Problem 1 — Intro

Studying $(\beta\partial)^n$ we can obtain that

$$(\beta\partial)^n = \sum_{k=1}^{n} S_k^n \beta^k \partial^k,$$

where S_k^n is the number of partitions of a set of n elements into k disjoint subsets.

To show this we do some combinatorics. Effectively we want to expand $\beta\partial(\beta\partial(\beta\cdots\partial(\beta\partial f)))$ for some function $f(\beta)$ in terms of $f^{(k)}(\beta)$. Using product rule we know that the coefficient multiplying $f^{(k)}$ is going to be $\beta\partial(\beta\partial(\beta\cdots\partial(\beta)))$ with k of the partials missing. Now we have $n-1$ positions to place $n-k$ partials in the string above, and if one is placed at position $1 \leq i \leq n-1$ then it can act only to the $n-i$ remaining β's.

We can depict this pictorially using two sets of dots one set for the available position to place the ∂ and another for each β that the ∂ can act on. Fig. 1 shows a representation of one of the terms in the product rule expansion when we place two derivatives on the first two slots.

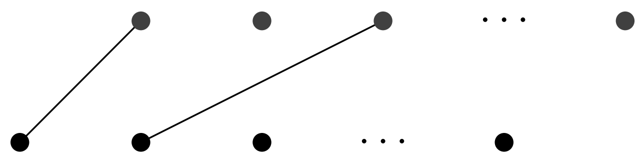

Figure 1: *Connecting the dots. The bottom dots represent places in the $\beta\partial(\beta\partial(\beta\cdots\partial(\beta)))$ string where ∂ can be placed, while the top dots represent the β that are there. A connected line means that a corresponding ∂ acts on that particular β. The particular configuration corresponds to the term in the expansion with two derivates acting like so $\beta(\partial\beta)\beta(\partial\beta)\cdots\beta$.*

Alternatively we can consider this as a set of n elements with $n-k$ possible connections. The connectionspartition that set of points into $n-(n-k+1)+1 = k$ subsets, and each subset partition corresponds to exactly one term in the product rule expansion of the coefficient in front of $f^{(k)}$, therefore each term will be multiplied by the number of partitions of a set of n elements into k subsets, S_k^n.

Now the hard part is done. Using the fact that $\partial^k e^{-\beta H} = (-H)^k e^{-\beta H}$ we can conclude that

$$\alpha(Z(\beta)) = \mathrm{Tr} \sum_{n=0}^{N} \alpha_n e^{-\beta H} \sum_{k=1}^{n} (-\beta H)^k S_k^n = \mathrm{Tr}\, e^{-\beta H} \sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} \alpha_{2n} T^n(-\beta H) + e^{-\beta H} \sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} \alpha_{2n+1} T^n(-\beta H).$$

If we call $\tilde{\beta} = \frac{4\pi^2}{\beta}$ we know that $Z(\beta) = Z(\tilde{\beta})$ and that $\beta\partial = -\tilde{\beta}\tilde{\partial}$ plugging these two equations we have that

$$\alpha(Z(\beta)) = \alpha(Z(\tilde{\beta})) = \mathrm{Tr}\, e^{-\tilde{\beta} H} \sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} (-1)^{2n} \alpha_{2n} T^n(-\tilde{\beta} H) + e^{-\tilde{\beta} H} \sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} (-1)^{2n+1} \alpha_{2n+1} T^n(-\tilde{\beta} H).$$

Subtracting the two expressions for $\beta = 2\pi = \tilde{\beta}$ we conclude that

$$\operatorname{Tr} \alpha(H) e^{-2\pi H} = \operatorname{Tr} \sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} (-1)^{2n+1} \alpha_{2n+1} T^n(-2\pi H) e^{-2\pi H} = 0.$$

Finally using the fact that $\det e^A = e^{\operatorname{tr} A}$ we can refine this expression to

$$\operatorname{Tr} \alpha(H) = -Z(2\pi),$$

which is different form what the problem set we should get, but the rest of the considerations must be fine. Still if $\alpha(E) > 0$ for all allowed E then the theory is excluded.

Problem 2 — Implementation

We know that a spectrum for a CFT is not valid if we can find a functional α such that $\operatorname{Tr} \alpha(H) \geq 0$. So we can solve the optimization problem for α constraining all energies above the ground to give positive contributions, and solve for what the maximum $\alpha(E_0)$ is. If it positive then we have found a functional that contradicts the assertion above and then we can reject that spectrum.

However, one can notice quickly that a possible solution to this problem, as stated in the original document, is to take $\alpha_n = 0$ for all n. Therefore, sdpb should always be able to find a polynomial that doesn't allow the theory to be rulled out.