Lecture Notes on Dark Bubble Cosmology

Daniel Panizo[a]

[a] *Somewhere*

E-mail: daniel.panizo@physics.uu.se

Abstract:

Contents

1 Introduction

2 Foundations of the Dark Bubble

2.1 Swampland conjectures

Chapter **??** was dedicated to providing a (quick) bird's eye view of the Stringy Kingdom's jurisdiction. We visited the prairies of braneworld constructions and learnt that some models lacked gravity, or did not take into account brane backreaction effects in the higher-dimensional picture or that the type of branes needed to induce a four-dimensional cosmological constant had to be supercritical. We then set off to the vast territory of the peaks and valleys of string flux compactifications. And here we learnt that this mechanism faces several fundamental problems. To build models where all the moduli are stabilised, equipped with scale separation and a small and positive cosmological constant Λ_4 seem to be unsolved issues [1, 2]. Even if some flux compactification constructions could avoid the above problems, any attempt to improve the model (i.e. to be able to reproduce some sectors of the standard model of particles) would require finding new solutions free from the problems discussed above.

These difficulties in string compactification have inspired researchers to go beyond the lands of the stringy kingdom, with the aim of **charting** its borders, if there are any. In order to do so, they proposed to take a different approach to the problem at hand; instead of aiming to obtain four-dimensional effective field theories with a positive cosmological constant Λ_4 from string theory, they thought that it could be more illuminating to identify, among all the existing lower-dimensional EFTs, those that can **actually** arise from quantum gravity. In order to avoid examining one by one the incalculable number of existing effective field theories, the idea is to find universal patterns that can be used as practical **criteria** to filter "good" EFTs from "bad" ones based on general characteristics. These screening criteria, which have not been proved yet,[1] are formulated in the form of **conjectures** that represent the very core of the **Swampland** programme [4], i.e. to identify the borders between the landscape of good EFTss and the **swamp**, those EFTss that do not meet the criteria to be obtainable from UV-complete theories.

The set of Swampland conjectures has grown since the programme was initiated in [4]. Interestingly, it has recently been shown that many of these conjectures are related, suggesting that they are no more than different aspects of some more fundamental principles of quantum gravity [5–10]. In this chapter we will not deal with the ever-growing number of swampland conjectures, but we willselect and present the basic concepts of some of them, which are relevant for the problem of finding de Sitter vacua in string theory and which are also the basis of the proposal to be presented in stage **??** of this work. The curious reader about the current state of the art of the Swampland is more than welcome to visit treferences presented above and the subsequent references therein.

This chapter is structured as follows: Section 2.1 discusses the difficulty of obtaining low-dimensional vacua with a positive cosmological constant. This is investigated by means of de Sitter conjecture(s), which set bounds on several features of de Sitter vacua from string theory. Although not related (at first sight) to the issue of positive vacua, we will examine in section 2.1 how gravity must always be the weakest force in any low energy description of quantum gravity with gauge forces. The consequences of the weakness of gravity for the fate of non-supersymmetric Anti-de Sitter spaces will be finally examined in section 2.1.

A watchtower to outlaw de Sitter theories

We have seen in chapter **??** how difficult it can be to construct de Sitter vacua in string theory. Furthermore, the few solutions that can be found are under scrutiny and are the source of debate

[1] See [3] for the first steps in proving one of these criteria.

about the components and/or mechanisms to build them [1,2,11]. One can then say that there is no full-fledged top-down de Sitter construction in a controlable regime of string theory. This suggests two different ways of thinking about this concern; on the one hand, it could be that these are just technical difficulties that will be overcome in the future as our understanding of the mathematical machinery used improves. On the other hand, this absence of de Sitter constructions points to more fundamental obstacles to the construction of de Sitter vacua in string theory. This is the route taken within the swampland programme.

We will now introduce the first swampland conjecture relevant to the present work, motivated by the difficulties of obtaining de Sitter vacua from string theory [12–14]:

de Sitter vacua, even if metastable, are excluded from quantum gravity.[2]

This de Sitter Conjecture (dSC) can also be read as: de Sitter belongs to the swampland. This very strong conjecture has been formulated as different bounds on the scalar potential of an effective field theory which is coupled to Einstein gravity. These read:

$$\min\left(\nabla_i\nabla_j V\right) \leq \frac{-c'}{M_{\rm Pl}^2}V, \qquad \text{and} \qquad |\nabla V| \geq \frac{c}{M_{\rm Pl}}V, \tag{2.1}$$

with c and c' positive constants of $\mathcal{O}(1)$ in Planck units. The $\min\left(\nabla_i\nabla_j V\right)$ represents the minimum possible eigenvalue of the mass matrix, i.e. the Hessian and $|\nabla V|$ is the norm of the vector derivatives with respect to all scalar fields in the theory. This condition provides a lower bound on the slope of the potential and states that if the potential is positive, it must be step enough not to allow for extrema points [12,15]. This last bound only needs to be imposed if the Hessian bound is violated, as proposed in the refined version of the de Sitter conjecture [16].

But these are not the only constraints that restrain the shape of the scalar potential. The Transplanckian Censorship Conjecture (TCC) states that any sub-Planckian quantum fluctuation should remain quantum. This implies that the expansion of the universe must slow down before any fluctuation become stretched beyond a Hubble size [17]. This restriction can be expressed in terms of the initial and final scale factors as follows:

$$\frac{a_f}{a_i} \times \ell_{(D)} < \frac{1}{H_f}, \tag{2.2}$$

where $\ell_{(D)}$ is the D-dimensional Planck length and H_f is the final Hubble parameter (i.e. the inverse Hubble radius at which the expansion should end to prevent quantum fluctions from becoming larger than the Hubble radius). Using both the classical Friedmann equations and the equations of motion for a scalar field ϕ coupled to gravity controlled by a potential $V(\phi)$, one can find a bound to the potential given by:

$$|\nabla V| \geq \frac{2}{M_{\rm Pl}\sqrt{(D-1)(D-2)}}V, \tag{2.3}$$

where the potential $V(\phi)$ is assumed to remain positive along the path connecting ϕ_f and ϕ_i in the moduli space.[3] The bound (2.3) appears at asymptotic regimes, i.e. at the asymptotic boundaries of the moduli space, when $\phi_i, \phi_f \to \infty$.

The similarities between the TCC bound (2.3) with the dSC bound (2.1) are not a coincidence. In fact, it can be shown that the undetermined constant c in the de Sitter bound (2.1) becomes fixed

[2] At least in the asymptotic regions of the moduli space. See the next footnote for a definition of such a space.

[3] The moduli space is the space of solutions to the equation of motion of all the fields $\Phi = \{\phi_1, \ldots, \phi_n\}$ in the theory. The asymptotic regions of this space contain the weak coupling limit theories. The values ϕ_f and ϕ_i are the coordinates of this space, and represent the final value of the scalar field when the scale factor is a_f and a_i, respectively.

to $2/\sqrt{(D-1)\,(D-2)}$ when one takes the values of the field ϕ to be in the asymptotic regimes of the moduli space. Using the Swampland Distance Conjecture (SDC),[4] one can then show that the asymptotic behaviour of the scalar potential V is the same as that predicted by the TCC.

Memorial to the weakness of gravity

Let us consider a D-dimensional effective field theory with a $U(1)$ gauge symmetry (i.e. a Maxwell theory) coupled to gravity, described by

$$S = \int \mathrm{d}^D X \sqrt{|g_{(D)}|}\left[\frac{1}{2\kappa_D}\, R^{(D)} - \frac{1}{4g^2}\mathfrak{F}_{MN}\mathfrak{F}^{MN}\right], \tag{2.4}$$

where g is the gauge coupling of the theory. The (electric) version[5] states that there must exist a particle in the theory with mass m and charge q that satisfies the inequality [19]:

$$m \leq \sqrt{\frac{D-2}{D-3}}\, g\, q\, \left(M_{\mathrm{Pl}}^{(D)}\right)^{\frac{D-2}{2}}, \tag{2.5}$$

where $M_{\mathrm{Pl}}^{(D)}$ is the D-dimensional Planck mass (**??**). This inequality becomes a strict inequality in the absence of supersymmetry.[6] Another interpretation of the inequality (2.5) is as follows:

Gravity is the weakest force of all in a theory of quantum gravity with gauge forces.

This conjecture may seem *ad hoc* at first sight, but it has deep foundations in black hole physics. Consider a four-dimensional charged (i.e. Reisner-Nordström) black hole with line invariant:

$$\mathrm{d}s^2_{\mathrm{RN}} = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2, \tag{2.6}$$

where

$$f(r) = 1 - \frac{2\, M_{\mathrm{ADM}}}{r} + \frac{2\, g^2\, Q^2}{r^2}, \tag{2.7}$$

where M_{ADM} is the mass of the black hole[7] and Q is its charge. The quadratic form of the function $f(r)$ gives two horizons for a Reisner-Nordström black hole as:

$$r_\pm = M_{\mathrm{ADM}} \pm \sqrt{M_{\mathrm{ADM}}^2 - 2g^2Q^2}. \tag{2.8}$$

The horizon is **degenerate** when $r_+ = r_-$. Furthermore, the horizon(s) require that the black hole mass satisfies an extremality bound in order to avoid the presence of naked singularities [21]. This reads:

$$M_{\mathrm{ADM}} \geq \sqrt{2}\, g\, Q\, M_{\mathrm{Pl}}, \tag{2.9}$$

where M_{Pl} has been restored to match dimensionality. A black hole that saturates the previous bound is called **extremal**. One can then see that the outer horizon shrinks towards the inner horizon as one approaches the extremality bound. The violation of this bound is associated with a naked singularity. Note that when $D = 4$, expression (2.5) and the lower bound (2.9) are the same with **opposite** orientation of the inequality. This coincidence is at the heart of black hole decay [19]. This can be understood by studying the evaporation process of a black hole.

[4] The SDC states that an infinite tower of states of an effective field theory coupled to gravity becomes exponentially light at any infinite field distance limit in the moduli space [13, 18]. As no tower of light states has been observed in our universe, we can conclude that we may not be living in such asymptotic regimes.
[5] As the magnetic version is not relevant for this work, we refer the reader to [19].
[6] According to the Coleman-Mandula theorem, the only symmetry that can relate the mass (related to Poincaré symmetry) and the charge (related to an internal symmetry) is supersymmetry [20].
[7] A definition of the ADM formalism can be found in section **??**.

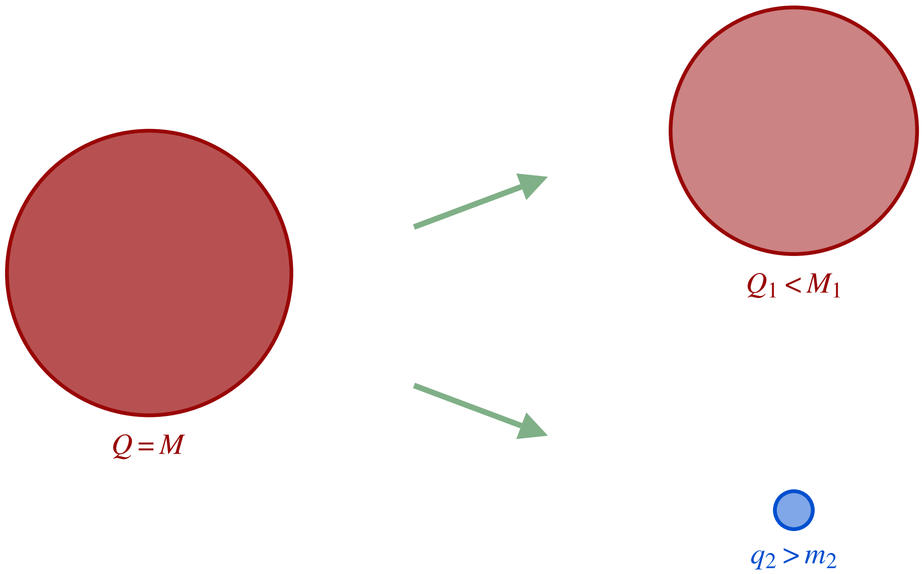

Figure 1: Decay of an extremal black hole. While one of the products can have a charge-to-mass ratio smaller than one to preserve the extremality bound, the other must have a charge-to-mass ratio greater than one. So the latter cannot be a black hole but a particle.

