Definition and existence of integral

6.1 Definition Let $[a, b]$ be a given interval. By a *partition P* of $[a, b]$ we mean a finite set of points $x_0, x_1, \ldots, x_n$, where

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

We write

$$\Delta x_i = x_i - x_{i-1} \qquad (i = 1, \ldots, n).$$

Now suppose f is a bounded real function defined on $[a, b]$. Corresponding to each partition P of $[a, b]$ we put

$$M_i = \sup f(x) \qquad (x_{i-1} \le x \le x_i),$$

$$m_i = \inf f(x) \qquad (x_{i-1} \le x \le x_i),$$

$$U(P, f) = \sum_{i=1}^{n} M_i \, \Delta x_i,$$

$$L(P, f) = \sum_{i=1}^{n} m_i \, \Delta x_i,$$

$$\overline{\int_a^b} f\,dx = \inf U(P, f), \tag{1}$$

$$\underline{\int_a^b} f\,dx = \sup L(P, f), \tag{2}$$

where the inf and the sup are taken over all partitions P of $[a, b]$. The left members of (1) and (2) are called the *upper* and *lower Riemann integrals* of f over $[a, b]$, respectively.

If the upper and lower integrals are equal, we say that f is *Riemann-integrable* on $[a, b]$, we write $f \in \mathscr{R}$ (that is, $\mathscr{R}$ denotes the set of Riemann-integrable functions), and we denote the common value of (1) and (2) by

$$\int_a^b f\,dx, \tag{3}$$

or by

$$\int_a^b f(x)\,dx. \tag{4}$$

This is the *Riemann integral* of f over $[a, b]$. Since f is bounded, there exist two numbers, m and M, such that

$$m \le f(x) \le M \qquad (a \le x \le b).$$

Hence, for every P,

$$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a),$$

so that the numbers $L(P, f)$ and $U(P, f)$ form a bounded set. This shows that *the upper and lower integrals are defined* for *every* bounded function f.

Problem When is f integrable ?

This problem can be investigated in a more general situation.

6.2 Definition Let α be a monotonically increasing function on $[a, b]$ (since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded on $[a, b]$). Corresponding to each partition P of $[a, b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$

It is clear that $\Delta\alpha_i \geq 0$. For any real function f which is bounded on $[a, b]$ we put

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \, \Delta\alpha_i,$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \, \Delta\alpha_i,$$

where M_i, m_i have the same meaning as in Definition 6.1, and we define

$$\overline{\int_a^b} f \, d\alpha = \inf U(P, f, \alpha), \tag{5}$$

$$\underline{\int_a^b} f \, d\alpha = \sup L(P, f, \alpha), \tag{6}$$

the inf and sup again being taken over all partitions.

If the left members of (5) and (6) are equal, we denote their common value by

$$\int_a^b f\,d\alpha \tag{7}$$

or sometimes by

$$\int_a^b f(x)\,d\alpha(x). \tag{8}$$

This is the *Riemann-Stieltjes integral* (or simply the *Stieltjes integral*) of f with respect to α, over $[a, b]$.

If (7) exists, i.e., if (5) and (6) are equal, we say that f is integrable with respect to α, in the Riemann sense, and write $f \in \mathscr{R}(\alpha)$.

By taking $\alpha(x) = x$, the Riemann integral is seen to be a special case of the Riemann-Stieltjes integral.

6.3 Definition We say that the partition P^* is a *refinement* of P if $P^* \supset P$ (that is, if every point of P is a point of P^*). Given two partitions, P_1 and P_2, we say that P^* is their *common refinement* if $P^* = P_1 \cup P_2$.

6.4 Theorem *If P^* is a refinement of P, then*

$$L(P, f, \alpha) \le L(P^*, f, \alpha) \tag{9}$$

and

$$U(P^*, f, \alpha) \le U(P, f, \alpha). \tag{10}$$

Proof To prove (9), suppose first that P^* contains just one point more than P. Let this extra point be x^*, and suppose $x_{i-1} < x^* < x_i$, where x_{i-1} and x_i are two consecutive points of P. Put

$$w_1 = \inf f(x) \qquad (x_{i-1} \le x \le x^*),$$
$$w_2 = \inf f(x) \qquad (x^* \le x \le x_i).$$

Clearly $w_1 \ge m_i$ and $w_2 \ge m_i$, where, as before,

$$m_i = \inf f(x) \qquad (x_{i-1} \le x \le x_i).$$

Hence

$$\begin{aligned} &L(P^*, f, \alpha) - L(P, f, \alpha) \\ &\qquad = w_1[\alpha(x^*) - \alpha(x_{i-1})] + w_2[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x_{i-1})] \\ &\qquad = (w_1 - m_i)[\alpha(x^*) - \alpha(x_{i-1})] + (w_2 - m_i)[\alpha(x_i) - \alpha(x^*)] \ge 0. \end{aligned}$$

If P^* contains k points more than P, we repeat this reasoning k times, and arrive at (9). The proof of (10) is analogous.

