Problem Books in Mathematics

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Dynamical Systems by Example

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Preface

This book is a large collection of problems, all with detailed solutions, on the core of the theory of dynamical systems. Besides the basic theory, the topics include topological dynamics, low-dimensional dynamics, hyperbolic dynamics, symbolic dynamics, and basic ergodic theory.

As in any other area of mathematics, it is important while learning dynamical systems to solve selected problems (always after a careful study of the material!), in particular to get a first working knowledge of the topics as well as of some less direct difficulties. It goes without saying that this helped substantially generations of mathematicians in getting a solid understanding of the theory and of its applications. Nevertheless, it is difficult to find large collections of problems on less basic subjects, at least other than on quite specific topics. Moreover, these problems often lack detailed solutions or even any solutions, while it would certainly be quite helpful for a student to have the possibility to study detailed solutions, especially if studying independently.

Certainly, it is not a good practice to study detailed solutions of exercises without first trying hard to solve them. Indeed, solving exercises is an important step toward learning a particular subject and ultimately toward becoming an independent mathematician, while perhaps also developing a personal style. On the other hand, it would be quite welcome to have available comprehensive sources of problems worked out in detail. In more advanced topics, these solutions can have the role of providing detailed arguments for comparison or even alternative views, possibly alerting to more direct approaches or to less obvious connections to other topics (and mathematics is full of such connections).

Our text is a contribution to fill this gap on selected topics of the theory of dynamical systems (as detailed below). It can be used as a companion to a textbook for a one-semester or two-semester course on dynamical systems at the advanced undergraduate or beginning graduate levels, or for independent study of those topics. Other than some basic pre-requisites from linear algebra, differential and integral calculus, complex analysis and topology, in each chapter we recall all notions and results (without proofs) that are necessary for the problems in that chapter, thus making the text self-contained.

The theory of dynamical systems is quite broad and active in terms of research. Hence, it was necessary to make a careful selection of the material. In this aspect, we followed closely our book [11], which gives an introduction to the theory of dynamical systems and to which the present book can be an excellent companion. The levels of the two texts are analogous, although the present book provides a more comprehensive working knowledge of the topics as well as a much larger variety of problems, also of various levels of difficulty (from simple to quite elaborate) and with a carefully planned interdependence when appropriate, so that the material is learned in successive steps.

We detail briefly the topics addressed in the book together with recommendations for further reading:

- Chapter I.1 considers the notion of a dynamical system, both for discrete and continuous time, invariant sets, orbits, periodic points, rotations and expanding maps of the circle, endomorphisms and automorphisms of the torus, autonomous differential equations and their flows, Poincaré sections, and Poincaré maps (see [3, 10, 20, 22]).
- Chapter I.2 studies continuous maps of a topological space and their topological properties, including the notions of α-limit set and of ω-limit set, recurrent points, nonwandering points, minimal sets, topological transitivity, topological mixing, and topological conjugacies, as well as topological entropy and its properties (see [16, 17, 48]).
- Chapter I.3 considers dynamical systems on low-dimensional spaces, including homeomorphisms of the circle, their lifts, orientation-preserving homeomorphisms and the notion of rotation number, continuous maps on the interval, and the Poincaré–Bendixson theory on the plane (see [2, 10, 22, 24]).
- Chapter I.4 studies hyperbolic dynamics, including the notion of a hyperbolic set, the Smale horseshoe, invariant families of cones, topological conjugacies, and invariant manifolds near a hyperbolic fixed point, as well as geodesic flows and their hyperbolicity (see [28, 40, 41, 50, 54]).
- Chapter I.5 considers selected topics of symbolic dynamics, including one-sided and two-sided shift maps, topological Markov chains, irreducible and transitive matrices, topological transitivity, topological mixing, topological entropy, and zeta functions (see [16, 33, 34]).
- Chapter I.6 studies selected basic topics of ergodic theory, including Poincaré's recurrence theorem, Birkhoff's ergodic theorem, and the notion of metric entropy (see [8, 23, 35, 46, 56]).

The book is conveniently separated into two parts. Part I (Chaps. I.1–I.6) recalls briefly in each chapter various notions and results that are needed for the problems of that chapter, also with the purpose of fixing notation, after which the problems are formulated without their solutions. In Part II (Chaps. II.1–II.6) the problems are restated but now with their solutions. In this way each problem can be solved without the risk of looking inadvertently at the solution. There are 240 problems ranging from simple to quite elaborate (with 40 per chapter), all with detailed solutions, on the basic theory, topological dynamics, low-dimensional dynamics, hyperbolic dynamics, symbolic dynamics, and basic ergodic theory. The text is complemented by 50 figures.

We also provide a list of important topics, certainly incomplete, that were left out of the book for reasons of scope together with recommendations for further reading:

- Holomorphic dynamics (see [13, 18, 36, 37, 38, 49]);
- Bifurcation theory and normal forms (see [5, 10, 21, 25]);
- Hamiltonian dynamics (see [1, 6, 10, 29, 39]);
- Discrete groups of isometries (see [4, 12, 31]);
- Dimension theory and multifractal analysis (see [7, 44]);
- Thermodynamic formalism and its applications (see [15, 32, 51]);
- Hyperbolicity and homoclinic bifurcations (see [42]);
- Hyperbolic dynamics and zeta functions (see [43]);
- Partial hyperbolicity and stable ergodicity (see [45]);
- Nonuniform hyperbolicity and smooth ergodic theory (see [9, 14, 47]);
- Hyperbolic systems with singularities and billiards (see [19, 30]);
- Algebraic dynamics and ergodic theory (see [52]);
- Infinite-dimensional dynamics (see [26, 27, 53, 55]).

Lisbon, Portugal
February 2019

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Contents

Part I
Theory and Problems

Chapter I.1
Basic Theory

In this chapter we consider the notion of a dynamical system, both for discrete and continuous time. In particular, we consider invariant sets, orbits, semiorbits, periodic points, rotations and expanding maps of the circle, endomorphisms and automorphisms of the torus, as well as autonomous ordinary differential equations and their flows. We also consider some basic constructions that determine new dynamical systems, including suspension semiflows, Poincaré sections, and Poincaré maps. We refer the reader to [3, 10, 20, 22] for additional topics.

Notions and Results

First we recall a few basic notions, including those of invariant set, orbit, and periodic point.

Definition 1.1 *A map* $f\colon X \to X$ *is called a* dynamical system *with discrete time. We define recursively*

$$f^n = f \circ f^{n-1}$$

for each $n \in \mathbb{N}$*, with the convention that* $f^0 = \mathrm{id}$. *When* f *is invertible, we also define* $f^{-n} = (f^{-1})^n$ *for each* $n \in \mathbb{N}$.

Given a map $f\colon X \to X$ and a set $A \subseteq X$, we write

$$f^{-1}A = \big\{x \in X : f(x) \in A\big\}.$$

Definition 1.2 *Given a map* $f\colon X \to X$*, a set* $A \subseteq X$ *is said to be:*

1. f-invariant *if* $f^{-1}A = A$*;*
2. forward f-invariant *if* $f(A) \subseteq A$*;*
3. backward f-invariant *if* $f^{-1}A \subseteq A$.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of all nonnegative integers.

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_1

Definition 1.3 *Given a map $f\colon X \to X$ and a point $x \in X$, the set*

$$\gamma^+(x) = \{f^n(x) : n \in \mathbb{N}_0\}$$

is called the positive semiorbit *of x. Moreover, when f is invertible, the sets*

$$\gamma^-(x) = \{f^{-n}(x) : n \in \mathbb{N}_0\}$$

and

$$\gamma(x) = \{f^n(x) : n \in \mathbb{Z}\}$$

are called, respectively, the negative semiorbit *of x and the* orbit *of x.*

Definition 1.4 *Given a map $f\colon X \to X$ and an integer $q \in \mathbb{N}$, a point $x \in X$ satisfying*

$$f^q(x) = x$$

*is called a q-*periodic point *of f. Moreover, $x \in X$ is called a* periodic point *if it is a q-periodic point for some $q \in \mathbb{N}$. The* fixed points *are the 1-periodic points, that is, those points $x \in X$ such that $f(x) = x$.*

A periodic point is said to have period *q if it is q-periodic, but is not p-periodic for any $p < q$.*

We also consider various classes of dynamical systems with discrete time.

Definition 1.5 *Let $\sim$ be the equivalence relation on $\mathbb{R}$ given by*

$$x \sim y \quad \Longleftrightarrow \quad x - y \in \mathbb{Z}.$$

The circle *S^1 is defined by $S^1 = \mathbb{R}/\!\sim$. The elements of S^1 are thus the equivalence classes*

$$[x] = \{x + m : m \in \mathbb{Z}\} \quad \textit{for } x \in \mathbb{R}.$$

Given $\alpha \in \mathbb{R}$, the rotation *$R_\alpha\colon S^1 \to S^1$ is defined by*

$$R_\alpha[x] = [x + \alpha].$$

Moreover, given an integer $m > 1$, the expanding map *$E_m\colon S^1 \to S^1$ is defined by*

$$E_m[x] = [mx].$$

Definition 1.6 *Given $n \in \mathbb{N}$, let $\sim$ be the equivalence relation on $\mathbb{R}^n$ given by*

$$x \sim y \quad \Longleftrightarrow \quad x - y \in \mathbb{Z}^n.$$

*The n-*torus *or simply the* torus *is defined by $\mathbb{T}^n = \mathbb{R}^n/\!\sim$. The elements of $\mathbb{T}^n$ are thus the equivalence classes*

$$[x] = \{x + y : y \in \mathbb{Z}^n\} \quad \textit{for } x \in \mathbb{R}^n.$$

Given an $n \times n$ matrix A with entries in $\mathbb{Z}$, the endomorphism of the torus induced by A *is the map $T_A \colon \mathbb{T}^n \to \mathbb{T}^n$ defined by*

$$T_A[x] = [Ax] \quad \textit{for } [x] \in \mathbb{T}^n.$$

When T_A is invertible, it is also called the automorphism of the torus induced by A.

Finally, we consider corresponding notions for continuous time.

Definition 1.7 *A family $\Phi = (\varphi_t)_{t \geqslant 0}$ of maps $\varphi_t \colon X \to X$, for $t \geqslant 0$, such that $\varphi_0 = \mathrm{id}$ and*

$$\varphi_{t+s} = \varphi_t \circ \varphi_s \quad \textit{for } t, s \geqslant 0$$

is called a semiflow on X. *Moreover, a family $\Phi = (\varphi_t)_{t \in \mathbb{R}}$ of maps $\varphi_t \colon X \to X$, for $t \in \mathbb{R}$, such that $\varphi_0 = \mathrm{id}$ and*

$$\varphi_{t+s} = \varphi_t \circ \varphi_s \quad \textit{for } t, s \in \mathbb{R}$$

is called a flow on X. *A family Φ of maps is called a* dynamical system *with continuous time if it is a flow or a semiflow.*

We denote by $v' = dv/dt$ the derivative of a function $v = v(t)$. It turns out that any differential equation $v' = f(v)$ with global unique solutions determines a flow.

Proposition 1.8 *Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that, given $v_0 \in \mathbb{R}^n$, the initial value problem*

$$\begin{cases} v' = f(v), \\ v(0) = v_0 \end{cases}$$

has a unique solution $t \mapsto v(t, v_0)$ defined on the whole $\mathbb{R}$. Then the family of maps $\varphi_t \colon \mathbb{R}^n \to \mathbb{R}^n$, for $t \in \mathbb{R}$, defined by $\varphi_t(v_0) = v(t, v_0)$ is a flow on $\mathbb{R}^n$.

A point $v_0 \in \mathbb{R}^n$ with $f(v_0) = 0$ is called a *critical point* of the equation $v' = f(v)$.

Definition 1.9 *Given a semiflow Φ on X, a set $A \subseteq X$ is said to be Φ-*invariant *if*

$$\varphi_t^{-1} A = A \quad \textit{for } t \geqslant 0.$$

*Moreover, given a flow Φ on X, a set $A \subseteq X$ is said to be Φ-*invariant *if*

$$\varphi_t^{-1} A = A \quad \textit{for } t \in \mathbb{R}.$$

Definition 1.10 *Given a semiflow Φ on X and a point $x \in X$, the set*

$$\gamma^+(x) = \{\varphi_t(x) : t \geqslant 0\}$$

is called the positive semiorbit *of x. Moreover, for a flow Φ on X, the sets*

$$\gamma^-(x) = \{\varphi_{-t}(x) : t \geqslant 0\}$$

and

$$\gamma(x) = \{\varphi_t(x) : t \in \mathbb{R}\}$$

are called, respectively, the negative semiorbit *of x and the* orbit *of x.*

In particular, for the flow in Proposition 1.8 associated with an autonomous differential equation $v' = f(v)$, if v_0 is a critical point of the equation, then $\gamma(v_0) = \{v_0\}$ is the orbit of v_0.

Poincaré sections give relations between the notions of a dynamical system for discrete and continuous time.

Definition 1.11 *A set $X \subsetneq Y$ is called a* Poincaré section *for a semiflow of maps $\varphi_t \colon Y \to Y$, for $t \geqslant 0$, if*

$$\tau(x) := \inf\bigl\{t > 0 : \varphi_t(x) \in X\bigr\} \in \mathbb{R}^+$$

for each $x \in X$, with the convention that $\inf\varnothing = +\infty$ *(see Figure I.1.1). The number $\tau(x)$ is called the* first return time *of x to X.*

Given a Poincaré section X for a semiflow $(\varphi_t)_{t\geqslant 0}$, its Poincaré map $f \colon X \to X$ *is defined by*

$$f(x) = \varphi_{\tau(x)}(x) \quad \textit{for } x \in X.$$

In particular, given a Poincaré section, one can construct a dynamical system with discrete time (the Poincaré map) from the semiflow. One can often study properties of the semiflow by studying properties of the Poincaré map and vice versa.

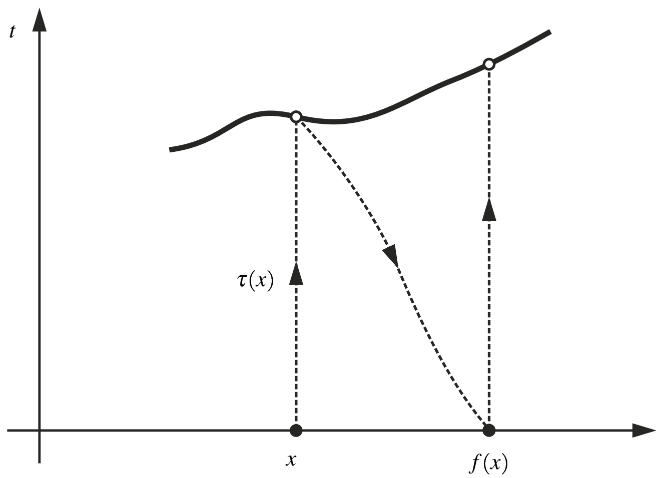

Fig. I.1.1 The first return time $\tau(x)$.

Problems

Problem 1.1 Determine all values of $a \in \mathbb{R}$ for which the map $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^4 - x$ has nonzero fixed points.

Problem 1.2 Determine all the periodic points of the map $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$.

Problem 1.3 Show that if a continuous map $f\colon \mathbb{R} \to \mathbb{R}$ has a periodic point with period 2, then it has at least one fixed point.

Problem 1.4 Show that a continuous map $f\colon [a,b] \to \mathbb{R}$ with $f([a,b]) \supseteq [a,b]$ has at least one fixed point.

Problem 1.5 Show that a continuous map $f\colon [a,b] \to [a,b]$ has at least one fixed point.

Problem 1.6 Consider the continuous map $f\colon [1,5] \to [1,5]$ with

$$f(1) = 3, \quad f(2) = 5, \quad f(3) = 4, \quad f(4) = 2 \quad \text{and} \quad f(5) = 1$$

such that f is linear on $[n, n+1]$ for $n = 1,2,3,4$ (see Figure I.1.2). Show that:

1. f has periodic points with period 5;
2. f^3 has no fixed points in $[1,3] \cup [4,5]$, but has a fixed point in $[3,4]$;
3. f has no periodic points with period 3.

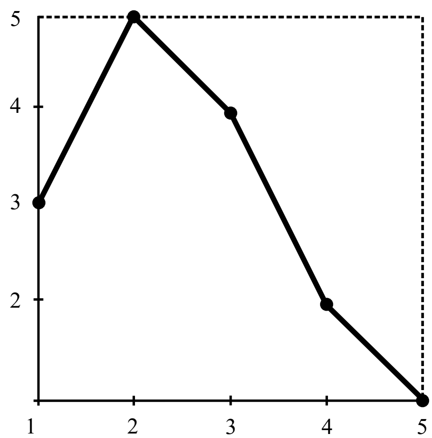

Fig. I.1.2 The map f in Problem 1.6.

Problem 1.7 Let $f\colon I \to I$ be a strictly increasing map on an interval $I \subseteq \mathbb{R}$. Show that any periodic point of f is a fixed point.

Problem 1.8 Let $f \colon I \to I$ be a strictly decreasing map on an interval $I \subseteq \mathbb{R}$. Show that any periodic point of f is either a fixed point or a periodic point with period 2.

Problem 1.9 Show that if x is a periodic point of a map f with period $2n$, then x is a periodic point of f^2 with period n.

Problem 1.10 Show that if x is a periodic point of f^2 with period n even, then x is a periodic point of f with period $2n$.

Problem 1.11 Show that if x is a periodic point of f^2 with period n odd, then x is a periodic point of f with period n or $2n$.

Problem 1.12 Given positive integers m and n, show that if x is a periodic point of f with period m, then it is a periodic point of f^n with period $m/(m,n)$ (recall that (m,n) denotes the greatest common divisor of m and n).

Problem 1.13 Given positive integers n and k, show that if x is a periodic point of f^n with period k, then it is a periodic point of f with period kn/l for some factor l of n with $(k,l) = 1$.

Problem 1.14 Let $f \colon I \to I$ be a map on a closed interval $I \subseteq \mathbb{R}$. Show that if the positive semiorbit $\gamma^+(x)$ of a point $x \in I$ is dense in I, then the set of points in I with a dense positive semiorbit is dense in I.

Problem 1.15 Let $f \colon X \to X$ be a continuous one-to-one map on a compact set. Show that if the set P of periodic points of f is dense in X, then f is a homeomorphism.

Problem 1.16 Let $f \colon X \to X$ be a continuous map on a compact set. Show that if all points of X are periodic points of f, then f is a homeomorphism.

Problem 1.17 Let $f \colon X \to X$ be a continuous map on a compact set. Show that if the set P of periodic points of f is dense in X and there exists $p \in \mathbb{N}$ such that all periodic points have period at most p, then f is a homeomorphism.

Problem 1.18 Consider an interval $I = (a,b) \subsetneq S^1$ with $0 < a < b < 1$. Show that there exists $n \in \mathbb{N}$ such that $E_2^n(I) = S^1$ (see Definition 1.5), identifying I with the set $\{[x] : x \in I\} \subsetneq S^1$ (see Figures I.1.3 and I.1.4).

Problem 1.19 Show that given $x \in S^1$ and $\delta > 0$, there exist $y \in (x-\delta, x+\delta) \subseteq S^1$ and $n \in \mathbb{N}$ such that

$$\left|E_2^n(x) - E_2^n(y)\right| \geqslant \frac{1}{4}.$$

Problem 1.20 Given $x \in S^1$, show that the union $\bigcup_{n=1}^{\infty} E_2^{-n} x$ is dense in S^1.

Problem 1.21 Consider the map $f \colon R \to R$ defined by $f(z) = z^3$ on the set

$$R = \{z \in \mathbb{C} : |z| = 1\}.$$

Show that the set of periodic points of f is dense in R.

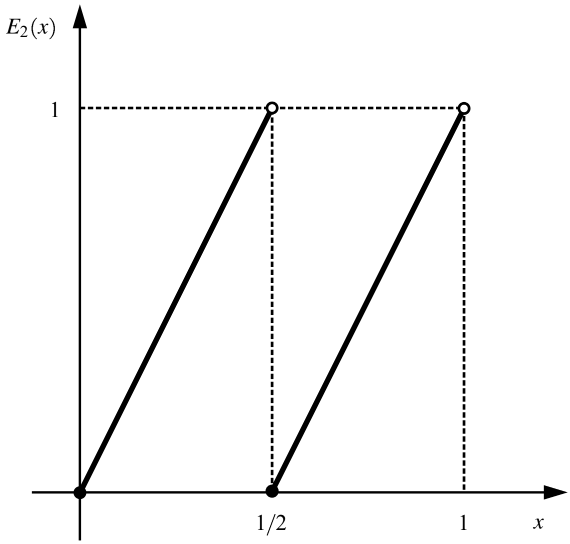

Fig. I.1.3 The expanding map E_2.

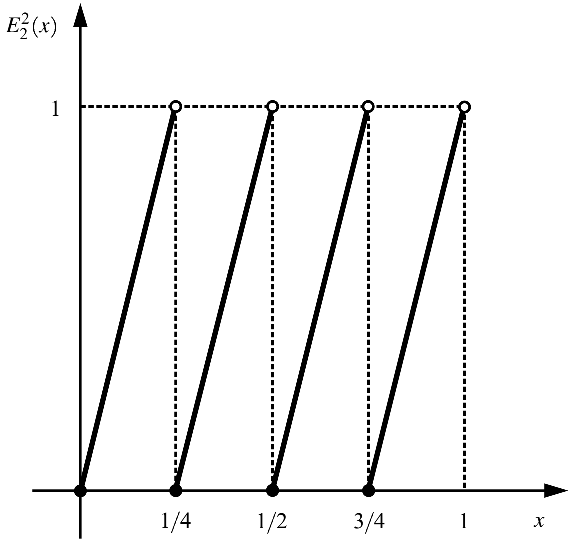

Fig. I.1.4 The map $E_2^2 = E_4$.

Problem 1.22 Consider the map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$f(x,y) = (E_m(x), E_m(y)).$$

Show that the set of periodic points of f is dense in $\mathbb{T}^2$.

Problem 1.23 Let $T_A\colon \mathbb{T}^n \to \mathbb{T}^n$ be an automorphism of the torus induced by a matrix A whose spectrum contains no root of 1. Show that the set of periodic points of T_A is $\mathbb{Q}^n/\mathbb{Z}^n$.

Problem 1.24 Let $T_A\colon \mathbb{T}^n \to \mathbb{T}^n$ be an endomorphism of the torus. Show that

$$\operatorname{card} T_A^{-1}x = |\det A| \quad \text{for all } x \in \mathbb{T}^n,$$

using the Smith normal form (which says that $A = PDQ$, for some matrices P, D and Q with integer entries such that $|\det P| = |\det Q| = 1$ and D is diagonal).

Problem 1.25 Let $T_A\colon \mathbb{T}^n \to \mathbb{T}^n$ be an automorphism of the torus induced by a matrix A whose spectrum contains no root of 1. Show that the number of q-periodic points of T_A is equal to $|\det(A^q - \mathrm{Id})|$.

Problem 1.26 Let $T_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the automorphism of the torus induced by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Show that the number of q-periodic points of T_A is equal to $\operatorname{tr}(A^q) - 2$.

Problem 1.27 Let $f\colon S^1 \to S^1$ be a C^1 map with nonvanishing derivative. Show that there exists $q \in \mathbb{N}$ such that

$$\operatorname{card} f^{-1}x = q \quad \text{for all } x \in S^1.$$

(The number q coincides with the degree $\deg f$ of f.)

Problem 1.28 Given a map $f\colon X \to X$, show that the complement of a backward f-invariant set is forward f-invariant.

Problem 1.29 Let $f\colon X \to X$ be a continuous map on a compact metric space. Moreover, let $A \subseteq X$ be a closed backward f-invariant set and define

$$\Lambda = \bigcap_{n=0}^{\infty} f^{-n}A.$$

Show that:

1. if U is an open neighborhood of Λ, then $f^{-n}A \subseteq U$ for any sufficiently large n;
2. the set Λ is f-invariant.

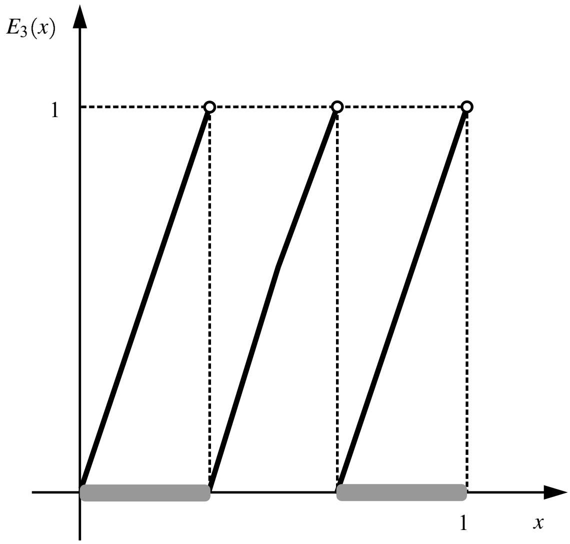

Fig. I.1.5 The set marked in gray is $[0,1/3] \cup [2/3,1]$.

Problem 1.30 Find the largest E_3-invariant set A contained in $J = [0,1/3] \cup [2/3,1]$ (see Figure I.1.5).

Problem 1.31 Consider the differential equation

$$\begin{cases} x' = 6y^5, \\ y' = -4x^3 \end{cases}$$

on $\mathbb{R}^2$. Show that for each set $I \subseteq \mathbb{R}_0^+$ the union

$$\bigcup_{a \in I} \{(x,y) \in \mathbb{R}^2 : y^6 + x^4 = a\}$$

is invariant under the flow determined by the differential equation.

Problem 1.32 Consider the differential equation

$$\begin{cases} x' = ax - xy, \\ y' = -y + x^2 - 2y^2, \end{cases}$$

on $\mathbb{R}^2$ for some $a > 0$. Show that the solutions starting in the sets

$$S_1 = \{(x,y) \in \mathbb{R}^2 : x = 0\} \quad \text{and} \quad S_2 = \left\{(x,y) \in \mathbb{R}^2 : y = \frac{x^2}{1+2a}\right\}$$

remain in these sets for all time.

Problem 1.33 Find the flow determined by the equation $x'' + x = 0$.

Problem 1.34 Find whether the equation $x' = x(x^2 + 1)$ determines a flow.

Problem 1.35 Show that the identity map is a Poincaré map for the differential equation $x'' + x = 0$ and compute the corresponding first return time.

Problem 1.36 Consider the differential equation

$$\begin{cases} x' = y, \\ y' = -\sin x. \end{cases}$$

Show that the set

$$X = \{(x,y) \in \mathbb{R}^2 : y = 0, x \in (0,\pi)\}$$

is a Poincaré section and determine the corresponding first return time and Poincaré map.

Problem 1.37 Show that there exists a homeomorphism $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ mapping the orbits $\varphi_t(x)$ determined by the equation $x' = -x$ on $\mathbb{R}^+$ (which has the phase portrait in Figure I.1.6) onto the orbits $\psi_t(x)$ determined by the equation $x' = -x^2$ on $\mathbb{R}^+$ (which has the phase portrait in Figure I.1.7), and a map $\tau\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with $t \mapsto \tau(x,t)$ increasing for each x such that

$$h(\varphi_{\tau(x,t)}(x)) = \psi_t(h(x)) \quad \text{for } x, t > 0.$$

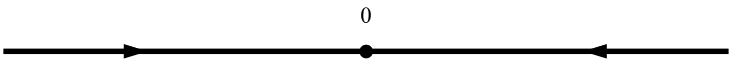

Fig. I.1.6 Phase portrait of the equation $x' = -x$.

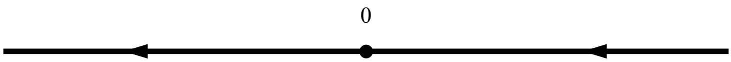

Fig. I.1.7 Phase portrait of the equation $x' = -x^2$.

Problem 1.38 Let $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ be C^1 functions with $f > 0$ such that

$$f(x+k, y+l) = f(x,y) \quad \text{and} \quad g(x+k, y+l) = g(x,y)$$

for all $x, y \in \mathbb{R}$ and $k, l \in \mathbb{Z}$. Then the differential equation

$$x' = f(x,y), \quad y' = g(x,y)$$

on $\mathbb{R}^2$ has unique solutions that are defined for all $t \in \mathbb{R}$. Let $\varphi_t \colon \mathbb{T}^2 \to \mathbb{T}^2$ be the corresponding flow (see Proposition 1.8). Each solution

$$\varphi_t(0,z) = (x(t),y(t)) = (x(t,z),y(t,z))$$

of the equation with $(x(0),y(0)) = (0,z)$ crosses infinitely often the line $x = 0$ (since f is uniformly bounded from below). The first intersection into the future occurs at the time

$$T_z = \inf\{t > 0 : x(t) = 1\}.$$

Show that the map $h\colon S^1 \to S^1$ defined by

$$h(z) = y(T_z,z)$$

(see Figure I.1.8) is invertible.

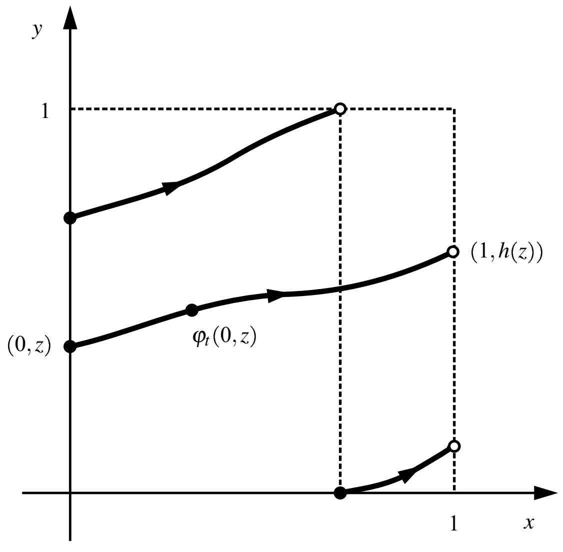

Fig. I.1.8 The map h in Problem 1.38.

Problem 1.39 Consider the differential equation $(x',y') = (\alpha,\beta)$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with $\alpha,\beta \neq 0$. Determine whether any orbit of the flow determined by this equation is dense.

Problem 1.40 Consider the differential equation $v' = f(v)$ on $\mathbb{R}^n$ for some map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of class C^1. Write $f = (f_1,\dots,f_n)$, $v = (v_1,\dots,v_n)$ and let

$$(L\varphi)(v) = \sum_{i=1}^{n} f_i(v) \frac{\partial \varphi(v)}{\partial v_i}$$

for each C^2 function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$. Show that each periodic orbit of the equation:

1. has at least two points in the set

$$A_\varphi = \{v \in \mathbb{R}^n : (L\varphi)(v) = 0\};$$

2. has at least one point in each of the sets

$$B_\varphi = \{v \in A_\varphi : (L^2\varphi)(v) \leqslant 0\} \quad \text{and} \quad C_\varphi = \{v \in A_\varphi : (L^2\varphi)(v) \geqslant 0\};$$

3. intersects A_φ transversally at each point in the sets

$$\{v \in A_\varphi : (L^2\varphi)(v) < 0\} \quad \text{and} \quad \{v \in A_\varphi : (L^2\varphi)(v) > 0\}.$$

Chapter I.2
Topological Dynamics

In this chapter we consider the class of topological dynamical systems, that is, the class of continuous maps on a topological space. In particular, we consider the notions of α-limit set and of ω-limit set, as well as various notions related to topological recurrence, including those of recurrent point, nonwandering point, and minimal set. We also consider the notions of topological transitivity, topological mixing, and topological conjugacy, as well as topological entropy. We refer the reader to [16, 17, 48] for additional topics.

Notions and Results

We first recall some notions and results from topological dynamics.

Definition 2.1 *A continuous map $f\colon X \to X$ is called a* topological dynamical system *with discrete time. When f is a homeomorphism, that is, a bijective continuous map with continuous inverse, it is also called an* invertible topological dynamical system.

Definition 2.2 *A flow Φ (respectively, a semiflow Φ) on X such that the map $(t,x) \mapsto \varphi_t(x)$ is continuous on its domain is called a* topological flow *(respectively, a* topological semiflow*). Any topological flow or semiflow is also called a* topological dynamical system *with continuous time.*

Definition 2.3 *Given a map $f\colon X \to X$, the* ω-limit set *of a point $x \in X$ is defined by*

$$\omega(x) = \bigcap_{n\in\mathbb{N}} \overline{\{f^m(x) : m \geqslant n\}}.$$

Moreover, when f is invertible, the α-limit set *of a point $x \in X$ is defined by*

$$\alpha(x) = \bigcap_{n\in\mathbb{N}} \overline{\{f^{-m}(x) : m \geqslant n\}}.$$

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_2

The following proposition gives a characterization of the ω-limit set.

Proposition 2.4 *Given a map $f\colon X \to X$ on a topological space, for each $x \in X$ the following properties hold:*

1. *$y \in \omega(x)$ if and only if there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$ such that $f^{n_k}(x) \to y$ when $k \to \infty$;*
2. *if f is continuous, then $\omega(x)$ is forward f-invariant.*

When f is invertible, one can formulate a corresponding result for the α-limit set.

Proposition 2.5 *Given an invertible map $f\colon X \to X$ on a topological space, for each $x \in X$ the following properties hold:*

1. *$y \in \alpha(x)$ if and only if there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$ such that $f^{-n_k}(x) \to y$ when $k \to \infty$;*
2. *if f^{-1} is continuous, then $\alpha(x)$ is backward f-invariant.*

We also consider corresponding notions for continuous time.

Definition 2.6 *Given a semiflow Φ on X, the* ω-limit set *of a point $x \in X$ is defined by*

$$\omega(x) = \bigcap_{t>0} \overline{\{\varphi_s(x) : s > t\}}.$$

Moreover, given a flow Φ on X, the α-limit set *of a point $x \in X$ is defined by*

$$\alpha(x) = \bigcap_{t<0} \overline{\{\varphi_s(x) : s < t\}}.$$

There are also corresponding versions of Propositions 2.4 and 2.5 for continuous time. In particular, given a semiflow Φ on a topological space X, for each $x \in X$ we have $y \in \omega(x)$ if and only if there exists a sequence $t_k \nearrow +\infty$ in $\mathbb{R}^+$ such that

$$\varphi_{t_k}(x) \to y \quad \text{when } k \to \infty.$$

Moreover, given a flow Φ on X, for each $x \in X$ we have $y \in \alpha(x)$ if and only if there exists a sequence $t_k \searrow -\infty$ in $\mathbb{R}^-$ such that

$$\varphi_{t_k}(x) \to y \quad \text{when } k \to \infty.$$

Now we recall a few notions related to recurrence.

Definition 2.7 *Given a map $f\colon X \to X$, a point $x \in X$ is said to be* (forward) recurrent *for f if $x \in \omega(x)$. The set of recurrent points for f is denoted by $R(f)$.*

By Proposition 2.4, a point $x \in X$ is recurrent if and only if there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$ such that $f^{n_k}(x) \to x$ when $k \to \infty$.

Definition 2.8 *Given a map $f\colon X \to X$ on a topological space, a point $x \in X$ is said to be* nonwandering *for f if for any open neighborhood U of x there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \varnothing$. The set of nonwandering points for f is denoted by $NW(f)$.*

Definition 2.9 *A map $f\colon X \to X$ on a topological space is said to be:*

1. topologically transitive *if given nonempty open sets $U, V \subsetneq X$, there exists an integer $n \in \mathbb{N}$ such that $f^{-n}U \cap V \neq \varnothing$;*
2. topologically mixing *if given nonempty open sets $U, V \subsetneq X$, there exists an integer $n \in \mathbb{N}$ such that $f^{-m}U \cap V \neq \varnothing$ for all $m \geqslant n$.*

The remaining notions and results concern the topological entropy of a continuous map on a compact metric space. It can be seen as a measure of the complexity of the dynamics.

Definition 2.10 *Let $f\colon X \to X$ be a continuous map on a compact metric space with distance d. For each $n \in \mathbb{N}$, we introduce a new distance $d_n = d_{n,f}$ on X by*

$$d_n(x,y) = \max\left\{d(f^k(x), f^k(y)) : 0 \leqslant k \leqslant n-1\right\}.$$

Let $N(n,\varepsilon) = N_f(n,\varepsilon)$ be the largest number of points $p_1, \ldots, p_m \in X$ such that

$$d_n(p_i, p_j) \geqslant \varepsilon \quad \text{whenever } i \neq j.$$

The topological entropy *of f is defined by*

$$h(f) = h_d(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n,\varepsilon).$$

Definition 2.11 *Two maps $f\colon X \to X$ and $g\colon Y \to Y$ on topological spaces are said to be* topologically conjugate *if there exists a homeomorphism $H\colon X \to Y$ such that*

$$H \circ f = g \circ H \quad \text{on } X,$$

that is, the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "H"'] & X \arrow[d, "H"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

is commutative. Then H is called a topological conjugacy *between f and g.*

A topological conjugacy can be seen as a dictionary between two dynamics and can sometimes be used to transfer a given property from one dynamics to the other. It turns out that topologically conjugate continuous maps have the same topological entropy.

Theorem 2.12 *Let $f\colon X \to X$ and $g\colon Y \to Y$ be continuous maps on compact metric spaces. If f and g are topologically conjugate, then $h(f) = h(g)$.*

Finally, we describe a few equivalents or alternative definitions of the topological entropy.

Definition 2.13 *Given $n \in \mathbb{N}$ and $\varepsilon > 0$:*

1. *let $M(n,\varepsilon) = M_f(n,\varepsilon)$ be the least number of points $p_1, \dots, p_m \in X$ such that for each $x \in X$ there exists $i \in \{1, \dots, m\}$ with $d_n(x, p_i) < \varepsilon$;*
2. *let $C(n,\varepsilon) = C_f(n,\varepsilon)$ be the least cardinality of a cover of X by sets $U_1, \dots, U_m$ with*

$$\sup\{d_n(x,y) : x,y \in U_i\} < \varepsilon \quad \text{for } i = 1, \dots, m.$$

Theorem 2.14 *If $f\colon X \to X$ is a continuous map on a compact metric space, then*

$$\begin{aligned} h(f) &= \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N(n,\varepsilon) \\ &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log M(n,\varepsilon) \\ &= \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log M(n,\varepsilon) \\ &= \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log C(n,\varepsilon). \end{aligned}$$

Problems

Problem 2.1 Show that if $f\colon X \to X$ is a continuous map on a compact metric space, then $f(\omega(x)) = \omega(x)$ for all $x \in X$.

Problem 2.2 Given a map $f\colon X \to X$ on a metric space without isolated points, show that if $\omega(x) \neq X$, then the positive semiorbit $\gamma^+(x)$ is not dense.

Problem 2.3 Let $f\colon X \to X$ be a continuous map. Given $x \in X$, show that if $\omega(x)$ contains infinitely many points, then it has no isolated points.

Problem 2.4 Let $f\colon I \to I$ be a continuous map on the interval $I = [0,1]$. Given $x \in I$, show that for any $a, b \in \omega(x)$ and any open neighborhood U of a, there exists an increasing sequence of positive integers $(k_i)_{i\in\mathbb{N}}$ such that $b \in \overline{\bigcup_{i\in\mathbb{N}} f^{k_i}(U)}$.

Problem 2.5 Let $f\colon X \to X$ be a continuous map on a compact metric space. Given $x \in X$, show that if $A = \omega(x)$, then for any nonempty closed subset $B \subsetneq A$ we have $B \cap \overline{f(A \setminus B)} \neq \varnothing$.

Problem 2.6 Compute the α-limit set and the ω-limit set of each point in $\mathbb{R}$ for the flow determined by the differential equation $x' = -x(1 + \cos^2 x)$ on $\mathbb{R}$.

Problem 2.7 Consider the differential equation

$$\begin{cases} r' = r(r-1), \\ \theta' = 1, \end{cases}$$

written in polar coordinates. Compute the α-limit set and the ω-limit set of each point in $\mathbb{R}^2$ for the flow determined by the differential equation.

Problem 2.8 Compute the α-limit set of each point $(x,y) \in \mathbb{R}^2$ with $|x| < 1$ for the flow determined by the differential equation

$$\begin{cases} x' = (x^2 - 1)(y - x), \\ y' = x. \end{cases}$$

Problem 2.9 Let $f \colon X \to X$ be a continuous map on a topological space and let $NW(f)$ be the set of nonwandering points. Show that:

1. $NW(f)$ is closed;
2. $NW(f)$ is forward f-invariant.

Problem 2.10 Let $f \colon X \to X$ be a continuous map on a topological space. Show that $\omega(x) \subseteq NW(f)$ for all $x \in X$.

Problem 2.11 Let $f \colon X \to X$ be a continuous map on a compact metric space. Show that

$$\lim_{m \to \infty} d(f^m(x), NW(f)) = 0$$

for every $x \in X$.

Problem 2.12 Let $f \colon X \to X$ be a continuous map. Show that $R(f)$ (the set of recurrent points) is forward f-invariant.

Problem 2.13 Given a map $f \colon X \to X$, show that $\overline{R(f)} \subseteq NW(f)$.

Problem 2.14 A nonempty closed forward invariant set without nonempty closed forward invariant proper subsets is said to be *minimal*. Show that any two distinct minimal sets for a map f must have empty intersection.

Problem 2.15 Give an example of an ω-limit set that is a minimal set.

Problem 2.16 Given a map $f \colon X \to X$ on a topological space and a nonempty closed forward f-invariant set $M \subseteq X$, show that the following properties are equivalent:

1. M is a minimal set;
2. $M = \overline{\gamma^+(x)}$ for all $x \in M$;
3. M is the ω-limit set of each of its points.

Problem 2.17 For a continuous map $f \colon X \to X$ show that f is topologically transitive if and only if for any nonempty open sets $U, V \subsetneq X$ there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$.

Problem 2.18 Let $f \colon X \to X$ be a continuous map. Show that if $\bigcup_{n=1}^{\infty} f^n(U)$ is dense for any nonempty open set $U \subseteq X$, then $\bigcup_{n=1}^{\infty} f^{-n}U$ is also dense for any nonempty open set $U \subseteq X$.

Problem 2.19 Given a continuous map $f \colon X \to X$, show that if f is topologically transitive, then any closed forward f-invariant proper subset of X has empty interior.

Problem 2.20 Given a continuous map $f\colon X \to X$ on a complete metric space with a countable basis and without isolated points, show that the following properties are equivalent:

1. f is topologically transitive;
2. the set of points with a dense positive semiorbit is dense.

Problem 2.21 Give an example of a continuous map $f\colon X \to X$ on a finite set with at least one dense positive semiorbit, but which is not topologically transitive.

Problem 2.22 Give an example of a continuous map $f\colon X \to X$ on an infinite set with at least one dense positive semiorbit, but which is not topologically transitive.

Problem 2.23 Show that the continuous map $f\colon [0,1] \cap \mathbb{Q} \to [0,1] \cap \mathbb{Q}$ defined by $f(x) = 1 - |2x - 1|$ has no dense positive semiorbits, but is topologically transitive.

Problem 2.24 Show that the map f in Problem 1.21 is topologically transitive.

Problem 2.25 Let $R_\alpha\colon S^1 \to S^1$ be a rotation of the circle with $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. Show that

$$\overline{\{R_\alpha^m(x) : m \in \mathbb{Z}\}} = S^1$$

for every $x \in S^1$.

Problem 2.26 Let $f\colon X \to X$ and $g\colon Y \to Y$ be topologically conjugate maps (see Definition 2.11). Show that for each $q \in \mathbb{N}$ the number of q-periodic points of f is equal to the number of q-periodic points of g.

Problem 2.27 Consider continuous maps $f, g\colon \mathbb{R} \to \mathbb{R}$ that are topologically conjugate via a homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ satisfying $h \circ f = g \circ h$. Show that if p is an attracting fixed point of f (which means that there exists an open neighborhood U of p such that for $x \in U$ we have $f^n(x) \to p$ when $n \to \infty$) if and only if $q = h(p)$ is an attracting fixed point of g.

Problem 2.28 Show that if two maps are topologically conjugate and one of them is topologically mixing, then the other is also topologically mixing.

Problem 2.29 Given a topologically mixing map $f\colon X \to X$, show that the map $f \times f\colon X \times X \to X \times X$ defined by

$$(f \times f)(x,y) = (f(x), f(y))$$

is also topologically mixing.

Problem 2.30 Let $f\colon X \to X$ be a continuous map on a compact metric space (X,d) such that

$$d(f(x), f(y)) \leqslant d(x,y) \quad \text{for all } x, y \in X.$$

Show that the topological entropy of f is zero.

Problem 2.31 Show that if $f\colon X \to X$ is a homeomorphism on a compact metric space (X,d), then

$$N_f(n,\varepsilon) = N_{f^{-1}}(n,\varepsilon)$$

for all $n \in \mathbb{N}$ and $\varepsilon > 0$.

Problem 2.32 Let $f\colon X \to X$ be a continuous map on a compact metric space. Given a closed forward f-invariant set $Y \subseteq X$, show that $h(f) \geqslant h(f|_Y)$.

Problem 2.33 Let $f\colon X \to X$ and $g\colon Y \to Y$ be continuous maps on compact metric spaces, respectively, (X,d_X) and (Y,d_Y) satisfying $f \circ H = H \circ g$ for some continuous onto map $H\colon Y \to X$. Show that $h(f) \leqslant h(g)$.

Problem 2.34 Let $f\colon X \to X$ and $g\colon Y \to Y$ be continuous maps on compact metric spaces satisfying $f \circ H = H \circ g$ for some homeomorphism $H\colon Y \to X$. Show that $h(f) = h(g)$.

Problem 2.35 Compute the topological entropy of the map f in Problem 1.21.

Problem 2.36 Determine whether there exists a continuous map $f\colon X \to X$ on a compact metric space with infinite topological entropy.

Problem 2.37 Show that if the distances d and d' generate both the topologies of a compact topological space X, then

$$h_d(f) = h_{d'}(f)$$

for any continuous map $f\colon X \to X$.

Problem 2.38 Let $f\colon X \to X$ be a continuous map on a compact metric space (X,d). Show that:

1. if $\mathcal{U}$ is a finite open cover of X, then letting

$$\mathcal{U}_n = \left\{ \bigcap_{k=0}^{n-1} f^{-k} U_k : U_0, \ldots, U_{n-1} \in \mathcal{U} \right\}$$

   and denoting by $N(\mathcal{U}_n)$ the smallest cardinality of the finite subcovers of $\mathcal{U}_n$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}_n)$$

   exists;
2. if $\mathcal{U}$ is a finite open cover of X with Lebesgue number δ (this is a number such that any open ball of radius δ is contained in some element of $\mathcal{U}$), then

$$M(n,\delta/2) \geqslant N(\mathcal{U}_n);$$

3. if $\mathcal{U}$ is a finite open cover of X with

$$\operatorname{diam}\mathcal{U} := \sup\left\{\operatorname{diam} U : U \in \mathcal{U}\right\} < \varepsilon,$$

where $\operatorname{diam} U = \{d(x,y) : x,y \in U\}$, then $N(n,\varepsilon) \leqslant N(\mathcal{U}_n)$;

4.
$$h(f) = \lim_{\operatorname{diam}\mathcal{U}\to 0} \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}_n).$$

Problem 2.39 Let $f\colon [0,1] \to [0,1]$ be a homeomorphism. Show that $h(f) = 0$.

Problem 2.40 Let $f\colon S^1 \to S^1$ be a homeomorphism. Show that $h(f) = 0$.

Chapter I.3
Low-Dimensional Dynamics

In this chapter we consider various dynamical systems on low-dimensional spaces (namely of dimension 1 for discrete time and of dimension 2 for continuous time). The reason for this separation is that the results and methods that can be used with these dynamical systems, because of specific topological properties, fail on higher-dimensional spaces. In particular, we consider homeomorphisms and diffeomorphisms of the circle, including the rotation number of an orientation-preserving homeomorphism, continuous maps on a compact interval, and flows defined by autonomous differential equations on the plane. We refer the reader to [2, 10, 22, 24] for additional topics.

Notions and Results

We start by recalling some notions and results associated to the study of dynamical systems on low-dimensional spaces.

Definition 3.1 *Consider a homeomorphism* $f\colon S^1 \to S^1$ *and let* $\pi\colon \mathbb{R} \to S^1$ *be the map defined by* $\pi(x) = [x]$*. A continuous function* $F\colon \mathbb{R} \to \mathbb{R}$ *is called a* lift *of* f *if*

$$f \circ \pi = \pi \circ F \quad \text{on } \mathbb{R}.$$

Proposition 3.2 *Let* $f\colon S^1 \to S^1$ *be a homeomorphism. Then:*

1. f *has lifts;*
2. *if* F *and* G *are lifts of* f*, then there exists* $k \in \mathbb{Z}$ *such that* $G - F = k$ *on* $\mathbb{R}$*;*
3. *any lift of* f *is a homeomorphism on* $\mathbb{R}$*.*

Definition 3.3 *A homeomorphism* $f\colon S^1 \to S^1$ *is said to be:*

1. orientation-preserving *if at least one of its lifts is an increasing function;*
2. orientation-reversing *if at least one of its lifts is a decreasing function.*

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_3

Now we consider the notion of the rotation number of an orientation-preserving homeomorphism.

Theorem 3.4 *Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism. If F is a lift of f, then for each $x \in \mathbb{R}$ the limit*

$$\rho(F) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} \in \mathbb{R}$$

exists and is independent of x. Moreover, if G is another lift of f, then

$$\rho(G) - \rho(F) \in \mathbb{Z}.$$

We denote by $\{\alpha\}$ the fractional part of a number $\alpha \in \mathbb{R}$.

Definition 3.5 *The* rotation number *of an orientation-preserving homeomorphism $f\colon S^1 \to S^1$ is defined by $\rho(f) = \{\rho(F)\}$ for any lift F of f.*

We note that the rotation number of an orientation-preserving homeomorphism f is well defined, that is, the number $\{\rho(F)\}$ is the same for all lifts F of f.

Theorem 3.6 *Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism. Then:*

1. *$\rho(f) \in \mathbb{Q}$ if and only if f has at least one periodic point;*
2. *if $\rho(f) = p/q$ with $(p,q) = 1$, then all periodic points of f have period q.*

We also introduce the notion of a function with bounded variation.

Definition 3.7 *Let P be the set of all partitions $\{x_k : k = 0,\dots,n\}$ of S^1 composed of points $x_0 < x_1 < \cdots < x_n$ in S^1, with $x_n = x_0$, for some $n \in \mathbb{N}$. A function $\varphi\colon S^1 \to \mathbb{R}$ is said to have* bounded variation *if*

$$\operatorname{Var}\varphi := \sup_P \sum_{k=1}^{n} |\varphi(x_k) - \varphi(x_{k-1})| < +\infty.$$

Theorem 3.8 (Denjoy) *Let $f\colon S^1 \to S^1$ be an orientation-preserving C^1 diffeomorphism whose derivative has bounded variation. If $\rho(f) \in \mathbb{R}\setminus\mathbb{Q}$, then f is topologically conjugate to the rotation $R_{\rho(f)}$.*

We also consider Sharkovsky's theorem for a continuous map on a compact interval. It relates the existence of periodic points with different periods with respect to a certain ordering on $\mathbb{N}$.

Definition 3.9 *The* Sharkovsky's ordering $\prec$ *on $\mathbb{N}$ is defined by*

$$\begin{aligned}
&1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^m \prec \cdots\\
&\ \cdots\\
&\ \prec \cdots \prec 2^m(2n+1) \prec \cdots \prec 2^m 7 \prec 2^m 5 \prec 2^m 3 \prec \cdots\\
&\ \cdots\\
&\ \prec \cdots \prec 2(2n+1) \prec \cdots \prec 2\cdot 7 \prec 2\cdot 5 \prec 2\cdot 3 \prec \cdots\\
&\ \prec \cdots \prec 2n+1 \prec \cdots \prec 7 \prec 5 \prec 3.
\end{aligned}$$

Theorem 3.10 (Sharkovsky) *Let $f\colon I \to I$ be a continuous map on a compact interval $I \subsetneq \mathbb{R}$. If f has a periodic point with period p and $q \prec p$, then f has a periodic point with period q.*

Finally, we formulate a result for autonomous differential equations on the plane. It can sometimes be used to establish the existence of periodic orbits.

Theorem 3.11 (Poincaré–Bendixson) *For the flow determined by a nonautonomous differential equation on $\mathbb{R}^2$, if the positive semiorbit $\gamma^+(x)$ of a point $x \in \mathbb{R}^2$ is bounded and $\omega(x)$ contains no critical points, then $\omega(x)$ is a periodic orbit.*

Problems

Problem 3.1 Show that the composition $R_\alpha \circ R_\beta$ of two rotations of the circle $R_\alpha, R_\beta : S^1 \to S^1$ is also a rotation of the circle.

Problem 3.2 Show that given $n \times n$ matrices A and B with entries in $\mathbb{Z}$, the composition of the endomorphisms of the torus $T_A, T_B\colon \mathbb{T}^n \to \mathbb{T}^n$ is again an endomorphism of the torus and determine a matrix C such that $T_A \circ T_B = T_C$.

Problem 3.3 Consider the homeomorphism of the circle defined by

$$f([x]) = \left[x + \frac{1}{2} + \frac{1}{4\pi}\sin(2\pi x)\right]$$

(see Figure I.3.1). Show that $\{0, 1/2\}$ is a periodic orbit of f and that $\rho(f) = 1/2$.

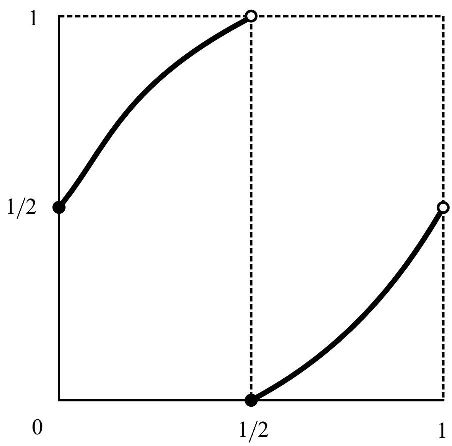

Fig. I.3.1 The map f in Problem 3.3.

Problem 3.4 Given $a \in (0, 1/(2\pi))$, consider the map $f\colon S^1 \to S^1$ defined by

$$f([x]) = [x + a\sin(2\pi x)]$$

(see Figure I.3.2). Show that each orbit starting in $(0, 1/2)$ is monotonous.

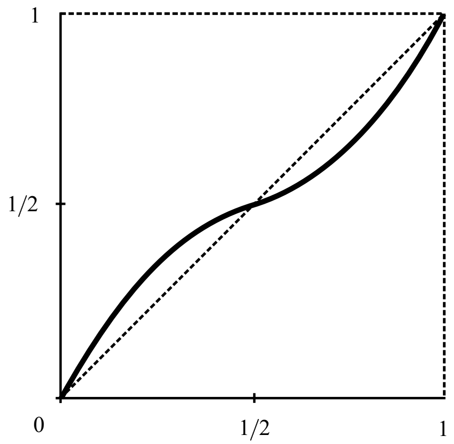

Fig. I.3.2 The map f in Problem 3.4.

Problem 3.5 Let $f\colon S^1 \to S^1$ be a homeomorphism with a single periodic orbit. Show that any other orbit is asymptotic to this periodic orbit (see Figure I.3.3 for an example).

Problem 3.6 Let $f\colon S^1 \to S^1$ be an orientation-reversing homeomorphism and let F be a lift of f. Show that $F(x+1) = F(x) - 1$ for all $x \in \mathbb{R}$ (see Figures I.3.4 and I.3.5).

Problem 3.7 Let $f\colon S^1 \to S^1$ be an orientation-reserving homeomorphism. Show that $\rho(f^2) = 0$.

Problem 3.8 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism with $\rho(f) = p/q$ for some integers $p, q \in \mathbb{N}$. Show that for each lift F of f there exists $x \in \mathbb{R}$ such that $F(x) - x - p/q \in \mathbb{Z}$.

Problem 3.9 Let $f, g\colon S^1 \to S^1$ be homeomorphisms. Show that if F and G are lifts, respectively, of f and g, then

$$\lim_{n\to\infty} \frac{F(G^n(x)) - G^n(x)}{n} = 0$$

for each $x \in \mathbb{R}$.

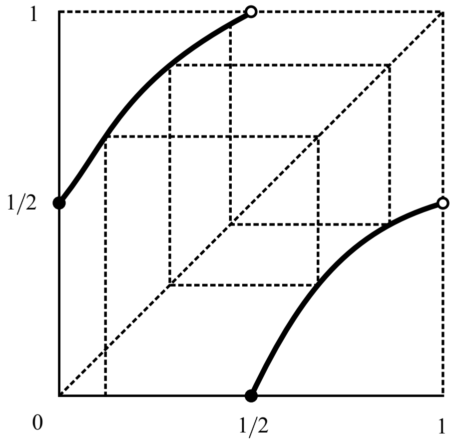

Fig. I.3.3 Homeomorphism of the circle with the single periodic orbit $\{0, 1/2\}$.

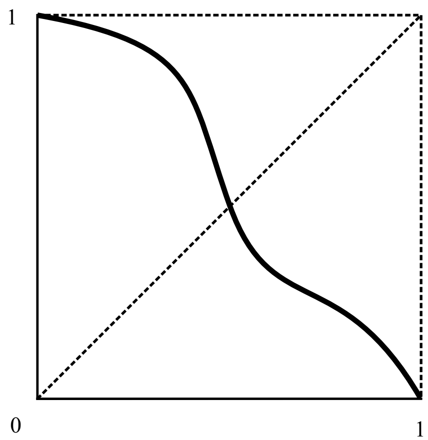

Fig. I.3.4 An orientation-reversing homeomorphism of the circle.

Problem 3.10 Let $f, g\colon S^1 \to S^1$ be homeomorphisms and let F and G be lifts, respectively, of f and g. Show that if f is an orientation-preserving homeomorphism, then:

1. for each $k \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n} = \lim_{n\to\infty} \frac{F^n(x_n) - x_n}{n}$$

for any $x, x_n \in \mathbb{R}$ with $|x - x_n| \leqslant k$ for $n \in \mathbb{N}$;

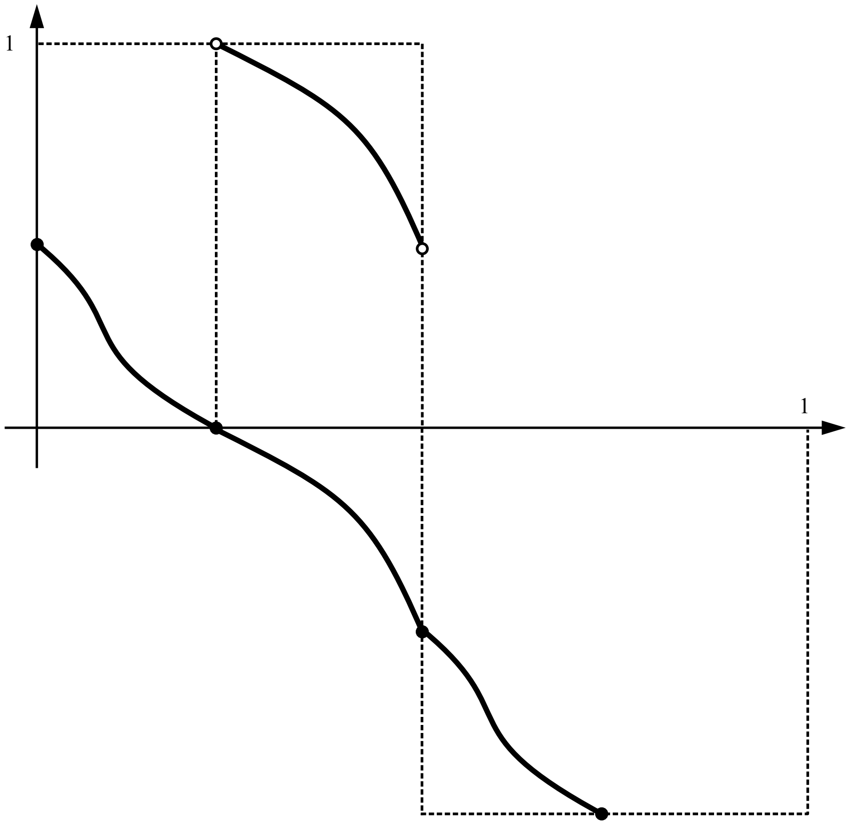

Fig. I.3.5 A lift of an orientation-reversing homeomorphism.