Black holes which are charged under a gauge $U(1)$ symmetry will induce an electric field around their horizon. This field allows for a discharge process analogous to Hawking radiation [22, 23]. While there are two different discharge channels, depending on the temperature-to-mass ratio of the black hole,[8] the one we are interested in is that of a cold black hole. This happens when $T_H \ll m$. In this case, the black hole discharges electrically through Schwinger[9] pair production.

Now consider an extremal black hole such that $M_{\rm BH}/Q_{\rm BH} = 1$ as in figure 1. A requirement for the discharge of this black hole is that the theory must contain a charged particle that can be emitted by the Schwinger process. If such particle exists, the black hole can decay into a pair of products such that one satisfies the extremality bound for black holes, i.e. $M_1 \geq Q_1$. By kinematic conservation (i.e. some of the total mass at rest of the initial black hole must be converted into kinetic energy), the second product **cannot** have a charge less than the mass, but greater than! This implies that the second object is not a smaller black hole, but a particle. This kinematic restriction can be derived from the conservation of energy and charge as follows. The mass $M_{\rm BH}$ of the initial black hole state must be greater than the sum of the decay product masses M_i. However, the charge $Q_{\rm BH}$ of the initial black hole should be equal to the sum of the products Q_i. This is

$$M_{\rm BH} \geq \sum_i M_i, \qquad Q_{\rm BH} = \sum_i q_i. \tag{2.10}$$

This entails

$$\frac{M_{\rm BH}}{Q_{\rm BH}} \geq \frac{1}{Q_{\rm BH}} \sum_i M_i = \frac{1}{Q_{\rm BH}} \sum_i \frac{M_i}{q_i} q_i \geq \frac{1}{Q_{\rm BH}} \left(\frac{M}{Q}\right)_{\rm min} \underbrace{\sum_i q_i}_{=Q_{\rm BH}} = \left(\frac{M}{Q}\right)_{\rm min} \tag{2.11}$$

[8]The Hawking temperature of a black hole is proportional to the difference between the outer and inner horizons. It is intrinsically related to the extremality of the black hole [24]. At $T_{\rm H} \gg m$, the black hole thermally discharges itself.

[9]A Schwinger effect is a predicted physical phenomenon that states that pairs of matter-antimatter particles can be created as a vacuum decay process in the presence of an electric field [25].

Therefore, the existence of a particle with a charge-to-mass ratio **greater** than that of the black hole is need for the latter to decay.

The previous discussion has been focused on a field theory with a $U(1)$ gauge symmetry. More generally, one can consider antisymmetric tensors or rank p, as the ones that charge objects such as the $(p-1)$-dimensional branes that we saw in the introduction of chapter **??**. It has been proposed that the WGC can be extended to these (black) branes.[10] Hence, given an abelian p-form with gauge coupling g, the WGC requires the existence of a charged $(p-1)$-brane of tension T and integer charge Q such that [19, 26]:

$$\frac{p(D-p-2)}{D-2}T^2 \leq \frac{Q^2}{\kappa_D^{2-D}}. \tag{2.12}$$

We will see that this has interesting consequences for the when applied to the stability of non-supersymmetric Anti-de Sitter vacua in string theory.

The necropolis of the fallen non-susy AdS vacua

In order to understand the implications of the weak gravity conjecture described above, we will consider a D-dimensional Anti-de Sitter vacuum supported by fluxes. As it was discussed in section **??**, the presence of fluxes (let us denote them by the letter f) will generate a potential that stabilises some of the moduli resulting from the compactification. Let us assume that the minimum corresponds to an AdS vacuum. These fluxes are Hodge dual to the top form gauge field strengths $\mathfrak{F}_D$, which are related to the gauge fields $\mathfrak{C}_{D-1}$. According to the WGC discussed above, this form will charge a $(D-2)$-dimensional brane, which will play the role of an hypersurface of co-dimension one (see appendix **??** for further details). This object will have a tension T, charge Q and will obey the rules defined in Eq. (2.12).

This hypersurface, located somewhere in the previously discussed D-dimensional AdS space, will interpolate between two different vacua with different values for the fluxes. Due to charge conservation the value of the flux[11] on one side of the brane will be f, while on the other side of the wall it will be equal to $f+Q$. If the vacuum hosting the brane is supersymmetric, then the brane will saturate the inequality (2.12) and remain steady at its position, separating two different supersymmetric configurations. However, if the vacuum is non-supersymmetric, then the weak gravity conjecture requires the strict inequality of equation (2.12). But such a co-dimensional one brane with tension less than the charge will correspond to an instability in the Anti-de Sitter space. Intuitively, we can see this process as simple conservation of energy: As the tension must be less than the charge, the energy cost of expanding the bubble will be less than the energy gain from the electric repulsion between different points on the wall. Therefore the brane will expand, mediating the decay of the vacuum with $f+Q$ to that inside the brane (i.e. the bubble) with vacuum f. We can then say that:

any non-supersymmetric AdS geometry supported by flux is unstable.

In other words, supersymmetry is the only mechanism to protect a vacuum decay from decaying into quantum gravity. This conjecture first appeared in [27, 28], but its foundations can be found in an older observation called *AdS fragmentation* in [29]. In this work, it was shown that in a D-dimensional Anti-de Sitter space containing a spacetime filling flux of the same rank, there will be $(D-2)$-branes charged with respect to the flux, which can nucleate and expand towards the boundary of the AdS space, leaving behind an AdS space with one less unit of flux. The viability

[10] The adjective black is added to create an analogy with black holes, as these objects are also surrounded by a horizon.

[11] Understand this value as the integral of F_p over some internal non-trivial p-cycle of the compact space.

of this process depends on the charge-to-tension ratio of the brane. In order to understand this process, let us construct the instanton solution[12] that mediates the vacuum decay.

Let us consider the following parametrisation of an AdS_D metric,

$$\mathrm{d}s^2 = L^2 \left(\cosh^2(r)\,\mathrm{d}\tau^2 + \mathrm{d}r^2 + \sinh^2(r)\,\mathrm{d}\Omega^2_{d-2}\right), \tag{2.13}$$

where L is the AdS radius. A spherical $(D-2)$-brane with radius $r(\tau)$ wrapping aroud S^{D-2} and evolving in the Euclidean time τ is described by the action [27]:

$$S = L^{D-1}\Omega_{D-2}\int \mathrm{d}\tau \left(T\sinh^{D-2}(r)\sqrt{\cosh^2(r) + \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2} - Q\sinh^{D-1}(r)\right), \tag{2.14}$$

where Ω_{D-2} is the volume of the unit $(D-2)$-sphere. One can then solve the Euclidean Einstein's equations, to find the radius for the spherical brane solution that extremises the action to be:

$$r_0 = \tanh^{-1}\left(\frac{T}{Q}\right), \tag{2.15}$$

in Planck units. In the Lorentzian signature, this is associated with the nucleation at rest of a bubble of radius r_0, which begins to expand, and mediates the decay of the vacua. Imposing the strict inequality of the WGC condition, we see that

$$r_0 < \tanh^{-1}(1). \tag{2.16}$$

This implies that a finite value of the Euclidean action can only be achieved with the strict inequality version of expression 2.12. On the contrary, if the vacuum is supersymmetric, we could satisfy the WGC by a BPS brane saturating the inequality (2.12), which implies an infinite radius solution, i.e. a non-expanding straight brane interpolating between two different supersymmetric vacua.

With this final conjecture we conclude our brief summary of the swampland program. Perhaps our dear reader is now wondering if a brane like the one mediating the aforementioned decay could be used to realise similar braneworld constructions to those discussed in section **??**. In contrast to de Sitter solutions, Anti-de Sitter ones are quite abundant in supergravity and have been studied in extension. If some of these spaces have broken supersymmetry, they should decay via nucleation of a co-dimension one brane as discussed in 2.1. Inspired by the Randal-Sundrum construction discussed in **??**, one could then find a way to induce a positive cosmological constant on the brane. However, this may not be an easy task, as the Randal-Sundrum construction requires a supercritical brane, i.e. $\sigma > Q$, while the aforementioned decay is mediated by subcritical ones $T < Q$.

At this point we would like to invite the reader to continue reading stage **??** of this work. There, we will jump into one of the nucleated branes discussed above and explore its induced cosmology.

2.2 Instantons: Coleman-de Lucia and Brown-Teitelboim

We will review two main types of instantons[13] in this appendix: Coleman-de Lucia [30] and Brown-Teitelboim [31] ones. These types of solutions are constitutional foundations of the dark bubble model proposed in part **??** of this work.

[12] The decay via brane nucleation will be a non-perturbative process, governed by the instanton (i.e. the brane) solution of the nucleated brane. More information about these solutions can be found in appendix **??**.

[13] Instantons are solutions to the classical Euclidean equations of motion which interpolate between real classical motions of the system, and thus provide a semiclassical "path" by which the system tunnels from one classical regime to the other.

Coleman-De Lucia instantons

Let us consider a theory with a single scalar field $\phi(t, \bar{x})$ which is controlled by a potential $V(\phi)$ in a D-dimensional spacetime. We further assume that this potential has *two* non-degenerate minima $V_{\pm}$, such that $V_{+}(\phi) > V_{-}(\phi)$. The shape of this potential is sketched in figure 2. The action governing the dynamics of such configuration is:

$$S = \int \mathrm{d}^D x \left(\tfrac{1}{2}\partial_M \phi \partial^M \phi - V(\phi)\right), \tag{2.17}$$

where M goes from 0 to $D-1$. From a classical point of view, if the scalar field ϕ is at rest in the local minimum V_{+}, it will not have enough energy to climb over the potential barrier and reach the global minimum V_{-}. However, if quantum mechanics is taken into account, there is a probability that the field will "tunnel" through the barrier and end up in the global minimum V_{-}. In this framework, we will say that the local minimum V_{-} is *metastable* and a *decay* can take place, so that the field configuration "traverses" from the *false* vacuum V_{+} to the *true* vacuum V_{-}.

The process described above was studied by Sidney Coleman and Curtis Callan in [32, 33]. The underlying physics of these decays is a first order transition[14] and it can be easily understood by a thermodynamic analogy.

