FFT:

1st Question: $f(x)=\sum_{i=0}^{n} a_i x^i$, $g(x)=\sum_{i=0}^{n} b_i x^i$ called coefficient representation

Def: $fg(x)=(\sum_{i=0}^{n} a_i x^i)(\sum_{i=0}^{n} b_i x^i)$, $O(n\times n)$

want cut complexity to $O(n\log n)$

2nd Preliminaries:

1. ① $\sum_{i=0}^{n} a_i x^i \to (a_0, \dots, a_n)$ called polynomial representation

② $\{(x_0, f(x_0)), \dots, (x_n, f(x_n))\}$ called point representation,

$O(n)$, since ~~solving by linear equation only need back substitution~~ representation only need n "$*$"

For f, g, if $T(f)=\{(x_0, f(x_0)), \dots, (x_n, f(x_n))\}$

$T(g)=\{(x_0, g(x_0)), \dots, (x_n, g(x_n))\}$,

$T[fg]=\{(x_0, f(x_0)\cdot g(x_0)), \dots, (x_n, f(x_n)\cdot g(x_n))\}$,

2. Unit root: $w_n^{(k)}=\cos\frac{2\pi k}{n}+i\sin\frac{2k\pi}{n}$ (sol of $z^n=1$)

obs: $w_n^0=w_n^n=1$,

Prop: ① $w_{rn}^{rk}=w_n^k$ ② $w_n^{k+\frac{n}{2}}=-w_n^k$

③ $\overline{w_n^k}=w_n^{n-k}$

3rd FFT:

1. $f(x)=(a_0+a_2x^2+\dots+a_{n-2}x^{n-2})+(a_1x+a_3x^3+\dots+a_{n-1}x^{n-1})$

$\triangleq f_1(x^2)+xf_2(x^2)$

Set $x=w_n^k$ $(k<\frac{n}{2})$, $f(w_n^k)=f_1(w_n^{2k})+w_n^k f_2(w_n^{2k})$

$=f_1(w_{\frac{n}{2}}^k)+w_n^k f_2(w_{\frac{n}{2}}^k)$ . . . . ①

Set $x=w_n^{k+\frac{n}{2}}$ $(k<\frac{n}{2})$, $f(w_n^{k+\frac{n}{2}})=f_1(w_n^{2k}\cdot w_n^n)-w_n^k f_2(w_n^{2k} \dots$

$=f_1(w_{\frac{n}{2}}^k)-w_n^k f_2(w_{\frac{n}{2}}^k)$ . . . . ②

By ①,②, solve $f_1(w_{\frac{n}{2}}^k)$, $f_2(w_{\frac{n}{2}}^k)$ can solve ①,②

By interactive $\Rightarrow$ $f_1(w_1^k)=f_2(w_1^k)=1$

Sol $f(w_n^k)$, $f(w_n^{k+\frac{n}{2}})$: $O(\log n)$, sol all n: $O(n)$ $\Rightarrow$ $O(n\log n)$

2. point representation → coefficient representation:

need IDFT:

represent DFT:

$y_i = f(W_n^i)$

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & (W_n^1)^1 & (W_n^1)^2 & & (W_n^1)^{n-1} \\ \cdots & \vdots & \vdots & & \\ 1 & \vdots & \vdots & & \\ 1 & (W_n^{n-1})^1 & (W_n^{n-1})^1 & & (W_n^{n-1})^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

$$x^n - 1 = (x-1)(x^{n-1} + \cdots + 1)$$

$$\begin{cases} x=1, & x^{n-1} + \cdots + 1 = n \\ & \end{cases}$$

$$(1, (W_n^{n-i})^1, \ldots, (W_n^{n-i})^{n-1}) \begin{pmatrix} 1 \\ (W_n^j)^1 \\ \vdots \\ (W_n^j)^{n-1} \end{pmatrix} = \sum_{k=0}^{n-1} (W_n^{n-i})^k (W_n^j)^k$$