南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

科目： Probability **单位：** Department of Mathematics
时间： 2023/12/9 09:00–12:00 **考场：** 107 in 3rd Teaching Building

题号	1	2	3	4	5
分值	25 分	15 分	20 分	20 分	20 分

本试卷共（5）大题，满分（100）分
（考试结束后请将试卷、答题卡、草稿纸一起交给监考老师）

Throughout denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space on which all the random variables (r.v.s) are defined, and denote by $\mathcal{B}$ the Borel σ-algebra on $\mathbb{R}$.

1. Suppose that $\mathcal{G} \subset \mathcal{F}$ is a sub σ-algebra and X is an $\mathbb{R}$-valued, almost surely (a.s.) nonnegative r.v. with $\mathbb{E}(X) = 1$. Define $\mathbb{Q}(A) := \mathbb{E}(X 1_A)$, $A \in \mathcal{G}$.

   (a) Show that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{G})$ with $\mathbb{Q} \ll \mathbb{P}$, i.e., $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$. [4 marks]

   (b) State Radon–Nikodym theorem and use Radon–Nikodym theorem to show that the conditional expectation $\mathbb{E}(X|\mathcal{G})$ exists uniquely. [6 marks]

   (c) Let $\mathbb{E}_{\mathbb{Q}}$ denote the expectation with respect to $\mathbb{Q}$. Show that $\mathbb{E}_{\mathbb{Q}}(Y) = \mathbb{E}(YX)$ for any r.v. Y such that XY is integrable. [4 marks]

   (d) Assume that $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ for $p \in (1, \infty)$. Prove that $\mathbb{E}|\mathbb{E}[X|\mathcal{G}]|^p \leq \mathbb{E}|X|^p$ and $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]Z] = \mathbb{E}[XZ]$ for all $Z \in L^{\frac{p}{p-1}}(\Omega, \mathcal{G}, \mathbb{P})$. [6 marks]

   (e) Let $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, satisfy $\lim\limits_{n\to\infty} \mathbb{P}(A_n) = 0$. Show that $\lim\limits_{n\to\infty} \int_{A_n} X \mathrm{d}\mathbb{P} = 0$. [5 marks]

   [Total for Question 1: 25 marks]

2. Let $(X_n)_{n\geq 1}$ be independent r.v.s. with $\mathbb{E}X_n = \mu < \infty$ for all $n \geq 1$. Denote by $S_n := \sum_{j=1}^n X_j$ and $\mathcal{F}_n := \sigma\{X_k : k \leq n\}$ for $n \geq 1$. For $n = 0$, let $S_0 := 0$ and $\mathcal{F}_0 := \{\emptyset, \Omega\}$ be the trivial σ-algebra.

   (a) Show that $(S_n - n\mu)_{n\geq 1}$ is an $(\mathcal{F}_n)_{n\geq 1}$-martingale. [8 marks]

(b) Assume that $\mu = 0$ and $\phi : \mathbb{R} \to \mathbb{R}$ is convex so that $\phi(S_n)$ is integrable for $n \in \mathbb{N}$. Show that $n \mapsto \mathbb{E}\phi(S_n)$ is a nondecreasing function. [7 marks]

[Total for Question 2: 15 marks]

3. Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be two measurable spaces, $g : \Omega \to E \times G$ and $f : E \times G \to \mathbb{R}$ be two maps. The section of $C \subset E \times G$ at $x \in E$ and $y \in G$ are defined, respectively, by $C(x) := \{y \in G : (x, y) \in C\}$ and $C(y) := \{x \in E : (x, y) \in C\}$.

(a) Suppose that $C \in \mathcal{E} \otimes \mathcal{G} := \sigma(\mathcal{E} \times \mathcal{G})$. Show that $C(x) \in \mathcal{G}$ for any $x \in E$ and $C(y) \in \mathcal{E}$ for any $y \in G$. [8 marks]

(b) Suppose that $f : (E \times G, \mathcal{E} \otimes \mathcal{G}) \to (\mathbb{R}, \mathcal{B})$ is measurable, then the sections $y \mapsto f(x, y)$ is $\mathcal{G}$-measurable for any $x \in E$ and $x \mapsto f(x, y)$ is $\mathcal{E}$-measurable for any $y \in G$. [6 marks]

(c) Show that $g : (\Omega, \mathcal{F}) \to (E \times G, \mathcal{E} \otimes \mathcal{G})$ is measurable if and only if $g_1 : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ and $g_2 : (\Omega, \mathcal{F}) \to (G, \mathcal{G})$ are measurable. [6 marks]
Hint: Use monotone class arguments.