6.5 Theorem $\underline{\int}_a^b f\,d\alpha \le \overline{\int}_a^b f\,d\alpha.$

Proof Let P^* be the common refinement of two partitions P_1 and P_2. By Theorem 6.4,

$$L(P_1, f, \alpha) \le L(P^*, f, \alpha) \le U(P^*, f, \alpha) \le U(P_2, f, \alpha).$$

Hence

$$L(P_1, f, \alpha) \le U(P_2, f, \alpha). \tag{11}$$

If P_2 is fixed and the sup is taken over all P_1, (11) gives

$$\underline{\int} f\,d\alpha \le U(P_2, f, \alpha). \tag{12}$$

The theorem follows by taking the inf over all P_2 in (12).

6.6 Theorem *$f \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that*

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon. \tag{13}$$

Proof For every P we have

$$L(P, f, \alpha) \le \underline{\int} f\, d\alpha \le \overline{\int} f\, d\alpha \le U(P, f, \alpha).$$

Thus (13) implies

$$0 \le \overline{\int} f\, d\alpha - \underline{\int} f\, d\alpha < \varepsilon.$$

Hence, if (13) can be satisfied for every $\varepsilon > 0$, we have

$$\overline{\int} f\, d\alpha = \underline{\int} f\, d\alpha,$$

that is, $f \in \mathscr{R}(\alpha)$.

Conversely, suppose $f \in \mathscr{R}(\alpha)$, and let $\varepsilon > 0$ be given. Then there exist partitions P_1 and P_2 such that

$$U(P_2, f, \alpha) - \int f\, d\alpha < \frac{\varepsilon}{2}, \tag{14}$$

$$\int f\, d\alpha - L(P_1, f, \alpha) < \frac{\varepsilon}{2}. \tag{15}$$

We choose P to be the common refinement of P_1 and P_2. Then Theorem 6.4, together with (14) and (15), shows that

$$U(P, f, \alpha) \le U(P_2, f, \alpha) < \int f\, d\alpha + \frac{\varepsilon}{2} < L(P_1, f, \alpha) + \varepsilon \le L(P, f, \alpha) + \varepsilon,$$

so that (13) holds for this partition P.

6.7 Theorem

(*a*) *If* (13) *holds for some* P *and some* ε, *then* (13) *holds* (*with the same* ε) *for every refinement of* P.

(*b*) *If* (13) *holds for* $P = \{x_0, \ldots, x_n\}$ *and if* s_i, t_i *are arbitrary points in* $[x_{i-1}, x_i]$, *then*

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \, \Delta\alpha_i < \varepsilon.$$

(*c*) *If* $f \in \mathscr{R}(\alpha)$ *and the hypotheses of* (*b*) *hold, then*

$$\left| \sum_{i=1}^{n} f(t_i) \, \Delta\alpha_i - \int_a^b f \, d\alpha \right| < \varepsilon.$$

Proof Theorem 6.4 implies (*a*). Under the assumptions made in (*b*), both $f(s_i)$ and $f(t_i)$ lie in $[m_i, M_i]$, so that $|f(s_i) - f(t_i)| \le M_i - m_i$. Thus

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \, \Delta\alpha_i \le U(P, f, \alpha) - L(P, f, \alpha),$$

which proves (*b*). The obvious inequalities

$$L(P, f, \alpha) \le \sum f(t_i) \, \Delta\alpha_i \le U(P, f, \alpha)$$

and

$$L(P, f, \alpha) \le \int f \, d\alpha \le U(P, f, \alpha)$$

prove (*c*).

6.8 Theorem *If f is continuous on $[a, b]$ then $f \in \mathscr{R}(\alpha)$ on $[a, b]$.*

Proof Let $\varepsilon > 0$ be given. Choose $\eta > 0$ so that

$$[\alpha(b) - \alpha(a)]\eta < \varepsilon.$$

Since f is uniformly continuous on $[a, b]$, there exists a $\delta > 0$ such that

$$|f(x) - f(t)| < \eta \tag{16}$$

if $x \in [a, b]$, $t \in [a, b]$, and $|x - t| < \delta$.