Consider a fluid, that is heated homogenously up to the point where it can start boiling. In its initial liquid phase, as much as you can try to have an even distribution of the temperature, there may be thermodynamic fluctuations at some given points where the temperature is slightly different from the surrounding volume. If this variation is favourable towards the phase transition temperature, a *bubble* of vapour phase will appear. This bubble can have two different fates: If its size is smaller that a certain threshold, so that the gain in energy density (i.e. the energy stored inside the bubble volume) is overcompensated by the loss of surface energy, then the bubble will collapse to nothing. On the contrary, a large enough bubble will have a favourable energy balance, which will cause the bubble to expand until all the liquid has undergone the phase transition to vapour.

This is a similar situation to the vacuum decay described some paragraphs above. We are now faced with a field decorating an empty spacetime. The configuration of the field is such that it starts in its *false* vacuum. In this case, not thermodynamical, but quantum fluctuations of the vacuum can occur, triggering the phase transition towards the *true* configuration in some specific region. This will happen through the nucleation of a spherically symmetric bubble of true vacuum.

The probability per unit of volume Γ that such an event will occur can be determined in the semi-classical approximation $\hbar \ll 1$ and it is given by:[15]

$$\Gamma \sim e^{-B}\,(1 + \mathcal{O}(\hbar)), \tag{2.18}$$

with $B = S_E(\phi_I) - S_E(\phi_{-})$. The term $S_E(\phi_{-})$ is the Euclidean action of the system evaluated in the false vacuum while $S_E(\phi_I)$ is the Euclidean action for the bounce solution. The bounce is the instanton solution corresponding to the nucleated bubble, i.e. solutions that extremise and give a finite value to the Euclidean version of the action that governs the dynamics of the system. Let us discuss this quantity in more detail.

[14] i.e. the amount of energy absorbed or released by the system is fixed.

[15] This can be deduced from the transition matrix element between the ground state $|0\rangle$ evolving in time. When $\hbar$-corrections are taken into account, the associated eigen-energy of the process is $E = \frac{1}{2}\omega\hbar + \hbar K\, e^{-B/\hbar}$, where $K \in \mathbb{C}$. Identifying the corrections with the non-Hermitian piece of the evolution operator $\mathcal{H}$ of the system, one can then relate the imaginary part of the energy E with the decay channel Γ.

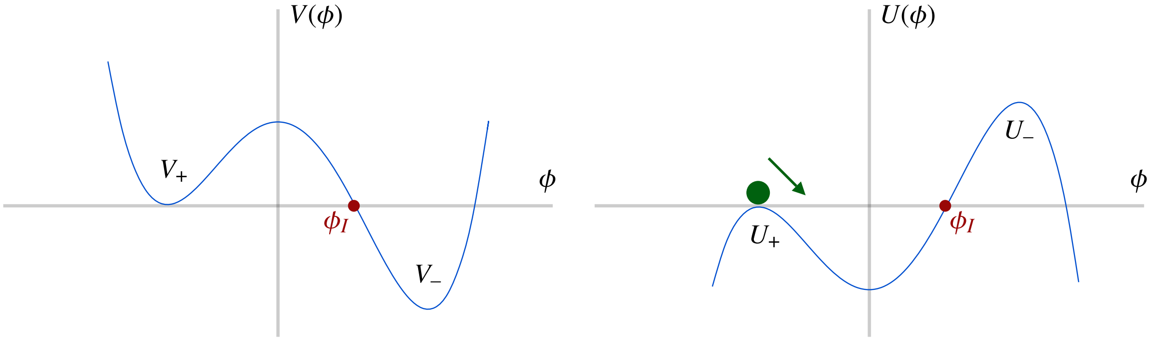

Figure 2: (Left): The potential $V(\phi)$ with its local and global minima. (Right): The inverted potential $U(\phi)$. This allows us to think in terms of the motion of a particle. Starting from ϕ_+, we then see that it can roll up to ϕ_I and bounce back. Hence the name for this type of solutions.

The bounce

In order to easily understand the bounce, we are going to work on three specific regimes that are the implicit ones in this work:

- The first one, by analytic continuation, we will perform all our computations in the Euclidean realm ($\tau = it$).
- The second one will affect the shape of the potential $V(\phi)$, as we will be working in the *thin* wall approximation. This is the case when the difference between the energy density of the two vacua is really small, so that one writes $|\Delta V(\phi)| = \varepsilon$ with $|\varepsilon| \ll 1$.
- The last regime will help us to think in terms of particle dynamics. If we invert the potential, i.e. $U(\phi) = -V(\phi)$, such as shown in figure 2, we can then use a motion analogy. In this case, a particle sitting on the "lower" hill could start rolling down, up to the point of the potential where its energy will be equal to that of the starting position. As we know from high school, this is the point that the particle will reach with zero velocity and would then *bounce* back down the valley. Restoring the sign, we then see the meaning of that bouncing point: it would be the final position at which the tunneling process ends. We will call this field position as ϕ_I, which is a solution of the equation of motion and extremise the value of the Euclidean action.

Given the previous assumptions and exploiting the implicit $O(D)$ symmetry of the Euclidean space of study, so that we can rewrite all coordinates $(\tau, \bar{x})$ in terms of a "radial" coordinate $r = \sqrt{\tau^2 + x_i x^i}$, it can be shown that the Euclidean version of action (2.17) is

$$\begin{aligned} S_E^{\text{total}} = \int \mathrm{d}^D x \left(\tfrac{1}{2}\partial_M \phi \partial^M \phi + U(\phi)\right) =& \,\Omega_{D-1} \int_R^\infty \mathrm{d}r\, r^{D-1} \left[\left(\frac{\mathrm{d}\phi_+}{\mathrm{d}r}\right)^2 + U_+(\phi)\right] \\ &+ \Omega_{D-1}\, R^{D-1} \int \mathrm{d}r\, \underbrace{\left[\left(\frac{\mathrm{d}\phi_I}{\mathrm{d}r}\right)^2 + U_+(\phi_+)\right]}_{S_1} \\ &+ \Omega_{D-1} \int_0^R \mathrm{d}r\, r^{D-1} \left[\left(\frac{\mathrm{d}\phi_-}{\mathrm{d}r}\right)^2 + U_-(\phi)\right], \end{aligned} \tag{2.19}$$

where Ω_{D-1} is the area of a unit-radius $(D-1)$-sphere given by:

$$\Omega_{D-1} = \frac{2\,\pi^{D/2}}{\Gamma(D/2)}. \tag{2.20}$$

Note that we have divided the integration regime into three different intervals:

- That corresponding to the *outside* of the bubble (first line in Eq. (2.19)). The phase transition has not yet reach to this region, so the field and associated potential $U(\phi)$ correspond to the *false* configuration.
- The *instanton* solution, which corresponds to the location of the bubble (second line). We will call this piece S_1, and it represents the Euclidean action for the bounce solution.
- The *inside* volume enclosed by it (last line). The decay has already taken place here, so we write ϕ_- and $U(\phi)$.

Let us now specify our computations for a potential such that $U(\phi_+) = 0$ and $U(\phi_-) \simeq -\varepsilon$. Plugging in these considerations into the action (2.19), we will obtain:

$$S_E^{\text{total}} = S_E(\phi_I) - S_E(\phi_-) = \Omega_{D-1}\left(R^{D-1}\, S_1 - \frac{R^D}{D}\varepsilon\right), \tag{2.21}$$

which corresponds to the expression in the argument of Eq. (2.18). If we now derive the action with respect to R, we will find the radius R that extremises the value of the action. This yields:

$$B = \frac{\Omega_{D-1}}{D}\left(\frac{D-1}{\varepsilon}\right)^{D-1} S_1^D. \tag{2.22}$$

In this way we have found the closed-form expression for the coefficient B in the thin-wall approximation. The explicit value of this term will depend on the potential shape integrand in S_1. In any case, this value ensures that the bubble nucleation probability is maximal.

When gravity wants to blow bubbles

The previous discussion was made without considering the role of gravity. When gravity is not present, any false vacuum in quantum field theory can decay. However, this is not the case when the theory is coupled to gravity. This was later studied by Coleman and de Luccia in [30], who showed that gravity can stabilise some false vacua, making them persistant.[16] Let us elaborate on this point:

The bubble, being an instanton solution, is formed at a radius R such that it minimises the Euclidean action. The formation of such a bubble has an energy cost, which is proportional to the bubble tension σ times the area of the bubble. But where does this energy come from? By conservation of energy, it must have come from reducing the vacuum energy inside its interior. This amount of energy is equal to the vacuum energy density times the volume enclosed by the bubble. This relationship between the energy contributions can be seen in Eq. (2.21). If the balance is *precise* so that one term compensates for the other, the bubble will be nucleated with the radius that minimises the action and remain at rest. In the case the bubble gains more energy than is needed to create it by reducing the vacuum energy inside it, the extra energy will provide the kinetics to expand the bubble. The bubble, which will nucleate at rest, is then accelerated outwards, asymptotically approaching the speed of light. This whole process can be understood by the expression

$$\Big|V_{\text{after}} - V_{\text{before}}\Big| = E_{\text{wall}} + E_{\text{kinetic}}, \tag{2.24}$$

[16] It can be shown that the bounce solution gets modified to by gravity as:

$$B_{\text{gravity}} = \frac{B}{(1 + (R_0/2\,\Lambda_D))^2}, \tag{2.23}$$

where R_0 is the radius that minimises the action (2.21) and Λ_D is the vacuum energy of the D-dimensional spacetime, i.e. a cosmological constant.

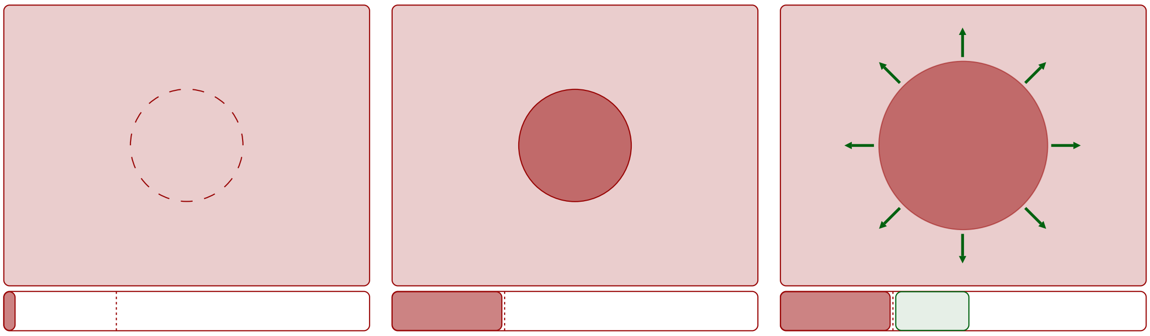

Figure 3: From left to right: Bubble nucleation requires a delicate balance between the surface energy (represented by the dashed line) and the vacuum energy difference (the bar under each rectangle). When the values are *exactly*equal to compensate each other, a bubble will nucleate at rest and remain there. However, if more energy is extracted from the decay than is required to form the bubble, the extra energy will be kinetic and used to expand the bubble.

where E_{kinetic} is the kinetic energy of the bubble if more energy was obtained by reducing the energy vacuum density inside. Note that when this term is zero, the tension σ, i.e. the "mass" of the bubble at rest is proportional to difference in vacuum energy discussed above.

Brown-Teitelboim instantons

The previous discussion only applies if the decay process occurs once. One can then wonder whether a tower of decays can be achieved within the precedent formalism. This was done by Brown and Teitelboim in [31], where they proved that one can have a tower of vacuum decays when antisymmetric tensor fields are considered in the bubble action.