[Total for Question 3: 20 marks]

4. Let X be a $\mathbb{R}$-valued r.v. on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with law $\mathbb{P}^X$. For $t \in \mathbb{R}$, define the characteristic function of X by

$$\phi_X(t) := \mathbb{E}e^{\mathbf{i}tX} = \int_{\mathbb{R}} e^{\mathbf{i}tx}\mathbb{P}^X(dx) = \mathbb{E}[\cos(tX) + \mathbf{i}\sin(tX)].$$

(a) Prove that $|\phi_X(u) - \phi_X(u+h)|^2 \leq 2(1 - \mathrm{Re}(\phi_X(h)))$, $u, h \in \mathbb{R}$. [6 marks]

(b) Assume that X is integrable. Show that $|\phi_X(u)| = 1$ for all $u \in \mathbb{R}$ if and only if X is a constant a.s. [6 marks]

(c) Let X_n be $N(\mu_n, \sigma_n^2)$ r.v.s, $n \geq 1$. Suppose that X_n converges in distribution to some r.v. X as $n \to \infty$. Show that μ_n and σ_n^2 have limits $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$ as $n \to \infty$, respectively, and that X is $N(\mu, \sigma^2)$ r.v. [8 marks]

[Total for Question 4: 20 marks]

5. Let $(X_n)_{n\geq 0}$ be an $(\mathcal{F}_n)_{n\geq 1}$-supermartingale and $S \leq T$ a.s. be two stopping times bounded by $N \in \mathbb{N}_+$.

(a) Show that X_T is integrable so that $E(X_T|\mathcal{F}_S)$ exists uniquely. [4 marks]

(b) Show that $EX_T \geq EX_N$ and $EX_S \geq EX_N$. [4 marks]

(c) Suppose that $T - S \leq 1$. Show that for any $A \in \mathcal{F}_S$, $\int_A E(X_T|\mathcal{F}_S)dP = \int_A X_T dP \leq \int_A X_S dP$. [6 marks]

Hint: Split $A \in \mathcal{F}_S$ into $A = \cup_{j=0}^{N} A_j$, where $A_j := A \cap \{S = j\} \cap \{T > j\}$, and compare $\int X_T dP$ and $\int X_S dP$ in A_j, $j \in \mathbb{N}$.

(d) Prove that $E(X_T|\mathcal{F}_S) \leq X_S$ a.s. (supermartingale case of Doob optional sampling theorem) so that $EX_T \leq EX_S$. [6 marks]

[Total for Question 5: 20 marks]

1. Solution

(a) $Q(\Omega) = E(X I_\Omega) = E(X) = 1$.

For disjoint A_i, $Q(\bigcup_{n=1}^{\infty} A_n) = E(X I_{\bigcup_{n=1}^{\infty} A_n}) = E[\sum_{n=1}^{\infty} X I_{A_n}] \overset{MCT}{=} \sum_{n=1}^{\infty} E(X I_{A_n}) = \sum_{n=1}^{\infty} Q(A_n)$

$P(A) = 0$,

(i) $X = I_B$, $Q(A) = E[I_B I_A] = P(A \cap B) \le P(A) = 0$ $\quad \therefore Q(A) = 0$

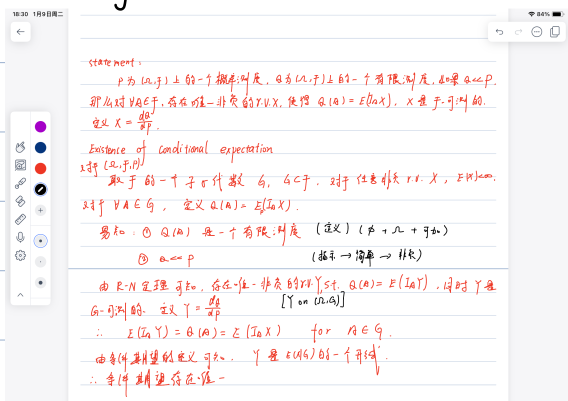
(ii) $X = \sum_{j=1}^{n} b_j I_{B_j}$. $Q(A) = \sum_{j=1}^{n} b_j E[I_{B_j} I_A] = \sum_{j=1}^{n} b_j P(A \cap B_j) \le \sum_{j=1}^{n} b_j P(A) = 0$ $\quad \therefore Q(A) = 0$

(iii) X is non-negative function. $\exists X_n \uparrow X$, X_n is non-negative simple ...

