Calculating the actual wave

The Problem

In the **Fundamental Solution** jupyter notebook we have analytically approximated the fundamental solution. Here we are going to use that solution in order to solve the following Cauchy Global Problem.

$$\begin{cases} \Delta \left(p(\vec{x},t) - \frac{1}{\omega_0}\frac{\partial}{\partial t}p(\vec{x},t)\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}p(\vec{x},t) = -\frac{\beta}{C_p}\frac{\partial E}{\partial t} & \forall (\vec{x},t) \in \mathbb{R}^3 \times \mathbb{R}^+ \\ p(\vec{x},0) = 0 & \forall \vec{x} \in \mathbb{R}^3 \\ \frac{\partial}{\partial t}p(\vec{x},0) = 0 & \forall \vec{x} \in \mathbb{R}^3 \end{cases}$$

Notation

This notation, even though it is the most accurate is pretentious and cumbersome to work with. Therefore, from now on we will employ a different notation that is cleaner and simpler to read and write.

First we define the linear differential operator $\mathcal{L}$ as such.

$$\mathcal{L}u(\vec{x},t) = \Delta \left(u(\vec{x},t) - \frac{1}{\omega_0}\frac{\partial}{\partial t}u(\vec{x},t)\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(\vec{x},t)$$

Now we can rewrite the cauchy problem like so:

$$\begin{cases} \mathcal{L}p(\vec{x},t) = -\frac{\beta}{C_p}\frac{\partial E}{\partial t} = \psi(\vec{x},t) & \forall (\vec{x},t) \in \mathbb{R}^3 \times \mathbb{R}^+ \\ p(\vec{x},0) = 0 & \forall \vec{x} \in \mathbb{R}^3 \\ p_t(\vec{x},0) = 0 & \forall \vec{x} \in \mathbb{R}^3 \end{cases}$$

This is undoubtedly better. Now let's try to find the solution.

Duhamel's Method

Duhamel's method is used in this case to simplify and solve the problem by solving the following set of problems: Let $F(\vec{x},t)$ be the *Fundamental Solution* solution defined by the following global cauchy problem:

$$ \begin{cases}

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\mathcal{L} F(\vec{x},t) = \delta(\vec{x})\delta(t) & \forall (\vec{x},t) \in \mathbb{R}^3 \times
\mathbb{R}^+\\
F(\vec{x},0) = 0 & \forall \vec{x} \in \mathbb{R}^3\\
F_t(\vec{x},0) = 0 & \forall \vec{x} \in \mathbb{R}^3
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\end{cases} $$

Now we can use it to find the solutions $u(\vec{x},t;s)$ to the following family of problems that arise from decomposing $\psi(\vec{x},t)$ in time. So $\forall s > 0$:

$$\begin{cases} \mathcal{L}u(\vec{x},t;s) = 0 & \forall (\vec{x},t) \in \mathbb{R}^3 \times (s,+\infty) \\ u(\vec{x},s;s) = 0 & \forall \vec{x} \in \mathbb{R}^3 \\ u_t(\vec{x},s;s) = \psi(\vec{x},s) & \forall \vec{x} \in \mathbb{R}^3 \end{cases}$$

Therefore the solution to our original PDE will be the following convolution

$$p(\vec{x},t) = \int_0^t \int_{\mathbb{R}^3} \psi(\vec{y},s)F(\vec{x}-\vec{y},t-s)\, d\vec{y}ds$$

Forcing Term

Now we need to find $\psi(\vec{x},t)$. To do this we need to think of the analytical form of the energy deposition of the particle in the fluid. Hence we introduce the Bethe-Bloch formula that will predict the spatial rate of energy deposition of the particle through the medium along the distance travelled from its path.

$$-\frac{dE}{dx} \approx -\left\langle \frac{dE}{dx} \right\rangle = \frac{4\pi}{m_e c^2}\frac{nz^2}{\beta^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \left[\ln\left(\frac{2m_e c^2\beta^2}{I\cdot(1-\beta^2)}\right) - \beta^2\right]$$

Now what we can do, is to assume that this energy is deposition will be Gaussian distributed over time and space in the medium. As a result the energy deposition in the medium is:

$$-\frac{dE}{dx}\cdot G(\vec{x},t)$$

where $G(\vec{x}, t)$ is a distribution in $\vec{x}$ and t:

$$G(\vec{x}, t) = \frac{1}{4\pi^2\sigma_x^3\sigma_t}\exp\left(-\frac{|\vec{x}|^2}{2\sigma_x^2}\right)\exp\left(-\frac{t^2}{2\sigma_t^2}\right)$$

Therefore we can write the forcing function $\psi(\vec{x}, t)$ like so.

$$\psi(\vec{x}, t) = -\frac{\beta_{Xe}}{C_p}\frac{dE}{dx}\frac{dx}{dt}G(\vec{x}, t)$$

Below we plot the energy envelope.