Let us now suppose a vacuum supported by a cosmological constant Λ_D which can be lowered in the presence of antisymmetric tensor field strength $F = \mathrm{d}A$, where A is the corresponding tensor field. In this case, the nucleated membrane will be charged under this field A. The Euclidean action to describing this type of instantons is given by:

$$\begin{aligned} S_E = &- \frac{1}{2\kappa_D}\int \mathrm{d}^D x\sqrt{|g|}\left({}^{(D)}R - 2\Lambda_D\right) + \sigma\int \mathrm{d}^{D-1}y\sqrt{|h|} + \frac{1}{\kappa_D}\int \mathrm{d}^{D-1}x\sqrt{|g|}\,K \\ &+ \frac{1}{(D-1)!}\int \mathrm{d}^D x\sqrt{|g|}\nabla_M\left(F^{M\cdots}A_{\cdots}\right) - \frac{1}{2D!}\int \mathrm{d}^D x\sqrt{|g|}\,|F|^2 \\ &+ \frac{q}{(D-1)!}\int \mathrm{d}^{D-1}y A_{M_1\cdots M_{D-1}}e^{M_1}_{m_1}\cdots e^{M_{D-1}}_{m_{D-1}}\epsilon^{m_1\cdots m_{D-1}}, \end{aligned} \tag{2.25}$$

with ϵ is the Levi-Civitta symbol. As already discussed in appendix **??**, Eq. (2.58), the first line represents the gravitational contribution of the whole space. In this case, the energy-momentum tensor of all matter living **on** the hypersurface (i.e. the membrane), is its tension σ. The second line (2.25) represents the presence of the field strength in the D-dimensional space and a boundary term for it, while the third line denotes the coupling of the brane to the tensor field A.

As F is a form of top degree, this means that it is proportional to the volume form:

$$F_{M_1\cdots M_D} = E(x)\epsilon_{M_1\cdots M_D}, \tag{2.26}$$

for some scalar function $E(x)$, i.e. the electric field. This can be shown to be constant away from the brane sources by examining the equation of motion for the tensor field A:

$$\partial_L E(x)\,\epsilon^{L\,M_1\cdots M_{D-1}} = -q\int \mathrm{d}^{D-1}y\,\delta\left(x^\alpha - x^\alpha(y)\right)e^{M_1}_{m_1}\cdots e^{M_{D-1}}_{m_{D-1}}\epsilon^{m_1\cdots m_{D-1}}. \tag{2.27}$$

This implies that the electric field $E(x)$ will jump one charge unit q across the brane.[17] This jump will be an additional contribution to lowering the vacuum energy via brane nucleation. Indeed, if one inserts Eq. (2.26) in the action (2.25) one can read off an effective cosmological constant:

$$\Lambda_{\text{eff}} = \Lambda_D + \tfrac{1}{2}\kappa_D\, E^2. \tag{2.28}$$

Thus, when the equation of motion (2.27) holds, we can rewrite the action (2.25) as:

$$\begin{aligned} S_E = &-\frac{1}{2\kappa_D}\int \mathrm{d}^D x\sqrt{|g|}\left({}^{(D)}R - 2\Lambda_D\right) + \frac{1}{2D!}\int \mathrm{d}^D x\sqrt{|g|}\,|F|^2 \\ &+\sigma\int \mathrm{d}^{D-1}y\sqrt{|h|} + \frac{1}{\kappa_D}\int \mathrm{d}^{D-1}x\sqrt{|g|}\,K\,, \end{aligned} \tag{2.29}$$

Let us now discuss the decay channel (2.18) associated with the nucleation of a single bubble of *true* vacuum[18] when the tensor field A permeates the vacuum. Here we will just highlight the most important results of [31]. The curious reader interested in the detailed steps of the derivation is referred to that paper.

It can be shown that the bounce of the action (2.29) is given by:

$$\begin{aligned} B = \sigma\,\Omega_{D-1}\,R^{D-1} + \frac{1}{\kappa_D}\Bigg|\Bigg[&\frac{2\Lambda_i}{(D-2)}\mathrm{Vol}_D\left(R,\epsilon_i,\Lambda_i\right) \\ &+(D-1)\,\epsilon_i\sqrt{\frac{1}{R^2}-\frac{2\Lambda_i}{(D-2)(D-1)}}\,\Omega_{D-1}\,R^{D-1}\Bigg]\Bigg|_+^-, \end{aligned} \tag{2.30}$$

where we have dropped the "eff" superscript of the effective cosmological constant Λ and ϵ_i represents the orientation choice for the normal n_μ of the hypersurface Σ, i.e. the membrane, as discussed in appendix **??**. The area covered by a unit-radius $(D-1)$-dimensional Euclidean membrane is Ω_{D-1} and is given by:

$$\Omega_{D-1} = \int \mathrm{d}^{D-1}\xi\sqrt{{}^{(D-1)}g} = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \tag{2.31}$$

where $\Gamma(x)$ is the usual gamma function. The volume of the inside and the complement[19] of the outside can be computed by:

$$\begin{aligned} V_D\left(R,\epsilon_{\mathrm{i}},\Lambda_{\mathrm{i}}\right) = \int_\pm \mathrm{d}^D x\sqrt{g} = &\left(\frac{(D-1)(D-2)}{2\,|\Lambda_{\mathrm{i}}|}\right)^{D/2}\Omega_{D-1} \\ &\times\left|\int_1^{\sigma_{\mathrm{i}}\left[1-2\Lambda_{\mathrm{i}}R^2/(D-2)(D-1)\right]^{1/2}} \mathrm{d}(\cos x)\sin^{D-2}x\right|. \end{aligned} \tag{2.32}$$

In the case that the effective cosmological constant is $\Lambda < 0$, the trigonometric functions in Eq. (2.32) should be replaced by hiperbolic trigonometric functions.

In the same spirit as in the case of Coleman-De Lucia instantons, one can find the value of R that extremises the action B by deriving it with respect to R. Substituting the resulting R_0 back into B gives the extremised nucleation probability of a Brown-Teitelboim bubble in D dimensions.

[17] Due to the conservation of charge, this charge difference will be carried by on the bubble's boundary, i.e. *the (mem)brane.*

[18] As the decay can occur repeatedly, we should speak of a *less false* vacuum. Let us stick to *true* to avoid confussion.

[19] This is the volume fraction of the background that is converted into the inside region when the bubble is created, i.e. the dashed volume in figure 3.

2.3 Hypersurfaces and junction conditions

The aim of this subsection is to provide the reader with a quick overview of the wonderful world of hypersurfaces, i.e. submanifolds of dimension dim $= D - d$ which are "slices" of D-dimensional manifolds. This subsection will follow along the lines of [34] and [35].

The mathematical definition of hypersurface can be cast in the following form:

$$\mathcal{S} = \{x^\alpha \in \mathcal{M} \,|\, \Phi(x^\alpha) = 0\} \subset \mathcal{M}. \tag{2.33}$$

Perhaps mathematical definitions will cause our dear reader to have a rash, so let us simplify the previous description in more "peasant" language. In addition, as we also care about our reader's sanity, we will restrict this study to hypersurfaces of *co-dimension* one. This is $d = 1$. More brave mathematical warriors, willing to fight through *co-dimension* d sub-manifolds, are welcome to read the exquisite selected literature on the topic [36–38].

As we said before, we define a hypersurface Σ as a "slice" of a higher-dimensional space with metric $g_{\mu\nu}$. This is something that has been known since the good old high school days. For example, one can define a two-dimensional sphere in a three-dimensional flat Euclidean space by

$$\Phi(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0, \tag{2.34}$$

with R as its radius. The **embedding** map Φ tells us how to "insert" the hypersurface Σ (the sphere) in the manifold $\mathcal{M}$, i.e. the three-dimensional Euclidean space. Note that this description of the sphere respects the definition given in expression (2.33). Its coordinates $\{x, y, z\}$ are contained in $\mathcal{M}$, the three-dimensional Euclidean space and Eq. (2.34) corresponds to the restriction in the second part of the definition. Eventually, one can also choose a new set of coordinates that are **intrinsic** to the sphere itself, such as $y^a = \{\phi, \theta\}$, so that we can relate the **extrinsic** coordinates $\{x, y, z\}$ to those of the sphere by the well-known parametric relations:

$$\begin{aligned} x &= R\cos\phi\sin\theta, \\ y &= R\ \sin\phi\sin\theta, \\ z &= R\cos\theta. \end{aligned} \tag{2.35}$$

This parametric equation can be written in a more general way as:

$$x^\alpha = x^\alpha(y^a). \tag{2.36}$$

Although it has been already specified in subsection **??**, the significance of the following notation demands to emphasise again that:

- **Extrinsic** coordinates will be denoted by **Greek** letters $\{\alpha, \beta, \gamma, \cdots\}$.
- **Intrinsic** coordinates will be denoted by **Latin** letters $\{a, b, c, \cdots\}$.

It is extremely important that this notation is crystal clear, as the aim of this subsection, and by extension this thesis, is to relate extrinsic and intrinsic properties of manifolds one another, so that we can get as much information as possible from both coordinate systems.

Returning to our simple spherical example, let us continue with more definitions. As a surface, it can be equipped with vectors. The ones of our interest are of two different types: The normal and the tangent vectors.

Normal vector

It is easy to think of a normal vector n^α in the spherical case described above: A stingy arrow pointing orthogonally (outside or inside) with respect to the surface Σ. The problem arises when

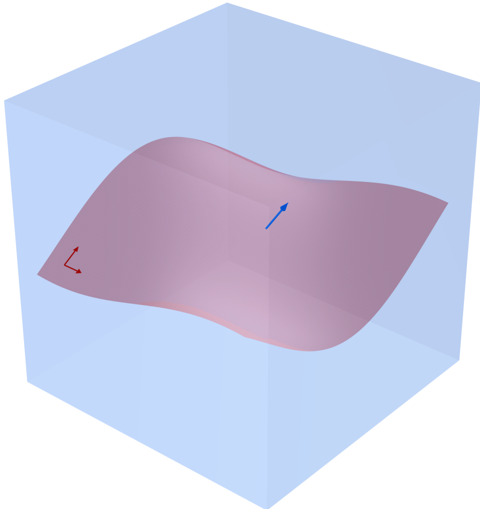

Figure 4: Two of the most basic elements to describe the embedding of a co-dimension one hypersurface Σ in a D-dimensional manifold are the normal vector n^μ and the tangent vectors e_a^α.

dealing with dimensions greater than three or signatures beyond the Euclidean one. How do you define the normal vector?

We can define a unit normal unit vector n^α imposing unitarity:

$$n^\alpha n_\alpha = \epsilon = \pm 1, \tag{2.37}$$

where $(+)$ corresponds to a timelike hypersurface and $(-)$ represents spacelike ones.[20] Furhtermore, we require that n^α points in the direction of increasing Φ. In the case we that we are looking at a spacelike surface, the normal vector will point in the direction of growing spatial sections, i.e. $n^\alpha \partial_\alpha \Phi > 0$. This implies that the normal vector can be defined as:

$$n_\alpha = \frac{\epsilon\, \partial_\alpha \Phi}{\sqrt{g^{\mu\nu}\, \partial_\mu \Phi\, \partial_\nu \Phi}}. \tag{2.38}$$

Note that for the two-dimensional sphere described above, as the metric $g^\mu_\nu = \mathbb{I}_{3\times 3}$, one recovers its usual euclidean definition.

