\left(r - r_0\right) + \frac{T_s}{k}\frac{1}{a(r)^3}\,\delta\left(x^a - x_0^a\right)\,\delta\left(r - r_0\right). \tag{5.7}$$

The first term of the previous expression is already known: It corresponds to the string stretching along the bulk, down to the position of the "frozen" brane. The second term is the interesting one; in fact, such scaling for the energy density localised on the brane $r = r_0$, is not new in this thesis. We discussed in section **??** that non-relativistic matter decays as $\rho_{\text{matter}} \sim a^{-3}$. This implies that the end point on the brane acts as a point particle in the induced expanding cosmology, with mass given by [?]:

$$M_{\text{particle}} = \frac{T_s}{k}. \tag{5.8}$$

This is our very first decoration of the induced cosmos. Radially stretching strings in the bulk will induce massive[60] particles on the bubble's wall. This can be thought as a point mass suspended by a string in the gravitational well of the AdS$_5$ space. As the universe expands and the brane moves upwards, the string pulls the point mass so that it can move along the brane, ensuring the aforementioned covariant conservation.

Our dear reader, though full of joy knowing that the induced cosmology will no longer be a wasteland, may not be at ease. Any type of worry caused by the point-particle mass on the brane is probably based on a simple everyday experience; If one places an object on a stretched out sheet, the sheet will sag down in response to the presence of such an object. Should this higher dimensional situation not be similar in concept? Could this somehow affect to the sign of the induced energy density? A thought experiment of this kind can even raise more fundamental questions: Where is gravity? Is it localised on the brane or does it extend along the entire bulk? These are questions to be answered in the following sections of this chapter. Let us start with the everyday example of bending.

[59] This is the analogous Bianchi identity to the right-hand side of Einstein's equation (**??**) so that the whole construction remains consistent.

[60] Note that given the scales in table 16, the mass of the induced particles will be of order $M_{\text{particle}} \sim M_{\text{Planck}}$. These would not be normal massive particles. To induce ordinary ones would need to have non-fundamental strings with end point on the brane and stretching up to the AdS boundary. An interesting possibility could be to have n coincidental D_3-branes along the radial direction r but at different positions in the compact dimensions S^5 in the string theory embedding discussed in chapter **??**. Strings stretching between different branes would have the potential to induce standard model-like particles [?, ?].

5.2 Tirando y Apoyando[61]

The situation described above is at the same heart of in the concept of the **backreaction**. The line invariant (5.1) represents a five-dimensional empty space with conformally flat slices along the throat direction r. Any kind of source settlng there, not only on the bubble, but in the whole ambient space will cause a deformation of the bulk. This new geometry, which contains information about the presence of the source is called **backreacted** geometry.[62]

Let us now consider perturbations of the bulk metric in the presence of induced matter on the brane, as a result of the presence of the string in the bulk. These would be described by two perturbation parameters on the brane:

1. How much matter density ρ you put on the brane.
2. How much the radial coordinate on the brane ξ would be deformed by the presence of this mass.

However, these aforementioned changes only apply to the geometry on the brane. If we also bring back the stretched sheet image, the brane should also bend along the throat direction [**?**, **?**]. And this change is correlated with how "big" the adjustment is in the ξ coordinate, i.e. the radial coordinate in the three-dimensional space. This can be expressed as $r = \tilde{r} + c(\xi)$. All these changes can be captured by following line invariant:

$$\mathrm{d}s^2_{\text{back}} = \mathrm{d}\tilde{r}^2 + a(\tilde{r})^2 \left[-\left(1 - \frac{4\rho}{\xi} + 2F(\xi)\right) \mathrm{d}t^2 + \left(1 + \frac{2\rho}{\xi} + 2F(\xi)\right) \mathrm{d}\xi^2 + \xi^2 \mathrm{d}\Omega_2^2 \right], \tag{5.9}$$

where $d\Omega_2^2$ is the line invariant of the two-sphere in the usual spatial sections and $F(\xi)$ is a function on ξ which can be written as

$$F(\xi) = k\, f(\xi) = \frac{\rho}{2\xi}, \tag{5.10}$$

so that the metric (5.9) is a solution of Einstein's equation, up to linear order in the perturbation expansion commented above.

The next step is to compute a good old friend: The Israel's junction condition (2.67). However, one has to be careful in this case; The induced energy-momentum tensor will not only be that of a brane (with $\rho = -p$), but will also have contributions coming from the induced matter on top, i.e. the end point of the string. This implies $S_{ab} = -\sigma h_{ab} + T_{ab}$. This yields:

$$\frac{1}{2\kappa_5} \left[\partial_r h_{ab}\right]^+_- + \frac{\sigma}{3} h_{ab} = -\left(T_{ab} - \frac{1}{3} T h_{ab}\right), \tag{5.11}$$

where we have exploited the fact that $K_{ab} = \frac{1}{2}\partial_r h_{ab}$ in the coordinates (5.1) and $[K]^-_+ = \frac{\kappa_5}{3}\left(4\sigma - T\right)$, with $T = h^{ab} T_{ab}$.

There is a problem in our previous line of reasoning; although Eq. (5.11) accounts for the presence of induced matter on the brane, the induced metric h_{ab} does not completely reflect any deformations of the bulk geometry. In other words: Eq. (5.11) is aware of the matter density ρ but not of the deformation induced on the brane. Any Gaussian normal perturbation on the bulk geometry of the form $g_{\alpha\beta} \to g_{\alpha\beta} + \delta g_{\alpha\beta}$ will induce a metric $a(r)^2\eta_{ab} \to a(r)^2\eta_{ab} + \hat{\gamma}_{ab}$, where $\hat{\gamma}_{ab}$

[61] In Spanish, to pull and to rest. Makes reference to two different ways of plucking guitar strings: "Tirando" means that the finger does not touch the string that is next lowest in pitch (physically higher) on the guitar, as is the case does with apoyando. Although the note is exactly the same, the colour and robustness of the sound varies according to the technique used.

[62] The derivation of the geometry to be discussed will not be presented in detail, as it is involved and full of technicalities. Explicit computations of backreaction geometries, more relevant to the results in this thesis can be found in chapter **??**. Of course, a detailed discussion of the present one can be found in [**?**].

is the induced perturbation piece, caused by the matter density χ and its radial (ξ) deformation. This implies that the correct junction condition to consider is:

$$\frac{1}{2\kappa_5}\left[\partial_r \hat{\gamma}_{ab}\right]_-^+ + \frac{\sigma}{3}\gamma_{ab} = -\left(T_{ab} - \frac{1}{3}T\eta_{ab}\right). \tag{5.12}$$

We are not done yet: due to the presence of matter ρ, in a similar way than the aforementioned sheet bending under the weight of the mass above, the bending of the brane in the radial direction will make the junction conditions hard to apply. The position of the brane will be given in terms of two different functions $r = f_\pm(x)$, depending on which side you look from. This requires us to apply two diffeomorphisms on the outside and inside to bring the brane back to $r = 0$, as it has been deformed due to the presence of the mass. This implies that $f(x) < 0$ represents the brane bending "down", towards the Poincaré horizon. Conversely, $f(x) > 0$ means that the brane bends "upwards", towards the AdS horizon. Requiring the brane to sit at $r = r_0 = 0$ in the new coordinates from both sides, one finds [**?**]:

$$r \to r - f(x), \quad x^a \to x^a + \frac{1}{2k}\left(1 - a^{-2}\right)\partial^a f(x) + w^a(x), \tag{5.13}$$

where $w^a(x)$ is an arbitrary function of the transverse coordinates. In these new coordinates, the induced metric perturbation piece γ_{ab} becomes:

$$\hat{\gamma}_{ab} = \gamma_{ab} + \frac{1}{k}\left(1 - a^2\right)\partial_a\partial_b f + 2kfh_{ab} - 2a^2\, w_{[a,b]}, \tag{5.14}$$

where $w_{[a,b]} = \partial_a q_b + \partial_b q_a$. Introducing Eq. (5.14) into the junction condition (5.12) evaluated at $r = r_0$ and imposing continuity of the induced metric [**?**] one finally obtains:

$$\frac{1}{2\kappa_5}\left[\partial_r \gamma_{ab}\right]_-^+ + \frac{\sigma}{3}\gamma_{ab} = -\Sigma_{ab} \tag{5.15}$$

where the tensor Σ_{ab} is given by:

$$\Sigma_{ab} = \left(T_{ab} - \frac{1}{3}T\eta_{ab}\right) - \frac{1}{\kappa_5}\left(\frac{1}{k_+} - \frac{1}{k_-}\right)\partial_a\partial_b F(\xi). \tag{5.16}$$

The expression above contains new information that needs to be discussed. While the first term on the right hand side of equation (5.16) has remained the same, the second term is a result of the deformation (5.14). This part contains the bending sample $F(\xi)$ of the backreaction (5.9). If we choose this bending gauge $F(\xi)$ such that Σ_{ab} becomes traceless, Eq. (5.16) becomes [**?**]:

$$T = -\frac{3}{\kappa_5}\left(\frac{1}{k_+} - \frac{1}{k_-}\right)\partial^a\partial_a F(\xi), \tag{5.17}$$

with $T = h_{ab}T^{ab}$ as the trace of the energy-momentum tensor on the brane. Recall that $F(\xi) \sim f(\xi)$, which was defined above as the "gauge" that identifies how much the brane bends in response to the presence of matter. In this sense, Eq. (5.17) can be used to determine how much and in which **direction** the brane bends with a non-zero energy-stress tensor T_{ab}. In fact, let us study three relevant cases:

- Any point mass on the brane has an energy density ρ_{Mass} given by $T_{00} \sim \delta(\xi)$. To compute the trace T, we raise an index so that $T \sim -\delta(\xi)$. This implies that $\partial^a\partial_a F(\xi) \sim \delta(\xi)$. Hence $f(\xi) < 0$. This means that the brane **sags down** in response to it.
- Similarly, a string extending from the AdS boundary and with one of its end points on the brane (i.e. Eq.(5.7)), will induce a contribution $T_{00} \sim -\delta(\xi)$. Going up one index, to compute the trace, we get $T \sim \delta(\xi)$. This brings us to $\partial^a\partial_a F(\xi) \sim -\delta(\xi)$. Therefore $f(\xi) > 0$. This means that the brane is **pulled up** in response to the presence of the string.

- As we can see from the string case, any content in the bulk between the AdS boundary and the brane that contributes with a positive trace T will pull the brane up.

Two of these cases are shown in figure 10 (see section 6.3 for further information on the third case). Let us recapitulate what we have done so far: We started with a brane that had conformally flat slices (5.1) along the radial direction. We then considered any deformations of the brane geometry that any mass **in** the bulk and **on** the brane could generate in Eq. (5.9). This is encoded in the gauge (5.10). In the case we study a situation like the one described in the second point above (i.e. a string with an end point on the brane), the gauge will tell us that the brane will be deformed upwards.[63] If we wanted to recover the flat description, we would need to **add** a mass to the end point of the string, to pull the string down, as described in the first bullet point above. This mass has already been computed in Eq. (5.8), demanding energy conservation. This proves that a string stretching outwards, pulling from the brane will induce a positive mass, yielding the expected density on the four-dimensional cosmos. On the other hand, matter sitting on the brane will produce negative masses.

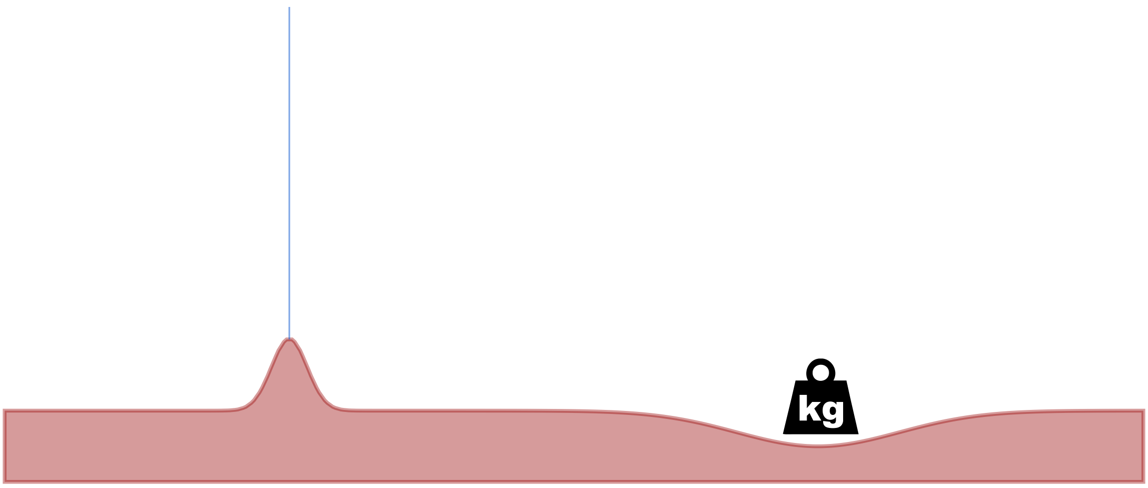

Figure 10: Sketch representation of the bubble, whose boundary responds to the presence of sources. Strings with end point on the bran will deform the conformally flat geometry and pull up from the brane. The opposite case would be generated by masses **on** the brane, which will sag down the brane, yielding a negative contribution to the four-dimensional energy-momentum tensor.

The previous discussion about the end points of strings acting as four-dimensional positive masses can be extended to reproduce an homogeneous and isotropic distribution of "dust" in the induced expanding four-dimensional cosmos. This will be achieved in section 6.1 by using an homogeneous and isotropic distribution of such strings around the bubble.

In fact, the presence of this cloud of strings is crucial to realise Einstein's gravity on the four-dimensional wall. Contrary to the case discussed in section **??**, gravity is **not** localised in the dark bubble scenario. The reason for this is that the presence of sources in the bulk (i.e. the strings) introduces non-normalisable modes of ζ, contrary to what we saw in section **??**. These non-normalisable solutions are required to have the right propagator of the graviton so that it can be interpreted as the right tensorial description of gravity in four dimensions. However, the scale of violation of localisation is controlled by the AdS length scale L and it is microscopic. We will not discuss these aspects further here, but we refer the curious reader to [**?**, **?**].

[63] Converserly, the opposite is true for branes stretching from the inside.

We will now continue delving into the interplay between the bulk behaviour due to the presence of any source in it and how this can affect the induced geometry on the brane. It is time to unmask the true power of Gauss-Codazzi equations (2.50).

5.3 Harmonious gravity and its dimensional transposition

We have previously studied how the behaviour of the brane depends on the position of the matter. The conformally flat direction r is bent *upwards* when the end point of a string is attached to it from the outside. The direction will be *downward* when matter is placed directly **on** top of it. Gravity is present in all bulk space and it is the force responsible for the bending behaviour described above. But what is the nature of the four-dimensional gravity on the braneworld? How does gravity behave on the bubble wall?

In order to gain a deeper understanding on the nature of gravity on and across the bulk space, our starting point will be the Gauss-Codazzi equation (2.50) with full tangential projection described in appendix ??. This expression can be modified to help us read off any induced energy-momentum tensor T_{ab} due to the contents of the bulk and on the brane [?]. The aim is to rewrite this Gauss-Codazzi equation (2.50) so that on one side of the equation we end up with some sort of four-dimensional geometry description. On the other side, something reminiscent of an induced energy-stress tensor. We can start this task by tracing Eq. (2.50),

$$\mathcal{J}_{ab} = R^{(5)}_{\alpha\beta\gamma\delta} e^{\alpha}_{c} e^{\beta}_{a} e^{\gamma}_{d} e^{\delta}_{b} h^{cd} = R^{(4)}_{ab} + \left(K_{ac}K^{c}_{b} - K^{c}_{c}K_{ab}\right), \tag{5.18}$$

where $\mathcal{J}_{ab}$ plays the role of the "pre-induced" energy-momentum tensor. One can continue to massage this term to write:

$$\begin{aligned} \mathcal{J}_{ab} &= e^{\beta}_{a} e^{\delta}_{b} \left(R^{(5)}_{\alpha\beta\gamma\delta} e^{\alpha}_{c} e^{\gamma}_{d} h^{cd}\right) \\ &= e^{\beta}_{a} e^{\delta}_{b} \left(R^{(5)}_{\beta\delta} - R_{\mu\beta\nu\delta} n^{\mu} n^{\nu}\right), \end{aligned} \tag{5.19}$$

where in the second step we have split the five-dimensional metric $g_{\mu\nu}$ into its normal and tangential pieces, with the help of the metric decomposition described in Eq. (2.42).

So far, we have only applied useful geometric relationships between tangential and normal coordinates. Let us now put physics to work for us. Although not specified in previous chapters, we have assumed ($k \gg H$) that the extrinsic curvature was mainly dominated by the AdS_5-vacua scale k. This implies that the extrinsic curvature K_{ab} can be written as

$$K_{ab} = k h_{ab} + \varsigma_{ab}, \tag{5.20}$$

where h_{ab} is the induced FRLW metric on the brane (3.11) and ς_{ab} contains subleading contributions with respect to k [?]. Inserting this approximation into Eq. (5.18) and accounting for the jump in the extrinsic curvature (i.e. the jump in the AdS_5-vacua scale k) yields:

$$\left[\frac{\mathcal{J}_{ab}}{k} - \frac{R^{(4)}_{ab}}{k}\right]^{-}_{+} = \left[3k h_{ab} + \left(2\varsigma_{ab} + \varsigma h_{ab}\right)\right]^{+}_{-} + \dots, \tag{5.21}$$

where ς is the trace of the subleading piece in K_{ab} and the dots ... represent the suppressed[64] contributions of ς/k. If we explicitly open the previous expression (5.21) we will see that:

$$\left(\frac{\mathcal{J}^{-}_{ab}}{k_{-}} - \frac{\mathcal{J}^{+}_{ab}}{k_{+}}\right) - R^{(4)}_{ab}\left(\frac{1}{k_{-}} - \frac{1}{k_{+}}\right) = 3\left(k_{+} - k_{-}\right) h_{ab} + 2\left(\varsigma^{+}_{ab} - \varsigma^{-}_{ab}\right) + \left(\varsigma^{+} - \varsigma^{-}\right) h_{ab}, \tag{5.22}$$

[64] These would go as $\mathcal{O}\left(\frac{\varsigma^2}{k^2}\right)$. we will comment on these terms later.

where we have dropped the suppressed contributions. Note that the right-hand side of Eq. (5.22) contains pieces that may vaguely resemble those inside the second Israel's junction condition (2.67). In fact, if one rewrites $\varsigma_{ab} = K_{ab} - k h_{ab}$ and rearranges the terms to leave $R^{(4)}_{ab}$ alone, it yields:

$$R^{(4)}_{ab} = \left(\frac{k_+ k_-}{k_- - k_+}\right)\Bigg[3\,(k_- - k_+)\,h_{ab} + \left(\frac{\mathcal{J}^+_{ab}}{k_+} - \frac{\mathcal{J}^-_{ab}}{k_-}\right) + \\ + \left(K^+ - K^-\right)h_{ab} + 2\,\left(K^+_{ab} - K^-_{ab}\right)\Bigg]. \tag{5.23}$$

On the left-hand side, we see the induced Ricci tensor. On the right, we find geometrical information about the bulk and how the brane is embedded. This can be trace-reversed to compute the four-dimensional Einstein tensor $G^{(4)}_{ab} = R^{(4)}_{ab} - \frac{R^{(4)}}{2}h_{ab}$, i.e.,

$$G^{(4)}_{ab} = \left(\frac{k_+ k_-}{k_- - k_+}\right)\Bigg[\left(\frac{\mathcal{J}^+_{ab}}{k_+} - \frac{\mathcal{J}^-_{ab}}{k_-}\right) - \frac{1}{2}h_{ab}\left(\frac{\mathcal{J}^+}{k_+} - \frac{\mathcal{J}^-}{k_-}\right) - \\ - 3\,(k_- - k_+)\,h_{ab} - 2\,\underbrace{\left[\left(K^-_{ab} - K^+_{ab}\right) - \left(K^- - K^+\right)h_{ab}\right]}_{\kappa_5\,S_{ab}}\Bigg]. \tag{5.24}$$

We will call this equation the *reversed* Gauss-Codazzi equation. Although it already has a very rich structure, it may be useful to give it a final massage to appreciate its power in all its glory. In fact, this can be done based on two facts:

- The "pre-induced" energy-momentum tensor $\mathcal{J}_{ab}$ contains a piece which represents the AdS$_5$ vacuum k as:
$$\frac{\mathcal{J}_{ab}}{k} = -3\,k\,h_{ab} + \tilde{\mathcal{J}}_{ab}, \tag{5.25}$$
where $\tilde{\mathcal{J}}_{ab}$ stands for any other extrinsic curvature contribution generated by $\chi(r, t, \ldots)$ in Eq. (3.9).
- The most general energy-momentum tensor S_{ab} is given by:
$$S_{ab} = -\sigma h_{ab} + T_{ab}, \tag{5.26}$$
with T_{ab} as the energy-momentum tensor of any field living **on** the brane.

These two points allow us to rewrite Eq. (5.24) as:

$$R^{(4)}_{ab} - \tfrac{1}{2}R^{(4)}h_{ab} + \Lambda_4 h_{ab} = T^{\text{extrinsic}}_{ab} - T^{\text{brane}}_{ab} = T^{\text{effective}}_{ab}, \tag{5.27}$$

where we have used Eqs. (3.20) and (3.23) to rewrite the cosmological constant $\Lambda_4 = \kappa_4 \rho_\Lambda$. The extrinsic energy-momentum tensor is given by [?, ?]:

$$T^{\text{extrinsic}}_{ab} = \left(\frac{k_+ k_-}{k_- - k_+}\right)\left(\frac{\tilde{\mathcal{J}}^+_{ab}}{k_+} - \frac{\tilde{\mathcal{J}}^-_{ab}}{k_-}\right) - \frac{1}{2}h_{ab}\left(\frac{\tilde{\mathcal{J}}^+}{k_+} - \frac{\tilde{\mathcal{J}}^-}{k_-}\right). \tag{5.28}$$

Expression (5.27) has a very rich structure, so let us extract the information out of it. First, it is **undeniable** that one recovers the right four-dimensional Einstein gravity when projecting down the bulk information throught the Gauss-Codazzi equation (2.50). Gravity, which extends all over the bulk space, will effectively behave as the usual Einstein gravity when restricted to the boundary of the bubble at $r = a(\tau)$. Any corrections coming from higher dimensions are suppressed. This can be easily observed by analysing the Friedmann equations of the system while keeping the subleading contributions ς_{ab} described in Eq. (5.21). These quadratic couplings will contribute with corrections of the form $\kappa_4^2 \rho^2\, k^{-2}$. Using the energy hierarchy of the embedding (7.62), this implies that corrections will only take place at "stringy" scales such as [?]:

$$\mathcal{O}\left(\frac{\varsigma^2}{k^2}\right): \quad \frac{\kappa_4^2\rho^2}{k^2} = \frac{\kappa_4 \rho}{k^2} \times \kappa_4 \rho \sim \ell_{10}^4 \rho \times \kappa_4 \rho \sim g_s\, \ell_s^4 \rho \times \kappa_4 \rho. \tag{5.29}$$

The whole Gauss-Codazzi projection (5.24) shows the usual effective low-energy four-dimensional description, which is only disturbed when densities are close to the ten-dimensional Planck scale. This proves that gravity effectively behaves as four-dimensional Einstein gravity on the brane, although it extends to the bulk.

The final comments of this section are devoted to the preservation of harmony on the dark bubble. The right-hand side of expression (5.27) shows the presence of two different contributions to energy-momentum tensor. As we have seen throughout stage **??** of this work, the dark bubble construction is based on two main sets of equations: Einstein's equations in the bulk (with the junction condition(2.67) playing the role of an equation of motion) together with the Gauss-Codazzi equations (2.50). Combining them, the four-dimensional Einstein equation (5.27) can be obtained. This equation can be thought of as the usual $G^{(4)}_{ab} = T^{(4)}_{ab}$ Einstein equation, which is covariantly conserved. This means that it will be conserved under time evolution and any choice of initial conditions will hence be maintained. For example, if we impose the initial condition to be

$$G_{(4)} = T_{\text{effective}} = T_{\text{brane}}, \tag{5.30}$$

this already indicates that $T_{\text{extrinsic}} = 2T_{\text{brane}}$ initially. The reason why this remains true for the subsequent time evolution is twofold:

1. T_{brane} is covariantly conserved on its own by its equations of motion derived from the brane action. Therefore, there cannot be any exchange of energy between T_{brane} and anything else.
2. The total energy-momentum tensor, involving $T_{\text{extrinsic}}$ is also covariantly conserved since it is equal to the 4D Einstein tensor (**??**). Hence, the Einstein equation in the form (5.30) is conserved throughout.

From a physical perspective, this can be understood as follows: let us assume matter fields living **on** the brane, i.e. those of the standard model.[65] We will call this the *visible* sector. Then we can set up initial conditions on the brane and in the bulk to make sure that the induced four-dimensional energy-momentum is exactly the expected one of the visible sector. By the covariant conservation cases discussed above, the previous time evolution will that of the visible sector coupled to four-dimensional gravity. However, in the case that there is an initial mismatch between the energy-momentum tensor of the *visible* sector and that of the induced four-dimensional energy-momentum tensor, we would interpret this as the presence of a *dark* sector.[66] This implies that the *dark* sector will couple to ordinary gravity from a four-dimensional point of view, coupled to the usual gravity in the same way as the visible sector [**?**]. For example, the best known representative of this sector has been extensively discussed in this work; this is dark energy. But this would not be the only constituent. In fact, the rich structure of the dark bubble model suggests that other more dynamical components associated with the extra dimensions could also be switched on. This would be a very interesting aspect to explore in within this research line.