2.

$$\lim_{n\to\infty}\frac{F^n(G^n(x))-G^n(x)}{n}=\lim_{n\to\infty}\frac{F^n(x)-x}{n}$$

for each $x\in\mathbb{R}$.

Problem 3.11 Let $f\colon S^1\to S^1$ be an orientation-preserving homeomorphism with

$$r/s<\rho(f)<p/q<1$$

for some positive integers r,s,p,q. Show that:

1. any lift F of f with $F(0)\in(0,1)$ satisfies

$$F^q(x)<x+p \quad \text{and} \quad F^s(x)>x+r$$

for all $x\in\mathbb{R}$;

2. if $g\colon S^1 \to S^1$ is an orientation-preserving homeomorphism with

$$d(f,g) := \max_{x\in S^1} d(f(x),g(x))$$

sufficiently small, then $r/s < \rho(g) < p/q$.

Problem 3.12 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism. Show that if $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$, then for any $m,n \in \mathbb{Z}$ with $m \neq n$ we have

$$S^1 = \bigcup_{k=0}^{\infty} f^{-k}I, \quad \text{for } I = [f^n(x), f^m(x)].$$

Problem 3.13 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism. Show that if $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$, then $\omega(x) = \omega(y)$ for any $x,y \in S^1$.

Problem 3.14 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism. Show that if there exists an open interval $J \subsetneq S^1$ such that the sets $f^k(J)$, for $k \in \mathbb{Z}$, are pairwise disjoint, then f is not conjugate to an irrational rotation.

Problem 3.15 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism with $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$. Show that if there exists a minimal set $K \neq S^1$ (see Problem 2.14 for the definition), then f is not conjugate to a rotation.

Problem 3.16 Show that the map $h\colon S^1 \to S^1$ given by

$$h(x) = \sin^2(\pi x/2)$$

is a topological conjugacy between the maps $f,g\colon S^1 \to S^1$ defined by

$$f(x) = 1 - |2x-1| \quad \text{and} \quad g(x) = 4x(1-x)$$

(see Figures I.3.6 and I.3.7).

Problem 3.17 Determine whether the maps E_4 and $R_{1/2}$ are topologically conjugate.

Problem 3.18 Determine for which values of $\alpha \in [0,1)$ the rotations R_α and $R_{2\alpha}$ are topologically conjugate.

Problem 3.19 Show that the map f in Problem 3.16 is topologically mixing.

Problem 3.20 Show that the map $g\colon S^1 \to S^1$ given by $g(x) = 4x(1-x)$ is topologically mixing.

Problem 3.21 Consider an interval $I = (a,b) \subsetneq S^1$ with $0 < a < b < 1$. For the map g in Problem 3.20, show that there exists $n \in \mathbb{N}$ such that $g^n(I) = S^1$, identifying I with $\{[x] : x \in I\} \subsetneq S^1$.

Problem 3.22 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism and let $NW(f)$ be the nonwandering set for f. Show that if $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$, then $NW(f)$ is a minimal set.

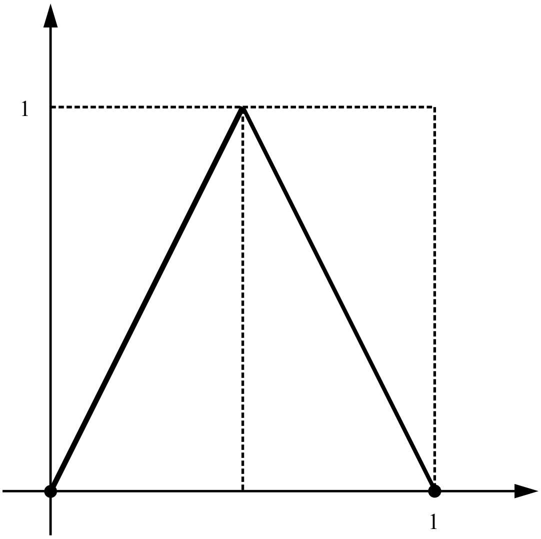

Fig. I.3.6 The map f in Problem 3.16.

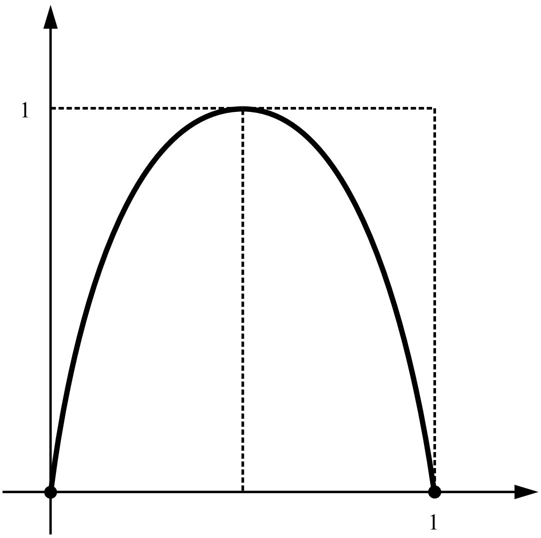

Fig. I.3.7 The map g in Problem 3.16.

Problem 3.23 Let $f: S^1 \to S^1$ be an orientation-preserving homeomorphism. Show that if $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$, then either $NW(f) = S^1$ or $NW(f)$ is a Cantor set (that is, a closed set with empty interior and without isolated points).

Problem 3.24 Let $f: S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number. Show that f has a unique minimal set $K \subseteq S^1$.

Problem 3.25 Let $f: S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number. Show that either $\omega(x)$ is nowhere dense or $\omega(x) = S^1$.

Problem 3.26 Let $f: S^1 \to S^1$ be a C^2 map. Show that its derivative has bounded variation.

Problem 3.27 Let $f: S^1 \to S^1$ be a C^1 diffeomorphism. Moreover, given $n \in \mathbb{N}$ and an interval $I \subseteq S^1$, let

$$D_n(I) = \sup_{x,y \in I} \log \frac{|(f^n)'(x)|}{|(f^n)'(y)|}.$$

Show that

$$D_n(I) \leqslant \sum_{i=0}^{n-1} D_1(f^i(I)).$$

Problem 3.28 Let $f: S^1 \to S^1$ be an orientation-preserving C^1 diffeomorphism with irrational rotation number. Show that if there exist a sequence $n_k \in \mathbb{N}$ with $n_k \nearrow \infty$ and a constant $C > 0$ such that

$$|(f^{n_k})'(x)| \cdot |(f^{-n_k})'(x)| > C$$

for all $x \in S^1$ and $k \in \mathbb{N}$, then every orbit of f is dense.

Problem 3.29 Consider the piecewise linear map $f: [1,3] \to [1,3]$ with

$$f(1) = 2, \quad f(2) = 3 \quad \text{and} \quad f(3) = 1$$

(see Figure I.3.8). Show that f has period points with all periods.

Problem 3.30 Show that the map $f: [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2], \\ 2(1-x) & \text{if } x \in [1/2, 1] \end{cases}$$

(see Figure I.3.6) has periodic points with all periods.

Problem 3.31 Show that the set $X = [-2,-1] \cup [1,2]$ does not have the property that any continuous map $f: X \to X$ satisfies Sharkovsky's ordering in Definition 3.9.

Problem 3.32 Show that a closed set X with the property that any continuous map $f: X \to X$ satisfies Sharkovsky's ordering in Definition 3.9 is connected.

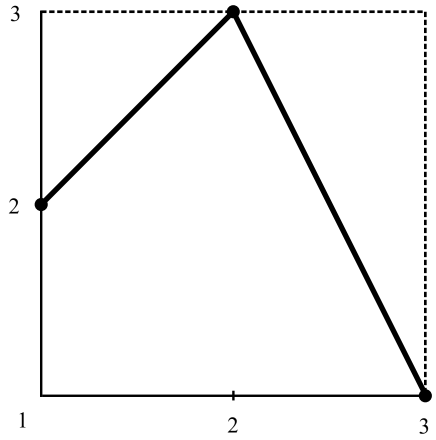

Fig. I.3.8 The map f in Problem 3.29.

Problem 3.33 Verify that the differential equation

$$\begin{cases} x' = xy - x^3, \\ y' = x + y - x^2 \end{cases}$$

has no periodic solutions contained in the second quadrant.

Problem 3.34 Verify that the differential equation

$$\begin{cases} r' = 4 - 3r\cos\theta - r^4, \\ \theta' = -1, \end{cases}$$

written in polar coordinates, has at least one periodic solution.

Problem 3.35 Verify that the differential equation

$$\begin{cases} r' = r(1-r), \\ \theta' = 1 + \cos^2\theta \end{cases}$$

has at least one periodic solution.

Problem 3.36 Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x,y) = \big(y, -x + y(1 - 3x^2 - 6y^2)\big)$$

and let

$$V(x,y) = x^2 + y^2 \quad \text{and} \quad \dot{V}(x,y) = \nabla V(x,y) \cdot f(x,y).$$

Show that:

1. $\dot{V}(x,y) \leqslant 0$ whenever $V(x,y) \geqslant 1/3$;
2. $\dot{V}(x,y) \geqslant 0$ whenever $V(x,y) \leqslant 1/6$;
3. there exists at least one periodic solution of the equation $(x',y') = f(x,y)$ in the set
$$D = \big\{(x,y) \in \mathbb{R}^2 : 1/6 \leqslant V(x,y) \leqslant 1/3\big\}.$$

Problem 3.37 Show that any flow determined by the differential equation does not have periodic solutions:

1. $v' = f(v)$ for some continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^n$, assuming that there exists a function $V\colon \mathbb{R}^n \to \mathbb{R}$ that is strictly decreasing along solutions;
2. $v' = -\nabla V(v)$ for some C^1 function $V\colon \mathbb{R}^n \to \mathbb{R}$.

Problem 3.38 Consider the differential equation

$$\begin{cases} x' = y, \\ y' = -f(x)y - g(x) \end{cases}$$

for some C^1 functions $f, g\colon \mathbb{R} \to \mathbb{R}$ with $f > 0$. Show that the equation has no periodic solutions.

Problem 3.39 Consider the differential equation

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y) \end{cases}$$

on $\mathbb{R}^2$. Show that if there exists a C^1 function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial(\varphi f)}{\partial x} + \frac{\partial(\varphi g)}{\partial y}$$

has the same sign almost everywhere on a simply connected open set $U \subseteq \mathbb{R}^2$, then the equation has no periodic orbits contained in U.

Problem 3.40 Show that the differential equation

$$\begin{cases} x' = y, \\ y' = -x - y + x^2 + y^2 \end{cases}$$

has no periodic solutions.

Chapter I.4
Hyperbolic Dynamics

In this chapter we consider various notions and results of hyperbolic dynamics. Besides the notion of a hyperbolic set, we consider the Smale horseshoe and some of its modifications, the characterization of a hyperbolic set in terms of invariant families of cones, as well as the construction of topological conjugacies near a hyperbolic fixed point. Moreover, we consider briefly geodesic flows on the upper half plane and their hyperbolicity. We refer the reader to [28, 40, 41, 50, 54] for additional topics.

Notions and Results

We start by recalling a few notions and results that are used in the chapter.

Definition 4.1 *Let $f\colon M\to M$ be a C^1 diffeomorphism. A compact f-invariant set $\Lambda\subseteq M$ is called a* hyperbolic set *for f if there exist $\lambda\in(0,1)$, $c>0$ and a splitting*

$$T_xM=E^s(x)\oplus E^u(x)$$

for each $x\in\Lambda$ such that:

1.
$$d_xfE^s(x)=E^s(f(x))\quad\text{and}\quad d_xfE^u(x)=E^u(f(x));$$

2 if $v\in E^s(x)$ and $n\in\mathbb{N}$, then

$$\|d_xf^nv\|\leqslant c\lambda^n\|v\|;$$

3. if $v\in E^u(x)$ and $n\in\mathbb{N}$, then

$$\|d_xf^{-n}v\|\leqslant c\lambda^n\|v\|.$$

The spaces $E^s(x)$ and $E^u(x)$ are called, respectively, the stable *and* unstable spaces *at x.*

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_4

Given $E \subseteq \mathbb{R}^p$ and $v \in \mathbb{R}^p$, let

$$d(v,E) = \min\{\|v-w\| : w \in E\}.$$

The distance between two subspaces $E,F \subseteq \mathbb{R}^p$ is defined by

$$d(E,F) = \max\left\{\max_{v\in E,\|v\|=1} d(v,F), \max_{w\in F,\|w\|=1} d(w,E)\right\}.$$

Theorem 4.2 *If $\Lambda \subsetneq \mathbb{R}^p$ is a hyperbolic set for a diffeomorphism $f\colon \mathbb{R}^p \to \mathbb{R}^p$, then the spaces $E^s(x)$ and $E^u(x)$ vary continuously with $x \in \Lambda$, that is, if $x_m \to x$ when $m \to \infty$, with $x_m, x \in \Lambda$ for each $m \in \mathbb{N}$, then*

$$d(E^s(x_m), E^s(x)) \to 0 \quad \textit{when } m \to \infty$$

and

$$d(E^u(x_m), E^u(x)) \to 0 \quad \textit{when } m \to \infty.$$

The following definition introduces the Smale horseshoe.

Definition 4.3 *Consider the horizontal strips*

$$H_1 = [0,1]\times[0,a] \quad \textit{and} \quad H_2 = [0,1]\times[1-a,1],$$

and the vertical strips

$$V_1 = [0,a]\times[0,1] \quad \textit{and} \quad V_2 = [1-a,1]\times[0,1],$$

for some constant $a \in (0,1/2)$. We assume that f is a C^1 diffeomorphism on an open neighborhood of the square $Q = [0,1]^2$ such that

$$f(H_1) = V_1 \quad \textit{and} \quad f(H_2) = V_2$$

(see Figure I.4.1), with the restriction of f to $H_1 \cup H_2$ given by

$$f(x,y) = \begin{cases} (ax, y/a) & \textit{if } (x,y) \in H_1, \\ (-ax+1, -y/a+1/a) & \textit{if } (x,y) \in H_2. \end{cases}$$

The set

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(Q) = \bigcap_{n\in\mathbb{Z}} f^n(H_1 \cup H_2)$$

is called Smale horseshoe.

We also consider the characterization of a hyperbolic set in terms of invariant families of cones.

Definition 4.4 *Let $f\colon M \to M$ be a C^1 diffeomorphism and let $\Lambda \subseteq M$ be a compact f-invariant set. For each $x \in \Lambda$, consider a splitting*

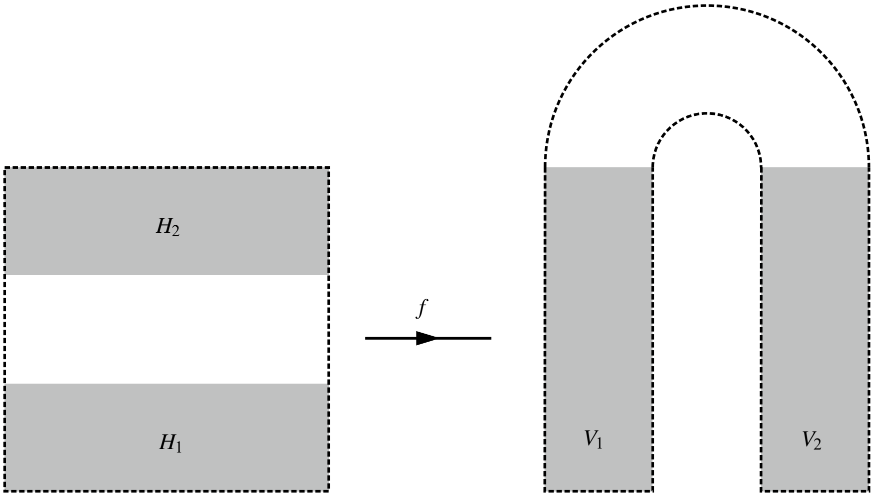

Fig. I.4.1 Horizontal and vertical strips.

$$T_xM = F^s(x) \oplus F^u(x)$$

and an inner product $\langle \cdot, \cdot \rangle' = \langle \cdot, \cdot \rangle'_x$ *on* T_xM *such that the dimensions* $\dim F^s(x)$ *and* $\dim F^u(x)$ *are independent of* x*. We denote by* $\|\cdot\|'$ *the norm induced by the inner product* $\langle \cdot, \cdot \rangle'$*. Given* $\gamma \in (0,1)$ *and* $x \in \Lambda$*, we define* cones *by*

$$C^s(x) = \{(v,w) \in F^s(x) \oplus F^u(x) : \|w\|' < \gamma \|v\|'\} \cup \{0\}$$

and

$$C^u(x) = \{(v,w) \in F^s(x) \oplus F^u(x) : \|v\|' < \gamma \|w\|'\} \cup \{0\}.$$

Theorem 4.5 *Let* $f : M \to M$ *be a* C^1 *diffeomorphism and let* $\Lambda \subseteq M$ *be a compact* f*-invariant set. Then* Λ *is a hyperbolic set for* f *if and only if there exist a splitting* $T_xM = F^s(x) \oplus F^u(x)$ *and an inner product* $\langle \cdot, \cdot \rangle'_x$ *on* T_xM*, for each* $x \in \Lambda$*, and there exist constants* $\mu, \gamma \in (0,1)$ *such that for each* $x \in \Lambda$*:*

1.
$$d_x f \overline{C^u(x)} \subsetneq C^u(f(x)) \quad \text{and} \quad d_x f^{-1} \overline{C^s(x)} \subsetneq C^s(f^{-1}(x));$$

2.
$$\|d_x f v\|' \geqslant \mu^{-1} \|v\|' \quad \text{for } v \in C^u(x);$$

3.
$$\|d_x f^{-1} v\|' \geqslant \mu^{-1} \|v\|' \quad \text{for } v \in C^s(x).$$

Now we formulate two main results of the theory. We first introduce the notion of a local topological conjugacy.

Definition 4.6 *Two maps $f\colon X \to X$ and $g\colon Y \to Y$ on topological spaces are said to be* locally topologically conjugate*, respectively, on open sets $U \subseteq X$ and $V \subseteq Y$ if there exists a homeomorphism $h\colon U \to V$ with $h(U) = V$ such that $h \circ f = g \circ h$ on U.*

The following result provides a local topological conjugacy to the linear part in the neighborhood of a *hyperbolic fixed point*, that is, a fixed point x such that $\{x\}$ is a hyperbolic set.

Theorem 4.7 (Grobman–Hartman) *Let $x \in \mathbb{R}^p$ be a hyperbolic fixed point of a C^1 diffeomorphism $f\colon \mathbb{R}^p \to \mathbb{R}^p$. Then the maps f and $d_x f$ are locally topologically conjugate, respectively, on open neighborhoods U of x and V of 0, that is, there exists a homeomorphism $h\colon U \to V$ with $h(U) = V$ such that*

$$h \circ f = d_x f \circ h \quad \text{on } U$$

(see Figures I.4.2 and I.4.3).

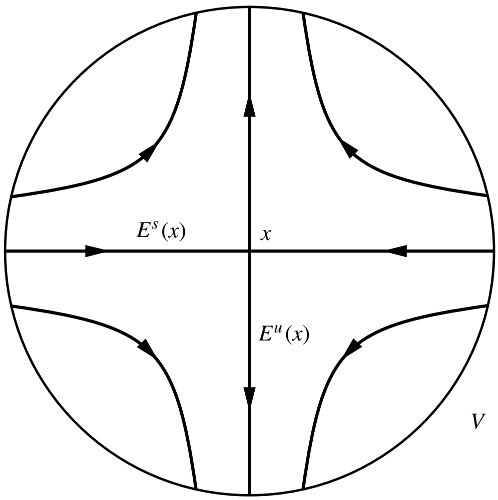

Fig. I.4.2 Behavior of the orbits of $d_x f$.

Let Λ be a hyperbolic set for a C^1 diffeomorphism $f\colon \mathbb{R}^p \to \mathbb{R}^p$. Given $\varepsilon > 0$, for each $x \in \Lambda$ we denote by $B(x,\varepsilon)$ the ball of radius ε centered at x and we consider the sets

$$V^s(x) = \left\{y \in B(x,\varepsilon) : \|f^n(y) - f^n(x)\| < \varepsilon \text{ for } n > 0\right\}$$

and

$$V^u(x) = \left\{y \in B(x,\varepsilon) : \|f^n(y) - f^n(x)\| < \varepsilon \text{ for } n < 0\right\}.$$

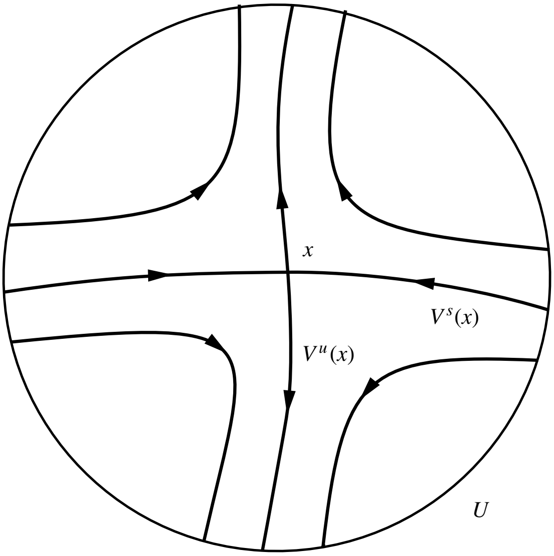

Fig. I.4.3 Behavior of the orbits of f.

Theorem 4.8 (Stable and unstable manifolds) *Let Λ be a hyperbolic set for a C^1 diffeomorphism $f\colon \mathbb{R}^p \to \mathbb{R}^p$. For any sufficiently small $\varepsilon > 0$, the following properties hold:*

1. *for each $x \in \Lambda$ the sets $V^s(x)$ and $V^u(x)$ are manifolds of class C^1 satisfying*

$$T_x V^s(x) = E^s(x) \quad \text{and} \quad T_x V^u(x) = E^u(x);$$

2. *there exist $\rho \in (0,1)$ and $C > 0$ such that*

$$\|f^n(y) - f^n(x)\| \leqslant C\rho^n \|y - x\| \quad \text{for } y \in V^s(x)$$

and

$$\|f^{-n}(y) - f^{-n}(x)\| \leqslant C\rho^n \|y - x\| \quad \text{for } y \in V^u(x),$$

for all $x \in \Lambda$ and $n \in \mathbb{N}$.

We also consider the notion of a hyperbolic set for a flow.

Definition 4.9 *Let Φ be a C^1 flow on a manifold M. A compact Φ-invariant set $\Lambda \subseteq M$ is called a* hyperbolic set *for Φ if there exist $\lambda \in (0,1)$, $c > 0$ and a splitting*

$$T_x M = E^s(x) \oplus E^0(x) \oplus E^u(x)$$

for each $x \in \Lambda$ such that:

1. *$E^0(x)$ is the 1-dimensional space generated by the vector $\frac{d}{dt}\varphi_t(x)|_{t=0}$;*

2. for each $t \in \mathbb{R}$ we have

$$d_x\varphi_t E^s(x) = E^s(\varphi_t(x)) \quad \text{and} \quad d_x\varphi_t E^u(x) = E^u(\varphi_t(x));$$

3. if $v \in E^s(x)$ and $t > 0$, then

$$\|d_x\varphi_t v\| \leqslant c\lambda^t \|v\|;$$

4. if $v \in E^u(x)$ and $t > 0$, then

$$\|d_x\varphi_{-t} v\| \leqslant c\lambda^t \|v\|.$$

The spaces $E^s(x)$ and $E^u(x)$ are called, respectively, the stable *and* unstable spaces *at x.*

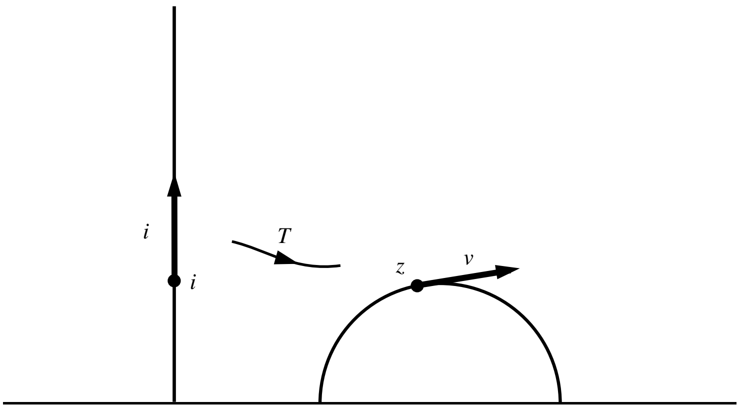

Fig. I.4.4 The Möbius transformation T.

Finally, we consider the upper half plane

$$\mathbb{H} = \left\{z \in \mathbb{C} : \operatorname{Im} z > 0\right\}$$

and its unit tangent bundle

$$S\mathbb{H} = \left\{(z,v) \in \mathbb{H} \times \mathbb{C} : |v|_z = 1\right\},$$

with the norm $|v|_z$ given by $|v|_z = |v|/\operatorname{Im} z$. Given $(s,v) \in S\mathbb{H}$, one can show that there exists a unique Möbius transformation

$$T(z) = \frac{az+b}{cz+d} \quad \text{with } ad-bc = 1$$

such that

$$T(i) = z \quad \text{and} \quad T'(i)i = v$$

(see Figure I.4.4). More precisely, let $x, y \in \mathbb{R} \cup \{\infty\}$ be, respectively, the limits of $\gamma(t) = T(ie^t)$ when $t \to -\infty$ and when $t \to +\infty$. Then:

1. for $x, y \in \mathbb{R}$ with $x < y$, we have

$$T(w) = \frac{\alpha y w + x}{\alpha w + 1}, \quad \text{with } \alpha = \left|\frac{z-x}{z-y}\right|;$$

2. for $x, y \in \mathbb{R}$ with $x > y$, we have

$$T(w) = \frac{yw - \alpha x}{w - \alpha}, \quad \text{with } \alpha = \left|\frac{z-y}{z-x}\right|;$$

3. for $x \in \mathbb{R}$ and $y = \infty$, we have

$$T(w) = \alpha w + x, \quad \text{with } \alpha = \operatorname{Im} z;$$

4. for $x = \infty$ and $y \in \mathbb{R}$, we have

$$T(w) = -\frac{\alpha}{w} + y, \quad \text{with } \alpha = \operatorname{Im} z.$$

Definition 4.10 *Let $\gamma(t) = T(ie^t)$ with T as above. The* geodesic flow $\varphi_t \colon S\mathbb{H} \to S\mathbb{H}$ *on the unit tangent bundle of the upper half plane is defined by*

$$\varphi_t(z, v) = (\gamma(t), \gamma'(t)).$$

Problems

Problem 4.1 Let Λ be a hyperbolic set with finitely many points. Show that Λ is composed of periodic points.

Problem 4.2 Show that the second and third conditions in Definition 4.1 can be replaced, respectively, by

$$\|d_x f^{-n} v\| \geqslant \frac{1}{c}\lambda^n \|v\| \quad \text{for } x \in \Lambda,\ v \in E^s(x),\ n \in \mathbb{N}$$

and

$$\|d_x f^n v\| \geqslant \frac{1}{c}\lambda^{-n} \|v\| \quad \text{for } x \in \Lambda,\ v \in E^u(x),\ n \in \mathbb{N}.$$

Problem 4.3 Show that if Λ is a hyperbolic set for a diffeomorphism f, then for all sufficiently large $k \in \mathbb{N}$ the set Λ is a hyperbolic set for f^k with constant $c = 1$.

Problem 4.4 Let Λ be a hyperbolic set for a diffeomorphism $f \colon \mathbb{R}^p \to \mathbb{R}^p$. Show that $\Lambda \times \Lambda$ is a hyperbolic set for the diffeomorphism $g \colon \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^p$ defined by

$$g(x, y) = (f(x), f^{-1}(y)).$$

Problem 4.5 Let A be a $p \times p$ matrix with real entries and without eigenvalues of modulus 1. Show that there exist a splitting $\mathbb{R}^p = E^s \oplus E^u$ and constants $\lambda \in (0,1)$ and $c > 0$ such that

$$\|A^n v\| \leqslant c\lambda^n \|v\| \quad \text{for } v \in E^s,\ n \in \mathbb{N}$$

and

$$\|A^{-n} v\| \leqslant c\lambda^n \|v\| \quad \text{for } v \in E^u,\ n \in \mathbb{N}.$$

Problem 4.6 For the Smale horseshoe Λ (see Definition 4.3), show that for any two-sided sequence $\omega = (\cdots i_{-1} i_0 i_1 \cdots)$ in $\{1,2\}^{\mathbb{Z}}$ the set

$$\Lambda_\omega = \bigcap_{n \in \mathbb{Z}} f^{-n} V_{i_n}$$

contains exactly one point.

Problem 4.7 For the Smale horseshoe Λ, show that the map $H \colon \{1,2\}^{\mathbb{Z}} \to \Lambda$ defined by

$$H(\cdots i_{-1} i_0 i_1 \cdots) = \bigcap_{n \in \mathbb{Z}} f^{-n} V_{i_n}$$

is bijective (by Problem 4.6 the map is well defined).

Problem 4.8 For the Smale horseshoe Λ, show that for each $m \in \mathbb{N}$ the number of m-periodic points of the map $f|_\Lambda$ is at least 2^m.

Problem 4.9 Consider horizontal strips

$$\bar{H}_i = \left\{(x,y) \in [0,1]^2 : \varphi_i(x) \leqslant y \leqslant \psi_i(x)\right\}$$

and vertical strips

$$\bar{V}_i = \left\{(x,y) \in [0,1]^2 : \bar{\varphi}_i(y) \leqslant x \leqslant \bar{\psi}_i(y)\right\},$$

for some functions $\varphi_i, \psi_i, \bar{\varphi}_i, \bar{\psi}_i \colon [0,1] \to [0,1]$, for $i = 1,2$, such that

$$\varphi_1 < \psi_1 < \varphi_2 < \psi_2$$

and

$$\bar{\varphi}_1 < \bar{\psi}_1 < \bar{\varphi}_2 < \bar{\psi}_2$$

(see Figure I.4.5). Moreover, let f be a C^1 diffeomorphism on an open neighborhood of the square $[0,1]^2$ such that $f(\bar{H}_i) = \bar{V}_i$ for $i = 1,2$ and

$$f(x,y) = (g(x), h(y)) \quad \text{for } (x,y) \in \bar{H}_1 \cup \bar{H}_2$$

and some C^1 functions g,h defined on compact subsets of $\mathbb{R}$. Show that if $\sup|g'| < 1$ and $\inf|h'| > 1$, then

$$\bar{\Lambda} = \bigcap_{n\in\mathbb{Z}} f^n(\bar{H}_1 \cup \bar{H}_2)$$

is a hyperbolic set for f.

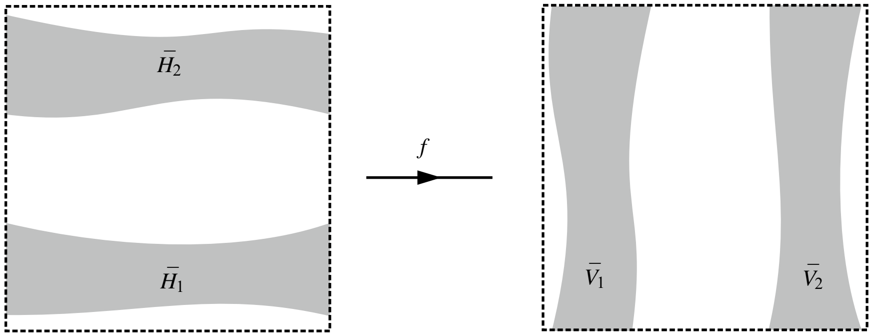

Fig. I.4.5 Horizontal and vertical strips in Problem 4.9.

Problem 4.10 Show that if a compact connected manifold M is a hyperbolic set for a diffeomorphism $f\colon M \to M$, then all stable spaces have the same dimension.

Problem 4.11 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1 map. Show that if there exist $\lambda \in (0,1)$ and $c > 0$ such that $|(f^n)'(x)| \leqslant c\lambda^n$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \frac{1}{n} \log |f^n(I)| < 0$$

for any compact interval $I \subsetneq \mathbb{R}$.

Problem 4.12 Show that there exists no diffeomorphism $f\colon S^2 \to S^2$ for which the whole S^2 is a hyperbolic set with $\dim E^u(x) = \{0\}$ for all $x \in S^2$.

Problem 4.13 Given functions $f, g\colon \mathbb{R} \to \mathbb{R}$, let

$$d(f,g) = \sup\bigl\{|f(x) - g(x)| : x \in \mathbb{R}\bigr\}.$$

Determine whether the set X of all bounded functions $f\colon \mathbb{R} \to \mathbb{R}$ with the property that $f(0) + f(1) = 0$ is a complete metric space with the distance d.

Problem 4.14 Show that the set of all C^1 functions $f\colon [a,b] \to \mathbb{R}$ is not a complete metric space with the distance

$$d(f,g) = \sup\bigl\{|f(x) - g(x)| : x \in [a,b]\bigr\}.$$

Problem 4.15 Construct a sequence of bounded C^1 functions $f_n\colon \mathbb{R} \to \mathbb{R}$ converging uniformly to a nondifferentiable function.

Problem 4.16 Let Λ be a compact set that is invariant under a C^1 diffeomorphism f and assume that there exist a splitting $T_xM = F^s(x) \oplus F^u(x)$ for each $x \in \Lambda$, an inner product $\langle \cdot, \cdot \rangle'_x$ on T_xM varying continuously with $x \in \Lambda$ and a constant $\gamma \in (0,1)$ such that

$$\|d_xfv\|' > \|v\|' \quad \text{for } x \in \Lambda, v \in \overline{C^u(x)} \setminus \{0\}.$$

Show that there exists a constant $\mu \in (0,1)$ such that

$$\|d_xfv\|' \geqslant \mu^{-1}\|v\|' \quad \text{for } x \in \Lambda, v \in \overline{C^u(x)}.$$

Problem 4.17 Consider the set

$$A = S^1 \times \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\right\}$$

and the map $f\colon A \to S^1 \times \mathbb{R}^2$ defined by

$$f([\theta],x,y) = \left([2\theta], \frac{1}{5}x + \frac{1}{2}\cos(2\pi\theta), \frac{1}{5}y + \frac{1}{2}\sin(2\pi\theta)\right).$$

Show that $f(A) \subsetneq A$ and that $f|_A$ is one-to-one.

Problem 4.18 For the map $f|_A$ in Problem 4.17, show that $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(A)$ is a hyperbolic set for f.

Problem 4.19 Show that if $\mathbb{T}^n$ is a hyperbolic set for an automorphism of the torus $T_A\colon \mathbb{T}^n \to \mathbb{T}^n$, then T_A has positive topological entropy.

Problem 4.20 For the maps $f, g\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax$ and $g(x) = bx$, with $a, b > 1$, show that there exists a homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ such that $h \circ f = g \circ h$.

Problem 4.21 Construct a topological conjugacy between the maps $f, g\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = (2x, 3y) \quad \text{and} \quad g(x,y) = (5x, 4y)$$

for $(x,y) \in \mathbb{R}^2$.

Problem 4.22 Determine the 2-periodic points of the automorphism of the torus T_A for the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Problem 4.23 Determine the 2-periodic points of the endomorphism of the torus T_B for the matrix

$$B = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

Problem 4.24 Show that the vector $X(x) = \frac{d}{dt}\varphi_t(x)\big|_{t=0}$ in Definition 4.9 is neither contracted nor expanded by $d_x\varphi_t$.

Problem 4.25 Let Λ be a hyperbolic set for a flow Φ on $\mathbb{R}^p$. Given a sequence $x_m \to x$ when $m \to \infty$, with $x_m, x \in \Lambda$, for each $m \in \mathbb{N}$, show that any sublimit of a sequence $v_m \in E^s(x_m) \subseteq \mathbb{R}^p$ with $\|v_m\| = 1$ is in $E^s(x)$.

Problem 4.26 For a sequence x_m as in Problem 4.25, show that there exists $m \in \mathbb{N}$ such that

$$\dim E^s(x_p) = \dim E^s(x_q) \quad \text{and} \quad \dim E^u(x_p) = \dim E^s(x_q)$$

for any $p, q > m$.

Problem 4.27 Find the stable and unstable invariant manifolds at the origin for the differential equation

$$\begin{cases} x' = x, \\ y' = -y + x^2. \end{cases}$$

Problem 4.28 Find the stable and unstable invariant manifolds at the origin for the differential equation

$$\begin{cases} x' = x(a - y), \\ y' = -y + x^2 - 2y^2, \end{cases}$$

for each given $a > 0$.

Problem 4.29 Given $a, b > 0$, show that the map $h \colon \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x^{b/a} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -|x|^{b/a} & \text{if } x < 0. \end{cases}$$

is a topological conjugacy between the flows determined by the differential equations

$$x' = ax \quad \text{and} \quad x' = bx.$$

In other words, if φ_t and ψ_t are the flows determined by the equations, then

$$h \circ \varphi_t = \psi_t \circ h \quad \text{for } t \in \mathbb{R}.$$

Problem 4.30 Let A and B be $n \times n$ matrices. Show that if there exists a topological conjugacy between the flows determined by the differential equations

$$v' = Av \quad \text{and} \quad v' = Bv$$

that is a diffeomorphism, then there exists also a topological conjugacy between the flows that is a linear map.

Problem 4.31 Let A and B be $n \times n$ matrices. Show that if there exists a topological conjugacy between the flows determined by the differential equations in Problem 4.30 that is a linear map, then the matrices A and B are conjugate, that is, there exists an invertible $n \times n$ matrix C such that $A = C^{-1}BC$.

Problem 4.32 Let φ_t and ψ_t be the flows determined, respectively, by the linear equations with matrices

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Verify that the map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(0,0) = 0$ and

$$h(x,y) = \sqrt{\frac{x^2+y^2}{x^2+xy+3y^2/2}} \left(x - \frac{y}{2} \log \frac{x^2+y^2}{2}, y \right)$$

for $(x,y) \neq (0,0)$ satisfies

$$h \circ \varphi_t = \psi_t \circ h.$$

Problem 4.33 For the differential equation

$$\begin{cases} x' = -x, \\ y' = y + x^2, \end{cases}$$

show that the map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$h(x,y) = \left(x, y + \frac{x^2}{3} \right)$$

is a topological conjugacy between the flows determined by the equation and its linearization at the origin.

Problem 4.34 For the differential equation

$$\begin{cases} x' = x, \\ y' = -y + x^n, \end{cases}$$

with $n \geqslant 2$, show that the map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$h(x,y) = \left(x, y - \frac{x^n}{n+1} \right)$$

is a topological conjugacy between the flows determined by the equation and its linearization at the origin.

Problem 4.35 Consider the Möbius transformation

$$T_A(z) = \frac{az+b}{cz+d}$$

associated with the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with } \det A = ad - bc = 1,$$

and let

$$T_{1,c}(z) = T_{\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)}(z) = z + c \quad \text{for } c \in \mathbb{R},$$

$$T_{2,c}(z) = T_{\left(\begin{smallmatrix} c & 0 \\ 0 & 1/c \end{smallmatrix}\right)}(z) = c^2 z \quad \text{for } c \in \mathbb{R}\setminus\{0\},$$

$$T_{3,c}(z) = T_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & c \end{smallmatrix}\right)}(z) = -\frac{1}{z+c} \quad \text{for } c \in \mathbb{R}.$$

Show that

$$T_A = \begin{cases} T_{1,ab} \circ T_{2,a} & \text{if } c = 0, \\ T_{1,a/c} \circ T_{2,1/c} \circ T_{3,d/c} & \text{if } c \neq 0. \end{cases}$$

Problem 4.36 Show that any straight line or circle is determined by an equation of the form

$$Az\bar{z} + Bz + \overline{B}\bar{z} + C = 0 \quad \text{with } A, C \in \mathbb{R},\ B \in \mathbb{C},$$

respectively, with $A = 0$ and with $A \neq 0$ and $AC < B\overline{B}$.

Problem 4.37 Show that the map $T(z) = 2z$ takes straight lines into straight lines and circles into circles.

Problem 4.38 Show that the map $T(z) = -1/(z+1)$ takes straight lines and circles into straight lines or circles.

Problem 4.39 Show that the components of the geodesic flow $\varphi_t \colon S\mathbb{H} \to S\mathbb{H}$ given by

$$\varphi_t(z, v) = (\gamma(t), \gamma'(t))$$

(see Definition 4.10), satisfy

$$(\operatorname{Re}\gamma'(t))' = \frac{2\operatorname{Re}\gamma'(t)\operatorname{Im}\gamma'(t)}{\operatorname{Im}\gamma(t)}$$

and

$$(\operatorname{Im}\gamma(t))' = \frac{(\operatorname{Im}\gamma'(t))^2 - (\operatorname{Re}\gamma'(t))^2}{\operatorname{Im}\gamma(t)}.$$

Problem 4.40 For the geodesic flow $\varphi_t \colon S\mathbb{H} \to S\mathbb{H}$, show that the quantities

$$H_1 = \frac{|\gamma'(t)|}{\operatorname{Im}\gamma(t)} \quad \text{and} \quad H_2 = \frac{\operatorname{Re}\gamma'(t)}{(\operatorname{Im}\gamma(t))^2}$$

are independent of t.

Chapter I.5
Symbolic Dynamics

In this chapter we consider various topics of symbolic dynamics, with emphasis on their relation to hyperbolic dynamics. This includes expanding maps, the Smale horseshoe, and its modifications. We also consider topological Markov chains and their periodic points, irreducible and transitive matrices, topological transitivity, and topological mixing as well as the notion of topological entropy. We refer the reader to [16, 33, 34] for additional topics.

Notions and Results

We first recall a few basic notions.

Definition 5.1 *Given an integer $k > 1$, consider the set $\Sigma_k^+ = \{1,\ldots,k\}^{\mathbb{N}}$ of one-sided sequences*

$$\omega = (i_1(\omega)i_2(\omega)\cdots)$$

with $i_n(\omega) \in \{1,\ldots,k\}$ for each $n \in \mathbb{N}$. The one-sided shift map $\sigma\colon \Sigma_k^+ \to \Sigma_k^+$ *is defined by*

$$\sigma(\omega) = (i_2(\omega)i_3(\omega)\cdots).$$

Given $\beta > 1$, for each $\omega, \omega' \in \Sigma_k^+$ let $n = n(\omega, \omega') \in \mathbb{N}$ be the smallest positive integer such that $i_n(\omega) \neq i_n(\omega')$ and define

$$d_\beta(\omega, \omega') = \begin{cases} \beta^{-n} & \text{if } \omega \neq \omega', \\ 0 & \text{if } \omega = \omega'. \end{cases}$$

One can show that the function d_β is a distance on Σ_k^+.

Proposition 5.2 *We have $h(\sigma|\Sigma_k^+) = \log k$ with respect to any distance d_β.*

We also consider the particular class of topological Markov chains.

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_5

Definition 5.3 *Given an integer $k > 1$, let $A = (a_{ij})$ be a $k \times k$ matrix with entries a_{ij} in $\{0,1\}$ and consider the set*

$$\Sigma_A^+ = \left\{\omega \in \Sigma_k^+ : a_{i_n(\omega)i_{n+1}(\omega)} = 1 \text{ for } n \in \mathbb{N}\right\}.$$

The restriction $\sigma|\Sigma_A^+ \colon \Sigma_A^+ \to \Sigma_A^+$ of $\sigma|\Sigma_k^+$ to Σ_A^+ is called the one-sided topological Markov chain *with* transition matrix A.

We denote by $\rho(A)$ the spectral radius of a square matrix A.

Theorem 5.4 *We have $h(\sigma|\Sigma_A^+) = \log\rho(A)$ with respect to any distance d_β.*

Now we consider corresponding two-sided versions of the former notions.

Definition 5.5 *Given an integer $k > 1$, consider the set $\Sigma_k = \{1,\dots,k\}^{\mathbb{Z}}$ of two-sided sequences*

$$\omega = (\cdots i_{-1}(\omega)i_0(\omega)i_1(\omega)\cdots).$$

The two-sided shift map $\sigma\colon \Sigma_k \to \Sigma_k$ *is defined by*

$$\sigma(\omega) = \omega', \quad \text{with } i_n(\omega') = i_{n+1}(\omega) \text{ for } n \in \mathbb{Z}.$$

Given $\beta > 1$, for each $\omega, \omega' \in \Sigma_k$ let $n = n(\omega,\omega') \in \mathbb{N}$ be the smallest integer such that

$$i_n(\omega) \neq i_n(\omega') \quad \text{or} \quad i_{-n}(\omega) \neq i_{-n}(\omega'),$$

and define

$$d_\beta(\omega,\omega') = \begin{cases} \beta^{-n} & \text{if } \omega \neq \omega', \\ 0 & \text{if } \omega = \omega'. \end{cases}$$

One can show that the function d_β is a distance on Σ_k.

Definition 5.6 *Given an integer $k > 1$, let $A = (a_{ij})$ be a $k \times k$ matrix with entries a_{ij} in $\{0,1\}$ and consider the set*

$$\Sigma_A = \left\{\omega \in \Sigma_k : a_{i_n(\omega)i_{n+1}(\omega)} = 1 \text{ for } n \in \mathbb{Z}\right\}.$$

The restriction $\sigma|\Sigma_A\colon \Sigma_A \to \Sigma_A$ of $\sigma|\Sigma_k$ to Σ_A is called the two-sided topological Markov chain *with* transition matrix A.

Finally, we recall a few additional notions and results, here formulated for simplicity only for the one-sided case.

Definition 5.7 *A $k \times k$ matrix A is called:*

1. irreducible *if for each $i, j \in \{1,\dots,k\}$ there exists $m = m(i,j) \in \mathbb{N}$ such that the entry (i,j) of A^m is positive;*
2. transitive *if there exists $m \in \mathbb{N}$ such that all entries of the matrix A^m are positive.*

Proposition 5.8 *If the matrix A is irreducible (respectively, transitive), then the topological Markov chain $\sigma|\Sigma_A^+$ is topologically transitive (respectively, topologically mixing).*

Proposition 5.9 *If the matrix A is transitive, then $h(\sigma|\Sigma_A^+) > 0$.*

We also recall the notion of the zeta function of a dynamical system with discrete time.

Definition 5.10 *Given a map $f\colon X \to X$ with*

$$a_n := \operatorname{card}\{x \in X : f^n(x) = x\} < +\infty$$

for each $n \in \mathbb{N}$, its zeta function *is defined by*

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{a_n z^n}{n}$$

for each $z \in \mathbb{C}$ such that the series converges.

Problems

Problem 5.1 Show that for each $\omega \in \Sigma_k^+$ and $r > 0$ the open ball

$$B_{d_\beta}(\omega, r) = \{\omega' \in \Sigma_k^+ : d_\beta(\omega', \omega) < r\}$$

is a cylinder, that is, there exist integers $i_1, \dots, i_n \in \{1, \dots, k\}$ with $n = n(r)$ such that

$$B_{d_\beta}(\omega, r) = C_{i_1 \cdots i_n} := \{\omega \in \Sigma_k^+ : i_j(\omega) = i_j \text{ for } j = 1, \dots, n\}.$$

Problem 5.2 Show that any cylinder $C_{i_1 \cdots i_n}$ (see Problem 5.1) is simultaneously open and closed.

Problem 5.3 Given $\beta, \beta' > 1$, show that the distances d_β and $d_{\beta'}$ generate the same topology on Σ_k^+.

Problem 5.4 Show that one can define a distance on Σ_k^+ by

$$d(\omega, \omega') = \sum_{n=1}^{\infty} \frac{1}{2^n} |i_n(\omega) - i_n(\omega')|$$

and that it generates the same topology as the distance d_β.

Problem 5.5 Show that one can define a distance on Σ_k^+ by

$$\bar{d}(\omega, \omega') = \begin{cases} 1/n(\omega, \omega') & \text{if } \omega \neq \omega', \\ 0 & \text{if } \omega = \omega' \end{cases}$$

and that it generates the same topology as the distance d_β.

Problem 5.6 Show that the distances d and $\bar{d}$ in Problems 5.4 and 5.5 are not equivalent.

Problem 5.7 Show that $(\Sigma_k^+, \bar{d})$ is a compact metric space.

Problem 5.8 Show that $(\Sigma_k^+, \bar{d})$ is a complete metric space.

Problem 5.9 Show that the shift map $\sigma|\Sigma_k$ is topologically transitive.

Problem 5.10 For a topological Markov chain $\sigma|\Sigma_A^+$ or $\sigma|\Sigma_A$ with transition matrix $A = (a_{ij})$, show that:

1. the number of blocks $(i_1 \cdots i_n)$ of length n with $a_{i_p i_{p+1}} = 1$ for $p = 1, \dots, n-1$ such that $i_1 = i$ and $i_n = j$ is equal to the (i, j) entry of the matrix A^n;
2. the number of n-periodic points is $\operatorname{tr}(A^n)$.

Problem 5.11 Given a $k \times k$ matrix A with entries a_{ij} in $\{0,1\}$, show that the following properties are equivalent:

1. there exists $n \in \mathbb{N}$ such that $a_{i_1 i_2} \cdots a_{i_{n-1} i_n} = 0$ for all $i_1, \dots, i_n \in \{1, \dots, k\}$;
2. $A^n = 0$ for some $n \in \mathbb{N}$;
3. $\Sigma_A = \varnothing$.

Problem 5.12 For a topological Markov chain $\sigma|\Sigma_A^+$, compute the number of periodic orbits with period p prime in terms of the transition matrix A.

Problem 5.13 For a topological Markov chain $\sigma|\Sigma_A^+$, compute the number of periodic orbits with period $p = p_1 p_2$, with p_1, p_2 prime, in terms of the transition matrix A.

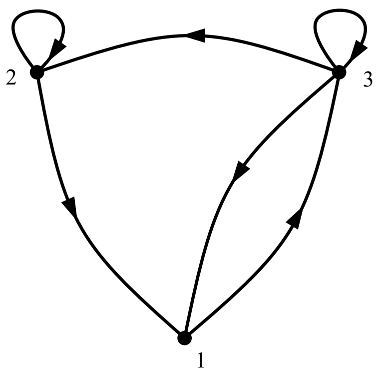

Fig. I.5.1 Graph associated to the matrix A in Problem 5.14.

Problem 5.14 Determine whether the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

is irreducible or transitive (see Figure I.5.1 for its graph).

Problem 5.15 Show that the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

is not transitive (see Figure I.5.2 for its graph).

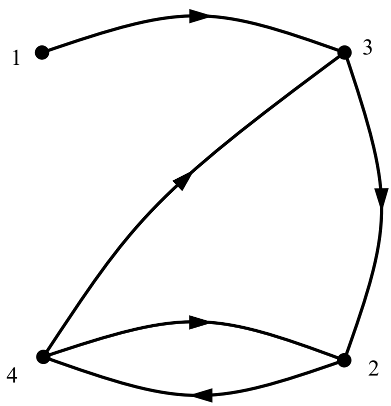

Fig. I.5.2 Graph associated to the matrix A in Problem 5.15.

Problem 5.16 Show that the topological Markov chain $\sigma|\Sigma_A^+$ with transition matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

has positive topological entropy.

Problem 5.17 Consider the topological Markov chains $\sigma|\Sigma_A^+$ and $\sigma|\Sigma_B^+$ with transition matrices, respectively,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

1. Compute $h(\sigma|\Sigma_A^+)$ and $h(\sigma|\Sigma_B^+)$.
2. Find the fixed points of $\sigma|\Sigma_A^+$ and $\sigma|\Sigma_B^+$.

Problem 5.18 Find topological Markov chains $\sigma|\Sigma_A^+$ and $\sigma|\Sigma_B^+$ both with zero topological entropy that are not topologically conjugate.

Problem 5.19 Find topological Markov chains $\sigma|\Sigma_A^+$ and $\sigma|\Sigma_B^+$ with

$$h(\sigma|\Sigma_A^+) = h(\sigma|\Sigma_B^+) \neq 0$$

that are not topologically conjugate.

Problem 5.20 For the set $\bar{\Lambda}$ in Problem 4.9, show that the correspondence

$$\omega = (\cdots i_{-1} i_0 i_1 \cdots) \mapsto \bar{\Lambda}_\omega = \bigcap_{n\in\mathbb{Z}} f^{-n} \bar{V}_{i_n}$$

defines a bijective map $H\colon \{1,2\}^{\mathbb{Z}} \to \bar{\Lambda}$.

Problem 5.21 Show that the bijection $H\colon \Sigma_2 \to \bar{\Lambda}$ in Problem 5.20 is a topological conjugacy between the shift map σ and the map f in Problem 4.9.

Problem 5.22 Show that the set

$$\{\omega \in \Sigma_k : \omega \text{ is } q\text{-periodic for some even } q \in \mathbb{N}\}$$

is dense in Σ_2.

Problem 5.23 Compute the zeta function of the shift map $\sigma|\Sigma_k$.

Problem 5.24 Compute the zeta function of the topological Markov chain $\sigma|\Sigma_A^+$ with transition matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Problem 5.25 Compute the zeta function of the automorphism of the torus $\mathbb{T}^2$ induced by the matrix

$$A = \begin{pmatrix} 5 & 1 \\ 9 & 2 \end{pmatrix}.$$

Problem 5.26 Given a nonempty finite set $\mathcal{F}$ in $\bigcup_{n\geqslant 2}\{1,\dots,k\}^n$ (that is, a set of blocks with finite length), consider the set $\mathcal{F}' \subsetneq \Sigma_k$ of all sequences in Σ_k containing no blocks in $\mathcal{F}$. Determine two sets $\mathcal{F}_1$ and $\mathcal{F}_2$ with $k=2$ such that $\mathcal{F}_1' = \mathcal{F}_2' \neq \varnothing$ with $\mathcal{F}_1 \neq \mathcal{F}_2$.

Problem 5.27 Show that $\mathcal{F}_1' \cap \mathcal{F}_2' = (\mathcal{F}_1 \cup \mathcal{F}_2)'$.

Problem 5.28 Given a nonempty finite set of blocks $\mathcal{F}$ in $\bigcup_{n\geqslant 2}\{1,\dots,k\}^n$, show that $\mathcal{F}' \subsetneq \Sigma_k$ is compact and σ-invariant.

Problem 5.29 Given a nonempty finite set of blocks $\mathcal{F}$ in $\bigcup_{n\geqslant 2}\{1,\ldots,k\}^n$, show that if $\mathcal{F}$ contains only blocks of length 2, then $\mathcal{F}' = \Sigma_A$ for some $k\times k$ matrix A with entries in $\{0,1\}$.

Problem 5.30 Show that for $k=2$ and $\mathcal{F}=\{(22)\}$, the matrix A in Problem 5.29 is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 5.31 Compute the number of n-periodic points of the restriction $\sigma|\mathcal{F}'$ of $\sigma\colon \Sigma_2 \to \Sigma_2$ to $\mathcal{F}'$ for the set $\mathcal{F}=\{(22)\}$.

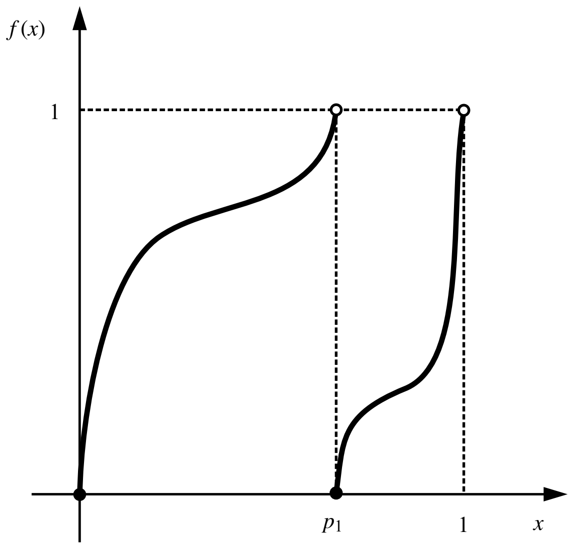

Fig. I.5.3 An expanding map $f\colon S^1 \to S^1$.

Problem 5.32 An orientation-preserving C^1 map $f\colon S^1 \to S^1$ is called an *expanding map* if

$$\lambda := \min_{x\in S^1} f'(x) > 1$$

(see Figure I.5.3 for an example). Let $i_0\cdots i_{n-1}$ be the base-2 representation of j and let $p_1,\ldots,p_{2^n}=p_0$ be the preimages of the unique fixed point p of f under f^n (see Figure I.5.4). Show that if $f\colon S^1\to S^1$ is an expanding map with $\deg f = 2$ (see Problem 1.2.7), then the intervals

$$I_{i_0\ldots i_{n-1}} = [p_j, p_{j+1}], \quad \text{for } j=0,\ldots,2^n-1,$$

satisfy:

1. $|I_{i_0 \cdots i_{n-1}}| \leqslant \lambda^{-n}$;
2. $I_{i_0 \ldots i_{n-1} i_n} \subsetneq I_{i_0 \ldots i_{n-1}}$; and
3. $f^m(I_{i_0 \cdots i_n}) = I_{i_m \cdots i_n}$ for each $m \leqslant n$.

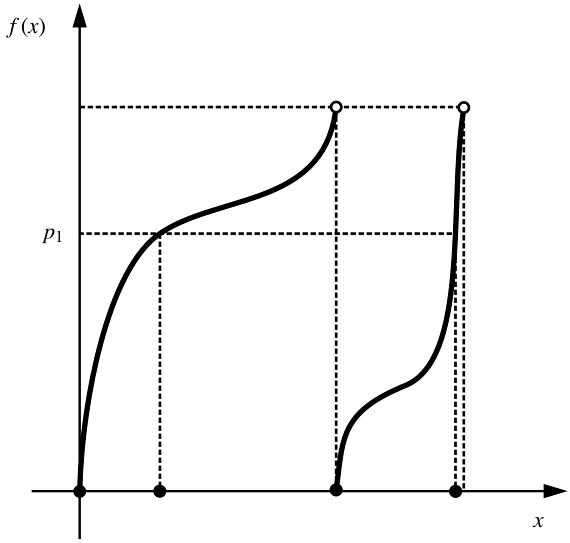

Fig. I.5.4 Preimages of the point p under f^2.

Problem 5.33 Let $f\colon S^1 \to S^1$ be an expanding map (see Problem 5.32) with $\deg f = 2$. Show that the set of all preimages $\bigcup_{n\in\mathbb{N}} f^{-n} p$ is dense in S^1.

Problem 5.34 Let $f\colon S^1 \to S^1$ be an expanding map with $\deg f = 2$. Show that there exists a continuous onto map $H\colon \Sigma_2^+ \to S^1$ such that $H \circ \sigma = f \circ H$ on Σ_2^+.

Problem 5.35 Let $f\colon S^1 \to S^1$ be an expanding map with $\deg f = k \geqslant 2$. Show that there exists a continuous onto map $H\colon \Sigma_k^+ \to S^1$ such that $H \circ \sigma = f \circ H$ on Σ_k^+.

Problem 5.36 Show that the map H in Problem 5.35 is not a homeomorphism.

Problem 5.37 Show that the periodic points of any expanding map $f\colon S^1 \to S^1$ are dense in S^1.

Problem 5.38 Show that any expanding map $f\colon S^1 \to S^1$ is topologically transitive.

Problem 5.39 Determine whether any expanding map $f\colon S^1 \to S^1$ is topologically mixing.

Problem 5.40 For the map f in Problem 5.35, show that $h(f) \leqslant \log k$.

Chapter I.6
Ergodic Theory

In this chapter we consider some basic topics of ergodic theory. This includes the notion of an invariant measure, Poincaré's recurrence theorem and Birkhoff's ergodic theorem. We also consider briefly the notion of metric entropy of an invariant probability measure. The prerequisites from measure theory and integration theory are briefly recalled in the initial section. We refer the reader to [8, 23, 35, 46, 56] for additional topics.

Notions and Results

We start by recalling some basic notions of measure theory. Let X be a set.