$\therefore Q(A) = E[X I_A] = E[\lim_{n\to\infty} X_n I_A] \overset{MCT}{=} \lim_{n\to\infty} E[X_n I_A] = \lim_{n\to\infty} 0 = 0$ $\quad \therefore Q(A) = 0$

(iv) X is general ....., $X = X^+ - X^-$, $\therefore$ ....

(b)



statement:

P 为 $(\Omega, \mathcal{F})$ 上的一个概率测度, Q 为 $(\Omega,\mathcal{F})$ 上的一个有限测度, 如果 $Q \ll P$.
那么对 $\forall A \in \mathcal{F}$, 存在唯一非负的 r.v. X, 使得 $Q(A) = E(I_A X)$, X 是 $\mathcal{F}$-可测的.
定义 $X = \frac{dQ}{dP}$.

Existence of conditional expectation
对于 $(\Omega, \mathcal{F}, P)$
取 $\mathcal{F}$ 的一个子 σ 代数 $\mathcal{G}$, $\mathcal{G} \subset \mathcal{F}$. 对于任意非负 r.v. X, $E(X) < \infty$.
对于 $\forall A \in \mathcal{G}$, 定义 $Q(A) = E_P(I_A X)$.
易知: ① $Q(A)$ 是一个有限测度 (定义) (ϕ + Ω + 可加)
② $Q \ll P$ (指示 → 简单 → 非负)

由 R-N 定理可知, 存在唯一非负的 r.v. Y, s.t. $Q(A) = E(I_A Y)$, 同时 Y 是 $\mathcal{G}$-可测的. 定义 $Y = \frac{dQ}{dP}$ [Y on $(\Omega, \mathcal{G})$]
$\therefore$ $E(I_A Y) = Q(A) = E(I_A X)$ for $A \in \mathcal{G}$.
由条件期望的定义可知, Y 是 $E(X|\mathcal{G})$ 的一个形式.
$\therefore$ 条件期望存在唯一

(c) (i) $Y = I_B$, $E_Q(Y) = E_Q(I_B) = Q(B) = E[X I_B] = E[XY]$

(ii) $Y = \sum_{j=1}^{n} b_j I_{B_j}$ .....

(iii) $Y \ge 0$, $Y_n \uparrow Y$. $E_Q(Y) = E_Q(\lim_{n\to\infty} Y_n) = \lim_{n\to\infty} E_Q(Y_n) = \lim_{n\to\infty} E[X Y_n] = E(XY)$

(iv) Y, $Y = Y^+ - Y^-$, ...

(d) By Jensen i...., $g(x) = |x|^p$, $\therefore g(E(x|\mathcal{G})) \le E[g(x)|\mathcal{G}]$, $(p>1)$

$\therefore E|E(x)|\mathcal{G}|^p = E[g(E(x)|\mathcal{G})] \le E[E(g(x)|\mathcal{G})] = E|x|^p$

$E[E(X|\mathcal{G}) Z] = E[E(ZX|\mathcal{G})] = E[ZX]$, and $Z \in L^{\frac{p}{p-1}}(\Omega, \mathcal{G}, P)$ $\quad \therefore E[|xz|] < \infty$

(e) $\because Q \ll P$, $\therefore \forall \varepsilon > 0$, $\exists \delta(\varepsilon)$, $P(A_n) < \delta(\varepsilon)$, we have $Q(A_n) < \varepsilon$. And $\lim_{n\to\infty} P(A_n) = 0$

$\therefore \exists N(\varepsilon)$, $n > N(\varepsilon)$, $P(A_n) < \delta(\varepsilon)$. Thus, for $\forall \varepsilon > 0$, $\exists N(\varepsilon)$, when $n > N(\varepsilon)$, we have

$Q(A_n) < \varepsilon$, i.e. $\lim_{n\to\infty} \int_{A_n} X dP = \lim_{n\to\infty} Q(A_n) = 0$

2. Solution.

(a) (i) For each n, $E|S_n - n\mu| \le E|S_n| + n\mu \le nE|X_1| + n\mu < \infty$

(ii) $S_n - n\mu$ is $\mathcal{F}_n$-measurable. For each n.