6 Decorating the Induced Expanding Cosmos

THINK HOW TO CONNECT WITH 4D COUNTERPART APPENDIX. OTHER THAN THAT, I THINK THE CHAPTER IS OK.

[65] We will explore in detail this with the electromagnetic field. Other fields, such as the weak and strong forces would require brane configurations a little bit more elaborated.

[66] This sector makes reference to roughly $\sim 95\%$ of the content of the universe that does not interact with the visible one. Dark matter $\sim 25\%$ could be massive fields weakly coupled to gravity. However, we have focused all our attention on another components of the dark sector in this work: Dark energy $\sim 70\%$. An excellent review of some of the aspects of the dark sector can be found in [**?**].

In this section you will learn...

- to populate the four-dimensional cosmos with different types of energy density ρ_i using bulk features
- how the previous process is encoded in the backreacted bulk geometry and projects down to four dimensions by the Gauss-Codazzi equation

In our attempt to enrich the empty cosmos obtained in the first chapters of stage **??**, chapter **??** was devoted to studying how the presence of strings stretching all the way up to the boundary of AdS_5 could induce massive point particles on the bubble wall. These were our first sketches to decorating the four-dimensional universe with more than just dark energy ρ_Λ. Moreover, such strings helped us to realise that their motion in the bulk also dictates the motion of the ending points in the form of geodesics on the four-dimensional cosmos. This encouraged us to delve in the interplay between five and four-dimensional gravity. Although not localised on the brane, the right nature of the four-dimensional Einstein equation arises as a projection of bulk features onto the brane.

In this chapter we will go one step further and try to "paint" the bubble wall with contents well known from our usual four-dimensional description: matter, radiation and waves. This will be achieved through the **backreaction** process studied in chapter **??**. In this way, when the bulk features are accurately encoded within the bulk geometry, so that the *extrinsic* curvature accounts for the deformation produced by them, the *reversed* Gauss-Codazzi equation (5.24) will give the induced *effective* four-dimensional energy momentum tensor T_{ab}^{eff}.

The organisation of this chapter is as follows: Inspired by section 5.2, we will start by populating the bulk geometry with radially stretching strings and matter in the bulk, along the lines [**?**]. We will then see, as predicted in the previous section, that the distribution of strings induces "dust" on the four-dimensional cosmos, while matter in the bulk will contribute with a radiation-like density. In section 6.2 we will review the main results obtained by Danielsson, Panizo and Tielemans in [**?**]. The aim will be to induce the same gravitational waves already discussed in section B from similar five-dimensional ones propagating in the bulk. Equivalently in section 6.3, following the work of Basile, Danielsson, Giri and Panizo [**?**], we will study how higher-dimensional gauge fields in the bulk will be responsible for decorating the induced expanding cosmos with the electromagnetic radiation introduced in B.

6.1 Glaze and splatter[67]

The embedding of the dark bubble into string theory discussed in chapter **??** required a five-dimensional bulk equipped with a non-extremal and unstable Reisser-Nordström hole, which would decay through the emission of D_3-branes. Although we presented the general form of the induced Friedmann-like equation (the potential V(a) in Eq. (7.45)), we focused on the fate of such cosmos.

Let us now turn our attention to the **contents** of such an induced universe. For simplicity, we will assume the black hole charge $Q = 0$, so that we obtain an asymptotic AdS-Schwarzschild geometry in the five-dimensional bulk [**?**]. In addition, we will include subcritical tension D_3-branes as the ones discussed in sec 7.6, so that one naturally obtains an induced dark energy density ρ_Λ on the bubble wall. The line invariant describing the geometry of such bulk is that of Eq. (3.8)

[67] Both make reference to painting techniques. To glaze consists of addying a glossy illumination effect to some layer of the paint, while to splatter is to soil with splashes of the desired color on a surface.

with the metric argument

$$f_{\pm}(r) = 1 + k_{\pm}^2 r^2 - \frac{2\kappa_5 M_{\pm}}{r^2}. \tag{6.1}$$

Inserting this geometry into Eq. (5.24), impossing a pure brane with $S_{ab} = -\sigma h_{ab}$ expanding in the radial direction ($r = a(\tau)$) and raising one index to the whole expression, we obtain:

$$\begin{aligned} {G^a_{\ b}}^{(4)} = &-\frac{6\kappa_5}{a(\tau)^4}\left(\frac{M_+k_- - M_-k_+}{k_- - k_+}\right)\left(\delta^a_0\delta^0_b - \tfrac{1}{3}\sum_{i=1}^{3}\delta^a_i\delta^i_b\right) \\ &- \underbrace{2\,k_+k_-\left(3 - \frac{\kappa_5}{k_- - k_+}\sigma\right)}_{\equiv\kappa_4\rho_\Lambda}\delta^a_b, \end{aligned} \tag{6.2}$$

where the second line has been rewritten using of the dark bubble identifications (3.23). Note that one can directly identify the induced energy-momentum tensor on the brane on the right hand side of the expression (6.2). The second line corresponds to the presence of dark energy ($p = \rho$), while the first line represents a homogeneous distribution of radiation with equation of state $p = \rho/3$, with energy density

$$\rho_{\mathrm{rad}} \sim 2\frac{M}{k\,a^4}, \tag{6.3}$$

where we have taken similar masses in and out, i.e. $M_+ = M_- = M$ and imposed the approximation (7.36). Note that this is only an *effective* form of radiation; the expansion of the bubble will be slowed down by the black hole in the bulk in the same way as a normal four–dimensional cosmology would be slowed down by the presence of radiation. The presence of the black hole does not imply a true form of radiation on the bubble cosmology, such as regular gravitational or electromagnetic radiation. These will be discussed in sections 6.2 and 6.3 respectively.

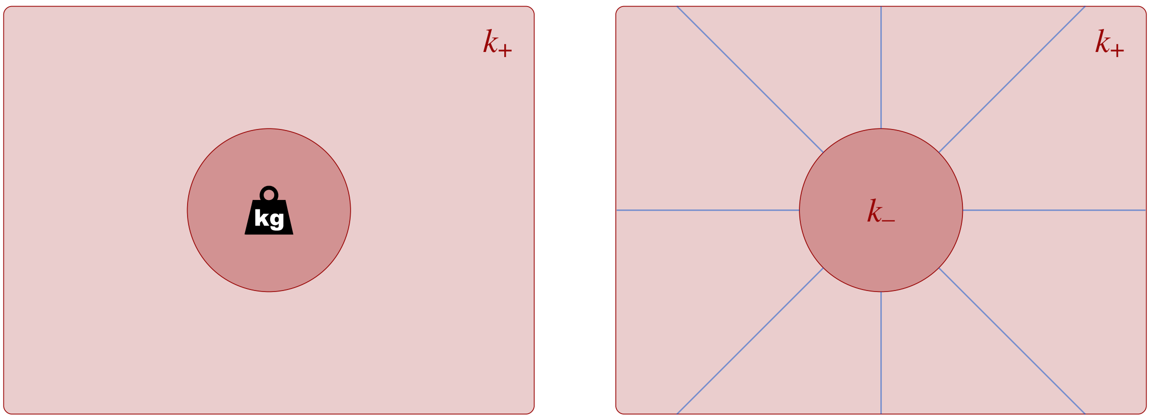

Figure 11: Left: A mass in the bulk, i.e. a black hole, will induce radiation on the bubble wall. Right: A homogeneous cloud of semi-infinite strings, extending radially towards the bulk's boundary, with end points on the brane, will induce an uniform a non-relativistic matter density on the expanding cosmology.

Let us now investigate how to induce non relativistic matter on the brane. This task is slightly more subtle, but we have already seen an introduction in section 5.1. A string stretched along the throat direction r, with its end point on the brane, will induce a positive mass $M_{\mathrm{particle}} = TL$ on the wall of the bubble. It is then natural to think that a homogeneous distribution of such strings ending on the brane will hence induce a uniform distribution of non-relativistic masses on the brane.

If we take the action (5.2) for N_s strings stretching along the throat direction of an empty AdS_5 geometry, we can see that the associated energy-stress tensor is:

$$T^{\mu\nu} = -\kappa_5 \, T_s \sum_i^{N_s} \int \mathrm{d}^2\xi \frac{1}{\sqrt{|g_{\mu\nu}|}} \sqrt{|\omega_{\alpha\beta}|} \, \omega^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \, \delta_i^5 \left(X - x_i\right). \tag{6.4}$$

Again, we can take the static gauge for all the strings (i.e. $\xi^0 = t$ and $\xi^1 = r$) and smear their energy-momentum tensor which would yield:

$$\mathcal{T}^\mu{}_\nu = \kappa_5 \int T^\mu_\nu \mathrm{d}^3\alpha = -\kappa_5 \frac{T_s N_s}{V_3} \frac{1}{r^3} \delta^\mu_\nu, \tag{6.5}$$

with non vanishing components $\mathcal{T}^t_t$ and $\mathcal{T}^r_r$. How do we encode this bulk information so that the *reversed* Gauss Codazzi equation (5.24) ends up projecting its features down to four dimensions? As we saw in section 5.3, it is through the bulk geometric information that we can gain insight into the induced four-dimensional geometry. This requires us to "cook up" a geometry that accounts for the presence of such a *cloud of strings*.[68] In other words: We need to find out how the geometry *reacts* to the presence of these strings. This is the **backreaction** of the original metric to the changes of the energy-momentum tensor sourcing it. The stretching strings will now be the source of the new geometry that we need to describe.

Spanning encore: Backreaction

Let us start by parametrising a backreacted Ansatz for this new geometry of the form:

$$\mathrm{d}^2_{\text{back}} = f_s(r)\mathrm{d}t^2 + f_s(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2, \tag{6.6}$$

where $f_s(r)$ is the metric argument account for the "cloud". This metric should yield an energy-momentum tensor such that its non-zero T^μ_ν components are given by those discussed in Eq. (6.5) plus the five-dimensonal dark energy density ones. Solving Einstein's equation one then finds:

$$f_s(r) = 1 + k^2 r^2 - \frac{2B}{r^2} - \frac{2\kappa_5 \, \alpha}{3r}, \tag{6.7}$$

where B a constant and $\alpha = T_s N_s / V_3$. B would be related to the ADM mass [?, ?] in the bulk space, if it existed. For the case in question, where only strings are present in the bulk, let $B = 0$.

One can now go back to the *reversed* Gauss-Codazzi equation (5.24) and insert this cloud of strings to obtain the induced energy-momentum tensor on the brane. This yields:

$$G^a_b{}^{(4)} = -\frac{2\kappa_5}{a(\tau)^3}\left(\frac{\alpha_+ k_- - \alpha_- k_+}{k_- - k_+}\right)\delta^a_0\delta^0_b - \underbrace{2\,k_+k_-\left(3 - \frac{\kappa_5}{k_- - k_+}\sigma\right)}_{\equiv \kappa_4 \rho_\Lambda}\delta^a_b, \tag{6.8}$$

where again, the second piece corresponds to the dark energy induced on the brane, while the first part represents an energy density ρ that dilutes as $\sim a^{-3}$ and pressureless. This corresponds to non-relativistic matter or dust, as presented in section ??.

Furthermore, the results shown in Eq. (5.8) can be recovered. Similary to what it was done above for matter in the bulk, one can approximate α and k to read off:

$$\rho_{\text{matter}} \sim \underbrace{\frac{T_s}{k}}_{M_{\text{part}}} \frac{N_s}{V_3} \frac{1}{a^3}, \tag{6.9}$$

[68] This metric was already constructed a long time ago. We will present the main derivation steps as a pedagogical resource in this thesis. See [?, ?, ?, ?] for further information.

which corresponds to the energy density distribution produced by N_s massive point particles distributed homogeneously and isotropically throught the four-dimensional cosmos.

There are a few physical notes to be made before the end of this section. Our dear reader has surely noticed the minus sign in front of the four-dimensional contributions M_- and α_- for both the bulk black hole and the string cloud. In fact, this is the same contribution with negative sign as any field placed **on** the brane. To understand why this is the physically correct sign, let us start again with the AdS-Schwarzschild background example described above. If we assume the same mass parameter M on both sides of the brane, we have shown that the induced cosmology is decorated with radiation, with the energy density $\rho_{\rm rad}$ positive and proportional to $M/k_+ - M/k_- > 0$. Now, as a thought experiment, let us imagine that the black hole is replaced by a shell with the same mass M. The metric outside of the spherically symmetric shell remains the same, as does the induced four-dimensional physics. If we increase the radius of the shell so that it touches the brane, then the bulk matter can be deposited on the braneworld with no change in the physics. Eventually, all the mass would have been placed on the brane. Therfore, the interior of the brane will be pure AdS. The term $-M/k_-$ should then be removed and replaced by the extra energy density on the brane, i.e. $\rho_{\rm brane}$. As we have shown in Eq. (5.27), it will contribute with the negative sign noted earlier. This shows what is going on. It is a mistake to conclude that adding matter to the brane will result in a negative energy density in four dimensions without any other consequences. What happens is that the added matter necessarily back reacts on the five-dimensional spacetime, yielding a contribution to the extrinsic curvature that gives an overall positive four-dimensional energy density ρ_i [**?**].

6.2 Shake and stir[69]

We have extensively discussed a uniform background of gravitational perturbations in section B. How can we induce such waves from the higher-dimensional embedding that the bubble represents? The most logical approach to this puzzle is to start by studying geometrical fluctuations in the AdS_5 bulk space. We will consider the most general possible perturbation: In addition to a time-dependent piece, one can also find variations along the throat-direction r [**?**]. These will later be required to behave in such a way that they evolve with the bubble, i.e. one can recover the four dimensional description (B.14) when "slicing" the five-dimensional ones to the top of brane, located at $r = a(\tau)$.

Let us start by studying the tranverse and traceless perturbations that propagates in the AdS_5 bulk. The line invariant describing this geometry is given by:[70]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\gamma_{ij} + \varepsilon P_{ij}(t, r, \{\alpha\})\right)\mathrm{d}\Omega_3^2, \tag{6.10}$$

where $\mathrm{d}\Omega_3^2$ is as in Eq. (B.7), γ_{ij} is the unit matrix for the three spatial directions and $P_{ij}(t, r, \{\alpha\})$ is the transverse and traceless tensorial perturbation and ε is a small parameter. It is easy to check that the induced metric on the brane in conformal coordinates then corresponds to (B.5).

As we did in section B, the transverse and traceless perturbations can be decomposed into S^3 harmonics (see appendix **??**). Inspired by the four-dimensional case, we will first solve for a single mode (i.e. a single wave with wave number n) $P_{ij} = \zeta_{5\mathrm{D}}(\eta, r)Y_{ij}$. The resulting gravitational wave

[69] To avoid the pigments of the painting to sit at the bottom of the container, so the applied layer is homogenous in color and bright.

[70] Note that in our conventions the coordinates t and r have the dimension of length and the coordinates α^i are dimensionless.

equation in this geometry is [?]:

$$\frac{\partial^2\zeta_{5\mathrm{D}}}{\partial t^2}-f^2\frac{\partial^2\zeta_{5\mathrm{D}}}{\partial r^2}-\frac{f}{r}\left(2+4k^2r^2+f\right)\frac{\partial\zeta_{5\mathrm{D}}}{\partial r}+\frac{n^2-1}{r^2}f\zeta_{5\mathrm{D}}=0, \tag{6.11}$$

where we assume empty[71] AdS_5 with $f(r)=1+k^2r^2$ and $\zeta_{5D}(\eta,r)$ represents the time and radial dependence of the wave. This determines the evolution of a gravitational wave throughout the AdS_5 bulk.

In order to make contact with the four-dimensional conformal time η (B.6) in the low-dimensional case (B.5), it will be useful to work with coordinates that provide a simple five-dimensional uplift of Eq. (B.11). These are:

$$w=\cos(kt)=\frac{\cos\eta}{\sqrt{1+\left(\frac{H}{k}\right)^2\sin^2\eta}}=\cos\eta+\mathcal{O}\left(\left(\tfrac{H}{k}\right)^2\sin^2\eta\right) \tag{6.12}$$

where the relation between bulk time and conformal time on the brane (3.12) has been used. If we want to connect back to the four-dimensional description, that is, $t\to\eta$, the relevant limit approximation implies $H/k\ll 1$, where one has $w\approx v$. In particular,

$$kt=\eta+\mathcal{O}\left(\left(\tfrac{H}{k}\right)^2\sin(2\eta)\right). \tag{6.13}$$

Note that this relation is only meaningful once the bubble has nucleated. This occurs at $\eta=-\pi/2$ as commented on the four-dimensional treatment (B.10). It corresponds to a bulk time $t\approx-\pi/2k$. The bulk time has in principle the full range $-\infty<t<+\infty$. However, one has to take into account the composite inside-outside geometry and the non-eternity of the bubble [?]. The time range of outside geometry is $-\infty<t_+<0$ where the limit $t_+\to 0$ corresponds to the bubble having eaten all of AdS_5^+. The inside geometry is only present once a bubble has nucleated after which it persists forever. Therefore $-\pi/2k<t_-<+\infty$.

Inserting Eq. (6.12) into the five-dimensional gravitational wave equation in the bulk (6.11) yields:

$$\begin{aligned}k^2\left[(1-w^2)\frac{\partial^2\zeta_{5\mathrm{D}}}{\partial w^2}-w\frac{\partial\zeta_{5\mathrm{D}}}{\partial w}\right]-f^2\frac{\partial^2\zeta_{5\mathrm{D}}}{\partial r^2}-\\-\frac{f}{r}\left(2+4k^2r^2+f\right)\frac{\partial\zeta_{5\mathrm{D}}}{\partial r}+\frac{n^2-1}{r^2}f\zeta_{5\mathrm{D}}=0.\end{aligned} \tag{6.14}$$

This previous equation will control the dynamics of the bulk gravitational perturbations both inside and outside the bubble. And also those of the induced gravitational waves on the brane. In fact, this equation needs to be complemented by suitable boundary conditions [?]. These are:

1. when $\zeta_{5\mathrm{D}}$ is restricted to the brane (i.e. the to-be "induced wave" ζ_{ind}), we require that ζ_{ind} coincides with the four-dimensional wave ζ (B.14), up to leading order in H/k. This requires to impose the boundary conditions at the location of the bubble $r=a(w)$, assuming $ka\gg 1$, i.e.

$$\zeta_{\mathrm{ind}}(w)\equiv\zeta_{5\mathrm{D}}\left(w,\tfrac{1}{H\sqrt{1-w^2}}\right)=\zeta(v)+\mathcal{O}\left(\left(\tfrac{H}{k}\right)^2\right). \tag{6.15}$$

2. Furthermore, as we did in the four-dimensional case, we require the waves to be sourceless. This means that the wave should fade out when approaching the centre of the bubble, i.e.

$$\lim_{r\to 0}\zeta_{5\mathrm{D}}(w,r)=0. \tag{6.16}$$

[71] We have assumed this for simplicity. One could also consider a five-dimensional bulk black hole as in chapter ??. As we have seen in section 6.1, each contribution will come separately in the expansion. To avoid long equations, we stick to an empty AdS and its gravitational perturbations.

Note that, although we are talking about boundary conditions for a wave, there are **two** different waves: inside and outside the bubble. So it seems natural to insist that *only* the inside wave decays and *only* the outside wave does not blow up as $r \to \infty$. Note however that their evolution is governed by the same wave equation (6.14), up to a difference in k. The first boundary condition at the location of the brane must be imposed for **both** of them. This boundary condition uniquely fixes the inside *and* the outside waves [**?**]. This means that if the outside wave were extrapolated in the supposed limit $r \to 0$, it would still vanish.

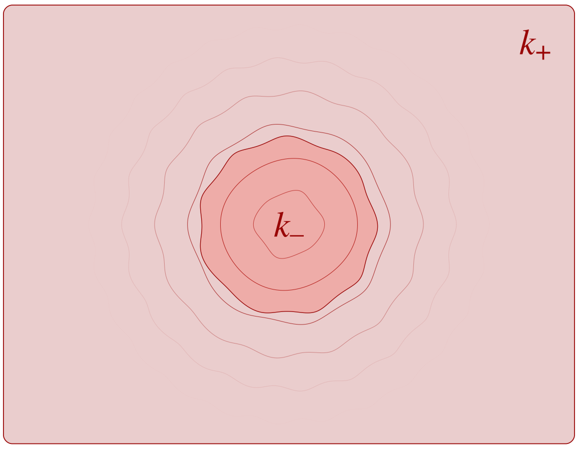

Figure 12: Gravitational waves in the bulk are sourceless, i.e. they vanish at the centre of the bubble. They "freeze out" at large r and behave like the induced wave when $r = \left(H\sqrt{1-w^2}\right)^{-1}$.

Solving for Eq. (6.14) and considering both boundary conditions (6.15, 6.16), re-inserting the global time t through Eq. (6.12) and massaging the resulting hypergeometric functions in terms of Chebyshev polynomials (A.19, A.20), we can write:

$$\begin{aligned}\zeta_{5\mathrm{D}}(t,r) &= D(r,n)\left[C(r,n)\,\cos\left((n+1)kt\right) + \sin(kt)\sin(nkt)\right],\\ \tilde{\zeta}_{5\mathrm{D}}(t,r) &= D(r,n)\left[C(r,n)\,\sin\left((n+1)kt\right) - \sin(kt)\cos(nkt)\right],\end{aligned} \tag{6.17}$$

where

$$D(r,n) = \frac{(kr)^{n-1}}{\left(1+k^2r^2\right)^{\frac{n-1}{2}}}, \qquad C(r,n) = \frac{\frac{1}{2}(1+n)(2-n)+k^2r^2}{(n+1)\left(1+k^2r^2\right)}. \tag{6.18}$$

Just as in the case of four-dimensional gravitational waves, one might also be tempted to use the large n, late time ($t \to \infty$) limit to find an uplift of the gravitational waves in a flat universe (B.14). However, one must also take into account that, to reach this flat limit, the wave must be considered close to the bubble. This implies an additional condition: the limit in which kr is large;

$$\begin{aligned}\zeta_{5\mathrm{D}}(t,z) &= \frac{-\frac{1}{2}n^2+k^2r^2}{nk^2r^2}\cos\left(nkt\right) + kt\sin(knt),\\ \tilde{\zeta}_{5\mathrm{D}}(t,z) &= \frac{-\frac{1}{2}n^2+k^2r^2}{nk^2r^2}\sin\left(nkt\right) - kt\cos(knt).\end{aligned} \tag{6.19}$$

Note that, when n is large, there are corrections beyond the four dimensional wave at high momentum (or energy carried by the wave) $p = n/r$. For example, at a fixed position in the throat, i.e. kr, there is a competition between n and k in the first term. The induced four-dimensional wave receives a modification when $kr \sim n$, which translates to a momentum p of the same order of magnitude than the AdS$_5$ scale, i.e. $p \sim n/r \sim k$. In this way, one can therefore conclude that the AdS-scale k represents a UV-scale where new physics appears on the four-dimensional cosmology [?]. This is agreement with the results that we previously discussed in section 7.7.

In the same spirit as we did for four-dimensional gravitational waves in section B, we can compose an uniform background of waves that fills in the five-dimensional bulk. Following the usual average-isotropy process,[72] we find an isotropic stress tensor consisting of three pieces; curvature, radiation, and flux:[73]

$$\langle T^{\mu}{}_{\nu}\rangle_{\rm iso} = \langle T^{\mu}{}_{\nu}\rangle_{\rm c} + \langle T^{\mu}{}_{\nu}\rangle_{\rm r} + \langle T^{\mu}{}_{\nu}\rangle_{\rm f}, \tag{6.20}$$

where each component is given by

$$\begin{gathered}\langle T^{\mu}{}_{\nu}\rangle_{\rm c} = \tfrac{1}{8\kappa_5 r^2}\left(7-\tfrac{n^4}{2k^4r^4}\right)\begin{pmatrix}1&0&0&0&0\\0&\frac{1}{3}&0&0&0\\0&0&\frac{1}{3}&0&0\\0&0&0&\frac{1}{3}&0\\0&0&0&0&1\end{pmatrix} + \mathbb{I}_{5\times5}\,\mathcal{O}\left(\tfrac{n^2}{k^2z^2}\right),\\ \langle T^{\mu}{}_{\nu}\rangle_{\rm r} = \tfrac{k^2n^2t^2}{4\kappa_5 r^2}\begin{pmatrix}-1&0&0&0&0\\0&\frac{1}{3}&0&0&0\\0&0&\frac{1}{3}&0&0\\0&0&0&\frac{1}{3}&0\\0&0&0&0&0\end{pmatrix},\qquad \langle T^{\mu}{}_{\nu}\rangle_{\rm f} = \tfrac{n^2}{8\kappa_5 r^2}\begin{pmatrix}0&0&0&0&-\frac{2t}{k^2r^3}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\2k^2tr&0&0&0&0\end{pmatrix},\end{gathered} \tag{6.21}$$

Note that the momentum of the wave is $p\sim n/r$, appears as a correction in the curvature contribution, but not in the radiation piece. The flux contribution shows how these gravitational waves represent a net flow of energy in the positive r bulk's direction.