Definition 6.1 *A set $\mathcal{A}$ of subsets of X is called a σ-*algebra *on X if:*

1. $\varnothing, X \in \mathcal{A}$;
2. $X \setminus B \in \mathcal{A}$ *whenever* $B \in \mathcal{A}$;
3. $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$ *whenever* $B_n \in \mathcal{A}$ *for all* $n \in \mathbb{N}$.

Given a set $\mathcal{A}$ of subsets of X, the *σ-algebra generated by* $\mathcal{A}$ is the smallest σ-algebra on X that contains all elements of $\mathcal{A}$. In particular, the *Borel σ-algebra on* $\mathbb{R}$, denoted by $\mathcal{B}$, is the σ-algebra generated by the open intervals of $\mathbb{R}$. We shall always assume that $\mathbb{R}$ is equipped with the Borel σ-algebra $\mathcal{B}$. More generally, the *Borel σ-algebra on* $\mathbb{R}^n$, denoted by $\mathcal{B}_n$, is the σ-algebra generated by the open rectangles $\prod_{i=1}^{n}(a_i, b_i)$, with $a_i < b_i$ for $i = 1, \ldots, n$. Clearly, $\mathcal{B}_1 = \mathcal{B}$.

Definition 6.2 *Given a σ-algebra $\mathcal{A}$ on X, a function $\mu \colon \mathcal{A} \to [0, +\infty]$ is called a* measure on X (with respect to $\mathcal{A}$) *if $\mu(\varnothing) = 0$ and*

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$$

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_6

for any pairwise disjoint sets $B_n \in \mathcal{A}$, for $n \in \mathbb{N}$. Then the triple $(X, \mathcal{A}, \mu)$ is called a measure space.

One can show that there exists a unique measure $\lambda \colon \mathcal{B}_n \to [0, +\infty]$ on $\mathbb{R}^n$ such that

$$\lambda\left(\prod_{i=1}^{n}(a_i, b_i)\right) = \prod_{i=1}^{n}(b_i - a_i)$$

for all $a_i < b_i$ and $i = 1, \ldots, n$. It is called the *Lebesgue measure* on $\mathbb{R}^n$.

We assume from now on in this section that we are given some measure space $(X, \mathcal{A}, \mu)$.

Definition 6.3 *A function $\varphi \colon X \to \mathbb{R}$ is said to be $\mathcal{A}$-*measurable *if $\varphi^{-1}B \in \mathcal{A}$ for every $B \in \mathcal{B}$ (where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$).*

To introduce the notion of integral of a measurable function, we first consider the class of simple functions. The *characteristic function* $\chi_B \colon X \to \{0, 1\}$ of a set $B \subseteq X$ is defined by

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases}$$

Definition 6.4 *Given sets $B_1, \ldots, B_n \in \mathcal{A}$ and numbers $a_1, \ldots, a_n \in \mathbb{R}$, the function*

$$s = \sum_{k=1}^{n} a_k \chi_{B_k}$$

is called a simple function.

Note that any simple function is measurable. Now we introduce the notion of integral of a nonnegative measurable function.

Definition 6.5 *The Lebesgue* integral *of a measurable function $\varphi \colon X \to \mathbb{R}_0^+$ is defined by*

$$\int_X \varphi \, d\mu = \sup\left\{ \sum_{k=1}^{n} a_k \mu(B_k) : \sum_{k=1}^{n} a_k \chi_{B_k} \leqslant \varphi \right\}.$$

The integral of an arbitrary measurable function can now be introduced as follows. Given a function $\varphi \colon X \to \mathbb{R}$, let

$$\varphi^+ = \max\{\varphi, 0\} \quad \text{and} \quad \varphi^- = \max\{-\varphi, 0\}.$$

Definition 6.6 *An $\mathcal{A}$-measurable function $\varphi \colon X \to \mathbb{R}$ is said to be μ-*integrable *if*

$$\int_X \varphi^+ \, d\mu < \infty \quad \textit{and} \quad \int_X \varphi^- \, d\mu < \infty.$$

The Lebesgue integral *of a μ-integrable function $\varphi \colon X \to \mathbb{R}$ is defined by*

$$\int_X \varphi \, d\mu = \int_X \varphi^+ \, d\mu - \int_X \varphi^- \, d\mu.$$

We also consider the class of measurable maps.

Definition 6.7 *A map $f\colon X \to X$ is said to be $\mathcal{A}$-*measurable *if $f^{-1}B \in \mathcal{A}$ for every $B \in \mathcal{A}$.*

Definition 6.8 *Given an $\mathcal{A}$-measurable map $f\colon X \to X$, a measure μ on X is said to be f-*invariant *if*

$$\mu(f^{-1}B) = \mu(B) \quad \text{for all } B \in \mathcal{A}.$$

Then we also say that f preserves μ.

Now we formulate two basic but also fundamental results of ergodic theory.

Theorem 6.9 (Poincaré's recurrence theorem) *Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map preserving a finite measure μ on X. For each set $A \in \mathcal{A}$, we have*

$$\mu\left(\left\{x \in A : f^n(x) \in A \text{ for infinitely many integers } n \in \mathbb{N}\right\}\right) = \mu(A).$$

The statement can be rephrased by saying that almost every point in A (that is, every point in $A \setminus B$ for some set B of measure zero) returns infinitely often to A.

The second result is instead of quantitative nature.

Theorem 6.10 (Birkhoff's ergodic theorem) *Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map preserving a probability measure μ on X. Given a μ-integrable function $\varphi\colon X \to \mathbb{R}$, the limit*

$$\varphi_f(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x))$$

exists for almost every $x \in X$, the function φ_f is μ-integrable and

$$\int_X \varphi_f \, d\mu = \int_X \varphi \, d\mu.$$

Finally, we introduce the notion of metric entropy of a measurable map with respect to an invariant probability measure.

Definition 6.11 *A finite set $\xi \subseteq \mathcal{A}$ is called a* partition *of X (with respect to μ) if:*

1. *$\mu\left(\bigcup_{C\in\xi} C\right) = 1$;*
2. *$\mu(C \cap D) = 0$ for any $C, D \in \xi$ with $C \neq D$.*

Given a partition ξ of X, we define

$$H_\mu(\xi) = -\sum_{C\in\xi} \mu(C) \log \mu(C),$$

with the convention that $0 \log 0 = 0$. Moreover, for each $n \in \mathbb{N}$ we consider the new partition ξ_n formed by the sets

$$C_1 \cap f^{-1}C_2 \cap \cdots \cap f^{-(n-1)}C_n,$$

with $C_1, \ldots, C_n \in \xi$.

Definition 6.12 *Given an $\mathcal{A}$-measurable map $f\colon X \to X$ preserving a probability measure μ on X, for each partition ξ of X let*

$$h_\mu(f,\xi) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu(\xi_n).$$

Definition 6.13 *Given an $\mathcal{A}$-measurable map $f\colon X \to X$ preserving a probability measure μ on X, the* metric entropy *of f with respect to μ is defined by*

$$h_\mu(f) = \sup_{n\in\mathbb{N}} h_\mu(f,\xi^{(n)})$$

for any sequence $\xi^{(n)}$ of partitions of X such that:

1. *given $C \in \xi^{(n)}$, there exist $C_1,\dots,C_m \in \xi^{(n+1)}$ satisfying*

$$\mu\left(C\backslash\bigcup_{i=1}^m C_i\right) = \mu\left(\bigcup_{i=1}^m C_i\backslash C\right) = 0;$$

2. *the union $\bigcup_{n\in\mathbb{N}} \xi^{(n)}$ generates the σ-algebra $\mathcal{A}$.*

One can show that the supremum giving $h_\mu(f)$ does not depend on each specific sequence of partitions $\xi^{(n)}$.

Problems

Problem 6.1 For a measure space $(X,\mathcal{A},\mu)$, show that:

1. if $A,B \in \mathcal{A}$ and $A \subseteq B$, then $\mu(A) \leqslant \mu(B)$;
2. if $B_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, then

$$\mu\left(\bigcup_{n=1}^\infty B_n\right) \leqslant \sum_{n=1}^\infty \mu(B_n).$$

Problem 6.2 For a measure space $(X,\mathcal{A},\mu)$, show that:

1. if $A,B \in \mathcal{B}$, then

$$\mu(A) + \mu(B) = \mu\big(A \cup B\big) + \mu\big(A \cap B\big);$$

2. if $A,B \in \mathcal{B}$ and

$$\mu\big((A\backslash B) \cup (B\backslash A)\big) = 0,$$

then $\mu(A) = \mu(B)$.

Problem 6.3 Show that the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ is generated by the set of all intervals:

1. $[a,b]$ with $a<b$;
2. $(a,b]$ with $a<b$.

Problem 6.4 Show that the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ is generated by the set of all intervals:

1. $(a,+\infty)$ with $a\in\mathbb{R}$;
2. $(-\infty,b)$ with $b\in\mathbb{R}$;
3. $[a,+\infty)$ with $a\in\mathbb{R}$;
4. $(-\infty,b]$ with $b\in\mathbb{R}$.

Problem 6.5 Show that the Borel σ-algebra on $\mathbb{R}$ coincides with the σ-algebra generated by the open sets in $\mathbb{R}$.

Problem 6.6 Given a set $\mathcal{C}$ generating the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$, show that a function $\varphi\colon X\to\mathbb{R}$ is $\mathcal{A}$-measurable if and only if $\varphi^{-1}B\in\mathcal{A}$ for every $B\in\mathcal{C}$.

Problem 6.7 Show that the square of an $\mathcal{A}$-measurable function $\varphi\colon X\to\mathbb{R}$ is also an $\mathcal{A}$-measurable function.

Problem 6.8 Show that any increasing function $\varphi\colon\mathbb{R}\to\mathbb{R}$ is $\mathcal{B}$-measurable.

Problem 6.9 Given $\mathcal{A}$-measurable functions $\varphi_n\colon X\to\mathbb{R}$, for $n\in\mathbb{N}$, show that $\sup_{n\in\mathbb{N}}\varphi_n$ and $\inf_{n\in\mathbb{N}}\varphi_n$ (here assumed to be finite everywhere) are also $\mathcal{A}$-measurable functions.

Problem 6.10 Given $\mathcal{A}$-measurable functions $\varphi_n\colon X\to\mathbb{R}$, for $n\in\mathbb{N}$, show that

$$\limsup_{n\to\infty}\varphi_n \quad\text{and}\quad \liminf_{n\to\infty}\varphi_n$$

(here assumed to be finite everywhere) are also $\mathcal{A}$-measurable functions.

Problem 6.11 For a measure space $(X,\mathcal{A},\mu)$ with $\mu(X)<+\infty$, let $\varphi_n\colon X\to\mathbb{R}$, for $n\in\mathbb{N}$, and $\varphi\colon X\to\mathbb{R}$ be $\mathcal{A}$-measurable functions such that

$$\varphi_n(x)\to\varphi(x) \quad\text{when } n\to\infty$$

for all $x\in X$. Show that given $\varepsilon>0$, there exists a set $B\in\mathcal{A}$ with $\mu(B)<\varepsilon$ such that

$$\varphi_n\to\varphi \quad\text{when } n\to\infty$$

uniformly on $X\setminus B$.

Problem 6.12 Let $\varphi\colon X\to\mathbb{R}$ be an $\mathcal{A}$-measurable function and let μ be a measure on X. Show that for each $t,c>0$ we have

$$\mu\big(\{x\in X:|\varphi(x)|\geqslant t\}\big)\leqslant\frac{1}{t^c}\int_X|\varphi|^c\,d\mu.$$

Problem 6.13 Show that the measure μ on S^1 obtained from the Lebesgue measure λ on $[0,1]$ is invariant under the expanding map $E_m\colon S^1 \to S^1$.

Problem 6.14 Given finitely many or countably many intervals $(a_i, b_i) \subsetneq [0,1]$, for $i \in I$, with $\sum_{i\in I}(b_i - a_i) = 1$, consider a map $f\colon [0,1] \to [0,1]$ such that

$$f(x) = \frac{x - a_i}{b_i - a_i}$$

for each $x \in (a_i, b_i)$ and $i \in I$ (see Figure I.6.1 for an example). Show that f preserves the Lebesgue measure λ on $[0,1]$.

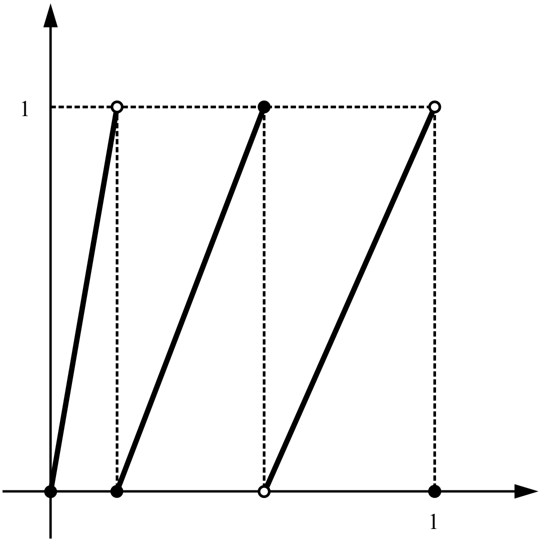

Fig. I.6.1 A map f as in Problem 6.14.

Problem 6.15 Show that the map $f\colon [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} x/(1-x) & \text{if } x \in [0,1/2], \\ (1-x)/x & \text{if } x \in [1/2,1] \end{cases}$$

(see Figure I.6.2) preserves the measure μ on $[0,1]$ defined by

$$\mu(A) = \int_A \frac{dx}{x}.$$

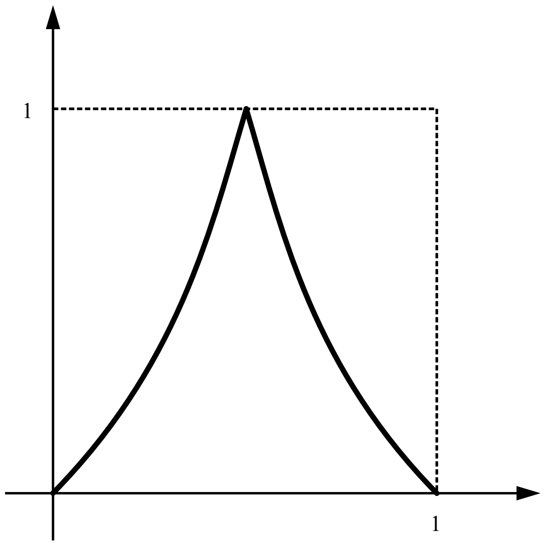

Fig. I.6.2 The map f in Problem 6.15.

Problem 6.16 Consider the Gauss map $f\colon [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} \{1/x\} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

(see Figure I.6.3). Show that f preserves the measure μ on $[0,1]$ defined by

$$\mu(A) = \int_A \frac{dx}{1+x}.$$

Problem 6.17 Show that the map $g\colon [0,1] \to [0,1]$ given by $g(x) = 4x(1-x)$ preserves the measure μ on $[0,1]$ defined by

$$\mu(A) = \int_A \frac{dx}{\sqrt{x(1-x)}}.$$

Problem 6.18 Given $\alpha \in \mathbb{R}$, show that the map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$(x,y) \mapsto (x+\alpha, x+y)$$

preserves the measure m on $\mathbb{T}^2$ obtained from the Lebesgue measure on $[0,1]^2$.

Problem 6.19 Let f be the Gauss map in Problem 6.16. Show that if

$$x = [a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

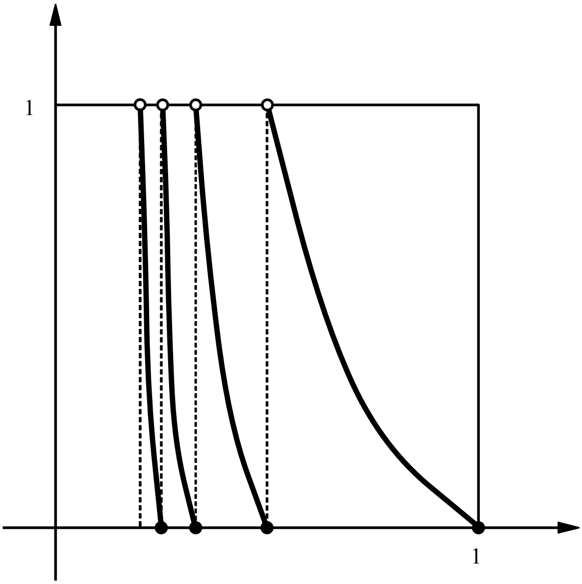

Fig. I.6.3 The Gauss map.

is the continued fraction of a number $x \in (0,1)$, then:

1. $f(x) = [a_2, a_3, \ldots]$;
2. letting
$$p_{-1} = 0, \quad p_0 = 1, \quad p_m = a_m p_{m-1} + p_{m-2}$$
and
$$q_{-1} = 1, \quad q_0 = 0, \quad q_m = a_m q_{m-1} + q_{m-2}$$
for $m \in \mathbb{N}$, we have
$$p_m q_{m-1} - p_{m-1} q_m = (-1)^{m-1} \quad \text{for } m \geqslant 0;$$
3. for $m \in \mathbb{N}$ we have
$$[a_1, a_2, \ldots, a_m] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_m}}}} = \frac{p_m}{q_m}.$$

Problem 6.20 Let $P = (p_{ij})$ be a $k \times k$ matrix with entries $p_{ij} \geqslant 0$ for $i, j = 1, \ldots, k$ such that
$$\sum_{j=1}^{k} p_{ij} = 1 \quad \text{for } j = 1, \ldots, k.$$

Moreover, take $p_1, \ldots, p_k \in (0,1)$ such that

$$\sum_{i=1}^{k} p_i = 1 \quad \text{and} \quad \sum_{i=1}^{k} p_i p_{ij} = p_j$$

for $j = 1, \ldots, k$. We define a measure μ on the σ-algebra on Σ_k^+ generated by the cylinders (see Problem 5.1) by

$$\mu(C_{i_1 \cdots i_n}) = p_{i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$$

for each $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$. Show that:

1. $\mu(C_{i_1 \cdots i_n}) = \sum_{j=1}^{k} \mu(C_{i_1 \cdots i_n j})$;
2. μ is σ-invariant;
3. $\mu(\Sigma_k^+) = 1$.

Problem 6.21 For the measure μ in Problem 6.20, consider the $k \times k$ matrix A with entries

$$a_{ij} = \begin{cases} 1 & \text{if } p_{ij} > 0, \\ 0 & \text{if } p_{ij} = 0. \end{cases}$$

Show that $\operatorname{supp} \mu = \Sigma_A^+$.

Problem 6.22 Let $f \colon X \to X$ be an $\mathcal{A}$-measurable map preserving a finite measure μ on X. Given a set $A \in \mathcal{A}$ with $\mu(A) > 0$, show that there exist integers $m, n \geqslant 0$ with $m \neq n$ such that

$$f^{-m} A \cap f^{-n} A \neq \varnothing.$$

Problem 6.23 Let $f \colon X \to X$ be an $\mathcal{A}$-measurable map preserving a finite measure μ on X. Given a set $A \in \mathcal{A}$ with $\mu(A) > 0$, show that the map $\tau_A \colon X \to \mathbb{N}$ given by

$$\tau_A(x) = \min\{n \in \mathbb{N} : f^n(x) \in A\}$$

is well defined for almost every $x \in A$.

Problem 6.24 Show that a function $\varphi \colon X \to \mathbb{R}$ satisfies $\varphi \circ f = \varphi$ for some map $f \colon X \to X$ if and only if the set $\varphi^{-1}\alpha$ is f-invariant for every $\alpha \in \mathbb{R}$.

Problem 6.25 Given $\alpha \in \mathbb{Q}$, consider the rotation $R_\alpha \colon S^1 \to S^1$. Let $U = [a, b]$ be an interval in S^1 with $0 \leqslant a < b \leqslant 1$ and define

$$T_n(x) = \operatorname{card}\bigl\{1 \leqslant k \leqslant n : R_\alpha^k(x) \in U\bigr\}.$$

Moreover, let μ be the measure on S^1 obtained from the Lebesgue measure on $[0, 1]$. Show that:

1. if $\varphi(x) = e^{2\pi i m x}$ with $m \in \mathbb{Z}$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(R_\alpha^k(x)) = \int_{S^1} \varphi \, d\mu \quad \text{for all } x \in S^1;$$

2. the former property holds for all continuous functions $\varphi\colon S^1 \to \mathbb{R}$;
3. for each $x \in S^1$ we have

$$\lim_{n\to\infty} \frac{T_n(x)}{n} = b - a.$$

Problem 6.26 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map. A probability measure μ on X is said to be *ergodic (with respect to f)* if any f-invariant set has either measure 0 or measure 1. Show that if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu\left(f^{-k}A \cap B\right) = \mu(A)\mu(B)$$

for any sets $A, B \in \mathcal{A}$, then the measure μ is ergodic.

Problem 6.27 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map. Show that an f-invariant measure μ on X is ergodic (see Problem 6.26) if and only if

$$\mu\left(\bigcup_{n=1}^{\infty} f^{-n}A\right) = 1$$

for any set $A \in \mathcal{A}$ with $\mu(A) > 0$.

Problem 6.28 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map preserving an ergodic probability measure μ on X. Given a set $A \in \mathcal{A}$ with $\mu(A) > 0$, show that

$$\int_A \tau_A \, d\mu = 1$$

with the function τ_A as in Problem 6.23.

Problem 6.29 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map. A probability measure μ on X is said to be *mixing (with respect to f)* if

$$\lim_{n\to\infty} \mu\left(f^{-n}A \cap B\right) = \mu(A)\mu(B)$$

for any sets $A, B \in \mathcal{B}$ and is said to be *weakly mixing (with respect to f)* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left|\mu\left(f^{-i}A \cap B\right) - \mu(A)\mu(B)\right| = 0$$

for any sets $A, B \in \mathcal{B}$. Show that if μ is mixing, then it is weakly mixing.

Problem 6.30 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map. Show that if a probability measure μ on X is weakly mixing (see Problem 6.29), then it is ergodic.

Problem 6.31 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map preserving a mixing probability measure μ on X (see Problem 6.29). Show that f is topologically mixing on the support $\operatorname{supp}\mu$ of μ.

Problem 6.32 Let μ be a probability measure on X. Show that if ξ is a partition of X, then

$$H_\mu(\xi) \leqslant \log \operatorname{card} \xi.$$

Problem 6.33 Let μ be a probability measure on X and let ξ and η be partitions of X. Show that if η is a *refinement* of ξ (that is, if for each $D \in \eta$ there exists $C \in \xi$ such that $\mu(D \backslash C) = 0$), then

$$H_\mu(\xi) \leqslant H_\mu(\eta).$$

Problem 6.34 Let μ be a probability measure on X. Show that if ξ and η are partitions of X and

$$\xi \vee \eta = \{C \cap D : C \in \xi, D \in \eta\},$$

then

$$H_\mu(\xi \vee \eta) \leqslant H_\mu(\xi) + H_\mu(\eta).$$

Problem 6.35 Let $f \colon X \to X$ be an $\mathcal{A}$-measurable map preserving a probability measure μ on X. Show that if ξ is a partition of X and

$$f^{-1}\xi = \{f^{-1}C : C \in \xi\},$$

then

$$H_\mu(f^{-1}\xi) = H_\mu(\xi).$$

Problem 6.36 Let $f \colon X \to X$ be an $\mathcal{A}$-measurable map preserving a probability measure μ on X. Show that for each partition ξ of X we have

$$h_\mu(f, \xi) = \inf_{n \in \mathbb{N}} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} f^{-i} \xi \right) = \lim_{n \to \infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} f^{-i} \xi \right).$$

Problem 6.37 Let μ be a probability measure on X and let ξ, ζ and η be partitions of X. Show that if ζ is a refinement of η (see Problem 6.33) and

$$H_\mu(\xi | \zeta) = - \sum_{C \in \xi, D \in \zeta} \mu(C \cap D) \log \frac{\mu(C \cap D)}{\mu(D)},$$

then

$$H_\mu(\xi | \zeta) \leqslant H_\mu(\xi | \eta).$$

Problem 6.38 Let μ be a probability measure on X. Show that if ξ, ζ and η are partitions of X, then

$$H_\mu(\xi \vee \zeta | \eta) = H_\mu(\xi | \zeta \vee \eta) + H_\mu(\zeta | \eta).$$

Problem 6.39 For the measure μ in Problem 6.20, compute $h_\mu(\sigma)$.

Problem 6.40 For the measure μ in Problem 6.20 with $p_{ij} = p_j$ for all $i, j = 1, \dots, k$, compute $h_\mu(\sigma)$.

Part II
Problems and Solutions

Chapter II.1
Basic Theory

Problem 1.1 Determine all values of $a \in \mathbb{R}$ for which the map $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^4 - x$ has nonzero fixed points.

Solution The fixed points of f are the solutions of the equation $f(x) = x$, that is,

$$ax^4 - x = x \quad \Longleftrightarrow \quad x(ax^3 - 2) = 0.$$

The (real) solutions are:

1. $x = 0$ when $a = 0$;
2. $x = 0$ and $x = (2/a)^{1/3}$ when $a \neq 0$.

Hence, the map f has nonzero fixed points if and only if $a \in \mathbb{R}\setminus\{0\}$.

Problem 1.2 Determine all the periodic points of the map $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$.

Solution First observe that since e^x takes only positive values, there are no periodic points in $\mathbb{R}_0^-$. Now consider the function

$$g(x) = e^x - x.$$

We have $g'(x) = e^x - 1 > 0$ for $x > 0$ and so

$$e^x - x = g(x) > g(0) > 0 \quad \text{for } x > 0.$$

This shows that f has no fixed points in $\mathbb{R}^+$. Finally, one can show by induction that

$$f^n(x) > f^{n-1}(x) > \cdots > x$$

for $x > 0$, which implies that f has no periodic points in $\mathbb{R}^+$. Summing up, f has no periodic points in $\mathbb{R}$.

Problem 1.3 Show that if a continuous map $f: \mathbb{R} \to \mathbb{R}$ has a periodic point with period 2, then it has at least one fixed point.

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_7

Solution Take p such that $f^2(p) = p$. We have $q = f(p) \neq p$. Now let

$$g(x) = f(x) - x.$$

Then

$$g(p) = f(p) - p = q - p$$

and

$$g(q) = f(q) - q = p - q.$$

Since $p \neq q$, the numbers $g(p)$ and $g(q)$ have different signs. Hence, by the intermediate value theorem there exists c in the interval determined by p and q with $g(c) = 0$, which thus is a fixed point of f.

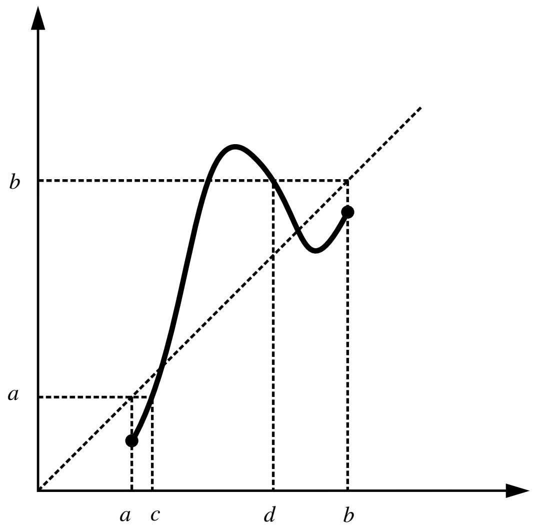

Fig. II.1.1 A map f satisfying $[a,b] \subseteq f([a,b])$.

Problem 1.4 Show that a continuous map $f\colon [a,b] \to \mathbb{R}$ with $f([a,b]) \supseteq [a,b]$ has at least one fixed point.

Solution Since f is continuous and $f([a,b]) \supseteq [a,b]$, there exist $c,d \in [a,b]$ such that

$$f(c) = a \leqslant c \quad \text{and} \quad f(d) = b \geqslant d$$

(see Figure II.1.1). Since

$$f(c) - c \leqslant 0 \quad \text{and} \quad f(d) - d \geqslant 0,$$

it follows from the continuity of f that there exists $\alpha \in [a,b]$ such that $f(\alpha) - \alpha = 0$, which thus is a fixed point of f.

Problem 1.5 Show that a continuous map $f\colon [a,b] \to [a,b]$ has at least one fixed point.

Solution Note that

$$f(a) \geqslant a \quad \text{and} \quad f(b) \leqslant b$$

(see Figure II.1.2). Proceeding as in Problem 1.4, it follows from the continuity of f that there exists $\alpha \in [a,b]$ such that $f(\alpha) - \alpha = 0$, which thus is a fixed point of f.

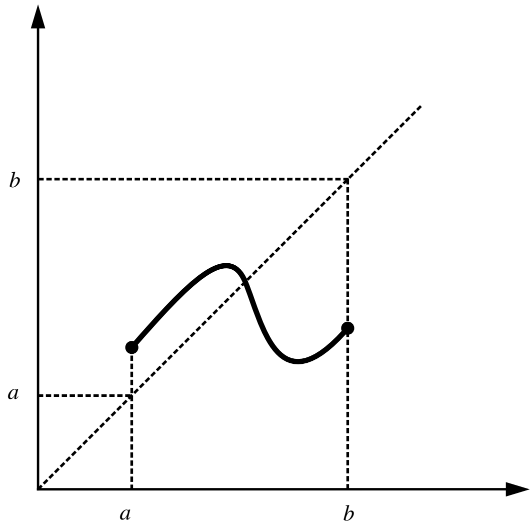

Fig. II.1.2 A map f satisfying $[a,b] \supseteq f([a,b])$.

Problem 1.6 Consider the continuous map $f\colon [1,5] \to [1,5]$ with

$$f(1) = 3, \quad f(2) = 5, \quad f(3) = 4, \quad f(4) = 2 \quad \text{and} \quad f(5) = 1$$

such that f is linear on $[n, n+1]$ for $n = 1,2,3,4$ (see Figure I.1.2). Show that:

1. f has periodic points with period 5;
2. f^3 has no fixed points in $[1,3] \cup [4,5]$, but has a fixed point in $[3,4]$;
3. f has no periodic points with period 3.

Solution 1. One can easily verify that

$$\{1,3,4,2,5\} \tag{II.1.1}$$

is a periodic orbit with period 5.

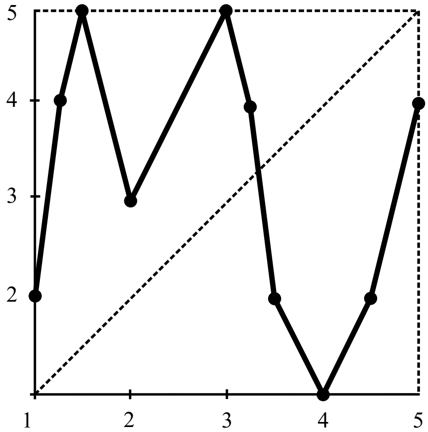

Fig. II.1.3 The map f^3.

2. Note that 1, 2, 3, 4 and 5 are not 3-periodic points, since they belong to the periodic orbit in (II.1.1). Moreover,

$$f^3([1,2]) = [2,3], \quad f^3([2,3]) = [3,5] \quad \text{and} \quad f^3([4,5]) = [1,4].$$

Hence, f^3 has no fixed points in the intervals $[1,2]$, $[2,3]$ or $[4,5]$. On the other hand, since

$$f^3([3,4]) = [1,5] \supsetneq [3,4],$$

it follows from Problem 1.4 that f^3 has a fixed point $\bar{x}$ in $[3,4]$. Alternatively, one could look at the graph of the map f^3 (see Figure II.1.3).

3. We need to verify that the point $\bar{x}$ in item 2 is a fixed point of f and not a periodic point with period 3. Observe that $f(\bar{x}) \in [2,4]$. If $f(\bar{x}) \in [2,3]$, then

$$f^2(\bar{x}) \in [4,5] \quad \text{and} \quad f^3(\bar{x}) \in [1,2],$$

which is impossible because $f^3(\bar{x}) = \bar{x} \in [3,4]$. Thus, we must have $f(\bar{x}) \in [3,4]$. Moreover, $f^2(\bar{x}) \in [3,4]$. Again, if $f^2(\bar{x}) \in [2,3]$, then $f^3(\bar{x}) \in [4,5]$, which is impossible. Therefore, the orbit of $\bar{x}$ is contained in $[3,4]$. But on this interval we have $f(x) = 10 - 2x$ and so $10/3$ is the only fixed point. Finally,

$$f^3(x) = 30 - 8x \quad \text{on } [3,4]$$

and so $10/3$ is also the only fixed point of f^3. This shows that f has no periodic points with period 3.

Problem 1.7 Let $f\colon I \to I$ be a strictly increasing map on an interval $I \subseteq \mathbb{R}$. Show that any periodic point of f is a fixed point.

Solution We proceed by contradiction. Assume that $x \in I$ is a periodic point of f that is not a fixed point. Then

$$x > f(x) \quad \text{or} \quad x < f(x).$$

We consider only the first case since the second case is completely analogous. So assume that $x > f(x)$. Since f is strictly increasing, proceeding inductively we obtain

$$x > f(x) > f^2(x) > \cdots > f^n(x) > \cdots$$

and so $x \neq f^n(x)$ for all $n \in \mathbb{N}$. But this is impossible because x was assumed to be a periodic point. This contradiction establishes the desired property.

Problem 1.8 Let $f\colon I \to I$ be a strictly decreasing map on an interval $I \subseteq \mathbb{R}$. Show that any periodic point of f is either a fixed point or a periodic point with period 2.

Solution Since f is decreasing, the map $g = f^2$ is increasing. Let x be a periodic point of f with even period. Then x is also a periodic point of g. By Problem 1.7, it is in fact a fixed point of g and so, all periodic points of f with even period have period 2.

Now let x be a periodic point of f with odd period greater than 2. Then x is also a periodic point of g, with the same odd period. However, by Problem 1.7, this is impossible and so all periodic points of f with odd period are fixed points. This establishes the desired property.

Problem 1.9 Show that if x is a periodic point of a map f with period $2n$, then x is a periodic point of f^2 with period n.

Solution Note that x has period $2n$ with respect to f if and only if

$$f^{2n}(x) = x \quad \text{and} \quad f^i(x) \neq x \text{ for } i = 1, \ldots, 2n-1.$$

Similarly, x has period n with respect to f^2 if and only if

$$f^{2n}(x) = x \quad \text{and} \quad f^{2i}(x) \neq x \text{ for } i = 1, \ldots, n-1.$$

Since

$$\left\{f^i(x) : i = 1, \ldots, 2n-1\right\} \supsetneq \left\{f^{2i}(x) : i = 1, \ldots, n-1\right\},$$

this yields the desired result.

Problem 1.10 Show that if x is a periodic point of f^2 with period n even, then x is a periodic point of f with period $2n$.

Solution Assume that x is a periodic point of f^2 with period n even. Then $f^i(x) \neq x$ for all even i with $1 < i < 2n$. We claim that $f^i(x) \neq x$ for all odd i dividing $2n$ (these are the possible periods of x with respect to f). Note that any such i satisfies $i < n$. If $f^i(x) = x$, then $f^{2i}(x) = x$. But $2i$ is even and less than $2n$, which gives a contradiction. Therefore, $f^i(x) \neq x$ for all odd i dividing $2n$ and so x has period $2n$ with respect to f.

Problem 1.11 Show that if x is a periodic point of f^2 with period n odd, then x is a periodic point of f with period n or $2n$.

Solution Assume that x is a periodic point of f^2 with period n odd. Then $f^i(x) \neq x$ for all even i with $1 < i < 2n$. To show that x has period n or $2n$ with respect to f, it suffices to verify that $f^i(x) \neq x$ for all odd $i \neq n$ dividing $2n$. Note that any such i satisfies $i < n$. This allows one to proceed as in Problem 1.10 to reach a contradiction. Hence, x has period n or $2n$ with respect to f.

Problem 1.12 Given positive integers m and n, show that if x is a periodic point of f with period m, then it is a periodic point of f^n with period $m/(m,n)$ (recall that (m,n) denotes the greatest common divisor of m and n).

Solution Let p be the period of x with respect to f^n. Then m divides np since

$$x = (f^n)^p(x) = f^{np}(x)$$

and so $m/(m,n)$ divides $np/(m,n)$. Since the integers $m/(m,n)$ and $n/(m,n)$ are coprime, we conclude that $m/(m,n)$ divides p. On the other hand,

$$(f^n)^{m/(m,n)}(x) = (f^m)^{n/(m,n)}(x) = x,$$

which shows that p divides $m/(m,n)$. Therefore, $p = m/(m,n)$.

Problem 1.13 Given positive integers n and k, show that if x is a periodic point of f^n with period k, then it is a periodic point of f with period kn/l for some factor l of n with $(k,l) = 1$.

Solution Since

$$x = (f^n)^k(x) = f^{kn}(x),$$

the period of x with respect to f is kn/l for some positive integer l. By Problem 1.12, we have

$$\frac{kn/l}{(kn/l,n)} = k.$$

Therefore,

$$\frac{n}{l} = (kn/l,n) = \left(\frac{n}{l}k, \frac{n}{l}l\right) = \frac{n}{l}(k,l),$$

which shows that $(k,l) = 1$ and so l divides n (since it divides kn).

Problem 1.14 Let $f\colon I \to I$ be a map on a closed interval $I \subseteq \mathbb{R}$. Show that if the positive semiorbit $\gamma^+(x)$ of a point $x \in I$ is dense in I, then the set of points in I with a dense positive semiorbit is dense in I.

Solution Since $\gamma^+(x)$ is dense in I and the interval has no isolated points, for each $n \in \mathbb{N}$ the set

$$\gamma^+(x) \setminus \{x, f(x), \ldots, f^{n-1}(x)\}$$

is also dense in I. In other words, the positive semiorbit of each element of $\gamma^+(x)$ is dense in I. This shows that the set of points with a dense positive semiorbit is dense in I.

Problem 1.15 Let $f\colon X \to X$ be a continuous one-to-one map on a compact set. Show that if the set P of periodic points of f is dense in X, then f is a homeomorphism.

Solution We first show that f is onto. Clearly,

$$f(X) \supseteq f(P) = P.$$

Since f is continuous and X is compact, the set $f(X)$ is also compact and so

$$f(X) \supseteq \overline{P} = X$$

because P is dense. This shows that f is onto.

To show that f is a homeomorphism it remains to verify that f is open, that is, that the image $f(U)$ of any open set $U \subseteq X$ is also open. Since f is onto and one-to-one, we have

$$f(X \backslash U) = X \backslash f(U)$$

and since f is continuous, the image $X \backslash f(U)$ of the compact set $X \backslash U$ is also compact. Therefore, $f(U)$ is an open set.

Problem 1.16 Let $f\colon X \to X$ be a continuous map on a compact set. Show that if all points of X are periodic points of f, then f is a homeomorphism.

Solution By Problem 1.15, it suffices to show that f is one-to-one. Take $x, y \in X$ with $x \neq y$. If $f(x) = f(y)$, then

$$f^j(x) = f^j(y) \quad \text{for all } j \in \mathbb{N}. \tag{II.1.2}$$

But since x and y are periodic points, denoting by n and m the periods, respectively, of x and y we obtain

$$x = f^{nm}(x) = f^{nm}(y) = y.$$

This contradiction shows that $f(x) \neq f(y)$ and so f is one-to-one.

Problem 1.17 Let $f\colon X \to X$ be a continuous map on a compact set. Show that if the set P of periodic points of f is dense in X and there exists $p \in \mathbb{N}$ such that all periodic points have period at most p, then f is a homeomorphism.

Solution Again by Problem 1.15, it suffices to show that f is one-to-one. Take $x, y \in X$ with $x \neq y$. If $f(x) = f(y)$, then property (II.1.2) holds. Moreover,

$$x = \lim_{n\to\infty} x_n \quad \text{and} \quad y = \lim_{n\to\infty} y_n$$

for some sequences $x_n, y_n \in P$. Since f is continuous, we conclude that

$$0 = f^j(x) - f^j(y) = \lim_{n\to\infty} \bigl(f^j(x_n) - f^j(y_n)\bigr)$$

for all $j \in \mathbb{N}$. Hence, given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$|f^j(x_n) - f^j(y_n)| \leqslant \varepsilon \quad \text{for } n \geqslant k \tag{II.1.3}$$

and $j = 1, \dots, p^2$. Denoting by n_j and m_j, respectively, the periods of x_j and y_j, we have $n_j m_j \leqslant p^2$ and it follows from (II.1.3) that

$$|x_j - y_j| = |f^{n_j m_j}(x_j) - f^{n_j m_j}(y_j)| \leqslant \varepsilon$$

for $j \geqslant k$. This implies that

$$x - y = \lim_{n\to\infty}(x_n - y_n) = 0,$$

which shows that f is one-to-one.

Problem 1.18 Consider an interval $I = (a,b) \subsetneq S^1$ with $0 < a < b < 1$. Show that there exists $n \in \mathbb{N}$ such that $E_2^n(I) = S^1$ (see Definition 1.5), identifying I with the set $\{[x] : x \in I\} \subsetneq S^1$ (see Figures I.1.3 and I.1.4).

Solution Consider the map $F \colon \mathbb{R} \to \mathbb{R}$ defined by $F(x) = 2x$. Note that $F^n(I) = (2^n a, 2^n b)$ is an open interval of length $2^n(b-a)$. Let n be the smallest positive integer such that

$$2^n(b-a) \geqslant 1.$$

Then the projection of $F^n(I)$ on S^1 is exactly S^1 and so $E_2^n(I) = S^1$.

Problem 1.19 Show that given $x \in S^1$ and $\delta > 0$, there exist $y \in (x-\delta, x+\delta) \subseteq S^1$ and $n \in \mathbb{N}$ such that

$$\left|E_2^n(x) - E_2^n(y)\right| \geqslant \frac{1}{4}.$$

Solution Take $x \in S^1$ and $\delta > 0$. By Problem 1.18, there exists $n \in \mathbb{N}$ such that

$$E_2^n\big((x-\delta, x+\delta)\big) = S^1.$$

In particular, there exist $u, v \in (x-\delta, x+\delta)$ such that $E_2^n(u) = 0$ and $E_2^n(v) = 1/2$. Since

$$\begin{aligned} \frac{1}{2} &= \left|E_2^n(u) - E_2^n(v)\right| \\ &\leqslant \left|E_2^n(u) - E_2^n(x)\right| + \left|E_2^n(x) - E_2^n(v)\right|, \end{aligned}$$

we conclude that

$$\left|E_2^n(u) - E_2^n(x)\right| \geqslant \frac{1}{4} \quad \text{or} \quad \left|E_2^n(x) - E_2^n(v)\right| \geqslant \frac{1}{4}.$$

Problem 1.20 Given $x \in S^1$, show that the union $\bigcup_{n=1}^\infty E_2^{-n}x$ is dense in S^1.

Solution Note that the set

$$A = \bigcup_{n=1}^{\infty} E_2^{-n}x$$

is dense in S^1 if and only if each open interval $I \subsetneq S^1$ contains a point $y \in A$. Now consider an interval $I = (a,b) \subsetneq S^1$ with $0 < a < b < 1$. By Problem 1.18, there exists $n \in \mathbb{N}$ such that $E_2^n(I) = S^1$. In particular, there exists $y \in I$ such that $E_2^n(y) = x$ and so $y \in A$. This shows that A is dense in S^1.

Problem 1.21 Consider the map $f\colon R \to R$ defined by $f(z) = z^3$ on the set

$$R = \{z \in \mathbb{C} : |z| = 1\}.$$

Show that the set of periodic points of f is dense in R.

Solution Note that $f^q(z) = z^{3^q}$ for each $q \in \mathbb{N}$. Indeed, proceeding inductively we obtain

$$f^q(z) = f(f^{q-1}(z)) = \left(z^{3^{q-1}}\right)^3 = z^{3^q}.$$

Hence, $z \in R$ is a q-periodic point of f if and only if

$$f^q(z) = z \quad \Longleftrightarrow \quad z^{3^q} = z \quad \Longleftrightarrow \quad z^{3^q-1} = 1.$$

Therefore, a point $z = e^{2\pi i\theta}$, with $\theta \in (0,1]$, is q-periodic if and only if

$$\theta = \frac{p}{3^q - 1} \quad \text{for } p = 1, \dots, 3^q - 1.$$

In particular, given $z \in R$ and $\varepsilon > 0$, by taking q sufficiently large we find that there exist periodic points ε-close to z. This implies that the set of periodic points of f is dense in R.

Problem 1.22 Consider the map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$f(x,y) = (E_m(x), E_m(y)).$$

Show that the set of periodic points of f is dense in $\mathbb{T}^2$.

Solution We have

$$f^q(x,y) = (E_m^q(x), E_m^q(y)) = (E_{m^q}(x), E_{m^q}(y)).$$

Hence, a point $(x,y) \in \mathbb{T}^2$ is q-periodic if and only if

$$m^q x - x = (m^q - 1)x = p_1$$

and

$$m^q y - y = (m^q - 1)y = p_2$$

for some $p_1, p_2 \in \mathbb{Z}$. This implies that the q-periodic points of f are

$$(x,y) = \left(\frac{p_1}{m^q - 1}, \frac{p_2}{m^q - 1}\right) \quad \text{for } p_1, p_2 = 1, 2, \dots, m^q - 1.$$

To verify that the set of periodic points of f is dense in $\mathbb{T}^2$, it suffices to note that $1/(m^q - 1)$ tends to zero when $q \to \infty$.

Problem 1.23 Let $T_A\colon \mathbb{T}^n \to \mathbb{T}^n$ be an automorphism of the torus induced by a matrix A whose spectrum contains no root of 1. Show that the set of periodic points of T_A is $\mathbb{Q}^n/\mathbb{Z}^n$.

Solution Let $[x] \in \mathbb{T}^n$ be a periodic point of T_A. Then there exist $q \in \mathbb{N}$ and $y \in \mathbb{Z}^n$ such that $A^q x = x + y$, that is,

$$(A^q - \mathrm{Id})x = y.$$

Since the spectrum of A contains no root of 1, the matrix $A^q - \mathrm{Id}$ is invertible and so

$$x = (A^q - \mathrm{Id})^{-1} y.$$

Since $A^q - \mathrm{Id}$ has only integer entries, the matrix $(A^q - \mathrm{Id})^{-1}$ has rational entries and so $x \in \mathbb{Q}^n$.

Now take $[x] \in \mathbb{Q}^n/\mathbb{Z}^n$. One can always write x in the form

$$x = \left(\frac{p_1}{r}, \dots, \frac{p_n}{r}\right) \tag{II.1.4}$$

for some integers $p_1, \dots, p_n \in \{0, 1, \dots, r-1\}$. Since A has only integer entries, for each $q \in \mathbb{N}$ we have

$$A^q x = \left(\frac{p_1'}{r}, \dots, \frac{p_n'}{r}\right) + y$$

for some $p_1', \dots, p_n' \in \{0, 1, \dots, r-1\}$ and $y \in \mathbb{Z}^n$. But since the number of points as in (II.1.4) is r^n, there exist $q_1, q_2 \in \mathbb{N}$ with $q_1 \neq q_2$ such that

$$A^{q_1} x - A^{q_2} x \in \mathbb{Z}^n.$$

Assuming, without loss of generality, that $q_1 > q_2$, we obtain

$$A^{q_1 - q_2}(A^{q_2} x) - A^{q_2} x \in \mathbb{Z}^n$$

and so

$$T_A^{q_1 - q_2}[A^{q_2} x] = [A^{q_2} x].$$

In other words, $T_A^{q_2}[x] = [A^{q_2} x]$ is a $(q_1 - q_2)$-periodic point. But since T_A is invertible, $[x]$ is also a periodic point of T_A.

Problem 1.24 Let $T_A\colon \mathbb{T}^n \to \mathbb{T}^n$ be an endomorphism of the torus. Show that

$$\operatorname{card} T_A^{-1} x = |\det A| \quad \text{for all } x \in \mathbb{T}^n,$$

using the Smith normal form (which says that $A = PDQ$, for some matrices P, D and Q with integer entries such that $|\det P| = |\det Q| = 1$ and D is diagonal).

Solution First note that $T_A = T_P T_D T_Q$. Since

$$|\det P| = |\det Q| = 1,$$

the maps T_P and T_Q are automorphisms of the torus and so

$$\operatorname{card} T_A^{-1} x = \operatorname{card} T_D^{-1}(T_{P^{-1}} x)$$

for all $x \in \mathbb{T}^n$. Hence, it suffices to show that

$$\operatorname{card} T_D^{-1} x = |\det D| \quad \text{for all } x \in \mathbb{T}^n.$$

Let $k_1, \ldots, k_n$ be the (integer) entries on the diagonal of D. The identity $T_D[y] = x$, with $x = [\bar{x}]$ and $\bar{x} \in [0,1)^n$, is equivalent to

$$Dy = \bar{x} + p \quad \text{with } p = (p_1, \ldots, p_n) \in \mathbb{Z}^n.$$

Therefore,

$$y = D^{-1}\bar{x} + \left(\frac{p_1}{k_1}, \ldots, \frac{p_n}{k_n}\right),$$

which gives

$$[y] = \left[D^{-1}\bar{x} + \left(\frac{p_1}{k_1}, \ldots, \frac{p_n}{k_n}\right)\right]$$

with $p_i = 1, \ldots, |k_i|$ for each $i = 1, \ldots, n$. Hence,

$$\operatorname{card} T_D^{-1} x = |k_1| \cdots |k_n| = |\det D|.$$

Problem 1.25 Let $T_A \colon \mathbb{T}^n \to \mathbb{T}^n$ be an automorphism of the torus induced by a matrix A whose spectrum contains no root of 1. Show that the number of q-periodic points of T_A is equal to $|\det(A^q - \mathrm{Id})|$.

Solution A point $[x] \in \mathbb{T}^n$ is q-periodic if and only if

$$[A^q x] = [x] \quad \Longleftrightarrow \quad [(A^q - \mathrm{Id})x] = 0.$$

Since the matrix $A^q - \mathrm{Id}$ is invertible (because by hypothesis the spectrum of A contains no root of 1), it induces the endomorphism of the torus $T_{A^q - \mathrm{Id}}$. Hence, the number of solutions of the equation

$$T_{A^q - \mathrm{Id}}[x] = [(A^q - \mathrm{Id})x] = 0$$

is equal to $\operatorname{card} T_{A^q - \mathrm{Id}}^{-1} 0$, which by Problem 1.24 is $|\det(A^q - \mathrm{Id})|$.

Problem 1.26 Let $T_A \colon \mathbb{T}^2 \to \mathbb{T}^2$ be the automorphism of the torus induced by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Show that the number of q-periodic points of T_A is equal to $\operatorname{tr}(A^q) - 2$.

Solution First note that the eigenvalues

$$\tau = (3 + \sqrt{5})/2 \quad \text{and} \quad \tau^{-1} = (3 - \sqrt{5})/2$$

of A are not roots of 1. Hence, by Problem 1.25, the number of q-periodic points of T_A is equal to $|\det(A^q - \mathrm{Id})|$. Moreover,

$$\begin{aligned}
|\det(A^q - \mathrm{Id})| &= \left|\det\begin{pmatrix} \tau^q - 1 & 0 \\ 0 & \tau^{-q} - 1 \end{pmatrix}\right| \\
&= |(\tau^q - 1)(\tau^{-q} - 1)| \\
&= \tau^q + \tau^{-q} - 2 \\
&= \mathrm{tr}(A^q) - 2.
\end{aligned}$$

Problem 1.27 Let $f\colon S^1 \to S^1$ be a C^1 map with nonvanishing derivative. Show that there exists $q \in \mathbb{N}$ such that

$$\operatorname{card} f^{-1}x = q \quad \text{for all } x \in S^1.$$

(The number q coincides with the degree $\deg f$ of f.)

Solution Since f' never vanishes, it is always positive or always negative. Hence, the map f is either increasing or decreasing. Therefore, as $x \in S^1$ increases its image $f(x)$ travels along S^1 always in the positive direction or always in the negative direction. When x returns to the initial point, the image $f(x)$ also returns to the initial image and since it travels always in the positive direction or always in the negative direction, all points in S^1 are attained the same number of times (more precisely, the number of times that $f(x)$ travels along S^1 either in the positive or in the negative direction). Finally, note that since $|f'|$ is continuous, it has a maximum and so the number of times that $f(x)$ travels along S^1 is finite.

Problem 1.28 Given a map $f\colon X \to X$, show that the complement of a backward f-invariant set is forward f-invariant.

Solution Let A be a backward f-invariant set. Given $x \in f(X \setminus A)$, there exists $y \in X \setminus A$ with $f(y) = x$. Since A is backward f-invariant, this implies that $x \notin A$ (otherwise $f(y) = x \in A$ and so $y \in A$). Hence,

$$f(X \setminus A) \subseteq X \setminus A.$$

Problem 1.29 Let $f\colon X \to X$ be a continuous map on a compact metric space. Moreover, let $A \subseteq X$ be a closed backward f-invariant set and define

$$\Lambda = \bigcap_{n=0}^{\infty} f^{-n}A.$$

Show that:

1. if U is an open neighborhood of Λ, then $f^{-n}A \subseteq U$ for any sufficiently large n;
2. the set Λ is f-invariant.

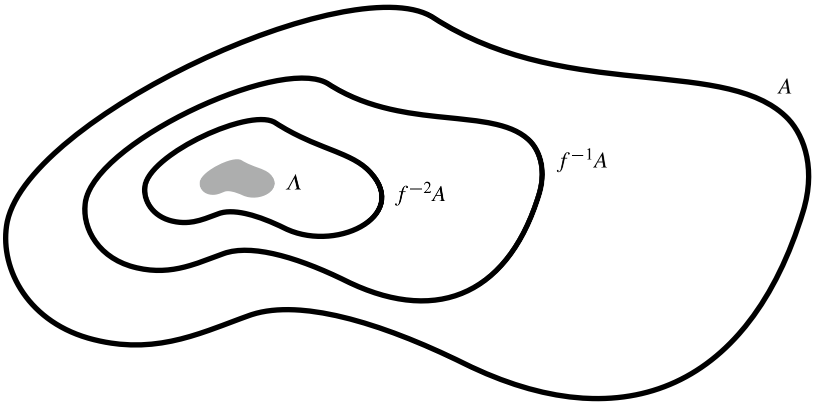

Fig. II.1.4 Preimages of the set A and their intersection.

Solution 1. Since A is backward f-invariant, the sets $B_n = f^{-n}A$ satisfy $B_{n+1} \subseteq B_n$ for all $n \in \mathbb{N}$ (see Figure II.1.4). Moreover,

$$\Lambda = \bigcap_{n=0}^{\infty} B_n.$$

The open sets $V_n = X \backslash B_n$ satisfy $V_{n+1} \supseteq V_n$ for all $n \in \mathbb{N}$ and together with U they cover X. By compactness, there exists a finite subcover, say $\{U, V_1, \dots, V_N\}$, and so $\{U, V_N\}$ is a cover of X. Hence,

$$f^{-n}A = B_n \subseteq U \quad \text{for all } n \geqslant N.$$

2. We have

$$f^{-1}\Lambda = \bigcap_{n=1}^{\infty} f^{-n}A.$$

But since the sequence $B_n = f^{-n}A$ is decreasing, we conclude that

$$\bigcap_{n=1}^{\infty} f^{-n}A = \bigcap_{n=0}^{\infty} f^{-n}A = \Lambda.$$

Problem 1.30 Find the largest E_3-invariant set A contained in $J = [0, 1/3] \cup [2/3, 1]$ (see Figure I.1.5).

Solution The expanding map $E_3 \colon S^1 \to S^1$ is given by

$$E_3(x) = \begin{cases} 3x & \text{if } x \in [0, 1/3), \\ 3x - 1 & \text{if } x \in [1/3, 2/3), \\ 3x - 2 & \text{if } x \in [2/3, 1). \end{cases}$$

Note that if A contains some point different from 0, then

$$E_3^{-1}A \cap (S^1 \setminus J) \neq \varnothing.$$

On the other hand, $E_3^{-1}\{0\} \neq \{0\}$ and so the set $\{0\}$ is not E_3-invariant. Hence, $\varnothing$ is the largest E_3-invariant set contained in J.

Problem 1.31 Consider the differential equation

$$\begin{cases} x' = 6y^5, \\ y' = -4x^3 \end{cases}$$

on $\mathbb{R}^2$. Show that for each set $I \subseteq \mathbb{R}_0^+$ the union

$$\bigcup_{a \in I} \left\{(x,y) \in \mathbb{R}^2 : y^6 + x^4 = a\right\}$$

is invariant under the flow determined by the differential equation.

Solution First note that the vector field is of class C^1, which ensures that each initial condition determines a unique solution. Each solution $(x,y) = (x(t),y(t))$ of the equation satisfies

$$\begin{aligned} (y^6 + x^4)' &= 6y^5y' + 4x^3x' \\ &= 6y^5(-4x^3) + 4x^3(6y^5) = 0. \end{aligned}$$

This implies that each orbit remains for all time in a set of the form

$$C_a = \left\{(x,y) \in \mathbb{R}^2 : y^6 + x^4 = a\right\}$$

for some $a \geqslant 0$ (see Figure II.1.5). Since the sets C_a are compact, this ensures that each solution is global and so that the equation determines a flow. Moreover, each set C_a and so also the union $\bigcup_{a \in I} C_a$ are invariant under the flow determined by the equation.

Problem 1.32 Consider the differential equation

$$\begin{cases} x' = ax - xy, \\ y' = -y + x^2 - 2y^2, \end{cases}$$

on $\mathbb{R}^2$ for some $a > 0$. Show that the solutions starting in the sets

$$S_1 = \left\{(x,y) \in \mathbb{R}^2 : x = 0\right\} \quad \text{and} \quad S_2 = \left\{(x,y) \in \mathbb{R}^2 : y = \frac{x^2}{1+2a}\right\}$$

remain in these sets for all time.

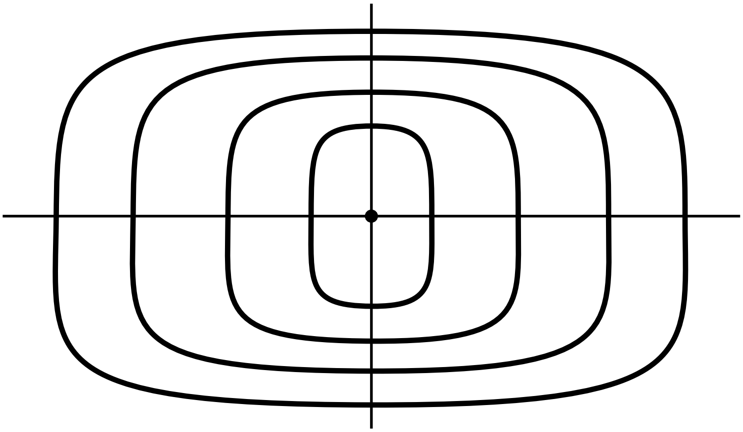

Fig. II.1.5 The sets $C_a = \{(u,v) \in \mathbb{R}^2 : y^6 + x^4 = a\}$.

Solution In the coordinates (x,y) the set S_1 is given by $x = 0$. Moreover, $x'|_{x=0} = 0$ on this set, which shows that any solution starting in S_1 remains in S_1 for all time.

Now consider the set S_2. We introduce new coordinates (x,z) with

$$z = y - \frac{x^2}{1+2a}.$$

This is indeed a coordinate change since the corresponding Jacobian matrix has determinant

$$\det\begin{pmatrix} 1 & 0 \\ -2x/(1+2a) & 1 \end{pmatrix} = 1 \neq 0.$$

We have

$$\begin{aligned} z'|_{z=0} &= \left(y - \frac{x^2}{1+2a}\right)'\Big|_{y=x^2/(1+2a)} \\ &= \left(-y + x^2 - 2y^2 - \frac{2x^2}{1+2a}(a-y)\right)\Big|_{y=x^2/(1+2a)} \\ &= -\frac{x^2}{1+2a} + x^2 - \frac{2x^4}{(1+2a)^2} - \frac{2x^2}{1+2a}\left(a - \frac{x^2}{1+2a}\right) = 0. \end{aligned}$$

This implies that each solution starting in S_2 remains in S_2 for all time.

Problem 1.33 Find the flow determined by the equation $x'' + x = 0$.