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Out[2]: <matplotlib.collections.QuadMesh at 0x7f9583388710>
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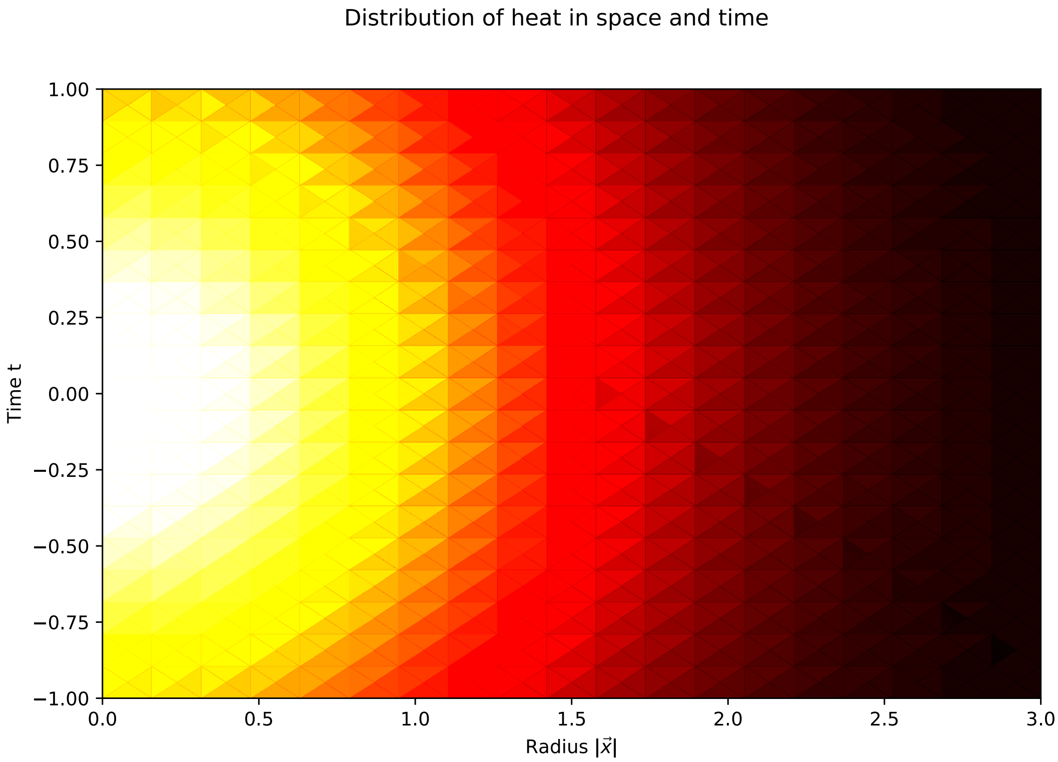

Integrating

Now to do this integral we need to first express it analytically. This should be fun!

$$p(\vec{x}, t) = \int_0^t \int_{\mathbb{R}^3} -\frac{\beta_{Xe}}{C_p}\frac{4\pi}{m_e c^2}\frac{nz^2}{\beta^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\left[\ln\left(\frac{2m_e c^2\beta^2}{I\cdot(1-\beta^2)}\right)-\beta^2\right]\frac{dz}{dt}\frac{1}{4\pi^2\sigma_x^3\sigma_t}e^{-\frac{|\vec{y}|^2}{2\sigma_x^2}}e^{-\frac{s^2}{2\sigma_t^2}}\Theta(t-s)\frac{8\pi^2 c}{|\vec{x}-\vec{y}|(t-s)}\sqrt{\frac{\pi\omega_0}{8c^2(t-s)}}\exp\left(-\frac{|\vec{x}-\vec{y}|^2\omega_0}{c^2(t-s)}-(t-s)\omega_0\right)\left[\exp\left(\frac{(|\vec{x}-\vec{y}|+c(t-s))^2\omega_0}{2c^2(t-s)}\right)-\exp\left(\frac{(|\vec{x}-\vec{y}|-c(t-s))^2\omega_0}{2c^2(t-s)}\right)\right]d\vec{y}ds$$

cool...

This keeps happening. But I think we can simplify the integral to a much better form. Frist of all, assuming the velocity of the particle to be constantnt we can reqwrite everything like this.

$$p(\vec{x}, t) = \mathcal{A}\int_0^t \int_{\mathbb{R}^3} \frac{1}{|\vec{x}-\vec{y}|(t-s)^{\frac{3}{2}}}\exp\left(-\frac{|\vec{y}|^2}{2\sigma_x^2}-\frac{s^2}{2\sigma_t^2}-\frac{|\vec{x}-\vec{y}|^2\omega_0}{c^2(t-s)}-(t-s)\omega_0\right)\left[\exp\left(\frac{(|\vec{x}-\vec{y}|+c(t-s))^2\omega_0}{2c^2(t-s)}\right)-\exp\left(\frac{(|\vec{x}-\vec{y}|-c(t-s))^2\omega_0}{2c^2(t-s)}\right)\right]d\vec{y}ds$$