Tangent vectors

Unlike normal vectors, tangent vectors live on the hypersurface. They are defined by:

$$e_a^\alpha = \frac{\partial x^\alpha}{\partial y^a}, \tag{2.39}$$

where again, x^α coordinates belong to the manifold $\mathcal{M}$ and y^a are coordinates of Σ, related by the parametric relation (2.36). Note that e_a^α will be a matrix, i.e. the Jacobian of the parametric transformation, with D rows and $D-1$ columns, as we are dealing with hypersurfaces of co-dimension one.

[20]Null hypersurfaces are trickier. Check [34] for further details.

As expected, tangent and normal vectors are orthogonal to each other, which means that their scalar product is null,

$$n_\alpha e^\alpha_a = 0. \tag{2.40}$$

Furthermore, tangent vectors, acting as "projectors" of D-dimensional coordinates x^α onto the hypersurface Σ, can be used to describe the **intrinsic** or **induced** line invariant on the surface. This is done by restricting the line element of the D-dimensional space to displacements confined to Σ, as:

$$\begin{aligned} ds_\Sigma &= g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta\Big|_\Sigma \\ &= g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a}\mathrm{d}y^a\right)\left(\frac{\partial x^\beta}{\partial y^b}\mathrm{d}y^b\right) \\ &= \underbrace{g_{\alpha\beta}\tfrac{\partial x^\alpha}{\partial y^a}\tfrac{\partial x^\beta}{\partial y^b}}_{h_{ab}}\,\mathrm{d}y^a\mathrm{d}y^b, \end{aligned} \tag{2.41}$$

where h_{ab} is the **induced** metric or first fundamental form. This will allow us to define the following completeness relation for the metric $g_{\alpha\beta}$ as:

$$g^{\alpha\beta} = \epsilon\, n^\alpha n^\beta + h^{ab}e^\alpha_a e^\beta_b. \tag{2.42}$$

This previous expression will be very useful in the following pages, as it relates the tangential and normal parts of the embedding in the line invariant hosting the hypersurface.

Let us now imagine a full tangential tensor $A^{\alpha\beta}$ defined on Σ, with no components in the normal directions (i.e. $A^{\alpha\beta}n_\alpha = 0$). Such tensor admits the decomposition

$$A^{\alpha\beta\cdots} = A^{ab\cdots}e^\alpha_a e^\beta_b\cdots = h^{ai}h^{bj}\cdots A_{ij\cdots}e^\alpha_a e^\beta_b\cdots \tag{2.43}$$

To understand how these tensors differenciate, we can simply apply the usual covariant derivative. However, the resulting information will depend on the chosen set of coordinates. If we choose the hypersurface coordinates y^a, it is easy to prove that:

$$\nabla_b A_a = \nabla_\beta A_\alpha e^\alpha_a e^\beta_b = \cdots = \partial_b A_a - \Gamma^i_{ab}A_i\,, \tag{2.44}$$

where $\cdots$ are intermediate steps of the computation. The expression (2.44) corresponds to the well-known *intrinsic* covariant differentiation. But this is not the end of the story. We can reproduce the same computation, but splitting the components of the metric $g_{\alpha\beta}$ into its normal and tangential pieces. For indices convenience, let us consider the vector $\nabla_\beta A_\alpha e^\beta_b$, whose tangential components are given by Eq. (2.44). This is:

$$\begin{aligned} \nabla_\beta A_\alpha e^\beta_b &= g_{\alpha\gamma}\,\nabla_\beta A^\gamma e^\beta_b \\ &= \left(\epsilon n_\alpha n_\gamma + h_{ij}e^i_\alpha e^j_\gamma\right)\,\nabla_\beta A^\gamma e^\beta_b \\ &= \epsilon\left(n_\gamma\,\nabla_\beta A^\gamma e^\beta_b\right)n_\alpha + h_{ij}\underbrace{\left(\nabla_\beta A^\gamma e^j_\gamma e^\beta_b\right)}_{\nabla_b A^j}e^i_\alpha, \end{aligned} \tag{2.45}$$

Note that the first term can be rewritten using the fact that $A^\gamma n_\gamma = 0$, as we assume that the tensor A_α is completely tangential. This allows us to rewrite the previous expression as:

$$\cdots = \nabla_b A_i e^i_\alpha - \epsilon A^i\underbrace{\left(\nabla_\beta n_\gamma e^\gamma_i e^\beta_b\right)}_{K_{bi}}n_\alpha, \tag{2.46}$$

where we have defined the symmetric *extrinsic* curvature of the hypersurface Σ or second fundamental form of the hypersurface as:

$$K_{ab} = \nabla_\beta n_\alpha e^\alpha_a e^\beta_b. \tag{2.47}$$

with trace computed after contraction against the induced metric h_{ab}

$$K = K^{ab}\, h_{ab} = \nabla_\alpha\, n^\alpha. \tag{2.48}$$

Note that the starting point in Eqs. (2.44) and (2.45) is the same; the covariant derivative of the tangent form A^α lives on Σ. Eq. (2.46) shows a pure tangential piece of the vector field (the first term) and its normal component (the second term). This piece carries geometrical information about how the hypersurface Σ is embedded in the hosting space $\mathcal{M}$ and hence, what kind of curvature it acquires. This term can only be zero if and only if the extrinsic curvature vanishes.

Gauss-Codazzi Equations

The next logical step in this discussion is to explore if the *intrinsic* Riemann tensor of the hypersurface Σ can also be expressed in terms of *extrinsic* information. Let us first recall the definition of a purely intrinsic curvature tensor as:

$$\left[\nabla_a, \nabla_b\right] A^c = R^c_{dba}\, A^d. \tag{2.49}$$

In the same spirit as in the previous computations, one can use the identities relating normal and tangent tensors in order to relate both *extrinsic* and *intrinsic* curvature tensors. This requires a modest amount of algebra and we refer the curious reader to [34]. Here we will only show the final result of the computation,

$$\begin{aligned} R_{\alpha\beta\gamma\delta} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d &= R_{abcd} + \epsilon \left(K_{ad}K_{bc} - K_{ac}K_{bd}\right), \\ R_{\mu\alpha\beta\gamma} e^\alpha_a e^\beta_b e^\gamma_c n^\mu &= \nabla_c K_{ab} - \nabla_c K_{ac}. \end{aligned} \tag{2.50}$$

These are known as the **Gauss-Codazzi** equations. They show that some components of the curvature tensor of any geometry in $\mathcal{M}$ can be decomposed into terms of the *intrinsic* and *extrinsic* curvature pieces of the hypersurface it may be hosting.

Although the Riemann tensor, in any of its forms, contains valuable information about the geometry it represents, more practical tensorial objects will be found in everyday physic computations. Let us then find expressions for both the Ricci tensor and scalar given the metric decomposition described in Eq. (2.42). For the Ricci tensor we find:

$$\begin{aligned} R_{\alpha\beta} &= g^{\mu\nu} R_{\mu\alpha\nu\beta} \\ &= \left(\epsilon n^\mu n^v + h^{mn} e^\mu_m e^v_n\right) R_{\mu\alpha\nu\beta} \\ &= \epsilon R_{\mu\alpha\nu\beta} n^\mu n^v + h^{mn} R_{\mu\alpha\nu\beta} e^\mu_m e^v_n, \end{aligned} \tag{2.51}$$

while the Ricci scalar gives:

$$\begin{aligned} R &= g^{\alpha\beta} R_{\alpha\beta} \\ &= \left(\epsilon n^\alpha n^\beta + h^{ab} e^\alpha_a e^\beta_b\right) \left(\epsilon R_{\mu\alpha\nu\beta} n^\mu n^\nu + h^{mn} R_{\mu\alpha\nu\beta} e^\mu_m e^\nu_n\right) \\ &= 2\epsilon h^{ab} R_{\mu\alpha\nu\beta} n^\mu e^\alpha_a n^\nu e^\beta_b + h^{ab} h^{mn} R_{\mu\alpha\nu\beta} e^\mu_m e^\alpha_a e^\nu_n e^\beta_b. \end{aligned} \tag{2.52}$$

It can be useful to make good use of relations (2.47) and (2.49) to further simplify the Ricci scalar expression (2.52). Some minutes of patience and algebra yield:

$$R = {}^{(D-1)}R + \epsilon \left(K^2 - K^{ab}K_{ab}\right) + 2\epsilon \nabla_\alpha \left(n^\beta \nabla_\beta n^\alpha - n^\alpha \nabla_\beta n^\beta\right). \tag{2.53}$$

This expression is the evaluation of the D-dimensional Ricci scalar on the $D-1$-dimensional hypersurface Σ. This result is extremely practical in the context of branes and hypersurfaces, especially when the action governing their dynamics requires to be split, as discussed in chapter **??**.

Junction conditions

The previous pages have been devoted to studying how the embedding of a co-dimension one hypersurface Σ can be used to provide a splitting of the hosting manifold $\mathcal{M}$ into its tangential and normal components. However, the physical concepts of such picture have not yet received our attention. For example, one can find the following situation in physics: Suppose that such a hypersurface Σ divides a spacetime geometry into two regions $\mathcal{V}^+$ and $\mathcal{V}^-$. Both regions have different metrics $g^{\pm}_{\alpha\beta}$. Furthermore, they are both solutions to the Einstein field equations. What conditions should be put on the metrics to ensure that both spaces join smoothly at Σ, so that the whole union of spaces becomes a solution to the Einstein equation? This set of requirements demanded on the geometric features of the spaces $\mathcal{M}^{\pm}$ and Σ are called **Junction conditions**. They were originally discussed in papers like [39–41].

Let us first imagine two D-dimensional manifold $\mathcal{M}_{\pm}$, described by different sets of coordinates $\{x^{\alpha}_{\pm}\}$ and equipped with metrics $g^{\pm}_{\alpha\beta}$. Furthermore, both spaces share a boundary $\partial\mathcal{M}$, which is a hypersurface Σ of co-dimension one described by a set of coordinates $\{y^a\}$. On top of all this, we assume that this composition space satisfies the D-dimensional Einstein equation. We can then try

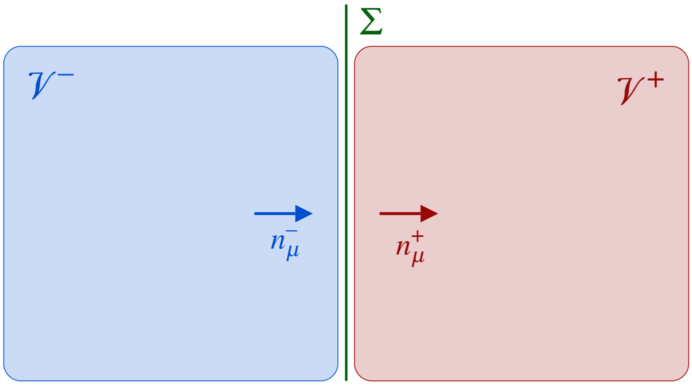

Figure 5: Pictorial representation of the composite space under study: Two different regions $\mathcal{V}^-$ and $\mathcal{V}^+$, with Σ as their common boundary. Note the choice of orientation of the normal vectors $n^{\pm}_{\mu}$. Different choices of normal orientation can be seen in figures 6 and **??**. In this work, blue will be associated to the inside/inside construction of Randall-Sundrum model in section **??**, while red will represent the inside/outside construction of the dark bubble model in part **??**.