Solution One can easily verify that the solutions of the equation are

$$x(t) = c_1 \cos t + c_2 \sin t \quad \text{for } c_1, c_2, t \in \mathbb{R}.$$

Letting $y = x'$, one can rewrite $x'' + x = 0$ in the form

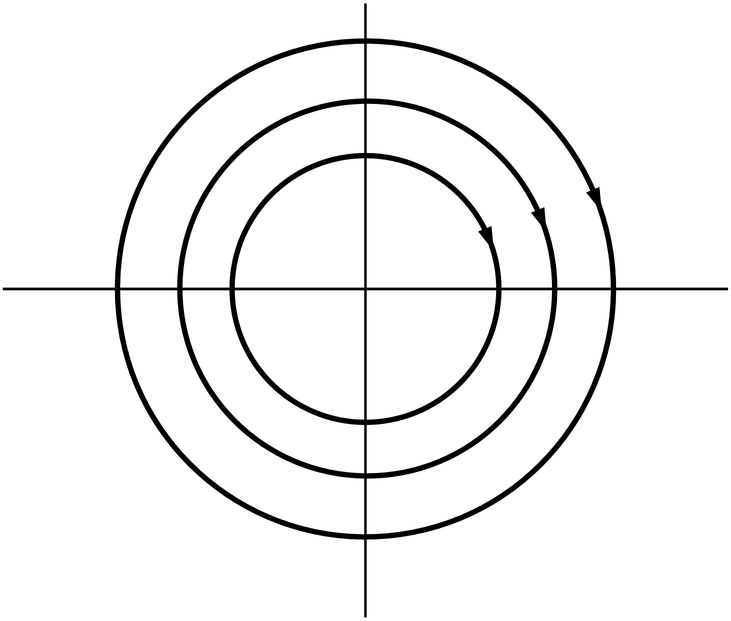

Fig. II.1.6 Phase portrait of equation (II.1.5).

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{II.1.5}$$

(which has the phase portrait in Figure II.1.6). To obtain the corresponding flow, we need to write each solution $(x(t), y(t))$ in terms of the initial condition $(x_0, y_0) = (x(0), y(0))$. We have

$$y(t) = x'(t) = -c_1 \sin t + c_2 \cos t$$

for $t \in \mathbb{R}$ and so

$$x_0 = c_1 \quad \text{and} \quad y_0 = c_2.$$

Therefore,

$$x(t) = x_0 \cos t + y_0 \sin t$$

and

$$y(t) = -x_0 \sin t + y_0 \cos t,$$

which gives the flow

$$\begin{aligned} \varphi_t(x_0, y_0) &= \begin{pmatrix} x_0 \cos t + y_0 \sin t \\ -x_0 \sin t + y_0 \cos t \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \end{aligned}$$

for $t \in \mathbb{R}$.

Problem 1.34 Find whether the equation $x' = x(x^2+1)$ determines a flow.

Solution First note that the equation has the phase portrait in Figure II.1.7. We have

$$\frac{x'}{x^3} = \frac{x^3+x}{x^3} > 1$$

for $x > 0$. Integrating we obtain

$$-\frac{1}{2}\left(\frac{1}{x(t)^2} - \frac{1}{x(0)^2}\right) \geqslant t$$

whenever $x(0) > 0$ and $t > 0$. Finally, letting $x(t) \to +\infty$ gives

$$t \leqslant \frac{1}{2x(0)^2},$$

which shows that the equation does not determine a flow because some solutions are not defined on the whole $\mathbb{R}$.

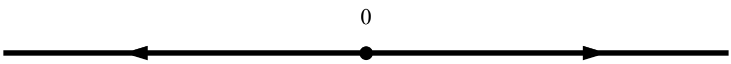

Fig. II.1.7 Phase portrait of the equation $x' = x(x^2+1)$.

Problem 1.35 Show that the identity map is a Poincaré map for the differential equation $x'' + x = 0$ and compute the corresponding first return time.

Solution It was shown in Problem 1.33 that the equation generates the flow

$$\varphi_t(x_0, y_0) = (x_0 \cos t + y_0 \sin t, -x_0 \sin t + y_0 \cos t)$$

for $(x_0, y_0) \in \mathbb{R}^2$ and $t \in \mathbb{R}$. The first return time of the point $(x_0, 0)$ to the half-line $\mathbb{R}^+ \times \{0\}$ is 2π, because $\varphi_{2\pi}(x_0, 0) = (x_0, 0)$ and

$$\varphi_t(x_0, 0) \notin \mathbb{R}^+ \times \{0\} \quad \text{for } t \in (0, 2\pi),$$

and so the corresponding Poincaré map is the identity map .

Problem 1.36 Consider the differential equation

$$\begin{cases} x' = y, \\ y' = -\sin x. \end{cases}$$

Show that the set

$$X = \{(x,y) \in \mathbb{R}^2 : y = 0, x \in (0,\pi)\}$$

is a Poincaré section and determine the corresponding first return time and Poincaré map.

Solution The critical points of the equation are $(x,y) = (n\pi, 0)$, for $n \in \mathbb{Z}$. Note that in the neighborhoods of the points

$$(x,y) = ((2n+1)\pi, 0) \quad \text{and} \quad (x,y) = (2n\pi, 0),$$

for $n \in \mathbb{Z}$, the behavior is different: the former points are centers while the latter are saddles (see Figure II.1.8). It follows readily from the phase portrait that the Poincaré map is the identity map.

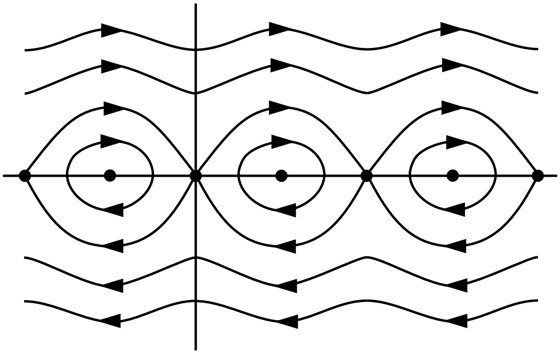

Fig. II.1.8 Phase portrait of the equation in Problem 1.36.

Now we compute the first return time or, equivalently, the periods of the periodic orbits. The solutions of the equation with initial condition $x(0) = x_0 \in (0,\pi)$ and $y(0) = 0$ satisfy

$$\frac{d}{dt}\left(\frac{1}{2}y^2 - \cos x\right) = 0.$$

Since $\cos x = 1 - 2\sin^2(x/2)$, we obtain

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 &= 2(\cos x - \cos x_0) \\ &= 4\left(\sin^2\left(\frac{x_0}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right).\end{aligned} \tag{II.1.6}$$

Now let

$$z = \sin\left(\frac{x}{2}\right), \quad k = \sin^2\left(\frac{x_0}{2}\right). \tag{II.1.7}$$

Note that $z(0) = \sqrt{k} \in (0,1)$. By (II.1.7) we obtain

$$\frac{dz}{dt} = \frac{1}{2}\frac{dx}{dt}\cos\left(\frac{x}{2}\right)$$

and

$$\begin{aligned}\left(\frac{dz}{dt}\right)^2 &= \frac{1}{4}\cos^2\left(\frac{x}{2}\right)\left(\frac{dx}{dt}\right)^2 \\ &= \frac{1}{4}\left(1-\sin^2\left(\frac{x}{2}\right)\right)\left(\frac{dx}{dt}\right)^2 \\ &= \frac{1}{4}(1-z^2)\left(\frac{dx}{dt}\right)^2.\end{aligned} \tag{II.1.8}$$

Using identities (II.1.7) and (II.1.8), it follows from (II.1.6) that

$$\left(\frac{dz}{dt}\right)^2 = (1-z^2)(k-z^2).$$

Hence, writing $w = z/\sqrt{k}$ we obtain

$$\left(\frac{dw}{dt}\right)^2 = (1-w^2)(1-kw^2).$$

Moreover, $w(0) = 1$ and $w'(0) = 0$. When a periodic orbit travels in the upper half plane, it completes a closed curve in the variables (w, w'). Since $w = 1/\sqrt{k}$ for $x = \pi$, the period is given by

$$\begin{aligned}T &= 4\int_1^{1/\sqrt{k}} \frac{ds}{\sqrt{(1-s^2)(1-ks^2)}} \\ &= -4\int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-ks^2)}} + 4\int_0^{1/\sqrt{k}} \frac{ds}{\sqrt{(1-s^2)(1-ks^2)}}.\end{aligned}$$

Let

$$K(m) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-ms^2)}}$$

and

$$F(\varphi, m) = \int_0^{\sin\varphi} \frac{ds}{\sqrt{(1-s^2)(1-ms^2)}}$$

be, respectively, the complete and incomplete elliptical integrals of the first kind. Note that

$$T = -4K(k) + 4F\left(\arcsin\frac{1}{\sqrt{k}}, k\right).$$

Problem 1.37 Show that there exists a homeomorphism $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ mapping the orbits $\varphi_t(x)$ determined by the equation $x' = -x$ on $\mathbb{R}^+$ (which has the phase portrait in Figure I.1.6) onto the orbits $\psi_t(x)$ determined by the equation $x' = -x^2$ on $\mathbb{R}^+$ (which has the phase portrait in Figure I.1.7), and a map $\tau\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with $t \mapsto \tau(x,t)$ increasing for each x such that

$$h(\varphi_{\tau(x,t)}(x)) = \psi_t(h(x)) \quad \text{for } x, t > 0.$$

Solution First note that $\varphi_t(x) = xe^{-t}$. To find the second flow, we solve the initial value problem

$$x' = -x^2, \quad x(0) = x_0.$$

One can easily verify that

$$x(t) = \frac{x_0}{1+tx_0}, \quad \text{that is,} \quad \psi_t(x) = \frac{x}{1+tx}.$$

Now we find h and τ such that

$$h(xe^{-\tau}) = \frac{h(x)}{1+th(x)}.$$

For example, taking $h(x) = x$, the equation becomes

$$xe^{-\tau} = \frac{x}{1+tx}.$$

Solving it for τ, we obtain

$$\tau(x,t) = \log(1+tx)$$

(note that $1+tx \geqslant 0$), which is an increasing function of t for each given x.

Problem 1.38 Let $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ be C^1 functions with $f > 0$ such that

$$f(x+k, y+l) = f(x,y) \quad \text{and} \quad g(x+k, y+l) = g(x,y)$$

for all $x, y \in \mathbb{R}$ and $k, l \in \mathbb{Z}$. Then the differential equation

$$x' = f(x,y), \quad y' = g(x,y)$$

on $\mathbb{R}^2$ has unique solutions that are defined for all $t \in \mathbb{R}$. Let $\varphi_t\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the corresponding flow (see Proposition 1.8). Each solution

$$\varphi_t(0,z) = (x(t), y(t)) = (x(t,z), y(t,z))$$

of the equation with $(x(0), y(0)) = (0, z)$ crosses infinitely often the line $x = 0$ (since f is uniformly bounded from below). The first intersection into the future occurs at the time

$$T_z = \inf\{t > 0 : x(t) = 1\}.$$

Show that the map $h\colon S^1 \to S^1$ defined by

$$h(z) = y(T_z, z)$$

(see Figure I.1.8) is invertible.

Solution Let $(\bar{x}(t,w), \bar{y}(t,w))$ be the solution of the equation

$$(\bar{x}', \bar{y}') = -(f,g)(\bar{x}, \bar{y}) \quad \text{with } (\bar{x}, \bar{y})(0) = (1, w).$$

Moreover, let

$$S_w = \sup\{t < 0 : \bar{x}(t,w) = 0\}$$

and consider the map $\bar{h}\colon S^1 \to S^1$ defined by $\bar{h}(w) = \bar{y}(S_w, w)$. We have

$$h \circ \bar{h} = \mathrm{id} \quad \text{and} \quad \bar{h} \circ h = \mathrm{id},$$

that is, $\bar{h}$ is the inverse of h.

Problem 1.39 Consider the differential equation $(x',y') = (\alpha,\beta)$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with $\alpha,\beta \neq 0$. Determine whether any orbit of the flow determined by this equation is dense.

Solution The solutions of the equation $(x',y') = (\alpha,\beta)$ on $\mathbb{R}^2$ are

$$(x(t),y(t)) = (x(0),y(0)) + (\alpha t,\beta t).$$

The corresponding solutions on the torus $\mathbb{T}^2$ (see Figure II.1.9) are periodic if and only if $(\alpha t,\beta t) \in \mathbb{Z}^2$ for some $t \neq 0$, which implies that

$$\alpha/\beta = (\alpha t)/(\beta t) \in \mathbb{Q}.$$

On the other hand, if $\alpha/\beta \in \mathbb{Q}$, then writing $\alpha/\beta = p/q$ with $p,q \in \mathbb{Z}$ we have

$$(\alpha t,\beta t) = (p,q) \in \mathbb{Z}^2 \quad \text{for } t = q/\beta.$$

That is, the solutions are periodic if and only if $\alpha/\beta \in \mathbb{Q}$. Clearly, a periodic solution is not dense and so one cannot have $\alpha/\beta \in \mathbb{Q}$.

On the other hand, when $\alpha/\beta \notin \mathbb{Q}$ all orbits are dense. Indeed, for the times $t = n/\alpha$ with $n \in \mathbb{Z}$ we have

$$(\alpha t,\beta t) = (n,\alpha t), \quad \text{with } \alpha = \alpha/\beta \in \mathbb{R}\backslash\mathbb{Q}.$$

This implies that the intersection of each orbit with a given horizontal line coincides with an orbit of a rotation with irrational rotation number α. This readily implies that when $\alpha/\beta \notin \mathbb{Q}$ each orbit of the equation $(x',y') = (\alpha,\beta)$ is dense.

Problem 1.40 Consider the differential equation $v' = f(v)$ on $\mathbb{R}^n$ for some map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of class C^1. Write $f = (f_1,\dots,f_n)$, $v = (v_1,\dots,v_n)$ and let

$$(L\varphi)(v) = \sum_{i=1}^{n} f_i(v)\frac{\partial\varphi(v)}{\partial v_i}$$

for each C^2 function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$. Show that each periodic orbit of the equation:

1. has at least two points in the set

$$A_\varphi = \{v \in \mathbb{R}^n : (L\varphi)(v) = 0\};$$

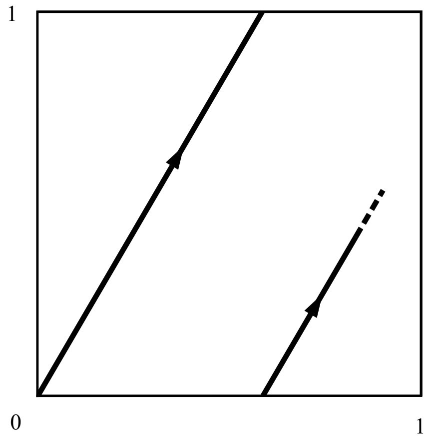

Fig. II.1.9 A solution of the equation $(x', y') = (\alpha, \beta)$ on $\mathbb{T}^2$.

2. has at least one point in each of the sets

$$B_\varphi = \{v \in A_\varphi : (L^2\varphi)(v) \leqslant 0\} \quad \text{and} \quad C_\varphi = \{v \in A_\varphi : (L^2\varphi)(v) \geqslant 0\};$$

3. intersects A_φ transversally at each point in the sets

$$\{v \in A_\varphi : (L^2\varphi)(v) < 0\} \quad \text{and} \quad \{v \in A_\varphi : (L^2\varphi)(v) > 0\}.$$

Solution 1. Note that if $v(t)$ is a periodic solution of the equation, then there exist $t = t_1$ and $t = t_2$ such that

$$\varphi(v(t_1)) \leqslant \varphi(v(t)) \leqslant \varphi(v(t_2))$$

(because any periodic orbit is a compact set). Since the points t_1 and t_2 are local extrema of the function $\varphi \circ v$, for $t = t_1$ and $t = t_2$ we have

$$\begin{aligned} (L\varphi)(v(t)) &= \langle f(v(t)), \nabla\varphi(v(t))\rangle \\ &= \langle v'(t), \nabla\varphi(v(t))\rangle \\ &= \frac{d}{dt}\varphi(v(t)) = 0. \end{aligned} \tag{II.1.9}$$

This shows that each periodic orbit has at least two points in A_φ.

2. Assume that $\varphi \circ v$ has a local maximum at $t = t_1$ and a local minimum at $t = t_2$. We have

$$
\begin{aligned}
\frac{d^2}{dt^2}\varphi(v(t)) &= \frac{d}{dt}(L\varphi)(v(t)) \\
&= \frac{d}{dt}\sum_{i=1}^{n} f_i(v(t))\frac{\partial\varphi}{\partial v_i}(v(t)) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial v_j} f_j \frac{\partial\varphi}{\partial v_i} + f_i \frac{\partial^2\varphi}{\partial v_i \partial v_j} f_j\right)|_{v=v(t)}
\end{aligned}
$$

and so

$$
\begin{aligned}
(L^2\varphi)(v(t)) &= \sum_{j=1}^{n} f_j(v(t))\frac{\partial(L\varphi)}{\partial v_j}(v(t)) \\
&= \sum_{i=j}^{n} f_j(v(t))\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial v_i}\frac{\partial\varphi}{\partial v_i} + f_i\frac{\partial^2\varphi}{\partial v_i \partial v_j}\right)|_{v=v(t)} \\
&= \frac{d^2}{dt^2}\varphi(v(t)).
\end{aligned}
$$

Therefore, $v(t_1) \in B_\varphi$ and $v(t_2) \in C_\varphi$.

3. It follows from (II.1.9) that

$$
(L^2\varphi)(v(t)) = \langle f(v(t)), \nabla\varphi(v(t))\rangle.
$$

Therefore, if $(L^2\varphi)(v(t)) \neq 0$, then $\nabla\varphi(v(t)) \neq 0$ and so the periodic orbit determined by the solution $v(t)$ intersects the surface $(L\varphi)(v) = 0$ transversally.

Chapter II.2
Topological Dynamics

Problem 2.1 Show that if $f: X \to X$ is a continuous map on a compact metric space, then $f(\omega(x)) = \omega(x)$ for all $x \in X$.

Solution Take $y \in \omega(x)$. By Proposition 2.4 there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$ satisfying

$$y = \lim_{k\to\infty} f^{n_k}(x).$$

Since f is continuous, we have

$$f(y) = \lim_{k\to\infty} f^{n_k+1}(x)$$

and, again by Proposition 2.4, $f(y) \in \omega(x)$. In other words,

$$f(\omega(x)) \subseteq \omega(x).$$

Now we show that $y = f(z)$ for some $z \in \omega(x)$. Since X is compact, the sequence $f^{n_k-1}(x)$ has a convergent subsequence, say $f^{m_k}(x)$, with limit $z \in \omega(x)$. But since f is continuous, we obtain

$$\begin{aligned} f(z) &= f\left(\lim_{k\to\infty} f^{m_k}(x)\right) \\ &= \lim_{k\to\infty} f^{m_k+1}(x) = y. \end{aligned}$$

Problem 2.2 Given a map $f: X \to X$ on a metric space without isolated points, show that if $\omega(x) \neq X$, then the positive semiorbit $\gamma^+(x)$ is not dense.

Solution Assume that $\overline{\gamma^+(x)} = X$. Given $y \in X$, since X has no isolated points, there exist sequences of positive integers $n_k \nearrow \infty$ and of positive numbers $r_k \searrow 0$ such that

$$f^{n_k}(x) \in B(y, r_k) \setminus B(y, r_{k+1}) \quad \text{for all } k \in \mathbb{N}$$

(see Figure II.2.1). Clearly, $f^{n_k}(x) \to y$ when $k \to \infty$ and so $y \in \omega(x)$. This shows that $\omega(x) = X$, which gives a contradiction.

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_8

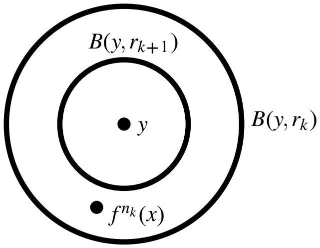

Fig. II.2.1 The points $f^{n_k}(x) \in B(y, r_k) \backslash B(y, r_{k+1})$.

Problem 2.3 Let $f\colon X \to X$ be a continuous map. Given $x \in X$, show that if $\omega(x)$ contains infinitely many points, then it has no isolated points.

Solution Assume that $\omega(x)$ contains infinitely many points and that $y \in \omega(x)$ is an isolated point. Then there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$ such that $f^{n_k}(x) \to y$ when $k \to \infty$. But since y is isolated, one must have $f_{n_k}(x) = y$ for infinitely many integers k. Hence, y must be periodic and so $\omega(x) = \gamma(y)$ is finite. This contradiction shows that $\omega(x)$ has no isolated points.

Problem 2.4 Let $f\colon I \to I$ be a continuous map on the interval $I = [0, 1]$. Given $x \in I$, show that for any $a, b \in \omega(x)$ and any open neighborhood U of a, there exists an increasing sequence of positive integers $(k_i)_{i\in\mathbb{N}}$ such that $b \in \overline{\bigcup_{i\in\mathbb{N}} f^{k_i}(U)}$.

Solution Since $a, b \in \omega(x)$, for any open neighborhood V of b there exist increasing sequences $n_i > m_i$, for $i \in \mathbb{N}$, and an integer $N \in \mathbb{N}$ such that

$$\lim_{i\to\infty} f^{m_i}(x) = a \quad \text{and} \quad \lim_{i\to\infty} f^{n_i}(x) = b,$$

with

$$f^{m_i}(x) \in U \quad \text{and} \quad f^{n_i}(x) \in V$$

for all $i \geqslant N$. For each $i \in \mathbb{N}$, let $z_i = f^{m_i}(x)$ and $k_i = n_i - m_i$. Clearly,

$$\lim_{i\to\infty} z_i = a \quad \text{and} \quad \lim_{i\to\infty} f^{k_i}(z_i) = b.$$

Since $z_i \in U$ for all $i \geqslant N$, we conclude that $b \in \overline{\bigcup_{i\in\mathbb{N}} f^{k_i}(U)}$.

Problem 2.5 Let $f\colon X \to X$ be a continuous map on a compact metric space. Given $x \in X$, show that if $A = \omega(x)$, then for any nonempty closed subset $B \subsetneq A$ we have $B \cap \overline{f(A \backslash B)} \neq \varnothing$.

Solution Assume that $B \cap \overline{f(A \backslash B)} = \varnothing$ for some nonempty closed subset $B \subsetneq A$. Then there exist open sets U and V such that

$$\overline{U} \cap \overline{V} = \varnothing, \quad B \subsetneq U \quad \text{and} \quad \overline{f(A \backslash B)} \subsetneq V. \tag{II.2.1}$$

Thus,

$$A\backslash B \subsetneq f^{-1}V = W,$$

with W an open set (by the continuity of f). So

$$f(\overline{W}) = \overline{f(W)} \subseteq \overline{V}$$

and it follows from (II.2.1) that $f(\overline{W}) \cap \overline{U} = \varnothing$. Since

$$A = \omega(x) = (A\backslash B) \cup B \subsetneq W \cup U,$$

by Proposition 2.4 there exists an integer $k_0 > 0$ such that $f^n(x) \in W \cup U$ for $n \geqslant k_0$. Moreover, $f^n(x) \in W$ and $f^{n+1}(x) \in U$ for infinitely many integers $n \geqslant k_0$ because both $A\backslash B$ and B are nonempty. Thus, there exists an increasing sequence $(n_i)_{i\in\mathbb{N}}$ such that

$$f^{n_i}(x) \in W, \quad f^{n_i+1}(x) \in U \quad \text{and} \quad y = \lim_{i\to\infty} f^{n_i}(x) \in \overline{W}.$$

Therefore,

$$f(y) = f\left(\lim_{i\to\infty} f^{n_i}(x)\right) = \lim_{i\to\infty} f^{n_i+1}(x) \in \overline{U}$$

and so $f(y) \in f(\overline{W}) \cap \overline{U}$, which yields a contradiction since $f(\overline{W}) \cap \overline{U} = \varnothing$. This shows that $B \cap \overline{f(A\backslash B)} \neq \varnothing$.

Problem 2.6 Compute the α-limit set and the ω-limit set of each point in $\mathbb{R}$ for the flow determined by the differential equation $x' = -x(1+\cos^2 x)$ on $\mathbb{R}$.

Solution Note that if $x = 0$, then $x' = 0$ and so

$$\omega(0) = \alpha(0) = \{0\}.$$

On the other hand, $x' < 0$ for $x > 0$ and $x' > 0$ for $x < 0$. Hence, $\omega(x) = \{0\}$ and $\alpha(x) = \varnothing$ for each $x \neq 0$ (see Figure II.2.2).

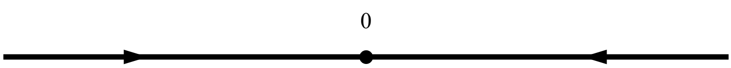

Fig. II.2.2 Phase portrait of the equation $x' = -x(1+\cos^2 x)$.

Problem 2.7 Consider the differential equation

$$\begin{cases} r' = r(r-1), \\ \theta' = 1, \end{cases}$$

written in polar coordinates. Compute the α-limit set and the ω-limit set of each point in $\mathbb{R}^2$ for the flow determined by the differential equation.

Solution Consider the sets

$$C_r = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\right\}$$

for $r \geqslant 0$. Any orbit starting in C_0 or C_1 remains, respectively, in C_0 or C_1. On the other hand, $r' < 0$ for $r \in (0,1)$ and $r' > 0$ for $r > 1$ (see Figure II.2.3). Hence, for each $p \in C_r$ we have

$$\begin{aligned}
&\alpha(p) = \omega(p) = \{(0,0)\} \quad \text{for } r = 0,\\
&\alpha(p) = C_1, \quad \omega(p) = \{(0,0)\} \quad \text{for } r \in (0,1),\\
&\alpha(p) = \omega(p) = C_1 \quad \text{for } r = 1,\\
&\alpha(p) = C_1, \quad \omega(p) = \varnothing \quad \text{for } r > 1.
\end{aligned}$$

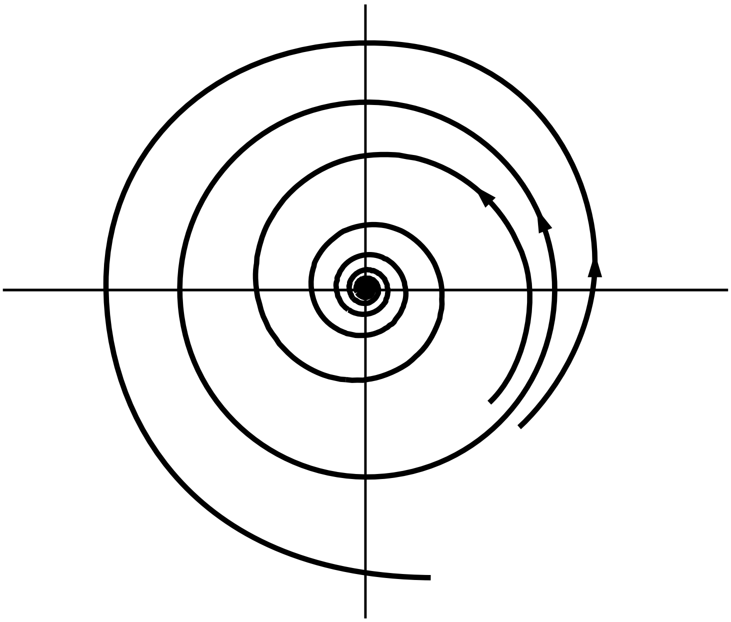

Fig. II.2.3 Phase portrait of the equation in Problem 2.7.

Problem 2.8 Compute the α-limit set of each point $(x,y) \in \mathbb{R}^2$ with $|x| < 1$ for the flow determined by the differential equation

$$\begin{cases} x' = (x^2 - 1)(y - x),\\ y' = x. \end{cases}$$

Solution The only critical point is the origin and is a repelling focus since the eigenvalues of the Jacobian matrix at the origin are $(1 \pm \sqrt{3}i)/2$. Moreover, for $|x| < 1$ the function

$$H(x,y) = \frac{y^2}{2} - \frac{1}{2}\log(1-x^2)$$

increases along solutions because

$$\dot{H}(x,y) = \nabla H(x,y) \cdot \big((x^2-1)(y-x), x\big) = x^2.$$

By sketching the phase portrait (see Figure II.2.4), we find that the α-limit set of any point (x,y) with $|x| < 1$ is the origin.

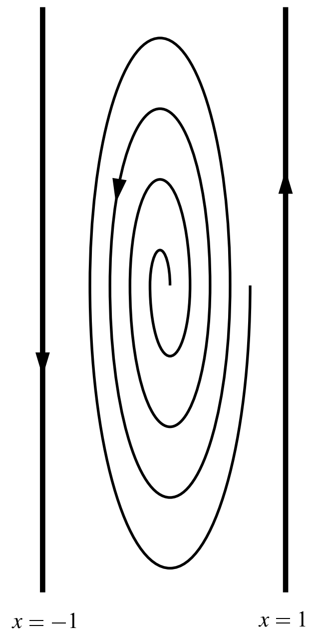

Fig. II.2.4 Phase portrait of the equation in Problem 2.8.

Problem 2.9 Let $f\colon X \to X$ be a continuous map on a topological space and let $NW(f)$ be the set of nonwandering points. Show that:

1. $NW(f)$ is closed;
2. $NW(f)$ is forward f-invariant.

Solution 1. We show that the complement of $NW(f)$ is open. If $x \notin NW(f)$, then there exists an open neighborhood U of x such that $f^n(U) \cap U = \varnothing$ for all $n \in \mathbb{N}$. Note that any point $y \in U$ is also outside of $NW(f)$. Hence, the complement of $NW(f)$ is open.

2. Take $x \in NW(f)$ and let V be an open neighborhood of $f(x)$. Then $U = f^{-1}V$ is an open neighborhood of x and there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \varnothing$. But then

$$f(f^n(U) \cap U) \subseteq f^n(V) \cap V,$$

which implies that $f^n(V) \cap V \neq \varnothing$. Hence, $f(x) \in NW(f)$ and so $NW(f)$ is forward f-invariant.

Problem 2.10 Let $f\colon X \to X$ be a continuous map on a topological space. Show that $\omega(x) \subseteq NW(f)$ for all $x \in X$.

Solution Let $x \in X$ and $y \in \omega(x)$. Moreover, let U be an open neighborhood of y. Since $y \in \omega(x)$, by Proposition 2.4 there exist positive integers $n_k \nearrow +\infty$ such that $f^{n_k}(x) \to y$. In particular, there exist $n_{k_0} < n_{k_1}$ such that

$$f^{n_{k_0}}(x) \in U \quad \text{and} \quad f^{n_{k_1}}(x) \in U.$$

Now let $z = f^{n_{k_0}}(x)$ and take $n = n_{k_1} - n_{k_0}$. Then

$$z \in f^n(U) \cap U \neq \varnothing,$$

which shows that $y \in NW(f)$.

Problem 2.11 Let $f\colon X \to X$ be a continuous map on a compact metric space. Show that

$$\lim_{m\to\infty} d(f^m(x), NW(f)) = 0$$

for every $x \in X$.

Solution We proceed by contradiction. If the property did not hold, then it would exist a compact set $K \subsetneq X$ containing infinitely many points of the positive semiorbit of x such that $K \cap NW(f) = \varnothing$. This implies that the semiorbit has an accumulation point $y \in K$. Then $y \in \omega(x)$, but on the other hand $y \notin NW(f)$, which contradicts to Problem 2.10.

Problem 2.12 Let $f\colon X \to X$ be a continuous map. Show that $R(f)$ (the set of recurrent points) is forward f-invariant.

Solution If there exist positive integers $n_k \nearrow \infty$ such that $f^{n_k}(x) \to x$ when $k \to \infty$, then

$$f^{n_k}(f(x)) = f(f^{n_k}(x)) \to f(x) \in \omega(x)$$

when $k \to \infty$ because f is continuous. This shows that $R(f)$ is forward f-invariant.

Problem 2.13 Given a map $f\colon X \to X$, show that $\overline{R(f)} \subseteq NW(f)$.

Solution Take $x \in \overline{R(f)}$. Then there exists a sequence $x_n \in \omega(x_n)$ such that $x_n \to x$ when $n \to \infty$. Now let U be an open neighborhood of x. Then there exists $m \in \mathbb{N}$ with $x_m \in U$. Since $x_m \in \omega(x_m)$, it also exists $n \in \mathbb{N}$ such that $f^n(x_m) \in U$. Hence, $f^n(U) \cap U \neq \varnothing$ and so $x \in NW(f)$, which establishes the desired property.

Problem 2.14 A nonempty closed forward invariant set without nonempty closed forward invariant proper subsets is said to be *minimal*. Show that any two distinct minimal sets for a map f must have empty intersection.

Solution Let M_1 and M_2 be minimal sets for the map f and assume that

$$A = M_1 \cap M_2 \neq \varnothing.$$

Note that A is closed and forward f-invariant since

$$f(A) \subseteq f(M_1) \cap f(M_2) \subseteq M_1 \cap M_2 = A.$$

But then A is a nonempty closed forward f-invariant proper subset both of M_1 and M_2, which contradicts the fact that M_1 and M_2 are minimal sets.

Problem 2.15 Give an example of an ω-limit set that is a minimal set.

Solution Consider a periodic orbit of a homeomorphism. Note that it is the ω-limit set of any point in the orbit and that it is a minimal set since it contains no proper nonempty closed invariant set.

Problem 2.16 Given a map $f\colon X \to X$ on a topological space and a nonempty closed forward f-invariant set $M \subseteq X$, show that the following properties are equivalent:

1. M is a minimal set;
2. $M = \overline{\gamma^+(x)}$ for all $x \in M$;
3. M is the ω-limit set of each of its points.

Solution ($1 \Rightarrow 2$). Assume that M is a minimal set and take $x \in M$. Then $\overline{\gamma^+(x)}$ is a nonempty closed forward f-invariant subset of M and so $\overline{\gamma^+(x)} = M$.
($2 \Rightarrow 3$). Now assume that $M = \overline{\gamma^+(x)}$ for all $x \in M$. Then for each $y \in M$ we have

$$\omega(y) = \bigcap_{x \in \gamma^+(y)} \overline{\gamma^+(x)} = \bigcap_{x \in \gamma^+(y)} M = M.$$

($3 \Rightarrow 1$). Finally, assume that $M = \omega(x)$ for every $x \in M$. Let N be a nonempty closed forward f-invariant subset of M and take $y \in N$. Then $M = \omega(y) \subseteq N$. Hence, $N = M$ and so M is a minimal set.

Problem 2.17 For a continuous map $f\colon X \to X$ show that f is topologically transitive if and only if for any nonempty open sets $U, V \subsetneq X$ there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$.

Solution First assume that f is topologically transitive. Let U and V be nonempty open sets. Then there exists $n \in \mathbb{N}$ such that $f^{-n}V \cap U \neq \varnothing$. Hence,

$$\begin{aligned}\varnothing &\neq f^n(f^{-n}V \cap U)\\ &\subseteq f^n(f^{-n}V) \cap f^n(U)\\ &\subseteq V \cap f^n(U)\end{aligned}$$

and so $V \cap f^n(U) \neq \varnothing$.

Now assume that for any nonempty open sets $U, V \subsetneq X$ there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$. Given $x \in f^n(U) \cap V$, there exists $y \in U$ such that $x = f^n(y)$. Hence, $y \in U \cap f^{-n}V$, which shows that $U \cap f^{-n}V \neq \varnothing$ and so f is topologically transitive.

Problem 2.18 Let $f: X \to X$ be a continuous map. Show that if $\bigcup_{n=1}^{\infty} f^n(U)$ is dense for any nonempty open set $U \subseteq X$, then $\bigcup_{n=1}^{\infty} f^{-n}U$ is also dense for any nonempty open set $U \subseteq X$.

Solution It suffices to show that for any nonempty open sets $U, V \subsetneq X$ there exists $n \in \mathbb{N}$ such that $U \cap f^{-n}V \neq \varnothing$. By the hypothesis, we have

$$U \cap \bigcup_{n=1}^{\infty} f^n(V) \neq \varnothing.$$

Hence, there exists $x \in U$ such that $x \in f^n(V)$ for some $n \in \mathbb{N}$. Therefore, there exists $y \in V$ such that $x = f^n(y)$ and so $y \in V \cap f^{-n}(U)$.

Problem 2.19 Given a continuous map $f: X \to X$, show that if f is topologically transitive, then any closed forward f-invariant proper subset of X has empty interior.

Solution We proceed by contradiction. Since f is topologically transitive, given nonempty open sets $U, V \subsetneq X$, there exists an integer $n \in \mathbb{N}$ such that $f^{-n}U \cap V \neq \varnothing$. Now let $C \subsetneq X$ be a nonempty closed f-invariant subset that has nonempty interior. Then there exist a nonempty open set $W \subsetneq C$ and $n \in \mathbb{N}$ such that

$$f^{-n}(X \backslash C) \cap W \neq \varnothing. \tag{II.2.2}$$

Take $x \in f^{-1}(X \backslash C)$. Then $f(x) \in X \backslash C$. Since C is forward f-invariant, this implies that $x \notin C$ (because $f(x) \in C$ whenever $x \in C$). Hence, $f^{-1}(X \backslash C) \subseteq X \backslash C$ and so also

$$f^{-n}(X \backslash C) \subseteq X \backslash C,$$

which contradicts to (II.2.2) (because $W \subsetneq C$). Thus, C has empty interior.

Problem 2.20 Given a continuous map $f: X \to X$ on a complete metric space with a countable basis and without isolated points, show that the following properties are equivalent:

1. f is topologically transitive;
2. the set of points with a dense positive semiorbit is dense.

Solution ($1 \Rightarrow 2$). Let $(U_n)_{n \in \mathbb{N}}$ be a basis for the topology. For each $n \in \mathbb{N}$, we consider the open set

$$V_n = \bigcup_{i \in \mathbb{N}} f^{-i} U_n.$$

Since f is topologically transitive, given a nonempty open set U, there exists $i \in \mathbb{N}$ such that $f^{-i}U_n \cap U \neq \varnothing$ and so $V_n \cap U \neq \varnothing$. Thus, each open set V_n is dense and their intersection $V = \bigcap_{n\in\mathbb{N}} V_n$ is also dense (because complete metric spaces have the property that any countable intersection of dense open sets is dense).

We show that V is in fact the set of points with a dense positive semiorbit. Indeed, $x \in V$ if and only if for each $n \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that $f^i(x) \in U_n$, that is, if and only if for each $n \in \mathbb{N}$ we have $\gamma^+(x) \cap U_n \neq \varnothing$, which is equivalent to $\overline{\gamma^+(x)} = X$.

$(2 \Rightarrow 1)$. Take a point x such that $\overline{\gamma^+(x)} = X$ and let $U, V \subsetneq X$ be nonempty open sets. Since X has no isolated points, $\gamma^+(x)$ visits infinitely often U and V. Hence, there exist $m, n \in \mathbb{N}$ with $m > n$ such that

$$f^m(x) \in U \quad \text{and} \quad f^n(x) \in V.$$

Therefore,

$$x \in f^{-m}U \cap f^{-n}V = f^{-n}(f^{-(m-n)}U \cap V)$$

and so the set $f^{-(m-n)}U \cap V$ is nonempty. Thus, f is topologically transitive.

Problem 2.21 Give an example of a continuous map $f: X \to X$ on a finite set with at least one dense positive semiorbit, but which is not topologically transitive.

Solution Consider the set $X = \{0, 1\}$ with the discrete topology and define a map $f: X \to X$ by $f(0) = f(1) = 0$. Then the positive semiorbit of the point 1 is the whole space, but f is not topologically transitive. Indeed, let $U = \{0\}$ and $V = \{1\}$. Then

$$f^n(U) = U \quad \text{and} \quad f^n(U) \cap V = \varnothing$$

for all $n \in \mathbb{N}$. Hence, it follows from Problem 2.17 that f is not topologically transitive.

Problem 2.22 Give an example of a continuous map $f: X \to X$ on an infinite set with at least one dense positive semiorbit, but which is not topologically transitive.

Solution Consider the set

$$X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$$

with the distance induced from the usual distance on $\mathbb{R}$ and define a map $f: X \to X$ by

$$f(0) = 0 \quad \text{and} \quad f(1/n) = 1/(n+1) \text{ for } n \in \mathbb{N}.$$

Note that f is continuous and that $\gamma^+(1)$ is the only dense positive semiorbit. On the other hand, f is not topologically transitive. For example, for the open sets $U = \{1/2\}$ and $V = \{1\}$ we have $f^{-n}U \cap V = \varnothing$ for all $n \in \mathbb{N}$.

Problem 2.23 Show that the continuous map $f: [0,1] \cap \mathbb{Q} \to [0,1] \cap \mathbb{Q}$ defined by $f(x) = 1 - |2x - 1|$ has no dense positive semiorbits, but is topologically transitive.

Solution Each point $x \in [0,1] \cap \mathbb{Q}$ has a finite positive semiorbit. Indeed, if $x = q/p$ with $p, q \in \mathbb{N}$, then $f(x) = q'/p$ for some $q' \in \mathbb{N}$. Therefore, the positive semiorbit of x has at most cardinality p and so it is not dense.

On the other hand, the graph of f^m consists of 2^{m-1} "tents" (see Figure II.2.5) and so given nonempty open intervals $I, J \subsetneq [0,1] \cap \mathbb{Q}$, there exists $m \in \mathbb{N}$ such that

$$f^{-m} I \cap J \neq \varnothing.$$

Hence, f is topologically transitive.

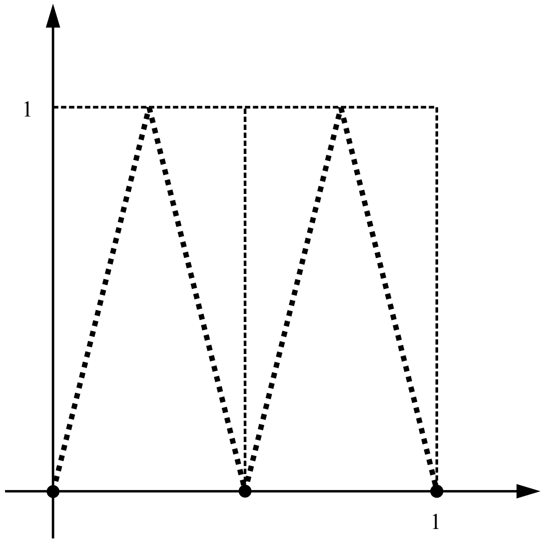

Fig. II.2.5 Graph of the map f^2.

Problem 2.24 Show that the map f in Problem 1.21 is topologically transitive.

Solution Let $U, V \subsetneq R$ be nonempty connected open sets. Take points $e^{2\pi i\theta} \in V$ and $e^{2\pi i\varphi} \in U$ with

$$\theta = 0.\theta_1\theta_2\cdots \quad \text{and} \quad \varphi = 0.\varphi_1\varphi_2\cdots$$

written in base-3. For any sufficiently large $n \in \mathbb{N}$, the point

$$p = \exp(2\pi i(0.\theta_1\theta_2\cdots\theta_n\varphi_1\varphi_2\cdots))$$

is also in V. Moreover, $f^n(p) = e^{2\pi i\varphi}$ and so $p \in f^{-n}U$. Therefore, $f^{-n}U \cap V \neq \varnothing$. This shows that the map f is topologically transitive.

Problem 2.25 Let $R_\alpha \colon S^1 \to S^1$ be a rotation of the circle with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Show that

$$\overline{\{R_\alpha^m(x) : m \in \mathbb{Z}\}} = S^1$$

for every $x \in S^1$.

Solution If $R_\alpha^{m_1}(x) = R_\alpha^{m_2}(x)$ for some integers $m_1 > m_2$, then

$$m_1\alpha - m_2\alpha = m$$

for some $m \in \mathbb{Z}$. But then $\alpha = m/(m_1 - m_2)$, which is impossible since $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Thus, the points $R_\alpha^m(x)$ are pairwise distinct.

Now assume that there exists a closest point $R_\alpha^m(x)$ to x with $m \neq 0$. Since the iterates $x_n = R_\alpha^{nm}(x)$ are equally spaced on S^1 and since $x_n \neq x$ for all $n \neq 0$, there exists $n \in \mathbb{Z}$ such that x_n is between $R_\alpha^m(x)$ and x. But this is impossible since then x_n would be closer to x than $R_\alpha^m(x)$. Hence, there exists a sequence $m_k \in \mathbb{Z}$ with $|m_k| \to \infty$ such that $R_\alpha^{m_k}(x) \neq x$ for all k and

$$R_\alpha^{m_k}(x) \to x \quad \text{when } k \to \infty. \tag{II.2.3}$$

Now consider the iterates $y_n = R_\alpha^{nm_k}(x)$ for a given k. Again since these are equally spaced on S^1, it follows from (II.2.3) that there are points of the orbit of x arbitrarily close to each given point $y \in S^1$. In other words, the orbit of x is dense in S^1.

Problem 2.26 Let $f\colon X \to X$ and $g\colon Y \to Y$ be topologically conjugate maps (see Definition 2.11). Show that for each $q \in \mathbb{N}$ the number of q-periodic points of f is equal to the number of q-periodic points of g.

Solution Let $H\colon X \to Y$ be a topological conjugacy between the maps f and g. Then

$$H \circ f = g \circ H \quad \text{on } X$$

and proceeding inductively we obtain

$$H \circ f^q = g^q \circ H \quad \text{on } X$$

for all $q \in \mathbb{N}$. Therefore, if $x \in X$ is a q-periodic point of f, that is, $f^q(x) = x$, then

$$H(x) = H(f^q(x)) = g^q(H(x)),$$

that is, $H(x) \in Y$ is a q-periodic point of g. Similarly, if $y \in Y$ is a q-periodic point of g, then

$$H^{-1}(y) = H^{-1}(g^q(y)) = f^q(H^{-1}(y)),$$

that is, $H^{-1}(y) \in X$ is a q-periodic point of f. Since the map H is bijective, this yields the desired result.

Problem 2.27 Consider continuous maps $f,g\colon \mathbb{R} \to \mathbb{R}$ that are topologically conjugate via a homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ satisfying $h \circ f = g \circ h$. Show that if p is an attracting fixed point of f (which means that there exists an open neighborhood U of p such that for $x \in U$ we have $f^n(x) \to p$ when $n \to \infty$) if and only if $q = h(p)$ is an attracting fixed point of g.

Solution Clearly, p is a fixed point of f if and only if $q = h(p)$ is a fixed point of g.

Now assume that p is an attracting fixed point of f. Then there exists an open neighborhood U of p such that for $x \in U$ we have $f^n(x) \to p$ when $n \to \infty$. Let $V = h(U)$. Then V is an open neighborhood of q (because h is a homeomorphism). Take $y \in V$ and let $x = h^{-1}(y) \in U$. Then $f^n(x) \to p$ when $n \to \infty$. Since h is continuous, we obtain

$$g^n(y) = g^n(h(x)) = h(f^n(x)) \to h(p) = q$$

when $n \to \infty$. Hence, q is an attracting fixed point of g. The desired statement follows now from interchanging the roles of f and g.

Problem 2.28 Show that if two maps are topologically conjugate and one of them is topologically mixing, then the other is also topologically mixing.

Solution Let $h\colon Y \to X$ be a homeomorphism such that

$$f \circ h = h \circ g$$

for some maps $f\colon X \to X$ and $g\colon Y \to Y$. Now assume that f is topologically mixing. Given nonempty open sets $U, V \subsetneq Y$, the images $h(U)$ and $h(V)$ are also open and so there exists $n \in \mathbb{N}$ such that

$$f^{-m}h(U) \cap h(V) \neq \varnothing \quad \text{for } m \geqslant n.$$

It follows from the identity

$$f^{-m} \circ h = h \circ g^{-m}$$

that

$$\begin{aligned} h(g^{-m}U \cap V) &= h(g^{-m}U) \cap h(V) \\ &= f^{-m}h(U) \cap h(V) \neq \varnothing \end{aligned}$$

and so

$$g^{-m}U \cap V \neq \varnothing \quad \text{for } m \geqslant n.$$

This shows that g is topologically mixing.

Problem 2.29 Given a topologically mixing map $f\colon X \to X$, show that the map $f \times f\colon X \times X \to X \times X$ defined by

$$(f \times f)(x,y) = (f(x), f(y))$$

is also topologically mixing.

Solution Let $U_1, U_2, V_1, V_2 \subseteq X$ be nonempty open sets and define $U = U_1 \times U_2$ and $V = V_1 \times V_2$. Since f is topologically mixing, there exists $m \in \mathbb{N}$ such that

$$f^{-n}U_1 \cap V_1 \neq \varnothing \quad \text{and} \quad f^{-n}U_2 \cap V_2 \neq \varnothing$$

for all $n \geqslant m$. Then

$$(f \times f)^{-n}(U_1 \times U_2) \cap (V_1 \times V_2) = (f^{-n}U_1 \cap V_1) \times (f^{-1}U_2 \cap V_2) \neq \varnothing$$

for all $n \geqslant m$. Since the sets of the form $U \times V$ with U and V open sets in X form a basis for the topology of $X \times X$, we conclude that the map $f \times f$ is topologically mixing.

Problem 2.30 Let $f\colon X \to X$ be a continuous map on a compact metric space (X,d) such that

$$d(f(x), f(y)) \leqslant d(x,y) \quad \text{for all } x,y \in X.$$

Show that the topological entropy of f is zero.

Solution It follows from the definition of the distance d_n in Definition 2.10 that $d_n = d$ for all $n \in \mathbb{N}$. Hence,

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(1,\varepsilon) = 0.$$

Problem 2.31 Show that if $f\colon X \to X$ is a homeomorphism on a compact metric space (X,d), then

$$N_f(n,\varepsilon) = N_{f^{-1}}(n,\varepsilon)$$

for all $n \in \mathbb{N}$ and $\varepsilon > 0$.

Solution Note that

$$d_{n,f}(x,y) = \max\bigl\{d(f^k(x), f^k(y)) : k = 0,\dots,n-1\bigr\}$$

and

$$\begin{aligned} &d_{n,f^{-1}}(f^{n-1}(x), f^{n-1}(y)) \\ &= \max\bigl\{d(f^{-k}(f^{n-1}(x)), f^{-k}(f^{n-1}(y))) : k = 0,\dots,n-1\bigr\} \\ &= d_{n,f}(x,y). \end{aligned}$$

Hence, if some points $x_1,\dots,x_p$, with $p = N_f(n,\varepsilon)$, satisfy $d_{n,f}(x_j,x_j) \geqslant \varepsilon$ for $i \neq j$, then

$$d_{n,f^{-1}}(f^{n-1}(x_i), f^{n-1}(x_j)) \geqslant \varepsilon \quad \text{for } i \neq j.$$

Therefore,

$$N_{f^{-1}}(n,\varepsilon) \geqslant p = N_f(n,\varepsilon).$$

Interchanging the roles of f and f^{-1} we also obtain

$$N_f(n,\varepsilon) \geqslant N_{f^{-1}}(n,\varepsilon)$$

and so $N_f(n,\varepsilon) = N_{f^{-1}}(n,\varepsilon)$.

Problem 2.32 Let $f\colon X \to X$ be a continuous map on a compact metric space. Given a closed forward f-invariant set $Y \subseteq X$, show that $h(f) \geqslant h(f|_Y)$.

Solution Note that $N_f(n,\varepsilon) \geqslant N_{f|_Y}(n,\varepsilon)$. Therefore,

$$\begin{aligned} h(f) &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_f(n,\varepsilon) \\ &\geqslant \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_{f|_Y}(n,\varepsilon) \\ &= h(f|_Y). \end{aligned}$$

Problem 2.33 Let $f\colon X \to X$ and $g\colon Y \to Y$ be continuous maps on compact metric spaces, respectively, (X, d_X) and (Y, d_Y) satisfying $f \circ H = H \circ g$ for some continuous onto map $H\colon Y \to X$. Show that $h(f) \leqslant h(g)$.

Solution It follows from the identity $f \circ H = H \circ g$ that

$$f^n \circ H = H \circ g^n$$

for all $n \in \mathbb{N}$. Since H is uniformly continuous, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_X(H(y), H(y')) < \varepsilon \quad \text{whenever } d_Y(y,y') < \delta.$$

Therefore, if

$$d_X(f^m(x'), f^m(x')) \geqslant \varepsilon$$

for $m = 0, \dots, n-1$, then

$$d_Y(g^m(H^{-1}(x)), g^m(H^{-1}(x'))) \geqslant \delta$$

for $m = 0, \dots, n-1$. This implies that $N_g(n,\delta) \geqslant N_f(n,\varepsilon)$ and so

$$\limsup_{n\to\infty} \frac{1}{n} \log N_g(n,\delta) \geqslant \limsup_{n\to\infty} \frac{1}{n} \log N_f(n,\varepsilon)$$

for each $\varepsilon > 0$. Letting $\varepsilon \to 0$, one can take $\delta \to 0$ and thus $h(g) \geqslant h(f)$.

Problem 2.34 Let $f\colon X \to X$ and $g\colon Y \to Y$ be continuous maps on compact metric spaces satisfying $f \circ H = H \circ g$ for some homeomorphism $H\colon Y \to X$. Show that $h(f) = h(g)$.

Solution It follows from Problem 2.33 that

$$h(f) \leqslant h(g) \quad \text{and} \quad h(g) \leqslant h(f).$$

Therefore, the maps f and g have the same topological entropy.

Problem 2.35 Compute the topological entropy of the map f in Problem 1.21.

Solution Given $n, k \in \mathbb{N}$, consider the points

$$x_j = e^{2\pi i (j/3^{n+k})} \quad \text{for } j = 0, \dots, 3^{n+k} - 1.$$

Then

$$\begin{aligned} d_n(x_j, x_{j+1}) &= d(f^{n-1}(x_j), f^{n-1}(x_{j+1})) \\ &= 3^{n-1} d(x_j, x_{j+1}) \\ &= 3^{-(k+1)} \end{aligned}$$

for $j = 0, \ldots, 3^{n+k} - 1$. Therefore,

$$d_n(x_i, x_j) \geqslant 3^{-(k+1)} \quad \text{for } i \neq j$$

and so

$$N(n, 3^{-(k+1)}) \geqslant 3^{n+k}. \tag{II.2.4}$$

On the other hand, given a set $X \subsetneq R$ with cardinality at least $3^{n+k} + 1$, there exist $x, y \in X$ with $x \neq y$ such that $d(x,y) < 3^{-(n+k)}$. Then $d_n(x,y) < 3^{-(k+1)}$ and so

$$N(n, 3^{-(k+1)}) \leqslant 3^{n+k}. \tag{II.2.5}$$

Combining (II.2.4) and (II.2.5) we obtain

$$N(n, 3^{-(k+1)}) = 3^{n+k} \quad \text{for } n, k \in \mathbb{N},$$

which gives

$$\begin{aligned} h(f) &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon) \\ &= \lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log N(n, 3^{-(k+1)}) \\ &= \lim_{k \to \infty} \limsup_{n \to \infty} \frac{n+k}{n} \log 3 \\ &= \log 3. \end{aligned}$$

Problem 2.36 Determine whether there exists a continuous map $f\colon X \to X$ on a compact metric space with infinite topological entropy.

Solution Consider the continuous map $f = E_2 \times E_3 \times \cdots$ on the compact topological space $X = \prod_{m \geqslant 2} S^1$ with the product topology. We note that X is metrizable. For example, let n be the smallest integer such that $x_n \neq y_n$, writing $x = (x_2, x_3, \ldots)$ and $y = (y_2, y_3, \ldots)$, and consider the distance

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-n} & \text{if } x \neq y. \end{cases}$$

For each $m \geqslant 2$ the set

$$X_m = \{x \in X : x_p = 0 \text{ for } p \neq m\}$$

is compact and forward f-invariant. Moreover, for the homeomorphism $H\colon X_m \to S^1$ defined by $H(x) = x_m$ we have

$$H \circ f = E_m \circ H \quad \text{on } X_m$$

and so H is a topological conjugacy between $f|X_m$ and E_m. Hence, it follows from Theorem 2.12 (or Problem 2.33) that

$$h(f) \geqslant h(f|X_m) = h(E_m) = \log m.$$

Since m is arbitrary, we conclude that $h(f) = \infty$.

Problem 2.37 Show that if the distances d and d' generate both the topology of a compact topological space X, then

$$h_d(f) = h_{d'}(f)$$

for any continuous map $f\colon X \to X$.

Solution Let

$$A_\varepsilon = \{(x,y) \in X \times X : d(x,y) \geqslant \varepsilon\}.$$

Note that

$$\delta(\varepsilon) := \min\{d'(x,y) : (x,y) \in A_\varepsilon\} \searrow 0$$

when $\varepsilon \searrow 0$. If $d_n(x,y) \geqslant \varepsilon$, then there exists an integer $i \in \{0,\dots,n-1\}$ such that $(f^i(x), f^i(y)) \in A_\varepsilon$ and so

$$d'_n(x,y) \geqslant d'(f^i(x), f^i(y)) \geqslant \delta(\varepsilon).$$

Therefore,

$$N_{d'}(n,\delta(\varepsilon)) \geqslant N_d(n,\varepsilon)$$

(making explicit the distance that is considered) and so

$$\begin{aligned} h_{d'}(f) &= \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_{d'}(n,\delta) \\ &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_{d'}(n,\delta(\varepsilon)) \\ &\geqslant \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_d(n,\varepsilon) \\ &= h_d(f). \end{aligned}$$

Interchanging d and d' we also obtain $h_d(f) \geqslant h_{d'}(g)$ and so $h_d(f) = h_{d'}(f)$.

Problem 2.38 Let $f\colon X \to X$ be a continuous map on a compact metric space (X,d). Show that:

1. if $\mathcal{U}$ is a finite open cover of X, then letting

$$\mathcal{U}_n = \left\{ \bigcap_{k=0}^{n-1} f^{-k} U_k : U_0, \ldots, U_{n-1} \in \mathcal{U} \right\}$$

and denoting by $N(\mathcal{U}_n)$ the smallest cardinality of the finite subcovers of $\mathcal{U}_n$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}_n)$$

exists;

2. if $\mathcal{U}$ is a finite open cover of X with Lebesgue number δ (this is a number such that any open ball of radius δ is contained in some element of $\mathcal{U}$), then

$$M(n, \delta/2) \geqslant N(\mathcal{U}_n);$$

3. if $\mathcal{U}$ is a finite open cover of X with $\operatorname{diam} \mathcal{U} := \sup\{\operatorname{diam} U : U \in \mathcal{U}\} < \varepsilon$, where

$$\operatorname{diam} U = \{d(x, y) : x, y \in U\},$$

then $N(n, \varepsilon) \leqslant N(\mathcal{U}_n)$;

4.
$$h(f) = \lim_{\operatorname{diam} \mathcal{U} \to 0} \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}_n).$$

Solution 1. Note that the elements of $\mathcal{U}_{n+m}$ are the open sets

$$\begin{aligned} \bigcap_{k=0}^{n+m-1} f^{-k} U_k &= \bigcap_{k=0}^{n-1} f^{-k} U_k \cap f^{-n} \left(\bigcap_{l=0}^{m-1} f^{-l} U_{l+n} \right) \\ &\in \left\{ U \cap f^{-n} V : U \in \mathcal{U}_n, V \in \mathcal{U}_m \right\}, \end{aligned}$$

with $U_0, \ldots, U_{n+m-1} \in \mathcal{U}$. Therefore,

$$N(\mathcal{U}_{n+m}) \leqslant N(\mathcal{U}_n) N(\mathcal{U}_m). \tag{II.2.6}$$

In other words, the sequence $c_n = \log N(\mathcal{U}_n)$ is subadditive, that is,

$$c_{n+m} \leqslant c_n + c_m \quad \text{for all } n, m \in \mathbb{N}. \tag{II.2.7}$$

We show that for any such sequence the limit

$$\lim_{n\to\infty} \frac{c_n}{n} = \inf_{n\in\mathbb{N}} \frac{c_n}{n} \tag{II.2.8}$$

exists. Given $n, k \in \mathbb{N}$, write $n = qk + r$ with $q \in \mathbb{N} \cup \{0\}$ and $r \in \{0, \ldots, q-1\}$. Then

$$\frac{c_n}{n} \leqslant \frac{c_{qk} + c_r}{qk + r} \leqslant \frac{qc_k + c_r}{qk + r}$$

and so

$$\limsup_{n\to\infty} \frac{c_n}{n} \leqslant \frac{c_k}{k}$$

since $q \to \infty$ when $n \to \infty$ (for a fixed k). Since k is arbitrary, this implies that

$$\limsup_{n\to\infty} \frac{c_n}{n} \leqslant \inf_{k\in\mathbb{N}} \frac{c_k}{k} \leqslant \liminf_{n\to\infty} \frac{c_n}{n},$$

which establishes property (II.2.8). It follows readily from (II.2.6) and (II.2.8) that the limit in item 1 exists.

