DEFINITIONS The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence

DEFINITION The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty.$$

Similarly, if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty.$$

DEFINITION A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \le M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \ge m$ for all n. The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

Definition A sequence $\{s_n\}$ of real numbers is said to be

(*a*) *monotonically increasing* if $s_n \le s_{n+1}$ $(n = 1, 2, 3, \ldots)$;
(*b*) *monotonically decreasing* if $s_n \ge s_{n+1}$ $(n = 1, 2, 3, \ldots)$.

Theorem *Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.*

Proof Suppose $s_n \leq s_{n+1}$ (the proof is analogous in the other case). Let E be the range of $\{s_n\}$. If $\{s_n\}$ is bounded, let s be the least upper bound of E. Then

$$s_n \leq s \qquad (n = 1, 2, 3, \ldots).$$

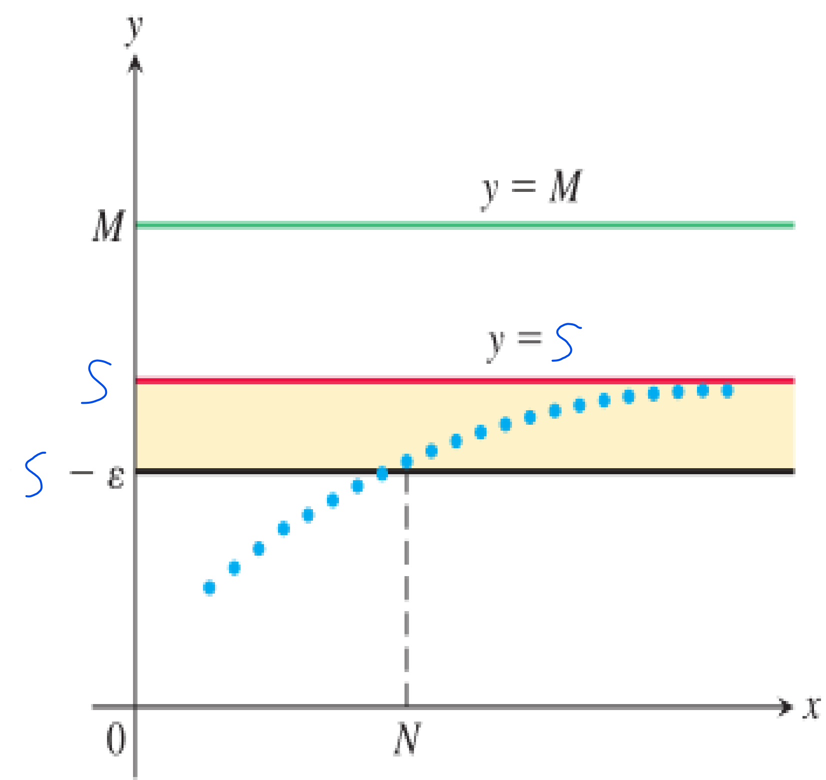
For every $\varepsilon > 0$, there is an integer N such that

$$s - \varepsilon < s_N \leq s,$$

for otherwise $s - \varepsilon$ would be an upper bound of E. Since $\{s_n\}$ increases, $n \geq N$ therefore implies

$$s - \varepsilon < s_n \leq s,$$

which shows that $\{s_n\}$ converges (to s).



Theorem *The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.*

Proof Let E^* be the set of all subsequential limits of $\{p_n\}$ and let q be a limit point of E^*. We have to show that $q \in E^*$.

Choose n_1 so that $p_{n_1} \neq q$. (If no such n_1 exists, then E^* has only one point, and there is nothing to prove.) Put $\delta = d(q, p_{n_1})$. Suppose $n_1, \ldots, n_{i-1}$ are chosen. Since q is a limit point of E^*, there is an $x \in E^*$ with $d(x, q) < 2^{-i}\delta$. Since $x \in E^*$, there is an $n_i > n_{i-1}$ such that $d(x, p_{n_i}) < 2^{-i}\delta$. Thus

$$d(q, p_{n_i}) \leq 2^{1-i}\delta$$

for $i = 1, 2, 3, \ldots$. This says that $\{p_{n_i}\}$ converges to q. Hence $q \in E^*$.

Definition Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$.