(iii) $E[S_n - n\mu \mid \mathcal{F}_{n-1}] = E[S_{n-1} - (n-1)\mu + X_n - \mu \mid \mathcal{F}_{n-1}] = S_{n-1} - (n-1)\mu + E(X_n \mid \mathcal{F}_{n-1}) - \mu$

$= S_n - (n-1)\mu + 0 = \cdots$

$\therefore$ ... - . -

(b) $\mu = 0$, $\{S_n\}_{n\ge1}$ is an $\{\mathcal{F}_n\}_{n\ge1}$-martingale.

$\therefore E[\phi(S_n)] = E[E(\phi(S_n)|\mathcal{F}_{n-1})] \ge E[\phi(E(S_n|\mathcal{F}_{n-1}))] = E[\phi(S_{n-1})]$

3. Solution.

(a) $C \in \mathcal{E} \times \mathcal{G}$, $C(x) = \{y \in G : (x,y) \in C\} = \{y \in G : x \in \mathcal{E}, y \in G\} = \{y \in G\} \subseteq \mathcal{G}$.

Let $H = \{C \in \mathcal{E} \otimes \mathcal{G} : C(x) \in \mathcal{G}\}$, $\mathcal{E} \times \mathcal{G} \subseteq H \subseteq \mathcal{E} \otimes \mathcal{G}$

(i) $C = \phi$, then $\phi \in \mathcal{E} \otimes \mathcal{G}$, and $C(x) = \phi \in \mathcal{G}$ $\therefore \phi \in H$

(ii) $C \in H$, $\therefore C \in \mathcal{E} \otimes \mathcal{G}$, $C(x) \in \mathcal{G}$ $\therefore C^c \in \mathcal{E} \otimes \mathcal{G}$ and $C^c(x) = [C(x)]^c \in \mathcal{G}$

$\therefore C^c \in H$

(iii) $C_1, \dots, C_n, \dots \in H$, $C_n \in \mathcal{E} \otimes \mathcal{G}$, $C_n(x) \in \mathcal{G}$ $\therefore \bigcup_{n=1}^{\infty} C_n \in \mathcal{E} \otimes \mathcal{G}$ and $(\bigcup_{n=1}^{\infty} C_n)(x) = \bigcup_{n=1}^{\infty} [C_n(x)]$

$\in H$ $\therefore \bigcup_{n=1}^{\infty} C_n \in H$ $\therefore H$ is a σ-algebra

$\sigma(\mathcal{E} \times \mathcal{G}) = H$ $\therefore C \in \mathcal{E} \otimes \mathcal{G}$, we have $C(x) \in \mathcal{G}$

Similarly, $C(y) \in \mathcal{E}$

(b) Let $g_x(y) = f(x,y)$, $\forall A \in \mathcal{B}$, $g_x^{-1}(A) = \{y \in G : g_x(y) = f(x,y) \in A\} = [g^{-1}(A)](x)$

f is $\mathcal{E} \otimes \mathcal{G}$-measurable. $\therefore$ By (a), $g_x(y)$ is $\mathcal{G}$-measurable.

Similarly. $f_y(x)$ ....

(c) "$\Rightarrow$" For any $A \in \mathcal{E}$, $g_1^{-1}(A) = g^{-1}(A \times G) \in \mathcal{F}$, ... $g_2^{-1}(B) \in \mathcal{F}$, $B \in \mathcal{G}$.

"$\Leftarrow$" For $A \times B \in \mathcal{E} \times \mathcal{G}$, $g^{-1}(A \times B) = g_1^{-1}(A) \cap g_2^{-1}(B) \in \mathcal{F}$

$\mathcal{H} = \{ C \in \mathcal{E} \otimes \mathcal{G} : g^{-1}(C) \in \mathcal{F} \}$, It is easy to know $\mathcal{H}$ is a σ-algebra.

$\mathcal{H} = \sigma(\mathcal{E} \times \mathcal{G})$ ......