to define a general metric $g_{\alpha\beta}$ that interpolates between two D-dimensional spaces. This can be written as:

$$g_{\alpha\beta} = \Theta(\lambda)g^+_{\alpha\beta} + \Theta(-\lambda)g^-_{\alpha\beta}, \tag{2.54}$$

where $\Theta(\pm\lambda)$ is the Heaviside distribution function and λ is an affine parameter describing geodesics connecting both regions, piercing through the hypersurface Σ. In this sense, when $\lambda > 0$, one could say to be placed in $\mathcal{M}^+$. Similarly, the minus sign represents a point of the geodesic in $\mathcal{M}^-$ and its null value is on the hypersurface Σ. Note that more complicated objects like the affine connection Γ or the Riemann tensor depend on derivatives of the metric. One has to be careful then, as we are dealing with distribution functions and combinations of them can give rise to non-distribution terms, which makes it difficult to find a physical interpretation. In fact, one finds oneself in such a situation by simply the deriving expression (2.54) with respect to any coordinate x^{γ}. This yields:

$$\partial_\gamma g_{\alpha\beta} = \Theta(\lambda)\,\partial_\gamma g^+_{\alpha\beta} + \Theta(-\lambda)\,\partial_\gamma g^-_{\alpha\beta} + \epsilon\delta(\lambda)\,n_\gamma\left[g^+_{\alpha\beta} - g^-_{\alpha\beta}\right], \tag{2.55}$$

where the last term comes from the derivative $\partial_\lambda \Theta(\lambda) = \delta(\lambda)$.[21] Note that this term will yield contributions of the form $\Theta(\pm\lambda)\delta(\pm\lambda)$ when computing the Christoffel symbols. But such a combination of distributions is not one![22] It is therefore necessary to get rid of such a term. In order to achieve this task, let us impose continuity of the metric accross the hypersurface Σ,

$$g^{+}_{\alpha\beta}|_\Sigma = g^{-}_{\alpha\beta}|_\Sigma. \tag{2.56}$$

This constraint can be further refined: The completness relation (2.42) allows us to split between tangential and normal components of the continuity relation. Here we note that $[n^\alpha]^+_- = n^+ - n^- = 0$.[23] Furthermore, coordinates $\{y^a\}$ are the same on both sides of the hypersurface Σ. This implies that tangent vectors are uniquely defined on it. With these two facts, one can rewrite Eq. (2.56) as:

$$h^{+}_{ab} = h^{-}_{ab}. \tag{2.57}$$

This is called the **first junction condition**, which forces the *induced* metric h_{ab} to be the same on both sides of Σ. This is an essential requirement for a well-defined geometry.

The derivation of the second junction can be done in different ways, with the previous mathematical approach requiring us to elaborate further along the lines discussed above.[24] While this approach can be formal and elegant, we would be need to introduce new concepts and complicated computations. Inspired by [35], a more physical approach will be presented in this section of the subsection, in line with the essence of this thesis.

We will first try to understand the geometry of a portion of the composition space presented above; A single manifold $\mathcal{M}$ and its boundary $\partial\mathcal{M}$ equipped with the metrics $g_{\mu\nu}$ and h_{ab}, respectively. The action describing the geometry and content of this space can be given as:

$$S_{\text{Total}} = S_{\mathcal{M}} + S_{\partial\mathcal{M}} + S_{\mathcal{L}_\Sigma}, \tag{2.58}$$

where $S_{\mathcal{M}}$ is the usual Einstein-Hilbert action as:

$$S_{\mathcal{M}} = \int_{\mathcal{M}} \mathrm{d}^D x \sqrt{|g|} \left(\tfrac{1}{2\kappa_D} {}^{(D)}R + \mathcal{L}_{\mathcal{M}} \right), \tag{2.59}$$

where ${}^{(D)}R$ the D-dimensional Ricci scalar and $\mathcal{L}_{\mathcal{M}}$ is the Lagrangian density for any kind of matter content in such space.

The action term $S_{\partial\mathcal{M}}$ in Eq. (2.58) is the Gibbons-Hawking-York term [42, 43] and is necessary for the propert definition of the variational principle since the Ricci scalar R is constructed from second derivatives of the metric. It describes how the submanifold is embedded as a boundary of $\mathcal{M}$. Hence, it can be expressed in terms of the extrinsic geometric pieces as:

$$S_{\partial\mathcal{M}} = \tfrac{\epsilon}{\kappa_D} \int_{\partial\mathcal{M}} \mathrm{d}^{D-1} y \sqrt{|h|} K, \tag{2.60}$$

with the induced metric h_{ab} and the trace of the extrinsic curvature K_{ab} as described in (2.41) and (2.47). Note the presence of the chosen normalisation ϵ of the normal vector n^α. Finally, the term $S_{\mathcal{L}_\Sigma}$ represents any type of matter content living **on** the boundary.

But this discussion is so far only valid for one manifold and its boundary. As described above, we will usually face situations where we find that the hypersurface Σ is the boundary of two different

[21]The required change of variables in the derivative follows from the fact that any displacement away from the hypersurface along one of the geodesics described above is given by $\mathrm{d}x^\alpha = n^\alpha \mathrm{d}\lambda$.
[22]Note that $\Theta(0) =$ indeterminate, while $\delta(0) = 1$. What is this?
[23]This requirement follows from footnote 2 plus the continuity of λ and x^α accross the hypersurface.
[24]These lines can be found in [34].

manifolds $\mathcal{M}^{\pm}$. Hence, we have to "duplicate" the previous action (2.58) and glue them together, along the boundary $\partial\mathcal{M}$ mediating between $\mathcal{M}^{-}$ and $\mathcal{M}^{+}$. This is assumed to be as the only boundary present in the construction.

Let us now derive the junction condition by applying the variational principle with respect to $g_{\mu\nu}$. In order to simplify this task and present the results in the tidiest way, we will carry out this computation piece by piece in the action. For each manifold $\mathcal{M}^{\pm}$ we find:

$$\begin{aligned}\delta S_{\mathcal{M}^{\pm}} = \tfrac{1}{2\kappa_D}\int_{\mathcal{M}_{\pm}} \mathrm{d}^D x\sqrt{|g|}\Big[&\left(G^{\pm}_{\mu\nu}-\kappa_D T^{\pm}_{\mu\nu}\right)\delta g^{\mu\nu}\\ &+\nabla_{\mu}\left(g_{\alpha\beta}\,\nabla^{\mu}\delta g^{\alpha\beta}-\nabla_{\alpha}\delta g^{\alpha\mu}\right)\Big],\end{aligned} \tag{2.61}$$

where $T_{\mu\nu}$ is the energy-momentum tensor corresponding to any matter content **in** the D-dimensional spaces

$$T_{\mu\nu} = \mathcal{L}_{\mathcal{M}}\, g_{\mu\nu} - 2\frac{\delta \mathcal{L}_{\mathcal{M}}}{\delta g^{\mu\nu}}. \tag{2.62}$$

The boundary term yields a variation of the form:

$$\begin{aligned}\delta\partial\mathcal{M}^{\pm} = \tfrac{\epsilon}{2\kappa_D}\int_{\partial\mathcal{M}^{\pm}} \mathrm{d}^{D-1} y\sqrt{|h|}\Big[&\left(K^{\pm}_{\mu\nu}-K^{\pm}g_{\mu\nu}\right)\delta g^{\mu\nu}\\ &+n_{\mu}\left(\nabla_{\alpha}\delta g_{\alpha\mu}-g_{\alpha\beta}\nabla^{\mu}\delta g^{\alpha\beta}\right)\Big].\end{aligned} \tag{2.63}$$

Note that the second line of each expression cancel each other out by the Gauss-Stokes theorem [34].

With the expressions (2.61) and (2.63) at hand and the action piece $S_{\mathcal{L}_{\Sigma}}$ in Eq. (2.58) representing matter content **on** the wall, one can then compute the dynamics of the whole composite space. However, one must be careful, as the normal n^{μ} is chosen to point in the direction of increasing volume in the transverse directions, i.e. from $\mathcal{M}_{-}$ to $\mathcal{M}_{+}$. This implies a change of the sign[25] for $n^{\mu}_{\pm}$. This will affect the definition of the extrinsic curvature $K_{\mu\nu}$, as it contains the normal vector n^{μ} inside (see Eq. (2.47) and figure 5). Consequently, we have:

$$n^{\mu} = -n^{\mu}_{+} = n^{\mu}_{-}. \tag{2.64}$$

The whole composition space (2.58) then reads:

$$\begin{aligned}\delta S_{\text{Total}} =\ & \delta S_{\mathcal{M}_{+}} + \delta S_{\mathcal{M}_{-}} - \delta S_{\partial\mathcal{M}_{-}} + \delta S_{\partial\mathcal{M}_{+}} + \delta S_{\mathcal{L}_{\Sigma}} =\\ =\ & \tfrac{1}{2\kappa_D}\int_{\mathcal{M}_{+}} \mathrm{d}^D x\sqrt{|g|}\left(G^{+}_{\mu\nu}-\kappa_D T^{+}_{\mu\nu}\right)\delta g^{\mu\nu}\\ &+\tfrac{1}{2\kappa_D}\int_{\mathcal{M}_{-}} \mathrm{d}^D x\sqrt{|g|}\left(G^{-}_{\mu\nu}-\kappa_D T^{-}_{\mu\nu}\right)\delta g^{\mu\nu}\\ &-\tfrac{\epsilon}{2\kappa_D}\int_{\Sigma} \mathrm{d}^{D-1} y\sqrt{|h|}\left(K^{+}g_{\mu\nu}-K^{+}_{\mu\nu}\right)\delta g^{\mu\nu}\\ &+\tfrac{\epsilon}{2\kappa_D}\int_{\Sigma} \mathrm{d}^{D-1} y\sqrt{|h|}\left(K^{-}g_{\mu\nu}-K^{-}_{\mu\nu}\right)\delta g^{\mu\nu}\\ &-\tfrac{1}{2\kappa_D}\int_{\Sigma} \mathrm{d}^{D-1} y\sqrt{|h|}\,\kappa_D\, S_{\mu\nu}\,\delta g^{\mu\nu},\end{aligned} \tag{2.65}$$

where $S_{\mu\nu}$ represents the energy-momentum tensor (2.62) for all matter content living **on** the hypersurface Σ. We can conveniently identify both Einstein equations for each manifold $\mathcal{M}_{\pm}$ in the

[25] but not in the value of the norm ϵ.

first two lines of expression (2.65). Both spaces are independent solutions of the Einstein equation, so they will not contribute to the equation of motion. The last three lines of previous expression correspond to the boundary geometric contributions $\partial M_{\pm}$ (where the orientation of the normal has already been taken into account) and any possible matter fields living on such hypersurface. For the whole expression (2.65) to be zero, it is required that these last three terms cancel, so

$$\kappa_D\, S_{\mu\nu} = \epsilon \left[\left(K^- g_{\mu\nu} - K^-_{\mu\nu} \right) - \left(K^+ g_{\mu\nu} - K^+_{\mu\nu} \right) \right] \tag{2.66}$$

A small massage and projection down to tangential components, as these are tangential tensors (2.43), we finally find:

$$\kappa_D\, S_{ab} = \epsilon \left([K_{ab}]^+_- - h_{ab}\, [K]^+_- \right) , \tag{2.67}$$

with $[A]^+_- = A^+ - A^-$. This is the (second) **junction condition**. Equation (2.67) states that the presence of a localised energy-momentum tensor S_{ab} **on** the hypersurface will source a jump discontinuity in the extrinsic curvature. Alternatively, one can read it backwards: a $D-1$-dimensional hypersurface Σ acting as the boundary of two different D-dimensional manifolds $\mathcal{M}_{\pm}$, which are independent solutions to the Einstein equation, must be equipped with an energy-momentum tensor S_{ab}, proportional to the jump in the extrinsic curvature of the embedding, such that the whole composite space is also solution to the Einstein equation.