Set

$$s^* = \sup E,$$
$$s_* = \inf E.$$

The numbers s^*, s_* are called the *upper* and *lower limits* of $\{s_n\}$; we use the notation

$$\limsup_{n\to\infty} s_n = s^*, \qquad \liminf_{n\to\infty} s_n = s_*.$$

Examples

(*a*) Let $\{s_n\}$ be a sequence containing all rationals. Then every real number is a subsequential limit, and

$$\limsup_{n\to\infty} s_n = +\infty, \qquad \liminf_{n\to\infty} s_n = -\infty.$$

(*b*) Let $s_n = (-1^n)/[1 + (1/n)]$. Then

$$\limsup_{n\to\infty} s_n = 1, \qquad \liminf_{n\to\infty} s_n = -1.$$

(*c*) For a real-valued sequence $\{s_n\}$, $\lim_{n\to\infty} s_n = s$ if and only if

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s.$$

Theorem *If $s_n \le t_n$ for $n \ge N$, where N is fixed, then*

$$\liminf_{n\to\infty} s_n \le \liminf_{n\to\infty} t_n,$$

$$\limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} t_n.$$

Theorem Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\} \subset \{s_n\}$. Let

$$s^* = \limsup x_n = \sup E,$$

then

(a) $s^* \in E$.

(b) *If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.*

Proof (a) If $s^* = +\infty$, then E is not bounded above; hence $\{s_n\}$ is not bounded above, and there is a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \to +\infty$.

If s^* is real, then E is bounded above, and at least one subsequential limit exists, so that $s^* \in E$.

If $s^* = -\infty$, then E contains only one element, namely $-\infty$, and there is no subsequential limit. Hence, for any real M, $s_n > M$ for at most a finite number of values of n, so that $s_n \to -\infty$.

This establishes (*a*) in all cases.

(b) Suppose there is a number $x > s^*$ such that $s_n \geq x$ for infinitely many values of n. In that case, there is a number $y \in E$ such that $y \geq x > s^*$, contradicting the definition of s^*.

Definition (Limit superior and limit inferior)**.** Suppose that $(a_n)_{n=m}^{\infty}$ is a sequence. We define a new sequence $(a_N^+)_{N=m}^{\infty}$ by the formula

$$a_N^+ := \sup(a_n)_{n=N}^{\infty}.$$

More informally, a_N^+ is the supremum of all the elements in the sequence from a_N onwards. We then define the *limit superior* of the sequence $(a_n)_{n=m}^{\infty}$, denoted $\limsup_{n\to\infty} a_n$, by the formula

$$\limsup_{n\to\infty} a_n := \inf(a_N^+)_{N=m}^{\infty}.$$

Similarly, we can define

$$a_N^- := \inf(a_n)_{n=N}^{\infty}$$

and define the *limit inferior* of the sequence $(a_n)_{n=m}^{\infty}$, denoted $\liminf_{n\to\infty} a_n$, by the formula

$$\liminf_{n\to\infty} a_n := \sup(a_N^-)_{N=m}^{\infty}.$$

Example Let $a_1, a_2, a_3, \ldots$ denote the sequence

$$1.1, -1.01, 1.001, -1.0001, 1.00001, \ldots.$$

Then $a_1^+, a_2^+, a_3^+, \ldots$ is the sequence

$$1.1, 1.001, 1.001, 1.00001, 1.00001, \ldots$$

and its infimum is 1. Hence the limit superior of this sequence is 1. Similarly, $a_1^-, a_2^-, a_3^-, \ldots$ is the sequence

$$-1.01, -1.01, -1.0001, -1.0001, -1.000001, \ldots$$

and the supremum of this sequence is -1. Hence the limit inferior of this sequence is -1. One should compare this with the supremum and infimum of the sequence, which are 1.1 and -1.01 respectively.

Example Let $a_1, a_2, a_3, \ldots$ denote the sequence

$$1, -2, 3, -4, 5, -6, 7, -8, \ldots$$

Then $a_1^+, a_2^+, \ldots$ is the sequence

$$+\infty, +\infty, +\infty, +\infty, \ldots$$

and so the limit superior is $+\infty$. Similarly, $a_1^-, a_2^-, \ldots$ is the sequence

$$-\infty, -\infty, -\infty, -\infty, \ldots$$

and so the limit inferior is $-\infty$.

Example Let $a_1, a_2, a_3, \ldots$ denote the sequence

$$1, -1/2, 1/3, -1/4, 1/5, -1/6, \ldots$$

Then $a_1^+, a_2^+, \ldots$ is the sequence

$$1, 1/3, 1/3, 1/5, 1/5, 1/7, \ldots$$

which has an infimum of 0, so the limit superior is 0. Similarly,

$a_1^-, a_2^-, \ldots$ is the sequence

$$-1/2, -1/2, -1/4, -1/4, -1/6, -1/6$$

which has a supremum of 0. So the limit inferior is also 0.

