

Solving Viscous Wave Equation

This notebook is used to solve the particular viscous wave equation for the setup of a muon passing through liquid Xenon. The goal is to find the pressure distribution in a container and its decay as a function of distance and time.

Wave equation

We can derive the viscous wave equation through the linearised Navier-Stokes Equations. The full derivation is in the notes, but the point is that we have added a single term that takes into account the viscous effects. The equation turns out to be as follows:

$$\Delta \left(p(\vec{x}, t) - \frac{1}{\omega_0} \frac{\partial}{\partial t} p(\vec{x}, t) \right) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p(\vec{x}, t)$$

where c is the speed of the wave in the medium, ω_0 is the coefficient responsible for damping, and $p(\vec{x}, t)$ is pressure as a function of space and time.

We now introduce the linear differential operator \mathcal{L} .

$$\mathcal{L}p := \Delta \left(p(\vec{x}, t) - \frac{1}{\omega_0} \frac{\partial}{\partial t} p(\vec{x}, t) \right) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p(\vec{x}, t)$$

Fundamental Solution

Now we search for the fundamental solution $F(\vec{x}, t)$ that satisfies the following.

$$\mathcal{L}F(\vec{x}, t) = \delta(\vec{x})\delta(t)$$

We can solve it using fourier methods and then we obtain the fundamental solution in terms of this integral over fourier momentum. Now we just need to evaluate this.

$$F(\vec{x}, t) = \Theta(t) \int_0^\infty 8\pi^2 c^2 k^2 e^{-\frac{c^2 k^2}{2\omega_0} t} \frac{e^{ikr} - e^{-ikr}}{2ikr} \frac{e^{ikt\sqrt{1-\frac{k^2 c^2}{4\omega_0}}} - e^{-ikt\sqrt{1-\frac{k^2 c^2}{4\omega_0}}}}{2ikt\sqrt{1-\frac{k^2 c^2}{4\omega_0}}} dk$$

Where $\Theta(t)$ is the Heaviside function and $r := |\vec{x}|$

Integration

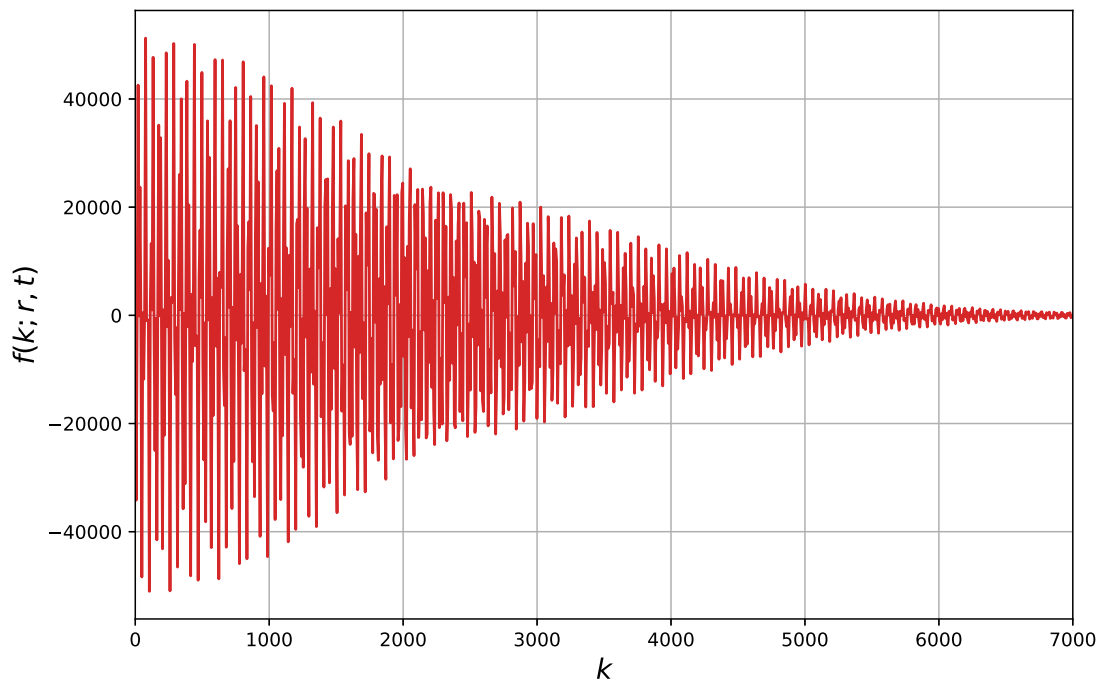
To numerically evaluate this expression we first need to study it. Therefore let's define the function f as the inner part of the integral like so.

$$F(\vec{x}, t) = \Theta(t) \int_0^\infty f(k; \vec{x}, t) dk$$

Now we realise that the function f is a piecewise function plotted below.