2.4 Randall-Sundrum braneworlds

One of the best known mechanisms for solving the issue of extra dimensions is to consider our universe as a three-dimensional hypersurface (brane) embedded in a higher-dimensional space. Our universe could then move freely along those extra dimensions, but any content of it would be confined to its volume, with no possibility of accessing the higher-dimensional picture.

First proposals of this type of construction date back to [44, 45]. In this toy model, where gravitational effects were not taken into account, matter was confined to a domain wall, the brane, which could move freely along an extra infinite dimension. Higher-dimensional equations of motion admit solutions close enough to the wall. However, gravity was missing, which could be a big problem when one wants to find a consistent theory of gravity which the graviton can propagate freely throught the full spacetime.

The first "braneworld" model which aimed to include gravity was discussed in [46]. This proposal consists of a three-dimensional membrane, in which all matter is confined and a set of extra dimensions which are required to be not infinitely large, but flat, closed (i.e. $y \sim y+1$) and small.[26] In this sense, matter is still localised on the brane, while now propagating throughout all spacetime. Although this model includes gravity, it does not take into account a subtle but important detail; As we saw in section **??**, any content in the universe will cause a backreaction in its geometry. In the same way, the presence of the brane carrying the matter in this model should cause changes in the higher-dimensional spacetime that contains it. This is known as *warping* and it describes how the brane deforms as it moves through the extra dimensions.

The first model to take into account both the presence of gravity and any possible deformations of the membrane which is embedded in the higher-dimensional space was that of Randall and Sundrum [47, 48]. This string theory inspired braneworld proposal, not only took into account those subtleties forgotten in the previously discussed models, but also proposed a solution to the gauge hierarchy problem in particle physics (i.e. why the gap between the electroweak scale and that of Planck is so big). This model requires two different branes, with positive and negative

[26] If extra dimensions were large in this configuration, the force of gravity could leak into them. This would cause modifications in Newton's law and lead to contradiction with experiments.

tensions respectively, both embedded in a five-dimensional Anti-de Sitter space. Thanks to the backreaction, which translates into curvature, i.e. *warping* in the extra dimensions, any mass trapped on the negative tension brane is much smaller than the Planck scale. This is known as RS-I model. In a later paper [48], a second model was proposed, called RS-II.[27]

Firstly, this second RS model does not require the presence of the negative tension brane. This model involves a single positive tension brane. This brane, as discussed above, will warp the geometry and it only involves an "infinitely" large extra dimension.[28] Although this may seem counterproductive, generating possible contradictions in Newtonian gravity at low energies on the brane, it is not. Randall and Sundrum managed to avoid this problem by forcing gravity to be localised **on** the brane. Let us review some of the key concepts of this construction.

Following [49], let us work in a five-dimensional effective theory of Einstein gravity with matter as

$$S = \frac{1}{2\kappa_5}\int \mathrm{d}^5x\sqrt{|g_{(5)}|}\left(R_{(5)} - 2\Lambda_5 + \mathcal{L}_\mathrm{m}\right) + \frac{1}{\kappa_5}\int \mathrm{d}^4x\sqrt{|h|}K, \tag{2.68}$$

where κ_5 is the Newton's constant (**??**) in five dimensions and the five-dimensional cosmological constant $\Lambda_5 < 0$. The matter Lagrangian is taken to be $\mathcal{L}_m$. As discussed in subsection **??**, the extrinsic curvature K contains information about how the matter, if localised on a hypersurface Σ, is embedded in the higher-dimensional spacetime. If we choose $\mathcal{L}_m = 0$, it can be shown that the metric Ansatz that solves the equation of motion is that of AdS_5 space[29]

$$\mathrm{d}s^2_{\mathrm{AdS}_5} = e^{2A(r)}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}r^2, \tag{2.69}$$

where the aforementioned warp is reflected in the factor $A(r) = \pm kr$ and $k = \sqrt{-\Lambda_5/6}$ is the Anti-de Sitter curvature. The AdS_5 radial coordinate, i.e. the *holographic* coordinate $r \in (-\infty, \infty)$ increases monotonically from the centre to the boundary of the Anti-de Sitter space.

Let us now imagine a (mem)brane with tension σ placed at $r = 0$.[30] This shell of matter will divide the five-dimensional space into two regions and it will hence generate a jump in the extrinsic curvature $K_{\mu\nu}$ on both sides of the brane (see subsection **??** for further information). This can be denoted in the warp factor as:

$$A(r) = \epsilon_+ k_+\,\Theta(r)\,r + \epsilon_- k_-\,\Theta(-r)\,r, \tag{2.70}$$

where $\epsilon_\pm$ represents the normalisation and orientation choice of the normal to the hypersurface Σ, $\Theta(r)$ is the Heaviside function and $k_\pm$ represent the curvatures of each side of both vacua. One of the most important key features of the Randall-Sundrum model is that the brane will act as a "mirror", i.e. a $\mathbb{Z}_2$ symmetry is imposed at $r = 0$. This forces us to identify both five-dimensional vacua across the brane;[31] In addition, the Randall-Sundrum model makes a specific choice of normal orientations, which, together with the aforementioned $\mathbb{Z}_2$ translate into:

$$\epsilon_- = -\epsilon_+ = 1 \qquad \text{and} \qquad k = k_+ = k_-. \tag{2.71}$$

This identification is often referred to as *inside-inside* construction. This results in a matter Lagrangian defined as:[32]

$$\mathcal{L}_\mathrm{m} = \frac{6}{\kappa_5}\,k\,\delta(r) = \sigma\delta(r). \tag{2.73}$$

[27] Again, physicists lack the imagination to nominate.

[28] In fact, this is an artefact of the choice of Poincaré coordinates. The presence of the $\mathbb{Z}_2$ and the warp factor indicate a finite volume, as we will see later.

[29] Some of these steps will be discussed in greater detail in chapter **??**.

[30] This object is not a fundamental one coming from string theory, hence the change of notation to σ.

[31] Hence the choice of k_- on either sides of the brane in figure 6.

[32] This can be obtained from the formal Ansatz

$$\mathcal{L}_\mathrm{m} = \frac{3}{\kappa_5}\left(\epsilon_+ k_+ - \epsilon_- k_-\right)\delta(r). \tag{2.72}$$

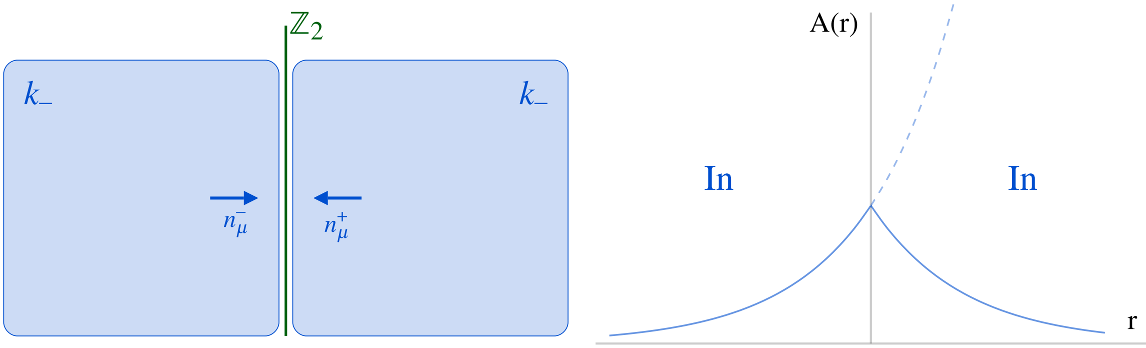

Figure 6: Left: A pictorial representation of the configuration under discussion (inside/inside). The presence of $\mathbb{Z}_2$ and the choice of normal orientations will have important consequences for this construction. Right: The presence of the "mirror" will cause the warp factor $A(r)$ to peak on the brane and to fade to 0 when $r \to \pm\infty$.

Imposing this matter configuration in the junction condition (2.67), while restricting to the case where $\epsilon_- = -\epsilon_+ = 1$ and $k = k_+ = k_-$ and leaving the warp factor $A(r)$ undetermined,[33] one can then induce an effective four-dimensional cosmological constant **on** the brane which reads:

$$\Lambda_4 = \frac{\kappa_5^2}{12}\sigma^2 - 3k^2. \tag{2.74}$$

The first thing to note is the absence of a cosmological constant when $\sigma = \frac{6}{\kappa_5}k$. This value corresponds *exactly* to that of the matter content defined in Eq. (2.73). This implies that the presence of a positive cosmological constant on the brane requires a brane with tension **greater** than the one discussed above. This is called a *supercritical* brane.

Let us now look at the presence of gravity in the model. Another crucial aspects of Randal-Sundrum models is that gravity still appears as four-dimensional to observers confined to the hypersurface. This can be studied by using first order transverse and traceless gravitational perturbations (**??**) in five dimensions. Following [50, 51] and in a similar way as we did in section **??**, one can find a differential equation of the form:

$$-\frac{\mathrm{d}^2\zeta}{\mathrm{d}\chi^2} + \underbrace{\left[\frac{3}{4}\left(\frac{A'}{A}\right)^2 + \frac{3}{2}\frac{A''}{A}\right]}_{V(\chi)}\zeta = 0, \tag{2.75}$$

where ζ is the radial piece of the wave and a prime denotes a derivative respect to χ, which is a convenient conformal coordinate of the form:

$$\chi = \int_0^r \frac{\mathrm{d}r'}{A(r')}. \tag{2.76}$$

The second derivative A'' is extremely important for identifying gravity on the braneworld. As the warp factor $A(\chi)$ involves an absolute value, its second derivative will be a delta function $\delta(r)$. Hence, the potential $V(\chi)$ will peak at the location of the brane. Furthermore, when $\Lambda_4 \geq 0$, one can always find a normalisable solution for ζ of the form:

$$\zeta(\chi) \propto A^{3/2}(\chi). \tag{2.77}$$

What happens when we remove the $\mathbb{Z}_2$ symmetry and choose the normal vectors to point in the direction of **always** increasing transverse volume is left to be discussed in chapter **??**.

[33] There will be a backreaction of the brane in the embedding space. The new warping factor is not relevant to the current discussion.

This implies that the wave peaks around the location of the brane and fades away when $\chi \to \pm\infty$. This solution can be interpreted as the *localisation* of gravity and it means that gravity uniquely lives **on** the brane [48,51]. In order to find how strong this four-dimensional gravity is, we can simply perform a dimensional reduction along the infinitely long fifth dimension r, to find that [52]:[34]

$$\kappa_5 = \kappa_4 \int_{-\infty}^{\infty} \mathrm{d}r e^{2[\epsilon_+ k_+ \, \theta(r)\, r + \epsilon_- k_- \, \theta(-r)\, r]} = \frac{\kappa_4}{k}, \tag{2.78}$$

where we have imposed the choices (2.71). Although the extra dimension r is not compact, it will still give a finite value for the dimensional reduction of the Newton's constant, as the volume along this direction is finite, due to the warping and the presence of the $\mathbb{Z}_2$ symmetry on the bubble wall.