2. Given $n \in \mathbb{N}$ and $\varepsilon > 0$, let $B_n(p_i, \varepsilon)$ be an open ball in the distance d_n. It follows from the definition of $M(n,\varepsilon)$ that

$$\bigcup_{i=1}^{M(n,\varepsilon)} B_n(p_i,\varepsilon) = X.$$

On the other hand, by the definition of d_n we have

$$B_n(p_i,\varepsilon) = \bigcap_{k=0}^{n-1} f^{-k} B(f^k(p_i),\varepsilon)$$

and so

$$X = \bigcup_{i=1}^{M(n,\varepsilon)} \bigcap_{k=0}^{n-1} f^{-k} B(f^k(p_i),\varepsilon).$$

Now let δ be a Lebesgue number of the open cover $\mathcal{U}$ (in particular, any open ball of radius $\delta/2$ is contained in some element of the cover). Then

$$B(f^k(p_i),\delta/2) \subseteq U_k^i$$

for some open set $U_k^i \in \mathcal{U}$. Therefore, for each $i = 1,\dots,M(n,\delta/2)$ we have

$$\bigcap_{k=0}^{n-1} f^{-k} B(f^k(p_i),\delta/2) \subseteq \bigcap_{k=0}^{n-1} f^{-k} U_k^i \in \mathcal{U}_n$$

and so $M(n,\delta/2) \geqslant N(\mathcal{U}_n)$.

3. Given $n \in \mathbb{N}$ and $\varepsilon > 0$, let $A_n \subsetneq X$ be a set with cardinality $N(n,\varepsilon)$ such that

$$d_n(p_i,p_j) \geqslant \varepsilon \quad \text{for all } p_i,p_j \in A_n \text{ with } i \neq j.$$

Since $\operatorname{diam}\mathcal{U} < \varepsilon$, for each $n \in \mathbb{N}$ no element of $\mathcal{U}_n$ can contain two distinct points $p_i, p_j \in A_n$ because

$$V_l := f^l\left(\bigcap_{k=0}^{n-1} f^{-k} U_k\right) \subseteq U_l$$

and so

$$\operatorname{diam} V_l \leqslant \operatorname{diam} U_l < \varepsilon.$$

Therefore, $N(n,\varepsilon) \leqslant N(\mathcal{U}_n)$.

4. Now let $\mathcal{U}$ be an open cover of X with Lebesgue number δ. By items 1 and 2 we have

$$\lim_{n\to\infty}\frac{1}{n}\log N(\mathcal{U}_n)\leqslant\limsup_{n\to\infty}\frac{1}{n}\log M(n,\delta/2).$$

Note that $\delta\to 0$ when $\operatorname{diam}\mathcal{U}\to 0$. Hence,

$$\begin{aligned}\limsup_{\operatorname{diam}\mathcal{U}\to 0}\lim_{n\to\infty}\frac{1}{n}\log N(\mathcal{U}_n)&\leqslant\limsup_{\operatorname{diam}\mathcal{U}\to 0}\limsup_{n\to\infty}\frac{1}{n}\log M(n,\delta/2)\\&=\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log M(n,\delta/2)\\&=h(f).\end{aligned}\tag{II.2.9}$$

Now let $\mathcal{U}$ be an open cover of X with $\operatorname{diam}\mathcal{U}<\varepsilon$. By items 1 and 3 we have

$$\limsup_{n\to\infty}\frac{1}{n}\log N(n,\varepsilon)\leqslant\lim_{n\to\infty}\frac{1}{n}\log N(\mathcal{U}_n)$$

and so

$$\limsup_{n\to\infty}\frac{1}{n}\log N(n,\varepsilon)\leqslant\inf_{\operatorname{diam}\mathcal{U}<\varepsilon}\lim_{n\to\infty}\frac{1}{n}\log N(\mathcal{U}_n).$$

Hence,

$$\begin{aligned}h(f)&=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N(n,\varepsilon)\\&\leqslant\lim_{\varepsilon\to 0}\inf_{\operatorname{diam}\mathcal{U}<\varepsilon}\lim_{n\to\infty}\frac{1}{n}\log N(\mathcal{U}_n)\\&=\liminf_{\operatorname{diam}\mathcal{U}\to 0}\lim_{n\to\infty}\frac{1}{n}\log N(\mathcal{U}_n).\end{aligned}\tag{II.2.10}$$

Combining (II.2.9) and (II.2.10), we conclude that the identity in item 4 holds.

Problem 2.39 Let $f\colon[0,1]\to[0,1]$ be a homeomorphism. Show that $h(f)=0$.

Solution Let $\mathcal{U}$ be an open cover of $[0,1]$ by open intervals (more precisely, by sets of the form $(a,b)\cap[0,1]$ with $a,b\in\mathbb{R}$). Since f is strictly increasing or strictly decreasing, the cover $f^{-1}\mathcal{U}$ is also composed of open intervals, with

$$\operatorname{card}f^{-1}\mathcal{U}=\operatorname{card}\mathcal{U}.$$

Hence, $\operatorname{card}\mathcal{U}_2\leqslant 2\operatorname{card}\mathcal{U}$ (see Problem 2.38). One can show by induction that

$$\operatorname{card}\mathcal{U}_n\leqslant n\operatorname{card}\mathcal{U}\tag{II.2.11}$$

and so

$$\lim_{n\to\infty}\frac{1}{n}\log\operatorname{card}\mathcal{U}_n=0.\tag{II.2.12}$$

By Problem 2.38, given $\varepsilon>0$ and an open cover $\mathcal{U}$ with $\operatorname{diam}\mathcal{U}<\varepsilon$, we have

$$N(n,\varepsilon) \leqslant N(\mathcal{U}_n) \leqslant \operatorname{card} \mathcal{U}_n$$

for each $n \in \mathbb{N}$. Hence, it follows from (II.2.12) that

$$\limsup_{n\to\infty} \frac{1}{n} \log N(n,\varepsilon) = 0$$

for each ε (taking a cover $\mathcal{U}$ by open intervals of length less than ε) and so $h(f) = 0$.

Problem 2.40 Let $f\colon S^1 \to S^1$ be a homeomorphism. Show that $h(f) = 0$.

Solution As in Problem 2.39, let $\mathcal{U}$ be an open cover of S^1 by open intervals. Since f is strictly increasing or strictly decreasing, the cover $f^{-1}\mathcal{U}$ is again composed of open intervals, with

$$\operatorname{card} f^{-1}\mathcal{U} = \operatorname{card} \mathcal{U}.$$

Hence, inequality (II.2.11) holds, which yields property (II.2.12). One can now show as in Problem 2.39 that $h(f) = 0$.

Chapter II.3
Low-Dimensional Dynamics

Problem 3.1 Show that the composition $R_\alpha \circ R_\beta$ of two rotations of the circle $R_\alpha, R_\beta \colon S^1 \to S^1$ is also a rotation of the circle.

Solution We have

$$R_\alpha[x] = [x+\alpha] \quad \text{and} \quad R_\beta[x] = [x+\beta]$$

for each $[x] \in S^1$. Hence,

$$(R_\alpha \circ R_\beta)[x] = R_\alpha[x+\beta] = [x+\alpha+\beta]$$

for each $[x] \in S^1$ and so $R_\alpha \circ R_\beta = R_{\alpha+\beta}$.

Problem 3.2 Show that given $n \times n$ matrices A and B with entries in $\mathbb{Z}$, the composition of the endomorphisms of the torus $T_A, T_B \colon \mathbb{T}^n \to \mathbb{T}^n$ is again an endomorphism of the torus and determine a matrix C such that $T_A \circ T_B = T_C$.

Solution We have

$$T_A[x] = [Ax] \quad \text{and} \quad T_B[x] = [Bx]$$

for each $[x] \in \mathbb{T}^n$. Hence,

$$(T_A \circ T_B)[x] = T_A[Bx] = [ABx]$$

for each $[x] \in \mathbb{T}^n$ and so $T_A \circ T_B = T_C$ with $C = AB$.

Problem 3.3 Consider the homeomorphism of the circle defined by

$$f([x]) = \left[x + \frac{1}{2} + \frac{1}{4\pi}\sin(2\pi x)\right]$$

(see Figure I.3.1). Show that $\{0, 1/2\}$ is a periodic orbit of f and that $\rho(f) = 1/2$.

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_9

Solution We have

$$f(0) = \frac{1}{2} \quad \text{and} \quad f\left(\frac{1}{2}\right) = 1.$$

Hence, $\{0, 1/2\}$ is a periodic orbit of f with period 2. On the other hand, the function $F\colon \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = x + \frac{1}{2} + \frac{1}{4\pi}\sin(2\pi x)$$

is a lift of f and satisfies

$$F^n(0) = \frac{n}{2} \quad \text{for } n \geqslant 0.$$

Indeed, $F(0) = 1/2$ and if $F^k(0) = k/2$, then

$$\begin{aligned} F^{k+1}(0) &= F(F^k(0)) = F(k/2) \\ &= \frac{k}{2} + \frac{1}{2} = \frac{k+1}{2}. \end{aligned}$$

Taking $x = 0$ in

$$\rho(F) = \lim_{n\to\infty} \frac{F^n(x) - x}{n},$$

we obtain

$$\rho(F) = \lim_{n\to\infty} \frac{F^n(0)}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

This implies that $\rho(f) = 1/2$.

Problem 3.4 Given $a \in (0, 1/(2\pi))$, consider the map $f\colon S^1 \to S^1$ defined by

$$f([x]) = [x + a\sin(2\pi x)]$$

(see Figure I.3.2). Show that each orbit starting in $(0, 1/2)$ is monotonous.

Solution Define a map $F\colon \mathbb{R} \to \mathbb{R}$ by

$$F(x) = x + a\sin(2\pi x).$$

For $0 < x < 1/2$ we have $\sin(2\pi x) > 0$ and so $F(x) > x$. Moreover,

$$F'(x) = 1 + 2\pi a\cos(2\pi x) > 1 - 2\pi a > 0.$$

Therefore, $F(x) < F(1/2) = 1/2$ and so

$$x < F(x) < \frac{1}{2}.$$

Proceeding inductively we obtain

$$0 < x < F(x) < F^2(x) < \cdots < F^n(x) < \frac{1}{2},$$

which implies that the orbit of $[x]$ is monotonous.

Problem 3.5 Let $f\colon S^1 \to S^1$ be a homeomorphism with a single periodic orbit. Show that any other orbit is asymptotic to this periodic orbit (see Figure I.3.3 for an example).

Solution Let x_0 be a periodic point of f with period p and let $U_0, \dots, U_{p-1}$ be the connected component of the set $S^1 \backslash \gamma(x_0)$. The sets U_i are open intervals and, without loss of generality, we can assume that

$$f(U_k) = U_{k+1} \text{ for } k = 0, \dots, p-1 \quad \text{and} \quad f(U_p) = U_0.$$

Hence, $f^p(U_k) = U_k$ for each k and since f is a homeomorphism, the map f^p is monotonous on each interval U_k. It follows from the continuity of f that

$$d(f^k(x), \gamma(x_0)) = \min\{d(f^k(x), f^l(x_0)) : l = 0, \dots, p-1\} \to 0$$

when $k \to \pm\infty$.

Problem 3.6 Let $f\colon S^1 \to S^1$ be an orientation-reversing homeomorphism and let F be a lift of f. Show that $F(x+1) = F(x) - 1$ for all $x \in \mathbb{R}$ (see Figures I.3.4 and I.3.5).

Solution Define a map $g\colon S^1 \to S^1$ by $g(x) = f(-x)$. Note that g is an orientation-preserving homeomorphism since it is the composition of two orientation-reversing homeomorphisms. Moreover, the map $G\colon \mathbb{R} \to \mathbb{R}$ given by $G(x) = F(-x)$ is a lift of g and so

$$G(x+1) = G(x) + 1 \quad \text{for all } x \in \mathbb{R}.$$

Therefore, for each $x \in X$ we have

$$F(x+1) = G(-x-1) = G(-x) - 1 = F(x) - 1.$$

Problem 3.7 Let $f\colon S^1 \to S^1$ be an orientation-reserving homeomorphism. Show that $\rho(f^2) = 0$.

Solution We first show that f has fixed points. Take $x \in [0,1)$ and let F be a lift of f with $F(0) \in [0,1)$. Note that $f(x) = x$ if and only if $F(x) - x \in \mathbb{Z}$. Since f is an orientation-reversing homeomorphism, it follows from Problem 3.6 that $F(1) = F(0) - 1$. Therefore, the continuous function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x) - x$ satisfies

$$g(1) = g(0) - 2. \tag{II.3.1}$$

Moreover, since F is decreasing (again because f is an orientation-reversing homeomorphism), the map g is also decreasing and so it follows from (II.3.1) that it takes exactly two integer values on $[0,1)$. This shows that f has fixed points.

Now let G be a lift of f^2. Since f has fixed points, the same happens with f^2. Hence, there exists $x \in S^1$ such that $G(x) - x \in \mathbb{Z}$ and so

$$G^n(x) = x + nk \quad \text{for all } n \in \mathbb{N}.$$

Therefore,

$$\rho(G) = \lim_{n\to\infty} \frac{G^n(x)-x}{n} = k,$$

which implies that $\rho(f^2) = \{\rho(G)\} = 0$.

Problem 3.8 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism with $\rho(f) = p/q$ for some integers $p,q \in \mathbb{N}$. Show that for each lift F of f there exists $x \in \mathbb{R}$ such that $F(x) - x - p/q \in \mathbb{Z}$.

Solution We proceed by contradiction. Assume that there exists a lift F of f such that $F(x) - x - p/q \in \mathbb{R}\setminus\mathbb{Z}$ for all $x \in \mathbb{R}$. Since F is continuous, there exists $k \in \mathbb{Z}$ such that

$$k < F(x) - x - \frac{p}{q} < k+1 \quad \text{for } x \in \mathbb{R}.$$

On the other hand,

$$F(x+1) - (x+1) = F(x) - x \quad \text{for } x \in \mathbb{R}$$

and since $[0,1]$ is compact, there exists $\varepsilon > 0$ such that

$$k + \varepsilon \leqslant F(x) - x - \frac{p}{q} \leqslant k + 1 - \varepsilon$$

for all $x \in \mathbb{R}$. Hence, it follows from the identity

$$F^n(x) - x = \sum_{i=0}^{n-1} [F(F^i(x)) - F^i(x)]$$

that

$$k + \frac{p}{q} + \varepsilon \leqslant \frac{F^n(x)-x}{n} \leqslant k + \frac{p}{q} + 1 - \varepsilon.$$

Thus,

$$\rho(f) = \left\{ \lim_{n\to\infty} \frac{F^n(x)-x}{n} \right\} \in \left[\frac{p}{q} + \varepsilon, \frac{p}{q} + 1 - \varepsilon \right],$$

which contradicts the hypothesis that $\rho(f) = p/q$. This shows that there exist $x \in \mathbb{R}$ such that $F(x) - x - p/q \in \mathbb{Z}$.

Problem 3.9 Let $f,g\colon S^1 \to S^1$ be homeomorphisms. Show that if F and G are lifts, respectively, of f and g, then

$$\lim_{n\to\infty} \frac{F(G^n(x)) - G^n(x)}{n} = 0$$

for each $x \in \mathbb{R}$.

Solution Note that

$$x_n := G^n(x) - \{G^n(x)\} \in [0,1]$$

for each $n \in \mathbb{N}$. Since F is a lift, we obtain

$$F(G^n(x)) - G^n(x) = F(x_n) - x_n.$$

On the other hand, since $[0,1]$ is compact, there exists a constant $M > 0$ (independent of x and n) such that

$$|F(G^n(x)) - G^n(x)| \leqslant |F(x_n)| + |x_n| \leqslant M + 1,$$

which readily yields the desired property.

Problem 3.10 Let $f, g \colon S^1 \to S^1$ be homeomorphisms and let F and G be lifts, respectively, of f and g. Show that if f is an orientation-preserving homeomorphism, then:

1. for each $k \in \mathbb{N}$ we have
$$\lim_{n\to\infty} \frac{F^n(x) - x}{n} = \lim_{n\to\infty} \frac{F^n(x_n) - x_n}{n}$$
for any $x, x_n \in \mathbb{R}$ with $|x - x_n| \leqslant k$ for $n \in \mathbb{N}$;
2. $$\lim_{n\to\infty} \frac{F^n(G^n(x)) - G^n(x)}{n} = \lim_{n\to\infty} \frac{F^n(x) - x}{n}$$
for each $x \in \mathbb{R}$.

Solution 1. If $|x - x_n| \leqslant k$ for some $k \in \mathbb{N}$, then

$$F(x) \leqslant F(x_n + k) = F(x_n) + k$$

and

$$F(x) \geqslant F(x_n - k) = F(x_n) - k.$$

Hence, $|F(x) - F(x_n)| \leqslant k$ and it follows by induction that

$$|F^m(x) - F^m(x_n)| \leqslant k$$

for each $m \in \mathbb{N}$. In particular, taking $m = n$ yields the inequality

$$|F^n(x) - F^n(x_n)| \leqslant k.$$

Since

$$\frac{F^n(x) - x}{n} - \frac{F^n(x_n) - x_n}{n} = \frac{F^n(x) - F^n(x_n)}{n} - \frac{x - x_n}{n},$$

we obtain

$$\left| \frac{F^n(x) - x}{n} - \frac{F^n(x_n) - x_n}{n} \right| \leqslant \frac{2k}{n} \to 0$$

when $n \to \infty$. This establishes the property in item 1.

2. Now take $x_n = \{G^n(x)\}$. For each $x \in \mathbb{R}$ we have

$$|x - x_n| \leqslant |x| + 1 =: k$$

and so it follows from item 1 that

$$\begin{aligned}\lim_{n\to\infty}\frac{F^n(G^n(x))-G^n(x)}{n} &= \lim_{n\to\infty}\frac{F^n(x_n)-x_n}{n}\\ &= \lim_{n\to\infty}\frac{F^n(x)-x}{n}.\end{aligned}$$

Problem 3.11 Let $f\colon S^1\to S^1$ be an orientation-preserving homeomorphism with

$$r/s<\rho(f)<p/q<1$$

for some positive integers r,s,p,q. Show that:

1. any lift F of f with $F(0)\in(0,1)$ satisfies
$$F^q(x)<x+p \quad\text{and}\quad F^s(x)>x+r$$
for all $x\in\mathbb{R}$;
2. if $g\colon S^1\to S^1$ is an orientation-preserving homeomorphism with
$$d(f,g):=\max_{x\in S^1} d(f(x),g(x))$$
sufficiently small, then $r/s<\rho(g)<p/q$.

Solution 1. Assume that $F^q(x)\geqslant x+p$ for some $x\in\mathbb{R}$. Since

$$F^q(x+p)=F^q(x)+p$$

and F is increasing, we obtain

$$\begin{aligned}F^{2q}(x)&=F^q(F^q(x))\geqslant F^q(x+p)\\ &=F^q(x)+p\geqslant x+2p\end{aligned}$$

and it follows by induction that

$$F^{nq}(x)\geqslant x+np \quad \text{for } n\in\mathbb{N}.$$

Thus,

$$\rho(F)=\lim_{n\to\infty}\frac{F^{nq}(x)-x}{nq}\geqslant\lim_{n\to\infty}\frac{np}{nq}=\frac{p}{q}. \tag{II.3.2}$$

On the other hand, since $F(0)\in(0,1)$, we have $\rho(F)\in[0,1)$ and so it follows from (II.3.2) that $\rho(f)\geqslant p/q$, which contradicts the fact that $\rho(f)<p/q$. This shows that $F^q(x)<x+p$. The second inequality in item 1 can be obtained analogously.

2. Since $F^q-\mathrm{id}$ is periodic and continuous, it has a maximum. Hence, it follows from the first inequality in item 1 that there exists $\delta>0$ such that

$$F^q(x)<x+p-\delta \quad \text{for all } x\in\mathbb{R}.$$

For $d(f,g)$ sufficiently small we have

$$G^q(x) < x + p - \frac{\delta}{2}$$

for all $x \in \mathbb{R}$ and some lift G of g with $G(0) \in (0,1)$. Therefore,

$$\begin{aligned}\rho(G) &= \lim_{n\to\infty} \frac{G^{qn}(x) - x}{nq} \\ &= \lim_{n\to\infty} \frac{1}{nq} \sum_{k=0}^{n-1} (G^q(G^{kq}(x)) - G^{kq}(x)) \\ &\leqslant \lim_{n\to\infty} \frac{1}{nq} n\Big(p - \frac{\delta}{2}\Big) \\ &= \frac{p}{q} - \frac{\delta}{2q} < \frac{p}{q},\end{aligned}$$

which implies that $\rho(g) < p/q$.

Analogously, it follows from the second inequality in item 1 that there exists $\mu > 0$ such that

$$F^s(x) > x + r + \mu \quad \text{for all } x \in \mathbb{R}.$$

Provided that $d(f,g)$ is sufficiently small, some lift G of g with $G(0) \in (0,1)$ satisfies

$$G^s(x) > x + r + \frac{\mu}{2}$$

for all $x \in \mathbb{R}$, which implies in a similar manner that $\rho(g) > r/s$.

Problem 3.12 Let $f: S^1 \to S^1$ be an orientation-preserving homeomorphism. Show that if $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$, then for any $m,n \in \mathbb{Z}$ with $m \neq n$ we have

$$S^1 = \bigcup_{k=0}^{\infty} f^{-k} I, \quad \text{for } I = [f^n(x), f^m(x)].$$

Solution We proceed by contradiction. Assume that $S^1 \neq \bigcup_{k=0}^{\infty} f^{-k} I$. Then

$$S^1 \neq \bigcup_{k=1}^{\infty} f^{-k(m-n)} I = \bigcup_{k=1}^{\infty} [f^{-km+(k+1)n}(x), f^{-(k-1)m+kn}(x)].$$

Since the intervals $f^{-k(m-n)}I$, for $k \in \mathbb{N}$, have adjacent endpoints, the sequence

$$x_k = f^{-k(m-n)}(f^n(x))$$

converges to some point $z \in S^1$ when $k \to \infty$. But z is a fixed point of f^{m-n} (since $f^{m-n}(x_k) = x_{k-1}$ has the same limit), thus contradicting the fact that $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$ (see Theorem 3.6).

Problem 3.13 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism. Show that if $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$, then $\omega(x) = \omega(y)$ for any $x, y \in S^1$.

Solution Take $x, y \in S^1$ and assume that $f^{m_n}(x) \to x_0 \in \omega(x)$ when $n \to \infty$ for some sequence $m_n \nearrow \infty$. By Problem 3.12, for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that

$$f^{k_n}(y) \in [f^{m_{n-1}}(x), f^{m_n}(x)].$$

Then $f^{k_n}(y) \to x_0$ when $n \to \infty$ and it follows from Proposition 2.4 that $\omega(x) \subseteq \omega(y)$. Interchanging x and y we also obtain $\omega(y) \subseteq \omega(x)$ and so $\omega(x) = \omega(y)$.

Problem 3.14 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism. Show that if there exists an open interval $J \subsetneq S^1$ such that the sets $f^k(J)$, for $k \in \mathbb{Z}$, are pairwise disjoint, then f is not conjugate to an irrational rotation.

Solution We proceed by contradiction. Let $h\colon S^1 \to S^1$ be a homeomorphism such that $f \circ h = h \circ R_\alpha$ for some irrational number α. If D is a dense subset of S^1, then $h(D)$ is also dense. Thus, if f is conjugate to R_α, then $\overline{\gamma^+(x)} = S^1$ for every $x \in S^1$ and so $\omega(x) = S^1$. On the other hand, if there exists an interval J as in the statement of the problem, then $\omega(x)$ has no interior points and so it cannot be S^1.

Problem 3.15 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism with $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$. Show that if there exists a minimal set $K \neq S^1$ (see Problem 2.14 for the definition), then f is not conjugate to a rotation.

Solution If f is conjugate to a rotation, then proceeding as in Problem 3.14 one can show that $\overline{\gamma^+(x)} = S^1$. Hence, by Problem 2.16, there exists no minimal set $K \neq S^1$.

Problem 3.16 Show that the map $h\colon S^1 \to S^1$ given by

$$h(x) = \sin^2(\pi x/2)$$

is a topological conjugacy between the maps $f, g\colon S^1 \to S^1$ defined by

$$f(x) = 1 - |2x - 1| \quad \text{and} \quad g(x) = 4x(1-x)$$

(see Figures I.3.6 and I.3.7).

Solution Clearly, h is continuous. Moreover, $h(0) = 0$, $h(1) = 1$ and since

$$h'(x) = \frac{\pi}{2}\sin(\pi x) > 0 \quad \text{for } x \in (0,1),$$

the map h is one-to-one and onto (hence invertible). Since its domain is a compact set, the images of closed sets are closed sets and so the inverse of h is continuous. Therefore, h is a homeomorphism. Moreover,

$$\begin{aligned}h(f(x)) &= \sin^2\left(\frac{\pi}{2}(1-|2x-1|)\right)\\ &= \cos^2\left(\frac{\pi}{2}|2x-1|\right)\\ &= \cos^2\left(\frac{\pi}{2}(2x-1)\right)\\ &= \cos^2\left(\pi x-\frac{\pi}{2}\right) = \sin^2(\pi x)\end{aligned}$$

and

$$\begin{aligned}g(h(x)) &= 4h(x)(1-h(x))\\ &= 4\sin^2\left(\frac{\pi}{2}x\right)\left(1-\sin^2\left(\frac{\pi}{2}x\right)\right)\\ &= 4\sin^2\left(\frac{\pi}{2}x\right)\cos^2\left(\frac{\pi}{2}x\right)\\ &= \left(2\sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}x\right)\right)^2 = \sin^2(\pi x),\end{aligned}$$

which shows that $h\circ f = g\circ h$.

Problem 3.17 Determine whether the maps E_4 and $R_{1/2}$ are topologically conjugate.

Solution By Problem 2.26, if the maps E_4 and $R_{1/2}$ were topologically conjugate, then they would have the same number of fixed points. But while the expanding map E_4 has fixed points, such as the origin, the rotation $R_{1/2}$ has none. Therefore, the maps are not topologically conjugate.

Problem 3.18 Determine for which values of $\alpha \in [0,1)$ the rotations R_α and $R_{2\alpha}$ are topologically conjugate.

Solution If R_α and $R_{2\alpha}$ are topologically conjugate, then $\rho(R_\alpha) = \rho(R_{2\alpha})$. Since

$$\rho(R_\alpha) = \{\alpha\} \quad \text{and} \quad \rho(R_\alpha) = \{2\alpha\},$$

we obtain

$$\alpha \equiv 2\alpha \bmod 1, \quad \text{that is,} \quad \alpha \equiv 0 \bmod 1.$$

But since $\alpha \in [0,1)$, we conclude that $\alpha = 0$. In particular, for $\alpha \in (0,1)$ the maps R_α and $R_{2\alpha}$ are not topologically conjugate. On the other hand, for $\alpha = 0$ we have

$$R_\alpha = R_{2\alpha} = \mathrm{id}$$

and so the maps are topologically conjugate.

Problem 3.19 Show that the map f in Problem 3.16 is topologically mixing.

Solution For each $n \in \mathbb{N}$ the graph of f^n is piecewise linear with

$$f^n\left(\frac{i}{2^n}\right) = \begin{cases}0 & \text{if } i = 0,2,\dots,2^n-2,\\ 1 & \text{if } i = 1,3,\dots,2^n-1\end{cases} \tag{II.3.3}$$

(see Figure II.3.1 for the graph of f^2). Now let $U \subsetneq S^1$ be a nonempty open set. Then there exists $n \in \mathbb{N}$ and $i \in \{0, 1, \dots, 2^n - 1\}$ such that

$$I = \left(\frac{i}{2^n}, \frac{i+1}{2^n} \right) \subseteq U.$$

In view of (II.3.3), for each $m \geqslant n$ we have $f^m(I) = S^1$ and so also $f^m(U) = S^1$. In particular,

$$f^m(U) \cap V \neq \varnothing$$

for any nonempty open set $V \subsetneq S^1$. Now take $x \in f^m(U) \cap V$. Then there exists $y \in U$ such that $f^m(y) = x$ and so

$$y \in f^{-m} V \cap U \neq \varnothing \quad \text{for all } m \geqslant n.$$

Hence, f is topologically mixing.

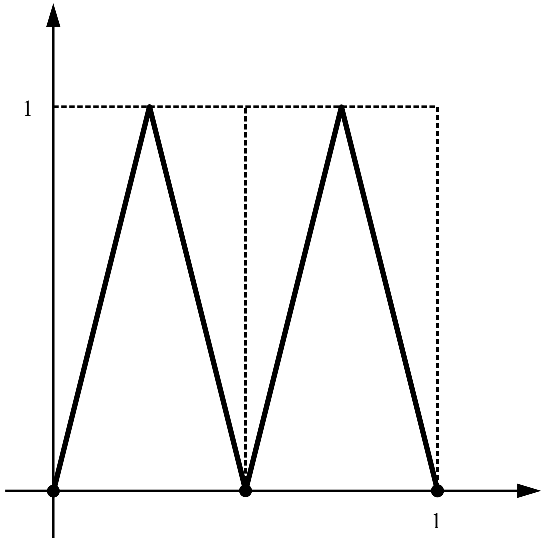

Fig. II.3.1 Graph of the map f^2.

Problem 3.20 Show that the map $g\colon S^1 \to S^1$ given by $g(x) = 4x(1-x)$ is topologically mixing.

Solution By Problem 3.16, the map g is topologically conjugate to the map f given by $f(x) = 1 - |2x - 1|$. On the other hand, by Problem 3.19, the latter is topologically mixing and so it follows from Problem 2.28 that g is also topologically mixing.

Problem 3.21 Consider an interval $I = (a,b) \subsetneq S^1$ with $0 < a < b < 1$. For the map g in Problem 3.20, show that there exists $n \in \mathbb{N}$ such that $g^n(I) = S^1$, identifying I with $\{[x] : x \in I\} \subsetneq S^1$.

Solution Again by Problem 3.16, the map g is topologically conjugate to the map f given by $f(x) = 1 - |2x - 1|$. Let $h\colon S^1 \to S^1$ be a homeomorphism such that

$$f \circ h = h \circ g \quad \text{on } S^1.$$

On the other hand, it is shown in Problem 3.19 that there exists $n \in \mathbb{N}$ such that $f^n(h(I)) = S^1$ (it suffices to take n such that $h(I)$ contains an interval of the form $(i/2^n, (i+1)/2^n)$ for some i). Then

$$S^1 = f^n(h(I)) = h(g^n(I))$$

and so $g^n(I) = S^1$ (recall that h is bijective).

Problem 3.22 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism and let $NW(f)$ be the nonwandering set for f. Show that if $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$, then $NW(f)$ is a minimal set.

Solution By Problem 2.9, $NW(f)$ is closed and forward f-invariant. To show that it is a minimal set, we must verify that it has no nonempty closed f-invariant proper subsets. We proceed by contradiction. Assume that $M \subsetneq NW(f)$ is a nonempty closed forward f-invariant subset. Then $S^1\backslash M$ is a nonempty open set. Let $I = (a,b)$ be a connected component of $S^1\backslash M$. We claim that

$$f^n(I) \cap I = \varnothing \quad \text{for all } n \in \mathbb{N}.$$

Otherwise it would exist $n \in \mathbb{N}$ such that $f^n(I) \cap I \neq \varnothing$ and so $f^{-n}(I) \cap I \neq \varnothing$. But since $S^1\backslash M$ is backward f-invariant, this implies that $f^{-n}(I) \subseteq I$ because otherwise I could not be a connected component. Then $f^n(I) \supseteq I$ and so it follows from Problem 1.5 that f has periodic points, which contradicts the fact that $\rho(f)$ is irrational. Therefore, $I \subseteq S^1\backslash NW(f)$ and since $S^1\backslash M$ is the union of its connected components, we obtain

$$S^1\backslash M \subseteq S^1\backslash NW(f).$$

Hence, $NW(f) \subseteq M$ and so $NW(f) = M$.

Problem 3.23 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism. Show that if $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$, then either $NW(f) = S^1$ or $NW(f)$ is a Cantor set (that is, a closed set with empty interior and without isolated points).

Solution It follows from Problems 2.16 and 3.22 that $NW(f)$ is the ω-limit set of each of its points. Together with Problem 2.9 this implies that $NW(f)$ is a closed set without isolated points. Indeed, $NW(f)$ has no periodic points (because $\rho(f)$ is irrational) and since

$$NW(f) = \omega(x) \quad \text{for all } x \in NW(f),$$

each point $y \in NW(f)$ is the limit of some sequence $f^{n_k}(x) \neq y$ with $n_k \nearrow \infty$ when $k \to \infty$. In other words, y is not an isolated point. Moreover, if $NW(f) \neq S^1$ and has nonempty interior, then for each connected component $U \subseteq NW(f)$ there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \varnothing$. But since $NW(f)$ is forward f-invariant, this implies that $f^n(U) \subseteq U$ because otherwise U could not be a connected component. Hence, it follows from Problem 1.5 that f has periodic points (notice that U is an interval). But this is impossible since $\rho(f)$ is irrational. Therefore, if $NW(f) \neq S^1$, then $NW(f)$ has empty interior.

Problem 3.24 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number. Show that f has a unique minimal set $K \subseteq S^1$.

Solution Take $x \in S^1$. By Problem 2.16, $\omega(x)$ is a minimal set. It is also f-invariant. Indeed, by Proposition 2.4, $y \in \omega(x)$ if and only if there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$ such that

$$y = \lim_{k\to\infty} f^{n_k}(x).$$

Since f is a homeomorphism, this is equivalent to

$$f(y) = \lim_{k\to\infty} f^{n_k+1}(x),$$

which again by Proposition 2.4 is equivalent to $f(y) \in \omega(x)$. This shows that the set $\omega(x)$ is f-invariant.

If $\omega(x) = S^1$, then there is nothing to prove. Now assume that $\omega(x) \neq S^1$ and write the open set $U = S^1 \setminus \omega(x)$ as a disjoint union of open intervals $U = \bigcup_{j\in\mathbb{Z}} I_j$. We claim that

$$\alpha(y), \omega(y) \subseteq \omega(x) \quad \text{for every } y \in U.$$

This follows from the fact that the orbit of y visits each interval I_j at most once. Indeed, since U is f-invariant, if $f^{k_1}(y) \in I_{j_1}$ and $f^{k_2}(y) \in I_{j_2}$, then

$$f^{k_2-k_1}\colon I_{j_1} \to I_{j_2} \text{ is a homeomorphism.}$$

If $j_1 = j_2$, then by Problem 1.5 the homeomorphism has a fixed point, thus contradicting the fact that $\rho(f)$ is irrational. The uniqueness of the minimal set is a consequence of Problem 2.16. Indeed, any other minimal set would intersect U and so in particular it could not be the ω-limit set of each one of its points.

Problem 3.25 Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number. Show that either $\omega(x)$ is nowhere dense or $\omega(x) = S^1$.

Solution Given $x \in S^1$, let $M = \omega(x)$. Note that $\omega(x) \neq \varnothing$. It follows from Problem 3.13 that M is independent of x. Hence, M is the ω-limit set of each of its points and so in view of Problems 2.16 and 3.24, it is the unique minimal set of f.

Note that the boundary of M is a closed subset of M that is also invariant (connected components of $S^1 \setminus M$ are taken by f to connected components of $S^1 \setminus M$ because M is f-invariant). Since M is a minimal set, this implies that

$$\partial M = M \quad \text{or} \quad \partial M = \varnothing,$$

which means that either M is nowhere dense or $M = S^1$.

Problem 3.26 Let $f\colon S^1 \to S^1$ be a C^2 map. Show that its derivative has bounded variation.

Solution Take $K > 0$ such that

$$|f''(x)| \leqslant K \quad \text{for } x \in S^1.$$

Moreover, take points $x_0 < x_1 < \cdots < x_n$ in S^1, with $x_n = x_0$, for some $n \in \mathbb{N}$. Then

$$|f'(x_i) - f'(x_{i-1})| = |f''(z_i)| \cdot |x_i - x_{i-1}|$$

for some $z_i \in (x_i, x_{i-1})$ and so

$$\begin{aligned}\sum_{i=1}^{n} |f'(x_i) - f'(x_{i-1})| &= \sum_{i=1}^{n} |f''(z_i)| \cdot |x_i - x_{i-1}| \\ &\leqslant \sum_{i=1}^{n} K(x_i - x_{i-1}) \leqslant K.\end{aligned}$$

Therefore, $\mathrm{Var}(f') < +\infty$.

Problem 3.27 Let $f\colon S^1 \to S^1$ be a C^1 diffeomorphism. Moreover, given $n \in \mathbb{N}$ and an interval $I \subseteq S^1$, let

$$D_n(I) = \sup_{x,y \in I} \log \frac{|(f^n)'(x)|}{|(f^n)'(y)|}.$$

Show that

$$D_n(I) \leqslant \sum_{i=0}^{n-1} D_1(f^i(I)).$$

Solution Since

$$(f^n)'(x) = f'(f^{n-1}(x)) f'(f^{n-2}(x)) \cdots f'(x),$$

we obtain

$$\begin{aligned}\log \frac{|(f^n)'(x)|}{|(f^n)'(y)|} &= \log \frac{\prod_{i=0}^{n-1} |f'(f^i(x))|}{\prod_{i=0}^{n-1} |f'(f^i(y))|} \\ &= \log \prod_{i=0}^{n-1} \frac{|f'(f^i(x))|}{|f'(f^i(y))|} \\ &= \sum_{i=0}^{n-1} \log \frac{|f'(f^i(x))|}{|f'(f^i(y))|}.\end{aligned} \tag{II.3.4}$$

For $x, y \in I$ we have $f^i(x), f^i(y) \in f^i(I)$ and so

$$\log \frac{|f'(f^i(x))|}{|f'(f^i(y))|} \leqslant D_1(f^i(I)).$$

Taking the supremum in (II.3.4) in x and y yields the desired inequality.

Problem 3.28 Let $f\colon S^1 \to S^1$ be an orientation-preserving C^1 diffeomorphism with irrational rotation number. Show that if there exist a sequence $n_k \in \mathbb{N}$ with $n_k \nearrow \infty$ and a constant $C > 0$ such that

$$|(f^{n_k})'(x)| \cdot |(f^{-n_k})'(x)| > C$$

for all $x \in S^1$ and $k \in \mathbb{N}$, then every orbit of f is dense.

Solution We proceed by contradiction. Assume that the orbit of some point $x \in S^1$ is not dense and let $X = \overline{\gamma(x)}$. Note that X is f-invariant. Indeed, $y \in \overline{\gamma(x)}$ if and only if there exists a sequence n_k in $\mathbb{Z}$ with $|n_k| \to \infty$ such that $f^{n_k}(x) \to y$ when $k \to \infty$, but since f is a homeomorphism, this is equivalent to $f^{n_k \pm 1}(x) \to f^{\pm 1}(y)$ when $k \to \infty$. Therefore, $y \in X$ if and only if $f^{\pm 1}(y) \in X$ and so X is f-invariant.

Now consider a connected component I of the nonempty open set $S^1 \setminus X$. We claim that the intervals $I_n = f^n(I)$, for $n \in \mathbb{Z}$, are disjoint. Indeed, if

$$f^i(I) \cap f^j(I) \neq \varnothing \quad \text{for some } i \neq j,$$

then $f^{j-i}(I) \cap I \neq \varnothing$ and so $f^{j-i}(I) = I$ since the set $S^1 \setminus X$ is f-invariant (because X is f-invariant). But by Problems 1.4 and 1.5, this implies that f has a periodic point, which is impossible because f has an irrational rotation number. This contradiction shows that the intervals I_n are disjoint. Therefore, denoting by $|I_n|$ the length of I_n, we obtain $\sum_{n \in \mathbb{Z}} |I_n| \leqslant 1$. So, necessarily $|I_n| \to 0$ when $|n| \to \infty$.

On the other hand,

$$\begin{aligned} |I_n| + |I_{-n}| &= \int_I \left(|(f^n)'(x)| + |(f^{-n})'(x)| \right) dx \\ &\geqslant 2 \int_I \sqrt{|(f^n)'(x)| \cdot |(f^{-n})'(x)|}\, dx \\ &\geqslant 2\sqrt{C}|I|, \end{aligned}$$

which contradicts the fact that $|I_n| \to 0$. Hence, every orbit of f is dense.

Problem 3.29 Consider the piecewise linear map $f\colon [1,3] \to [1,3]$ with

$$f(1) = 2, \quad f(2) = 3 \quad \text{and} \quad f(3) = 1$$

(see Figure I.3.8). Show that f has period points with all periods.

Solution Note that f is continuous and that it has a periodic orbit with period 3 (namely $\{1,2,3\}$). It follows from Sharkovsky's theorem (Theorem 3.10) that f has periodic points with all periods.

Problem 3.30 Show that the map $f\colon [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,1/2], \\ 2(1-x) & \text{if } x \in [1/2,1] \end{cases}$$

(see Figure I.3.6) has periodic points with all periods.

Solution Note that f is continuous and that $\{2/9, 4/9, 8/9\}$ is a periodic orbit of f with period 3. Hence, by Sharkovsky's theorem (Theorem 3.10), the map f has periodic points with all periods.

Problem 3.31 Show that the set $X = [-2,-1] \cup [1,2]$ does not have the property that any continuous map $f\colon X \to X$ satisfies Sharkovsky's ordering in Definition 3.9.

Solution Consider the continuous map $f\colon X \to X$ defined by $f(x) = -x$. Clearly, f has periodic points with period two but it has no fixed points. Hence, it does not satisfy Sharkovsky's ordering.

Problem 3.32 Show that a closed set X with the property that any continuous map $f\colon X \to X$ satisfies Sharkovsky's ordering in Definition 3.9 is connected.

Solution We proceed by contradiction. Assume that X is disconnected. Then $X = A \cup B$ with A and B nonempty closed sets with $A \cap B = \varnothing$. Take points $p \in A$ and $q \in B$, and define a continuous map $f\colon X \to X$ by

$$f(x) = \begin{cases} q & \text{if } x \in A, \\ p & \text{if } x \in B. \end{cases}$$

Note that p and q are periodic points of f with period two, but that f has no fixed points. This shows that the map does not satisfy Sharkovsky's ordering.

Problem 3.33 Verify that the differential equation

$$\begin{cases} x' = xy - x^3, \\ y' = x + y - x^2 \end{cases}$$

has no periodic solutions contained in the second quadrant.

Solution In the second quadrant we have $x < 0$ and $y > 0$. Hence,

$$x' = xy - x^3 < 0.$$

This implies that the first component of each solution whose orbit is contained in the second quadrant is strictly decreasing and so the solution cannot be periodic (neither constant nor periodic nonconstant).

Problem 3.34 Verify that the differential equation

$$\begin{cases} r' = 4 - 3r\cos\theta - r^4, \\ \theta' = -1, \end{cases}$$

written in polar coordinates, has at least one periodic solution.

Solution Note that

$$r' = 4 - 3r\cos\theta - r^4 > 0$$

for any sufficiently small r and that

$$r' = 4 - 3r\cos\theta - r^4 < 0$$

for any sufficiently large r. This implies that for any sufficiently small $r_1 > 0$ and any sufficiently large $r_2 > 0$, the positive semiorbit of any point in the set

$$D = \left\{(x,y) \in \mathbb{R}^2 : r_1 < \sqrt{x^2+y^2} < r_2\right\} \tag{II.3.5}$$

is contained in D and so in particular it is bounded. Moreover, D contains no critical points (the origin is the only critical point). Thus, we have the qualitative behavior in Figure II.3.2. It follows from the Poincaré–Bendixson theorem (Theorem 3.11) that the ω-limit set of any point in D is a periodic orbit.

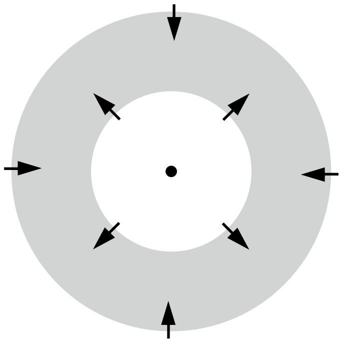

Fig. II.3.2 Behavior on the boundary of the set D.

Problem 3.35 Verify that the differential equation

$$\begin{cases} r' = r(1-r), \\ \theta' = 1 + \cos^2\theta \end{cases}$$

has at least one periodic solution.

Solution Since $\theta' > 0$, it follows from the first equation that the origin is the only critical point. Moreover,

$$r(1-r) > 0 \quad \text{for any sufficiently small } r > 0$$

and

$$r(1-r) < 0 \quad \text{for any sufficiently large } r > 0.$$

Hence, the positive semiorbit of any point in the set D in (II.3.5) with r_1 sufficiently small and r_2 sufficiently large is contained in D. Again, we have the qualitative behavior in Figure II.3.2. It follows from the Poincaré–Bendixson theorem (Theorem 3.11) that the ω-limit set of any point in D is a periodic orbit.

Problem 3.36 Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x,y) = \big(y, -x + y(1-3x^2-6y^2)\big)$$

and let

$$V(x,y) = x^2 + y^2 \quad \text{and} \quad \dot{V}(x,y) = \nabla V(x,y) \cdot f(x,y).$$

Show that:

1. $\dot{V}(x,y) \leqslant 0$ whenever $V(x,y) \geqslant 1/3$;
2. $\dot{V}(x,y) \geqslant 0$ whenever $V(x,y) \leqslant 1/6$;
3. there exists at least one periodic solution of the equation $(x',y') = f(x,y)$ in the set

$$D = \big\{(x,y) \in \mathbb{R}^2 : 1/6 \leqslant V(x,y) \leqslant 1/3\big\}.$$

Solution 1. Note that

$$\begin{aligned}\dot{V}(x,y) &= 2xy + 2y(-x + y(1-3x^2-6y^2)) \\ &= 2y^2(1-3x^2-6y^2) \\ &\leqslant 2y^2(1-3x^2-3y^2).\end{aligned}$$

Hence, if $V(x,y) \geqslant 1/3$, then $\dot{V}(x,y) \leqslant 0$.

2. Now observe that

$$\begin{aligned}\dot{V}(x,y) &= 2y^2(1-3x^2-6y^2) \\ &\geqslant 2y^2(1-6x^2-6y^2).\end{aligned}$$

Hence, if $V(x,y) \leqslant 1/6$, then $\dot{V}(x,y) \geqslant 0$.

3. For a solution $(x(t),y(t))$ with $(x(0),y(0)) = p$ we have

$$\begin{aligned}\dot{V}(p) &= \nabla V(p) \cdot f(p) \\ &= \nabla V(x(t),y(t)) \cdot f(x(t),y(t))\big|_{t=0} \\ &= \nabla V(x(t),y(t)) \cdot (x'(t),y'(t))\big|_{t=0} \\ &= \frac{d}{dt} V(x(t),y(t))\big|_{t=0}.\end{aligned}$$

By item 1, if $\|p\|^2 = 1/3$, then $\dot{V}(p) \leqslant 0$ and so

$$\|(x(t), y(t))\|^2 \leqslant 1/3$$

for all sufficiently small $t > 0$. Similarly, by item 2, if $\|p\|^2 = 1/6$, then $\dot{V}(p) \geqslant 0$ and so

$$\|(x(t), y(t))\|^2 \geqslant 1/6$$

for all sufficiently small $t > 0$. Therefore, any solution starting in D remains in D for all positive time. Moreover, one can easily verify that 0 is the only critical point and so D contains no critical points. Hence, it follows from the Poincaré–Bendixson theorem (Theorem 3.11) that there exists at least one periodic orbit in D.

Problem 3.37 Show that any flow determined by the differential equation does not have periodic solutions:

1. $v' = f(v)$ for some continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^n$, assuming that there exists a function $V\colon \mathbb{R}^n \to \mathbb{R}$ that is strictly decreasing along solutions;
2. $v' = -\nabla V(v)$ for some C^1 function $V\colon \mathbb{R}^n \to \mathbb{R}$.

Solution 1. We proceed by contradiction. Assume that $v\colon \mathbb{R} \to \mathbb{R}^n$ is a periodic solution with period T. Then $V(v(T)) = V(v(0))$, but by hypothesis we also have

$$V(v(T)) < V(v(0)).$$

This contradiction yields the desired result.

2. Assume once more that $v\colon \mathbb{R} \to \mathbb{R}^n$ is a periodic solution with period T. Then $V(v(T)) = V(v(0))$, but on the other hand

$$\begin{aligned} V(v(T)) - V(v(0)) &= \int_0^T \frac{d}{dt} V(v(t))\,dt \\ &= \int_0^T \nabla V(v(t)) \cdot v'(t)\,dt \\ &= -\int_0^T \|v'(t)\|^2\,dt < 0. \end{aligned}$$

This contradiction implies that there are no periodic solutions.

Problem 3.38 Consider the differential equation

$$\begin{cases} x' = y, \\ y' = -f(x)y - g(x) \end{cases}$$

for some C^1 functions $f, g\colon \mathbb{R} \to \mathbb{R}$ with $f > 0$. Show that the equation has no periodic solutions.

Solution Note that

$$\operatorname{div}(y,-f(x)y-g(x)) = \frac{\partial}{\partial x}y + \frac{\partial}{\partial y}(-f(x)y-g(x)) = -f(x)$$

is negative on $\mathbb{R}^2$. Therefore, by Green's theorem, if there exists a periodic solution $(x,y)\colon \mathbb{R}\to\mathbb{R}^2$ with period T and interior D (in the sense of Jordan's curve theorem), then

$$\begin{aligned}
0 &> \int_D \operatorname{div}(y,-f(x)y-g(x))\,dx\,dy \\
&= \int_{\partial D} (f(x)y+g(x))\,dx + y\,dy \\
&= \int_{\partial D} -y'\,dx + x'\,dy \\
&= \int_0^T [-y'(t)x'(t)+x'(t)y'(t)]\,dt = 0.
\end{aligned}$$

This contradiction shows that there are no periodic solutions.

Problem 3.39 Consider the differential equation

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y) \end{cases}$$

on $\mathbb{R}^2$. Show that if there exists a C^1 function $\varphi\colon \mathbb{R}^2\to\mathbb{R}$ such that

$$\frac{\partial(\varphi f)}{\partial x} + \frac{\partial(\varphi g)}{\partial y}$$

has the same sign almost everywhere on a simply connected open set $U\subseteq\mathbb{R}^2$, then the equation has no periodic orbits contained in U.

Solution Without loss of generality, we assume that

$$\frac{\partial(\varphi f)}{\partial x} + \frac{\partial(\varphi g)}{\partial y} > 0$$

on some simply connected open set (the other case can be treated in a similar manner). Let C be a periodic orbit of the equation with period T and let D be its interior (in the sense of Jordan's curve theorem). By Green's theorem, we have

$$\begin{aligned}
0 < \int_D \left(\frac{\partial(\varphi f)}{\partial x} + \frac{\partial(\varphi g)}{\partial y}\right)dx\,dy &= \int_C (-\varphi g\,dx + \varphi f\,dy) = \int_C \varphi(-y'\,dx + x'\,dy) \\
&= \int_0^T \varphi(x(t),(t))[-y'(t)x'(t)+x'(t)y'(t)]\,dt = 0.
\end{aligned}$$

This contradiction shows that there are no periodic orbits contained in U.

Problem 3.40 Show that the differential equation

$$\begin{cases} x' = y, \\ y' = -x - y + x^2 + y^2 \end{cases}$$

has no periodic solutions.

Solution Consider the C^1 function $\varphi(x,y) = e^{-2x}$. Then

$$\begin{aligned} \frac{\partial}{\partial x}(e^{-2x}y) + \frac{\partial}{\partial y}\big(e^{-2x}(-x - y + x^2 + y^2)\big) &= -2e^{-2x}y - e^{-2x} + 2e^{-2x}y \\ &= -e^{-2x}, \end{aligned}$$

which is negative everywhere. Hence, it follows from Problem 3.39 that there are no periodic solutions.

Chapter II.4
Hyperbolic Dynamics

Problem 4.1 Let Λ be a hyperbolic set with finitely many points. Show that Λ is composed of periodic points.

Solution It suffices to note that since Λ is invariant, all its points are periodic and so Λ is the union of finitely many periodic points.

Problem 4.2 Show that the second and third conditions in Definition 4.1 can be replaced, respectively, by

$$\|d_x f^{-n} v\| \geqslant \frac{1}{c}\lambda^n \|v\| \quad \text{for } x \in \Lambda,\, v \in E^s(x),\, n \in \mathbb{N}$$

and

$$\|d_x f^n v\| \geqslant \frac{1}{c}\lambda^{-n} \|v\| \quad \text{for } x \in \Lambda,\, v \in E^u(x),\, n \in \mathbb{N}.$$

Solution We first consider the condition on the unstable space

$$\|d_x f^{-n} v\| \leqslant c\lambda^n \|v\| \quad \text{for } x \in \Lambda,\, v \in E^u(x),\, n \in \mathbb{N}.$$

Take $x \in \Lambda$ and $v \in E^u(x)$. For all $n \in \mathbb{N}$ we have $y = f^n(x) \in \Lambda$ and $w = d_x f^n v \in E^u(f^n(x))$. Therefore,

$$\|d_y f^{-n} w\| \leqslant c\lambda^n \|w\|$$

if and only if

$$\|d_{f^n(x)} f^{-n} d_x f^n v\| \leqslant c\lambda^n \|d_x f^n v\|.$$

By the chain rule, the last inequality is equivalent to

$$\|d_x f^n v\| \geqslant \frac{1}{c}\lambda^{-n} \|v\|.$$

One can show in an analogous manner that the condition on the stable space

$$\|d_x f^n v\| \leqslant c\lambda^n \|v\| \quad \text{for } x \in \Lambda,\, v \in E^s(x),\, n \in \mathbb{N}$$

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_10

is equivalent to

$$\|d_x f^{-n} v\| \geqslant \frac{1}{c}\lambda^n \|v\| \quad \text{for } x \in \Lambda,\, v \in E^s(x),\, n \in \mathbb{N}.$$

Problem 4.3 Show that if Λ is a hyperbolic set for a diffeomorphism f, then for all sufficiently large $k \in \mathbb{N}$ the set Λ is a hyperbolic set for f^k with constant $c = 1$.

Solution Since Λ is hyperbolic set for f, there exist $\lambda \in (0,1)$, $d \geqslant 1$ and a splitting $T_xM = E^s(x) \oplus E^u(x)$ for each $x \in \Lambda$ such that

$$d_x f E^s(x) = E^s(f(x)), \quad d_x f E^u(x) = E^u(f(x)), \tag{II.4.1}$$

$$\|d_x f^n v\| \leqslant d\lambda^n \|v\| \quad \text{for } v \in E^s(x), n \in \mathbb{N}, \tag{II.4.2}$$

and

$$\|d_x f^{-n} v\| \leqslant d\lambda^n \|v\| \quad \text{for } v \in E^u(x), n \in \mathbb{N}. \tag{II.4.3}$$

To show that Λ is a hyperbolic set for f^k with $c = 1$, note first that by (II.4.1) we have

$$d_x f^k E^s(x) = d_{f^{k-1}(x)} f \cdots d_x f E^s(x) = E^s(f^k(x))$$

and

$$d_x f^k E^u(x) = d_{f^{k-1}(x)} f \cdots d_x f E^u(x) = E^u(f^k(x)).$$

Moreover, it follows from (II.4.2) with $n = km$ that

$$\|d_x f^{km} v\| \leqslant d\lambda^{km} \|x\| \quad \text{for } v \in E^s(x),\ m \in \mathbb{N},$$

that is,

$$\|d_x (f^k)^m v\| \leqslant d(\lambda^k)^m \|x\| \quad \text{for } v \in E^s(x),\ m \in \mathbb{N}.$$

Similarly, it follows from (II.4.3) with $n = km$ that

$$\|d_x (f^{-k})^m v\| \leqslant d(\lambda^k)^m \|x\| \quad \text{for } v \in E^u(x),\ m \in \mathbb{N}.$$

Note that $\mu := d\lambda^k < 1$ for all sufficiently large k. Since

$$d(\lambda^k)^m \leqslant d^m (\lambda^k)^m = \mu^m,$$

we conclude that for all sufficiently large k the set Λ is a hyperbolic set for the diffeomorphism f^k with constant $c = 1$.

Problem 4.4 Let Λ be a hyperbolic set for a diffeomorphism $f\colon \mathbb{R}^p \to \mathbb{R}^p$. Show that $\Lambda \times \Lambda$ is a hyperbolic set for the diffeomorphism $g\colon \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^p$ defined by

$$g(x,y) = (f(x), f^{-1}(y)).$$

Solution First note that $\Lambda \times \Lambda$ is compact and g-invariant since

$$g^{-1}(\Lambda \times \Lambda) = f^{-1}(\Lambda) \times f(\Lambda) = \Lambda \times \Lambda.$$

Now we consider the splittings

$$T_x\mathbb{R}^p = E^u(x) \oplus E^s(x), \quad \text{for } x \in \Lambda,$$

associated with the hyperbolic set Λ. Although $E^u(x)$ and $E^s(x)$ are subspaces of $T_x\mathbb{R}^p$, there is a linear isomorphism between $E^u(x) \oplus E^s(x)$ and $E^u(x) \times E^s(x)$. Hence, writing $z = (x, y)$ and letting

$$E^u(z) = E^u(x) \times E^s(y) \quad \text{and} \quad E^s(z) = E^s(x) \times E^u(y),$$

we obtain

$$\begin{aligned} T_z(\mathbb{R}^p \times \mathbb{R}^p) &= (T_x\mathbb{R}^p) \times (T_y\mathbb{R}^p) \\ &= (E^u(x) \oplus E^s(x)) \times (E^u(y) \oplus E^s(y)) \\ &\simeq (E^u(x) \times E^s(y)) \oplus (E^s(x) \times E^u(y)) \\ &= E^u(z) \oplus E^s(z). \end{aligned}$$

Since

$$d_z g = \begin{pmatrix} d_x f & 0 \\ 0 & d_y f^{-1} \end{pmatrix}, \tag{II.4.4}$$

we have

$$\begin{aligned} d_z g E^u(z) &= d_{(x,y)} g(E^u(x) \times E^s(y)) \\ &= d_x f E^u(x) \oplus d_y f^{-1} E^s(y) \\ &= E^u(f(x)) \oplus E^s(f^{-1}(y)) = E^u(g(z)) \end{aligned}$$

and, analogously,

$$d_z g E^s(z) = E^s(g(z)).$$

Finally, we establish exponential bounds along the spaces $E^u(z)$ and $E^s(z)$. Since Λ is a hyperbolic set, there exist $\lambda \in (0,1)$ and $c > 0$ such that for each

$$w = (u, v) \in E^s(x) \times E^u(y)$$

we have

$$\|d_x f^n u\| \leqslant c\lambda^n \|u\| \quad \text{and} \quad \|d_y f^{-n} v\| \leqslant c\lambda^n \|v\|$$

for all $n \in \mathbb{N}$. On the other hand, by (II.4.4), we have

$$d_z g^n = \begin{pmatrix} d_x f^n & 0 \\ 0 & d_y f^{-n} \end{pmatrix}$$

and so, considering the norm $\|(u,v)\| = \|u\| + \|v\|$, we obtain

$$\begin{aligned} \|d_z g^n w\| &= \|(d_x f^n u, d_y f^{-n} v)\| \\ &= \|d_x f^n u\| + \|d_y f^{-n} v\| \\ &\leqslant c\lambda^n \|u\| + c\lambda^n \|v\| = c\lambda^n \|w\| \end{aligned}$$

for $n \in \mathbb{N}$. Similarly, for each

$$w = (u, v) \in E^u(z) = E^u(x) \times E^s(y)$$

we have

$$\|d_x f^{-n} u\| \leqslant c\lambda^n \|u\| \quad \text{and} \quad \|d_y f^n v\| \leqslant c\lambda^n \|v\|$$

for $n \in \mathbb{N}$, which gives

$$\|d_z g^{-n} w\| \leqslant c\lambda^n \|w\| \quad \text{for } n \in \mathbb{N}.$$

This shows that $\Lambda \times \Lambda$ is a hyperbolic set for the diffeomorphism g.