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Out[2]: [<matplotlib.lines.Line2D at 0x7f925451f2e8>]
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Integrand plot vs k



Integration

Now that we have the plot we need to somehow integrate it. To do this efficiently we need to find when does the wave decay over a specific amount. As a result we need to find its envelope. To do this we have to study each of the pieces of the piecewise function separately.

Envelope

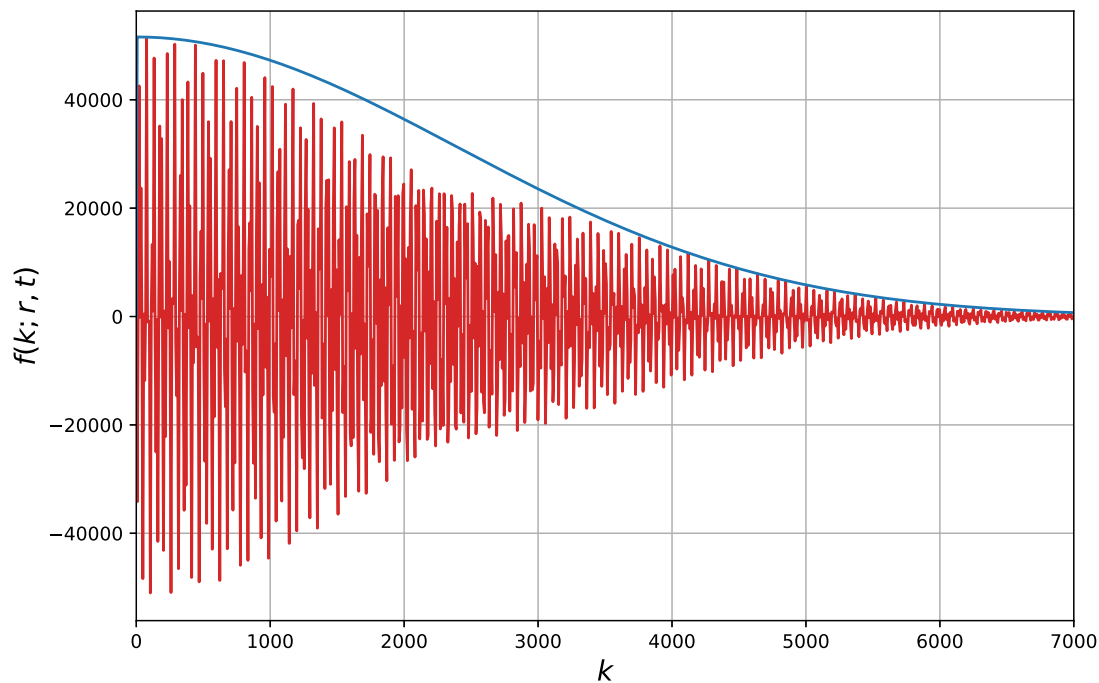
Now based on the analytic expression of the integrand we can conclude that the envelope function $\varepsilon(k; r, t)$ is the following.

$$\varepsilon(k; r, t) = \begin{cases} \frac{8\pi^2 c^2 e^{-\frac{c^2 k^2}{2\omega_0} t}}{rct \sqrt{1 - \frac{k^2 c^2}{4\omega_0}}} & \text{if } k \leq \frac{2\omega_0}{c} \\ 8\pi^2 c^2 e^{-\frac{c^2 k^2}{2\omega_0} t} \frac{\sinh\left(kct \sqrt{\frac{k^2 c^2}{4\omega_0} - 1}\right)}{rct \sqrt{\frac{k^2 c^2}{4\omega_0} - 1}} & \text{if } k > \frac{2\omega_0}{c} \end{cases}$$

Let's try to plot one against the other

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Out[3]: (-56122.79871165482, 56375.24228145919)
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Integrand plot vs k with envelope



Approximations

Now to analytically calculate the integral of that equation we need to make certain approximations. To do this we need some intuitive understanding of the order of magnitude of the constants we are using.

Constants

We have two defining constants in the problem. The decay frequency ω_0 and the speed of sound in LXe c . From the wave equation we have derived that the speed of sound is:

$$c = \sqrt{\frac{K}{\rho_0}}$$

Where K is the bulk modulus and ρ_0 is the rest density of Liquid Xenon. Now the biggest question comes from trying to find values for these constants. From [nist](#) we obtain the following list of constants:

- Bulk Modulus K [$\frac{kg}{s^2 m}$]: 1.2667×10^9
- Rest Density ρ_0 [$\frac{kg}{m^3}$]: 2.9663×10^3

Therefore we calculate the speed of sound in LXe to be:

$$c = \sqrt{\frac{K}{\rho_0}} = 653.47 \frac{m}{s}$$

Now for the attenuation frequency ω_0 we have derived in the equation that it is the ratio of the Bulk modulus to the viscosity. Similarly to the speed of sound we have obtained the following values from [nist](#).