Although the Randall-Sundrum construction uses elements that can easily be provided from string theory, it is not yet clear whether a phenomenological model can be obtained at all. The main point is that the type of brane required in order to induce a positive cosmological constant Λ_4, i.e. supercritical branes, are not those that can be found in string theory [53]. Moreover, the proposal discussed above seems to be in contradiction with the Maldacena-Nuñez no-go theorem [54].[35] Similarly, it was found in [55, 56] that the spacetime transverse to a domain wall interpolating between two different vacuum values always approaches the boundary of AdS. This implies that Randall-Sundrum four-dimensional de Sitter braneworld constructions are forbidden in string theory.

In any case, several iterations and variations of this model with interesting cosmological consequences have recently been studied in [57, 58]. The curious reader is welcome to get a better grasp of the proposals by reading these papers. In addition, we will extensively revisit braneworld constructions in stage **??** of this work. For now, we will take a quick look to the most famous mechanism for hiding extra dimensions: Compactifications.

3 Inducing Dark Energy on a Bubble

4 Quantum Bubbles

5 Gravity and Strings of the Dark Bubble

6 Decorating the Induced Expanding Cosmos

7 Stringy Embedding of the Dark Bubble and its Phenomenology

8 Conclusions

Acknowledgement

We would like to thank someone. [11]

References

[1] U. H. Danielsson and T. Van Riet, *What if string theory has no de Sitter vacua?*, Int. J. Mod. Phys. D **27** (2018), no. 12, 1830007 [1804.01120].

[34] From action (2.68), perform a dimensional reduction along r and identify the Planck mass in five and four dimensions.

[35] The Maldacena-Nuñez no-go theorem states that de Sitter solutions are not allowed in regular warped compactifications at tree level, i.e. de Sitter solutions would require from perturbative and/or non-perturbative corrections and/or the presence of O_3 planes, non-dynamical objects similar to branes, with mirror properties and negative tension.

[2] M. Cicoli, S. De Alwis, A. Maharana, F. Muia and F. Quevedo, *De Sitter vs Quintessence in String Theory*, Fortsch. Phys. **67** (2019), no. 1-2, 1800079 [1808.08967].

[3] B. Heidenreich and M. Lotito, *Proving the Weak Gravity Conjecture in Perturbative String Theory, Part I: The Bosonic String*, 2401.14449.

[4] C. Vafa, *The String landscape and the swampland*, hep-th/0509212.

[5] N. Gendler and I. Valenzuela, *Merging the weak gravity and distance conjectures using BPS extremal black holes*, JHEP **01** (2021) 176 [2004.10768].

[6] E. Palti, *The Swampland: Introduction and Review*, Fortsch. Phys. **67** (2019), no. 6, 1900037 [1903.06239].

[7] M. van Beest, J. Calderón-Infante, D. Mirfendereski and I. Valenzuela, *Lectures on the Swampland Program in String Compactifications*, Phys. Rept. **989** (2022) 1–50 [2102.01111].

[8] T. D. Brennan, F. Carta and C. Vafa, *The String Landscape, the Swampland, and the Missing Corner*, PoS **TASI2017** (2017) 015 [1711.00864].

[9] N. B. Agmon, A. Bedroya, M. J. Kang and C. Vafa, *Lectures on the string landscape and the Swampland*, 2212.06187.

[10] M. Graña and A. Herráez, *The Swampland Conjectures: A Bridge from Quantum Gravity to Particle Physics*, Universe **7** (2021), no. 8, 273 [2107.00087].

[11] U. H. Danielsson, S. S. Haque, P. Koerber, G. Shiu, T. Van Riet and T. Wrase, *De Sitter hunting in a classical landscape*, Fortsch. Phys. **59** (2011) 897–933 [1103.4858].

[12] G. Obied, H. Ooguri, L. Spodyneiko and C. Vafa, *De Sitter Space and the Swampland*, 1806.08362.

[13] H. Ooguri, E. Palti, G. Shiu and C. Vafa, *Distance and de Sitter conjectures on the Swampland*, Physics Letters B **788** (jan, 2019) 180–184.

[14] S. K. Garg and C. Krishnan, *Bounds on slow roll and the de Sitter Swampland*, Journal of High Energy Physics **2019** (Nov., 2019).

[15] S. K. Garg, C. Krishnan and M. Zaid Zaz, *Bounds on Slow Roll at the Boundary of the Landscape*, JHEP **03** (2019) 029 [1810.09406].

[16] D. Andriot, *New Constraints on Classical de Sitter: Flirting with the Swampland*, Fortschritte der Physik **67** (Jan., 2019).

[17] A. Bedroya and C. Vafa, *Trans-Planckian Censorship and the Swampland*, Journal of High Energy Physics **2020** (Sept., 2020).

[18] H. Ooguri and C. Vafa, *On the geometry of the string landscape and the swampland*, Nuclear Physics B **766** (Mar., 2007) 21–33.

[19] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, *The string landscape, black holes and gravity as the weakest force*, Journal of High Energy Physics **2007** (jun, 2007) 060–060.

[20] S. Coleman and J. Mandula, *All Possible Symmetries of the S Matrix*, Phys. Rev. **159** (Jul, 1967) 1251–1256.

[21] R. Penrose, *Gravitational Collapse: The Role of General Relativity*, General Relativity and Gravitation **34** (2002), no. 7, 1141–1165.

[22] T. Banks, M. Johnson and A. Shomer, *A note on gauge theories coupled to gravity*, Journal of High Energy Physics **2006** (Sept., 2006) 049–049.

[23] G. Gibbons, *Vacuum polarization and the spontaneous loss of charge by black holes*, Communications in Mathematical Physics **44** (10, 1975).

[24] S. Gubser and F. Pretorius, *The Little Book of Black Holes*. Science Essentials. Princeton University Press, 2017.

[25] J. Schwinger, *On Gauge Invariance and Vacuum Polarization*, Phys. Rev. **82** (Jun, 1951) 664–679.

[26] B. Heidenreich, M. Reece and T. Rudelius, *Sharpening the weak gravity conjecture with dimensional reduction*, Journal of High Energy Physics **2016** (Feb., 2016).

[27] H. Ooguri and C. Vafa, *Non-supersymmetric AdS and the Swampland*, Adv. Theor. Math. Phys. **21** (2017) 1787–1801 [1610.01533].

[28] B. Freivogel and M. Kleban, *Vacua Morghulis*, 1610.04564.

[29] J. Maldacena, J. Michelson and A. Strominger, *Anti-de Sitter fragmentation*, Journal of High Energy Physics **1999** (Feb., 1999) 011–011.

[30] S. Coleman and F. De Luccia, *Gravitational effects on and of vacuum decay*, Phys. Rev. D **21** (Jun, 1980) 3305–3315.

[31] J. D. Brown and C. Teitelboim, *Neutralization of the Cosmological Constant by Membrane Creation*, Nucl. Phys. B **297** (1988) 787–836.

[32] S. Coleman, *Fate of the false vacuum: Semiclassical theory*, Phys. Rev. D **15** (May, 1977) 2929–2936.

[33] C. G. Callan and S. Coleman, *Fate of the false vacuum. II. First quantum corrections*, Phys. Rev. D **16** (Sep, 1977) 1762–1768.

[34] E. Poisson, *A relativist's toolkit: the mathematics of black-hole mechanics*. Cambridge university press, 2004.

[35] L. Avilés, H. Maeda and C. Martínez, *Junction conditions in scalar–tensor theories*, Classical and Quantum Gravity **37** (Mar., 2020) 075022.

[36] J.-Y. Shen, C. Peng and L.-X. Li, *Multiway junction conditions I: booklets and webs*, 2402.00694.

[37] C.-S. Chu and H. S. Tan, *Generalized Darmois-Israel junction conditions*, 2103.06314.

[38] A. Bayntun, C. P. Burgess and L. van Nierop, *Codimension-2 brane–bulk matching: examples from six and ten dimensions*, New Journal of Physics **12** (jul, 2010) 075015.

[39] W. Israel, *Singular hypersurfaces and thin shells in general relativity*, Nuovo Cim. B **44S10** (1966) 1 [Erratum: Nuovo Cim.B 48, 463 (1967)].

[40] K. Lanczos, *Bemerkung zur de Sitterschen Welt*, Physikalische Zeitschrift **23** (Jan., 1922) 539.

[41] G. Darmois, *Les équations de la gravitation einsteinienne*. Gauthier-Villars et. cie., 1927.

[42] G. W. Gibbons and S. W. Hawking, *Action integrals and partition functions in quantum gravity*, Phys. Rev. D **15** (May, 1977) 2752–2756.

[43] J. W. York, *Role of Conformal Three-Geometry in the Dynamics of Gravitation*, Phys. Rev. Lett. **28** (Apr, 1972) 1082–1085.

[44] V. Rubakov and M. Shaposhnikov, *Do we live inside a domain wall?*, Physics Letters B **125** (1983), no. 2, 136–138.

[45] M. Pavšič, *On the quantisation of gravity by embedding spacetime in a higher dimensional space*, Classical and Quantum Gravity **2** (12, 1985) 869.

[46] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *The hierarchy problem and new dimensions at a millimeter*, Physics Letters B **429** (1998), no. 3, 263–272.

[47] L. Randall and R. Sundrum, *Large Mass Hierarchy from a Small Extra Dimension*, Physical Review Letters **83** (Oct, 1999) 3370–3373 [9905221].

[48] L. Randall and R. Sundrum, *An Alternative to Compactification*, Physical Review Letters **83** (Dec, 1999) 4690–4693 [9906064].

[49] R. C. Myers, *Dielectric-branes*, Journal of High Energy Physics **1999** (Dec., 1999) 022–022.

[50] N. Alonso-Alberca, P. Meessen and T. Ortin, *Supersymmetric brane-worlds*, Physics Letters B **482** (June, 2000) 400–408.

[51] A. Karch and L. Randall, *Locally localized gravity*, Journal of High Energy Physics **2001** (May, 2001) 008–008.

[52] S. Banerjee, U. Danielsson and S. Giri, *Dark Bubble: FAQs. Misconceptions, and why it is not Randall-Sundrum*, 2212.14004.

[53] P. Kraus, *Dynamics of anti-de Sitter domain walls*, Journal of High Energy Physics **1999** (Dec., 1999) 011–011.

[54] J. Maldancena and C. Nuñez, *Supergravity description of field theories on curved manifolds and a no-go theorem*, International Journal of Modern Physics A **16** (Feb., 2001) 822–855.

[55] K. Behrndt and M. Cvetic, *Anti-de Sitter vacua of gauged supergravities with eight supercharges*, Phys. Rev. D **61** (2000) 101901 [hep-th/0001159].

[56] R. Kallosh and A. D. Linde, *Supersymmetry and the brane world*, JHEP **02** (2000) 005 [hep-th/0001071].

[57] A. Karch and L. Randall, *Geometries with mismatched branes*, Journal of High Energy Physics **2020** (Sept., 2020).

[58] L. Randall, *The Boundaries of KKLT*, Fortschritte der Physik **68** (Feb., 2020).