Problem 4.5 Let A be a $p \times p$ matrix with real entries and without eigenvalues of modulus 1. Show that there exist a splitting $\mathbb{R}^p = E^s \oplus E^u$ and constants $\lambda \in (0,1)$ and $c > 0$ such that

$$\|A^n v\| \leqslant c\lambda^n \|v\| \quad \text{for } v \in E^s,\ n \in \mathbb{N}$$

and

$$\|A^{-n} v\| \leqslant c\lambda^n \|v\| \quad \text{for } v \in E^u,\ n \in \mathbb{N}.$$

Solution Let $F^s \subseteq \mathbb{C}^p$ be the subspace generated by all vectors $v \in \mathbb{C}^p$ satisfying

$$(A - \tau \,\mathrm{Id})^k v = 0 \tag{II.4.5}$$

for some $k \in \mathbb{N}$ and some eigenvalue τ of A with $|\tau| < 1$. Similarly, let $F^u \subseteq \mathbb{C}^p$ be the subspace generated by all vectors $v \in \mathbb{C}^p$ satisfying (II.4.5) for some $k \in \mathbb{N}$ and some eigenvalue τ of A with $|\tau| > 1$. Since A is invertible, it follows from the identities

$$(A - \tau \,\mathrm{Id})^k A = A(A - \tau \,\mathrm{Id})^k$$

and

$$(A - \tau \,\mathrm{Id})^k A^{-1} = A^{-1}(A - \tau \,\mathrm{Id})^k$$

that

$$AF^s = F^s \quad \text{and} \quad AF^u = F^u. \tag{II.4.6}$$

Moreover, since A has no eigenvalues of modulus 1, we have $\mathbb{C}^p = F^s \oplus F^u$. Therefore, letting

$$E^s = F^s \cap \mathbb{R}^p \quad \text{and} \quad E^u = F^u \cap \mathbb{R}^p,$$

we obtain $\mathbb{R}^p = E^s \oplus E^u$ and it follows from (II.4.6) that

$$AE^s = E^s \quad \text{and} \quad AE^u = E^u.$$

Given $\varepsilon > 0$, let $R_j = \lambda_j \operatorname{Id} + N$ for some eigenvalue λ_j of B and let

$$N = \begin{pmatrix} 0 & \varepsilon & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon \\ 0 & & & 0 \end{pmatrix}.$$

Moreover, let

$$S^{-1}BS = \begin{pmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_k \end{pmatrix}$$

be a Jordan canonical form of the matrix $B = A|_{F^s}$, for some invertible matrix $S = S(\varepsilon)$ with complex entries. Then

$$B^n = S \begin{pmatrix} R_1^n & & 0 \\ & \ddots & \\ 0 & & R_k^n \end{pmatrix} S^{-1}$$

and

$$R_j^n = \sum_{i=0}^{n-1} \binom{n}{i} \lambda_j^{n-i} N^i.$$

Therefore,

$$\begin{aligned} \|B^n\| &\leqslant \|S\| \cdot \|S^{-1}\| \sum_{i=0}^{n-1} \binom{n}{i} \sigma(B)^{n-i} \|N\|^i \\ &\leqslant D_1 \sum_{i=0}^{n} \binom{n}{i} \sigma(B)^{n-i} (D_2\varepsilon)^i \\ &= D_1 \big(\sigma(B) + D_2\varepsilon\big)^n \end{aligned}$$

for some constants $D_1, D_2 > 0$ with D_2 independent of ε. Since $\sigma(B) < 1$, one can take ε sufficiently small such that $\sigma(B) + D_2\varepsilon < 1$. Then

$$\|A^n|_{E^s}\| \leqslant \|B^n\| \leqslant D_1 \big(\sigma(B) + D_2\varepsilon\big)^n.$$

A similar argument applies to $A^{-n}|_{E^u}$. Summing up, there exist $\lambda \in (0,1)$ and $c > 0$ as desired.

Problem 4.6 For the Smale horseshoe Λ (see Definition 4.3), show that for any two-sided sequence $\omega = (\cdots i_{-1} i_0 i_1 \cdots)$ in $\{1,2\}^{\mathbb{Z}}$ the set

$$\Lambda_\omega = \bigcap_{n \in \mathbb{Z}} f^{-n} V_{i_n}$$

contains exactly one point.

Solution For each $\omega = (\cdots i_{-1} i_0 i_1 \cdots)$ and $n \in \mathbb{N}$, consider the set

$$R_n(\omega) = \bigcap_{k=-n+1}^{n} f^{-k} V_{i_k}$$

(see Figure II.4.1). Due to the contraction and expansion of f on $H_1 \cup H_2$ (see Definition 4.3), each set $R_n(\omega)$ is a square of side a^n and so $\operatorname{diam} R_n(\omega) \to 0$ when $n \to \infty$. This implies that each intersection

$$\begin{aligned}\Lambda_\omega &= \bigcap_{n \in \mathbb{Z}} f^{-n} V_{i_n} \\ &= \bigcap_{n \in \mathbb{N}} R_n(\omega)\end{aligned}$$

contains at most one point. On the other hand, since $R_n(\omega)$ is a decreasing sequence of nonempty closed sets, each intersection is nonempty. This shows that

$$\operatorname{card} \Lambda_\omega = 1 \quad \text{for each } \omega \in \{1,2\}^{\mathbb{Z}}.$$

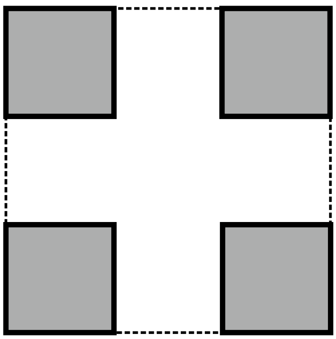

Fig. II.4.1 Sets $R_1(\omega)$ (in gray).

Problem 4.7 For the Smale horseshoe Λ, show that the map $H\colon \{1,2\}^{\mathbb{Z}} \to \Lambda$ defined by

$$H(\cdots i_{-1} i_0 i_1 \cdots) = \bigcap_{n \in \mathbb{Z}} f^{-n} V_{i_n}$$

is bijective (by Problem 4.6 the map is well defined).

Solution It follows from the construction of the Smale horseshoe that

$$\begin{aligned}\Lambda &= \bigcap_{n\in\mathbb{Z}} f^{-n}(V_1 \cup V_2)\\ &= \bigcup_{\omega\in\{1,2\}^{\mathbb{Z}}} \bigcap_{n\in\mathbb{Z}} f^{-n}V_{i_n}\\ &= \bigcup_{\omega\in\{1,2\}^{\mathbb{Z}}} H(\omega)\end{aligned}$$

and so H is onto. To show that it is one-to-one, take $\omega, \omega' \in \{1,2\}^{\mathbb{Z}}$ with $\omega \neq \omega'$. Then there exists $m \in \mathbb{Z}$ such that $i_m(\omega) \neq i_m(\omega')$ and so

$$V_{i_m(\omega)} \cap V_{i_m(\omega')} = \varnothing.$$

Hence,

$$H(\omega) \cap H(\omega') = \left(\bigcap_{n\in\mathbb{Z}} f^{-n}V_{i_n(\omega)}\right) \cap \left(\bigcap_{n\in\mathbb{Z}} f^{-n}V_{i_n(\omega')}\right) = \varnothing,$$

which shows that $H(\omega) \neq H(\omega')$. Therefore, H is bijective.

Problem 4.8 For the Smale horseshoe Λ, show that for each $m \in \mathbb{N}$ the number of m-periodic points of the map $f|_\Lambda$ is at least 2^m.

Solution Consider the sequence $\omega(p) = (\cdots ppp \cdots) \in \{1,2\}^{\mathbb{Z}}$ obtained from repeating indefinitely a vector $p \in \{1,2\}^m$. For $\omega = \omega(p)$, the set Λ_ω in Problem 4.6 satisfies

$$\begin{aligned}f^m\Lambda_\omega &= \bigcap_{n\in\mathbb{Z}} f^{-n+m}V_{i_n}\\ &= \bigcap_{n\in\mathbb{Z}} f^{-n+m}V_{i_{n-m}} = \Lambda_\omega.\end{aligned}$$

Hence, the unique element x_p of $\Lambda_{\omega(p)}$ (see Problem 4.6) is an m-periodic point. On the other hand, by Problem 4.7, for $p, q \in \{1,2\}^m$ with $p \neq q$ we have $x_p \neq x_q$. Therefore, since the set $\{1,2\}^m$ has exactly 2^m elements, the map $f|_\Lambda$ has at least 2^m m-periodic points.

Problem 4.9 Consider horizontal strips

$$\bar{H}_i = \left\{(x,y) \in [0,1]^2 : \varphi_i(x) \leqslant y \leqslant \psi_i(x)\right\}$$

and vertical strips

$$\bar{V}_i = \left\{(x,y) \in [0,1]^2 : \bar{\varphi}_i(y) \leqslant x \leqslant \bar{\psi}_i(y)\right\},$$

for some functions $\varphi_i, \psi_i, \bar{\varphi}_i, \bar{\psi}_i \colon [0,1] \to [0,1]$, for $i = 1,2$, such that

$$\varphi_1 < \psi_1 < \varphi_2 < \psi_2$$

and

$$\bar{\varphi}_1 < \bar{\psi}_1 < \bar{\varphi}_2 < \bar{\psi}_2$$

(see Figure I.4.5). Moreover, let f be a C^1 diffeomorphism on an open neighborhood of the square $[0,1]^2$ such that $f(\bar{H}_i) = \bar{V}_i$ for $i = 1,2$ and

$$f(x,y) = (g(x), h(y)) \quad \text{for } (x,y) \in \bar{H}_1 \cup \bar{H}_2$$

and some C^1 functions g, h defined on compact subsets of $\mathbb{R}$. Show that if $\sup|g'| < 1$ and $\inf|h'| > 1$, then

$$\bar{\Lambda} = \bigcap_{n\in\mathbb{Z}} f^n(\bar{H}_1 \cup \bar{H}_2)$$

is a hyperbolic set for f.

Solution Note that $\bar{\Lambda}$ is f-invariant. Indeed,

$$\begin{aligned} f^{-1}\bar{\Lambda} &= f^{-1}\bigcap_{n\in\mathbb{Z}} f^n(\bar{H}_1 \cup \bar{H}_2) \\ &= \bigcap_{n\in\mathbb{Z}} f^{n-1}(\bar{H}_1 \cup \bar{H}_2) = \bar{\Lambda}. \end{aligned}$$

Now observe that

$$d_{(x,y)}f = \begin{pmatrix} g'(x) & 0 \\ 0 & h'(y) \end{pmatrix} \quad \text{for } (x,y) \in \bar{H}_1 \cup \bar{H}_2. \tag{II.4.7}$$

Let $E^s(x,y)$ and $E^u(x,y)$ be, respectively, the horizontal and vertical axes. For each $(x,y) \in \bar{\Lambda}$ we consider the decomposition

$$\mathbb{R}^2 = E^s(x,y) \oplus E^u(x,y).$$

Since the matrices in (II.4.7) are diagonal, we have

$$d_x f E^s(x,y) = E^s(f(x,y)) \quad \text{and} \quad d_x f E^u(x,y) = E^u(f(x,y))$$

for all $(x,y) \in \bar{\Lambda}$. Moreover, it follows from (II.4.7) that

$$\|d_{(x,y)}fv\| \leqslant \sup|g'| \cdot \|v\| \quad \text{for } v \in E^s(x,y)$$

and

$$\|d_{(x,y)}f^{-1}v\| \leqslant (\inf|h'|)^{-1}\|v\| \quad \text{for } v \in E^u(x,y).$$

Since $\sup|g'| < 1$ and $\inf|h'| > 1$, we conclude that $\bar{\Lambda}$ is a hyperbolic set for f with constants $c = 1$ and

$$\lambda = \max\{\sup|g'|, (\inf|h'|)^{-1}\} \in (0,1).$$

Problem 4.10 Show that if a compact connected manifold M is a hyperbolic set for a diffeomorphism $f\colon M \to M$, then all stable spaces have the same dimension.

Solution Since the stable and unstable spaces $E^s(x)$ and $E^u(x)$ vary continuously with x, their dimensions are locally constant. More precisely, for each $x \in M$ there exists an open neighborhood U_x of x such that

$$\dim E^s(y) = \dim E^s(x) \quad \text{and} \quad \dim E^u(y) = \dim E^u(x)$$

for all $y \in U_x$. Since M is compact, there exists a finite cover of M by sets U_x. Hence, it follows from the connectedness of M, that all stable spaces have the same dimension.

Alternatively, let

$$A_k = \{x \in M : \dim E^s(x) = k\}$$

and

$$B_k = \{x \in M : \dim E^s(x) \neq k\}$$

for $k = 0, 1, \ldots, \dim M$. Note that

$$M = A_k \cup B_k \quad \text{and} \quad A_k \cap B_k = \varnothing.$$

We show that A_k is closed. Take $x_n \in A_k$ such that $x_n \to x \in M$ when $n \to \infty$. By the continuity of the stable spaces, there exists $N \in \mathbb{N}$ such that

$$\dim E^s(x_n) = \dim E^s(x) \quad \text{for } n > N. \tag{II.4.8}$$

Hence, $\dim E^s(x) = k$ and so $x \in A_k$, which shows that A_k is closed. Now take $x_n \in B_k$ such that $x_n \to x \in M$ when $n \to \infty$. Since $x_n \in B_k$, we have

$$\dim E^s(x_n) \neq k \quad \text{for all } n.$$

Hence, it follows from (II.4.8) that $\dim E^s(x) \neq k$ and so $x \in B_k$. This shows that B_k is closed. Since M is connected, we conclude that for each k either $A_k = \varnothing$ or $B_k = \varnothing$. Hence, there exists $k \in \{0, 1, \ldots, \dim M\}$ such that $A_k = M$.

Problem 4.11 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1 map. Show that if there exist $\lambda \in (0,1)$ and $c > 0$ such that $|(f^n)'(x)| \leqslant c\lambda^n$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \frac{1}{n} \log |f^n(I)| < 0$$

for any compact interval $I \subsetneq \mathbb{R}$.

Solution We have

$$d(f^n(x), f^n(y)) = \left| \int_x^y (f^n)'(z)\, dz \right| \leqslant c\lambda^n d(x,y) \tag{II.4.9}$$

for all $n \in \mathbb{N}$ and $x, y \in I$. Therefore,

$$|f^n(I)| \leqslant c\lambda^n |I|$$

and so

$$\limsup_{n\to\infty} \frac{1}{n} \log|f^n(I)| < 0.$$

Problem 4.12 Show that there exists no diffeomorphism $f\colon S^2 \to S^2$ for which the whole S^2 is a hyperbolic set with $\dim E^u(x) = \{0\}$ for all $x \in S^2$.

Solution Assume that $\dim E^u(x) = \{0\}$ for all $x \in S^2$. Then one can proceed as in (II.4.9) to conclude that

$$d(f^n(x), f^n(y)) \leqslant c\lambda^n d(x,y) \to 0 \quad \text{when } n \to \infty \tag{II.4.10}$$

for all $x, y \in S^1$. But since f is a diffeomorphism, we have $f(S^2) = S^2$, which contradicts to (II.4.10) because the area of any open set must eventually contract when we apply f successively. This contradiction shows that no such diffeomorphism exists.

Problem 4.13 Given functions $f, g\colon \mathbb{R} \to \mathbb{R}$, let

$$d(f,g) = \sup\bigl\{|f(x) - g(x)| : x \in \mathbb{R}\bigr\}.$$

Determine whether the set X of all bounded functions $f\colon \mathbb{R} \to \mathbb{R}$ with the property that $f(0) + f(1) = 0$ is a complete metric space with the distance d.

Solution We show that X is complete with the distance d. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Note that it is also a Cauchy sequence in the set of all bounded continuous functions equipped with the distance d, which is a complete metric space. Hence, there exists a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $d(f_n, f) \to 0$ when $n \to \infty$. Moreover, it follows from the inequality

$$|f_n(x) - f(x)| \leqslant d(f_n, f)$$

that

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for each } x \in \mathbb{R}.$$

In order to show that $f \in X$, note that

$$\begin{aligned} f(0) + f(1) &= \lim_{n\to\infty} f_n(0) + \lim_{n\to\infty} f_n(1) \\ &= \lim_{n\to\infty} (f_n(0) + f_n(1)) = 0 \end{aligned}$$

since $f_n \in X$ for each $n \in \mathbb{N}$.

Problem 4.14 Show that the set of all C^1 functions $f\colon [a,b] \to \mathbb{R}$ is not a complete metric space with the distance

$$d(f,g) = \sup\bigl\{|f(x) - g(x)| : x \in [a,b]\bigr\}.$$

Solution We show that the limit of C^1 functions need not be differentiable. For an example, take $[a,b] = [-1,1]$ and for each $n \in \mathbb{N}$ let

$$f_n(x) = \sqrt{(x-1/2)^2 + 1/n} \quad \text{for } x \in [-1,1]$$

(see Figure II.4.2). Moreover, define $f\colon [-1,1] \to \mathbb{R}$ by

$$f(x) = |x-1/2|.$$

Then $d(f_n, f) \to 0$ when $n \to \infty$, but the function f is not differentiable at $1/2$.

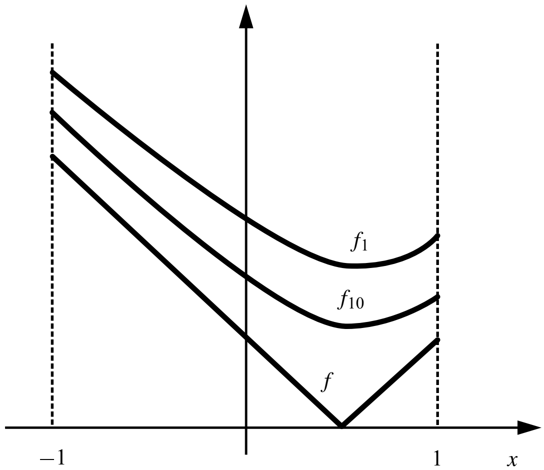

Fig. II.4.2 The functions f_n and f in Problem 4.14.

Problem 4.15 Construct a sequence of bounded C^1 functions $f_n\colon \mathbb{R} \to \mathbb{R}$ converging uniformly to a nondifferentiable function.

Solution For each $n \in \mathbb{N}$, let

$$g_n(x) = \begin{cases} -x - \frac{1}{2n} & \text{if } x \leqslant -\frac{1}{n}, \\ \frac{n}{2}x^2 & \text{if } -\frac{1}{n} \leqslant x \leqslant \frac{1}{n}, \\ x - \frac{1}{2n} & \text{if } x \geqslant \frac{1}{n} \end{cases}$$

and define a function $f_n\colon \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} -\arctan(x+1) + 1 - \frac{1}{2n} & \text{if } x \leqslant -1, \\ g_n(x) & \text{if } x \in [-1,1], \\ \arctan(x-1) + 1 - \frac{1}{2n} & \text{if } x > 1 \end{cases}$$

(see Figure II.4.3). Moreover, define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} -\arctan(x+1)+1 & \text{if } x \leqslant -1, \\ |x| & \text{if } x \in [-1,1], \\ \arctan(x-1)+1 & \text{if } x \geqslant 1. \end{cases}$$

One can easily verify that f_n is a sequence of bounded C^1 functions converging uniformly to f. On the other hand, f is not differentiable at 0.

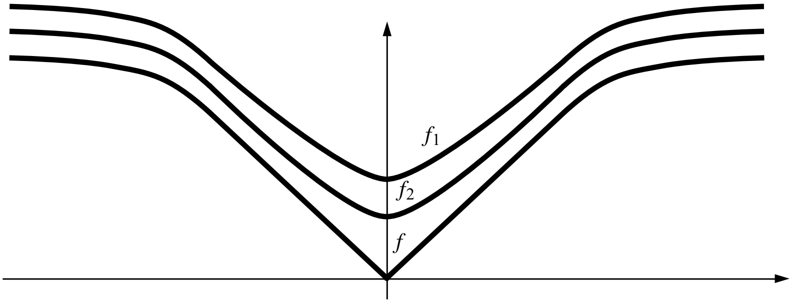

Fig. II.4.3 The functions f_n and f in Problem 4.15.

Problem 4.16 Let Λ be a compact set that is invariant under a C^1 diffeomorphism f and assume that there exist a splitting $T_xM = F^s(x) \oplus F^u(x)$ for each $x \in \Lambda$, an inner product $\langle\cdot,\cdot\rangle'_x$ on T_xM varying continuously with $x \in \Lambda$ and a constant $\gamma \in (0,1)$ such that

$$\|d_xfv\|' > \|v\|' \quad \text{for } x \in \Lambda, v \in \overline{C^u(x)} \setminus \{0\}.$$

Show that there exists a constant $\mu \in (0,1)$ such that

$$\|d_xfv\|' \geqslant \mu^{-1}\|v\|' \quad \text{for } x \in \Lambda, v \in \overline{C^u(x)}.$$

Solution Take $x \in \Lambda$ and $v \in \overline{C^u(x)} \setminus \{0\}$ such that $\|v\|' = 1$. Let

$$S_x = \overline{C^u(x)} \cap \{w \in F^u(x) : \|w\|' = 1\}.$$

By the hypothesis we have

$$\|d_xfv\|' > 1 \quad \text{for } v \in S_x.$$

Since the set $\{(x,v) : x \in \Lambda, v \in S_x\}$ is compact, there exists $\mu \in (0,1)$ such that

$$\|d_xfv\|' \geqslant \mu^{-1} \quad \text{for } x \in \Lambda,\ v \in S_x.$$

Now take $v \in \overline{C^u(x)} \setminus \{0\}$. Then

$$\left\| d_x f \frac{v}{\|v\|'} \right\|' \geqslant \mu^{-1}$$

and so

$$\|d_x f v\|' \geqslant \mu^{-1} \|v\|' \quad \text{for } v \in \overline{C^u(x)} \setminus \{0\}.$$

This establishes the desired property.

Problem 4.17 Consider the set

$$A = S^1 \times \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\right\}$$

and the map $f\colon A \to S^1 \times \mathbb{R}^2$ defined by

$$f([\theta],x,y) = \left([2\theta], \frac{1}{5}x + \frac{1}{2}\cos(2\pi\theta), \frac{1}{5}y + \frac{1}{2}\sin(2\pi\theta)\right).$$

Show that $f(A) \subsetneq A$ and that $f|_A$ is one-to-one.

Solution We have

$$\begin{aligned}
&\left(\frac{1}{5}x + \frac{1}{2}\cos(2\pi\theta)\right)^2 + \left(\frac{1}{5}y + \frac{1}{2}\sin(2\pi\theta)\right)^2 \\
&= \frac{1}{25}x^2 + \frac{1}{25}y^2 + \frac{1}{4} + \frac{1}{5}x\cos(2\pi\theta) + \frac{1}{5}y\sin(2\pi\theta) \\
&\leqslant \frac{1}{25}(x^2+y^2) + \frac{2}{5} + \frac{1}{4} < 1
\end{aligned} \tag{II.4.11}$$

and so $f(A) \subsetneq A$. To show that f is one-to-one, assume that

$$f([\theta_1],x_1,y_1) = f([\theta_2],x_2,y_2).$$

Then

$$\begin{aligned}
2\theta_1 &\equiv 2\theta_2 \bmod 1, \\
\frac{1}{5}x_1 + \frac{1}{2}\cos(2\pi\theta_1) &= \frac{1}{5}x_2 + \frac{1}{2}\cos(2\pi\theta_2), \\
\frac{1}{5}y_1 + \frac{1}{2}\sin(2\pi\theta_1) &= \frac{1}{5}y_2 + \frac{1}{2}\sin(2\pi\theta_2).
\end{aligned} \tag{II.4.12}$$

It follows from the first relation that either $\theta_1 = \theta_2$ or $\theta_1 = \theta_2 + 1/2$.

When $\theta_1 = \theta_2$ it follows from the second and third relations in (II.4.12) that $x_1 = x_2$ and $y_1 = y_2$. Now assume that $\theta_1 = \theta_2 + 1/2$. In this case the second and third relations in (II.4.12) become

$$\frac{1}{5}(x_1 - x_2) = \cos(2\pi\theta_1) \quad \text{and} \quad \frac{1}{5}(y_1 - y_2) = \sin(2\pi\theta_1).$$

Therefore,

$$
\begin{aligned}
1 &= \cos^2(2\pi\theta_1) + \sin^2(2\pi\theta_1) \\
&= \frac{1}{25}(x_1 - x_2)^2 + \frac{1}{25}(y_1 - y_2)^2
\end{aligned}
$$

and so

$$
\begin{aligned}
1 &\leqslant \frac{1}{25}[(x_1 - x_2)^2 + (y_1 - y_2)^2] \\
&\leqslant \frac{1}{25}\left[(|x_1| + |x_2|)^2 + (|y_1| + |y_2|)^2\right] \\
&\leqslant \frac{8}{25} < 1
\end{aligned}
$$

since $|x_i| \leqslant 1$ and $|y_i| \leqslant 1$ for $i = 1, 2$. This contradiction shows that one cannot have $\theta_1 = \theta_2 + 1/2$ and so $\theta_1 = \theta_2$.

Problem 4.18 For the map $f|_A$ in Problem 4.17, show that $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(A)$ is a hyperbolic set for f.

Solution The map f is continuous on the compact set A and so $f(A)$ is also compact. Moreover, it follows from (II.4.11) that $f(A)$ is contained in the interior of A. Therefore, Λ is the intersection of a decreasing sequence of compact sets and so it is nonempty and compact. One can also verify that f is a diffeomorphism on some open neighborhood of Λ. Finally,

$$
\begin{aligned}
f^{-1}\Lambda &= f^{-1}\left(\bigcap_{n\in\mathbb{N}} f^n(A)\right) \\
&= \bigcap_{n\geqslant 0} f^n(A) = A \cap \bigcap_{n\in\mathbb{N}} f^n(A) \\
&= \bigcap_{n\in\mathbb{N}} f^n(A) = \Lambda
\end{aligned}
$$

and so Λ is f-invariant.

Now we consider the splitting

$$
T_z\Lambda = F^u(z) \oplus F^s(z) = \mathbb{R} \times \mathbb{R}^2
$$

for each $z = (\theta, x, y)$. Since

$$
d_{([\theta],x,y)} f(u, v, w) = \left(2u, -\frac{1}{2}u\sin(2\pi\theta) + \frac{1}{5}v, \frac{1}{2}u\cos(2\pi\theta) + \frac{1}{5}w\right),
$$

we obtain

$$
d_z f F^s(z) = F^s(f(z)). \tag{II.4.13}
$$

Moreover, for each $z \in A$ and $(0, v, w) \in F^s(z)$ we have

$$\|d_z f(0,v,w)\| = \left\| \left(0, \frac{1}{5}v, \frac{1}{5}w\right) \right\| \leqslant \frac{1}{5}\|(0,v,w)\|,$$

which together with (II.4.13) implies that

$$\|d_z f^n(0,v,w)\| \leqslant \frac{1}{5^n}\|(0,v,w)\| \quad \text{for } n \in \mathbb{N}. \tag{II.4.14}$$

Now we show that the cones

$$C^u(z) = \left\{(u,v,w) : v^2 + w^2 < \frac{1}{4}u^2\right\} \cup \{(0,0,0)\}$$

are df-invariant. Assume that $v^2 + w^2 \leqslant \frac{1}{4}u^2$ and let

$$\bar{u} = 2u, \quad \bar{v} = -\frac{1}{2}u\sin(2\pi\theta) + \frac{1}{5}v, \quad \bar{w} = \frac{1}{2}u\cos(2\pi\theta) + \frac{1}{5}w.$$

Then

$$\begin{aligned}
\bar{v}^2 + \bar{w}^2 &= \frac{u^2\sin^2(2\pi\theta)}{4} + \frac{1}{25}v^2 - \frac{1}{5}uv\sin(2\pi\theta) \\
&\quad + \frac{u^2\cos^2(2\pi\theta)}{4} + \frac{1}{5}uw\cos(2\pi\theta) + \frac{1}{25}w^2 \\
&= \frac{u^2}{4} + \frac{1}{25}(v^2+w^2) + \frac{1}{5}u(w\cos(2\pi\theta) - v\sin(2\pi\theta)) \\
&\leqslant \frac{u^2}{4} + \frac{1}{25}\cdot\frac{1}{4}u^2 + \frac{1}{5}u(w\cos(2\pi\theta) - v\sin(2\pi\theta)).
\end{aligned}$$

We have

$$\begin{aligned}
w\cos(2\pi\theta) - v\sin(2\pi\theta) &\leqslant w + v \\
&\leqslant \sqrt{2}\sqrt{v^2+w^2} \\
&\leqslant \frac{1}{\sqrt{2}}u
\end{aligned}$$

and so

$$\bar{v}^2 + \bar{w}^2 \leqslant \left(\frac{1}{4} + \frac{1}{100} + \frac{1}{5\sqrt{2}}\right)u^2 < \frac{1}{4}\bar{u}^2.$$

This shows that

$$d_z f\overline{C^u(z)} \subsetneq C^u(f(z)) \tag{II.4.15}$$

(see Figure II.4.4).

Moreover,

$$\|d_z f(u,v,w)\|^2 > 4u^2 \geqslant \frac{4}{1+1/4}\|(u,v,w)\|^2 \tag{II.4.16}$$

for $(u,v,w) \in \mathbb{R}^3$ with $u \neq 0$. Now consider a 2-dimensional space

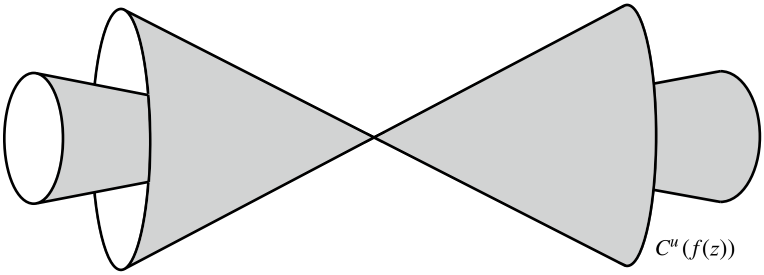

Fig. II.4.4 The cone $C^u(f(z))$ and its image $d_z f\overline{C^u(z)}$.

$$E^u(z) \subseteq G^u(z) := \bigcap_{n\geqslant 0} d_{f^{-n}(z)} f^n \overline{C^u(f^{-n}(z))}$$

for each $z \in \Lambda$. It exists because by (II.4.15) the set $G^u(z)$ is the intersection of a decreasing sequence of closed sets and so it is nonempty. Moreover, since the closed unit ball S^3 in $\mathbb{R}^3$ is compact, the set $G^u(z) \cap S^3$ is also compact and so any sequence of linearly independent vectors

$$v_1^m, v_2^m \in \bigcap_{n=0}^{m} d_{f^{-n}(z)} f^n \overline{C^u(f^{-n}(z))},$$

for $m \in \mathbb{N}$, has a converging subsequence to some linearly independent vectors $v_1, v_2 \in G^u(z)$. Therefore, $G^u(z)$ contains a 2-dimensional space. We show that in fact

$$E^u(z) = G^u(z).$$

Indeed, if there exists $(u,v,w) \in G^u(z) \backslash E^u(z)$, then writing

$$(u,v,w) = v_u + v_s \quad \text{with } v_u \in F^u(z) \text{ and } v_s \in E^s(z) \backslash \{0\},$$

it follows from (II.4.16) that

$$\|d_z f^{-n}((u,v,w) - v_u)\| \leqslant \left(\frac{16}{5}\right)^{-n} \big(\|(u,v,w)\| + \|v_u\|\big) \to 0$$

when $n \to \infty$. But since

$$(u,v,w) - v_u = v_s \in E^s(z),$$

we also have

$$\|d_z f^{-n}((u,v,w) - v_u)\| \geqslant \frac{1}{5^n} \|(u,v,w) - v_u\| \to +\infty$$

when $n \to \infty$ (because $v_s \neq 0$). This contradiction shows that $E^u(z) = G^u(z)$. Since

$$\begin{aligned}
d_z f G^u(z) &= \bigcap_{n\geqslant 0} d_z f d_{f^{-n}(z)} f^n \overline{C^u(f^{-n}(z))} \\
&= \bigcap_{n\geqslant 0} d_{f^{-(n+1)}(f(z))} f^{n+1} \overline{C^u(f^{-(n+1)}(f(z)))} \\
&= \bigcap_{n\geqslant -1} d_{f^{-(n+1)}(f(z))} f^{n+1} \overline{C^u(f^{-(n+1)}(f(z)))}
\end{aligned}$$

(because the sequence is decreasing), we obtain

$$d_z f G^u(z) = G^u(f(z))$$

and so also

$$d_z f E^u(z) = E^u(f(z)).$$

Putting everything together, we conclude that Λ is a hyperbolic set with stable and unstable spaces $F^s(z)$ and $E^u(z)$ at each $z \in \Lambda$ (see (II.4.14) and (II.4.16)).

Problem 4.19 Show that if $\mathbb{T}^n$ is a hyperbolic set for an automorphism of the torus $T_A\colon \mathbb{T}^n \to \mathbb{T}^n$, then T_A has positive topological entropy.

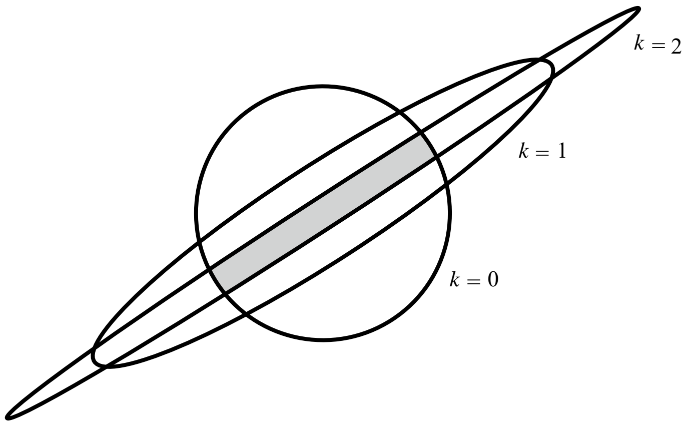

Fig. II.4.5 Sets $T_A^{-k}B(T_A^k(p_i), \varepsilon)$ for $k = 0, 1, 2$ and their intersection $B_2(p_i, \varepsilon)$ (in gray).

Solution We consider a cover of $\mathbb{T}^n$ by d_m-open balls

$$B_m(p_i, \varepsilon) := \bigcap_{k=0}^{m-1} T_A^{-k} B(T_A^k(p_i), \varepsilon)$$

(see Figure II.4.5). It follows from results in linear algebra that for each sufficiently small $\delta > 0$ there exists $C > 0$ such that

$$C^{-1}(|\lambda_i|^{-1}-\delta)^k\|v\| \leqslant \|A^{-k}v\| \leqslant C(|\lambda_i|^{-1}+\delta)^k\|v\|$$

for each $k \in \mathbb{N}$ and each vector $v \in \mathbb{C}^n$ in the generalized eigenspace associated to the eigenvalue λ_i. Hence, there exists $D > 0$ (independent of m, ε and i) such that

$$\text{the } n\text{-volume of the ball } B_m(p_i,\varepsilon) \text{ is at most } D\prod_{|\lambda_i|>1}(|\lambda_i|^{-1}+\delta)^m\varepsilon^n.$$

Let d be the dimension of the unstable spaces and let

$$\lambda = \min\{|\lambda_i| : |\lambda_i| > 1\} > 1.$$

Then

$$\begin{aligned} M(m,\varepsilon) &\geqslant D^{-1}\prod_{|\lambda_i|>1}(|\lambda_i|^{-1}+\delta)^{-m}\varepsilon^{-n} \\ &\geqslant D^{-1}(\lambda^{-1}+\delta)^{-dm}\varepsilon^{-n}. \end{aligned}$$

Hence, it follows from Theorem 2.14 that

$$h(T_A) = \lim_{\varepsilon\to 0}\liminf_{m\to\infty}\frac{1}{m}\log M(m,\varepsilon) \geqslant -d\log(\lambda^{-1}+\delta).$$

Letting $\delta \to 0$ we conclude that

$$h(T_A) \geqslant d\log\lambda > 0.$$

Problem 4.20 For the maps $f,g\colon \mathbb{R}\to\mathbb{R}$ defined by $f(x) = ax$ and $g(x) = bx$, with $a,b > 1$, show that there exists a homeomorphism $h\colon \mathbb{R}\to\mathbb{R}$ such that $h\circ f = g\circ h$.

Solution We assume that h is differentiable on $\mathbb{R}^+$ with $h(\mathbb{R}^+) = \mathbb{R}^+$. It follows from the identity $h\circ f = g\circ h$ that

$$h\circ f^n = g^n\circ h$$

for all $n\in\mathbb{N}$, that is, for each $x > 0$ we have

$$h(a^n x) = b^n h(x) \quad \text{for } n\in\mathbb{N}. \tag{II.4.17}$$

Now consider property (II.4.17) extended to $t\in\mathbb{R}$, that is,

$$h(a^t x) = b^t h(x) \quad \text{for } t\in\mathbb{R}. \tag{II.4.18}$$

Taking derivatives in (II.4.18) with respect to t, we obtain

$$h'(a^t x)a^t x\log a = b^t\log b h(x).$$

Moreover, taking $t = 0$ yields the identity

$$h'(x)x\log a = \log b h(x)$$

and so $h(x) = x^{\log b/\log a}$, up to a multiplicative constant. This suggests considering the function $h\colon \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x^{\log b/\log a} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -|x|^{\log b/\log a} & \text{if } x < 0. \end{cases} \tag{II.4.19}$$

One can easily verify that h is a homeomorphism, with inverse $h^{-1}\colon \mathbb{R} \to \mathbb{R}$ given by

$$h^{-1}(x) = \begin{cases} x^{\log a/\log b} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -|x|^{\log a/\log b} & \text{if } x < 0. \end{cases}$$

Moreover,

$$\begin{aligned} h(f(x)) &= (ax)^{\log b/\log a} \\ &= e^{\log b} x^{\log b/\log a} \\ &= bx^{\log b/\log a} \\ &= g(h(x)) \end{aligned}$$

for $x > 0$. One can show analogously that the same identity also holds for $x \leqslant 0$.

Problem 4.21 Construct a topological conjugacy between the maps $f, g\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = (2x, 3y) \quad \text{and} \quad g(x,y) = (5x, 4y)$$

for $(x,y) \in \mathbb{R}^2$.

Solution Consider the maps

$$h_1(x) = \begin{cases} x^{\log 5/\log 2} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -|x|^{\log 5/\log 2} & \text{if } x < 0 \end{cases}$$

and

$$h_2(x) = \begin{cases} x^{\log 4/\log 3} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -|x|^{\log 4/\log 3} & \text{if } x < 0 \end{cases}$$

(see (II.4.19)). Moreover, let $f_a(x) = ax$. By Problem 4.20, we have

$$h_1 \circ f_2 = f_5 \circ h_1 \quad \text{and} \quad h_2 \circ f_3 = f_4 \circ h_2,$$

Now consider the map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$h(x,y) = (h_1(x), h_2(y)),$$

which is a homeomorphism, with inverse $h^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$h^{-1}(x,y) = (h_1^{-1}(x), h_2^{-1}(y)).$$

We have

$$\begin{aligned} h(f(x,y)) &= \big((2x)^{\log 5/\log 2}, (3y)^{\log 4/\log 3}\big) \\ &= \big(e^{\log 5} x^{\log 5/\log 2}, e^{\log 4} y^{\log 4/\log 3}\big) \\ &= \big(5x^{\log 5/\log 2}, 4y^{\log 4/\log 3}\big) \\ &= g(h(x,y)) \end{aligned}$$

for $x > 0$. One can show analogously that the same identity holds for $x \leqslant 0$ and so

$$h \circ f = g \circ h.$$

Therefore, h is a topological conjugacy between f and g.

Problem 4.22 Determine the 2-periodic points of the automorphism of the torus T_A for the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Solution The 2-periodic points of T_A are obtained solving the equation

$$\begin{aligned} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

on $\mathbb{T}^2$, that is,

$$\begin{cases} 5x + 3y \equiv x \bmod 1, \\ 3x + 2y \equiv y \bmod 1. \end{cases}$$

This gives

$$y \equiv -3x \bmod 1, \quad 5x \equiv 0 \bmod 1$$

and so the 2-periodic points of T_A are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/5 \\ 2/5 \end{pmatrix}, \begin{pmatrix} 2/5 \\ 4/5 \end{pmatrix}, \begin{pmatrix} 3/5 \\ 1/5 \end{pmatrix}, \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}.$$

Problem 4.23 Determine the 2-periodic points of the endomorphism of the torus T_B for the matrix

$$B = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

Solution The 2-periodic points of T_B are obtained solving the equation

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

on $\mathbb{T}^2$, that is,

$$\begin{cases} 10x + 4y \equiv x \bmod 1, \\ 4x + 2y \equiv y \bmod 1. \end{cases}$$

This gives

$$y \equiv -4x \bmod 1, \quad 7x \equiv 0 \bmod 1$$

and so the 2-periodic points of T_B are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/7 \\ 3/7 \end{pmatrix}, \begin{pmatrix} 2/7 \\ 6/7 \end{pmatrix}, \begin{pmatrix} 3/7 \\ 2/7 \end{pmatrix}, \begin{pmatrix} 4/7 \\ 5/7 \end{pmatrix}, \begin{pmatrix} 5/7 \\ 1/7 \end{pmatrix}, \begin{pmatrix} 6/7 \\ 4/7 \end{pmatrix}.$$

Problem 4.24 Show that the vector $X(x) = \frac{d}{dt}\varphi_t(x)\big|_{t=0}$ in Definition 4.9 is neither contracted nor expanded by $d_x\varphi_t$.

Solution Since $(\varphi_t)_{t\in\mathbb{R}}$ is a flow, we have $\varphi_{t\circ s} = \varphi_t \circ \varphi_s$ for all $t, s \in \mathbb{R}$. Hence,

$$\frac{d}{dt}\varphi_{t+s}(x)\big|_{t=0} = \frac{d}{dt}\varphi_t(\varphi_s(x))\big|_{t=0} = X(\varphi_s(x))$$

and

$$\begin{aligned} X(\varphi_s(x)) &= \frac{d}{dt}\varphi_{t+s}(x)\big|_{t=0} \\ &= \frac{d}{dt}\varphi_s(\varphi_t(x))\big|_{t=0} \\ &= d_{\varphi_t(x)}\varphi_s\big|_{t=0}\frac{d}{dt}\varphi_t(x)\big|_{t=0} \\ &= d_x\varphi_s X(x). \end{aligned}$$

Therefore,

$$\|d_x\varphi_s X(x)\| = \|X(\varphi_s(x))\|,$$

which yields the desired property.

Problem 4.25 Let Λ be a hyperbolic set for a flow Φ on $\mathbb{R}^p$. Given a sequence $x_m \to x$ when $m \to \infty$, with $x_m, x \in \Lambda$, for each $m \in \mathbb{N}$, show that any sublimit of a sequence $v_m \in E^s(x_m) \subseteq \mathbb{R}^p$ with $\|v_m\| = 1$ is in $E^s(x)$.

Solution Since the closed unit sphere in $\mathbb{R}^p$ is compact, the sequence v_m has sublimits. Moreover, since $v_m \in E^s(x_m)$, there exist constants $c > 0$ and $\lambda \in (0,1)$ such that

$$\|d_{x_m}\varphi_t v_m\| \leqslant c\lambda^t \|v_m\|$$

for all $m \in \mathbb{N}$ and $t \geqslant 0$. Letting $m \to \infty$, we obtain

$$\|d_x\varphi_t v\| \leqslant c\lambda^t \|v\|$$

for all $t \geqslant 0$ and any sublimit v of the sequence v_m. This implies that v has no component in $E^0(x) \oplus E^u(x)$ and so $v \in E^s(x)$.

Problem 4.26 For a sequence x_m as in Problem 4.25, show that there exists $m \in \mathbb{N}$ such that

$$\dim E^s(x_p) = \dim E^s(x_q) \quad \text{and} \quad \dim E^u(x_p) = \dim E^s(x_q)$$

for any $p, q > m$.

Solution Since the dimensions $\dim E^s(x_m)$ and $\dim E^u(x_m)$ take only finitely many values, there exists a subsequence y_m of x_m such that the numbers $\dim E^s(y_m)$ and $\dim E^u(y_m)$ are independent of m. Let $k = \dim E^s(y_m)$ (which is independent of m) and let

$$v_{1m}, \dots, v_{km} \in E^s(y_m)$$

be an orthonormal basis for $E^s(y_m)$. Since the closed unit sphere in $\mathbb{R}^p$ is compact, the sequence $(v_{1m}, \dots, v_{km})$ has sublimits and each sublimit $(v_1, \dots, v_k)$ is an orthonormal set. It follows from Problem 4.25 that

$$v_1, \dots, v_k \in E^s(x) \quad \text{and so } \dim E^s(x) \geqslant k.$$

Proceeding analogously for the unstable spaces, we obtain $\dim E^u(x) \geqslant p - 1 - k$. Since

$$\mathbb{R}^p = E^s(x) \oplus E^0(x) \oplus E^u(x),$$

this implies that

$$\dim E^s(x) = k \quad \text{and} \quad \dim E^u(x) = p - 1 - k. \tag{II.4.20}$$

In particular, the vectors $v_1, \dots, v_k$ generate $E^s(x)$. If z_m is another subsequence of x_m such that $\dim E^s(z_m)$ and $\dim E^u(z_m)$ are independent of m, respectively, with values l and $p - 1 - l$, then

$$\dim E^s(x) = l \quad \text{and} \quad \dim E^u(x) = p - 1 - l.$$

Comparing with (II.4.20), we conclude that $l = k$. This establishes the desired property.

Problem 4.27 Find the stable and unstable invariant manifolds at the origin for the differential equation

$$\begin{cases} x' = x, \\ y' = -y + x^2. \end{cases}$$

Solution Note that the origin is the only critical point of the differential equation. Since the matrix of the linearization at the origin

$$\begin{cases} x' = x, \\ y' = -y, \end{cases}$$

has eigenvalues ± 1, the origin is a hyperbolic fixed point. The stable and unstable spaces are, respectively,

$$E^s = \{(0,a) : a \in \mathbb{R}\} \quad \text{and} \quad E^u = \{(a,0) : a \in \mathbb{R}\}$$

(the vertical and horizontal axes). Moreover, for each $b \in \mathbb{R}$ the set

$$\left\{ (x,y) \in \mathbb{R}^2 : y - \frac{x^2}{3} = \frac{b}{x} \right\} \tag{II.4.21}$$

is invariant under the flow determined by the differential equation. Indeed,

$$\begin{aligned} \left(y - \frac{x^2}{3} - \frac{b}{x} \right)' \Big|_{y-x^2/3=b/x} &= \left(-y + x^2 - \frac{2x}{3}x' + \frac{b}{x^2}x' \right) \Big|_{y-x^2/3=b/x} \\ &= \left(-y + x^2 - 2\frac{x^2}{3} + \frac{b}{x} \right) \Big|_{y-x^2/3=b/x} \\ &= -\frac{x^2}{3} - \frac{b}{x} + x^2 - 2\frac{x^2}{3} + \frac{b}{x} = 0. \end{aligned}$$

For $b = 0$ the set in (II.4.21) is tangent to E^u at the origin. Hence, it is the unstable invariant manifold at the origin.

Furthermore, the line $x = 0$ is invariant under the flow determined by the differential equation and is tangent to E^s at the origin. Hence, it is the stable invariant manifold at the origin.

Problem 4.28 Find the stable and unstable invariant manifolds at the origin for the differential equation

$$\begin{cases} x' = x(a-y), \\ y' = -y + x^2 - 2y^2, \end{cases}$$

for each given $a > 0$.

Solution The origin is a critical point of the differential equation. The linearization at the origin is

$$\begin{cases} x' = ax, \\ y' = -y \end{cases}$$

with $a > 0$, and so $(0,0)$ is a hyperbolic fixed point for each $a > 0$. The stable and unstable spaces are, respectively,

$$E^s = \{(0,b) : b \in \mathbb{R}\} \quad \text{and} \quad E^u = \{(b,0) : b \in \mathbb{R}\}.$$

On the other hand, by Problem 1.32, the manifolds defined by

$$x = 0 \quad \text{and} \quad y = \frac{x^2}{1+2a}$$

are invariant under the flow determined by the differential equation. Since they are tangent, respectively, to E^s and E^u at the origin, they are the stable and unstable invariant manifolds at the origin.

Problem 4.29 Given $a, b > 0$, show that the map $h\colon \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x^{b/a} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -|x|^{b/a} & \text{if } x < 0. \end{cases}$$

is a topological conjugacy between the flows determined by the differential equations

$$x' = ax \quad \text{and} \quad x' = bx.$$

In other words, if φ_t and ψ_t are the flows determined by the equations, then

$$h \circ \varphi_t = \psi_t \circ h \quad \text{for } t \in \mathbb{R}.$$

Solution We have

$$\varphi_t(x) = e^{at}x \quad \text{and} \quad \psi_t(x) = e^{bt}x.$$

Moreover, h is a homeomorphism, with inverse $h^{-1}\colon \mathbb{R} \to \mathbb{R}$ given by

$$h^{-1}(x) = \begin{cases} x^{a/b} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -|x|^{a/b} & \text{if } x < 0. \end{cases}$$

Finally, we have

$$\begin{aligned} h(\varphi_t(x)) &= h(e^{at}x) \\ &= \left(e^{3t}x\right)^{b/a} \\ &= e^{bt}x^{b/a} \end{aligned}$$

and

$$\psi_t(h(x)) = \psi_t(x^{b/a}) = e^{bt}x^{b/a}$$

for $x > 0$, with analogous identities for $x \leqslant 0$.

Problem 4.30 Let A and B be $n \times n$ matrices. Show that if there exists a topological conjugacy between the flows determined by the differential equations

$$v' = Av \quad \text{and} \quad v' = Bv$$

that is a diffeomorphism, then there exists also a topological conjugacy between the flows that is a linear map.

Solution The flows determined by the differential equations are, respectively,

$$\varphi_t(v) = e^{At}v \quad \text{and} \quad \psi_t(v) = e^{Bt}v.$$

Now let $h\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that

$$h(e^{At}v) = e^{Bt}h(v).$$

Taking derivatives with respect to v, we obtain

$$d_{e^{At}v}he^{At} = e^{Bt}d_vh.$$

Letting $v = 0$ yields the identity

$$Ce^{At} = e^{Bt}C, \quad \text{with } C = d_0h. \tag{II.4.22}$$

Finally, taking derivatives in the identity $h^{-1}(h(v)) = v$ we obtain

$$d_{h(0)}h^{-1}C = \text{ Id},$$

which shows that the matrix C is invertible. Now we consider the invertible linear map $g\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $g(v) = Cv$. It follows from (II.4.22) that

$$g(e^{At}v) = e^{Bt}g(v) \tag{II.4.23}$$

for every $t \in \mathbb{R}$ and $v \in \mathbb{R}^n$, that is,

$$g \circ \varphi_t = \psi_t \circ g.$$

Problem 4.31 Let A and B be $n \times n$ matrices. Show that if there exists a topological conjugacy between the flows determined by the differential equations in Problem 4.30 that is a linear map, then the matrices A and B are conjugate, that is, there exists an invertible $n \times n$ matrix C such that $A = C^{-1}BC$.

Solution Assume that (II.4.23) holds for some invertible linear map $g\colon \mathbb{R}^n \to \mathbb{R}^n$ and all $t \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Writing $g(v) = Cv$ for some invertible $n \times n$ matrix C, we obtain

$$Ce^{At} = e^{Bt}C.$$

Finally, taking $t = 0$ yields the identity $CA = BC$, which establishes the desired property.

Problem 4.32 Let φ_t and ψ_t be the flows determined, respectively, by the linear equations with matrices

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Verify that the map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(0,0) = 0$ and

$$h(x,y) = \sqrt{\frac{x^2+y^2}{x^2+xy+3y^2/2}}\left(x - \frac{y}{2}\log\frac{x^2+y^2}{2}, y\right)$$

for $(x,y) \neq (0,0)$ satisfies

$$h \circ \varphi_t = \psi_t \circ h.$$

Solution One can easily verify that

$$e^{At} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} \qquad \text{and} \qquad e^{Bt} = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
&h(e^{At}(x,y)) \\
&= h(e^{-t}x, e^{-t}y) \\
&= \sqrt{\frac{e^{-2t}x^2+e^{-2t}y^2}{e^{-2t}x^2+e^{-2t}xy+3e^{-2t}y^2/2}}\left(e^{-t}x - \frac{e^{-t}y}{2}\log\frac{e^{-2t}x^2+e^{-2t}y^2}{2}, e^{-t}y\right) \\
&= \sqrt{\frac{x^2+y^2}{x^2+xy+3y^2/2}}\left(e^{-t}x + te^{-t}y - \frac{e^{-t}y}{2}\log\frac{x^2+y^2}{2}, e^{-t}y\right) \\
&= e^{Bt}\sqrt{\frac{x^2+y^2}{x^2+xy+3y^2/2}}\left(x - \frac{y}{2}\log\frac{x^2+y^2}{2}, y\right) \\
&= e^{Bt}h(x,y).
\end{aligned}$$

Problem 4.33 For the differential equation

$$\begin{cases} x' = -x, \\ y' = y + x^2, \end{cases}$$

show that the map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$h(x,y) = \left(x, y + \frac{x^2}{3}\right)$$

is a topological conjugacy between the flows determined by the equation and its linearization at the origin.

Solution The solution of the equation with $(x(0), y(0)) = (x_0, y_0)$ is given by

$$x(t) = x_0 e^{-t}, \quad y(t) = y_0 e^t + \frac{x_0^2}{3}(e^t - e^{-2t}) \tag{II.4.24}$$

with $t \in \mathbb{R}$. The linearization at the origin of the equation is

$$\begin{cases} x' = -x, \\ y' = y, \end{cases}$$

whose solution with $(x(0), y(0)) = (x_0, y_0)$ is given by

$$x(t) = x_0 e^{-t}, \quad y(t) = y_0 e^t \tag{II.4.25}$$

with $t \in \mathbb{R}$. By (II.4.24) and (II.4.25), the flows determined by the two equations are, respectively,

$$\varphi_t(x_0, y_0) = \left(x_0 e^{-t}, y_0 e^t + \frac{x_0^2}{3}(e^t - e^{-2t}) \right)$$

and

$$\psi_t(x_0, y_0) = (x_0 e^{-t}, y_0 e^t).$$

On the other hand, h is a homeomorphism, with inverse given by

$$h^{-1}(x, y) = \left(x, y - \frac{x^2}{3} \right).$$

Since

$$\begin{aligned} \psi_t(h(x_0, y_0)) &= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 + x_0^2/3 \end{pmatrix} \\ &= \begin{pmatrix} x_0 e^{-t} \\ e^t (y_0 + x_0^2/3) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} h(\varphi_t(x_0, y_0)) &= h\left(x_0 e^{-t}, y_0 e^t + \frac{x_0^2}{3}(e^t - e^{-2t}) \right) \\ &= \left(x_0 e^{-t}, y_0 e^t + \frac{x_0^2}{3}(e^t - e^{-2t}) + \frac{(x_0 e^{-t})^2}{3} \right) \\ &= \left(x_0 e^{-t}, e^t (y_0 + \frac{x_0^2}{3}) \right), \end{aligned}$$

we conclude that

$$h \circ \varphi_t = \psi_t \circ h \quad \text{for } t \in \mathbb{R}. \tag{II.4.26}$$

Problem 4.34 For the differential equation

$$\begin{cases} x' = x, \\ y' = -y + x^n, \end{cases}$$

with $n \geqslant 2$, show that the map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$h(x,y) = \left(x, y - \frac{x^n}{n+1}\right)$$

is a topological conjugacy between the flows determined by the equation and its linearization at the origin.

Solution The solution of the equation with $(x(0), y(0)) = (x_0, y_0)$ is given by

$$x(t) = x_0 e^t, \quad y(t) = y_0 e^{-t} - \frac{x_0^n}{n+1}(e^{-t} - e^{nt}) \tag{II.4.27}$$

with $t \in \mathbb{R}$. The linearization at the origin of the equation is

$$\begin{cases} x' = x, \\ y' = -y, \end{cases}$$

whose solution with $(x(0), y(0)) = (x_0, y_0)$ is given by

$$x(t) = x_0 e^t, \quad y(t) = y_0 e^{-t} \tag{II.4.28}$$

with $t \in \mathbb{R}$. By (II.4.27) and (II.4.28), the flows determined by the two equations are, respectively,

$$\varphi_t(x_0, y_0) = \left(x_0 e^t, y_0 e^{-t} - \frac{x_0^n}{n+1}(e^{-t} - e^{nt})\right)$$

and

$$\psi_t(x_0, y_0) = (x_0 e^t, y_0 e^{-t}).$$

Moreover, one can easily verify that h is a homeomorphism. Since

$$\begin{aligned} \psi_t(h(x_0, y_0)) &= \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 - x_0^n/(n+1) \end{pmatrix} \\ &= \begin{pmatrix} x_0 e^t \\ e^{-t}(y_0 - x_0^n/(n+1)) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} h(\varphi_t(x_0, y_0)) &= h\left(x_0 e^t, y_0 e^{-t} - \frac{x_0^n}{n+1}(e^{-t} - e^{nt})\right) \\ &= \left(x_0 e^t, y_0 e^{-t} - \frac{x_0^n}{n+1}(e^{-t} - e^{nt}) - \frac{(x_0 e^t)^n}{n+1}\right) \\ &= \left(x_0 e^t, e^{-t}(y_0 - \frac{x_0^n}{n+1})\right), \end{aligned}$$

we find that property (II.4.26) holds.

Problem 4.35 Consider the Möbius transformation

$$T_A(z) = \frac{az+b}{cz+d}$$

associated with the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with } \det A = ad - bc = 1,$$

and let

$$\begin{aligned} T_{1,c}(z) &= T_{\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)}(z) = z + c \quad \text{for } c \in \mathbb{R}, \\ T_{2,c}(z) &= T_{\left(\begin{smallmatrix} c & 0 \\ 0 & 1/c \end{smallmatrix}\right)}(z) = c^2 z \quad \text{for } c \in \mathbb{R}\setminus\{0\}, \\ T_{3,c}(z) &= T_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & c \end{smallmatrix}\right)}(z) = -\frac{1}{z+c} \quad \text{for } c \in \mathbb{R}. \end{aligned}$$

Show that

$$T_A = \begin{cases} T_{1,ab} \circ T_{2,a} & \text{if } c = 0, \\ T_{1,a/c} \circ T_{2,1/c} \circ T_{3,d/c} & \text{if } c \neq 0. \end{cases}$$

Solution If $c = 0$, then

$$\begin{aligned} T_A(z) &= \frac{az+b}{d} \\ &= \frac{a}{d}z + \frac{b}{d} \\ &= a^2 z + ab \\ &= (T_{1,ab} \circ T_{2,a})(z). \end{aligned}$$

On the other hand, if $c \neq 0$, then

$$\begin{aligned} T_A(z) &= \frac{c(az+b)}{c(cz+d)} \\ &= \frac{caz+cb}{c^2 z + cd} \\ &= \frac{caz + ad - ad + bc}{c^2 z + cd} \\ &= \frac{a(cz+d)}{c(cz+d)} - \frac{1}{c(cz+d)} \\ &= \frac{a}{c} - \frac{1/c^2}{z + d/c} \\ &= (T_{1,a/c} \circ T_{2,1/c} \circ T_{3,d/c})(z). \end{aligned}$$

Problem 4.36 Show that any straight line or circle is determined by an equation of the form

$$Az\bar{z} + Bz + \overline{B}\bar{z} + C = 0 \quad \text{with } A, C \in \mathbb{R},\ B \in \mathbb{C},$$

respectively, with $A = 0$ and with $A \neq 0$ and $AC < B\overline{B}$.