Trembling encore: Backreaction

In section B, we reviewed how the geometry can be modified when you go up to first order in the gravitational perturbation expansion (B.1). We had a geometry at leading order and on top of it, we put a small perturbation of this, representing a propagating wave (B.14). One can then ask how the whole geometry will be deformed by the presence of these waves, similar to what we did in section 6.1. Had we not thought of the possible changes in the geometry caused by the energy density of gravitational waves, we would have said that they were *probe*. However, let us explore how the geometry *backreacts* to its presence. For example, an isotropic and homogeneous background of gravitational waves (6.20) can be captured by a backreacted geometry with Ansatz [?]:

$$\mathrm{d}s^2_{\rm back} = \left(g^{(0)}_{\mu\nu} + \varepsilon^2\, g^{(2)}_{\mu\nu}\right)\mathrm{d}x^\mu \mathrm{d}x^\nu \approx -f_1(t,r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f_2(t,r)} + r^2\mathrm{d}\Omega_3^2. \tag{6.22}$$

with

$$\begin{aligned} f_1(t,r) &= 1+k^2r^2+\varepsilon^2\left(q_1-q_2k^2t^2\right),\\ f_2(t,r) &= 1+k^2r^2+\varepsilon^2\left(q_1-q_3k^2t^2\right).\end{aligned} \tag{6.23}$$

The set of coefficients $\{q_i\}\in\mathbb{Z}$ will be determined later. Note that when the expansion parameter $\varepsilon\to0$, we recover the AdS$_5$ background (6.10). The *backreaction* piece $g^{(2)}_{\mu\nu}$ is given by the ε^2 coefficient in the small ε expansion, in particular

$$g^{(2)}_{tt} = q_1 - q_2k^2t^2, \qquad\qquad g^{(2)}_{rr} = \frac{q_3k^2t^2-q_1}{(1+k^2r^2)^2}. \tag{6.24}$$

[72] This is discussed in section B.
[73] Note that the asymmetry in the indices is caused the choice of covariant-contravariant. The tensor is symmetric when both are of the same type.

How do you fix the q_i-coefficients? As we want the Ansatz (6.22) to capture the gravitational waves energy-momentum tensor (6.20), one then has to compute the second order Einstein tensor (B.4) and compare to Eq. (6.20). This yields

$$q_1 = -\frac{7}{24}, \qquad q_2 = -\frac{n^2}{6}, \qquad q_3 = \frac{q_2}{2} = -\frac{n^2}{12}. \tag{6.25}$$

The change in the bulk geometry described by the Ansatz (6.22) will affect the dynamics of the expanding bubble. In fact, as this backreacted geometry captures the presence of the gravitational waves (6.19), the junction condition 3.13 will be sensitive to this and reflect these changes through the projection of the induced energy-momentum tensor T_{ab}. By computing the extrinsic curvature (2.47), inserting this in the junction condition (3.13) and massaging the terms to the left and right, one then finds (in the large k limit) that the induced Friedmann-like equation on the brane is:

$$\begin{aligned}\delta^a_b\left(3H^2+\frac{1}{a^2}\right)+\frac{2}{a^2}\delta^a_0\delta^0_b &= \Lambda_4\delta^a_b+\kappa_4\sigma+\\ +\varepsilon^2 &\underbrace{\left(\frac{7}{24\kappa_4 a^2}(3\delta^a_0\delta^0_b+\delta^a_i\delta^i_b)-\frac{n^2}{4\kappa_4 H^2 a^4}\left(\delta^a_0\delta^0_b-\frac{1}{3}\delta^a_i\delta^i_b\right)\right)}_{=T^a_b},\end{aligned} \tag{6.26}$$

where σ corresponds to the tension of the brane. Note that the second line is the induced four-dimensional energy-momentum tensor T^a_b which we found in Eq. (B.16). Alternatively, one can use the Gauss-Codazzi equation (5.24) to obtain the same result in the form [?]

$$\begin{aligned}G^{(4)\,a}_b =& -\Lambda_4\delta^a_b-\\ &-\varepsilon^2\left(\frac{7}{24\kappa_4 a^2}(3\delta^a_0\delta^0_b+\delta^a_i\delta^i_b)-\frac{n^2}{4\kappa_4 H^2 a^4}\left(\delta^a_0\delta^0_b-\frac{1}{3}\delta^a_i\delta^i_b\right)\right).\end{aligned} \tag{6.27}$$

This already shows the power of the junction condition: First, it is aware of changes in the bulk geometry, i.e. the Ansatz to capture the backreaction of the five-dimensional spacetime to the presence of an uniform background of gravitational perturbations. This translates to changes in the extrinsic curvature of the brane K_{ab}, which are reflected in the projection of bulk features onto the brane surface, i.e. the induced expanding cosmology. This implies that the junction conditions take care of gravitational perturbations in the bulk, providing a clear connection between the cosmological features of the bulk and and the boundary bubble. But as we will see in the next section, gravitational waves bulk-boundary dynamics are not the only feature controlled by the Gauss-Codazzi equation. Any feature, living on the brane or in the bulk, will pass through the Gauss-Codazzi toll, modifying the bubble dynamics and hence, the induced geometry on its boundary. Gravity, which sees *everything*, will be the conductor of such ballet.

6.3 Chiaroscuro[74]

Let us now study how gauge fields in the AdS_5 bulk, which also enjoy a ten-dimensional supergravity interpretation, can give rise to their four dimensional counterparts (B.27) on the boundary of the bubble [?].

While Ramond-Ramond fields may seem appealing candidates for generating gauge fields on the brane,[75] there is an even more straightforward candidate for this task; the gauge field $\mathfrak{F}_{\mu\nu}$ represented in the DBI-piece of the D_3-action (7.16):

$$S_{\mathrm{DBI}} = -T_3\int \mathrm{d}^4\xi\sqrt{|\left(P[G+2\pi\alpha'\mathfrak{F}]\right)|}. \tag{6.28}$$

[74]Painting technique, used to create strong contrast between light and dark tones.
[75]We will not consider any possible contribution they may make in this section.

If we explicitly expand (6.28) up to second order for small α', we obtain:[76]

$$S_{\rm DBI} = -\int d^4x\sqrt{|g_{(4)}|}\left(T_3 + \underbrace{T_3\,\frac{(2\pi\alpha')^2}{4}}_{1/(4g^2)}\mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu} + \mathcal{O}\left(\mathfrak{F}\mathfrak{F}\right)^2\right), \tag{6.29}$$

where $g^2 = 2\pi g_s$ is the gauge coupling. The second piece in Eq. (6.29) can then be identified with the action of electromagnetism in four dimensions (B.17). In turn, the gauge field on the brane can be written as:

$$2\pi\alpha'\,\mathfrak{F}_{\mu\nu} = 2\pi\alpha'\,F_{\mu\nu} + \mathfrak{B}_{\mu\nu}, \tag{6.30}$$

where $F_{\mu\nu}$ is the usual Maxwell field strength and $\mathfrak{B}_{\mu\nu}$ is the antisymmetric two-form under which extended objects in string theory can be charged. Note that any possible change in the gauge of this field $\mathfrak{B}_{\mu\nu}$ should be accordingly compensated for a shift in $F_{\mu\nu}$. In this way, $\mathfrak{F}$ remains gauge invariant. The bulk field strength $\mathfrak{H} = d\mathfrak{B}$ is of course also gauge invariant.

If we now perform a variation of the DBI-action (6.29) with respect to $\mathfrak{B}$, we then obtain a source term for the field strength $\mathfrak{H}$ given simply by $\mathfrak{F}$. This can be understood as follows: The D_3-expanding brane is equipped with the gauge invariant $\mathfrak{F}$ field strength. This will be the source of the field strength in the bulk $\mathfrak{H}_{\mu\nu\rho}$. To make our computations easier, we can make two simplifications [?]:

1. There is no $\mathfrak{H}_{\mu\nu\rho}$ flux inside the space enclosed by the D_3-brane.
2. The outer $\mathfrak{H}_{\mu\nu\rho}$ flux is purely due to the presence of the electromagnetic field $\mathfrak{F}$ on the brane.

With this description in hand, the total action describing the dynamics of the tandem bulk-brane in the presence of the source field strength $\mathfrak{F}$ and the bulk flux $\mathfrak{H}$ can be given by:

$$S_5[g_{\mu\nu}, \mathfrak{F}_{\mu\nu}, \mathfrak{H}_{\mu\nu\rho}] = S_{\rm bulk}[g_{\mu\nu}, \mathfrak{H}_{\mu\nu\rho}] + S_{\rm DBI}[P[G_{\mu\nu}, \mathfrak{F}_{\mu\nu}]], \tag{6.31}$$

with

$$\begin{aligned} S_{\rm bulk}[g_{\mu\nu}, \mathfrak{H}_{\mu\nu\rho}] &= \tfrac{1}{2\kappa_5}\int d^5x\sqrt{|g_{(5)}|}\left(R - \tfrac{1}{12 g_s}\mathfrak{H}^2\right), \\ S_{\rm DBI}[P[G,\mathfrak{F}]] &= -T_3\int d^5x\,\delta(r - a[\eta])\,\sqrt{|\left(g_{(4)} + 2\pi\alpha'\mathfrak{F}\right)|}. \end{aligned} \tag{6.32}$$

Here we will work in the five-dimensional Einstein frame, with fixed dilaton coupling $e^{-\phi} = g_s^{-1}$. The compact dimension moduli are assumed to be stabilised. Furthermore, note that the D_3-brane will be localised at $r = a(\eta)$, where η the conformal time on the brane. The line invariant describing the AdS$_5$ geometry is that of the Poincaré patch (3.5):[77] This implies the following relationship of the metric determinants:

$$\sqrt{|g_{(5)}|} = \tfrac{1}{kr}\sqrt{|g_{(4)}|} \tag{6.33}$$

If we expand for small α', as we did in expression (6.29) and then vary the action with respect to $\mathfrak{B}$, we find the equations of motion

$$\partial_r \mathfrak{H}^{r\mu\nu} = \frac{2\kappa_5\,kr}{\alpha'\,\pi^2}\,\mathfrak{F}^{\mu\nu}\,\delta\left(r - a[\eta]\right), \tag{6.34}$$

[76] Note that we will be using Greek indices even though we refer to the four usual coordinates to describe the expanding cosmos, normally represented by Latin indices. This is because we will study the D_3-brane and its contents from a five-dimensional point of view in this section.

[77] The computations in this section are performed for large ka-limit, which means late-time induced cosmologies. This implies that the curvature of the bubble, therefore that of the bulk space, is neglegible.

which integrated across the brane at $r = a(\eta)$ gives

$$\Delta \mathfrak{H}^{r\mu\nu}\big|_{r=a} = \mathfrak{H}^{r\mu\nu}\big|_{r=a} = \frac{2\kappa_5\, ka}{\alpha'\, \pi^2}\; \mathfrak{F}^{\mu\nu}\big|_{r=a}\,. \tag{6.35}$$

This equation describes that the D_3-brane is equipped with the invariant gauge field strength $\mathfrak{F}_{\mu\nu}$ and mediates the decay of the two AdS_5 vacua. Furthermore, it is also responsible of sourcing the field strength $\mathfrak{H}_{\mu\nu\rho}$ located *outside* the bubble. And how do we identify the field strength (6.30) with the usual Maxwell one described in (B.25)? They are just the same with different coordinate parametrisations, i.e. conformal in the four-dimensional case and global in the five-dimensional spacetime, related by expression (6.12).

We can then reproduce our computations from section B. To be specific, we focus on electromagnetic waves propagating along z, with the electric field polarized along x, and the magnetic field polarized along y. Given a wave propagating along z direction, and $\mathbf{E}, \mathbf{B}$ respectively polarised along $\{x, y\}$ directions, we have

$$\mathfrak{F}^{tx} = \frac{\mathcal{E}(t,z)}{k^2 a^4}, \quad \mathfrak{F}^{xz} = \frac{\mathcal{E}(t,z)}{k^2 a^4}, \tag{6.36}$$

where $\mathcal{E}$ is a dimensional function that determines the amplitude of the wave. These waves act as sources for the $\mathfrak{H}_{\mu\nu\rho}$ field strength in the bulk. In fact, imposing the absence of this inside of the bubble, the equation of motion (6.35) yields [?]:

$$\begin{aligned} \mathfrak{H}^{rtx}\big|_{r=a} &\equiv \mathfrak{h}_1(t,a,z) = \frac{2\kappa_5\, ka}{\alpha'\, \pi^2}\mathfrak{F}^{tx} = \frac{2\kappa_5}{\alpha'\, \pi^2 k}\frac{\mathcal{E}(t,z)}{a^3}, \\ \mathfrak{H}^{rxz}\big|_{r=a} &\equiv \mathfrak{h}_2(t,a,z) = \frac{2\kappa_5\, ka}{\alpha'\, \pi^2}\mathfrak{F}^{xz} = \frac{2\kappa_5}{\alpha'\, \pi^2 k}\frac{\mathcal{E}(t,z)}{a^3}, \end{aligned} \tag{6.37}$$

In figure 13 we can see how the electromagnetic waves (hence the $\mathfrak{F}$ field strength) on the brane source the $\mathfrak{H}$-field in the bulk above the brane. The $\mathfrak{H}$-field must solve the equation of motion (6.37)

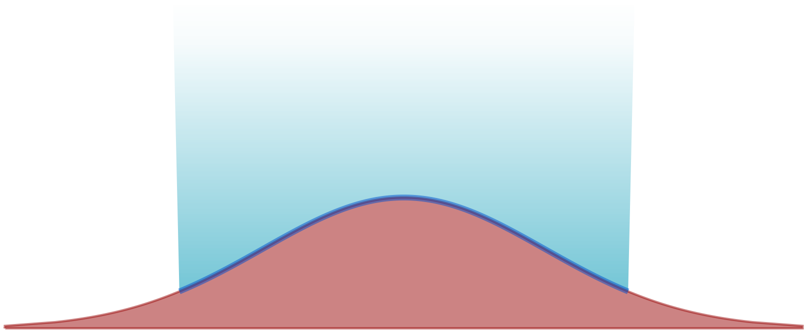

Figure 13: A sketch of a portion of the dark bubble carrying an electromagnetic field strength $\mathfrak{F}$, which sources the field strength $\mathfrak{H}$ in the bulk. The junction conditions imply a jump in the $\mathfrak{H}$-field encoded in Eq. (6.35). The bending is caused by the backreaction of the $\mathfrak{H}$-field (see section 5.2 for further details).

and the Bianchi identities associated with such a gauge field, away from the brane.[78] The Bianchi identities imply:

$$\begin{aligned} \star \mathrm{d} \star \mathfrak{H} = 0 &\to \nabla_\alpha \mathfrak{H}^{\alpha\beta\gamma} = 0\,, \\ \star \mathrm{d}\mathfrak{H} = 0 &\to \epsilon^{\alpha\mu\nu\lambda}{}_{\kappa}\partial_\alpha \mathfrak{H}_{\mu\nu\lambda} = 0\,, \end{aligned} \tag{6.38}$$

[78] Again, for simplicity, we assume that the dilaton as well as any other moduli of the background are fixed.

where $\star$ is the five-dimensional Hodge star and d the usual exterior derivative. In order to make the presentation of the results as smooth as possible to spare our dear reader pain and tears, we introduce a non-trivial time dependence[79] in the bulk field strength $\mathfrak{H}$. This can be done by adding an extra "leg" to this field, in the form of:

$$\mathfrak{H}^{txz} \equiv \mathfrak{h}_3(t,r,z) \tag{6.39}$$

Inserting Eq. (6.39), together with Eq. (6.36) into the equation of motion (6.35) yields

$$\begin{aligned}\left(\partial_r + \frac{3}{r}\right)\mathfrak{h}_1 + \partial_z \mathfrak{h}_3 = 0,\\ \left(\partial_r + \frac{3}{r}\right)\mathfrak{h}_2 + \partial_t \mathfrak{h}_3 = 0,\\ \partial_z \mathfrak{h}_2 - \partial_t \mathfrak{h}_1 = 0,\end{aligned} \tag{6.40}$$

while the Bianchi identity (6.38) becomes

$$r^2(\partial_t \mathfrak{h}_2 - \partial_z \mathfrak{h}_1) + \, k^4\,\partial_r\left(r^6 \mathfrak{h}_3\right) = 0 \tag{6.41}$$

Let us now explicitely solve for the $\mathfrak{H}$-field strength. We make the ansatz:

$$\mathfrak{h}_1 = \frac{f_1(t,\Xi,z)}{k^2 r^3}, \qquad \mathfrak{h}_2 = \frac{f_2(t,\Xi,z)}{k^2 r^3}, \qquad \mathfrak{h}_3 = \frac{f_3(t,\Xi,z)}{k^3 r^4}, \tag{6.42}$$

with $f_i(t,\Xi,z)$ functions

$$f_i(t,\Xi,z) \equiv \alpha_i(t,\Xi)\,\sin\left(kn\,(t+z)\right) + \beta_i(t,\Xi)\,\cos\left(kn\,(t+z)\right), \tag{6.43}$$

where the functions α_i and β_i are dimensionless. The parameter Ξ is given by:

$$\Xi = \frac{n}{2k^3 t r^2}. \tag{6.44}$$

This whole massage of the coordinates and the Ansatz itself deserves a good explanation [?]:

1. First, note that an observer on the brane, with conformal time $\eta = kt = -(Hr)^{-1}$, will experience the following relation for the argument within the trigonometric pieces of Eq. (6.43):
$$\frac{1}{knt} \sim \frac{rH}{n} \sim \frac{H}{p}, \tag{6.45}$$
where p is the energy of the wave. Similar to gravitational waves (see section 6.2), the momentum of the wave is greater than the Hubble horizon $p \gg H$, so $(knt)^{-1} \ll 1$ becomes a natural variable for expansion.

2. Furthermore, the choice of the new variable Ξ follows the same argument, as:
$$\Xi = \frac{n}{2k^3 t r^2} \sim \frac{nH}{k^2 r} \sim \frac{pH}{k^2} \sim \frac{H}{k} \cdot \frac{p}{k} \sim \frac{p}{M_{pl}}, \tag{6.46}$$
where in the last step, we have used the relations from section 7.7, exactly Eq. (7.61) and $H \sim \rho_\Lambda^{1/2}$ with ρ_Λ as in Eq. (7.63). M_{pl} is the Planck mass in four dimensions, restored to the correct dimensions in the previous expressions. This implies that Ξ is a small parameter compared to the four-dimensional Planck mass, provided that $H \ll p \ll M_{\text{pl}}$. This is the regime that is relevant for our computations.

[79]Science is trial and error. In the early stages of this research, we observed that the absence of such a non-trivial dependence would cause the four-dimensional observer to detect an unphysical change in the energy density of the brane. As the aim of the dark bubble model is to make contact with reality, we rejected this possibility. See [?] for further information.

By imposing both restrictions discussed above, one can expand for small Ξ and small momentum p and solve the equations of motion (6.40) and Bianchi identities (6.38) order by order. In this way, we find the solution, up to order $(knt)^{-1}$ but in all orders in Ξ to be given by:

$$\begin{aligned}\mathfrak{h}_1(t,\Xi,z) &= \frac{\alpha}{k^2r^3}\sin\left(\vartheta(t,z)\right) + \frac{2\beta}{k^2r^3}\left[\sin\left(\tfrac{\Xi}{2}+\vartheta(t,z)\right) - \sin\left(\vartheta(t,z)\right)\right],\\ \mathfrak{h}_2(t,\Xi,z) &= \mathfrak{h}_1(t,\Xi,z) - \frac{\beta\,\Xi}{k^3ntr^3}\cos\left(\tfrac{\Xi}{2}\right)\sin\left(\vartheta(t,z)\right),\\ \mathfrak{h}_3(t,\Xi,z) &= \frac{2\beta\,\Xi}{nk^3r^4}\sin\left(\tfrac{\Xi}{2}+\vartheta(t,z)\right),\end{aligned} \tag{6.47}$$

where $\vartheta(t,z) = kn(t+z)$ and α and β are now constants.[80] We note that only $\mathfrak{h}_2$ requires a correction at $(knt)^{-1}$ to the equations of motion up to leading order in $(knt)^{-1}$.

With solutions (6.47) at hand, we only need to perform one extra step to find the energy-momentum tensor associated with the $\mathfrak{H}_{\mu\nu\rho}$ field in the bulk. Varying the action (6.31) with respect to $g_{\mu\nu}$ we find:

$$T_{\mu\nu} = \frac{1}{\kappa_5}\left(-\frac{1}{2}g_{\mu\nu}\mathfrak{H}^2 + 3\,\mathfrak{H}_{\mu\rho\omega}\mathfrak{H}_{\nu}{}^{\rho\omega}\right). \tag{6.48}$$

Similar to what we have already done for all previous homogeneous backgrounds in sections B and 6.2, we must consider for all possible waves travelling in all directions, incoming and outgoing ones with any wavenumber n. Homogenising (A.30) and isotropising (A.30) yields:

$$\langle T^{\mu}_{\nu}\rangle_{\rm EM} = \frac{3\alpha^2}{\kappa_5\,k^2r^4}\begin{pmatrix} -1 & 0 & 0 & 0 & \frac{-\beta}{\alpha\,tk^4r^3}\\ 0 & \frac{1}{3} & 0 & 0 & 0\\ 0 & 0 & \frac{1}{3} & 0 & 0\\ 0 & 0 & 0 & \frac{1}{3} & 0\\ \frac{\beta\,r}{\alpha t} & 0 & 0 & 0 & 0\end{pmatrix}. \tag{6.49}$$

It is really easy to see the similarities and the differences respect to that of the gravitational waves (6.20). Contrary to that case, here we do not observe any contribution in the form of curvature. We do identify two main pieces: That of the diagonal, corresponding to the electromagnetic radiation, with an energy density diluting as $\rho_{\rm EM}\ \sim a^{-4}$ and an off-diagonal piece showing an energy flux. This flux represents how the expanding bubble, equipped with the $\mathfrak{F}_{\mu\nu}$ field strength that sources the $\mathfrak{H}_{\mu\nu\rho}$ **pushes** the latter, carrying it out in the expansion in the AdS$_5$-bulk [**?**].

Glowing encore: Backreaction

In the eventuality that our dear reader has got lost conceptually in the previous paragraphs, we would like to remind them what the aim of this whole chapter is: We would like to decorate the induced expanding four dimensional cosmology by the action of bulk features "projected down" to the D_3-brane geometry. In this particular case, the aim is to reproduce the energy-momentum tensor associated with the homogeneous background radiation described in Eq. (B.29).

So far, we have understood how the correspondence between the source field-strength $\mathfrak{F}_{\mu\nu}$ of the D_3-brane and the bulk field $\mathfrak{H}_{\mu\nu\rho}$ works, but we still need to induce the expected energy-momentum tensor T_{ab} (B.29) on the brane. According to section 6.1, we need to find a backreacted metric that takes into account for the presence of the homogeneous and isotropic background of electromagnetic waves described in Eq. (6.49).

The backreacted geometry Ansatz that captures the presence of the background radiation can be written as [**?**]:

$$\mathrm{d}s^2_{\rm back} \approx -j_1(t,r)\mathrm{d}t^2 + j_2(t,r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2, \tag{6.50}$$

[80] This follows from the expansion of $\{\alpha_i,\beta_i\}$-function for small ε and p.

where

$$j_i(t,r) \equiv k^2 r^2 - \frac{\varepsilon^2 \left(\log(-\varrho k^2 t r) + q_i^{\text{EM}}\right)}{k^2 r^2} \tag{6.51}$$

with $\left\{q_i^{\text{EM}}\right\}$ as undetermined numbers and ϱ as a free parameter. This line invariant yields an energy-momentum tensor of the form:

$$\langle T^{\mu}{}_{\nu}\rangle_{\text{back}} = \frac{3\,\varepsilon^2}{2\,\kappa_5\,k^2\,r^4} \begin{pmatrix} -1 & 0 & 0 & 0 & \frac{-1}{k^4 r^3 t} \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ \frac{r}{t} & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.52}$$

where $q_1^{\text{EM}} - q_2^{\text{EM}} = 1/4$ has already been "fixed"[81] to resemble (6.49). Coefficients $\{\alpha, \beta\}$ in Eq. (6.49) can also be related to the expansion parameter ε and the amplitude $\mathcal{E}$ of the four-dimensional electromagnetic wave in Eq. (6.35) as:

$$\alpha = \beta = \frac{\epsilon}{\sqrt{2}} = \frac{2\kappa_5 k}{\alpha' \pi^2}\mathcal{E}. \tag{6.53}$$

Note that ϱ does not appear in (6.52). However, this does not make it an irrelevant term; this parameter can activate an AdS-Schwarzschild background behaviour in the calculation of the induced four-dimensional cosmology, in the same way as in section 6.1.

Finally, let us study how the backreaction of the bulk geometry (6.50), due to the presence of the $\mathfrak{H}_{\mu\nu\rho}$ field-strength, changes the motion of the bubble wall, and hence the induced energy-momentum tensor T_{ab}. As in previous sections, one can equivalently arrive to the same result by applying Israel's second junction condition (2.67) or the reversed Gauss-Codazzi equation (5.24). However, one must be careful with the following computation; Whenever we explored how the bulk's content induces certain energy densities in the expanding cosmology, with the presence of strings and matter in section 6.1 and gravitational waves in section 6.2, we assumed a **pure** brane, i.e. with nothing on top apart except the pure tension of the brane σ. This left the Israel's junction condition (2.67) to be as simple as in Eq. (3.13).

This is not the current case at hand, as we have initially equipped the brane with a gauge field strength $\mathfrak{F}_{\mu\nu}$. This requires us to account for an additional gauge field $\mathfrak{F}_{\mu\nu}$ contribution living **on** the brane. This can be written as:

$$S_{ab} = -\sigma\, h_{ab} + T_{ab}^{\text{EM}}. \tag{6.54}$$

In this way, the resulting energy-momentum tensor T_{ab} induced on the brane will have two main contributions:

1. A *positive* contribution coming from the extrinsic curvature.
2. A piece that comes from the gauge field **on** the brane, which **adds** to the tension and thus gives a *negative* contribution.

This can be read off from the Friedmann-like equation governing the dynamics of the induced cosmology which the second Israel's condition (2.67) spits out as:

$$\begin{aligned}\left(\frac{\dot a}{a}\right)^2 \delta^a_b = \frac{\Lambda_4}{3}\delta^a_b + \frac{\kappa_4}{3}\left(\mathfrak{F}^{ac}\mathfrak{F}_{bc} - \frac{1}{4}\,\delta^a_b \mathfrak{F}_{ij}\mathfrak{F}^{ij}\right) \\ + \varepsilon^2 \frac{\kappa_4}{6\kappa_5 k_+^3 a^4}\left(3\delta^a_0\delta^0_b - \delta^a_i\delta^i_b\right)\left(q_2^{\text{EM}} + \log\left[-\varrho\frac{k_+}{H}\right]\right).\end{aligned} \tag{6.55}$$

[81] We will definitely settle them to a certain value later on.