Example Let $a_1, a_2, a_3, \ldots$ denote the sequence

$$1, 2, 3, 4, 5, 6, \ldots$$

Then $a_1^+, a_2^+, \ldots$ is the sequence

$$+\infty, +\infty, +\infty, \ldots$$

so the limit superior is $+\infty$. Similarly, $a_1^-, a_2^-, \ldots$ is the sequence

$$1, 2, 3, 4, 5, \ldots$$

which has a supremum of $+\infty$. So the limit inferior is also $+\infty$.

Proposition *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let L^+ be the limit superior of this sequence, and let L^- be the limit inferior of this sequence (thus both L^+ and L^- are extended real numbers).*

(a) *For every $x > L^+$, there exists an $N \geq m$ such that $a_n < x$ for all $n \geq N$. (In other words, for every $x > L^+$, the elements of the sequence $(a_n)_{n=m}^{\infty}$ are eventually less than x.) Similarly, for every $y < L^-$ there exists an $N \geq m$ such that $a_n > y$ for all $n \geq N$.*

(b) *For every $x < L^+$, and every $N \geq m$, there exists an $n \geq N$ such that $a_n > x$. (In other words, for every $x < L^+$, the elements of the sequence $(a_n)_{n=m}^{\infty}$ exceed x infinitely often.) Similarly, for every $y > L^-$ and every $N \geq m$, there exists an $n \geq N$ such that $a_n < y$.*

(c) *We have $\inf(a_n)_{n=m}^{\infty} \leq L^- \leq L^+ \leq \sup(a_n)_{n=m}^{\infty}$.*

Theorem

(*a*) *If* $p > 0$, *then* $\lim_{n \to \infty} \frac{1}{n^p} = 0$.

(*b*) *If* $p > 0$, *then* $\lim_{n \to \infty} \sqrt[n]{p} = 1$.

(*c*) $\lim_{n \to \infty} \sqrt[n]{n} = 1$.

(*d*) *If* $p > 0$ *and* α *is real, then* $\lim_{n \to \infty} \frac{n^\alpha}{(1+p)^n} = 0$.

(*e*) *If* $|x| < 1$, *then* $\lim_{n \to \infty} x^n = 0$.

Proof

(*d*) Let k be an integer such that $k > \alpha$, $k > 0$. For $n > 2k$,

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}.$$

Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k} \qquad (n > 2k).$$

Since $\alpha - k < 0$, $n^{\alpha - k} \to 0$, by (*a*).

DEFINITIONS Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
&\vdots \\
s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\
&\vdots
\end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

From the Cauchy convergence criterion for sequences, we have the following

Theorem *Σa_n converges if and only if for every $\varepsilon > 0$ there is an integer N such that*

$$\left|\sum_{k=n}^{m} a_k\right| \leq \varepsilon$$

if $m \geq n \geq N$.

Theorem A series of nonnegative terms converges if and only if its partial sums form a bounded sequence.

Theorem

(a) *If $|a_n| \le c_n$ for $n \ge N_0$, where N_0 is some fixed integer, and if Σc_n converges, then Σa_n converges.*

(b) *If $a_n \ge d_n \ge 0$ for $n \ge N_0$, and if Σd_n diverges, then Σa_n diverges.*

Theorem *Suppose $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

Proof Let

$$s_n = a_1 + a_2 + \cdots + a_n,$$

$$t_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k}.$$

For $n < 2^k$,

$$\begin{aligned} s_n &\le a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\le a_1 + 2a_2 + \cdots + 2^k a_{2^k} \qquad = t_k, \end{aligned}$$

so that

$$s_n \le t_k.$$

On the other hand, if $n > 2^k$,

$$\begin{aligned} s_n &\ge a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\ge \tfrac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} \\ &= \tfrac{1}{2}t_k, \end{aligned}$$

so that

$$2s_n \ge t_k.$$

Thus, $\{s_n\}$ and $\{t_k\}$ are either both bounded or both unbounded. $\square$

THEOREM **The Integral Test**

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

If $p > 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges; if $p \leq 1$, the series diverges.

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Here $n! = 1 \cdot 2 \cdot 3 \cdots n$ if $n \geq 1$, and $0! = 1$.

Since

$$s_n = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdots n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3,$$

the series converges, and the definition makes sense.