Solution Writing $z = x + iy$ and $B = B_1 + iB_2$, the equation in the problem becomes

$$A(x^2 + y^2) + 2B_1x - 2B_2y + C = 0. \tag{II.4.29}$$

When $A = 0$, clearly (II.4.29) determines a straight line. On the other hand, when $A \neq 0$ one can rewrite (II.4.29) in the form

$$A\left(x + \frac{B_1}{A}\right)^2 + A\left(y - \frac{B_2}{A}\right)^2 - \frac{B_1^2}{A} - \frac{B_2^2}{A} + C = 0,$$

that is,

$$\left(x + \frac{B_1}{A}\right)^2 + \left(y - \frac{B_2}{A}\right)^2 = \frac{B_1^2}{A^2} - \frac{B_2^2}{A^2} - \frac{C}{A} = \frac{B\overline{B}}{A^2} - \frac{C}{A}. \tag{II.4.30}$$

In order that (II.4.30) determines a circle we must have

$$\frac{B\overline{B}}{A^2} - \frac{C}{A} > 0.$$

Problem 4.37 Show that the map $T(z) = 2z$ takes straight lines into straight lines and circles into circles.

Solution If the equation in Problem 4.36 holds, then

$$z = T^{-1}(w) = w/2$$

satisfies

$$\begin{aligned} 0 &= Aw\overline{w}/4 + Bw/2 + \overline{B}\overline{w}/\sqrt{2} + C \\ &= A'w\overline{w} + B'w + \overline{B}'\overline{w} + C', \end{aligned}$$

with

$$A' = A/4, \quad B' = B/2, \quad C' = C.$$

Note that $A', C' \in \mathbb{R}$. Moreover, $A' = 0$ if and only if $A = 0$ and so T maps straight lines into straight lines. Finally, if $A \neq 0$ and $AC < B\overline{B}$, then

$$A'C' = AC/4 < B\overline{B}/4 = B'\overline{B}'$$

and so T maps circles into circles.

Problem 4.38 Show that the map $T(z) = -1/(z+1)$ takes straight lines and circles into straight lines or circles.

Solution If the equation in Problem 4.36 holds, then

$$z = T^{-1}(w) = -1 - 1/w$$

satisfies

$$\begin{aligned}
0 &= A\left(-1-\frac{1}{w}\right)\left(-1-\frac{1}{\overline{w}}\right)+B\left(-1-\frac{1}{w}\right)+\overline{B}\left(-1-\frac{1}{\overline{w}}\right)+C \\
&= \frac{1}{w\overline{w}}A(w+1)(\overline{w}+1)+B(-w\overline{w}-\overline{w})+\overline{B}(-w\overline{w}-w)+C \\
&= \frac{1}{w\overline{w}}(A'w\overline{w}+B'w+\overline{B}'\overline{w}+C'),
\end{aligned}$$

with

$$A' = A-B-\overline{B}, \quad B' = A-\overline{B}, \quad C' = C+A.$$

Note that $A', C' \in \mathbb{R}$. If $A' = 0$, then T maps straight lines and circles into straight lines. On the other hand, if $A' \neq 0$, then T maps straight lines and circles into circles (clearly $A'C' < B'\overline{B}'$ since T is bijective).

Problem 4.39 Show that the components of the geodesic flow $\varphi_t\colon S\mathbb{H} \to S\mathbb{H}$ given by

$$\varphi_t(z,v) = (\gamma(t), \gamma'(t))$$

(see Definition 4.10), satisfy

$$(\operatorname{Re}\gamma'(t))' = \frac{2\operatorname{Re}\gamma'(t)\operatorname{Im}\gamma'(t)}{\operatorname{Im}\gamma(t)}$$

and

$$(\operatorname{Im}\gamma(t))' = \frac{(\operatorname{Im}\gamma'(t))^2-(\operatorname{Re}\gamma'(t))^2}{\operatorname{Im}\gamma(t)}.$$

Solution Let

$$T(z) = \frac{az+b}{cz+d}.$$

We have

$$\gamma(t) = T(ie^t) = \frac{aie^t+b}{cie^t+d}.$$

Since $ad-bc=1$, we obtain

$$\begin{aligned}
\gamma'(t) &= \frac{i(bc-ad)e^t}{(ce^t-id)^2} \\
&= \frac{-ie^t}{(ce^t-id)^2} \\
&= \frac{ie^t}{(cie^t+d)^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}\gamma'(t) &= \frac{ie^t}{(cie^t+d)^2} \\ &= \frac{ie^t(d-cie^t)^2}{(d^2+c^2e^{2t})^2} \\ &= \frac{2cde^{2t}}{(d^2+c^2e^{2t})^2} + i\frac{e^t(d^2-c^2e^{2t})}{(d^2+c^2e^{2t})^2},\end{aligned} \tag{II.4.31}$$

while

$$\begin{aligned}\gamma(t) &= \frac{aie^t+b}{cie^t+d} = \frac{(aie^t+b)(d-cie^t)}{d^2+c^2e^{2t}} \\ &= \frac{bd+ace^{2t}}{d^2+c^2e^{2t}} + i\frac{(ad-bc)e^t}{d^2+c^2e^{2t}} \\ &= \frac{bd+ace^{2t}}{d^2+c^2e^{2t}} + i\frac{e^t}{d^2+c^2e^{2t}}.\end{aligned} \tag{II.4.32}$$

It follows from (II.4.31) and (II.4.32) that

$$\operatorname{Re}\gamma'(t) = \frac{2cde^{2t}}{(d^2+c^2e^{2t})^2}, \quad \operatorname{Im}\gamma'(t) = \frac{e^t(d^2-c^2e^{2t})}{(d^2+c^2e^{2t})^2}, \tag{II.4.33}$$

and

$$\operatorname{Re}\gamma(t) = \frac{bd+ace^{2t}}{d^2+c^2e^{2t}}, \quad \operatorname{Im}\gamma(t) = \frac{e^t}{d^2+c^2e^{2t}}. \tag{II.4.34}$$

Therefore,

$$\begin{aligned}(\operatorname{Re}\gamma'(t))' &= \frac{d}{dt}\left(\frac{2cde^{2t}}{(d^2+c^2e^{2t})^2}\right) \\ &= \frac{4cd(d^2-c^2e^{2t})e^{2t}}{(d^2+c^2e^{2t})^3}\end{aligned}$$

and

$$\begin{aligned}\frac{2\operatorname{Re}\gamma'(t)\operatorname{Im}\gamma'(t)}{\operatorname{Im}\gamma(t)} &= 2\frac{2cde^{2t}}{(d^2+c^2e^{2t})^2}\cdot\frac{e^t(d^2-c^2e^{2t})}{(d^2+c^2e^{2t})^2}\left(\frac{e^t}{d^2+c^2e^{2t}}\right)^{-1} \\ &= \frac{4cd(d^2-c^2e^{2t})e^{2t}}{(d^2+c^2e^{2t})^3}.\end{aligned}$$

Finally, we also have

$$\begin{aligned}(\operatorname{Im}\gamma'(t))' &= \frac{d}{dt}\left(\frac{e^t(d^2-c^2e^{2t})}{(d^2+c^2e^{2t})^2}\right) \\ &= \frac{e^t(d^4-6c^2d^2e^{2t}+c^4e^{4t})}{(d^2+c^2e^{2t})^3}\end{aligned}$$

and

$$\begin{aligned}\frac{(\operatorname{Im}\gamma'(t))^2-(\operatorname{Re}\gamma'(t))^2}{\operatorname{Im}\gamma(t)}&=\frac{e^{2t}(d^4-6c^2d^2e^{2t}+c^4e^{4t})}{(d^2+c^2e^{2t})^4}\left(\frac{e^t}{d^2+c^2e^{2t}}\right)^{-1}\\&=\frac{e^t(d^4-6c^2d^2e^{2t}+c^4e^{4t})}{(d^2+c^2e^{2t})^3}.\end{aligned}$$

This establishes the desired identities.

Problem 4.40 For the geodesic flow $\varphi_t\colon S\mathbb{H}\to S\mathbb{H}$, show that the quantities

$$H_1=\frac{|\gamma'(t)|}{\operatorname{Im}\gamma(t)}\quad\text{and}\quad H_2=\frac{\operatorname{Re}\gamma'(t)}{(\operatorname{Im}\gamma(t))^2}$$

are independent of t.

Solution It follows from (II.4.33) and (II.4.34) that

$$\begin{aligned}|\gamma'(t)|^2&=(\operatorname{Re}\gamma'(t))^2+(\operatorname{Im}\gamma'(t))^2\\&=\frac{(d^2+c^2e^{2t})^2e^{2t}}{(d^2+c^2e^{2t})^4}\\&=\frac{e^{2t}}{(d^2+c^2e^{2t})^2}\end{aligned}$$

and

$$(\operatorname{Im}\gamma(t))^2=\frac{e^{2t}}{(d^2+c^2e^{2t})^2}.$$

Hence,

$$H_1=\frac{|\gamma'(t)|}{\operatorname{Im}\gamma(t)}=1$$

On the other hand, it also follows from (II.4.33) that

$$H_2=\frac{\operatorname{Re}\gamma'(t)}{(\operatorname{Im}\gamma(t))^2}=\frac{2dce^{2t}}{e^{2t}}=2cd.$$

Chapter II.5
Symbolic Dynamics

Problem 5.1 Show that for each $\omega \in \Sigma_k^+$ and $r > 0$ the open ball

$$B_{d_\beta}(\omega, r) = \{\omega' \in \Sigma_k^+ : d_\beta(\omega', \omega) < r\}$$

is a cylinder, that is, there exist integers $i_1, \ldots, i_n \in \{1, \ldots, k\}$ with $n = n(r)$ such that

$$B_{d_\beta}(\omega, r) = C_{i_1 \cdots i_n} := \{\omega \in \Sigma_k^+ : i_j(\omega) = i_j \text{ for } j = 1, \ldots, n\}.$$

Solution Note that $\omega' \in B_{d_\beta}(\omega, r)$ if and only if $d_\beta(\omega, \omega') < r$. By the definition of d_β this is equivalent to require that

$$\beta^{-n(\omega,\omega')} < r \quad \iff \quad n(\omega, \omega') > -\frac{\log r}{\log \beta}$$

(again $n(\omega, \omega') \in \mathbb{N}$ is the smallest positive integer n such that $i_n(\omega) \neq i_n(\omega')$). Therefore, $\omega' \in B_{d_\beta}(\omega, r)$ if and only if

$$i_j(\omega') = i_j(\omega) \quad \text{for } j \leqslant n(r) := \left\lfloor -\frac{\log r}{\log \beta} \right\rfloor,$$

which is the same as $\omega' \in C_{i_1(\omega) \cdots i_{n(r)}(\omega)}$.

Problem 5.2 Show that any cylinder $C_{i_1 \cdots i_n}$ (see Problem 5.1) is simultaneously open and closed.

Solution By Problem 5.1, any cylinder is an open ball.

To show that any cylinder is closed, note first that $\omega'' \notin C_{i_1 \cdots i_n}$ if and only if $i_j(\omega) \neq i_j$ for some $j \in \{1, \ldots, n\}$. If $\omega' \in C_{i_1 \cdots i_n}$, then

$$i_j(\omega') = i_j \quad \text{for } j \in \{1, \ldots, n\}$$

and so $d_\beta(\omega'', \omega') \geqslant \beta^{-n}$, which shows that

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_11

$$B_{d_\beta}(\omega'', \beta^{-n}) \cap C_{i_1 \cdots i_n} = \varnothing.$$

Therefore, the complement of each cylinder $C_{i_1 \cdots i_n}$ is open.

Problem 5.3 Given $\beta, \beta' > 1$, show that the distances d_β and $d_{\beta'}$ generate the same topology on Σ_k^+.

Solution It suffices to verify that given $\omega \in \Sigma_k^+$ and $r > 0$, for each $\omega' \in B_{d_\beta}(\omega, r)$ there exists $r' > 0$ such that

$$B_{d_{\beta'}}(\omega', r') \subseteq B_{d_\beta}(\omega, r). \tag{II.5.1}$$

Indeed, one can then interchange the roles of β and β' to also conclude that given $\omega \in \Sigma_k^+$ and $r > 0$, for each $\omega' \in B_{d_{\beta'}}(\omega, r)$ there exists $r' > 0$ such that

$$B_{d_\beta}(\omega', r') \subseteq B_{d_{\beta'}}(\omega, r).$$

This implies that the two distances d_β and $d_{\beta'}$ generate the same topology on Σ_k^+.

Take $\omega' \in B_{d_\beta}(\omega, r)$. By Problem 5.1 we have

$$B_{d_\beta}(\omega, r) = C_{i_1 \cdots i_{n(r)}}, \quad \text{with } n = n(r),$$

for some integers $i_1, \ldots, i_{n(r)} \in \{1, \ldots, k\}$. Therefore, $\omega' \in B_{d_\beta}(\omega, r)$ if and only if $i_n(\omega') = i_n$ for $n \leqslant n(r)$ and so

$$B_{d_{\beta'}}(\omega', r') \subseteq B_{d_\beta}(\omega, r) \quad \text{for all } r' \leqslant \beta^{-n(r)}.$$

This establishes property (II.5.1).

Problem 5.4 Show that one can define a distance on Σ_k^+ by

$$d(\omega, \omega') = \sum_{n=1}^{\infty} \frac{1}{2^n} |i_n(\omega) - i_n(\omega')|$$

and that it generates the same topology as the distance d_β.

Solution Clearly, $d(\omega, \omega') \geqslant 0$ and $d(\omega, \omega') = 0$ if and only if $i_n(\omega) = i_n(\omega')$ for all $n \in \mathbb{N}$, that is, if and only if $\omega = \omega'$. Moreover, $d(\omega, \omega') = d(\omega', \omega)$ and

$$d(\omega, \omega') \leqslant d(\omega, \omega'') + d(\omega'', \omega').$$

Hence, d is a distance on Σ_k^+.

It remains to verify that the distances d and d_β generate the same topology. As in Problem 5.3, it suffices to verify that given $\omega \in \Sigma_k^+$ and $r > 0$, for each $\omega' \in B_{d_\beta}(\omega, r)$ there exists $r_1 > 0$ such that

$$B_d(\omega', r_1) \subseteq B_{d_\beta}(\omega, r) \tag{II.5.2}$$

and for each $\omega' \in B_d(\omega, r)$ there exists $r_2 > 0$ such that

$$B_{d_\beta}(\omega', r_2) \subseteq B_d(\omega, r). \tag{II.5.3}$$

For the first property note that, by Problem 5.1, $\omega' \in B_{d_\beta}(\omega, r)$, that is, $d_\beta(\omega, \omega') < r$ if and only if there exists $n(r) \in \mathbb{N}$ such that

$$i_n(\omega) = i_n(\omega') \quad \text{for } n \leqslant n(r). \tag{II.5.4}$$

If $\omega'' \in B_d(\omega', r_1)$, that is, $d(\omega'', \omega') < r_1$ for some $r_1 < 1/2^{n(r)}$, then

$$i_n(\omega'') = i_n(\omega') \quad \text{for } n \leqslant n(r) \tag{II.5.5}$$

since otherwise we would have

$$d(\omega'', \omega') \geqslant \frac{1}{2^n}|i_n(\omega'') - i_n(\omega')| \geqslant \frac{1}{2^n} > r_1$$

for some $n \leqslant n(r)$. This establishes property (II.5.2).

Now take $\omega' \in B_d(\omega, r)$. Again by Problem 5.1, the ball $B_{d_\beta}(\omega', r_2)$ is a cylinder $C_{i_1 \cdots i_m}$ with $m \to \infty$ when $r_2 \to 0$. Hence, for $\omega'' \in B_{d_\beta}(\omega', r_2)$ we have

$$\begin{aligned} d(\omega'', \omega) &= \sum_{n=1}^{\infty} \frac{1}{2^n}|i_n(\omega'') - i_n(\omega)| \\ &\leqslant \sum_{n=m+1}^{\infty} \frac{1}{2^n}(k-1) = \frac{k-1}{2^m} < r \end{aligned}$$

provided that r_2 is sufficiently small. This shows that $\omega'' \in B_d(\omega, r)$, which establishes property (II.5.3).

Problem 5.5 Show that one can define a distance on Σ_k^+ by

$$\bar{d}(\omega, \omega') = \begin{cases} 1/n(\omega, \omega') & \text{if } \omega \neq \omega', \\ 0 & \text{if } \omega = \omega' \end{cases}$$

and that it generates the same topology as the distance d_β.

Solution Clearly, $\bar{d}(\omega, \omega') \geqslant 0$ and $\bar{d}(\omega, \omega') = 0$ if and only if $\omega = \omega'$. Moreover, $\bar{d}(\omega, \omega') = \bar{d}(\omega', \omega)$. For the triangle inequality, given $\omega, \omega', \omega'' \in \Sigma_k^+$, let n_1, n_2 and n_3 be, respectively, the smallest positive integers such that

$$i_{n_1}(\omega) \neq i_{n_1}(\omega''), \quad i_{n_2}(\omega) \neq i_{n_2}(\omega'), \quad i_{n_3}(\omega') \neq i_{n_3}(\omega''). \tag{II.5.6}$$

Then

$$\bar{d}(\omega, \omega'') = \frac{1}{n_1}, \quad \bar{d}(\omega, \omega') = \frac{1}{n_2}, \quad \bar{d}(\omega', \omega'') = \frac{1}{n_3}.$$

Observe that if $n_2 > n_1$ and $n_3 > n_1$, then $i_{n_1}(\omega) = i_{n_1}(\omega') = i_{n_1}(\omega'')$, which contradicts (II.5.6). Hence, $n_2 \leqslant n_1$ or $n_3 \leqslant n_1$, which gives

$$\frac{1}{n_1} \leqslant \frac{1}{n_2} \quad \text{or} \quad \frac{1}{n_1} \leqslant \frac{1}{n_3}.$$

This shows that

$$\bar{d}(\omega,\omega') \leqslant \bar{d}(\omega,\omega'') + \bar{d}(\omega'',\omega')$$

and so $\bar{d}$ is a distance on Σ_k^+.

Now we verify that the distances $\bar{d}$ and d_β generate the same topology. As in Problem 5.3, it suffices to verify that given $\omega \in \Sigma_k^+$ and $r > 0$, for each $\omega' \in B_{d_\beta}(\omega, r)$ there exists $r_1 > 0$ satisfying (II.5.2) and for each $\omega' \in B_{\bar{d}}(\omega, r)$ there exists $r_2 > 0$ satisfying (II.5.3). For the first property note that, by Problem 5.1, $\omega' \in B_{d_\beta}(\omega, r)$ if and only if there exists $n(r) \in \mathbb{N}$ satisfying (II.5.4). If $\omega'' \in B_{\bar{d}}(\omega', r_1)$ for some $r_1 < 1/n(r)$, then property (II.5.5) holds since otherwise we would have

$$\bar{d}(\omega'',\omega') \geqslant \frac{1}{n} \geqslant \frac{1}{n(r)} > r_1$$

for some $n \leqslant n(r)$. This establishes property (II.5.2).

Now take $\omega' \in B_{\bar{d}}(\omega, r)$. Again by Problem 5.1, the ball $B_{d_\beta}(\omega', r_2)$ is a cylinder $C_{i_1\cdots i_m}$ with $m \to \infty$ when $r_2 \to 0$. Hence, for $\omega'' \in B_{d_\beta}(\omega', r_2)$ we have

$$\bar{d}(\omega'',\omega) < \frac{1}{m} < r$$

provided that r_2 is sufficiently small. This shows that $\omega'' \in B_{\bar{d}}(\omega, r)$, which establishes property (II.5.3).

Problem 5.6 Show that the distances d and $\bar{d}$ in Problems 5.4 and 5.5 are not equivalent.

Solution Consider the constant sequence $\omega = (111\cdots)$ and for each $m \in \mathbb{N}$ define a new sequence $\omega_m \in \Sigma_k^+$ by

$$i_j(\omega_m) = \begin{cases} 1 & \text{if } j < m, \\ 2 & \text{if } j \geqslant m. \end{cases}$$

Note that the smallest positive integer j such that $i_j(\omega) \neq i_j(\omega_m)$ is $n(\omega, \omega_m) = m$. Now assume that d and $\bar{d}$ are equivalent. Then there exists $c > 0$ such that

$$c^{-1} d(\omega,\omega_m) \leqslant \bar{d}(\omega,\omega_m) \leqslant c\,d(\omega,\omega_m) \quad \text{for } m \in \mathbb{N}$$

or, equivalently,

$$c^{-1}\frac{1}{2^{m-1}} \leqslant \frac{1}{m} \leqslant c\frac{1}{2^{m-1}} \quad \text{for } m \in \mathbb{N}.$$

Multiplying by 2^{m-1} and letting $m \to \infty$ yields a contradiction. Therefore, the distances d and $\bar{d}$ are not equivalent.

Problem 5.7 Show that $(\Sigma_k^+, \bar{d})$ is a compact metric space.

Solution First note that the open balls in the distance $\bar{d}$ are again cylinders. Indeed, $\omega' \in B_{\bar{d}}(\omega, r)$ if and only if

$$\frac{1}{n(\omega,\omega')} < r \quad \Longleftrightarrow \quad n(\omega,\omega') > \frac{1}{r}$$

(again $n(\omega,\omega') \in \mathbb{N}$ is the smallest positive integer n such that $i_n(\omega) \neq i_n(\omega')$). Therefore, $\omega' \in B_{\bar{d}}(\omega, r)$ if and only if

$$i_j(\omega') = i_j(\omega) \quad \text{for } j \leqslant n(r) := \left\lfloor \frac{1}{r} \right\rfloor.$$

To show that Σ_k^+ is compact, we first equip $\{1,\dots,k\}$ with the topology in which all subsets of $\{1,\dots,k\}$ are open. Then the product topology on $\Sigma_k^+ = \{1,\dots,k\}^{\mathbb{N}}$ coincides with the topology generated by the cylinders. It follows from Tychonoff's theorem that $(\Sigma_k^+, \bar{d})$ is a compact topological space because it is the product of compact topological spaces, with the product topology.

Problem 5.8 Show that $(\Sigma_k^+, \bar{d})$ is a complete metric space.

Solution We first give an argument using the fact that $(\Sigma_k^+, \bar{d})$ is a compact metric space. Let $(\omega_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in Σ_k^+. Since Σ_k^+ is compact, there exists a subsequence k_n with $\omega_{k_n} \to \omega$ when $n \to \infty$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\bar{d}(\omega_m, \omega_{k_n}) < \varepsilon/2 \quad \text{for } m, k_n > N$$

and

$$\bar{d}(\omega_{k_n}, \omega) < \varepsilon/2 \quad \text{for } k_n > N.$$

Therefore,

$$\bar{d}(\omega, \omega_m) \leqslant \bar{d}(\omega, \omega_{k_n}) + \bar{d}(\omega_{k_n}, \omega_m) \leqslant \varepsilon \quad \text{for } k \geqslant N$$

and so the sequence $(\omega_m)_{m\in\mathbb{N}}$ converges.

Now we give an alternative argument without using the compactness of the space. Again, let $(\omega_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in Σ_k^+. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\bar{d}(\omega_i, \omega_j) < \varepsilon \quad \text{for } i, j \geqslant N.$$

But $\bar{d}(\omega_i, \omega_j) < \varepsilon$ if and only if $1/n(\omega_i, \omega_j) < \varepsilon$. Let $\varepsilon = 1/p$. Then $n(\omega_i, \omega_j) > p$. Hence, for each $p \in \mathbb{N}$ there exists $N(p) \in \mathbb{N}$ such that $n(\omega_i, \omega_j) > p$ for all $i, j \geqslant N(p)$, that is,

$$i_l(\omega_i) = i_l(\omega_j) \quad \text{for all } l \in \{1,\dots,p\}.$$

We define a sequence $\omega \in \Sigma_k^+$ by

$$i_p(\omega) = i_p(\omega_{N(p)}) \quad \text{for } p \in \mathbb{N}.$$

Then $i_p(\omega) = i_p(\omega_m)$ for all $m \geqslant N(p)$ and so $\bar{d}(\omega, \omega_m) < 1/p$. Letting $m \to \infty$ we obtain

$$\limsup_{m\to\infty} \bar{d}(\omega, \omega_m) \leqslant \frac{1}{p}$$

and so letting $p \to \infty$ gives

$$\limsup_{m\to\infty} \bar{d}(\omega, \omega_m) = 0.$$

This shows that $\omega_m \to \omega$ when $m \to \infty$.

Problem 5.9 Show that the shift map $\sigma|\Sigma_k$ is topologically transitive.

Solution First note that the sets

$$D_{j_{-m}\cdots j_m} = \left\{\omega \in \Sigma_k : i_n(\omega) = j_n \text{ for } |n| \leqslant m\right\},$$

with $j_{-m}, \dots, j_m \in \{1, \dots, k\}$, are open balls in the distance d_β and that they generate the topology. Thus, it suffices to consider these sets in the notion of topological transitivity. Given two sets $D_{j_{-m}\cdots j_m}$ and $D_{k_{-m}\cdots k_m}$, we claim that there exists $p \in \mathbb{N}$ such that

$$\sigma^{-p} D_{j_{-m}\cdots j_m} \cap D_{k_{-m}\cdots k_m} \neq \varnothing.$$

Given $\omega = (\cdots i_{-1} i_0 i_1 \cdots) \in D_{j_{-m}\cdots j_m}$, take $p \geqslant 2m + 2$ and

$$\omega' = (\cdots k_{-m} \cdots k_m l_1 \cdots l_{p-2m-1} i_{-m} \cdots i_m \cdots) \in D_{k_{-m}\cdots k_m}$$

with center at k_0, for some $l_1, \dots, l_{p-2m-1} \in \{1, \dots, k\}$. Then $\sigma^p(\omega) \in D_{j_{-m}\cdots j_m}$ and so

$$\omega' \in \sigma^{-p} D_{j_{-m}\cdots j_m} \cap D_{k_{-m}\cdots k_m}.$$

This shows that the map $\sigma|\Sigma_k$ is topologically transitive.

Problem 5.10 For a topological Markov chain $\sigma|\Sigma_A^+$ or $\sigma|\Sigma_A$ with transition matrix $A = (a_{ij})$, show that:

1. the number of blocks $(i_1 \cdots i_n)$ of length n with $a_{i_p i_{p+1}} = 1$ for $p = 1, \dots, n-1$ such that $i_1 = i$ and $i_n = j$ is equal to the (i, j) entry of the matrix A^n;
2. the number of n-periodic points is $\operatorname{tr}(A^n)$.

Solution 1. Note that $a_{i_p i_{p+1}} = 1$ for $p = 1, \dots, n-1$ if and only if $a_{i_1 i_2} \cdots a_{i_{n-1} i_n} = 1$. Hence, the number of blocks of length n with $i_1 = 1$ and $i_n = j$ is equal to

$$\sum_{i_2 \cdots i_{n-1}} a_{i i_2} \cdots a_{i_{n-1} j} = (A^n)_{ij},$$

the (i, j) entry of A^n.

2. The number of n-periodic points of $\sigma|\Sigma_A^+$ or $\sigma|\Sigma_A$ is the number of blocks of length n with $i = j$ for some $n \in \{1, \ldots, k\}$. Hence, it follows from item 1 that

$$\sum_{i=1}^{k}(A^n)_{ii} = \mathrm{tr}(A^n).$$

Problem 5.11 Given a $k \times k$ matrix A with entries a_{ij} in $\{0,1\}$, show that the following properties are equivalent:

1. there exists $n \in \mathbb{N}$ such that $a_{i_1 i_2} \cdots a_{i_{n-1} i_n} = 0$ for all $i_1, \ldots, i_n \in \{1, \ldots, k\}$;
2. $A^n = 0$ for some $n \in \mathbb{N}$;
3. $\Sigma_A = \varnothing$.

Solution $(1 \Rightarrow 2)$. We have

$$(A^n)_{i_1 i_n} = \sum_{i_2 \cdots i_{n-1}} a_{i_1 i_2} \cdots a_{i_{n-1} i_n} = 0$$

for all $i_1, i_n \in \{1, \ldots, k\}$ and so $A^n = 0$.

$(2 \Rightarrow 3)$. If $A^n = 0$, then

$$0 = (A^n)_{i_1 i_n} = \sum_{i_2 \cdots i_{n-1}} a_{i_1 i_2} \cdots a_{i_{n-1} i_n}$$

and since $a_{i_1 i_2} \cdots a_{i_{n-1} i_n} \geqslant 0$, we conclude that

$$a_{i_1 i_2} \cdots a_{i_{n-1} i_n} = 0 \quad \text{for all } i_1, i_n \in \{1, \ldots, k\}.$$

Therefore, no block $(i_1 \cdots i_n)$ of length n can occur in an element of Σ_A. In other words $\Sigma_A = \varnothing$.

$(3 \Rightarrow 1)$. If $a_{i_1 i_2} \cdots a_{i_{n-1} i_n} = 1$ for some $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$ with $i_1 = i_n$, then $\omega = (\cdots ppp \cdots)$, with $p = (i_1 \cdots i_n)$, would be an element of Σ_A. But since $\Sigma_A = \varnothing$ we must have

$$a_{i_1 i_2} \cdots a_{i_{n-1} i_1} = 0$$

for all $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$. In particular, for blocks $(i_1 \cdots i_k)$ of length k, there exist $m, n \in \{1, \ldots, k\}$ with $m < n$ such that $i_m = i_n$ and so

$$a_{i_1 i_2} \cdots a_{i_{k-1} i_k} = 0 \quad \text{because } a_{i_m i_{m+1}} \cdots a_{i_{n-1} i_n} = 0.$$

This establishes property 1.

Problem 5.12 For a topological Markov chain $\sigma|\Sigma_A^+$, compute the number of periodic orbits with period p prime in terms of the transition matrix A.

Solution By Problem 5.10, the number of p-periodic points of $\sigma|\Sigma_A^+$ is equal to $\operatorname{tr}(A^p)$ while the number of fixed points of $\sigma|\Sigma_A^+$ is equal to $\operatorname{tr}A$. Since p is prime, the periodic points with period p are exactly the p-periodic points that are not fixed points. Hence, the number of periodic orbits with period p is equal to

$$\frac{\operatorname{tr}(A^p) - \operatorname{tr}A}{p}.$$

Problem 5.13 For a topological Markov chain $\sigma|\Sigma_A^+$, compute the number of periodic orbits with period $p = p_1 p_2$, with p_1, p_2 prime, in terms of the transition matrix A.

Solution If $p = p_1 p_2$, with p_1, p_2 prime, then by Problem 5.10 the number of p-periodic points is equal to $\operatorname{tr}(A^p)$, the number of p_i-periodic points is equal to $\operatorname{tr}(A^{p_i})$, and the number of fixed points is equal to $\operatorname{tr}A$. Hence, the number of periodic points with period p is equal to

$$\begin{aligned}
&\operatorname{tr}(A^p) - (\operatorname{tr}(A^{p_1}) - \operatorname{tr}A) - (\operatorname{tr}(A^{p_2}) - \operatorname{tr}A) - \operatorname{tr}A \\
&= \operatorname{tr}(A^p) - \operatorname{tr}(A^{p_1}) - \operatorname{tr}(A^{p_2}) + \operatorname{tr}A,
\end{aligned}$$

which implies that the number of periodic orbits with period p is equal to

$$\frac{\operatorname{tr}(A^p) - \operatorname{tr}(A^{p_1}) - \operatorname{tr}(A^{p_2}) + \operatorname{tr}A}{p}.$$

Problem 5.14 Determine whether the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

is irreducible or transitive (see Figure I.5.1 for its graph).

Solution Note that

$$A^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

Since all entries of A^2 are positive, the matrix A is transitive and so it is also irreducible (since any transitive matrix is irreducible).

Problem 5.15 Show that the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

is not transitive (see Figure I.5.2 for its graph).

Solution Let $a_{ij}^{(n)}$ be the entries of A^n. We show by induction on n that $a_{k1}^{(n)} = 0$ for all $n \geqslant 0$ and $1 \leqslant k \leqslant 4$. This is clear for $n = 1$. Now assume that the property holds for some $n \in \mathbb{N}$. Then

$$a_{k1}^{(n+1)} = \sum_{j=1}^{4} a_{kj}^{(1)} a_{j1}^{(n)} = 0 \quad \text{for } 1 \leqslant k \leqslant 4.$$

In particular, the matrix A is not transitive.

Alternatively, one can note that the graph in Figure I.5.2 has no arrows pointing to 1. Hence, it follows from item 1 in Problem 5.10 that no power of A can have all entries positive.

Problem 5.16 Show that the topological Markov chain $\sigma|\Sigma_A^+$ with transition matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

has positive topological entropy.

Solution Note that

$$A^2 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 3 & 2 & 2 & 1 \\ 2 & 2 & 3 & 1 \\ 2 & 3 & 2 & 2 \end{pmatrix}.$$

Therefore, the matrix A is transitive and it follows from Proposition 5.9 that the topological Markov chain has positive topological entropy.

Problem 5.17 Consider the topological Markov chains $\sigma|\Sigma_A^+$ and $\sigma|\Sigma_B^+$ with transition matrices, respectively,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

1. Compute $h(\sigma|\Sigma_A^+)$ and $h(\sigma|\Sigma_B^+)$.
2. Find the fixed points of $\sigma|\Sigma_A^+$ and $\sigma|\Sigma_B^+$.

Solution 1. By Theorem 5.4, we have

$$h(\sigma|\Sigma_A^+) = \log \rho(A) = \log \frac{1+\sqrt{5}}{2}$$

and

$$h(\sigma|\Sigma_B^+) = \log \rho(B) = \log 1 = 0.$$

2. To determine the fixed points, note that $\operatorname{tr} A = 1$ and $\operatorname{tr} B = 2$. By Problem 5.10, $\sigma|\Sigma_A^+$ has one fixed point, which is $(11\cdots)$, while $\sigma|\Sigma_B^+$ has two fixed points, namely, $(11\cdots)$ and $(22\cdots)$.

Problem 5.18 Find topological Markov chains $\sigma|\Sigma_A^+$ and $\sigma|\Sigma_B^+$ both with zero topological entropy that are not topologically conjugate.

Solution Consider the topological Markov chains $\sigma|\Sigma_A^+$ and $\sigma|\Sigma_B^+$ with transition matrices, respectively,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$h(\sigma|\Sigma_A^+) = \log \rho(A) = \log 1 = 0$$

and

$$h(\sigma|\Sigma_B^+) = \log \rho(B) = \log 1 = 0.$$

On the other hand, the topological Markov chains $\sigma|\Sigma_A^+$ and $\sigma|\Sigma_B^+$ are not topologically conjugate because Σ_A^+ contains the single sequence $(11\cdots)$, while Σ_B^+ contains the two sequences $(11\cdots)$ and $(22\cdots)$.

Problem 5.19 Find topological Markov chains $\sigma|\Sigma_A^+$ and $\sigma|\Sigma_B^+$ with

$$h(\sigma|\Sigma_A^+) = h(\sigma|\Sigma_B^+) \neq 0$$

that are not topologically conjugate.

Solution Consider the topological Markov chains $\sigma|\Sigma_A^+$ and $\sigma|\Sigma_B^+$ with transition matrices, respectively,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since

$$\rho(A) = \rho(B) = \frac{1+\sqrt{5}}{2},$$

it follows from Proposition 5.4 that

$$h(\sigma|\Sigma_A^+) = h(\sigma|\Sigma_B^+) = \log \frac{1+\sqrt{5}}{2}.$$

On the other hand, the number of fixed points of $\sigma|\Sigma_A^+$ is $\operatorname{tr} A = 2$, while the number of fixed points of $\sigma|\Sigma_B^+$ is $\operatorname{tr} B = 3$. Hence, the topological Markov chains are not topologically conjugate.

Problem 5.20 For the set $\bar{\Lambda}$ in Problem 4.9, show that the correspondence

$$\omega = (\cdots i_{-1} i_0 i_1 \cdots) \mapsto \bar{\Lambda}_\omega = \bigcap_{n\in\mathbb{Z}} f^{-n} \bar{V}_{i_n}$$

defines a bijective map $H\colon \{1,2\}^{\mathbb{Z}} \to \bar{\Lambda}$.

Solution As in Problem 4.6, the intersection $\bar{\Lambda}_\omega$ is a nonempty closed set for each $\omega \in \{1,2\}^{\mathbb{Z}}$ since it is the intersection of the decreasing sequence of nonempty closed sets

$$\bar{R}_n(\omega) = \bigcap_{m=-n+1}^{n} f^{-m} \bar{V}_{i_m}. \tag{II.5.7}$$

On the other hand, the projections of $\bar{R}_n(\omega)$ on the horizontal and vertical axes are bounded, respectively, by $\sup|g'|^n$ and $(\inf|h'|)^{-n}$. Since

$$\sup|g'| < 1 \quad \text{and} \quad \inf|h'| > 1,$$

the diameter of $\bar{R}_n(\omega)$ tends to 0 when $n \to \infty$ and so the intersection

$$\bar{\Lambda}_\omega = \bigcap_{n=1}^{\infty} \bar{R}_n(\omega)$$

contains exactly one point.

To show that the map H is injective, take $\omega, \omega' \in \{1,2\}^{\mathbb{Z}}$ with $\omega \neq \omega'$. Writing

$$\omega = (\cdots i_{-1} i_0 i_1 \cdots) \quad \text{and} \quad \omega' = (\cdots i'_{-1} i'_0 i'_1 \cdots),$$

we have $i_j \neq i'_j$ for some $j \in \mathbb{Z}$. Hence, $\bar{V}_{i_j} \neq \bar{V}_{i'_j}$ and so also

$$\bigcap_{n\in\mathbb{Z}} f^{-n} \bar{V}_{i_n} \neq \bigcap_{n\in\mathbb{Z}} f^{-n} \bar{V}_{i'_n},$$

that is, $H(\omega) \neq H(\omega')$. Finally, we have

$$\begin{aligned}
\bar{\Lambda} &= \bigcap_{n\in\mathbb{Z}} f^{-n+1}(\bar{H}_1 \cup \bar{H}_2) \\
&= \bigcap_{n\in\mathbb{Z}} f^{-n}(\bar{V}_1 \cup \bar{V}_2) \\
&= \bigcup_{\omega\in\{1,2\}^{\mathbb{Z}}} \bigcap_{n\in\mathbb{Z}} f^{-n} \bar{V}_{i_n} \\
&= \bigcup_{\omega\in\{1,2\}^{\mathbb{Z}}} \Lambda_\omega = H\big(\{1,2\}^{\mathbb{Z}}\big),
\end{aligned}$$

which shows that the map H is onto.

Problem 5.21 Show that the bijection $H\colon \Sigma_2 \to \bar{\Lambda}$ in Problem 5.20 is a topological conjugacy between the shift map σ and the map f in Problem 4.9.

Solution To show that H is a homeomorphism, it suffices to verify that H is continuous. Indeed, since Σ_2 and $\bar{\Lambda}$ are compact and H is a continuous bijection, it is automatically a homeomorphism.

Note that if

$$i_k(\omega) = i_k(\omega') \quad \text{for } k \in \{-m+1, \dots, m\},$$

then $\bar{R}_m(\omega) = \bar{R}_m(\omega')$ (see (II.5.7)). Let

$$\lambda = \max\left\{\sup|g'|, (\inf|h'|)^{-1}\right\} < 1.$$

Since $\bar{R}_m(\omega)$ is contained in a square of side λ^m, we obtain

$$d(H(\omega), H(\omega')) \leqslant \sqrt{2}\lambda^m.$$

Given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $\sqrt{2}\lambda^m < \varepsilon$ and so if $d(\omega, \omega') < \beta^{-m}$, then $\bar{R}_m(\omega) = \bar{R}_m(\omega')$ and

$$d(H(\omega), H(\omega')) \leqslant \sqrt{2}\lambda^m < \varepsilon.$$

Finally, we have

$$\begin{aligned} H(\sigma(\omega)) &= \bigcap_{n\in\mathbb{Z}} f^{-n}\bar{V}_{i_{n+1}(\omega)} \\ &= \bigcap_{n\in\mathbb{Z}} f^{1-n}\bar{V}_{i_n(\omega)} \\ &= f(H(\omega)). \end{aligned}$$

Problem 5.22 Show that the set

$$X = \left\{\omega \in \Sigma_k : \omega \text{ is } q\text{-periodic for some even } q \in \mathbb{N}\right\}$$

is dense in Σ_2.

Solution Given $\omega \in \Sigma_2$ and $p \in \mathbb{N}$, consider the sequence

$$\omega_p = \left(\cdots i_{-p+1} i_{-p+2} \cdots i_p i_{-p+1} i_{-p+2} \cdots i_p \cdots\right)$$

with $i_j = i_j(\omega)$ for $j = -p+1, \dots, p$. Clearly, ω_p is $2p$-periodic for the shift map $\sigma|\Sigma_2$ and $n(\omega, \omega_p) > p$. Therefore,

$$d(\omega, \omega_p) < \beta^{-p} \to 0 \quad \text{when } p \to \infty.$$

This shows that the set X is dense in Σ_2.

Problem 5.23 Compute the zeta function of the shift map $\sigma|\Sigma_k$.

Solution We have

$$a_n = \operatorname{card}\{\omega \in \Sigma_k : \sigma^n(\omega) = \omega\} = k^n$$

and so

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{k^n z^n}{n}.$$

Since

$$\left(\sum_{n=1}^{\infty} \frac{k^n z^n}{n}\right)' = \sum_{n=1}^{\infty} k^n z^{n-1} = \frac{k}{1-kz} \tag{II.5.8}$$

for $|z| < 1/k$, it follows that

$$\zeta(z) = \exp\bigl(-\log(1-kz)\bigr) = \frac{1}{1-kz}$$

for $|z| < 1/k$.

Problem 5.24 Compute the zeta function of the topological Markov chain $\sigma|\Sigma_A^+$ with transition matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution By Problem 5.10, the number of n-periodic points of $\sigma|\Sigma_A^+$ is $\operatorname{tr}(A^n)$. Since

$$\det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = -\lambda^3 + 3\lambda^2 - \lambda = -\lambda(\lambda^2 - 3\lambda + 1),$$

the eigenvalues of A are

$$0, \quad \tau = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \tau^{-1} = \frac{3-\sqrt{5}}{2}.$$

Hence,

$$\begin{aligned} \zeta(z) &= \exp \sum_{n=1}^{\infty} \frac{(\tau^n + \tau^{-n}) z^n}{n} \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{(\tau z)^n}{n} + \sum_{n=1}^{\infty} \frac{(\tau^{-1} z)^n}{n}\right). \end{aligned}$$

Using (II.5.8) with $k = \tau$ and $k = \tau^{-1}$, we conclude that

$$\begin{aligned}\zeta(z) &= \exp\bigl(-\log(1-\tau z)-\log(1-\tau^{-1}z)\bigr)\\ &= \frac{1}{(1-\tau z)(1-\tau^{-1}z)}\\ &= \frac{1}{1-3z+z^2}\end{aligned}$$

for $|z|<\min\{\tau^{-1},\tau\}=(3-\sqrt{5})/2$.

Problem 5.25 Compute the zeta function of the automorphism of the torus $\mathbb{T}^2$ induced by the matrix

$$A=\begin{pmatrix}5 & 1\\ 9 & 2\end{pmatrix}.$$

Solution First observe that the eigenvalues of A are

$$\tau=\frac{7+3\sqrt{5}}{2}\quad\text{and}\quad\tau^{-1}=\frac{7-3\sqrt{5}}{2}.$$

By Problem 1.26 we have

$$\begin{aligned}a_n &= \operatorname{card}\bigl\{x\in\mathbb{T}^2 : T_A^n(x)=x\bigr\}\\ &= \operatorname{tr}(A^n)-2\\ &= \tau^n+\tau^{-n}-2\end{aligned}$$

and so

$$\begin{aligned}\zeta(z) &= \exp\sum_{n=1}^{\infty}\frac{(\tau^n+\tau^{-n}-2)z^n}{n}\\ &= \exp\left(\sum_{n=1}^{\infty}\frac{(\tau z)^n}{n}+\sum_{n=1}^{\infty}\frac{(\tau^{-1}z)^n}{n}-2\sum_{n=1}^{\infty}\frac{z^n}{n}\right).\end{aligned}$$

Using (II.5.8) with $k=\tau$, $k=\tau^{-1}$ and $k=1$, we conclude that

$$\zeta(z)=\exp\bigl(-\log(1-\tau z)-\log(1-\tau^{-1}z)+2\log(1-z)\bigr)$$

for $|z|<\min\{\tau^{-1},\tau,1\}$. Since $\tau>1$, we obtain

$$\zeta(z)=\frac{(1-z)^2}{(1-\tau z)(1-\tau^{-1}z)}=\frac{(1-z)^2}{1-7z+z^2}$$

for $|z|<(7-3\sqrt{5})/2$.

Problem 5.26 Given a nonempty finite set $\mathcal{F}$ in $\bigcup_{n\geqslant 2}\{1,\ldots,k\}^n$ (that is, a set of blocks with finite length), consider the set $\mathcal{F}'\subsetneq\Sigma_k$ of all sequences in Σ_k containing no blocks in $\mathcal{F}$. Determine two sets $\mathcal{F}_1$ and $\mathcal{F}_2$ with $k=2$ such that $\mathcal{F}_1'=\mathcal{F}_2'\neq\varnothing$ with $\mathcal{F}_1\neq\mathcal{F}_2$.

Solution Take

$$\mathcal{F}_1 = \{(22)\} \quad \text{and} \quad \mathcal{F}_2 = \{(221), (222)\}.$$

We show that $\mathcal{F}_1' = \mathcal{F}_2'$. A sequence $\omega \in \mathcal{F}_1'$ does not contain the block (22) and so it also cannot contain the blocks (221) or (222). Hence, $\omega \in \mathcal{F}_2'$. On the other hand, a sequence $\omega \in \mathcal{F}_2'$ does not contain the block (22), since otherwise it would also contain (221) or (222), which is impossible. Hence, $\mathcal{F}_1' = \mathcal{F}_2'$.

Problem 5.27 Show that $\mathcal{F}_1' \cap \mathcal{F}_2' = (\mathcal{F}_1 \cup \mathcal{F}_2)'$.

Solution Take $\omega \in \mathcal{F}_1' \cap \mathcal{F}_2'$. Since $\omega \in \mathcal{F}_1'$, it contains no block in $\mathcal{F}_1$ and since $\omega \in \mathcal{F}_2'$, it contains no block in $\mathcal{F}_2$. Therefore, ω contains no block in $\mathcal{F}_1 \cup \mathcal{F}_2$, that is, $\omega \in (\mathcal{F}_1 \cup \mathcal{F}_2)'$.

Now take $\omega \in (\mathcal{F}_1 \cup \mathcal{F}_2)'$. Then ω contains no block in $\mathcal{F}_1 \cup \mathcal{F}_2$. This implies that it contains no block in $\mathcal{F}_i$ for $i = 1, 2$ and so $\omega \in \mathcal{F}_1' \cap \mathcal{F}_2'$.

Problem 5.28 Given a nonempty finite set of blocks $\mathcal{F}$ in $\bigcup_{n \geqslant 2} \{1, \ldots, k\}^n$, show that $\mathcal{F}' \subsetneq \Sigma_k$ is compact and σ-invariant.

Solution The invariance is clear because the image and preimage of any sequence under the shift map $\sigma\colon \Sigma_k \to \Sigma_k$ contains exactly the same blocks as the original sequence.

To show that $\mathcal{F}'$ is compact, it suffices to show that it is closed. Take $\omega \in \Sigma_k \backslash \mathcal{F}'$ containing a block in $\mathcal{F}$ of length p and without loss of generality centered at the origin. Let

$$U = B_{d_\beta}(\omega, \beta^{-p}) = \{\omega' \in \Sigma_k : d_\beta(\omega, \omega') < \beta^{-p}\}.$$

Then $d_\beta(\omega, \omega') < \beta^{-p}$ if and only if $n(\omega, \omega') > p$ (again $n(\omega, \omega')$ is the smallest positive integer n such that $i_n(\omega) \neq i_n(\omega')$ or $i_{-n}(\omega) \neq i_{-n}(\omega')$). In particular, no $\omega' \in U$ is in $\mathcal{F}'$ and so U is an open neighborhood of ω that does not intersect $\mathcal{F}'$. This shows that $\mathcal{F}'$ is closed.

Problem 5.29 Given a nonempty finite set of blocks $\mathcal{F}$ in $\bigcup_{n \geqslant 2} \{1, \ldots, k\}^n$, show that if $\mathcal{F}$ contains only blocks of length 2, then $\mathcal{F}' = \Sigma_A$ for some $k \times k$ matrix A with entries in $\{0, 1\}$.

Solution Consider the $k \times k$ matrix A with entries $a_{ij} \in \{0, 1\}$ such that $a_{ij} = 0$ if and only if $(ij) \in \mathcal{F}$. Then clearly $\Sigma_A = \mathcal{F}'$.

Problem 5.30 Show that for $k = 2$ and $\mathcal{F} = \{(22)\}$, the matrix A in Problem 5.29 is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solution The entries a_{ij} of the matrix A satisfy $a_{ij} = 0$ if and only if $(ij) \in \mathcal{F}$. Hence, the only null entry of A is a_{22}.

Problem 5.31 Compute the number of n-periodic points of the restriction $\sigma|\mathcal{F}'$ of $\sigma\colon \Sigma_2 \to \Sigma_2$ to $\mathcal{F}'$ for the set $\mathcal{F} = \{(22)\}$.

Solution By Problem 5.30, the number of n-periodic points of $\sigma|\mathcal{F}'$ is equal to the number of n-periodic points of the topological Markov chain $\sigma|\Sigma_A$ with transition matrix A as in Problem 5.30. On the other hand, by Problem 5.10, the latter is equal to $\operatorname{tr}(A^n)$.

Note that the eigenvalues of A are

$$\lambda = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \mu = \frac{1-\sqrt{5}}{2}.$$

Considering the eigenvectors, respectively, $(\lambda, 1)$ and $(\mu, 1)$ we obtain

$$A = \begin{pmatrix} \lambda & \mu \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\mu \\ -1 & \lambda \end{pmatrix}$$

and so

$$A^n = \begin{pmatrix} \lambda & \mu \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\mu \\ -1 & \lambda \end{pmatrix}.$$

Therefore,

$$\operatorname{tr}(A^n) = \lambda^n + \mu^n = \lambda^n + (-1)^n \lambda^{-n}.$$

Problem 5.32 An orientation-preserving C^1 map $f\colon S^1 \to S^1$ is called an *expanding map* if

$$\lambda := \min_{x \in S^1} f'(x) > 1$$

(see Figure I.5.3 for an example). Let $i_0 \cdots i_{n-1}$ be the base-2 representation of j and let $p_1, \dots, p_{2^n} = p_0$ be the preimages of the unique fixed point p of f under f^n (see Figure I.5.4). Show that if $f\colon S^1 \to S^1$ is an expanding map with $\deg f = 2$ (see Problem 1.27), then the intervals

$$I_{i_0 \dots i_{n-1}} = [p_j, p_{j+1}], \quad \text{for } j = 0, \dots, 2^n - 1,$$

satisfy:

1. $|I_{i_0 \cdots i_{n-1}}| \leqslant \lambda^{-n}$;
2. $I_{i_0 \dots i_{n-1} i_n} \subsetneq I_{i_0 \dots i_{n-1}}$;
3. $f^m(I_{i_0 \cdots i_n}) = I_{i_m \cdots i_n}$ for each $m \leqslant n$.

Solution 1. For each $n \in \mathbb{N}$ we have

$$\deg(f^n) = (\deg f)^n = 2^n,$$

that is, all points have exactly 2^n preimages under f^n. We denote these preimages by p_j for $j = 1, \dots, 2^n$, numbering them in anticlockwise direction on S^1 so that $p_{2^n} = p_0 = p$. These points determine 2^n intervals

$$I_{i_0 \dots i_{n-1}} = [p_j, p_{j+1}],$$

with $i_0 \cdots i_{n-1}$ as in the statement of the problem. Clearly,

$$f^n([p_j, p_{j+1}]) = f^n(I_{i_0\cdots i_{n-1}}) = S^1 \tag{II.5.9}$$

and so

$$1 = \int_{p_j}^{p_{j+1}} (f^n)'(x)\,dx \geqslant \lambda^n |I_{i_0\cdots i_{n-1}}|.$$

This yields property 1.

2. Take $q \in f^{-1}p \backslash \{p\}$. By (II.5.9), in each interval $I_{i_0\cdots i_{n-1}}$ there exists a unique point q_j such that $f^n(q_j) = q$. Since

$$f^{n+1}(q_j) = f(f^n(q_j)) = f(q) = p$$

and

$$f^{n+1}(p_j) = f(f^n(p_j)) = f(p) = p,$$

it follows that q_j and p_j are endpoints of some intervals $I_{j_0\cdots j_n}$. Since $q_j \in [p_j, p_{j+1}]$, we conclude that

$$I_{i_0\cdots i_{n-1}0} = [p_j, q_j] \quad \text{and} \quad I_{i_0\cdots i_{n-1}1} = [q_j, p_{j+1}],$$

which establishes property 2.

3. For the last property, note that the image of each interval $I_{i_0\cdots i_n} = [p_j, p_{j+1}]$ under f is also an interval whose endpoints are preimages of p under f^{n-1} (since the endpoints of $I_{i_0\cdots i_n}$ belong to $f^{-n}p$). More precisely, let p'_k, for $k = 0, \ldots, 2^{n-1} - 1$, be the elements of $f^{-(n-1)}p$. Then

$$f(I_{i_0\cdots i_n}) = [f(p_j), f(p_{j+1})]$$

(since f is orientation preserving) with

$$f(p_j) = p'_k \quad \text{for } k \equiv j \bmod 2^{n-1}.$$

For $j = i_0 \cdots i_n$ we have

$$j \equiv i_1 \cdots i_n \bmod 2^{n-1}$$

and so $f(I_{i_0\cdots i_n}) = I_{i_1\cdots i_n}$. The last property follows now by induction.

Problem 5.33 Let $f\colon S^1 \to S^1$ be an expanding map (see Problem 5.32) with $\deg f = 2$. Show that the set of all preimages $\bigcup_{n\in\mathbb{N}} f^{-n}p$ is dense in S^1.

Solution By Problem 5.32, the endpoints of each interval $I_{i_0\cdots i_{n-1}}$ belong to $f^{-n}p$. Moreover,

$$|I_{i_0\cdots i_{n-1}}| \leqslant \lambda^{-n} \to 0 \quad \text{when } n \to \infty$$

and so the distance between two consecutive elements of $f^{-n}p$ tends to zero when $n \to \infty$. This readily implies that the set $\bigcup_{n\in\mathbb{N}} f^{-n}p$ is dense in S^1.

Problem 5.34 Let $f\colon S^1 \to S^1$ be an expanding map with $\deg f = 2$. Show that there exists a continuous onto map $H\colon \Sigma_2^+ \to S^1$ such that $H \circ \sigma = f \circ H$ on Σ_2^+.

Solution We define a map $H\colon \Sigma_2^+ \to S^1$ as follows. Given $\omega = (j_1 j_2 \cdots) \in \Sigma_2^+$, let $i_l = j_l - 1$ for each l and define

$$H(\omega) = \bigcap_{n\in\mathbb{N}} I_{i_1\cdots i_n},$$

using the same notation as in Problem 5.32. By item 2 in that problem the intervals $I_{i_1\cdots i_n}$ form a decreasing sequence of closed sets and so their intersection is nonempty. Moreover, since

$$|I_{i_1\cdots i_n}| \to 0 \quad \text{when } n \to \infty,$$

the set $\bigcap_{n\in\mathbb{N}} I_{i_1\cdots i_n}$ contains exactly one point, which we continue to denote by $H(\omega)$.

Now we show that H is continuous. Assume that $d_\beta(\omega,\omega') = \beta^{-n}$. Writing

$$\omega = (j_1 j_2 \cdots) \quad \text{and} \quad \omega' = (j_1' j_2' \cdots),$$

this means that n is the smallest positive integer such that $j_n \neq j_n'$. Finally, letting $j_l = j_l' = i_l + 1$ for $l < n$, we obtain

$$d_\beta(H(\omega), H(\omega')) \leqslant |I_{i_1\cdots i_{n-1}}| \leqslant \lambda^{-n} \to 0$$

when $\beta^{-n} \to 0$ and so the map H is continuous. To show that it is onto, take $x \in S^1$ and for each $n \in \mathbb{N}$ let

$$i_n = \begin{cases} 1 & \text{if } f^{n-1}(x) \in [p,q), \\ 2 & \text{otherwise,} \end{cases}$$

with p and q as in Problem 5.32. Then $x \in I_{i_1\cdots i_n}$ for all $n \in \mathbb{N}$ and so $x = h(i_1 i_2 \cdots)$.

Finally, we show that

$$h \circ \sigma = f \circ H \quad \text{on } \Sigma_2^+.$$

Let $x = H(\omega)$. Then $x \in I_{i_1\cdots i_n}$ for all $n \in \mathbb{N}$. By Problem 5.32 we have

$$f(I_{i_1\cdots i_n}) = I_{i_2\cdots i_n}$$

and so

$$f(x) = \bigcap_{n\in\mathbb{N}} I_{i_2\cdots i_n}. \tag{II.5.10}$$

Therefore,

$$f(H(\omega)) = f(x) = H(\sigma(\omega)). \tag{II.5.11}$$

Problem 5.35 Let $f\colon S^1 \to S^1$ be an expanding map with $\deg f = k \geqslant 2$. Show that there exists a continuous onto map $H\colon \Sigma_k^+ \to S^1$ such that $H \circ \sigma = f \circ H$ on Σ_k^+.

Solution Let $i_0 \cdots i_{n-1}$ be the base-k representation of j and let $p_1, \ldots, p_{k^n} = p_0$ be the preimages of a given fixed point p of f under f^n. Proceeding in a similar manner to that in Problem 5.32, we find that the intervals

$$I_{i_0\ldots i_{n-1}} = [p_j, p_{j+1}], \quad \text{for } j = 0, \ldots, k^n - 1,$$

satisfy:

1. $|I_{i_0\cdots i_{n-1}}| \leqslant \lambda^{-n}$;
2. $I_{i_0\ldots i_{n-1}i_n} \subsetneq I_{i_0\ldots i_{n-1}}$;
3. $f^m(I_{i_0\cdots i_n}) = I_{i_m\cdots i_n}$ for each $m \leqslant n$.

As in Problem 5.34, let $i_l = j_l - 1$ for each l and define a map $H\colon \Sigma_k^+ \to S^1$ by

$$H(j_1 j_2 \cdots) = \bigcap_{n\in\mathbb{N}} I_{i_1\cdots i_n}.$$

It follows from the former properties that the map H is well defined, continuous, and onto. Finally, we show that $H \circ \sigma = f \circ H$. Let $x = H(\omega)$. Then $x \in I_{i_1\cdots i_n}$ and

$$f(I_{i_1\cdots i_n}) = I_{i_2\cdots i_n}$$

for all $n \in \mathbb{N}$. This implies that properties (II.5.10) and (II.5.11) hold.

Problem 5.36 Show that the map H in Problem 5.35 is not a homeomorphism.

Solution Note that Σ_k^+ is the union of two disjoint nonempty closed sets, but that the same does not happen with S^1.

Problem 5.37 Show that the periodic points of any expanding map $f\colon S^1 \to S^1$ are dense in S^1.

Solution Let $k = \deg f$. By Problem 5.35, there exists a continuous onto map $H\colon \Sigma_k^+ \to S^1$ such that

$$H \circ \sigma = f \circ H.$$

Hence, if $\sigma^n(\omega) = \omega$, then

$$f^n(H(\omega)) = H(\sigma^n(\omega)) = H(\omega).$$

That is, the image of an n-periodic point of σ under H is an n-periodic point of f. Since the periodic points of σ are dense in Σ_k^+ and H is continuous and onto, the periodic points of f are also dense.

Problem 5.38 Show that any expanding map $f\colon S^1 \to S^1$ is topologically transitive.