The left-hand side represents the usual geometrical evolution of the scale factor for late time cosmologies (i.e. there is no contribution from the curvature term) in the Friedmann equations. The two contributions on the right-hand side correspond to the energy-momentum tensor associated with the induced cosmology. Note that by setting $a = b = 0$, we recover the first Friedmann equation. It is crucial to note that the gauge field on the brane, in the first line, gives a **negative** contribution to the energy density, as does the brane tension. This needs to be corrected to a positive contribution by the second term due to the extrinsic curvature. In other words, this is the gravitational backreaction of the bulk.

We can do this by properly adjusting the free parameters ϱ and $q_2^{\rm EM}$, which are degenerate and can be shifted into each other. First, the parameter ϱ can be chosen such that there is no logarithmic contribution in Eq. (6.55), i.e. $-\varrho\, k/H = 1$. Furthermore, if we impose that the aforementioned combination of contributions to the energy-momentum tensor must be **that** of the electromagnetic tensor (B.29), we would find [**?**]:

$$\varepsilon^2 \overbrace{\left(\frac{q_2^{\rm EM}}{2a^4 k_+^3 \kappa_5}\right)\left(3\delta^a_0\delta^0_b - \delta^a_i\delta^i_b\right)}^{\text{backreaction piece}} + \underbrace{\left(\mathfrak{F}^{ac}\mathfrak{F}_{bc} - \tfrac{1}{4}\,\delta^a_b\mathfrak{F}_{ij}\mathfrak{F}^{ij}\right)}_{\langle T^a{}_b\rangle_{\rm iso}} \equiv -\langle T^a{}_b\rangle_{\rm iso}. \tag{6.56}$$

One can then solve for $q_2^{\rm EM}$ imposing that the right-hand side corresponds to the usual **positive** electromagnetic energy density (recall that in our conventions $T^0_0 = -\rho$). Using relations (6.53) and (7.60) we can fix $q_2^{\rm EM}$ to be:

$$q_2^{EM} = \frac{k_+^2 M_{pl}^2}{2^7\, T_{D_3}\, g_s}\, N\,\pi, \tag{6.57}$$

where N is the number of D_3-branes in the ten-dimensional background (7.65) and T_{D_3} is the fundamental tension of one of them.

The sharp mind of our dear reader may have noticed that the imposition on the right-hand side of Eq. (6.56) to be rather *ad-hoc*. What would happen if one does not impose it? What if the backreaction is not **exact** enough to compensate for the negative contribution to the energy density from the fields on the brane? That would correspond to a mismatch between the induced energy-momentum tensor and the observed one, Would it not? Would it not be seen as a failure of the model to predict the right energy-momentum tensor? Dear Padawan, these are questions that only the *dark* ~~*side*~~ *sector* can answer.

As discussed in section 5.3, any initial mismatch between the energy-momentum tensor of the *visible* sector, i.e. Eq. (B.29) and the effective induced four-dimensional one (6.55) can be interpreted as the presence of a *dark* sector. This layer would also couple to four-dimensional gravity like the visible one, but it will not interact with visible fields. This tempting proposal should be further explored in future works about the dark bubble scenario.

7 Stringy Embedding of the Dark Bubble and its Phenomenology

CHERRY ON TOP. SHOULD THIS CHAPTER CONTAIN SOME FURTHER MOTIVATION FOR OTHER EMBEDDINGS??

In this chapter you will learn...

- the first stringy embedding of the dark bubble and its dual field theory description
- which the main ten-dimensional characters are and their lower-dimensional interpretation
- how to induce four-dimensional $\Lambda_4 > 0$ on the bubble boundary from ten-dimensional supergravity and the associated energy hierarchy to this construction

If our dear reader has read this work from the very beginning and/or is a seasoned explorer of the bewitched lands described in stage **??** of our jorney, they should already be aware of the difficulty in finding metastable vacua solutions in string theory that are equipped with a positive cosmological Λ. The mechanism of flux compactification has always led the string phenomenology community to challenging (yet illuminating) situations, giving them a hard time when aiming to stabilise all moduli arising from the compactification [**?**, **?**, **?**] or unleashing intractable instabilities when breaking supersymmetry [**?**, **?**, **?**, **?**].

Previous chapter **??** introduced a new braneworld technology for obtaining four-dimensional cosmologies equipped with a positive cosmological constant Λ from higher dimensions. We did use a language based on string theory and related constituents, such as the WGC and the branes that mediate the decay of non-supersymmetric AdS$_5$-vacua. This chapter is a subsequent continuation of the precending discussion with one single objective:

To embed the dark bubble model in string theory.

In this chapter, we will elaborate the first stringy embedding of the dark bubble cosmology proposed by Danielsson, Henriksson and Panizo and its phenomenological consequences for the associated energy hierarchy to the embedding [**?**, **?**]. In order to easily navigate through such a voluminous chapter, let us introduce the content of its sections:

Section 7.1 will present a small introduction to the dual field theory (DFT) description of the setup used and explains its two faces (DFT and gravitational description) relate to each other. We will then visit the near horizon geometry (a.k.a. the pit) which is sourced by a stack of D_3 branes in section 7.2 and then follow all the path of the brane that (almost) escaped from it in section 7.3. Its dynamics will not be unfamiliar to us, as they were previously introduced in chapter **??**. Differences and similarities between the two approaches (Junction conditions vs Hamiltonian) in describing that motion will be commented in section 7.4. We will discover that the brane will eventually fall down in section 7.5; This will cause the induced cosmology to bounce back, failing on reproducing a cosmological era of accelerated expansion. This will force us to explore stringy corrections (a.k.a the hero) that can support the escaping brane providing the final boost to make the right jump in 7.6. Out of the well in section 7.7, we will realise how the pit's shadow tendrils extend beyond and control all phenomenological energy scales, providing the construction with a novel scale hierarchy.

7.1 *Alter ego* field theory

We will start by reviewing the dual language of the string background used in [**?**] to embed the dark bubble model. The statement is based on one of the most extensively studied examples of the AdS/CFT[82] correspondence [**?**, **?**, **?**]. Condensing the duality in some lines, it proposes that type IIB supergravity with AdS$_5 \times S^5$ geometry sourced in the near horizon limit of a stack of N D_3-branes can also be described by a strongly coupled $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills theory when $N \to \infty$.

[82] We recommend reading [**?**] for an exquisite introduction to the topic.

The construction used in the dark bubble embedding is a modification[83] of the original work. The gravitational side of the duality comprises of a stack of N black branes, with angular momentum in some of their transverse directions. The near horizon geometry sourced by such a stack will resemble that of a ten-dimensional Kerr black hole; This is because the rotation along some of the compact directions will mix the time and angular components of the metric. As in the original case, the whole ten-dimensional geometry can be decomposed asymptotically into $\mathrm{AdS}_5 \times S^5$. If we decided to reduce on the compact dimensions S^5, the resulting five-dimensional geometry would resemble that of a Reissner-Nordström black hole.

In order to understand how such effective charge appears in the five-dimensional space, we are required to examine the dual field theory description. This is the low energy effective description of the stack of N D_3-branes. These rotating branes in the compact dimensions S^5 exhibit an $SO(6)$ rotational symmetry. This set of isometries can be "isomorphically" translated to an $SU(4)$ global R-symmetry acting on the fields of the $\mathcal{N} = 4$ super Yang-Mills theory.

Among the fields in the $\mathcal{N} = 4$ super Yang-Mills description, we will find a gauge field A_μ,[84] some fermions and six scalars, all in the adjoint representation of the $SU(N)$ group. The eigenvalue scalars, which correspond to directions in a $\mathbb{R}^6$ space, will span a moduli space. The position of the eigenvalues in this moduli space will describe what the configuration of the gravitational description is. If the distance between each of the eigenvalues is small, one can determine the ten-dimensional geometry. But one single isolated eigenvalue points to the existence of a probe brane which has been embedded into the geometry sourced by the stack of branes.

These isolated eigenvalue points can appear when the $\mathcal{N} = 4$ super Yang-Mills description is decorated with additional background fields. These represent chemical potentials μ_i and a non-zero temperature $T \neq 0$. The presence of these fields will cause two relevant changes to the system:

- The configuration of eigenvalues in the $\mathbb{R}^6$ moduli space will start to rotate. This will represent states with non-zero R-charge. The rotational velocity is determined by the chemical potentials μ_i.
- The background fields will introduce an effective negative mass contribution in the Lagrangian describing the interaction of the scalar fields. This will signal to an instability of the system, i.e. the rotational configuration of the previous point may eventually change to the "intruder" one if the rotational velocity reaches a threshold, i.e. the chemical potentials reach such threshold.

The presence of the chemical potentials μ_i and the non-zero temperature $T \neq 0$ indicates to a *Higgsing* instability of the gauge group $SU(N) \to SU(N - \Delta N) \times U(\Delta N)$. This results in a spontaneous color symmetry **breaking**.

What does all this mean from the gravitational perspective? Remember that the ten-dimensional geometry is the near-horizon limit sourced by the stack of N D_3-branes. The angular velocities in the compact directions of S^5 correspond to those of the rotating moduli in the $\mathbb{R}^6$ space described above which is also related to the chemical potentials μ_i. When these potentials μ_i cause a change in the rotational configuration of the moduli and the higgsing process occurs, this is replicated by the emission of ΔN branes on the gravity side, which will escape from the stack that is the source of the Kerr geometry. This brane *nucleation*[85] implies that the membrane(s) will tunnel through the outer horizon of the black hole to minimise its energy. The new born brane(s) will

[83] This family of solutions was originally introduced in [?] as a consistent truncation to five-dimensions of $\mathcal{N} = 2$ supergravity. The whole ten-dimensional uplift can be found in [?].

[84] This field will be relevant to describe the coordinate-mixing in the gravitational description later on.

[85] This decay process is exponentially suppressed $\sim e^{-N}$ [?].

then start to expand, in order to reach the mininum, carrying with them a portion of the total angular momentum J associated with the rotational motion of the background in the compact directions. If we "zoom-out" to only see the AdS$_5$-space, we will observe a similar configuration, i.e. expanding brane(s) nucleated from the outer horizon of the Reisser-Nordström black hole which carry an R-charge Q, related to angular momentum J in the compact directions.

This is, *grosso modo*,[86] the descripion of the two different aspects of the same coin. It can be inferred that this set-up represents a suitable supergravity configuration to embed the dark bubble model discussed in chapter **??**. In order to simplify our computations, we will from now restric ourselves to an isotropic case of the aforementioned chemical potentials. This means that $\mu_1 = \mu_2 = \mu_3 = \mu$.

7.2 The pit

Let us start by defining the parametric coordinates[87] that will be used to describe the ten dimensional rotating background:

$$X^M = \{t, \alpha, \beta, \gamma, z, \theta, \psi, \phi_1, \phi_2, \phi_3\}, \tag{7.1}$$

where t represents the ten-dimensional ambient space time while $\{\alpha, \beta, \gamma\}$ represent the three non-compact usual dimensions. The coordinate z refers to the AdS$_5$-throat. Directions in the compact S^5 are $\{\theta, \psi, \phi_1, \phi_2, \phi_3\}$. The line invariant of this ten-dimensional geometry is given by following expression:

$$\mathrm{d}s_{10}^2 = \mathrm{d}s_5^2 + L^2 \sum_{i=1}^{3} \left\{ \mathrm{d}\sigma_i^2 + \sigma_i^2 \left(\mathrm{d}\phi_i + \tfrac{1}{L} A(z) \right)^2 \right\}, \tag{7.2}$$

where $A(z)$ is a ten-dimensional one-form that will be later discussed and the σ_i-functions are combinations of trigonometric functions of two angles of the S^5 as:

$$\sigma_1 = \sin\theta \ , \qquad \sigma_2 = \cos\theta \sin\psi \ , \qquad \sigma_3 = \cos\theta\cos\psi \ . \tag{7.3}$$

As stated in chapter **??**, L represents the AdS$_5$ radius in Eq. (7.2). It is important to note that this radius also determines the scale of the extra compact dimensions of S^5. This could be seen as a problematic feature, as it points to lack of scale separation, as discussed in **??**. However, this will be a key characteristic of the dark bubble embedding into supegravity, as this model achieves the aforementioned scale separation via tension-to-charge ratio instead [**?**, **?**]. We will further examine this feature in section 7.7.

Refocusing our attention on the line invariant (7.2), we can define the five-dimensional asymptotically-AdS metric ds$_5$ as follows:

$$\mathrm{d}s_5^2 = -j(z)^{-2} s(z)\mathrm{d}t^2 + j(z)\left[s(z)^{-1}\mathrm{d}z^2 + z^2\mathrm{d}\Omega_3^2\right] \ , \tag{7.4}$$

where $d\Omega_3^2$ is the usual unit metric of the 3-sphere (B.7). The radial functions $j(z)$ and $s(z)$ are expressed as

$$\begin{aligned} j(z) &= 1 + \frac{q^2}{z^2} \ , \\ s(z) &= 1 - \frac{m}{z^2} + \frac{z^2}{L^2} j(z)^3, \end{aligned} \tag{7.5}$$

[86]In Latin, "roughly speaking".

[87]Remember that Latin indices will represent the usual four-dimensional coordinates. Greek indices denote bulk coordinates and in this chapter, we will introduce capital Latin indices $\{M, N\}$ for the ten-dimensional coordinates. For more information, see appendix **??**.

where m and q are representing respectively the mass and charge of the five-dimensional Reisser-Nordström black hole discussed in section 7.1. Let us then perform a "massage" to Eqs. (7.4, 7.5) to provide a more familiar Reisser-Nordström black hole line invariant [?]. To begin this task, one can perform a straightforward change of coordinates by defining a new radial coordinate r as:

$$r^2 = z^2 j(z) = q^2 + z^2. \tag{7.6}$$

This change of coordinates will substantially transform Eq. (7.4), resulting in:

$$\begin{aligned} \mathrm{d}s_5^2 &= -g(r)\mathrm{d}t^2 + g(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2, \\ g(r) &= \underbrace{1 + k^2 r^2}_{f(r)} - \frac{2\kappa_5 M}{r^2} + \frac{\kappa_5^2 Q^2}{r^4}, \end{aligned} \tag{7.7}$$

with

$$M = \frac{1}{2\kappa_5}\left(m + 2q^2\right), \qquad Q^2 = \frac{1}{\kappa_5^2} q^2 (m + q^2), \tag{7.8}$$

where $\kappa_5 = 8\pi G_5$. It is now easy to see that the change of variables as described in Eq. (7.6) transforms the line invariant (7.4) into the familiar form[88] of a five dimensional Reissner-Nordström black hole living in an AdS vacuum. It should be noted that, for a small black hole, with $r_H \ll L$, we recover a flat space description and we need $Q < M$ to get a horizon and no naked singularity. However, if we want to study a horizon larger that the AdS$_5$ radius, this immediately implies $Q \ll M$.

Now let's review the interpretation of both ten and-five dimensional geometries. In the ten-dimensional approach, we will observe a Kerr black hole rotating in three of the compact directions of S^5 (i.e. $\{\phi_i\}$). When we zoom out and the five-dimensional sphere is no longer resolved, the black hole no longer rotates. It becames "static" from the point of view of a five-dimensional observer, but it acquires a charge q due to its motion in the compact directions (further explaination will be provide later). This charge becomes effective (i.e. charge Q) when the change of coordinates (7.6) is performed to recover the more familiar description of a Reissner-Nordström black hole living in AdS [?].

As it is well known, this type of black hole has two different horizons [?]: The *outer* horizon (i.e. event horizon) and the *inner* one (i.e. the Cauchy horizon). When these two horizons become degenerate (i.e. $r_H = r_h$), this corresponds to a description of an extremal black hole. We will not further discuss charged black holes and their connection to the weak gravity conjecture here (see section 2.1 for more details), but, for computational purposes in the incoming sections, we find more convenient to express (7.7) in terms of the two aforementioned horizons: $\{r_h, r_H\}$ as the inner and outer horizons [?]. In this way, we can express $g(r)$ in Eq. (7.7) with the following Ansatz:

$$g(r) = \frac{k^2}{r^4}\left(r^2 + c\right)\left(r^2 - r_h^2\right)\left(r^2 - r_H^2\right), \tag{7.9}$$

with $c \in \mathbb{R}$. A simple match between Eqs. (7.7, 7.9) shows that:

$$\begin{aligned} M &= \frac{1}{2\kappa_5}\left[r_h^2 + r_H^2 + k^2\left(r_h^4 + r_h^2 r_H^2 + r_H^4\right)\right], \\ Q^2 &= \frac{r_h^2\, r_H^2}{\kappa_5^2}\left[1 + k^2\left(r_h^2 + r_H^2\right)\right], \\ c &= r_h^2 + r_H^2 + \tfrac{1}{k^2}. \end{aligned} \tag{7.10}$$

[88] The patch described by the metric (7.4) corresponds to the requirement $r \geqslant q$. This is actually not an issue since the horizon r_H will be covered by the patch.

which simplifies when the black hole is extremal. Let us keep in mind these identifications for future sections and return our attention back to the one-form potential $A(z)$ in Eq. (7.2). This form acts as a gauge field from the five-dimensional point of view. In term of the old coordinates (i.e. $\{t, \alpha, \beta, \gamma, z\}$) it can be written as:

$$A(z) = \frac{q}{z_H^2 + q^2} \sqrt{(q^2 + z_H^2) + \frac{z_H^4}{L^2} j(z_H)^3} \left(1 - \frac{z_H^2 + q^2}{z^2 + q^2}\right) \mathrm{d}t. \tag{7.11}$$

Note that gauge freedom has been used to add a constant that sets $A(z_H) = 0$. This is required as a $A(z)$ is a temporal gauge potential, and must vanish at the horizon. This potential can be expressed in a more handable way in two steps:

1. Part of the argument of the root in Eq. (7.11) is the mass m of the black hole.[89] This is:

$$m = z_H^2 \left(1 + \frac{z_H^2}{L^2} j(z_H)^3\right). \tag{7.12}$$

2. Furthermore, performing the change of variables (7.6) and making use of the identifications (7.8) to further simplify in the new coodinate system, one obtains:

$$A(r) = \kappa_5 Q \left(\frac{1}{r_H^2} - \frac{1}{r^2}\right). \tag{7.13}$$

The self-dual $\mathfrak{F}_5$ Ramond-Ramond (RR) field strength is sourced by the stack of rotating D_3-branes in the ten-dimensional background,[90] which can be expressed in terms of the radial function $j(z)$ and the one-form $A(z)$. However, what we will need for future computations is the corresponding four-form potential $\mathfrak{C}_4$, with $\mathrm{d}\mathfrak{C}_4 = \mathfrak{F}_5$. This is:

$$\begin{aligned} \mathfrak{C}_4(z, z_H, q) = & \frac{1}{L}\left[(z^2 + q^2)^2 - (z_H^2 + q^2)^2\right] \mathrm{d}t \wedge \epsilon_3 \\ & + L^2 q \sqrt{z_H^2 + q^2 + \frac{z_H^4}{L^2} j(z_H)^3} \sum_i \sigma_i^2 \, \mathrm{d}\phi_i \wedge \epsilon_3 \,, \end{aligned} \tag{7.14}$$

where ϵ_3 corresponds to the volume-form associated with $\mathrm{d}\Omega_3$ in Eq. (7.4). Following the recipe describe above, one cast the four-form $\mathfrak{C}_4$ in terms of the new variables as:

$$\mathfrak{C}_4(r, r_H, Q) = \underbrace{\frac{1}{L}\left[r^4 - r_H^4\right] \mathrm{d}t \wedge \epsilon_3}_{(\mathfrak{C}_4)_t} + \underbrace{L^2 \frac{Q}{\kappa_5} \sum_i \sigma_i^2 \, \mathrm{d}\phi_i \wedge \epsilon_3}_{(\mathfrak{C}_4)_\phi = \sum_i (\mathfrak{C}_4)_{\phi_i}}. \tag{7.15}$$

Note that we have split the four-form $\mathfrak{C}_4$ into its time-dependent and ϕ_i-dependent pieces. This notation will simplify further computations.

With this we have completed the description of the black hole in ten and five-dimensional perspectives. Let us now explore how the D_3-brane is explicitly embedded in the ten-dimensional ambient space.

7.3 The brane that (almost) escaped from the pit

Let us now move on and study the motion of a nucleated D_3-brane.[91] The starting point of such enterprise is the action governing its dynamics (**??**):

$$S_{D_3} = S_{\mathrm{DBI}} + S_{\mathrm{WZ}} = -T_3 \int \mathrm{d}^4\xi \sqrt{|\left(P[G + 2\pi\alpha' \mathfrak{F}]\right)|} + T_3 \int P[\mathfrak{C}_4]. \tag{7.16}$$

[89] This can be easily obtained by computing the outer horizon in Eq. (7.4).
[90] Ten-dimensional fields will be represented with a gothic calligraphy. See appendix **??** for more notation conventions.
[91] Note that we will denote the tension of the brane with T_3 and not T_{D_3} as in Eq. (**??**). This will be explained later in the chapter.

The Dirac-Born-Infeld (DBI) term describes both the geometrical features of the embedding and the coupling of the brane to Neveu-Schwarz (NS) gauge fields living in the bulk,[92] while the Wess-Zumino (WZ) term with $\mathfrak{C}_4$ captures the non-gravitational forces on the brane due to fields external to it. $P[\dots]$ denotes the pullback of fields to the w of spacetimeorld volume of the brane. This is no more than the projection of ten-dimensional coordinates onto the three-dimensional geometry of the D_3-brane, with tension T_3. These, the embedding functions of the brane will be denoted by capital and $\mathfrak{gothic}$ versions of the coordinates of spacetime $\mathfrak{X}^M(\xi)$. For simplicity, we will set the dilaton ϕ constant.

The dynamical description of the branes is as follows: The stack of D_3-branes, and hence the nucleated one, wrap the three-sphere $\mathrm{d}\Omega_3$ inside the AdS$_5$ piece of the metric, with the radial r-coordinate (hence $\mathfrak{R}$) depending only on the time component. The black hole in the background has ten dimensions and rotates in the azimuthal ϕ_i-directions on S^5. This implies that a D_3-brane escaping from the stack will preserve a portion of the angular momentum of this. Therefore, it will also move in those directions. For the sake of simplicity in the construction of this model, set the three angular momenta associated with each ϕ_i-directions equal in the background. This implies that the brane will rotate with the same angular velocity in all ϕ_i directions. Finally, we will assume a constant position in the remaining coordinates of the compact S^5, i.e. $\{\Psi = \psi_0, \Theta = \theta_0\}$. The embedding functions of the brane are then given by:

$$\begin{aligned} &\mathfrak{T} = \mathfrak{T}(\tau)\,, \qquad \mathfrak{R} = \mathfrak{R}(\tau)\,, \qquad \Theta = \theta_0, \qquad \Psi = \psi_0\,, \\ &\Phi_1 = \Phi_2 = \Phi_3 = \Phi(\tau)\ . \end{aligned} \tag{7.17}$$

Here, although τ is used as a time parameter on the induced geometry, we do not yet specify whether it denotes its proper time. We will later see that this is case. From the point of view of an observer living in AdS$_5$, the rotation translates into an effective charge Q under the gauge field A. When a brane nucleates, carrying a portion of the charge, it will also effectively reduce the charge of the remaining black hole.

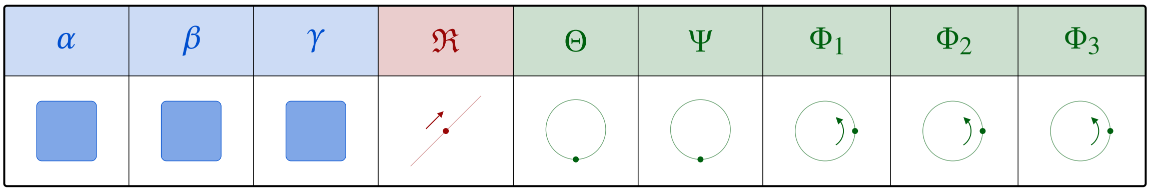

Figure 14: Schematic table to describe the position of the nucleated D_3-brane. The brane wraps around the three standard spatial directions, given by coordinates $\{\alpha, \beta, \gamma\}$. It will move radially along the $\mathfrak{R}$ direction of the throat. Green coordinates represent the compact directions of S^5. The brane is assumed to be fixed at $\{\Theta, \Psi\}$, while moving with the same angular momentum J_i along the three other directions $\{\Phi_i\}$.

Before analysing the action (7.16), we need some more practical notation. Derivatives with respect to τ will be denoted by a dot. The components of the ten-dimensional metric (7.1) will be denoted by G_{MN}. It will also be useful to define the combination of the azimuthal-time and azimuthal-azimuthal components in the compact piece of Eq. (7.1) as $G_{t\phi} = \sum_i G_{t\phi_i}$ and $G_{\phi\phi} = \sum_i G_{\phi_i\phi_i}$ respectively.

[92]The construction described in this chapter does not include NS gauge fields, meaning that $\mathfrak{F} = \mathfrak{F}_{MN} = 0$ in the bulk. Further discussion on this topic can be found in section 6.3.