- Bulk Modulus $K [\frac{kg}{s^2m}]$: 1.2667×10^9
- viscosity $\mu [Pa.s]$: 5.1701×10^{-4}

We therefore obtain that the attenuation frequency is:

$$\omega_0 = \frac{K}{\mu} = 2.4500 \times 10^{12} Hz$$

Now that we have the values we can start estimating the possible evolution of the system, by using some very useful approximations.

Near... Far... wherever you are!

We see that there is a special point in f at which the root becomes imaginary. That happens when $k = \frac{2\omega_0}{c}$. If we evaluate we see that the graph will be attenuated before that point. Therefore we can approximate the function by just considering the following case where $k \leq \frac{2\omega_0}{c}$. We obtain the following expression for $f(k; r, t)$.

$$f(k; r, t) = \frac{8\pi^2 c e^{-\frac{c^2 k^2}{2\omega_0} t}}{r t \sqrt{1 - \frac{k^2 c^2}{4\omega_0}}} \sin(kr) \sin\left(kct \sqrt{1 - \frac{k^2 c^2}{4\omega_0}}\right)$$

Now that's a way prettier equation. However it is still a pain in the *** to integrate. Therefore we can consider one more simplification. Let's look at the Taylor series of the following function in x centered at $0 \forall \alpha > 0$.

$$\sqrt{1 - \alpha x^2} = 1 - \frac{\alpha x^2}{2} - \frac{\alpha^2 x^4}{8} + \mathcal{O}(x^6)$$

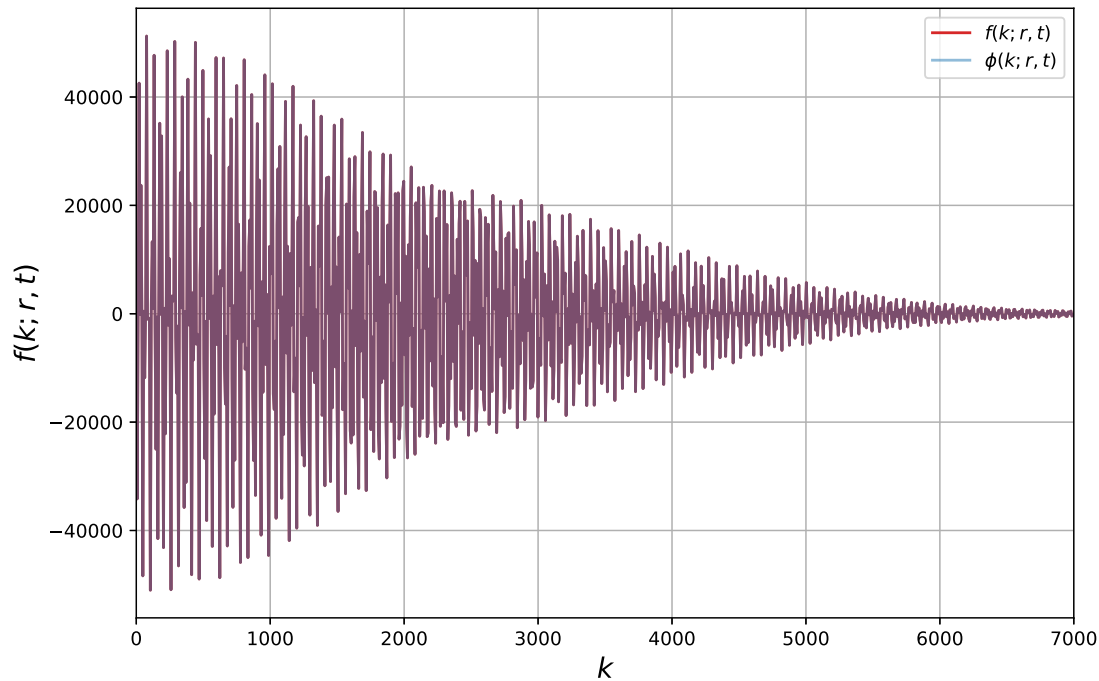
in our case we notice how tiny α is and hence we drop all the terms except the first in our expansion. Finally we obtain a much more manageable expression for the integral in question. We call that new function ϕ .

$$f(k; r, t) \approx \phi(k; r, t) = \frac{8\pi^2 c}{r t} e^{-\frac{c^2 k^2}{2\omega_0} t} \sin(kr) \sin(kct)$$

In fact we can plot both $f(k; r, t)$ and $\phi(k; r, t)$ to see their differences.