4. Solution

(a) $|\phi_X(u) - \phi_X(u+h)|^2 = \left|\int_R e^{iux}[1-e^{ihx}]\, p(dx)\right|^2 \le \int_R |e^{iux}|^2 p(dx) \int_R |1-e^{ihx}|^2 p(dx)$

$= \int_R (1-\cos hx)^2 + (\sin hx)^2\, p(dx) = 2\int_R (1-\cos hx)\, p(dx) = 2[1 - Re(\phi_X(h))]$

(b) $|\phi_X(u)|^2 = \left|\int_R e^{iux} p(dx)\right|^2 \le \int_R |e^{iux}|^2 p(dx) = 1$ ("=" holds $\Leftrightarrow$ $e^{iux} = c$)

$\therefore E[e^{iuX}] = e^{iuX}$, $\therefore E(X) = X$ a.s

$|\phi_X(u)| = |E[\cos uX] + iE[\sin uX]| = \sqrt{(E\cos uX)^2 + (E\sin uX)^2} \le \sqrt{E(\cos^2 uX + \sin^2 uX)} = 1$

$\therefore \sin uX = C_1$, $\cos uX = C_2$ $\therefore e^{iuX} = c$ $\therefore$ ......

(c) $\varphi_{X_n}(t) = \exp\{i\mu_n t - \frac{1}{2}\sigma_n^2 t^2\} \longrightarrow \varphi_X(t)$

$\therefore \mu_n \longrightarrow Im(\ln \varphi_X(1)) = \mu$ $\quad \sigma_n^2 \longrightarrow Re(-2\ln\varphi_X(1)) = \sigma^2$

$\therefore \varphi_{X_n}(t) = \exp\{i\mu_n t - \frac{1}{2}\sigma_n^2 t^2\} \longrightarrow \exp\{i\mu t - \frac{1}{2}\sigma^2 t^2\} = \varphi_X(t)$, $X \sim N(\mu, \sigma^2)$

5. Solution

(a) $E[|X_T|] = E\left[\sum_{n=0}^{N} |X_n| I_{\{T=n\}}\right] \le \sum_{n=0}^{N} E[|X_n|] < \infty$

By the definition of conditional expectation , $E(X_T|\mathcal{F}_S)$ exists uniquely.

(b) $E[X_T] = E\left[\sum_{n=0}^{N} X_n I_{\{T=n\}}\right] = \sum_{n=0}^{N} E(X_n I_{\{T=n\}}) \ge \sum_{n=0}^{N} E[E(X_N|\mathcal{F}_n) I_{\{T=n\}}]$

$= \sum_{n=0}^{N} E[E(I_{\{T=n\}} X_N | \mathcal{F}_n)] = E\left(\sum_{n=0}^{N} I_{\{T=n\}} X_N\right) = E(X_N)$

Similarly , $E[X_S] \ge E(X_N)$

(c) Let $A_j = A \cap \{S=j\} \cap \{T>j\}$, then A_j are disjoint and $\bigcup_{j=0}^{N} A_j = A$.

$\because T - S \le 1$, if $T = S$, it holds. .

if $T = S+1$, in A_j, $X_j \ge E(X_{j+1}|\mathcal{F}_j)$

$\therefore \int_{A_j} E(X_T|\mathcal{F}_S)\,dp = \int_{A_j} E(X_{j+1}|\mathcal{F}_j)\,dp \le \int_{A_j} X_j\,dp = \int_{A_j} X_S\,dp$

$\therefore \int_A E(X_T|\mathcal{F}_S)\,dp = \int_{\bigcup_{j=0}^{N} A_j} E(X_T|\mathcal{F}_S)\,dp = \sum_{j=1}^{N} \int_{A_j} E(X_T|\mathcal{F}_S)\,dp \le \sum_{j=1}^{N} \int_{A_j} X_S\,dp = \int_A X_S\,dp$

$\therefore$ For any $A \in \mathcal{F}_s$, $\int_A E(X_T|\mathcal{F}_s)dp = \int_A X_T\, dp \leq \int_A X_s\, dp$ $\quad (T - S \leq 1)$

(d) By (c), we know, for any $A \in \mathcal{F}_s$ $\quad (s \leq T)$ $[\exists m \leq N,\ T \leq S+m]$

$$\int_A X_{s+1} dp = \int_A E(X_{s+1}|\mathcal{F}_s)dp \leq \int X_s\, dp \quad , \quad (T - (S+m-1) \leq 1)$$

$\therefore \int_A E(X_T|\mathcal{F}_s)\,dp \leq \int_A X_{s+m-1} dp \leq \int_A X_{s+m-2} dp \leq \cdots \leq \int_A X_{s+1} dp \leq \int_A X_s dp$

$\therefore$ $E(X_T|\mathcal{F}_s) \leq X_s$, a.s.