Constrained optimization Full:

From the examples given, we find optimization has sth. to do with eigen-value, boundary.

Theo 1: λ is eigen-value of A, $A^T = A$,

prove: $m = \min\{x^TAx, \|x\|=1\} \le \lambda \le M = \max\{x^TAx, \|x\|=1\}$

proof: For eigen-value, $x^TAx = \lambda x^Tx = \lambda$

$m \le x^TAx \le M, \Rightarrow m \le \lambda \le M$

Th2: $A^T = A$, M, m def as Th1,

M is the greatest eigen-value, m is the least,

proof: set P s.t. $A = PDP^{-1}$, we know: $x^TAx = y^TDy$

where $x = Py$. Also $\|x\| = \|Py\| = \|y\|$ for all y,

In particular $\|y\| = 1 \Leftrightarrow \|x\| = 1$

WLTh: $D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, $a \ge b \ge c$, $y = [u_1, u_2, u_3]$

$y^TDy = ay_1^2 + by_2^2 + cy_3^2 \le a$. $M = \max\{x^TAx, \|x\|=1\} = \max\{y^TDy, \|y\|=1\}$
$\le a$

[Since $y^TDy = a \Leftrightarrow y = [1,0,0]^T$] $\Rightarrow M = a$

eg: $\begin{cases} 4x^2 + 9y^2 = 36 \\ \max, \min\ xy. \end{cases}$ sol: $x_1 = \frac{x}{3}$, $x_2 = \frac{y}{2} \Rightarrow$ $x_1^2 + x_2^2 = 1$

$q((3x_1, 2x_2) = 6x_1x_2$. It equals to $\begin{cases} \max, \min\ 6x_1x_2 \\ \text{while } x_1^Tx = 1,\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{cases}$

Note: $6x_1x_2 = x^TAx$, $A = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$, $\Rightarrow \lambda_{1,2} = \pm 3$ correspond to

$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Singular Value decomposition:

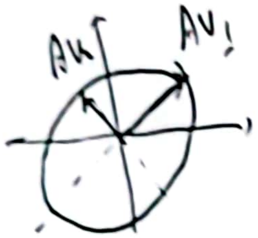
let $A \in M_{m \times n}(\mathbb{R})$, $\{v_1, \dots, v_n\}$ is orthonormal basis of $\mathbb{R}^n$ and eigen-vector of A^TA.

$v_k \leftrightarrow \lambda_k$ eigen-value. For $1 \le i \le n$

$\|Av_i\|^2 = (Av_i)^T Av_i = v_i A^T A v_i = \lambda_i$

eigen-value of A^TA are all non-negative

def: singular value of A: $\sigma_i = \sqrt{\lambda_i}$, $i \in [1, n]$.



The singular value decomposition (SVD):

$A \in M_{m \times n}(K)$, rank $A = r$. $\exists\ \Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, $\Sigma \in M_{m \times n}(K)$ has $\sigma_1 \ge \dots \ge \sigma_r > 0$ in D $\left(\sigma_1 \dots \sigma_r\right)$. $\exists\ U \in O_m(K)$, $V \in O_n(K)$ s.t. $A = U \Sigma V^T$.

proof: $\lambda_k \leftrightarrow v_k$ are eigen for A^TA

$\Rightarrow \{Av_1, \dots, Av_r\}$ are orthogonal basis for Col A.

Set $u_i = \dfrac{Av_i}{\|Av_i\|} = \dfrac{Av_i}{\sigma_i} \Rightarrow Av_i = \sigma_i u_i \ (i \in [1, r])$

Extend $\{u_1, \dots, u_r\} \to \underbrace{\{u_1, \dots, u_m\}}_{\text{basis}}$ in $\mathbb{R}^m$

$U = [u_1, \dots, u_m]$, $V = [v_1, \dots, v_n]$

$AV = [Av_1, \dots, Av_r, 0, \dots, 0] = [\sigma_1 u_1, \dots, \sigma_r u_r, 0, \dots, 0]$

$r < m$

$U\Sigma = [u_1, \dots, u_m] \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{pmatrix} = [\sigma_1 u_1, \dots, \sigma_r u_r, 0, \dots, 0]$

$= AV \Rightarrow A = U \Sigma V^{-1} = U \Sigma V^T$

Find SVD: 1° Find A^TA and $\lambda_i, v_i \Rightarrow \sigma_i$, D

2° $A \Rightarrow \dfrac{Av_i}{\sigma_i} = u_i$

* Count the number of non-zero singular value is a reliable way to count rank of A