Let us start the analysis of Eq. (7.16). The pullback $P[G]$ is no more than the induced $g_{\tau\tau}$-metric entry of the expanding D_3-brane. This is:

$$P[G] = g_{\tau\tau} = G_{tt}\dot{\mathfrak{T}}^2 + G_{rr}\dot{\mathfrak{R}}^2 + 2G_{t\phi}\dot{\Phi}\dot{\mathfrak{T}} + G_{\phi\phi}\dot{\Phi}^2. \tag{7.18}$$

This is also the square of the velocity that a comoving observer living on the brane will experience,

$$\dot{X}^2 = \dot{X}_M\dot{X}^M = G_{tt}\dot{\mathfrak{T}}^2 + G_{rr}\dot{\mathfrak{R}}^2 + 2G_{t\phi}\dot{\Phi}\dot{\mathfrak{T}} + G_{\phi\phi}\dot{\Phi}^2 \ , \tag{7.19}$$

with

$$\dot{X} \equiv \frac{\partial X^M}{\partial \tau}\partial_M = \dot{\mathfrak{T}}\partial_t + \dot{\mathfrak{R}}\,\partial_r + \dot{\Phi}\sum_{i=1}^{3}\partial_{\phi_i} \ . \tag{7.20}$$

The induced line element on the brane worldvolume is:

$$\mathrm{d}s_4^2 = \dot{X}^2\mathrm{d}\tau^2 + \mathfrak{R}(\tau)^2\mathrm{d}\Omega_3^2 \ . \tag{7.21}$$

Now one can see that picking τ to be the proper time of an observer living on the brane would set $\dot{X}^2 = -1$, and the induced metric takes the usual FLRW-form 3.11, with $\mathfrak{R}(\tau)$ acting as the scale factor.

Similarly, the pullback of the four-form $\mathfrak{C}_4$ is:

$$P[\mathfrak{C}_4] = \left((\mathfrak{C}_4)_t\frac{\partial\mathfrak{T}}{\partial\tau} + \sum_i(\mathfrak{C}_4)_\phi\frac{\partial\Phi_i}{\partial\tau}\right)\mathrm{d}\tau\wedge\epsilon_3. \tag{7.22}$$

Finally, the action (7.16) can be rewritten as:

$$S_{D_3} \equiv \int d\tau\mathcal{L}_{D_3} = -2\pi^2T_3\int d\tau\left\{\mathfrak{R}^3\sqrt{-\dot{X}^2} - (\mathfrak{C}_4)_t\dot{\mathfrak{T}} - (\mathfrak{C}_4)_\phi\dot{\Phi}\right\}, \tag{7.23}$$

where $\mathcal{L}_{D_3}$ represents the Lagrangian of the D_3-brane. Note that integration over the S^3 has already been taken into account. As can be observed, the Lagrangian does not depend on the time component $\mathfrak{T}$ or the angular coordinates Φ_i, only their derivatives. This points to conserved quantities within the system. To make our computations as simple as possible, let us define the conserved angular momentum

$$J \equiv \frac{1}{2\pi^2T_3}\frac{\mathrm{d}\mathcal{L}_{D_3}}{\mathrm{d}\dot{\Phi}} = \frac{\mathfrak{R}^3}{\sqrt{-\dot{X}^2}}\left(G_{t\phi}\dot{\mathfrak{T}} + G_{\phi\phi}\dot{\Phi}\right) + (\mathfrak{C}_4)_\phi. \tag{7.24}$$

Using Legendre transformation, we can substitute $\dot{\Phi}$ for J, which automatically satisfies Φ's EoM. Additionaly, let's define $J_C = J - (\mathfrak{C}_4)_\phi$, which will be given a physical interpretation later. By selecting the positive root, so that the gauge force can compensate the tension, we then obtain:

$$\dot{\Phi} = -\frac{G_{t\phi}}{G_{\phi\phi}}\dot{\mathfrak{T}} \pm J_C\sqrt{\frac{\left(\frac{G_{t\phi}^2}{G_{\phi\phi}^2} - \frac{G_{tt}}{G_{\phi\phi}}\right)\dot{\mathfrak{T}}^2 - \frac{G_{rr}}{G_{\phi\phi}}\dot{\mathfrak{R}}^2}{(\mathfrak{R}^6\,G_{\phi\phi} + J_C^2)}}. \tag{7.25}$$

and the Legendre transformation becomes:

$$\begin{aligned}\mathcal{L}_{D_3}^J &= \mathcal{L}_{D3} - \dot{\Phi}J\\ &= -2\pi^2T_3\dot{\mathfrak{T}}\left\{-(\mathfrak{C}_4)_t - \frac{G_{t\phi}}{G_{\phi\phi}}J_C + \sqrt{-\left(\mathfrak{R}^6 + \frac{J_C^2}{G_{\phi\phi}}\right)\left(\underbrace{G_{tt} - \frac{G_{t\phi}^2}{G_{\phi\phi}}}_{B} + G_{rr}\mathfrak{R}'^2\right)}\right\},\end{aligned} \tag{7.26}$$

Note that the derivative is now represented by a prime instead of a dot. For computational convenience, one may choose to write $\frac{\partial R}{\partial \tau} = \dot{\mathfrak{T}} \frac{\partial R}{\partial \mathfrak{T}}$, which implies that prime is a derivative with respect to the time-coordinate of spacetime. Introducing the components of the ten-dimensional line invariant (7.2), Eq.(7.26) can be expressed as

$$\mathcal{L}_{D_3}^{J} = -2\pi^2 T_3 \dot{\mathfrak{T}} \left\{ -(\mathfrak{C}_4)_t - \frac{J_C}{L} A_t + \sqrt{-\left(\mathfrak{R}^6 + \frac{J_C^2}{L^2}\right)\left(\underbrace{g_{tt}}_{B} + g_{rr}\mathfrak{R}'^2\right)} \right\}, \tag{7.27}$$

where $B = g_{tt}$ results after simplification of the metric entries. The linear term J_C in (7.27) shows that the brane carries charge and couples to the bulk gauge field. Note that J_C rather than J measures the charge [?]. This relative shift can be explained by frame dragging caused by the rotation in the extra dimensions.[93]

Let us elaborate on the conceptual picture that the square root term in Eq. (7.27) represents: In the absence of J_C, This is simply an effective five-dimensional DBI-term. The presence of J_C can be understood as an additional flux representing an uniform density of charged particles over the D3-brane. This dissolved particle density spread over the brane can be interpreted as the charge Q from a five-dimensional point of view.[94]

Lastly, the $(\mathfrak{C}_4)_t$-term can be studied as a five-dimensional WZ-term; One can always define an effective five-dimensional flux $F_5^{\text{eff}} = d\big[(\mathfrak{C}_4)_t \mathrm{d}t \wedge \epsilon_3\big]$ with F_5^{eff} proportional to the volume form of the line invariant $\mathrm{d}s_5^2$.

Our D_3-dynamics equation (7.27) can be written in a simpler expression in terms of the Hamiltonian

$$H = 2\pi^2 T_3 \dot{\mathfrak{T}} \left(\frac{-g_{tt}\sqrt{\mathfrak{R}^6 + \frac{J_C^2}{L^2}}}{\sqrt{-(g_{tt} + g_{rr}\mathfrak{R}'^2)}} - (\mathfrak{C}_4)_t - \frac{J_C}{L} A_t \right), \tag{7.29}$$

given in terms of the proper time of the brane. Careful here! It is of extreme importance to realize that this is **not** the time variable relevant for a four-dimensional observer like our dear reader [?]. Such a witness moves along with the nucleated brane in AdS_5, but does not move together with the brane in its compact dimensions motion. This is a similar argument to that Kaluza-Klein (KK) compactifications to four dimensions. Any observer, made out of KK particles, is governed by a time component which does **not** involve the internal momenta or velocities of their particles. We will refer to the relevant four-dimensional proper time as $\tilde{\tau}$, which is computed as

$$\mathrm{d}\tilde{\tau} = \mathrm{d}t\sqrt{-(g_{tt} + g_{rr}\mathfrak{R}'^2)}. \tag{7.30}$$

This allows us to rewrite the Hamiltonian (7.29) in terms of the proper time of the four-dimensional observer as:

$$\begin{aligned} H = 2\pi^2 T_3 \frac{d\mathfrak{T}}{d\tilde{\tau}} \Bigg[& \sqrt{\left(\mathfrak{R}^6 + \frac{J_C^2}{L^2}\right)}\sqrt{g(\mathfrak{R}) + \dot{\mathfrak{R}}^2} \\ & - \frac{1}{L}\left(\mathfrak{R}^4 - r_H^4\right) - \frac{\kappa_5\, J_C\, Q}{L}\left(\frac{1}{r_H^2} - \frac{1}{\mathfrak{R}^2}\right) \Bigg], \end{aligned} \tag{7.31}$$

[93] The right angular momentum of the expading bubble is obtained when its rotation in the compact direction is measured with respect to a "static" reference system of the background. Hence the relative shift.

[94] One can formally collapse the brane to a point (i.e. set $\mathfrak{R} = 0$), to see Eq. (7.27) become:

$$-\frac{2\pi^2 T_3 J_C}{L}\dot{\mathfrak{T}}\sqrt{-(g_{tt} + g_{rr}\mathfrak{R}'^2)}. \tag{7.28}$$

In the absence of the dissolved particles, the total mass of the D_3 approaches zero in this limit. In this case the mass converges to $\frac{2\pi^2 T_3 J_C}{L}$. This can be interpreted as the total mass of the dissolved particles.

where Eq. (7.30) has been used to rewrite the first term in the Hamiltonian (7.29), while its second and third pieces correspond to expressions (7.15) and (7.13) respectively. Recall that $g(\mathfrak{R})$ is the function to describe the AdS$_5$-Reissner-Nordström geometry (7.7).

This Hamiltonian can be equated to a conserved energy E,[95] while the constant in $(\mathfrak{C}_4)_t$ can be considered a gauge choice that also shifts E [?]. In order to fix such conserved energy, it is important to know that the bubble, acting as a Coleman-de Lucia instanton, will nucleate at *rest*. This results into zero total energy in the approximately extremal background.[96] Imposing $E = 0$, one can then solve Eq. (7.31) to find the equation of motion controlling the dynamics of the brane as:

$$\dot{\mathfrak{R}}^2 = -f(\mathfrak{R}) - k^2\mathfrak{R}^2 + \frac{k^2 \left(\mathfrak{R}^4 - r_H^4 + \frac{Q\,\kappa_5\,J_c}{r_H^2}\left(1 - \frac{r_H^2}{\mathfrak{R}^2}\right)\right)^2}{k^2\,J_c^2 + \mathfrak{R}^6}, \tag{7.32}$$

where $f(\mathfrak{R}) = g(\mathfrak{R}) - k^2\mathfrak{R}^2$ and terms in green and blue will be discussed in the next section. Eq. (7.32) still contains both geometrical information, through its first DBI term (i.e. $g(\mathfrak{R})$) and WZ information (i.e the last term). This equation of motion is the furthest we can go simplifying the action of the D_3-brane (7.16) into the ten-dimensional background. In order to understand why the embedding of the dark bubble into string **works**, we will compare Eq. (7.32) to the equation governing a four-dimensional expanding cosmology derived from the junction conditions, as shown in Eq. (3.22). This will be the aim of the next section.

7.4 We are not so different after all, said Israel to Hamilton

As stated above, the five-dimensional picture is given by an AdS$_5$-Reissner-Nordström geometry. The black hole (i.e. the stack of D_3-branes), is unstable due to the presence of the large chemical potential μ and one or several can escape (i.e. nucleate) from the outer horizon r_H. Let's begin our comparison by considering one single nucleated brane. It has tunneled from the outer horizon r_H at radius $r > r_H$ and starts to expand. Similarly to the simplest example described in section **??**, this shell will mediate the decay between two different vacua: The *outside*, which has no idea yet that neither vacuum scale k, nor the black hole charge Q and M have changed and the *inside*, where this information has already been taken into account, resulting in a change in the geometry. Therefore, the inner and outer metric argument $g(r)$ can be written as:

$$g_\pm(r) = k_\pm^2 r^2 + \underbrace{1 - \frac{2\kappa_5 M_\pm}{r^2} + \frac{\kappa_5^2 Q_\pm^2}{r^4}}_{f_\pm(r)}, \tag{7.33}$$

In the same spirit as in section **??**, the junction condition (3.17) becomes:

$$T_3 = \frac{3}{\kappa_5}\left(\sqrt{\frac{g_-(a)}{a^2} + \frac{\dot{a}^2}{a^2}} - \sqrt{\frac{g_+(a)}{a^2} + \frac{\dot{a}^2}{a^2}}\right). \tag{7.34}$$

Here T_3 represents the tension of the brane (we will later see why T_3 and not σ). The location of the brane along the AdS$_5$-throat is $r = a(\tau)$.[97]

[95] Although we did not derive this conserved quantity, we saw that the Lagrangian density (7.23) does not depend on the coordinate $\mathfrak{T}$. This implies E to be conserved.

[96] As described in the introduction of this chapter, the ten-dimensional ambient space has broken supersymmetry, due to the presence of close-to-zero temperature T and chemical potential μ.

[97] This τ is the **real** proper time discussed in Eq. (7.30). The tilde was dropped to obtain a cleaner notation.

In order to obtain a similar equation to (7.32) to compare with, one can solve exactly for $\dot{a}^2$ to obtain a Friedmann-like equation:

$$\begin{aligned}\dot{a}^2 = \frac{1}{4\sigma^2 a^2}\left[\left(k_-^2 - k_+^2\right)a^2 + \left(f_-(a) - f_+(a)\right)\right]^2 + \\ + \frac{\sigma^2}{4}a^2 - \frac{1}{2}\left[\left(f_-(a) + f_+(a)\right) + \left(k_-^2 + k_+^2\right)a^2\right],\end{aligned} \tag{7.35}$$

As a handy notation, let us refer to the *critical* tension of this brane (3.19) as $\sigma = T_3\frac{\kappa_5}{3} = (k_- - k_+) = \Delta k$. In order to compare with (7.32), it is important to ensure that we are in the correct regime, i.e. that of the probe approximation. This means that we must work at linear order in the perturbations when making expansions of the form:

$$k_\pm = k \mp \tfrac{1}{2}\Delta k, \qquad Q_\pm = Q \mp \tfrac{1}{2}\Delta Q, \qquad M_\pm = M \mp \tfrac{1}{2}\Delta M. \tag{7.36}$$

Note that $\Delta Q = Q_- - Q_+ < 0$. This means that when the brane leaves the stack, it carries mass (i.e. tension) and a little bit of charge (the angular momentum J_c from the ten-dimensional perspective) with it. In this way, one ensures that the physical picture is respected by the approximation.

With approximations (7.36) at hand, let us substitute them in Eq. (7.35) to read:

$$\dot{a}^2 = -f(a) - k^2a^2 + \frac{k^2}{a^6}\left(a^4 + \frac{a^2}{2k\Delta k}\left(f_-(a) - f_+(a)\right)\right)^2, \tag{7.37}$$

where only linear order terms have been considered. Here $f(a)$ condenses zeroth order information about the mass M and charge Q, while $f_-(a) - f_+(a)$ encodes linear terms in ΔQ and ΔM. The structure of Eq. (7.37) starts to resemble that of the embedding potential (7.32). But we aim for a closer match. Notice that this expression can be simplified even further. If one evaluates order by order the junction condition (2.57) and $g(r)$ at the horizon r_H, accounting that $g_\pm(r)$ musth vanish at the horizon r_H, it results in the following constraints at the zeroth and first order:

$$\begin{aligned}0^{\text{th}}: &\quad k^2a^2 + g(a) = 0, \\ 1^{\text{st}}: &\quad k\,\Delta k\, a_H^2 - \frac{\kappa_5\Delta M}{a_H^2} + \frac{\kappa_5 Q\Delta Q}{a_H^4} = 0.\end{aligned} \tag{7.38}$$

Substitute (7.38) in Eq. (7.37) results in:

$$\dot{a}^2 = -f(a) - k^2a^2 + \frac{k^2}{a^6}\left(a^4 - a_H^4 - \frac{\kappa_5^2 Q\Delta Q}{a_H^2\, k\Delta k}\left(1 - \frac{a_H^2}{a^2}\right)\right)^2. \tag{7.39}$$

Overall, the whole structure of Eq. (7.39) is identical to that of expression (7.32). Recall that, although we had chosen capital letters $\mathfrak{X}$ to denote the embedding functions of the brane, the five-dimensional picture uses lowercase letters. In any case, both represent the same coordinates. Let us discuss the terms highlighted in colours:

- fraction: The comparison of both terms easily yields:

$$J_C = -\frac{\kappa_5\Delta Q}{k\,\Delta k}. \tag{7.40}$$

  This has already been discussed; It is the realisation that the brane's effective angular momentum in the compact directions is equivalent to an effective charge ΔQ carried by the brane when observing the system from a five-dimensional perspective.

- denominator: At first glance, one can observe the absence of a J_C^2 term in Eq. (7.39). This is because one important character, present in the string theory brane action, but not in the junction conditions, was not taken into account; The flux related to the dissolved charge density discussed above. Opposite to the DBI + WZ action, the junction condition (7.34) did not account for the presence of this dissolved flux. In fact, this can be used to enhance the junction conditions. Replacing T_3 in expression (7.34) by $T_3\sqrt{1+k^2\,J_C^2 a^{-6}}$ the result is the same as in Eq. (7.32).

After having related the two problematic terms, the match is complete. One can easily see that, the equation of motion for the D_3-brane given by Eq. (7.16) and the junction conditions across the brane (7.34) are the **same**[98] expression. This just tells us that the construction from the DBI+WZ action (7.16) in the ten-dimensional stringy ambient space description produces the same dynamics as the junction condition for the co-dimension one shell in the five-dimensional AdS_5 bulk. This can just point to one simple fact:

The dark bubble model has been embedded into string theory.

7.5 Why do we fall?[99] A collapsing cosmology

There is one little detail we have not commented on yet: This embedding of the dark bubble model within string theory is not able to reproduce a late time cosmology dominated by a cosmological constant (yet). This implies that the curvature of the universe will eventually become dominant, reaching a maximum value for the scale factor $a(\tau)$ and collapsing back, in the form of a bouncing cosmology. To realise this, one simply needs to check the behaviour of the Friedmann-like expression (7.32) for late-time cosmologies (i.e. $k\,\mathfrak{R}\to\infty$). In this regime, the $k^2\,\mathfrak{R}(\tau)^2$ cancels, indicating that the brane's tension is equal to the critical one, as discussed in section **??**. But let us examine this issue from the perspective of the brane tension T_3.

Junction conditions are formulated from a general relativity perspective; If both vacua (inside and outside) are standing-alone solutions to Einstein equation, what induced energy momentum tensor (3.13) should the brane be equipped with such that the whole is a solution t of spacetimeo the Einstein equation? This implies that Newton's constant G_5 is **constant** from this perspective. This does not imply that any other combination of variables to describe this constant cannot change, as long as the resulting Newton's variable G_5 remains constant.

Given the AdS/CFT dictionary [**?**], one finds that:

$$G_5=\frac{\pi L^3}{2N^2}, \tag{7.41}$$

where again L is the curvature radius of the compact direction and the AdS-throat, while N is the number of branes sitting in the stack of D_3-branes sourcing the geometry of the ambient space. Imposing the Newton's constant to be constant across the brane (i.e. $\Delta G_5=0$), we get:

$$\frac{\Delta L}{L}=\frac{2}{3}\frac{\Delta N}{N}, \tag{7.42}$$

where ΔL represent the difference in the curvature radius across the brane and ΔN encodes how many D_3-branes have nucleated together, mediating the decay. This relation (7.42) can be used to rewrite the critical tension of the brane as:

$$\begin{aligned} T_3&=\frac{3\Delta k}{8\pi G_5}\to-\frac{3}{8\pi G_5}\frac{\Delta L}{L^2}=-\frac{1}{4\pi G_5 L}\frac{\Delta N}{N}=-\frac{1}{2\pi^2L^4}N\,\Delta N\\ &=-\frac{\Delta N}{(2\pi)^3 g_s\ell_s^4}=-\Delta N\,T_{D_3}. \end{aligned} \tag{7.43}$$

[98] This will be a recurrent scheme in this part of the thesis. We will find that the equations of motion of the bubble obtained from the action are also reproduced by the junction conditions.

[99] So we can learn to pick our bubble up.

Note that relation (7.42) has been used to reach the third step. The fourth equality is obtained using Eq. (7.41), and the final expression just requires us to use Eq. (??) and another AdS/CFT dictionary key:

$$L^4 = 4\pi g_s N \ell_s^4. \tag{7.44}$$

When $\Delta N = N_{\text{final}} - N_{\text{initial}} = -1$ (i.e. just one nucleated brane), Eq. (7.43) states that the critical tension is exactly that of a D_3-brane. This means that the induced cosmological constant will be zero [?]. Before we decide to delve into the intricacies of inducing an effective positive, yet small four-dimensional dark energy density ρ_Λ in the model, let us first analyse how the induced four-dimensional cosmology, driven to the failure by the absence of a late-time cosmological constant dominance, behaves under the parameters controlling its dynamics.

For starters, let us express Eq. (7.39) in terms of the horizons $\{a_h, a_H\}$ (recall that from a four-dimensional perspective $\mathfrak{R}(\tau) = r(\tau) = a(\tau)$) by substituting Eq. (7.10). This gives:

$$\begin{aligned}\dot{a}^2 =& -\frac{k^2(a_H^2 - a^2)(a_h^2 - a^2)\left(a_h^2 + a_H^2 + \frac{1}{k^2} + a^2\right)}{a^4} + \\ &+ \frac{k^2\left(a^4 - a_H^4 + \frac{a_h\, J_c\sqrt{1+\left(k^2 a_H^2 + a_h^2\right)}}{a_H}\left(1 - \frac{a_H^2}{a^2}\right)\right)^2}{k^2 J_c^2 + a^6}.\end{aligned} \tag{7.45}$$

The first property to note is that $\dot{a} = 0$ when $a = a_H$. As mentioned above, the brane nucleates at rest, which not only the junction condition, but also the DBI action from string theory remark in this construction. When both horizons coincide (i.e. $r_h = r_H$), one recovers the extremal description of the Reissner-Nordström black hole living in the AdS$_5$ bulk. Figure 15 will allow us to further explore other critical points of this construction.

There are three (four) particular points to consider in figure 15:

- Both horizons $\{a_h, a_H\}$ (first two roots in the non-extremal case): They correspond to the inner and outer horizons of the Reisser-Nordström black hole respectively. In the extremal case, both horizons coincide, which corresponds to the first root of the blue plot.
- The nucleation point $\{a_{\text{nuc}}\}$ (2^{nd} or 3^{rd} roots, respectively): This is the value of a at which the D_3 brane (i.e. the four-dimensional universe) will nucleate. The region between a_H and a_{nuc} is the classical forbidden region through which the brane must tunnel. See chapter **??** for further information about the nucleation event.
- The maximum size $\{a_b\}$ of the collapsing universe: This is given by the last root of the potential $V(a)$. At this point, the size of the bubble is maximal, and the four-dimensional universe, without positive dark energy density, but equipped with positive curvature, will reach its maximum size and start contracting, in the form of a bouncing cosmology.

Which different features are controlled by which parameters? By construction, the potential (7.45) goes to 1 when the scale factor $a \to \infty$. The AdS scale k^{-1} is in charge of controlling the overall value of the potential (i.e. basically scaling the y-axis). The angular momentum of the brane J_C will regulate the barrier of the potential.

The potential described in Eq. (7.45) lacks a sustained cosmological constant and experiences only a fraction of an e-folding just after creation. The expansion rate vanishes at nucleation, the universe then accelerates for a short time after which it enters a decelerated phase. As the curvature is positive, and there is no dark energy in the game, the universe will eventually stop expanding and start contracting, entering a bouncing phase [?].

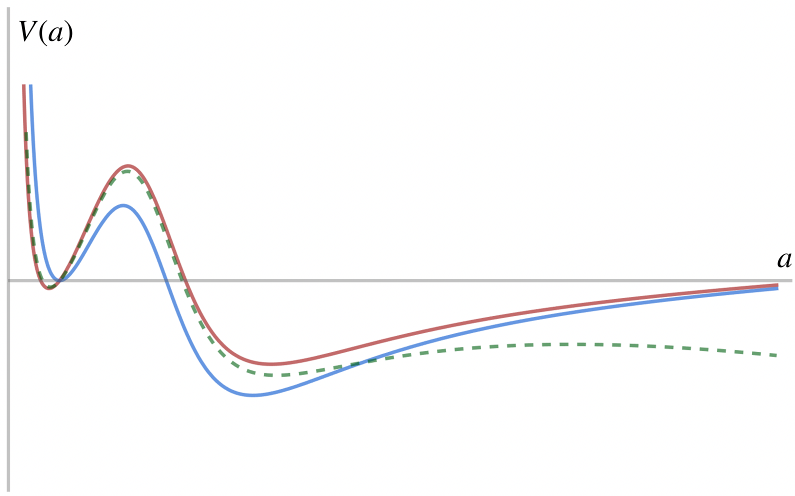

Figure 15: The potential controlling the dynamics of the four-dimensional cosmology riding on the D_3-brane. Non-extremal and Extremal cases, with respective non-degenerate and degenerate horizons, will ultimately bounce back due to the absence of an effective positive cosmological constant (observe the asymptotic behaviour towards one when $a \to \infty$). When the brane has induced dark energy (see section 7.6), the (dashed) potential will dictacte a never-ending expansion of the four-dimensional cosmos.

Of course, this is not the desired fate for the dark bubble model in its stringy embedding. Our goal is to find a low-dimensional model capable of describing dark energy derived from string theory. Do previous computations indicate that the dark bubble model embedding should be ruled out? Absolutely not. All stringy computations so far have been performed assuming that the brane is a BPS one, although the ten-dimensional ambient space hosting it has broken supersymmetry. This will be taken into account in the incoming sections.