Theorem *e is irrational.*

Proof. Assume that there are $p, q \in \mathbb{N}$ such that

$$e = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \frac{q}{p}$$

then

$$p! + \frac{p!}{2!} + \cdots + \frac{p!}{p!} + \frac{p!}{(p+1)!} + \cdots = q(p-1)!$$

Since

$$\forall k = 1, 2, \cdots, p, \quad k! \mid p!,$$

We have

$$\frac{p!}{(p+1)!} + \frac{p!}{(p+2)!} + \cdots \in \mathbb{Z}^+$$

This is impossible, since

$$\begin{aligned} & \frac{p!}{(p+1)!} + \frac{p!}{(p+2)!} + \cdots \\ = & \frac{1}{p+1} + \frac{1}{(p+1)(p+2)} + \cdots \\ < & \frac{1}{p+1} + \frac{1}{(p+1)^2} + \cdots \\ = & \frac{\frac{1}{p+1}}{1 - \frac{1}{p+1}} \\ = & \frac{1}{p} \leq 1. \end{aligned}$$

Theorem (Root Test) *Given* Σa_n, *put* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

Then

(*a*) *if* $\alpha < 1$, Σa_n *converges;*
(*b*) *if* $\alpha > 1$, Σa_n *diverges;*
(*c*) *if* $\alpha = 1$, *the test gives no information.*

Proof If $\alpha < 1$, we can choose β so that $\alpha < \beta < 1$, and an integer N such that

$$\sqrt[n]{|a_n|} < \beta$$

for $n \geq N$. That is, $n \geq N$ implies

$$|a_n| < \beta^n.$$

Since $0 < \beta < 1$, $\Sigma \beta^n$ converges. Convergence of Σa_n follows now from the comparison test.

If $\alpha > 1$, there is a subsequence $\{a_{n_k}\}$ such that

$$\sqrt[n_k]{|a_{n_k}|} \to \alpha.$$

Hence $|a_n| > 1$ for infinitely many values of n, so that the condition $a_n \to 0$, necessary for convergence of Σa_n, does not hold

To prove (*c*), we consider the series

$$\sum \frac{1}{n}, \sum \frac{1}{n^2}.$$

For each of these series $\alpha = 1$, but the first diverges, the second converges.

Theorem (Ratio Test) *The series* Σa_n

(*a*) *converges if* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(*b*) *diverges if* $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *for all* $n \geq n_0$, *where* n_0 *is some fixed integer.*

Proof If condition (*a*) holds, we can find $\beta < 1$, and an integer N, such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta$$

for $n \geq N$. In particular,

$$\begin{aligned} &|a_{N+1}| < \beta|a_N|, \\ &|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &|a_{N+p}| < \beta^p|a_N|. \end{aligned}$$

That is,

$$|a_n| < |a_N|\beta^{-N}\cdot\beta^n$$

for $n \geq N$, and (*a*) follows from the comparison test, since $\Sigma\beta^n$ converges.

If $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$, it is easily seen that the condition $a_n \to 0$ does not hold, and (*b*) follows.

Theorem *For any sequence $\{c_n\}$ of positive numbers,*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n}, \quad \limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof We shall prove the second inequality. Put $\alpha = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$.

We can assume $\alpha < \infty$ and take any $\beta > \alpha$.

There is an integer N such that

$$\frac{c_{n+1}}{c_n} \le \beta$$

for $n \ge N$. In particular, for any $p > 0$,

$$c_{N+k+1} \le \beta c_{N+k} \qquad (k = 0, 1, \dots, p-1).$$

Multiplying these inequalities, we obtain

$$c_{N+p} \le \beta^p c_N,$$

$$c_n \le c_N \beta^{-N} \cdot \beta^n \qquad (n \ge N).$$

Hence

$$\sqrt[n]{c_n} \le \sqrt[n]{c_N \beta^{-N}} \cdot \beta,$$

so that

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \beta,$$

Since this is true for every $\beta > \alpha$, we have

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \alpha.$$

Definition Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a *power series*. The numbers c_n are called the *coefficients* of the series; z is a complex number.

Theorem *Given the power series* $\Sigma c_n z^n$, *put*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \qquad R = \frac{1}{\alpha}.$$

(*If* $\alpha = 0$, $R = +\infty$; *if* $\alpha = +\infty$, $R = 0$.) *Then* $\Sigma c_n z^n$ *converges if* $|z| < R$, *and diverges if* $|z| > R$.

Proof Put $a_n = c_n z^n$, and apply the root test:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}.$$

Note: R is called the radius of convergence of $\Sigma c_n z^n$.

Theorem *Suppose the radius of convergence of $\Sigma c_n z^n$ is 1, and suppose $c_0 \geq c_1 \geq c_2 \geq \cdots$, $\lim_{n\to\infty} c_n = 0$. Then $\Sigma c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.*

Proof Put $a_n = z^n$, $b_n = c_n$. Observe that A_n is bounded, since

$$|A_n| = \left|\sum_{m=0}^{n} z^m\right| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \leq \frac{2}{|1 - z|},$$

if $|z| = 1$, $z \neq 1$.

The theorem follows. $\square$