Solution First note that any map as in the statement of the problem has degree $\deg f \geqslant 2$. Hence, the hypotheses of Problems 5.34 and 5.35 are satisfied. Given nonempty open sets $U, V \subsetneq S^1$, take intervals

$$I_{i_0\cdots i_{n-1}} \subseteq U \quad \text{and} \quad I_{i'_0\cdots i'_{n-1}} \subseteq V \tag{II.5.12}$$

as introduced in those problems. Then

$$\begin{aligned} U \cap f^n V &\supseteq I_{i_0\cdots i_{n-1}} \cap f^n(I_{i'_0\cdots i'_{n-1}}) \\ &= I_{i_0\cdots i_{n-1}} \cap S^1 \neq \varnothing. \end{aligned}$$

It follows from Problem 2.17 that f is topologically transitive.

Problem 5.39 Determine whether any expanding map $f\colon S^1 \to S^1$ is topologically mixing.

Solution Given nonempty open sets $U, V \subsetneq S^1$, take intervals $I_{i_0\cdots i_{n-1}}$ and $I_{i'_0\cdots i'_{n-1}}$ as in (II.5.12). Note that for each $m > n$ we have

$$f^{-m} I_{i'_0\cdots i'_{n-1}} \supseteq I_{i_0\cdots i_{n-1} j_1\cdots j_{m-n} i'_0\cdots i'_{n-1}}$$

for any $j_1, \ldots, j_{m-n} \in \{1, \ldots, k\}$. Hence,

$$\begin{aligned} U \cap f^{-m} V &\supseteq I_{i_0\cdots i_{n-1}} \cap f^{-m} I_{i'_0\cdots i'_{n-1}} \\ &\supseteq I_{i_0\cdots i_{n-1} j_1\cdots j_{m-n} i'_0\cdots i'_{n-1}} \neq \varnothing, \end{aligned}$$

which shows that f is topologically mixing.

Problem 5.40 For the map f in Problem 5.35, show that $h(f) \leqslant \log k$.

Solution By Problem 5.35, there exists a continuous onto map $H\colon \Sigma_k^+ \to S^1$ such that $H \circ \sigma = f \circ H$ on Σ_k^+. Hence, it follows from Problem 2.33 that

$$h(f) \leqslant h(\sigma | \Sigma_k^+) = \log k.$$

Chapter II.6
Ergodic Theory

Problem 6.1 For a measure space $(X, \mathcal{A}, \mu)$, show that:

1. if $A, B \in \mathcal{A}$ and $A \subseteq B$, then $\mu(A) \leqslant \mu(B)$;
2. if $B_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leqslant \sum_{n=1}^{\infty} \mu(B_n).$$

Solution 1. Since

$$B = A \cup (B \setminus A) \quad \text{and} \quad A \cap (B \setminus A) = \varnothing,$$

it follows from Definition 6.2 that

$$\mu(B) = \mu(A) + \mu(B \setminus A).$$

Therefore, $\mu(B) \geqslant \mu(A)$.

2. Let

$$A_1 = B_1 \quad \text{and} \quad A_n = B_n \setminus \bigcup_{i=1}^{n-1} B_i \quad \text{for } n > 1.$$

Then $A_n \cap A_m = \varnothing$ for $n \neq m$ and so

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \leqslant \sum_{n=1}^{\infty} \mu(B_n).$$

Problem 6.2 For a measure space $(X, \mathcal{A}, \mu)$, show that:

1. if $A, B \in \mathcal{B}$, then

$$\mu(A) + \mu(B) = \mu\big(A \cup B\big) + \mu\big(A \cap B\big);$$

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3_12

2. if $A, B \in \mathcal{B}$ and
$$\mu\big((A\backslash B) \cup (B\backslash A)\big) = 0,$$
then $\mu(A) = \mu(B)$.

Solution 1. Note that

$$A \cup B = A \cup (B\backslash A) \quad \text{and} \quad B = (B\backslash A) \cup (A \cap B)$$

are both disjoint unions. Hence,

$$\mu(A \cup B) = \mu(A) + \mu(B\backslash A)$$

and so
$$\begin{aligned}\mu(A \cup B) + \mu(A \cap B) &= \mu(A) + \mu(B\backslash A) + \mu(A \cap B)\\ &= \mu(A) + \mu(B).\end{aligned}$$

2. It follows from the hypothesis that

$$\mu(A\backslash B) = \mu(B\backslash A) = 0.$$

Hence,
$$\mu(A) = \mu(A\backslash B) + \mu(A \cap B) = \mu(A \cap B)$$
and
$$\mu(B) = \mu(B\backslash A) + \mu(A \cap B) = \mu(A \cap B),$$
which shows that $\mu(A) = \mu(B)$.

Problem 6.3 Show that the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ is generated by the set of all intervals:

1. $[a, b]$ with $a < b$;
2. $(a, b]$ with $a < b$.

Solution 1. Since $\mathcal{B}$ is generated by the open intervals, it suffices to note that

$$(a, b) = \bigcup_{n=1}^{\infty}\left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

and
$$[a, b] = \bigcap_{n=1}^{\infty}\left(a - \frac{1}{n}, b + \frac{1}{n}\right)$$

for all $a, b \in \mathbb{R}$. Indeed, this implies that the σ-algebra generated by the closed intervals coincides with $\mathcal{B}$.

2. Similarly, it suffices to note that

$$(a, b) = \bigcup_{n=1}^{\infty}\left(a, b - \frac{1}{n}\right]$$

and

$$(a,b] = \bigcap_{n=1}^{\infty}\left(a, b+\frac{1}{n}\right)$$

for all $a, b \in \mathbb{R}$.

Problem 6.4 Show that the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ is generated by the set of all intervals:

1. $(a, +\infty)$ with $a \in \mathbb{R}$;
2. $(-\infty, b)$ with $b \in \mathbb{R}$;
3. $[a, +\infty)$ with $a \in \mathbb{R}$;
4. $(-\infty, b]$ with $b \in \mathbb{R}$.

Solution 1. As in Problem 6.3, it suffices to note that

$$(a,b) = (a,+\infty)\setminus\bigcap_{n=1}^{\infty}\left(b-\frac{1}{n}, +\infty\right)$$

and

$$(a,+\infty) = \bigcup_{n=1}^{\infty}(a, a+n)$$

for all $a, b \in \mathbb{R}$.

2. Similarly, we have

$$(a,b) = (-\infty,b)\setminus\bigcap_{m=1}^{\infty}\left(-\infty, a+\frac{1}{m}\right)$$

and

$$(-\infty,b) = \bigcup_{n=1}^{\infty}(-n, b)$$

for all $a, b \in \mathbb{R}$.

3. In this case, note that

$$(a,b) = \bigcup_{n=1}^{\infty}\left[a+\frac{1}{n}, +\infty\right)\setminus[b,+\infty)$$

and

$$[a,+\infty) = \bigcup_{n=1}^{\infty}\bigcap_{m=1}^{\infty}\left(a-\frac{1}{m}, n\right)$$

for all $a, b \in \mathbb{R}$.

4. Finally, we have

$$(a,b) = \bigcup_{n=1}^{\infty} \Big(-\infty, b-\frac{1}{n}\Big]\backslash(-\infty,a]$$

and

$$(-\infty,b] = \bigcup_{n=1}^{\infty}\bigcap_{m=1}^{\infty} \Big(-n, b+\frac{1}{m}\Big)$$

for all $a,b \in \mathbb{R}$.

Problem 6.5 Show that the Borel σ-algebra on $\mathbb{R}$ coincides with the σ-algebra generated by the open sets in $\mathbb{R}$.

Solution Clearly, the Borel σ-algebra on $\mathbb{R}$ is contained in the σ-algebra $\mathcal{C}$ generated by the open sets in $\mathbb{R}$. Hence, it suffices to show that any open set $A \subseteq \mathbb{R}$ is a countable union of open intervals (since then any set in $\mathcal{C}$ is also in the Borel σ-algebra).

Let $A \subseteq \mathbb{R}$ be an open set. For each $x \in A \cap \mathbb{Q}$, let $I_x \subseteq A$ be the largest open interval containing x (that is, the union of all open intervals $I \subseteq A$ containing x). Then

$$A = \bigcup_{x\in A\cap\mathbb{Q}} I_x$$

is a countable union of open intervals.

Problem 6.6 Given a set $\mathcal{C}$ generating the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$, show that a function $\varphi\colon X \to \mathbb{R}$ is $\mathcal{A}$-measurable if and only if $\varphi^{-1}B \in \mathcal{A}$ for every $B \in \mathcal{C}$.

Solution Clearly, if φ is $\mathcal{A}$-measurable, then $\varphi^{-1}B \in \mathcal{A}$ for every $B \in \mathcal{C}$. Now assume that

$$\mathcal{F} := \big\{B \in \mathcal{B} : \varphi^{-1}B \in \mathcal{A}\big\} \supseteq \mathcal{C}.$$

Then

1. $\varnothing, \mathbb{R} \in \mathcal{F}$;
2. if $B \in \mathcal{F}$, then

$$\varphi^{-1}(\mathbb{R}\backslash B) = \mathbb{R}\backslash\varphi^{-1}(B) \in \mathcal{A}$$

   and so $\mathbb{R}\backslash\mathcal{B} \in \mathcal{F}$;
3. given sets $B_n \in \mathcal{F}$, for $n \in \mathbb{N}$, we have

$$\varphi^{-1}\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \varphi^{-1}B_n \in \mathcal{A}$$

   and so $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{F}$.

Therefore, $\mathcal{F}$ is a σ-algebra containing $\mathcal{C}$. Since $\mathcal{C}$ generates the Borel σ-algebra on $\mathbb{R}$, we conclude that $\mathcal{F} \supseteq \mathcal{B}$ and so in particular φ is $\mathcal{A}$-measurable.

Problem 6.7 Show that the square of an $\mathcal{A}$-measurable function $\varphi\colon X \to \mathbb{R}$ is also an $\mathcal{A}$-measurable function.

Solution For each $a > 0$ we have

$$\begin{aligned}\{x \in X : \varphi(x)^2 < a\} &= \{x \in X : -\sqrt{a} < \varphi(x) < \sqrt{a}\} \\ &= \varphi^{-1}\big(-\sqrt{a}, \sqrt{a}\big).\end{aligned}$$

Hence, it follows from Problems 6.4 and 6.6 that the function φ^2 is $\mathcal{A}$-measurable.

Problem 6.8 Show that any increasing function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}$-measurable.

Solution By Problems 6.4 and 6.6, it suffices to show that the set

$$B_a = \{x \in \mathbb{R} : \varphi(x) > a\}$$

belongs to the Borel σ-algebra $\mathcal{B}$ for each $a \in \mathbb{R}$. Let $b = \inf B_a$. Then either $b \in B_a$ and so $B_a = [b, +\infty)$ (see Figure II.6.1 for an example) or $b \notin B_a$ and so $B_a = (b, +\infty)$. In both cases one has $B_a \in \mathcal{B}$.

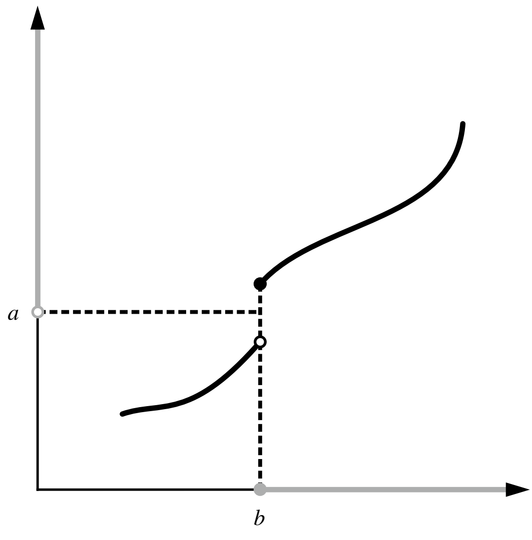

Fig. II.6.1 Preimage of the interval $(a, +\infty)$ under a discontinuous increasing function.

Problem 6.9 Given $\mathcal{A}$-measurable functions $\varphi_n\colon X \to \mathbb{R}$, for $n \in \mathbb{N}$, show that $\sup_{n\in\mathbb{N}} \varphi_n$ and $\inf_{n\in\mathbb{N}} \varphi_n$ (here assumed to be finite everywhere) are also $\mathcal{A}$-measurable functions.

Solution We have

$$\left\{x \in X : \sup_{n\in\mathbb{N}} \varphi_n(x) > a\right\} = \{x \in X : \varphi_n(x) > a \text{ for some } n \in \mathbb{N}\}$$
$$= \bigcup_{n\in\mathbb{N}} \{x \in X : \varphi_n(x) > a\} \in \mathcal{A}$$

since each function φ_n is $\mathcal{A}$-measurable. Hence, it follows from Problems 6.4 and 6.6 that the function $\sup_{n\in\mathbb{N}} \varphi_n$ is $\mathcal{A}$-measurable.

To show that the infimum is $\mathcal{A}$-measurable it suffices to note that

$$\inf_{n\in\mathbb{N}} \varphi_n = -\sup_{n\in\mathbb{N}}(-\varphi_n).$$

Problem 6.10 Given $\mathcal{A}$-measurable functions $\varphi_n : X \to \mathbb{R}$, for $n \in \mathbb{N}$, show that

$$\limsup_{n\to\infty} \varphi_n \quad \text{and} \quad \liminf_{n\to\infty} \varphi_n$$

(here assumed to be finite everywhere) are also $\mathcal{A}$-measurable functions.

Solution Note that

$$\limsup_{n\to\infty} \varphi_n = \inf_{n\geqslant 1} \sup_{m\geqslant n} \varphi_m \quad \text{and} \quad \liminf_{n\to\infty} \varphi_n = \sup_{n\geqslant 1} \inf_{m\geqslant n} \varphi_m.$$

Hence, the desired property follows readily from Problem 6.9.

Problem 6.11 For a measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) < +\infty$, let $\varphi_n : X \to \mathbb{R}$, for $n \in \mathbb{N}$, and $\varphi : X \to \mathbb{R}$ be $\mathcal{A}$-measurable functions such that

$$\varphi_n(x) \to \varphi(x) \quad \text{when } n \to \infty$$

for all $x \in X$. Show that given $\varepsilon > 0$, there exists a set $B \in \mathcal{A}$ with $\mu(B) < \varepsilon$ such that

$$\varphi_n \to \varphi \quad \text{when } n \to \infty$$

uniformly on $X \setminus B$.

Solution For each $n, q \in \mathbb{N}$, let

$$B_n(q) = \bigcup_{k=n}^{\infty} \left\{x \in X : |\varphi_k(x) - \varphi(x)| \geqslant \frac{1}{q}\right\}.$$

Note that for each q the sequence $B_n(q)$ decreases with n and that

$$\bigcap_{n=1}^{\infty} B_n(q) = \varnothing.$$

Now let

$$A_n = B_n(q) \backslash B_{n+1}(q) \quad \text{for } n \geqslant 1.$$

Note that $A_n \cap A_m = \varnothing$ for $n \neq m$ and that

$$B_n(q) = \bigcup_{m=n}^{\infty} A_m$$

for $n \in \mathbb{N}$. Since $\mu(X) < +\infty$, we obtain

$$\sum_{m=n}^{\infty} \mu(A_m) = \mu\left(\bigcup_{m=n}^{\infty} A_m\right) = \mu(B_n(q)) < +\infty$$

for each $n \in \mathbb{N}$. Therefore,

$$\lim_{n\to\infty} \mu(B_n(q)) = \lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(A_m) = 0.$$

Given $\varepsilon > 0$ and $q \in \mathbb{N}$, take $n_q \in \mathbb{N}$ such that $\mu(B_{n_q}(q)) < \varepsilon/2^q$ and let

$$B = \bigcup_{q=1}^{\infty} B_{n_q}(q).$$

By Problem 6.1 we obtain

$$\mu(B) \leqslant \sum_{q=1}^{\infty} \mu\big(B_{n_q}(q)\big) < \sum_{q=1}^{\infty} \frac{\varepsilon}{2^q} = \varepsilon.$$

On the other hand, for $n > n_q$ and $x \in X \backslash B$ we have

$$|\varphi_n(x) - \varphi(x)| < \frac{1}{q}.$$

Therefore, $\varphi_n \to \varphi$ when $n \to \infty$ uniformly on $X \backslash B$.

Problem 6.12 Let $\varphi \colon X \to \mathbb{R}$ be an $\mathcal{A}$-measurable function and let μ be a measure on X. Show that for each $t, c > 0$ we have

$$\mu\big(\{x \in X : |\varphi(x)| \geqslant t\}\big) \leqslant \frac{1}{t^c} \int_X |\varphi|^c \, d\mu.$$

Solution Let

$$B_t = \{x \in X : |\varphi(x)| \geqslant t\}.$$

Then

$$t^c \mu(B_t) = \int_X t^c \chi_{B_t} \, d\mu \leqslant \int_X |\varphi|^c \, d\mu,$$

which yields the desired inequality.

Problem 6.13 Show that the measure μ on S^1 obtained from the Lebesgue measure λ on $[0,1]$ is invariant under the expanding map $E_m\colon S^1 \to S^1$.

Solution We identify S^1 with $[0,1]$. For each set $B \subseteq [0,1]$ in $\mathcal{B}$, let

$$B_i = \left\{\left[\frac{x+i}{m}\right] : x \in B\right\}$$

for $i = 1, \ldots, m$. Then

$$E_m^{-1}B = \bigcup_{i=1}^{m} B_i.$$

Since $B_i \cap B_j = \varnothing$ for $i \neq j$, we obtain

$$\begin{aligned}\mu(E_m^{-1}B) &= \sum_{i=1}^{m} \lambda(B_i)\\ &= \sum_{i=1}^{m} \frac{\lambda(B)}{m}\\ &= \lambda(B) = \mu(B)\end{aligned}$$

and so the measure μ is E_m-invariant.

Problem 6.14 Given finitely many or countably many intervals $(a_i, b_i) \subsetneq [0,1]$, for $i \in I$, with $\sum_{i\in I}(b_i - a_i) = 1$, consider a map $f\colon [0,1] \to [0,1]$ such that

$$f(x) = \frac{x - a_i}{b_i - a_i}$$

for each $x \in (a_i, b_i)$ and $i \in I$ (see Figure I.6.1 for an example). Show that f preserves the Lebesgue measure λ on $[0,1]$.

Solution For each $a \in [0,1]$ we have

$$f^{-1}(0,a) = \bigcup_{i\in I}(a_i, a_i + a(b_i - a_i))$$

and the union is disjoint. Hence,

$$\begin{aligned}\lambda\big(f^{-1}(0,a)\big) &= \sum_{i\in I} \lambda\big((a_i, a_i + a(b_i - a_i))\big)\\ &= \sum_{i\in I} a(b_i - a_i)\\ &= a = \lambda\big((0,a)\big).\end{aligned}$$

Since the intervals $(0,a)$ generate the Borel σ-algebra on $[0,1]$, it follows that the measure λ is f-invariant.

Problem 6.15 Show that the map $f\colon [0,1]\to[0,1]$ defined by

$$f(x)=\begin{cases} x/(1-x) & \text{if } x\in[0,1/2],\\ (1-x)/x & \text{if } x\in[1/2,1]\end{cases}$$

(see Figure I.6.2) preserves the measure μ on $[0,1]$ defined by

$$\mu(A)=\int_A \frac{dx}{x}.$$

Solution Note that

$$\mu([a,b])=\int_a^b \frac{dx}{x}=\log b-\log a.$$

Since

$$f^{-1}[a,b]=\left[\frac{a}{1+a},\frac{b}{1+b}\right]\cup\left[\frac{1}{1+b},\frac{1}{1+a}\right]$$

(and the intervals on the right-hand side are disjoint), we obtain

$$\begin{aligned}\mu\bigl(f^{-1}[a,b]\bigr)&=\mu\left(\left[\frac{a}{1+a},\frac{b}{1+b}\right]\right)+\mu\left(\left[\frac{1}{1+b},\frac{1}{1+a}\right]\right)\\ &=\log\frac{b}{1+b}-\log\frac{a}{1+a}+\log\frac{1}{1+a}-\log\frac{1}{1+b}\\ &=\log b-\log a=\mu([a,b]).\end{aligned}$$

Since the intervals $[a,b]$ generate the Borel σ-algebra on $[0,1]$ (see Problem 6.3), it follows that the measure μ is f-invariant.

Problem 6.16 Consider the Gauss map $f\colon[0,1]\to[0,1]$ defined by

$$f(x)=\begin{cases}\{1/x\} & \text{if } x\neq 0,\\ 0 & \text{if } x=0\end{cases}$$

(see Figure I.6.3). Show that f preserves the measure μ on $[0,1]$ defined by

$$\mu(A)=\int_A\frac{dx}{1+x}.$$

Solution First note that it suffices to consider the intervals $A=[0,a)$ for $a\in(0,1)$ since these generate the Borel σ-algebra on $[0,1]$. We have

$$f^{-1}[0,a)=\bigcup_{n=1}^{\infty}\left[\frac{1}{n+a},\frac{1}{n}\right)$$

and the union is disjoint. Hence,

$$\begin{aligned}
\mu\big(f^{-1}[0,a)\big) &= \sum_{n=1}^{\infty} \mu\Big(\Big[\frac{1}{n+a},\frac{1}{n}\Big)\Big) \\
&= \sum_{n=1}^{\infty} \int_{1/(n+a)}^{1/n} \frac{dx}{1+x} \\
&= \sum_{n=1}^{\infty} \left(\log\Big(1+\frac{1}{n}\Big) - \log\Big(1+\frac{1}{n+a}\Big)\right) \\
&= \sum_{n=1}^{\infty} \left(\log\frac{n+1}{n+1+a} - \log\frac{n}{n+a}\right) \\
&= -\log\frac{1}{1+a} = \int_0^a \frac{dx}{1+x} = \mu([0,a))
\end{aligned}$$

and so the measure μ is f-invariant.

Problem 6.17 Show that the map $g\colon [0,1] \to [0,1]$ given by $g(x) = 4x(1-x)$ preserves the measure μ on $[0,1]$ defined by

$$\mu(A) = \int_A \frac{dx}{\sqrt{x(1-x)}}.$$

Solution Note that

$$\left(\arcsin\Big(\frac{x}{a}\Big)\right)' = \frac{1/a}{\sqrt{1-x^2/a^2}} = \frac{1}{\sqrt{a^2-x^2}}$$

for $a > 0$. Since

$$x(1-x) = \frac{1}{4} - \left(x-\frac{1}{2}\right)^2,$$

we obtain

$$\begin{aligned}
(\arcsin(2x-1))' &= \left(\arcsin\Big(\frac{x-1/2}{1/2}\Big)\right)' \\
&= \frac{1}{\sqrt{1/4-(x-1/2)^2}} \qquad \text{(II.6.1)} \\
&= \frac{1}{\sqrt{x(1-x)}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu([a,b]) &= \int_a^b \frac{dx}{\sqrt{x(1-x)}} \\
&= \arcsin(2b-1) - \arcsin(2a-1).
\end{aligned}$$

On the other hand, since

$$g^{-1}[a,b]=\left[\frac{1}{2}-\frac{1}{2}\sqrt{1-a},\frac{1}{2}-\frac{1}{2}\sqrt{1-b}\right]\cup\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-b},\frac{1}{2}+\frac{1}{2}\sqrt{1-a}\right]$$

(and the intervals on the right-hand side are disjoint), we obtain

$$\begin{aligned}\mu\left(g^{-1}[a,b]\right)&=\mu\left(\left[\frac{1}{2}-\frac{1}{2}\sqrt{1-a},\frac{1}{2}-\frac{1}{2}\sqrt{1-b}\right]\right)\\&\quad+\mu\left(\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-b},\frac{1}{2}+\frac{1}{2}\sqrt{1-a}\right]\right)\\&=\arcsin(-\sqrt{1-b})-\arcsin(-\sqrt{1-a})\\&\quad+\arcsin(\sqrt{1-a})-\arcsin(\sqrt{1-b})\\&=2\arcsin(\sqrt{1-a})-2\arcsin(\sqrt{1-b}).\end{aligned}$$

Now observe that

$$(\arcsin(\sqrt{1-x}))'=\frac{(\sqrt{1-x})'}{\sqrt{1-(1-x)}}=-\frac{1}{2\sqrt{x(1-x)}}.$$

Comparing with (II.6.1) and taking $x=1$, we conclude that

$$\arcsin(2x-1)+2\arcsin(\sqrt{1-x})=\frac{\pi}{2}$$

for all $x\in[0,1]$. Therefore,

$$\begin{aligned}\mu\left(g^{-1}[a,b]\right)&=2\arcsin(\sqrt{1-a})-2\arcsin(\sqrt{1-b})\\&=\arcsin(2b-1)-\arcsin(2a-1)\\&=\mu([a,b]).\end{aligned}$$

Since the intervals $[a,b]$ generate the Borel σ-algebra on $[0,1]$ (see Problem 6.3), it follows that the measure μ is g-invariant.

Problem 6.18 Given $\alpha\in\mathbb{R}$, show that the map $f\colon\mathbb{T}^2\to\mathbb{T}^2$ defined by

$$(x,y)\mapsto(x+\alpha,x+y)$$

preserves the measure m on $\mathbb{T}^2$ obtained from the Lebesgue measure on $[0,1]^2$.

Solution Let

$$h\colon(x,y)\mapsto(x,x+y)\quad\text{and}\quad g\colon(x,y)\mapsto(x+\alpha,y).$$

Then $f=g\circ h$. Since the map $H(x,y)=(x,x+y)$ on $\mathbb{R}^2$ has Jacobian

$$\det d_{(x,y)}H=\det\begin{pmatrix}1&0\\1&1\end{pmatrix}=1,$$

for each $B\subseteq[0,1]^2$ in $\mathcal{B}_2$ we have

$$m(h(B)) = \int_{h(B)} 1 = \int_B |\det d_{(x,y)} H| \, dxdy = \int_B 1 = m(B)$$

and so the measure m is h-invariant. On the other hand, since g is a translation, m is also g-invariant. Hence, for each set $A \subseteq [0,1]^2$ in $\mathcal{B}_2$ (identifying $\mathbb{T}^2$ with $[0,1]^2$) we have

$$\begin{aligned} m\big(f^{-1}A\big) &= m\big((g \circ h)^{-1}A\big) \\ &= m\big(h^{-1}(g^{-1}A)\big) \\ &= m(g^{-1}A) \\ &= m(A) \end{aligned}$$

and so the measure m is f-invariant.

Problem 6.19 Let f be the Gauss map in Problem 6.16. Show that if

$$x = [a_1, a_2, \ldots] = \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

is the continued fraction of a number $x \in (0,1)$, then:

1. $f(x) = [a_2, a_3, \ldots]$;
2. letting

$$p_{-1} = 0, \quad p_0 = 1, \quad p_m = a_m p_{m-1} + p_{m-2}$$

and

$$q_{-1} = 1, \quad q_0 = 0, \quad q_m = a_m q_{m-1} + q_{m-2}$$

for $m \in \mathbb{N}$, we have

$$p_m q_{m-1} - p_{m-1} q_m = (-1)^{m-1} \quad \text{for } m \geqslant 0;$$

3. for $m \in \mathbb{N}$ we have

$$[a_1, a_2, \ldots, a_m] := \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_m}}}} = \frac{p_m}{q_m}.$$

Solution 1. It is clear that $f(x) = [a_2, a_3, \ldots]$.

2. The identity is immediate for $m = 0$. Now assume that it holds with m replaced by $m - 1$. Then

$$\begin{aligned} p_m q_{m-1} - p_{m-1} q_m &= (a_m p_{m-1} + p_{m-2}) q_{m-1} - p_{m-1}(a_m q_{m-1} + q_{m-2}) \\ &= -(p_{m-1} q_{m-2} - p_{m-2} q_{m-1}) \\ &= -(-1)^{m-2} = (-1)^{m-1}, \end{aligned}$$

which establishes the identity.

3. Again the identity is immediate for $m = 1$. Now assume that it holds with m replaced by $m-1$ and note that

$$[a_1, \dots, a_m] = \left[a_1, \dots, a_{m-1} + \frac{1}{a_m}\right].$$

By the formulas in item 2 with m replaced by $m-1$ and with a_{m-1} replaced by $a_{m-1} + 1/a_m$, we obtain

$$\begin{aligned}
[a_1, \dots, a_m] &= \frac{\left(a_{m-1} + \frac{1}{a_m}\right) p_{m-2} + p_{m-3}}{\left(a_{m-1} + \frac{1}{a_m}\right) q_{m-2} + q_{m-3}} \\
&= \frac{a_m(a_{m-1} p_{m-2} + p_{m-3}) + p_{m-2}}{a_m(a_{m-1} q_{m-2} + q_{m-3}) + q_{m-2}} \\
&= \frac{a_m p_{m-1} + p_{m-2}}{a_m q_{m-1} + a_{m-2}} \\
&= \frac{p_m}{q_m}.
\end{aligned}$$

Problem 6.20 Let $P = (p_{ij})$ be a $k \times k$ matrix with entries $p_{ij} \geqslant 0$ for $i, j = 1, \dots, k$ such that

$$\sum_{j=1}^{k} p_{ij} = 1 \quad \text{for } j = 1, \dots, k.$$

Moreover, take $p_1, \dots, p_k \in (0, 1)$ such that

$$\sum_{i=1}^{k} p_i = 1 \quad \text{and} \quad \sum_{i=1}^{k} p_i p_{ij} = p_j$$

for $j = 1, \dots, k$. We define a measure μ on the σ-algebra on Σ_k^+ generated by the cylinders (see Problem 5.1) by

$$\mu(C_{i_1 \cdots i_n}) = p_{i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$$

for each $n \in \mathbb{N}$ and $i_1, \dots, i_n \in \{1, \dots, k\}$. Show that:

1. $\mu(C_{i_1 \cdots i_n}) = \sum_{j=1}^{k} \mu(C_{i_1 \cdots i_n j})$;
2. μ is σ-invariant; and
3. $\mu(\Sigma_k^+) = 1$.

Solution 1. We have

$$\begin{aligned}
\sum_{j=1}^{k} \mu(C_{i_1 \cdots i_n j}) &= \sum_{j=1}^{k} p_{i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n} p_{i_n j} \\
&= p_{i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n} \\
&= \mu(C_{i_1 \cdots i_n}).
\end{aligned}$$

2. Note that

$$\sigma^{-1}C_{i_1\cdots i_n} = \bigcup_{j=1}^{k} C_{ji_1\cdots i_n}$$

and the union is disjoint. Hence,

$$\begin{aligned}\mu\big(\sigma^{-1}C_{i_1\cdots i_n}\big) &= \sum_{j=1}^{k} \mu(C_{ji_1\cdots i_n}) \\ &= \sum_{j=1}^{k} p_j p_{ji_1}\cdots p_{i_{n-1}i_n} \\ &= p_{i_1}p_{i_1i_2}\cdots p_{i_{n-1}i_n} \\ &= \mu(C_{i_1\cdots i_n}).\end{aligned}$$

Since the cylinders generate the σ-algebra on Σ_k^+, this implies that the measure μ is σ-invariant.

3. We have

$$\mu(\Sigma_k^+) = \sum_{i=1}^{k} \mu(C_i) = \sum_{i=1}^{k} p_i = 1.$$

Problem 6.21 For the measure μ in Problem 6.20, consider the $k \times k$ matrix A with entries

$$a_{ij} = \begin{cases} 1 & \text{if } p_{ij} > 0, \\ 0 & \text{if } p_{ij} = 0. \end{cases}$$

Show that $\operatorname{supp}\mu = \Sigma_A^+$.

Solution Given $(i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$, we have

$$\mu(C_{i_1\cdots i_n}) = p_{i_1}p_{i_1i_2}\cdots p_{i_{n-1}i_n} > 0$$

(by the definition of A). Since the cylinders generate the topology on Σ_A^+, we conclude that $(i_1 i_2 \cdots) \in \operatorname{supp}\mu$ and so $\Sigma_A^+ \subseteq \operatorname{supp}\mu$.

On the other hand, given $(i_1 i_2 \cdots) \in \Sigma_k^+ \setminus \Sigma_A^+$, there exists $n \in \mathbb{N}$ with $a_{i_n i_{n+1}} = 0$ and thus,

$$p_{i_n i_{n+1}} = 0 \quad \text{and} \quad \mu(C_{i_1\cdots i_{n+1}}) = 0.$$

Since $(i_1 i_2 \cdots) \in C_{i_1\cdots i_{n+1}}$, we conclude that $(i_1 i_2 \cdots) \notin \operatorname{supp}\mu$ and so $\operatorname{supp}\mu \subseteq \Sigma_A^+$.

Problem 6.22 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map preserving a finite measure μ on X. Given a set $A \in \mathcal{A}$ with $\mu(A) > 0$, show that there exist integers $m, n \geqslant 0$ with $m \neq n$ such that

$$f^{-m}A \cap f^{-n}A \neq \varnothing.$$

Solution We proceed by contradiction. Assume that

$$f^{-m}A \cap f^{-n}A = \varnothing$$

for any integers $m,n \geqslant 0$ with $m \neq n$. Since μ is f-invariant, we obtain

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} f^{-i}A\right) &= \sum_{i=1}^{\infty} \mu\left(f^{-i}A\right)\\ &= \sum_{i=1}^{\infty} \mu(A) = +\infty,\end{aligned}$$

but this is impossible because the measure μ is finite. This contradiction yields the desired property.

Problem 6.23 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map preserving a finite measure μ on X. Given a set $A \in \mathcal{A}$ with $\mu(A) > 0$, show that the map $\tau_A\colon X \to \mathbb{N}$ given by

$$\tau_A(x) = \min\{n \in \mathbb{N} : f^n(x) \in A\}$$

is well defined for almost every $x \in A$.

Solution By Poincaré's recurrence theorem (Theorem 6.9), we have

$$\mu\left(\left\{x \in A : f^n(x) \in A \text{ for infinitely many integers } n \in \mathbb{N}\right\}\right) = \mu(A).$$

In particular, the set of points in A that eventually return to A applying f successively has the same measure as the set A. This shows that the number $\tau_A(x)$ is well defined for almost every $x \in A$.

Problem 6.24 Show that a function $\varphi\colon X \to \mathbb{R}$ satisfies $\varphi \circ f = \varphi$ for some map $f\colon X \to X$ if and only if the set $\varphi^{-1}\alpha$ is f-invariant for every $\alpha \in \mathbb{R}$.

Solution The set $\varphi^{-1}\alpha$ is f-invariant if and only if

$$\varphi^{-1}\alpha = f^{-1}(\varphi^{-1}\alpha) = (\varphi \circ f)^{-1}\alpha.$$

This is equivalent to require that

$$x \in \varphi^{-1}\alpha \quad \text{if and only if} \quad x \in (\varphi \circ f)^{-1}\alpha$$

or, equivalently,

$$\varphi(x) = \alpha \quad \text{if and only if} \quad \varphi(f(x)) = \alpha.$$

Therefore, $\varphi^{-1}\alpha$ is f-invariant for every $\alpha \in \mathbb{R}$ if and only if

$$\varphi(f(x)) = \varphi(x)$$

for every $x \in X$, that is, if and only if φ is f-invariant.

Problem 6.25 Given $\alpha \in \mathbb{Q}$, consider the rotation $R_\alpha\colon S^1 \to S^1$. Let $U = [a,b]$ be an interval in S^1 with $0 \leqslant a < b \leqslant 1$ and define

$$T_n(x) = \operatorname{card}\{1 \leqslant k \leqslant n : R_\alpha^k(x) \in U\}.$$

Moreover, let μ be the measure on S^1 obtained from the Lebesgue measure on $[0,1]$. Show that:

1. if $\varphi(x) = e^{2\pi imx}$ with $m \in \mathbb{Z}$, then
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(R_\alpha^k(x)) = \int_{S^1} \varphi \, d\mu \quad \text{for all } x \in S^1;$$
2. the former property holds for all continuous functions $\varphi\colon S^1 \to \mathbb{R}$; and
3. for each $x \in S^1$ we have
$$\lim_{n\to\infty} \frac{T_n(x)}{n} = b - a.$$

Solution 1. Let $a_m = e^{2\pi im\alpha}$. For the function $\varphi(x) = e^{2\pi imx}$ we have

$$\begin{aligned} \varphi(R_\alpha(x)) &= e^{2\pi im(x+\alpha)} \\ &= e^{2\pi im\alpha} e^{2\pi imx} \\ &= a_m \varphi(x). \end{aligned}$$

It follows by induction that

$$\varphi(R_\alpha^k(x)) = a_m^k \varphi(x) \quad \text{for } k \in \mathbb{N}.$$

For $m \neq 0$ we have

$$\sum_{k=1}^{n} \varphi(R_\alpha^k(x)) = \varphi(x) \sum_{k=1}^{n} a_m^k = \varphi(x) a_m \frac{1 - a_m^n}{1 - a_m}$$

and so

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(R_\alpha^k(x)) = 0. \tag{II.6.2}$$

On the other hand, for $m = 0$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(R_\alpha^k(x)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1 = 1. \tag{II.6.3}$$

Finally, using the formula $e^{iy} = \cos y + i \sin y$ and integrating we obtain

$$\int_{S^1} \varphi \, d\mu = \begin{cases} 0 & \text{if } m \neq 0, \\ 1 & \text{if } m = 0. \end{cases}$$

Comparing with (II.6.2) and (II.6.3), it follows that the desired property holds for the function $\varphi(x) = e^{2\pi imx}$.

2. First recall that the set of all linear combinations of the functions $e^{2\pi imx}$ is dense in the set of all continuous functions $\varphi\colon S^1 \to S^1$ equipped with the distance

$$\|\varphi - \psi\| = \sup\{d(\varphi(x), \psi(x)) : x \in S^1\}.$$

Hence, given $\varepsilon > 0$, there exists such a linear combination ψ with $\|\varphi - \psi\| < \varepsilon/2$. Letting

$$\psi_1(x) = \psi(x) - \frac{\varepsilon}{2} \quad \text{and} \quad \psi_2(x) = \psi(x) + \frac{\varepsilon}{2}$$

for $x \in S^1$, we have

$$\psi_1(x) \leqslant \varphi(x) \leqslant \psi_2(x)$$

and

$$\int_{S^1} (\psi_2 - \psi_1)\, d\mu = \varepsilon.$$

Since $\varphi \leqslant \psi_2$, we obtain

$$\int_{S^1} (\varphi - \psi_1)\, d\mu \leqslant \int_{S^1} (\psi_2 - \psi_1)\, d\mu = \varepsilon$$

and so

$$\begin{aligned}\int_{S^1} \varphi\, d\mu - \varepsilon &\leqslant \int_{S^1} \psi_1\, d\mu \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \psi_1(R_\alpha^k(x)) \\ &\leqslant \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(R_\alpha^k(x))\end{aligned} \tag{II.6.4}$$

(it follows readily from item 1 that the property also holds for linear combinations of the functions $e^{2\pi imx}$). One can show in a similar manner that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(R_\alpha^k(x)) \leqslant \int_{S^1} \varphi\, d\mu + \varepsilon. \tag{II.6.5}$$

Since ε is arbitrary, it follows from (II.6.4) and (II.6.5) that the property in item 1 holds for all continuous functions $\varphi\colon S^1 \to \mathbb{R}$.

3. First observe that

$$T_n(x) = \sum_{k=1}^{n} \chi_U(R_\alpha^k(x)).$$

Given $\varepsilon > 0$, there exist continuous functions $\psi_1, \psi_2\colon S^1 \to \mathbb{R}$ with

$$\psi_1 \leqslant \chi_U \leqslant \psi_2$$

and

$$\int_{S^1} (\psi_2 - \psi_1)\, d\mu < \varepsilon.$$

Proceeding as before, we find that the property in item 1 also holds with φ replaced by χ_U. Hence,

$$\lim_{n\to\infty} \frac{T_n(x)}{n} = \int_{S^1} \chi_U\, d\mu = b - a \quad \text{for all } x \in S^1.$$

Problem 6.26 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map. A probability measure μ on X is said to be *ergodic (with respect to f)* if any f-invariant set has either measure 0 or measure 1. Show that if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu\left(f^{-k}A \cap B\right) = \mu(A)\mu(B)$$

for any sets $A, B \in \mathcal{A}$, then the measure μ is ergodic.

Solution Let $A \subseteq X$ be an f-invariant set. By the hypothesis we obtain

$$\begin{aligned}
\mu(A)\mu(X\backslash A) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu\left(f^{-k}A \cap (X\backslash A)\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu\left(A \cap (X\backslash A)\right) = 0
\end{aligned}$$

because $f^{-k}A = A$ for every $k \geqslant 0$. Thus, either $\mu(A) = 0$ or $\mu(X\backslash A) = 0$ and so the measure μ is ergodic.

Problem 6.27 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map. Show that an f-invariant measure μ on X is ergodic (see Problem 6.26) if and only if

$$\mu\left(\bigcup_{n=1}^{\infty} f^{-n}A\right) = 1$$

for any set $A \in \mathcal{A}$ with $\mu(A) > 0$.

Solution First assume that μ is ergodic and consider the set

$$B = \bigcup_{n=1}^{\infty} f^{-n}A. \tag{II.6.6}$$

Clearly, $f^{-n}B \subseteq B$ for all $n \in \mathbb{N}$. Since the measure μ is f-invariant, we conclude that $\mu(B\backslash f^{-n}B) = 0$ for all $n \in \mathbb{N}$. Now we consider the set

$$C = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} f^{-k}B \subseteq B.$$

Clearly, $f^{-1}C = C$ and since μ is ergodic, either $\mu(C) = 0$ or $\mu(C) = 1$. Note that

$$B\backslash C = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} B\backslash f^{-k}B$$

and so $\mu(B\backslash C) = 0$. Moreover, since

$$\mu(f^{-n}A) = \mu(A) > 0$$

and $B \supseteq f^{-n}A$, we conclude that

$$\mu(C) = \mu(B) > 0,$$

which implies that $\mu(C) = 1$. Therefore,

$$\mu(B) = \mu(C) = 1.$$

Now assume that the set B in (II.6.6) has full measure for any set $A \in \mathcal{A}$ with $\mu(A) > 0$. Given an f-invariant set $A \in \mathcal{A}$ with $\mu(A) > 0$, we have

$$f^{-n}A = A \quad \text{for all } n \in \mathbb{N}.$$

Then $B = A$, which implies that

$$\mu(A) = \mu(B) = 1.$$

Problem 6.28 Let $f \colon X \to X$ be an $\mathcal{A}$-measurable map preserving an ergodic probability measure μ on X. Given a set $A \in \mathcal{A}$ with $\mu(A) > 0$, show that

$$\int_A \tau_A \, d\mu = 1$$

with the function τ_A as in Problem 6.23.

Solution By Problem 6.23 the function τ_A is well defined for almost every $x \in A$ and so the integral is also well defined.

Consider the sets

$$A_n = \left\{x \in A : \tau_A(x) = n\right\}$$

and

$$B_n = \left\{x \in X : f^n(x) \in A, f^k(x) \notin A \text{ for } k = 1, \ldots, n-1\right\}$$

for each $n \in \mathbb{N}$. Clearly, $A_n \subseteq B_n$. Moreover,

$$A_n \cap A_m = \varnothing \quad \text{and} \quad B_n \cap B_m = \varnothing$$

for any $n, m \in \mathbb{N}$ with $n \neq m$. We have

$$f^{-k}A\setminus\bigcup_{n=1}^{\infty}B_n=\varnothing \quad \text{for all } k\in\mathbb{N}.$$

On the other hand, by Problem 6.27, $\mu(\bigcup_{n=1}^{\infty}f^{-n}A)=1$ and so also

$$\mu\left(\bigcup_{n=1}^{\infty}B_n\right)=1.$$

Now observe that $f^{-1}(B_n\setminus A_n)=B_{n+1}$ and

$$\begin{aligned}\mu(B_n)&=\mu(A_n)+\mu(B_n\setminus A_n)\\&=\mu(A_n)+\mu(f^{-1}(B_n\setminus A_n))\\&=\mu(A_n)+\mu(B_{n+1}).\end{aligned}$$

Proceeding inductively, we obtain

$$\mu(B_n)=\sum_{k\geqslant n}\mu(A_k)$$

and so

$$\begin{aligned}1=\mu\left(\bigcup_{n=1}^{\infty}B_n\right)&=\sum_{n=1}^{\infty}\mu(B_n)\\&=\sum_{n=1}^{\infty}\sum_{k\geqslant n}\mu(A_k)=\sum_{k=1}^{\infty}\sum_{n=1}^{k}\mu(A_k)\\&=\sum_{k=1}^{\infty}k\mu(A_k)=\int_A\tau_A\,d\mu.\end{aligned}$$

Problem 6.29 Let $f\colon X\to X$ be an $\mathcal{A}$-measurable map. A probability measure μ on X is said to be *mixing (with respect to f)* if

$$\lim_{n\to\infty}\mu\bigl(f^{-n}A\cap B\bigr)=\mu(A)\mu(B)$$

for any sets $A,B\in\mathcal{B}$ and is said to be *weakly mixing (with respect to f)* if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\bigl|\mu\bigl(f^{-i}A\cap B\bigr)-\mu(A)\mu(B)\bigr|=0$$

for any sets $A,B\in\mathcal{B}$. Show that if μ is mixing, then it is weakly mixing.

Solution Assume that μ is mixing. Given $\varepsilon>0$, there exists $p\in\mathbb{N}$ such that

$$\bigl|\mu\bigl(f^{-i}A\cap B\bigr)-\mu(A)\mu(B)\bigr|<\varepsilon$$

for all $i \geqslant p$. Therefore,

$$\frac{1}{n}\sum_{i=0}^{n-1}\left|\mu\left(f^{-i}A\cap B\right)-\mu(A)\mu(B)\right| \leqslant \frac{2p}{n}+\frac{n-p}{n}\varepsilon$$

(separating the terms with $i < p$ and $i \geqslant p$). Letting $n \to \infty$ we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\left|\mu\left(f^{-i}A\cap B\right)-\mu(A)\mu(B)\right| \leqslant \varepsilon$$

and since ε is arbitrary, it follows that μ is weakly mixing.

Problem 6.30 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map. Show that if a probability measure μ on X is weakly mixing (see Problem 6.29), then it is ergodic.

Solution Assume that μ is weakly mixing and let $A \in \mathcal{B}$ be an f-invariant set. It follows from the definition of weakly mixing measure with $A = B$ that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\left|\mu(A)-\mu(A)^2\right| = 0$$

and so

$$\mu(A) = \mu(A)^2.$$

Hence, either $\mu(A) = 0$ or $\mu(A) = 1$, which shows that the measure μ is ergodic.

Problem 6.31 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map preserving a mixing probability measure μ on X (see Problem 6.29). Show that f is topologically mixing on the support $\operatorname{supp}\mu$ of μ.

Solution Let $U, V \subsetneq X$ be nonempty open sets intersecting the support $\operatorname{supp}\mu$. Then $\mu(U), \mu(V) > 0$ and since μ is mixing, we have

$$\lim_{n\to\infty}\mu\left(f^{-n}U\cap V\right) = \mu(U)\mu(V) > 0.$$

Hence,

$$\mu(f^{-n}U\cap V) > 0$$

for all sufficiently large n, which implies that

$$f^{-n}U\cap V \neq \varnothing$$

also for all sufficiently large n. Therefore, f is topologically mixing on $\operatorname{supp}\mu$.

Problem 6.32 Let μ be a probability measure on X. Show that if ξ is a partition of X, then

$$H_\mu(\xi) \leqslant \log\operatorname{card}\xi.$$

Solution Let $\psi\colon X \to \mathbb{R}$ be the continuous function defined by

$$\psi(x) = \begin{cases} x\log x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{II.6.7}$$

Note that

$$H_\mu(\xi) = -\sum_{C\in\xi} \psi(\mu(C)).$$

Since

$$\psi''(x) = 1/x > 0 \quad \text{for } x \neq 0,$$

the function ψ is strictly convex and so

$$\psi\left(\sum_{i=1}^p a_i x_i\right) \leqslant \sum_{i=1}^p a_i \psi(x_i) \tag{II.6.8}$$

for any numbers $x_1,\ldots,x_p,a_1,\ldots,a_p \in [0,1]$ with $\sum_{i=1}^p a_i = 1$. Moreover, (II.6.8) is an equality if and only if all numbers x_i corresponding to a nonzero coefficient a_i are equal. Therefore,

$$\begin{aligned} H_\mu(\xi) &= -\sum_{C\in\xi} \psi(\mu(C)) \\ &= -\operatorname{card}\xi \sum_{C\in\xi} \frac{1}{\operatorname{card}\xi}\psi(\mu(C)) \\ &\leqslant -\operatorname{card}\xi\, \psi\left(\sum_{C\in\xi} \frac{\mu(C)}{\operatorname{card}\xi}\right) \\ &= -\operatorname{card}\xi\, \psi\left(\frac{1}{\operatorname{card}\xi}\right) \\ &= -\log\frac{1}{\operatorname{card}\xi} = \log\operatorname{card}\xi. \end{aligned}$$

Problem 6.33 Let μ be a probability measure on X and let ξ and η be partitions of X. Show that if η is a *refinement* of ξ (that is, if for each $D \in \eta$ there exists $C \in \xi$ such that $\mu(D\backslash C) = 0$), then

$$H_\mu(\xi) \leqslant H_\mu(\eta).$$

Solution Note that

$$\begin{aligned} H_\mu(\eta) &= -\sum_{D\in\eta} \psi(\mu(D)) \\ &= -\sum_{C\in\xi}\sum_{D\subseteq C} \psi(\mu(D)) \end{aligned} \tag{II.6.9}$$

with ψ as in (II.6.7). For each $C \in \xi$, we have

$$\sum_{D \subseteq C} \mu(D) = \mu(C)$$

and so

$$\begin{aligned}
\psi(\mu(C)) &= \psi\left(\sum_{D \subseteq C} \mu(D)\right) \\
&= \sum_{D \subseteq C} \mu(D) \log \sum_{D \subseteq C} \mu(D) \\
&\geqslant \sum_{D \subseteq C} \mu(D) \log \mu(D) \\
&= \sum_{D \subseteq C} \psi(\mu(D)).
\end{aligned}$$

Therefore, it follows from (II.6.9) that

$$\begin{aligned}
H_\mu(\xi) &= -\sum_{C \in \xi} \psi(\mu(C)) \\
&\leqslant -\sum_{C \in \xi} \sum_{D \subseteq C} \psi(\mu(D)) = H_\mu(\eta).
\end{aligned}$$

Problem 6.34 Let μ be a probability measure on X. Show that if ξ and η are partitions of X and

$$\xi \vee \eta = \{C \cap D : C \in \xi, D \in \eta\},$$

then

$$H_\mu(\xi \vee \eta) \leqslant H_\mu(\xi) + H_\mu(\eta).$$

Solution We have

$$\begin{aligned}
H_\mu(\xi \vee \eta) &= -\sum_{C \in \xi, D \in \eta} \mu(C \cap D) \log \mu(C \cap D) \\
&= -\sum_{C \in \xi, D \in \eta} \mu(C \cap D)\left[\log \frac{\mu(C \cap D)}{\mu(C)} + \log \mu(C)\right] \\
&= -\sum_{D \in \eta} \sum_{C \in \xi} \mu(C) \psi\left(\frac{\mu(C \cap D)}{\mu(C)}\right) \\
&\quad - \sum_{C \in \xi, D \in \eta} \mu(C \cap D) \log \mu(C)
\end{aligned}$$

again with ψ as in (II.6.7). Since the function ψ is convex (see Problem 6.32), we obtain

$$
\begin{aligned}
H_\mu(\xi \vee \eta) &\leqslant -\sum_{D\in\eta} \psi\left(\sum_{C\in\xi} \mu(C)\frac{\mu(C\cap D)}{\mu(C)}\right) - \sum_{C\in\xi} \mu(C)\log\mu(C) \\
&= -\sum_{D\in\eta} \psi(\mu(D)) + H_\mu(\xi) \\
&= H_\mu(\eta) + H_\mu(\xi).
\end{aligned}
$$

Problem 6.35 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map preserving a probability measure μ on X. Show that if ξ is a partition of X and

$$f^{-1}\xi = \{f^{-1}C : C \in \xi\},$$

then

$$H_\mu(f^{-1}\xi) = H_\mu(\xi).$$

Solution Since the measure μ is f-invariant, we obtain

$$
\begin{aligned}
H_\mu(f^{-1}\xi) &= -\sum_{C\in\xi} \mu(f^{-1}C)\log\mu(f^{-1}C) \\
&= -\sum_{C\in\xi} \mu(C)\log\mu(C) = H_\mu(\xi).
\end{aligned}
$$

Problem 6.36 Let $f\colon X \to X$ be an $\mathcal{A}$-measurable map preserving a probability measure μ on X. Show that for each partition ξ of X we have

$$h_\mu(f,\xi) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} f^{-i}\xi\right) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} f^{-i}\xi\right).$$

Solution First note that

$$\xi_n = \bigvee_{i=0}^{n-1} f^{-i}\xi \quad \text{for each } n \in \mathbb{N}.$$

Hence, by Problems 6.34 and 6.35, for each $n, m \in \mathbb{N}$ we have

$$
\begin{aligned}
H_\mu(\xi_{n+m}) &= H_\mu(\xi_n \vee f^{-n}\xi_m) \\
&\leqslant H_\mu(\xi_n) + H_\mu(f^{-n}\xi_m) \\
&= H_\mu(\xi_n) + H_\mu(\xi_m).
\end{aligned}
$$

This shows that the sequence $c_n = H_\mu(\xi_n)$ is subadditive (see (II.2.7)). Therefore, by (II.2.8), the limit

$$\lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_n) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu(\xi_n) = h_\mu(f,\xi)$$

exists.

Problem 6.37 Let μ be a probability measure on X and let ξ, ζ, and η be partitions of X. Show that if ζ is a refinement of η (see Problem 6.33) and

$$H_\mu(\xi|\zeta) = -\sum_{C\in\xi, D\in\zeta} \mu(C\cap D)\log\frac{\mu(C\cap D)}{\mu(D)},$$

then

$$H_\mu(\xi|\zeta) \leqslant H_\mu(\xi|\eta).$$

Solution We have

$$\begin{aligned}
H_\mu(\xi|\zeta) &= -\sum_{C\in\xi, D\in\zeta} \mu(C\cap D)\log\frac{\mu(C\cap D)}{\mu(D)}\\
&= -\sum_{C\in\xi, D\in\zeta, E\in\eta} \mu(D\cap E)\frac{\mu(C\cap D)}{\mu(D)}\log\frac{\mu(C\cap D)}{\mu(D)}\\
&= -\sum_{C\in\xi, E\in\eta} \mu(E)\sum_{D\in\zeta}\frac{\mu(D\cap E)}{\mu(E)}\psi\left(\frac{\mu(C\cap D)}{\mu(D)}\right).
\end{aligned}$$

It follows from the convexity of ψ (see Problem 6.32) that

$$H_\mu(\xi|\zeta) \leqslant -\sum_{C\in\xi, E\in\eta} \mu(E)\psi\left(\sum_{D\in\zeta}\frac{\mu(D\cap E)}{\mu(E)}\cdot\frac{\mu(C\cap D)}{\mu(D)}\right).$$

Finally, since ζ is a refinement of η, we obtain

$$\sum_{D\in\zeta}\frac{\mu(D\cap E)}{\mu(E)}\cdot\frac{\mu(C\cap D)}{\mu(D)} = \sum_{D\in\zeta, D\subseteq E}\frac{\mu(C\cap D)}{\mu(E)} = \frac{\mu(C\cap E)}{\mu(E)}$$

and so

$$\begin{aligned}
H_\mu(\xi|\zeta) &\leqslant -\sum_{C\in\xi, E\in\eta} \mu(E)\psi\left(\frac{\mu(C\cap E)}{\mu(E)}\right)\\
&= -\sum_{C\in\xi, E\in\eta} \mu(C\cap E)\log\frac{\mu(C\cap E)}{\mu(E)}\\
&= H_\mu(\xi|\eta).
\end{aligned}$$

Problem 6.38 Let μ be a probability measure on X. Show that if ξ, ζ, and η are partitions of X, then

$$H_\mu(\xi\vee\zeta|\eta) = H_\mu(\xi|\zeta\vee\eta) + H_\mu(\zeta|\eta).$$

Solution We have

$$
\begin{aligned}
H_\mu(\xi \vee \zeta|\eta) &= -\sum_{C\in\xi, D\in\zeta, E\in\eta} \mu(C\cap D\cap E)\log\frac{\mu(C\cap D\cap E)}{\mu(D)} \\
&= -\sum_{C\in\xi, D\in\zeta, E\in\eta} \mu(C\cap D\cap E)\log\frac{\mu(C\cap D\cap E)}{\mu(D\cap E)} \\
&\quad -\sum_{C\in\xi, D\in\zeta, E\in\eta} \mu(C\cap D\cap E)\log\frac{\mu(D\cap E)}{\mu(D)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
H_\mu(\xi \vee \zeta|\eta) &= H_\mu(\xi|\zeta\vee\eta) - \sum_{D\in\zeta, E\in\eta} \mu(D\cap E)\log\frac{\mu(D\cap E)}{\mu(D)} \\
&= H_\mu(\xi|\zeta\vee\eta) + H_\mu(\zeta|\eta).
\end{aligned}
$$

Problem 6.39 For the measure μ in Problem 6.20, compute $h_\mu(\sigma)$.

Solution Consider the partition $\xi = \{C_1, \dots, C_k\}$. Then

$$
\begin{aligned}
H_\mu(\xi_n) &= -\sum_{i_1\cdots i_n} \mu(C_{i_1\cdots i_n})\log\mu(C_{i_1\cdots i_n}) \\
&= -\sum_{i_1\cdots i_n} p_{i_1}p_{i_1i_2}\cdots p_{i_{n-1}i_n}\log\bigl(p_{i_1}p_{i_1i_2}\cdots p_{i_{n-1}i_n}\bigr) \\
&= -\sum_{i_1\cdots i_n} p_{i_1}p_{i_1i_2}\cdots p_{i_{n-1}i_n}\log p_{i_1} \\
&\quad -\sum_{i_1\cdots i_n} p_{i_1}p_{i_1i_2}\cdots p_{i_{n-1}i_n}\sum_{j=1}^{n-1}\log p_{i_ji_{j+1}} \\
&= -\sum_{i=1}^k p_i\log p_i - (n-1)\sum_{i=1}^k\sum_{j=1}^k p_ip_{ij}\log p_{ij}.
\end{aligned}
$$

Now consider the partitions $\xi^{(n)} = \xi_n$. We have $\xi_m^{(n)} = \xi_{n+m-1}$ and so

$$
\begin{aligned}
h(\sigma,\xi^{(n)}) &= \inf_{m\in\mathbb{N}} \frac{1}{m}H_\mu(\xi_{n+m-1}) \\
&= \inf_{m\in\mathbb{N}} -\frac{1}{m}\left(\sum_{i=1}^k p_i\log p_i + (n+m-2)\sum_{i=1}^k\sum_{j=1}^k p_ip_{ij}\log p_{ij}\right) \\
&= -\sum_{i=1}^k\sum_{j=1}^k p_ip_{ij}\log p_{ij}.
\end{aligned}
$$

Since the partitions $\xi^{(n)}$ satisfy the hypotheses of Definition 6.13, we obtain

$$h_\mu(\sigma) = \sup_{n\in\mathbb{N}} h(\sigma, \xi^{(n)}) = -\sum_{i=1}^{k}\sum_{j=1}^{k} p_i p_{ij} \log p_{ij}. \tag{II.6.10}$$

Problem 6.40 For the measure μ in Problem 6.20 with $p_{ij} = p_j$ for all $i, j = 1, \ldots, k$, compute $h_\mu(\sigma)$.

Solution It follows from (II.6.10) that

$$\begin{aligned} h_\mu(\sigma) &= -\sum_{i=1}^{k}\sum_{j=1}^{k} p_i p_{ij} \log p_{ij} \\ &= -\sum_{i=1}^{k}\sum_{j=1}^{k} p_i p_j \log p_j \\ &= -\sum_{j=1}^{k} p_j \log p_j. \end{aligned}$$

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Index

L. Barreira and C. Valls, *Dynamical Systems by Example*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-15915-3