7.6 A hero can be anyone: Curvature corrections to obtain Λ_4

Looking carefully at Eqs. (7.43) and (7.45), it is easy to see the absence of an induced dark energy density ρ_Λ, at least, at leading order in the brane action. At the beginning of this section, we argued that the angular momentum of the stack of branes[100] breaks supersymmetry. We also discussed the implications that the WGC would have in non-supersymmetric backgrounds in section 2.1. Broken supersymmetry leads to a non-saturated inequality (i.e. $Q_{D_p} < T_{D_p}$) for extended objects in string theory. This would require the presence of corrections to the D_p-brane action, going beyond the rough sketch provided by the leading order, specially in the description of the brane's tension, making it subcritical. In the dual field theory description of this embedding, the nucleation of a D_3-brane can be understood as if some of the $\mathcal{N} = 4$ SYM scalar fields obtained a non-vanishing expectation value. This results in a Higgsing of the gauge group as $SU(N) \to SU(N-1) \times U(1)$. Coming back to the gravity side, this process can be described as corrections of the order $1/N$, where N is the number of D_3-branes in the stack.[101]

If this is the case, What are the options for correcting the action? There are two types of corrections (up to a power of the string coupling g_s) that can contribute roughly at the same order

[100] The high chemical potential μ and the non-zero temperature T in the dual field theory.

[101] One could argue that the expected corrections should be of order $1/N^2$, because all fields would belong to the adjoint representation. However, the relevant description of the group is **after** nucleation, hence the fundamental representation of $SU(N-1)$, with the expected correction translation to $1/N$.

of magnitude $\frac{1}{N}$.

- **"Stringy" corrections**: These corrections at $\mathcal{O}(\alpha'^2)$ can be obtained for both relevant parts of the D_p-brane action. On the one hand, curvature corrections for the DBI action were found in [?]. On simple grounds, each term in the effective action is required to be consistent with those terms of order $\mathcal{O}(\alpha'^2)$ of the corresponding disk-level scattering amplitude [?]. On the other hand, the WZ curvature corrections can be found by imposing that the chiral anomaly on the world volume of intersecting D-branes cancels with the anomalous variation of the WZ action [?]. These additional terms would schematically represent a correction to the tension $\delta\sigma$ of the form:
$$\delta\sigma \sim T_{D_3}\frac{\alpha'^2}{L^4} \sim T_{D_3}\frac{1}{g_s N} \sim \frac{1}{g_s L\, G_5\, N^2} \sim \frac{1}{g_s L^4}, \tag{7.46}$$
where we have used same set of relations as in Eq. (7.43). Note that, since the length scale $\ell_s^2 = \alpha'$ is a classical scale set by the tension of the string, these contributions can be interpreted as a tree level effect.

- **Quantum loop corrections**: Quantum loop corrections can be written as:
$$T_{D_3}\frac{\ell_{10}^4}{L^4} \sim \frac{1}{g_s \ell_s^4}\frac{g_s \ell_s^4}{L^4} \sim \frac{1}{L^4}, \tag{7.47}$$
with ℓ_{10} as the ten-dimensional Planck length. The Planck length is obtained from the classical Newton's constant by introducing Planck's constant, showing that this correction comes from quantum loops [?]. A renormalized one-loop calculation for a massive field of mass m, leads to a contribution to the vacuum energy of order m^4. Hence, we see that the loops above would be generated by fields with $m \sim 1/L$, which are present in the theory. [102]

So we have two possible candidates. It is not hard to see that the most prominent correction is that of higher curvature corrections (7.46), since we want to obtain results for vacua that are weakly coupled (i.e. $g_s \ll 1$). Let us then dissect the individual contributions to the D_p-brane action.

Corrections to the DBI term

As discussed before, string theory corrections to the DBI-action at order α'^2 due to curvature were found in [?]:
$$\begin{aligned} S_{\rm DBI}^{\alpha'^2} = -\, T_{D_3}\frac{\pi^2\alpha'^2}{48}\int d^4\xi\, e^{-\Phi}\sqrt{|g|}\big[\,(R_T)_{abcd}(R_T)^{abcd} \\ -\,2(R_T)_{ab}(R_T)^{ab} - (R_N)_{abij}(R_N)^{abij} + 2(\bar{R})_{ij}(\bar{R})^{ij}\,\big]. \end{aligned} \tag{7.48}$$

Note that $(R_T)_{abcd} = R^{(4)}_{abcd}$ is the induced curvature on the brane. The curvature of the normal bundle is given by $(R_N)_{abij}$. Both are related[103] to the bulk and extrinsic curvatures through the Gauss-Codazzi equation (2.50). However, there is one term that cannot be found in appendix **??**. This is $(\bar{R})_{ij}$, which can be defined as:
$$\bar{R}^{ij} = h^{ab}R_{ab}^{(4+d)\;ij} + h^{ab}h^{cd}\left(K^i_{ac}K^j_{bd} - \chi K^i_{ab}K^j_{cd}\right). \tag{7.49}$$

Contrary to other corrections to the action, Eq. (7.49) has no clear geometrical interpretation. Let us focus our attention on its two terms containing extrinsic curvature pieces. There is an important story here.

[102] Note that this also is the mass-scale of the neutrino.

[103] Note that $\{i,j\}$ indices represent directions that are co-dimensions of the embedding space. We will later drop these indices as we are dealing with a co-dimension one hypersurface.

The first one, $K^{i}_{ac}K^{ac|j}$, was derived for the first time in [?]. The second term, $K_a^{a\,i}K_b^{b|j}$, has an undetermined coefficient χ, which could not be determined from the scattering amplitudes in [?]. How do we then fix its value? In fact, this term seems to be the responsible for contributing in a non-trivial way to the correction of the DBI piece of the dark bubble's action (7.16). In short: T and S-dualities of string theory can be used in order to fix $\chi = 1$, as was done in [?]. One has to be careful with terms that can be mixed up by the action of S-duality, but fortunately, none of them are mixed up in the computation.[104] Therefore the result $\chi = 1$ from [?] can be trusted.

Let us now use this non-trivial correction to shift the tension of the brane in the DBI piece (7.48). As the cosmological constant we would like to induce will only be relevant at late-time descriptions of the expanding cosmology, it will be enough to study a pure AdS$_5$-background. Any matter component induced by the Reissner-Nordström piece of the metric (7.2) can be ignored. Furthermore, as the cosmological constant is small compared to all other scales in the model, we can ignore the expansion of the dark bubble itself when calculating its value.[105] With this in mind, let us assume that the brane sits at a fixed position along the AdS$_5$-throat, (i.e. $r = r_0$) and on the S^5. With $AdS_5 \times S^5$ as a direct product, and the brane positioned within the AdS$_5$, all contributions to the extrinsic curvature $K^{|i}_{ab}$ with $i \neq r$ vanish. In this sense, we recover the co-dimension one shell extensively which is discussed in detail in appendix ??. With such single extra dimension, one can start to simplify the terms. By definition, $(R_N)_{ab}{}^{rr} = 0$, as it is anti-symmetric. R_T will also vanish, as the brane is flat in the late-time cosmological approximation that has been taken. Hence, the only surviving term in (7.48) is $\bar{R}^{ij}$. Trusting $\chi = 1$ in Eq. (7.49), as derived in [?], it gives:

$$\bar{R}^{rr} = -16k^2 g^{rr}. \tag{7.50}$$

Substituting this result back into Eq. (7.48), and simplifying factors, it is easy to see:

$$T_{D_3} \to T_{D_3}\left(1 - \frac{32}{3}\pi^2\alpha'^2\, k^4\right). \tag{7.51}$$

As expected, there has been a shift in the tension, making it *smaller* than the original leading contribution. This is in line with WGC implications, as discussed above. Let us now turn our attention back to the Wess-Zumino piece of the action.

Corrections to the WZ term

To discuss the possible correction contributions to the Wess-Zumino piece of the D_3-action (7.16), let us first examine its most general expression. It describes the coupling of the brane to Ramond-Ramond fields of different dimensions (represented by $\mathfrak{C}$). It is given by:

$$S_{\rm WZ} = T_{D_p}\int \mathrm{d}^{p+1}x\, \mathfrak{C} \wedge \sqrt{\frac{\mathfrak{A}(4\pi^2\alpha'^2 R_T)}{\mathfrak{A}(4\pi^2\alpha'^2 R_N)}}. \tag{7.52}$$

Here $\mathfrak{A}$ represents the amplitude computation that contains tangential or normal terms. In our specific case, since R_T and R_N vanish, there are no relevant corrections from this expression. However, there is a potential contribution related to the above, as it was shown in [?]. This is:

$$S_{WZ} \supset -\frac{\pi^2\alpha'^2}{12}\int d^4x e^{-\Phi}\sqrt{|g|}\epsilon^{a_0a_1a_2a_3}\tfrac{1}{(p+1)!}\partial_r \mathfrak{F}^{(5)}_{ra_0a_1a_2a_3}\bar{R}^{rr}, \tag{7.53}$$

where $\mathfrak{F}^{(5)}$ is the flux sourced by the presence of the D_3-brane along the non-compact directions and ϵ is the Levi-Civita symbol. This is the only relevant term of all those derived in [?] for the case

[104] We refer the reader to [?], in order to discover the subtle details of such derivation.

[105] Correction coming from induced curvature itself will be sub-sub-leading and irrelevant for the computations to be shown.

studied. NOte the presence of $\bar{R}^{rr}$, which contains the trace of the extrinsic curvature discussed above. Unfortunately, it is not possible to fully trust the results derived in [?].

However, one can argue the following: Eq. (7.53) is equipped with a term of the form

$$\partial_r \mathfrak{F}^{(5)}_{ra_0a_1a_2a_3}. \tag{7.54}$$

In the analysis of [?], assuming that the trace of the extrinsic curvature vanishes, there is no difference between ∂ and the full covariant derivative $\nabla = \partial + ...$ One could therefore argue that the correct expression should be (7.53) with a *covariant* derivative. Then the correction to the WZ will simply vanish as a result of the equations of motion for $\mathfrak{F}$ [?]! However, an explicit scattering calculation is needed to confirm these results.

Surprisingly, the previous argument only leaves the correction as the simple term derived in Eq. (7.51). What consequences can this have for the embedding of the dark bubble? Let us explore them.

7.7 N, the parameter in the shadows controlling scales

As previously discussed in chapter **??**, regular compactifications already impose severe restrictions on the size of compact dimensions, so that gravitational effects have no effect on the four-dimensional physics. But the dark bubble scenario presented above is not a regular compactification model. Could it be that its energy hierarchy holds surprises? Let us explore this in detail.

Although we may have enjoyed our short stay in the four-dimensional world, in order to obtain an explicit value for the induced dark energy, we must climb up back to the ten-dimensional ambient space described at the beginning of this chapter **??**.

This ten-dimensional rotating background,[106] sourced by the presence of the famous D_3-brane stack, is equipped with a ten-dimensional Planck length ℓ_{10} described as:

$$\ell_{10}^8 = 2^6 \pi^7 g_s^2 \alpha'^4, \tag{7.55}$$

where again, g_s is the string coupling and α' is the Regge slope parameter. Although the determination of this parameter will be key to understanding the novelty provided by the embedding of the dark bubble, it will be more convenient to refer to its associated value, the length scale of the string [107] (i.e. how long a string is), given by:

$$\ell_s = \sqrt{\alpha'} \tag{7.56}$$

Let us go down to five dimensions: Here our rotating black hole is no longer moving along some of the compact dimensions of S^5. It is a regular Reissner Nordström black hole and the secret information about its hidden angular momentum is encoded in its charge Q. Following the discussion in chapter **??**, the five-dimensional Planck length is given by

$$\ell_5^3 = \frac{\ell_{10}^8}{\mathrm{Vol}\,(S^5)} = 2^6 \pi^4 g_s^2 \frac{\alpha'^4}{L^5}\,. \tag{7.57}$$

This can be rewritten in terms of the number of branes located in the D_3-stack. Using Eq. (7.44), Eq. (7.57) becomes:

$$L^3 = \frac{N^2 \ell_5^3}{4\pi^2}\,. \tag{7.58}$$

[106] From now on and until we say otherwise, we will work in natural units $\hbar = c = 1$. We will restore them later to talk about the energy scales.

[107] This section will be an interesting dance between lengths, masses and energies. It may be appropiate to have in mind the definition of Newton's constant (**??**).

If you have a basic understanding of the correspondence discussed in [?], you may be concerned that this model has neither a low coupling nor a high number of branes N. We will see later that the number of branes in the stack is large enough for the correspondence to hold.[108] For the moment, let us take a leap of faith and assume that $g_s N \gg 1$. This implies the following hierarchy of scales:

$$\underbrace{(N/2\pi)^{2/3}\,\ell_5}_{L} \gg \underbrace{\pi^{3/8}\,(N/2\pi)^{5/12}\,\ell_5}_{\ell_{10}} \gg \ell_5, \tag{7.59}$$

Down to this point, there is nothing new. A regular compactification of a ten-dimensional space down to a five-dimensional one. All the energy scales are the expected ones. But now we have a toll ahead: the junction conditions. As discussed in chapter **??**, the presence of the brane which mediates the decay between the vacua imposes a jump in the extrinsic curvature. This leads to an induced energy-momentum tensor S_{ab}, so that the whole system remains a solution of the Einstein equation. In this section we have obtained an extremely important expression (3.23) which will be used in our next task: To find the four-dimensional Planck length in this construction.

Making use of Eq. (7.42) to rewrite Eq. (3.23), it yields:

$$\kappa_4 = 2\frac{k_- k_+}{k_- - k_+}\kappa_5 = \frac{3k^2}{-\Delta N k}N\kappa_5 = 3\frac{N}{L}\kappa_5\,, \tag{7.60}$$

where we have written $k_- \simeq k_+$ (recall the relevant regime described in Eq. (7.36)) and $\Delta N = -1$. Again, we can play the game and rewrite the AdS-length (7.58) in terms of $\ell_4 = \sqrt{\kappa_4}$ as:

$$L = \frac{N^{1/2}}{2\sqrt{3}\pi}\ell_4, \tag{7.61}$$

which allow us to massage the hierarchy (7.59) in terms of ℓ_4 instead:

$$\underbrace{\frac{N^{1/2}}{2\sqrt{3}\,\pi}\,\ell_4}_{L} \gg \underbrace{\frac{N^{1/4}}{2^{3/4}\sqrt{3}\,\pi^{3/8}}\,\ell_4}_{\ell_{10}} \gg \ell_4 \gg \underbrace{\frac{N^{-1/6}}{\sqrt{3}(2\pi)^{1/3}}\,\ell_4}_{\ell_5}. \tag{7.62}$$

And it is at this point when surprises come in. It is of crucial importance to notice how unusual the hierarchy $\ell_5 \ll \ell_4$ is. In regular compactifications, $L \gg \ell_5$ immediately leads to $\ell_5 \gg \ell_4$. In this way, gravity is stronger in the compact dimensions and hence the upper bound on the volume of those closed directions to avoid conflict with observations. But this is not the case for our bubbly construction [?]. This can be already deduced from Eq. (7.60). The presence of the large factor N is the term responsible for this inversion in the hierarchy. With a weaker gravity in the extra dimensions, there is no strict upper bound to be imposed. In this way, the dark bubble scenario acquires its own energy hierarchy not through the usual scale separation as in regular compactifications, but through charge-to-tension ratio (i.e. the sub-extremality of the bubble).

Let us now find the explicit value for the dark energy density and the energy scale associated with this embedding. From Eqs. (7.51) and (3.23), we see that:

$$\rho_\Lambda = \frac{32}{3}\frac{\pi^2\alpha'^2}{L^4}T_{D_3} \quad \rightarrow \quad \rho_\Lambda = \frac{4}{3\pi g_s\, L^4}, \tag{7.63}$$

where Eq. (7.44) has been used. Rewriting for one final time expression (7.61), by using equations (**??**), (7.58) and (7.60) we find:

$$L^2 = \frac{2}{3\pi}NG_4. \tag{7.64}$$

[108] If every brane was worth a single \$, we would be hobnobbing with Mr.Bezos and Mr. Musk.

Finally, Eq. (7.64) can be used in Eq. (7.63) to write:

$$N = \frac{\sqrt{3\pi}}{G_4\sqrt{g_s \rho_\Lambda}}. \tag{7.65}$$

The total number of branes N sitting in the stack which source the ten-dimensional geometry (7.2) can then be determined by the observed dark energy density in the low-dimensional geometry, its Newton's constant and the string coupling g_s. In order to find the low-energy value of the string coupling, we take a result to be discussed[109] in chapter **??**, Eq. (5.8):

$$M_{\text{particle}} = T_s\, L, \tag{7.66}$$

which corresponds to the mass of a four-dimensional particle represented by the end point of a fundamental string. This string has charge e and tension $T_s = 1/2\pi\alpha'$ and it is attached to the D_3-brane [?]. The end point mass will induce a four-dimensional Reisser-Nordström geometry with line invariant:

$$\mathrm{d}^2_{\mathrm{RN}_{4\mathrm{D}}} = -g(r)_{\mathrm{RN}}\mathrm{d}t^2 + g(r)^{-1}_{\mathrm{RN}}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2, \tag{7.67}$$

where

$$g(r)_{\mathrm{RN}} = 1 - \frac{2cM_{\mathrm{RN}}}{8\pi\hbar r}\ell_4^2 + \underbrace{\frac{e^2}{4\pi\epsilon_0 c\hbar}}_{\alpha_{\mathrm{EM}}}\frac{\ell_4^2}{8\pi r^2}, \tag{7.68}$$

with $\alpha_{\mathrm{EM}} \simeq \frac{1}{137}$, is the fine structure constant. Since the string is fundamental (i.e. a BPS object saturating the WGC, $q = m$), the induced geometry will be that of an entremal Reisser-Nordström geometry. With this in mind, one can obtain the following relationship between the string coupling g_s and the fine-structure constant α_{EM}:

$$M_{\text{particle}} = M_{\mathrm{RN}} \quad \rightarrow \quad g_s = \frac{2}{3}\alpha_{\mathrm{EM}}. \tag{7.69}$$

Electromagnetism is the only long-range force, apart from gravity, and it is therefore natural that the low energy value of the string coupling is determine by it.

With relation (7.69), we have all the relevant ingredients to obtain the explicit energy hierarchy of the dark bubble. Introducing this result in expression (7.65), together with $\rho_\Lambda \simeq 6.8 \times 10^{-27}\mathrm{kg/m}^3$ [?], we obtain (**now** with $\hbar$ and c reinserted):

$$N = 1.2 \times 10^{63}, \tag{7.70}$$

which fixes the hierarchy of scales (7.62) of the dark bubble construction as described in table (16). All the scales previously discussed are unambiguously fixed. Throughout the calculations, we have used the reduced Planck scale for convenience. In the table we have chosen instead to use $G_d = \tilde{\ell}_d^{d-2}$.

Let us look at some of the implications of these results:

- As shown above, the dark bubble model does not enjoy the usual scale separation required in regular compactifications. It is not the size L of these extra dimensions (including the curvature of the AdS-throat), but the charge-to-tension ratio, controlled by N, the key feature to obtain the hierarchy presented in table (16). It is noteworthy that, even though any value of L could have been possible, the model predicts modifications of gravity slightly beyond our current observations [?]. Furthermore, although this order of magnitude is not new in the literature [?], it is obtained by computations based on first principles and from a completely different approach that the aforementioned reference.

[109]This should be the only placeholder used in this thesis, i.e. the only result to be taken from a future rather than a past chapter.

Scale	Length (m)	Energy
L	5.1×10^{-5}	$3.8\,\mathrm{meV}$
$\sqrt{\alpha'}$	1.8×10^{-20}	$11.2\,\mathrm{TeV}$
$\tilde{\ell}_{10}$	1.4×10^{-20}	$13.7\,\mathrm{TeV}$
$\tilde{\ell}_{5}$	3.9×10^{-45}	$5.1 \times 10^{28}\,\mathrm{TeV}$

Figure 16: New hierarchy of scales associated to the embedding of the dark bubble in string theory.

- Also surprising are the resulting values for both the string and the ten-dimensional scales $\{\sqrt{\alpha'}, \tilde{\ell}_{10}\}$. They are also marginally above our current observations. These new scales could be detected in the form of string excitations related to photons or any other light known particles.[110]

We finish this chapter with these impressive results. Let us summarise how we did got to this point:

We started with a well-defined ten-dimensional ambient space, with its geometry sourced as a near horizon limit to the rotating stack of D_3-branes. From a five-dimensional point of view, the geometry is that of an Reisser-Nordström black hole living in an AdS space. Due to the presence of a high chemical potential μ and non-zero temperature T in the dual field theory of the model, supersymmetry is broken. This implies an unstable configuration of the stack, allowing branes to escape through the horizon r_H of the black hole and start to expand. At leading order, these escape branes are BPS objects, which translate in the absence of an induced dark energy density in the expanding four-dimensional cosmology they accomodate. However, this fact was changed when considered for higher curvature corrections to the D_3-action. This lowered the tension of the exiled brane, causing it to be non-critical, hence decorating the construction with the sought-after cosmological constant. Furthermore, feeding in our observational dark energy density to the model, resulted in a novel energy hierarchy, with precise new scales marginally beyond our sharpest observations. Is it not monumental that such a simple string theory construction yields an explicit and novel hierarchy of scales, just around the corner from our current observations?

8 Conclusions and Outlook

Acknowledgement

We would like to thank someone. [?]

A Notation and Useful Formulae

This appendix will outline the basic notation and conventions used in this work.

The metric and its coordinates

The metric can be written in a X^M-coordinate chart as:

$$g = g_{MN}(X)\ \mathrm{d}X^M \otimes \mathrm{d}X^N, \tag{A.1}$$

[110] It is important to emphasise that the conditions for such a test of the like may be more complicated than just a threshold of energy. The luminosity of each beam and the pair-production bounds may also affect to the detection of such excitations, pushing back the testability of the predictions in time.

with a Lorentzian/Euclidean signature of mostly plus entries

$$g = \mathrm{diag}(-,+,\cdots,\cdots,+), \qquad\qquad g^{\mathrm{E}} = \mathrm{diag}(+,+,\cdots,\cdots,+). \tag{A.2}$$

The determinant of the metric will be represented by $|g| = \mp \det g$. As we are going to deal with Lorentzian metrics of different dimensionalities, it could be adequate to specify the index notation used to describe these coordinates. This can be summarised as:

- Four-dimensional spacetime coordinates: Latin lower case letters

$$\{a, b, c, \cdots\}. \tag{A.3}$$

- Spatial coordinates in three dimensions or more directions:

$$\{i, j, k\}. \tag{A.4}$$

- Five-dimensional spacetime and external coordinates of higher dimensional spaces: Greek lower case letters

$$\{\mu, \nu, \gamma\}. \tag{A.5}$$

- Ten or D-dimensional spacetime coordinates: Latin capital letters

$$\{M, N, P\}. \tag{A.6}$$

In addition, it could be useful to define how coordinates and/or tensors are represented in different dimensionalities.

- Generic coordinates in four or five dimensions are defined as x.
- Generic coordinates that make reference to induced or internal geometries are indicated with y.
- Generic coordinates from ten or D-dimensions are given by X.
- Ten or D-dimensional non-geometrical tensors (like field strengths) are written with upper and lower case $\mathfrak{GothiC}$ letters.

Finally, it could be convenient to specify the form and signs of the energy-momentum tensor T_{MN}. A perfect fluid is described by:

$$T_{MN} = (\rho + p) u_M u_N + p\, g_{MN}, \tag{A.7}$$

where $u_M = (-N(t), 0, 0, 0)$ is the fluid four-velocity (assuming no relativistic motion of this), and (ρ, p) its energy density and pressure. To easily read the right entries of the energy-momentum tensor, one needs the tensor to have one contravariant index and a covariant one as:

$$T^a_b = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{A.8}$$

where we have a four-dimensional case as an example.