Out[5]: <matplotlib.legend.Legend at 0x7f9250952f98>

Integrand plot with Approximation



As we can see the two plots are very similar too! This is amazing because when it comes to estimate the integral, $\phi(k; r, t)$ is so much better than the ugly thingy that we had before. Let's try to do this integral.

Analytic Solution

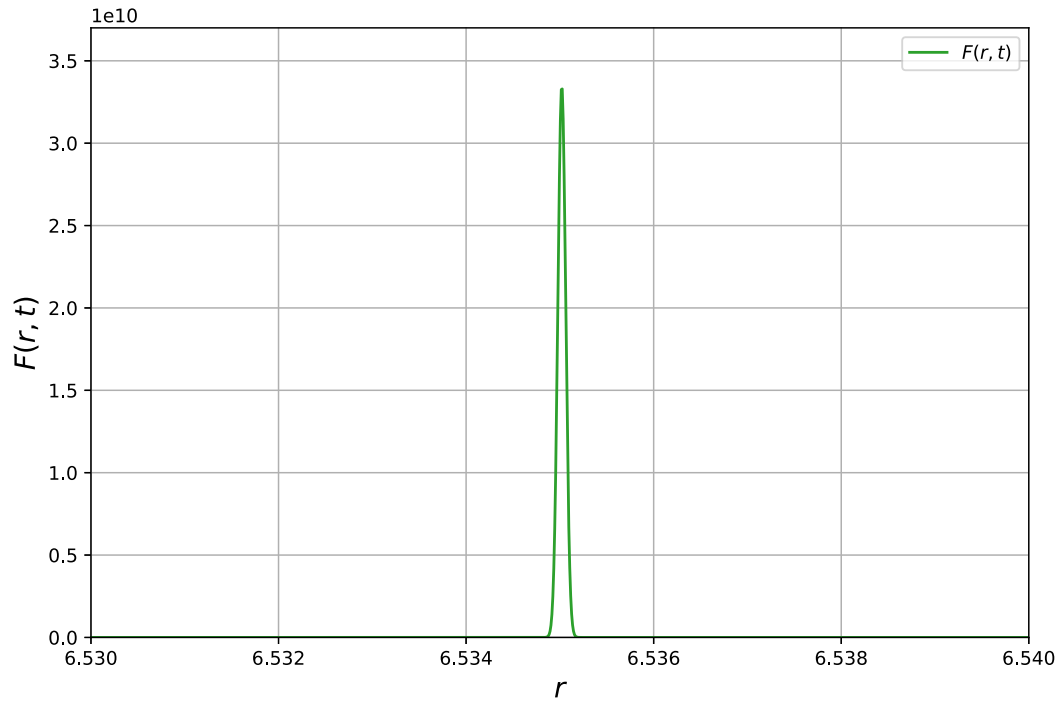
Finally, after a lot of plots and months of sweat here is the final solution. We can just analyse this in terms of exponentials and after realising that for any realistic value of (r, t) we are at the constant part of the error function integral, we can represent the solution like so.

$$F(r, t) = \Theta(t) \frac{8\pi^2 c}{rt} \sqrt{\frac{\pi\omega_0}{8c^2 t}} \exp\left(-\frac{r^2\omega_0}{c^2 t} - t\omega_0\right) \left[\exp\left(\frac{(r+ct)^2\omega_0}{2c^2 t}\right) - \exp\left(\frac{(r-ct)^2\omega_0}{2c^2 t}\right) \right]$$

Now I know this looks complicated but in reality it isn't. Let's examine a few features. Just like any solution to the wave equation we observe that the fundamental solution is indeed a superposition of two waves as functions of $r \pm ct$. Also we see a characteristic exponential decay term dependent on the radius and time. Let's plot and animate this to make sure.

Out[6]: <matplotlib.legend.Legend at 0x7f9250921fd0>

Snapshot of Fundamental Solution



Conclusion

It is important to recognise that the final solution is the solution to the following equation

$$\mathcal{L}F(\vec{x}, t) = \delta(\vec{x})\delta(t)$$

Which means that the solution is a distribution. Therefore the numbers on the y axis of the above plot do not mean much in terms of the actual thing. HOWEVER! It is possible to express every solution as a convolution with this. Specifically let's consider the following equation.

$$\mathcal{L}p(\vec{x}, t) = \psi(\vec{x}, t)$$

Where $\psi(\vec{x}, t)$ is any forcing functions according to the previous note, we can analytically express the solution in the form of a convolution. Specifically the solution for the pressure can be expressed as.

$$p(\vec{x}, t) = \psi * F = \int_{\mathbb{R}^3} \int_{\mathbb{R}} \psi(\vec{y}, s) F(\vec{x} - \vec{y}, t - s) ds d\vec{y}$$

Well now we are doing business. Almost there now we need to replace ψ with the Bethe-Bloch formula and we can get the analytic formula for the pressure due to this. Shouldn't be very hard at this point.