Differential forms

A p-form in a coordinate chart X^M is expressed in component as:

$$\omega^{(p)} = \frac{1}{p!}\omega^{(p)}_{M_1\ldots M_p}\,\mathrm{d}X^{M_1}\wedge\ldots\wedge\mathrm{d}X^{M_p}, \tag{A.9}$$

where $\wedge$ is the wedge product. This product gives a $(p+q)$-form of the factors p-form and a q-form, with $p+q\leq D$, with D the dimensionality of the space on which these forms are defined on. It is defined to be totally antisymmetric and satisfies

$$\omega^{(p)}\wedge\omega^{(q)} = (-)^{pq}\omega^{(q)}\wedge\omega^{(p)}. \tag{A.10}$$

Continuing with operations on forms, when two p-forms of the same rank are contracted, we will use the notation:

$$\begin{aligned}|\omega^{(p)}| &= \frac{1}{p!}\,\omega_{M_1\ldots M_p}\,\omega^{M_1\ldots M_p}.\\ |\omega^{(p)}|_{MN} &= \frac{1}{(p-1)!}\omega_{MM_2\ldots M_p}\,\omega_N^{\;M_2\ldots M_p}.\end{aligned} \tag{A.11}$$

Among the forms, one of the most important ones is the canonical volume form of a manifold $\mathcal{M}$. This tensor is given by the combination of the Levi-Civita symbol $\epsilon^{(LC)}_{M_1\ldots M_p}$ and the Jacobian $\sqrt{|g|}$ as:

$$\epsilon_{M_1\ldots M_p} = \sqrt{|g|}\,\epsilon^{(LC)}_{M_1\ldots M_p}, \tag{A.12}$$

where

$$\epsilon^{(LC)}_{M_1M_2\ldots M_D} = \begin{cases} +1 & \text{if } M_1M_2\ldots M_D \text{ is even permutation of } 01\ldots D,\\ -1 & \text{if } M_1M_2\ldots M_D \text{ is odd permutation of } 01\ldots D,\\ 0 & \text{else .}\end{cases} \tag{A.13}$$

This tensor allows us to define the **Hodge** operator as:

$$\star\left(\mathrm{d}X^{M_1}\wedge\ldots\wedge\mathrm{d}X^{M_p}\right) = \frac{1}{(D-p)!}\epsilon_{N_1\ldots N_{D-p}}{}^{M_1\ldots M_p}\mathrm{d}X^{N_1}\wedge\ldots\wedge\mathrm{d}X^{N_{D-p}}. \tag{A.14}$$

Finally, we define the exterior derivative d as a linear map from p-forms to $(p+1)$-forms of the form:

$$\mathrm{d}\omega^{(p)} = \frac{1}{p!}\partial_{[N}\omega^{(p)}_{M_1\ldots M_p]}\mathrm{d}X^N\wedge\mathrm{d}X^{M_1}\wedge\ldots\wedge\mathrm{d}X^{M_p}. \tag{A.15}$$

Hypergeometric functions and Chebyshev polynomials

Hypergeometric differential equations are of the form:

$$z(1-z)\frac{d^2w}{dz^2} + [c-(a+b+1)z]\frac{dw}{dz} - abw = 0, \tag{A.16}$$

with three singular points at $z=\{0,1,\infty\}$. Solutions are given by hypergeometric functions as:

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots \tag{A.17}$$

When $b=-a$, with $a\in\mathbb{Z}$ and $c=1/2$, Eq. (A.17) can be expressed as:

$${}_2F_1(a,-a,1/2;z) = T_n\left(1-2z\right). \tag{A.18}$$

Here $T_n(x)$ represents the Chebyshev polynomial of first kind, given by:

$$T_n(x) = \cos\left(n\arccos(x)\right), \tag{A.19}$$

and second kind $U_n(x)$:

$$U_n(x)^2 = \frac{T_{n+1}(x)^2-1}{x^2-1}. \tag{A.20}$$

Vector and tensor spherical harmonic decomposition

Our dear reader is probably aware of the joy and relief that scalar spherical harmonics provide. This celebration can be continued with the decomposition of vectors and tensors in such a delightful basis. We will not explictly describe each of its components in this section (see [**?**]). Here we will only present its basic properties.

The spatial geometry will be that of S^3 as described in Eq. (B.7). Any **vector** can be decomposed as:

$$V^a = g^{ab} \sum_{k=k_{\min}}^{\infty} \sum_{\ell=\ell_{\min}}^{k} \sum_{m=-\ell}^{\ell} V^{k\ell m} Y^{k\ell m}_{(1)b}, \tag{A.21}$$

where $V^{k\ell m}$ is a Fourier coefficients with time dependence. $Y^{k\ell m}_{b}$ is the spherical harmonic part, that can be composed in three different ways. For convenience, we will only show here the first way in Lindblom's paper [**?**].

$$Y^{k\ell m}_{(1)a} = \frac{1}{\sqrt{\ell(\ell+1)}} \epsilon_a{}^{bc} \nabla_b Y^{k\ell m} \nabla_c \cos\gamma, \tag{A.22}$$

where $\epsilon_a{}^{bc}$ is the full antisymmetric tensor volume and ∇ the covariant derivative with respect to S^3 geometry. This harmonic decomposition satisfies the following eigenvalue relation and divergence identity:

$$\begin{aligned} \nabla^b \nabla_b Y^{k\ell m}_{(1)a} &= 1 - k(k+2) Y^{k\ell m}_{(1)a}, \\ \nabla^a Y^{klm}_{(1)a} &= 0. \end{aligned} \tag{A.23}$$

Tensors allow for richer and more involved compositions (up to five different ways). In this thesis we will only present the fourth one from Lindblom:

$$T_{ab} = g^{ab} \sum_{k=k_{\min}}^{\infty} \sum_{\ell=\ell_{\min}}^{k} \sum_{m=-\ell}^{\ell} T^{k\ell m} Y^{k\ell m}_{(4)ab}, \tag{A.24}$$

where

$$\begin{aligned} Y^{klm}_{(4)ab} =& \sqrt{\frac{(\ell-1)(\ell+2)}{2k(k+2)}} \left\{ \frac{1}{2} E^{k\ell} \left(\nabla_a F^{\ell m}_b + \nabla_b F^{\ell m}_a \right) \right. \\ & \left. + \csc^2\gamma \left[\frac{1}{2}(\ell-1)\cos\gamma E^{k\ell} + C^{k\ell} \right] \left(F^{\ell m}_a \nabla_b \cos\gamma + F^{\ell m}_b \nabla_a \cos\gamma \right) \right\}. \end{aligned} \tag{A.25}$$

The quantity $E^{k\ell}$ is given by:

$$E^{\ell\ell}(\gamma) = -\frac{2\cos\gamma}{\ell-1} C^{\ell\ell}(\gamma), \tag{A.26}$$

if $k = \ell$. For $k > \ell$:

$$E^{k\ell}(\gamma) = -\frac{2(k+2)\cos\gamma}{(\ell-1)(\ell+2)} C^{k\ell}(\gamma) + \frac{2\sqrt{(k+1)(k-\ell)(k+\ell+1)}}{(\ell-1)(\ell+2)\sqrt{k}} C^{k-1\ell}(\gamma), \tag{A.27}$$

where $C^{k\ell}$ are Gegenbauer polynomials [**?**]. Finally, $F^{\ell m}_a$ is:

$$F^{\ell m}_a = \frac{1}{\sqrt{\ell(\ell+1)}} \epsilon_a{}^{bc} \nabla_b \left(\sin^\ell \gamma Y^{\ell m} \right) \nabla_c \cos\gamma, \tag{A.28}$$

As harmonic functions, they satisfy the following relationships (the eigenvalue, divergence and traceless equations):

$$\begin{aligned} \nabla^c \nabla_c Y^{k\ell m}_{(4)ab} &= 2 - k(k+2) Y^{k\ell m}_{(4)ab}, \\ \nabla^a Y^{klm}_{(4)ab} &= 0, \\ g^{ab} Y^{klm}_{(4)ab} &= 0. \end{aligned} \tag{A.29}$$

Isotropy and homogeneity

To integrate over all possibles phases of a wave (chapters **??** and **??**).

$$\langle T^i{}_j \rangle = \lim_{L\to\infty} \left(\tfrac{1}{L} \int_1^\infty \mathrm{d}\theta\, T^i{}_j \right). \tag{A.30}$$

To isotropise:

$$\langle T^i{}_j \rangle_{\text{Total}} = \tfrac{1}{3} \left(\langle T^i{}_j \rangle_{\theta_1} + \langle T^i{}_j \rangle_{\theta_2} + \langle T^i{}_j \rangle_{\theta_3} \right) \tag{A.31}$$

Some Riemannian geometry relations

$$[\nabla_a \,, \nabla_b]\, X^{cd} = R^c{}_{iab}\, X^{id} + R^d{}_{jab}\, X^{dj} \tag{A.32}$$

$$\nabla_a v^a = \frac{1}{\sqrt{|g|}} \partial_a \left(\sqrt{|g|} v^a \right) \tag{A.33}$$

B Aspects of Four-Dimensional Cosmology

Decorating the cosmos with waves

As we have seen above, the universe is not empty. Although almost to 95% of its content is made up of the unknown (dark energy and dark matter), there is still a tiny fraction that we know very well how to describe dynamically. In this part of the chapter, we will focus on two radiation fields that fill the spacetime: These are gravitational and electromagnetic waves. As the aim of this thesis is to be as pedagogical as possible, the purpose of this section is doublefold:

1. To present these fields acting and propagating in a well-known enviroment, where the dynamics can be easily computed and imagined.
2. To present important techniques and methods that will be used later in this thesis.

These fields will be again revisited when the higher-dimensional approach to cosmology is studied in chapter **??**.

Shake it!

When gravity, in its full glory, is introduced in a course on general relativity for the first time, it is normally presented at leading order. The geometry of spacetime is presented pure, without any possible little perturbations that may mess up in our computations. But the universe is not so pure. Albert was already aware of this fact back in 1916, when he presented his general theory of relativity [**?**, **?**]. One shocking prediction of GR was the existence of gravitational waves (GW): Ripples in the fabric of spacetime, causing ondulations that would propagate at the speed of light c in all directions, away from the source causing it. The culprits of these events could be cataclysmic events, such as supernovae or other extremely massive accelerating objects, such as a pair of black holes orbiting around each other. Even the very beginning of the universe could have left imprinted these waves in its spacetime, but extremely weak and diluted to be detected by the current technology. It took 100 years to get observational evidence of gravitational waves, but it was very well worthed the Nobel prize for Barry Barish, Kip Thorne and Rainer Weiss as leaders of the LIGO project that detect them for the first time [**?**].

We will describe gravitational perturbations in general relativity[111] as a perturbation of the metric g_{ab} as

$$g_{ab} = g^{(0)}_{ab} + \varepsilon\, g^{(1)}_{ab} + \varepsilon^2\, g^{(2)}_{ab} + \mathcal{O}(\varepsilon^3), \tag{B.1}$$

[111] In four dimensions. Generalisations to D dimensions are straightforward.

with $|\varepsilon| \ll 1$ as the formal expansion parameter. The conventional procedure is to solve Einstein's equation order by order in ε. If one plugs Eq. (B.1) into Einstein equation, it yields:

$$\begin{aligned} G_{ab} + \Lambda g_{ab} =& \varepsilon^0 \left(G^{(0)}_{ab}[g^{(0)}] + \Lambda g^{(0)}_{ab} \right) \\ &+ \varepsilon^1 \left(G^{(1)}_{ab}[g^{(1)}] + \Lambda g^{(1)}_{ab} \right) \\ &+ \varepsilon^2 \left(G^{(1)}_{ab}[g^{(2)}] + G^{(2)}_{ab}[g^{(1)}] + \Lambda g^{(2)}_{ab} \right) + \mathcal{O}(\varepsilon^3) = 0, \end{aligned} \tag{B.2}$$

where $G^{(i)}_{ab}[g^{(j)}]$ denotes the i-th order variation of the Einstein tensor evaluated on the j-th order metric perturbation. Formally this is a quantity of order $\max\{i, j\}$ in ε.

At zeroth order, Einstein's equation simply yields the background geometry $g^{(0)}_{ab}$. This is the usual result, already discussed in section ??. Gravitational waves appear at first order in ε through the linearized Einstein equation (the GW equation), which can be schematically understood as solutions to

$$G^{(1)}_{ab}[g^{(1)}] + \Lambda g^{(1)}_{ab} = 0. \tag{B.3}$$

The second order has a more subtle meaning. It can be written as:

$$G^{(1)}_{ab}[g^{(2)}] + \Lambda g^{(2)}_{ab} = \underbrace{-G^{(2)}_{ab}[g^{(1)}]}_{\kappa_4 T^{(2)}_{ab}}, \tag{B.4}$$

where we have moved the quadratic term $G^{(2)}_{ab}[g^1]$ to the right hand side. In this way, this term can be interpreted as the energy-momentum tensor T_{ab} generated by the presence of the waves that fill the spacetime geometry. These perturbations will "decorate" the cosmos with specific energy densities representing its nature.

One could then continue with higher and higher order corrections, exploring how further subleading perturbations would create further contributions to the energy-momentum tensor and how these would affect the whole geometry up to any order in the perturbation series. In order to keep our reader's sanity in its finest, we will just stick up to the second order correction.

Let us then consider gravitational waves in an expanding FLRW cosmology with spherical spatial sections. We will describe gravitational waves using the metric

$$\mathrm{d}s^2 = g^{(0)}_{ab} + \varepsilon\, g^{(1)}_{ab} = a(\eta)^2 \left(-\mathrm{d}\eta^2 + \mathrm{d}\Omega_3^2 \right) + \varepsilon\, a(\eta)^2 p_{ij}(\eta, \{\theta\})\, \mathrm{d}\Omega_3^2, \tag{B.5}$$

where η represents the conformal time,

$$\eta = \int \frac{dt}{a(t)}, \tag{B.6}$$

$\mathrm{d}\Omega_3$ the spatial piece of the line invariant (??) with positive curvature $k = 1$

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\alpha^2 + \sin^2 \alpha \mathrm{d}\beta^2 + \sin^2 \alpha \sin^2 \beta \mathrm{d}\gamma^2, \tag{B.7}$$

and the tensor $p_{ij}(\eta, \{\theta\})$ with $\theta = \{\alpha, \beta, \gamma\}$ represents the perturbation of the spatial sections of the metric. This tensor is a transverse and traceless contribution in the form of gravitational spherical harmonics.[112] For a single mode $p_{ij}(\eta, \{\theta\}) = \zeta(\eta) Y^n_{ij}(\{\theta\})$, labelled by some discrete wave number n, the GW equation is

$$\frac{\mathrm{d}^2\zeta}{\mathrm{d}\eta^2} + 2\mathtt{H}\frac{\mathrm{d}\zeta}{\mathrm{d}\eta} + (n^2 - 1)\zeta = 0. \tag{B.8}$$

[112] In summary, every vector and tensor in the spatial sections can be decomposed into spherical harmonics. Further information can be found in appendix ??.

$\mathtt{H}$ is the conformal Hubble parameter, given by[113]

$$\mathtt{H} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\eta} = -\cot\eta. \tag{B.10}$$

To simplify the integration of Eq. (B.8), it is useful to redefine the time coordinate to

$$v = \cos\eta, \qquad\qquad \text{with } \ v \ \in \ [0,1). \tag{B.11}$$

Inserting this definition in Eq. (B.8) yields

$$\left(1-v^2\right)\frac{\mathrm{d}^2\zeta}{\mathrm{d}v^2} + v\frac{\mathrm{d}\zeta}{\mathrm{d}v} + (n^2-1)\zeta = 0. \tag{B.12}$$

Three regular singular points: $(v = -1, +1, \infty)$... This smells like an hypergeometric differential equation (A.16). Solutions can be written in the form (A.17)

$$\zeta(v) = \ {}_2F_1\left(-\frac{n+1}{2}, \frac{n-1}{2}, -\frac{1}{2}; 1-v^2\right), \tag{B.13a}$$

$$\tilde{\zeta}(v) = \left(1-v^2\right)^{3/2} \ {}_2F_1\left(1-\frac{n}{2}, 1+\frac{n}{2}, \frac{5}{2}; 1-v^2\right). \tag{B.13b}$$

Luckily, as the wave number n is an integer, these hypergeometrics can be rewritten in terms of the Chebyshev polynomials $\{T_n, U_n\}$ (A.19, A.20):

$$\zeta(v) = vT_n\left(v\right) - \frac{n}{n+1}T_{n+1}\left(v\right), \tag{B.14a}$$

$$\tilde{\zeta}(v) = \sqrt{1-v^2}\left[vU_{n-1}\left(v\right) - \frac{n}{n+1}U_n\left(v\right)\right], \tag{B.14b}$$

Restoring the conformal time η coordinate, it finally yields:

$$\zeta(\eta) = \frac{1}{n+1}\cos\left((n+1)\eta\right) + \sin\eta\sin(n\eta) \tag{B.14c}$$

$$\tilde{\zeta}(\eta) = \frac{1}{n+1}\sin\left((n+1)\eta\right) - \sin\eta\cos(n\eta), \tag{B.14d}$$

Note that at late times, i.e. $\eta \to 0$, gravitational waves with large wave number n become like plane waves in a flat universe as [114]

$$\zeta(\eta) = -\eta\cos(n\eta) + \frac{1}{n}\sin(n\eta). \tag{B.15}$$

This means that the wave will freeze out to a constant at very late times. We will also be able to read this from the energy-momentum tensor T_{ab} for these waves. For the sake of simplicity let us consider a uniform background of gravitational waves.[115] This implies a combination of waves that are homogeneously and isotropically distributed at late time cosmologies and with large wave number n. This can be done in three easy steps:

1. Consideration of ingoing and outgoing waves through a given spatial direction.

[113]One can obtain an explicit expression for the scale factor $a(\eta)$ by solving the first Friedmann equation of (B.5). This is

$$a(\eta) = -\frac{1}{\mathtt{H}\sin\eta}, \tag{B.9}$$

with $\eta \in [-\pi/2, 0)$.

[114]Late time cosmologies take place at large scale factor a in an expanding universe. This implies that the curvature term in the first Friedmann Eq. (??) can be neglected and spatial sections can be seen as flat.

[115]This convenience will manifest itself in chapter ??.

2. Averaging the previous contribution over all possible phases that the wave may have, integrating as in Eq. (A.30).
3. Finally, isotropising, i.e. taking into account all possible directions in the spatial sections, i.e. applying Eq. (A.31).

By following the previous recipe, one finds the isotropic energy-momentum tensor (B.4) to be:[116]

$$\langle T^a{}_b \rangle_{\rm iso} = \frac{7}{8\kappa_4}\frac{1}{a^2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} + \frac{n^2}{4\kappa_4 H^2}\frac{1}{a^4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{B.16}$$

Note that the first term corresponds to the curvature! This can be seen as its energy is diluted as $\rho \sim 1/a^2$. It can also be read from the trace[117] its equation of state $p = -\rho/3$. The second term has the equation of state $p = \rho/3$ and a dilution $\rho \sim 1/a^4$, which corresponds to radiation. If the wavelength of the gravitational waves is larger than the horizon (hence the wave number is smaller) it becomes frozen, and the curvature component is all that remains.

It is interesting to see that an isotropic and homogeneous distribution of perturbations in the spacetime geometry will end up contributing with both radiation and curvature to the content of the universe. Let us now see if this is also the case for electromagnetic waves.

Charge it!

We have already presented the Einstein-Hilbert action (**??**) in section **??**, which can account for matter and/or fields within the geometry of spacetime. If the field of our choice is that of electromagnetism, this is described by a four-dimensional Einstein-Hilbert-Maxwell theory:

$$S[g_{ab}, A_a] = \int \mathrm{d}^4x\sqrt{|g|}\left(R - 2\Lambda - \frac{1}{4g^2}F_{ab}F^{ab}\right), \tag{B.17}$$

where F^{ab} is the field strength associated with the $U(1)$ gauge field A^a, and g is the gauge coupling constant. For convenience, we assume this field is sourceless: it is just there, permeating the spacetime, without no nearby source emitting it. Electromagnetic wave solutions in this background are solutions to sourceless Maxwell's equations, which can be formulated in three equivalent ways:

1. The first way is simply to vary the action with respect to the gauge field A_a, which yields the standard Maxwell equations:

$$*\mathrm{d}*F = 0 = \nabla_b F^{ab}, \qquad \mathrm{d}F = 0 = \nabla_{[a}F_{bc]}. \tag{B.18}$$

2. One can go further and combine both previous equations to obtain a wave equation for the field strength F_{ab} as:

$$\nabla^i\nabla_i F_{ab} = -2R_{abcd}F^{cd} + R_a{}^cF_{cb} + F_a{}^cR_{cb}\,, \tag{B.19}$$

where relation (A.32) has been used. This points to a remarkable feature of curved spacetime: electromagnetic waves couple to the curvature of spacetime. This implies that the curvature can act as a source of electromagnetic waves: F_{ab} can be decomposed along the world line of an observer (see [**?**, **?**] for further details)

$$F_{ab} = 2u_{[a|}E_{b]} + \epsilon_{abc}B^c\,, \tag{B.20}$$

[116] Note that we have expanded the Einstein equation up to second order in ε, but the pertubative metric (B.5) only sum up to first order. Hence, the first term on the left hand side of Eq. (B.4) is 0!

[117] We will mainly display the energy-momentum tensor and the related tensor with one index up and one down along this thesis. See appendix **??** for more information.

where $u^a = \mathrm{d}x^a/\mathrm{d}\tau$ is the 4-velocity tangent to the observer's worldline. This implies that the electric and magnetic fields observed by our witness are given by the projections $E_a = F_{ab}u^b$, and $H_a = \epsilon_{abc}F^{bc}/2$. Projecting (B.19) along u^a gives the wave equation for the electric field E^a, and its dual gives the wave equation for the magnetic field.

3. Finally, Maxwell's equations can also be written for the gauge field A_a instead of the field strength F_{ab}. Eq. (B.19) This becomes:

$$\nabla_i \nabla^i A^a - \nabla^a \nabla_i A^i - R^a{}_i A^i = 0\,. \tag{B.21}$$

The extra curvature term comes again from the commutation of the covariant derivatives (A.32). This can be further simplified if imposing the Lorenz gauge $\nabla_i A^i = 0$. With this choice, the left-hand side becomes the de Rham operator $\Delta_{(\mathrm{dR})}$, which is the ordinary Laplacian supplemented by the Ricci curvature. The resulting wave equation is also referred to as the de Rham equation,

$$\Delta_{(\mathrm{dR})} A^a = \nabla_i \nabla^i A^a - R^a{}_i A^i = 0\,. \tag{B.22}$$

Although the three formulations above are equivalent in content, choosing wisely which one to use can save us pain and time. Note that Eqs. (B.19) and (B.21) are inherently second order in the field strength and the gauge field respectively. On the other hand, the first formulation in (B.18) involves only a single derivative on F^{ab}. This will be the approach taken in this section.

Let us solve for electromagnetic waves in a homogeneous and isotropic universe with flat geometry described by

$$\mathrm{d}s^2 = a(\eta)^2 \left(-\mathrm{d}\eta^2 + \mathrm{d}x_i \mathrm{d}x^i\right), \tag{B.23}$$

with $x^i = \{x, y, z\}$. In case we want to solve for a closed geometry, one only needs to start from the non-perturbed part of Eq. (B.5), decompose the vector field A^a into vector harmonics (see appendix **??**) and solve. But the main interest of this thesis is to explain late time cosmologies, hence a conformally flat geometry like (B.23) is enough. The scale factor associated to this geometry is given by

$$a(\eta) = \frac{-1}{H\eta}. \tag{B.24}$$

We can then decompose F^{ab} in a basis of plane waves, under the following Ansantz:

$$F^{ab} = e^{i n_\lambda x^\lambda} \Upsilon^{ab}(\eta) \tag{B.25}$$

where n^λ is the wave vector and Υ is the tensorial part with conformal time dependence. Again, we will compute what the solution of one monochromatic wave (i.e. a single wave number n) is by solving Eqs. (B.18) for a wave travelling in the z-direction. This yields:

$$\Upsilon^{t,z} = \eta \Upsilon'^{\eta,i} - 4\Upsilon^{\eta,i} - in\,\eta \Upsilon^{i,z} = -\eta \Upsilon'^{i,z} + 4\Upsilon^{i,z} + in\,\eta \Upsilon^{\eta,i} = 0\,, \tag{B.26}$$

where a Υ' denotes a conformal time derivative and $x^i = \{x, y\}$-directions. Inserting this into the Ansatz (B.25) we get

$$\begin{aligned}
F^{\eta,x} &= \frac{e^{inz}}{a^4}\left(c_1 \cos(n\,\eta) + c_2 \sin(n\,\eta)\right),\\
F^{\eta,y} &= \frac{e^{inz}}{a^4}\left(c_3 \cos(n\,\eta) + c_4 \sin(n\,\eta)\right),\\
F^{x,z} &= \frac{e^{inz}}{a^4} i\left(c_2 \cos(n\,\eta) - c_1 \sin(n\,\eta)\right),\\
F^{y,z} &= \frac{e^{inz}}{a^4} i\left(c_4 \cos(n\,\eta) - c_3 \sin(n\,\eta)\right).
\end{aligned} \tag{B.27}$$

The energy momentum tensor carried by the electromagnetic wave is obtained by varying the action (B.17) with respect to A^a, resulting in

$$T^a{}_b = F^{ac}F_{bc} - \frac{1}{4}\delta^a{}_b F^{cd}F_{cd}\,. \tag{B.28}$$

Finally, in the same spirit as for the gravitational waves in section B, a uniform background of electromagnetic radiation requires an average of all possible phases and directions of the waves with different polarizations. This results in an isotropic energy momentum tensor of the form:

$$\langle T^a{}_b\rangle_{\text{iso}} = \frac{\mathcal{E}^2}{2a^4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}, \tag{B.29}$$

where $\mathcal{E}$ accounts for the amplitude of the wave, encoded by the constants c_i in eq (B.27). Note that, in contrast to the gravitational case in the previous sections, electromagnetic waves contribute only with a radiation piece with $\rho \sim a^{-4}$. Furthermore, no electromagnetic wave can become frozen, no matter how large its wavelength is.

With these results, we close this section. They will return later in chapter **??**, acting as "boundary conditions" for the higher-dimensional view of cosmology to be presented.

Quantum cosmology

The previous chapter was devoted to presenting the observable universe and its dynamics from a classical perspective. General relativity seemed to work like a charm in the usual regimes where the Λ-CDM can be applied. However, we have also seen in section **??** that it has its flaws. Without delving into too much detail, some of these issues can be solved assuming that the universe underwent a period of exponential expansion called *inflation* inmediately after the Big Bang [**?**,**?**,**?**,**?**,**?**,**?**]. Althought this proposal elegantly solves some of the Λ-CDM problems such as the horizon or flatness problem, it raises new complicated questions hard to answer. For example, one may wonder what the initial conditions for any fields controlling such exponential enlargement were. This will undoubtedly lead us towards the *fine tuning* problem [**?**]. Moreover, it is still necessary to find reliable observational evidence which can prove that such process took place at the beginning [**?**,**?**].

But there is an even more fundamental problem lying behind that of initial conditions. As Lemaître already pointed out in 1933 [**?**] and discussed in chapter **??**, the effective potential divergence in Eq. (**??**) when $a \to 0$ signals a breakdown of our classical understanding of cosmology. The *primeval atom* proposed by the priest in [**?**] could be defined as a metastable and quantum state where the notion of spacetime ceases to exist. The key word to put the focus on here is *quantum*.

As we saw in the introduction **??**, one of the main priorities of the physics community has been to find a formalism to describe all the fundamental interactions within the same framework. While electromagnetism and both the strong and weak nuclear forces can be studied within the framework of the standard model of particle physics [**?**, **?**], gravity remains as a troublemaker to our wishes, unleashing infinities that cannot be tamed. The unified description of the three forces mentioned above can work out in energetic regimes where gravity is not strong enough to affect their interactions. However, this was not what the weather looked like at the very beginning of the universe ($\sim 10^{30}\,K$). In this regime, the Compton wavelength of a particle is more or less equal to its gravitational (Schwarzschild) radius. Hence, any quantum fluctuation would "blur" the classical concept of spacetime, pointing to a breakdown of the classical description of gravity.

Although the dream is to achieve a consistent theory of quantum gravity, the rentless efforts of the physics community have not yet provided the desired result. String theory, the best candidate we have for this title, will be discussed in stage **??** of this work. In this chapter, we will prepare our minds for this task by taking a more modest approach. Instead of quantising a non-perturbative renormalisable theory as gravity, we will try to describe the *whole* state of the universe with a semi-classical description, via the canonical quantisation of General Relativity. The premise is that this semi-classical approximation should coincide with the semi-classical low-energy description of the yet-to-be found theory of quantum gravity. This is the main aim of *Quantum* Cosmology [**?**, **?**].

The idea is somehow simple: One takes one's favorite universe, described by the rules of General Relativity and proceeds to quantise canonically by following the Dirac's method [**?**] as if it was an usual quantum mechanics system. This implies identifying what the canonical variables are and to introduce a quantum wavefunction ψ (i.e. a quantum state $|\psi\rangle$ living in a Hilbert space) to represent the state of the universe. When the canonical variables and their conjugated counterparts have been promoted to operators, an Schrödinger-like equation can be defined to describe the evolution of the state of the universe. Finally, one would need to solve for the specific quantum state $|\psi\rangle$ that solves the aforamentioned equation. This requires us to provide the right set of boundary conditions. An interesting question to ask here would be: What is the right choice of boundary conditions? How can we define a "boundary" for the quantum system to be studied if we have never left it? We will see in this chapter that this still remains as a source of debate.

In this chapter we will not provide a complete review of the current state of quantum cosmology, but we will introduce the basic notions and framework that will be latter discussed (from a higher dimensional point of view) in this thesis. We will start with a discussion of the quantisation procedure for the most general four-dimensional cosmological configuration in section B. We will then realise that the amount of information to be handled is overwhelming, which will require us to drastically reduce the number of variables controlling the system. A simple, yet powerful toy model will be considered in section B, where we will also discuss about the physical implications of the most well-known boundary condition proposals.

Scene reconstruction of the quantum cosmos

Before we start with the contents of this section, we would like to invite our dear reader to enjoy the hypersurface discussion in appendix **??**, so that the lecture of this chapter will be a more pleasant experience after having acquired some familiarity with the to be used geometrical notation.

Let us start by choosing a Lorentzian manifold $\mathcal{M}$ that accepts a global time coordinate. This type of manifold always accepts time-orientability, which allows us to simplify the computation by decomposing the spacetime components. This will consist in separating the space slices from the global time coordinate t. Each spacelike hypersurface of constant time t will be denoted by Σ_t. This is is the ADM formalism, named after Arnowitt, Deser and Misner [**?**].

The set of coordinates used to describe the decomposition foliation is given by:

$$x^a = (t, x^i), \tag{B.30}$$

and the most general expression of the metric on the manifold $\mathcal{M}$ in these coordinates is:

$$\begin{aligned} \mathrm{d}s^2 &= g_{ab}\mathrm{d}x^a\mathrm{d}x^b \\ &= -\left(N^2 - N_i N^i\right)\,\mathrm{d}t^2 + 2N_i\mathrm{d}x^i\mathrm{d}t + \gamma_{ij}(t,x)\,\mathrm{d}x^i\mathrm{d}x^j, \end{aligned} \tag{B.31}$$

where $N \equiv N(t)$ represents the lapse function, as discussed in chapter **??**. The function N_i is called the shift function, and measures the path difference between the same point p on the hypersurface Σ_t at different "slices" of time t. When $N_i = 0$, one recovers the usual description in comoving spatial coordinates. We will see that these two functions will play the role of constraints when we study the dynamics of the system. The metric h_{ij} represents the spatial sections of the four-dimensional geometry, i.e. the metric induced on them.

The dynamics of these slices will be controlled by the classical Einstein-Hilbert action (**??**) enhanced by the Gibbons-Hawking-York boundary term [**?**, **?**]. This extra piece accounts for any extrinsic contribution, i.e. how the the Σ_t slices is embedded in the whole four-dimensional space. The total action is given by:

$$S[g,h,\Phi] = \frac{1}{2\kappa_4}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}\left({}^{(4)}R - 2\Lambda_4\right) + \frac{\epsilon}{\kappa_4}\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}K + S_m[\Phi], \tag{B.32}$$

where $|h| = \det h_{ab}$ the determinant of the induced metric on the boundary $\partial\mathcal{M}$. ϵ represents the norm of the normal vector n_a and K is the trace of the extrinsic curvature (see appendix **??**). Finally, any matter fields are encoded in the action term $S_m[\Phi] = S_m[\phi_0, \cdots \phi_n]$. This four-dimensional action can be broken down into its $3+1$ slice decomposition. By use of Eq. (2.53) one can write:[118]

$$S[h,\Phi] = \int \mathrm{d}L = \frac{1}{2\kappa_4}\int_{\mathcal{M}} \mathrm{d}t\,\mathrm{d}^3x\,\sqrt{|h|}\,N\,\left({}^{(3)}R - K_{ij}K^{ij} - K^2 - 2\Lambda\right) + S_m[\Phi], \tag{B.33}$$

with the extrinsic curvature explicitely given by:

$$K_{ij} = \frac{1}{2N}\left(\partial_t h_{ij} + \nabla_i N_j - \nabla_j N_i\right). \tag{B.34}$$

Although it may seem appealing to compute the equations of motion directly from the action (B.33), it will be more illustrative to perform a Legendre transformation to the Lagrangian and obtain the Hamiltonian, as proposed by the ADM formalism. This requires us to identify the canonical coordinates[119] in the system: $\{h_{ij}, N, N_i, \Phi\}$. Hence, the canonical momenta can be computed in the standard way [**?**]:

$$\begin{aligned} \pi_{ij} &= \frac{\delta L}{\delta \dot{h}_{ij}} = -\frac{\sqrt{|h|}}{2\kappa_4}\left(K_{ij} - h_{ij}K\right), \qquad \pi_i = \frac{\delta L}{\partial \dot{N}_i} = 0, \\ \pi_{\phi_n} &= \frac{\delta L}{\delta \dot{\phi}_n} = \frac{\sqrt{|h|}}{N}\left(\dot{\phi}_n - N^i\partial_i\phi_n\right), \qquad \pi_N = \frac{\delta L}{\partial \dot{N}} = 0. \end{aligned} \tag{B.35}$$

Note that the conjugated momenta associated to the lapse N and shift N_i functions are zero. This implies that we are dealing with Dirac's primary constraints [**?**]. Perhaps our reader has never heard of such constraints. Let us rephrase them in a more "peasant" language. To do this, we then write the Hamiltonian as:

$$S = \int \mathrm{d}t\,\mathrm{d}^3x\left(\pi_N\dot{N} + \pi^i\dot{N}_i - N\mathcal{H} - N_i\mathcal{H}^i\right), \tag{B.36}$$

where $\mathcal{H}_m$ represents the Hamiltonian piece for the matter fields ϕ_i and

$$\begin{aligned} \mathcal{H} &= 2\kappa_4 G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{h}}{2\kappa_4}\left({}^{(3)}R - 2\Lambda_4\right) + \mathcal{H}_m, \\ \mathcal{H}_i &= -2\nabla^j\pi_{ij} + \partial_i\mathcal{H}_m. \end{aligned} \tag{B.37}$$

Note that the derivatives of Eq. (B.36) with respect to the lapse N and shift N_i act as *Lagrange multipliers*, which will result in severe constraints as:

$$\mathcal{H}_i = 0, \qquad\qquad \mathcal{H} = 0. \tag{B.38}$$

[118]Note that the last term in Eq. (2.53) is a total derivative.

[119]Note that the choice of canonical variables is not unique. Different choices will lead to different quantum theories upon quantisation. Here we will stick to the ADM choice [**?**].

From now on, we will refer to $\mathcal{H}$ as *The* Hamiltonian. This Hamiltonian will govern the evolution of the state of the universe along the space of configurations it can have. In order to have a good notion of distances and the geometry of such territory, we will define G_{ijkl}. This object receives the name of DeWitt metric [?] and it is formulated as

$$G_{ijkl} = \frac{1}{2} h^{-1/2} \left(h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl} \right), \tag{B.39}$$

which characterises the geometry of the *superspace*. Formally, one can define this space by:

$$S(\Sigma) \equiv \{ h_{ij}(x), \phi(x) \mid x \in \Sigma \} / \operatorname{Diff}_0(\Sigma). \tag{B.40}$$

As mentioned above, the *superspace* contains all possible different metrics h_{ij} and matter field configurations that the universe can have. It is infinitely dimensional, with a finite number of coordinates $\{h_{ij}(x), \Phi(x)\}$ at each point x of the three-dimensional surface Σ.

As we know from our quantum mechanics course, the quantum state of a system is represented by the wave function ψ associated to it [?]. This object is a functional $\psi[h_{ij}, \Phi]$ of the superspace, which provides the amplitude to find a particular hypersurface Σ_t of the universe with a given three-dimensional metric h_{ij} and matter field configuration Φ. Note the absence of any explicit time dependence. This should not be a cause of concern; At the end of the day, we are aiming to quantise general relativity, so there is an implicit time dependence in the spatial h_{ij} information.

Let us now quantise the system. According to Dirac's quantisation procedure [?], substituting the canonical momenta (B.35) by operators[120]

$$\pi^{ij} = -i \frac{\delta}{\delta h_{ij}}, \qquad \pi_n = -i \frac{\delta}{\delta \phi_n}, \tag{B.41}$$

which yields the following equations for ψ:

$$\mathcal{H}_i \psi = \mathcal{H} \psi = 0. \tag{B.42}$$

The first constraint forces the wavefunction ψ to be invariant under any three-dimensional diffeomorphisms. This will not be of much relevance in this work, as we will restrict ourselves to *comoving* frame (So $N_i = 0$). For the second constraint we especifically have:

$$\left[2\kappa_4 G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} + \frac{\sqrt{h}}{2\kappa_4} \left({}^{(3)}R - 2\Lambda \right) - \mathcal{H}_m \right] \psi = 0. \tag{B.43}$$

This equation is known as the *Wheeler-De Witt* equation [?] and it will be the main object of study in quantum cosmology. It describes the dynamical evolution of the wavefunction of the universe, hence its state. It also ensures the explicit **time independence** of the "timeless" wavefunction ψ. This equation can then be thought of a zero-energy analogue of the Schrödinger equation [?] due to its similarities. At the end of the day, it describes the temporal evolution of a quantum system.

We would not like to finish this section of the chapter without commenting on one of the most studied and reliable forms of solving Schrödinger-like equations in quantum mechanics: The path integral [?, ?]. This object provides the probability amplitude for a system (i.e. a particle or a universe) to move between two different states within a give time interval. The trajectory it will follow will not be deterministic, as the uncertainty principle will limit our precision in calculating pairs

[120] Note that we have neither π_i nor π_N. This is because the to-be operator version of the constraints should act like them, which is the case of the lapse and shift functions. This implies that the wavefunction ψ is independent of them.

of physical quantities [?]. In this case, we are required then to sum over *all* possible configurations that interpolate between the final and initial states of study. Morever, as our object of study is a gravitational system with potential topological changes,[121] this would require us to account for them along the path. This can be cast as:

$$Z = \left\langle h_{ij}^{(\mathrm{f})},\, \phi_{\mathrm{f}}\,;\, \Sigma_{\mathrm{f}} \,\middle|\, h_{ij}^{(\mathrm{i})},\, \phi_{\mathrm{i}}\,;\, \Sigma_{\mathrm{i}} \right\rangle = \sum_m \int \mathcal{D}g\mathcal{D}\phi e^{iS[g,\phi]}, \tag{B.44}$$

where the sum $\sum_m$ takes into account all possible topologies that the four-dimensional geometry can have. The integration is performed over all possible g_{ab} and ϕ_i configurations, represented by $\{\mathcal{D}g, \mathcal{D}\phi\}$. The action $S[g,\phi]$ is that described in Eq. (B.32). Note that this would imply a strongly oscillating integrand which could suffer from convergence issues when integrating. One might, in principle, think that an analytic continuation to the Euclidean description (i.e. $r = it$) would tame such a problem. Nothing further from reality. Divergences will continue appearing due to the non-renormalisable nature of gravity [?, ?, ?]. Furthermore, the non-perturbative aspect of this force will lead to an unbounded from below action (B.32) [?, ?].

All in all, despite the difficulties, the path integral of gravity has been proved to be an extremely useful tool in the semi-classical (i.e. the quantum cosmology) approximation. In this regime, the path integral is a weighted sum over all solutions that extremise the action (B.32). This ease the computation, and allows us to define the wavefunction ψ describing the state of the universe as:

$$\psi[h_{ij}, \phi\,;\, \Sigma] = \int \mathcal{D}g\, \mathcal{D}\phi\, e^{i\, S[g,\phi]}. \tag{B.45}$$

This general wavefunction ψ satisfies the Wheeler-DeWitt equation (B.43). However,there is a subtle catch here; the path integral formalism does not provide a specific initial configuration state $|h_{ij}, \phi; \Sigma\rangle$. This brings us back to the initial condition issue discussed at the beginning of this chapter. We have solved for the most general solution of the wavefunction (B.45) and in order to select the *specific* wave that describes the evolution of the universe, we need to impose a set of boundary conditions on the countour of integration. From the persepective of a quantum mechanics course, this is easy. You are given a potential with some boundary conditions, impose them and pick out the solution. As an external observer of the system, you have an idea of the "shape" of the studied system.[122] Nevertheless, within the framework of quantum cosmology, where the observer is part of the system, the choice of boundary conditions to be imposed is not so clear. Ideally, such a choice should be provided by the physics of the system. However, from a four-dimensional quantum cosmology point of view proposals and debates about the choice of boundary conditions are all we have to work with. This will change when we approach this problem from a string theory point of view in chapter **??**, using the dark bubble framework [?].

Before embarking ourselves on the study of the two most common proposals for the aforamentioned discussion, let us first drastically reduce the number of degrees of freedom and limit ourselves to a reduced set of them to have a *concrete* description of a simple wavefunction describing the evolution of the universe. Then, we will be able to easily impose the two different boundary choices and delve into their physical implications.

When you have eliminated all that is impossible

Let us start by simplifying our superspace. Its infinite dimensionality does not help with computations. So the best way to proceed is to do what physicists do best; to approximate the system with

[121] Recall that we consider *all* possible intermediate states.

[122] Another way of thinking about this is to try to explain the concept of phase transition from gas to liquid, but when the only conceptual physical understanding available is that of the liquid phase.

a toy model. In this case, we will restrict our attention to just a few individual variables of the superspace and freeze any other degrees of freedom. The resulting configuration of the superspace is called *minisuperspace*.[123] This simplification will allow us to have a *tractable* set of degrees of freedom, which will facilitate any explicit computation.

The toy model that we have proposed considers the quantisation of an empty four-dimensional universe, with closed spatial section and a positive cosmological constant Λ_4. As we saw in chapter **??**, this can be described by a Friedmann-Robertson-Leimatre-Walker metric (**??**) with $k = 1$. As our aim is to obtain the dynamics controlling its evolution, we need to substitute Eq. (**??**) in the action (B.32) to obtain the Lagrangian:

$$S = \frac{\mathrm{Vol}_{S^3}}{\kappa_4} \int \mathrm{d}t N \left(-\frac{3a\dot{a}^2}{N^2} + 3a - \Lambda_4 a^3 \right), \tag{B.46}$$

where $\mathrm{Vol}_{S^3} = 2\pi^2$ appears after integrating over the closed spatial directions $x = \{\alpha, \beta, \gamma\}$. Note that the only dynamical variable present in the previous action is the scale factor a. This implies that we have reduced the minisuperspace to only one dimension. The canonical coordinates of the Lagrangian (B.46) are then given by (a, π_a), where π_a is the conjugated momentum as

$$\pi_a = \frac{\delta L}{\delta \dot{a}} = -\frac{6\mathrm{Vol}_{S^3}}{\kappa_4\, N} a\dot{a}. \tag{B.47}$$

The corresponding Legendre transformation will yield *The* classical Hamiltonian for this toy model, which is:

$$\mathcal{H} = -\frac{\kappa_4}{\mathrm{Vol}_{S^3}} \frac{\pi_a^2}{12\, a} + \frac{\mathrm{Vol}_{S^3}}{\kappa_4}\, a \left(\Lambda_4\, a^2 - 3 \right). \tag{B.48}$$

If we now quantise the system as described in section B, we need to replace the conjugated momentum π_a by $-i\partial_a$ and *The* Hamiltonian constraint $\mathcal{H} = 0$ by the Wheeler-DeWitt equation (B.43). Simplifying and rearranging terms so that the above equation resembles that of Schrödinger with an effective potential $V(a)$, we get:

$$\left[-\frac{1}{2} \frac{\partial^2}{\partial a^2} + \frac{\mathrm{Vol}_{S^3}^2}{\kappa_4^2} \underbrace{\left(6\, a^2 \left(3 - \Lambda_4\, a^2 \right) \right)}_{V(a)} \right] \psi_{4D} = 0. \tag{B.49}$$

Note that the effective potential in figure 17 has two roots at $a_0 = 0$ and $a_* = 3/\sqrt{\Lambda_4}$. Returning to the Schrödinger equation analogy, we can think of our cosmos system as being driven by the effective potential $V(a)$, which has two clear regions separated by the *turning point* a_* when $\Lambda_4 > 0$. These would be called *quantum* region when $V(a) > 0$ and the *classical* region when $V(a) < 0$.

Given the "tameness" of the potential $V(a)$ in the Wheeler-DeWitt equation (B.49), the wavefunction solution can be found using the semi-classical Wentzel-Kramers-Brillouin (WKB) approximation [**?**]. For simplicity of notation, let us define:

$$S(a_f, a_i) = \frac{\mathrm{Vol}_{S^3}}{\kappa_4} \int_{a_i}^{a_f} \mathrm{d}a' \sqrt{2|V(a')|}, \tag{B.50}$$

which is the argument of the exponents in the wavefunction solution:

$$\psi(a) = \frac{1}{|V(a)|^{1/4}} \begin{cases} \mathcal{A} e^{S(a,0)} + \mathcal{B} e^{-S(a,0)} & \text{if } a < a_*, \\ \mathcal{C} e^{iS(a,a_*)} + \mathcal{D} e^{-iS(a,a_*)}, & \text{if } a > a_*. \end{cases} \tag{B.51}$$

[123]No, physicists are not the best at naming things.

The pairs $\{\mathcal{A},\mathcal{B}\}$ and $\{\mathcal{C},\mathcal{D}\}\in\mathbb{C}$ and can be related by the WKB formulas as:

$$\begin{cases}\mathcal{A}=\frac{1}{2}e^{-S(a,0)}\left(\mathcal{C}e^{i\frac{\pi}{4}}+\mathcal{D}e^{-i\frac{\pi}{4}}\right)\\ \mathcal{B}=e^{S(a,0)}\left(\mathcal{C}e^{-i\frac{\pi}{4}}+\mathcal{D}e^{i\frac{\pi}{4}}\right)\end{cases}\qquad\begin{cases}\mathcal{C}=\frac{1}{2}\mathcal{B}e^{-S_0+i\frac{\pi}{4}}+\mathcal{A}e^{S_0-i\frac{\pi}{4}}\\ \mathcal{D}=\frac{1}{2}\mathcal{B}e^{-S_0-i\frac{\pi}{4}}+\mathcal{A}e^{S_0+i\frac{\pi}{4}}\end{cases}\tag{B.52}$$

Note that the wavefunction Eq. (B.51) and its undetermined coefficients $\{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\}$ clearly point to those infinitely many possible solutions to the Wheeler-DeWitt Eq. (B.49). Here we will use the two most common boundary condition proposals introduced in section B.

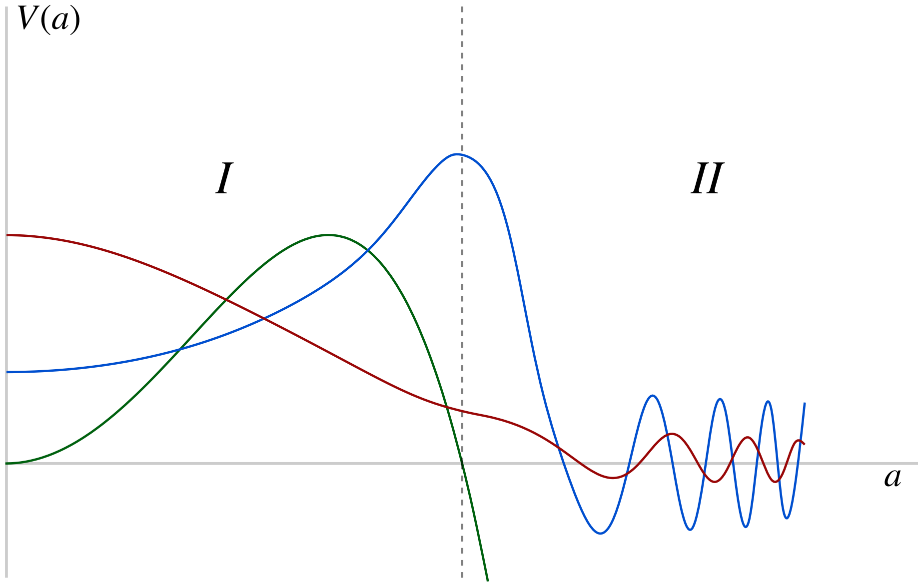

Figure 17: Plot of the potential controlling the dynamics of the wavefunction describing the evolution of the universe. The real parts of the Vilenkin and Hartle-Hawking wavefunctions are also shown. Region I corresponds to the quantum or Euclidean region, while Region II represents the classical part of the potential.

*

No-boundary proposal This proposal, argued by Hartle and Hawking [**?**, **?**], suggests that the Euclidean version of the wavefunction (B.45) should be restricted to the integration on compact four-dimensional Euclidean manifolds. This implies that the slice Σ where ψ is defined is the **only** boundary to the system. From a more physical perspective, this can be translated into the claim that the universe had **no** singular **boundary** in the past. Hence the name of the proposal.

The interpretation of the no-boundary wavefunction is that the geometry arises from nothing. Translated this to the initial condition problem this would lead to conditions on $h_{ij}(x)$ and $\phi(x)$ and its derivatives in the imaginary time component. The full discussion of how to fix these restrictions can be found in [**?**]. This implies a choice of the coefficients for region I to be such that only the increasing exponential part of the wave exists, i.e. $(\mathcal{A},\mathcal{B})=(1,0)$. Making use of the relations (B.52) one can also obtain the parametrical dependence in region II. The entire wavefunction (B.51) under the no-boundary condition proposal reads:

$$\psi_{\text{HH}}(a)=\frac{1}{|V(a)|^{1/4}}\begin{cases}e^{S(a,0)} & \text{Region I}\\ 2e^{S(a_*,0)}\cos\left(S(a,a_*)-\frac{\pi}{4}\right) & \text{Region II}\end{cases}.\tag{B.53}$$

*

Tunneling The second well-known boundary proposal is that of Vilenkin [**?**, **?**, **?**]. This proposal requires the wavefunction ψ to be everywhere bounded, and at singular boundaries of superspace, ψ includes only outgoing modes. This can be thought of as a analogy to quantum tunneling in quantum mechanics. The boundary condition imposed there is an statement about outgoing modes at ∞. From a more physical point of view, the idea behind this proposal is that any possible state described by ψ should not include universe's states contracting down from an infinite size, i.e. only expanding states from "nothing". Given the simplicity of the minisuperspace and behaviour of the wavefunction solution in the classical region, is easy to see that $(\mathcal{C}, \mathcal{D}) = (0, 1)$. Using relations (B.52), and imposing $\mathcal{A} \sim 0$, as it is exponentially supressed, we find:

$$\psi_{\mathrm{V}}(a) \approx \frac{1}{|V(R)|^{1/4}} \begin{cases} e^{S(a_*,0)} e^{-S(a,0)+i\frac{\pi}{4}} & \text{Region I} \\ e^{-iS(a,a_*)} & \text{Region II} \end{cases}, \tag{B.54}$$

The explicit form of both the Hartle-Hawking and Vilenkin wavefunction expressions (B.53) and (B.54) allow us to extract what the nucleation probability of a universe with a cosmological constant Λ_4 is. This is no more than the amplitude of the wave under the *quantum* region as:

$$P_{HH} \propto \exp\left(+\frac{24\pi^2}{\kappa_4 \Lambda_4}\right), \qquad P_V \propto \exp\left(-\frac{24\pi^2}{\kappa_4 \Lambda_4}\right). \tag{B.55}$$

Note how the Hartle-Hawking probability favours the nucleation of universes with small positive cosmological constant Λ_4, while the opposite is true for the Vilenkin case. However, the question remains; what is the best set of boundary conditions to describe the beginning of our cosmos? Although recent developments and claims [**?**] and counterclaims [**?**] have been made in the last years, we will take a different approach to settle this discussion. This will involve using string theory and its extra dimensions, which means a higher-dimensional view of the problem and letting the UV-complete theory operating in that description to fix the boundary conditions in the four-dimensional interpretation for us. We will then revisit this boundary choice discussion, but from a string theory perspective, using the dark bubble construction [**?**] in chapter **??**. But first we will venture into the realm of string theory and its extra dimensions in stage **??**.