If P is any partition of $[a, b]$ such that $\Delta x_i < \delta$ for all i, then (16) implies that

$$M_i - m_i \leq \eta \qquad (i = 1, \ldots, n) \tag{17}$$

and therefore

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\, \Delta\alpha_i$$

$$\leq \eta \sum_{i=1}^{n} \Delta\alpha_i = \eta[\alpha(b) - \alpha(a)] < \varepsilon.$$

By Theorem 6.6, $f \in \mathscr{R}(\alpha)$.

6.9 Theorem *If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then $f \in \mathscr{R}(\alpha)$.*

Proof Let $\varepsilon > 0$ be given. For any positive integer n, choose a partition P such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \qquad (i = 1, \ldots, n).$$

This is possible since α is continuous.

We suppose that f is monotonically increasing (the proof is analogous in the other case). Then

$$M_i = f(x_i), \qquad m_i = f(x_{i-1}) \qquad (i = 1, \ldots, n),$$

so that

$$U(P, f, \alpha) - L(P, f, \alpha) = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \cdot [f(b) - f(a)] < \varepsilon$$

if n is taken large enough. By Theorem 6.6, $f \in \mathscr{R}(\alpha)$.

6.10 Theorem *Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.*

6.11 Theorem *Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$.*

6.11 Theorem *Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$.*

Proof Choose $\varepsilon > 0$. Since ϕ is uniformly continuous on $[m, M]$, there exists $\delta > 0$ such that $\delta < \varepsilon$ and $|\phi(s) - \phi(t)| < \varepsilon$ if $|s - t| \le \delta$ and $s, t \in [m, M]$.

Since $f \in \mathscr{R}(\alpha)$, there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2. \tag{18}$$

Let M_i, m_i have the same meaning as in Definition 6.1, and let M_i^*, m_i^* be the analogous numbers for h. Divide the numbers $1, \ldots, n$ into two classes: $i \in A$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \ge \delta$.

For $i \in A$, our choice of δ shows that $M_i^* - m_i^* \le \varepsilon$.

For $i \in B$, $M_i^* - m_i^* \le 2K$, where $K = \sup|\phi(t)|$, $m \le t \le M$. By (18), we have

$$\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i)\, \Delta\alpha_i < \delta^2$$

so that $\sum_{i \in B} \Delta\alpha_i < \delta$. It follows that

$$\begin{aligned} U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i \in A} (M_i^* - m_i^*)\, \Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*)\, \Delta\alpha_i \\ &\le \varepsilon[\alpha(b) - \alpha(a)] + 2K\delta < \varepsilon[\alpha(b) - \alpha(a) + 2K]. \end{aligned}$$

6.12 Theorem

(*a*) *If* $f_1 \in \mathscr{R}(\alpha)$ *and* $f_2 \in \mathscr{R}(\alpha)$ *on* $[a, b]$, *then*

$$f_1 + f_2 \in \mathscr{R}(\alpha),$$

$cf \in \mathscr{R}(\alpha)$ *for every constant* c, *and*

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha,$$

$$\int_a^b cf\, d\alpha = c \int_a^b f\, d\alpha.$$

(*b*) *If* $f_1(x) \le f_2(x)$ *on* $[a, b]$, *then*

$$\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha.$$

(*c*) *If* $f \in \mathscr{R}(\alpha)$ *on* $[a, b]$ *and if* $a < c < b$, *then* $f \in \mathscr{R}(\alpha)$ *on* $[a, c]$ *and on* $[c, b]$, *and*

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha.$$

(*d*) *If* $f \in \mathscr{R}(\alpha)$ *on* $[a, b]$ *and if* $|f(x)| \le M$ *on* $[a, b]$, *then*

$$\left| \int_a^b f\, d\alpha \right| \le M[\alpha(b) - \alpha(a)].$$

(*e*) *If* $f \in \mathscr{R}(\alpha_1)$ *and* $f \in \mathscr{R}(\alpha_2)$, *then* $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ *and*

$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2\,;$$

if $f \in \mathscr{R}(\alpha)$ *and* c *is a positive constant, then* $f \in \mathscr{R}(c\alpha)$ *and*

$$\int_a^b f\, d(c\alpha) = c \int_a^b f\, d\alpha.$$

Proof If $f = f_1 + f_2$ and P is any partition of $[a, b]$, we have

$$(20) \qquad L(P, f_1, \alpha) + L(P, f_2, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P, f_1, \alpha) + U(P, f_2, \alpha).$$

If $f_1 \in \mathscr{R}(\alpha)$ and $f_2 \in \mathscr{R}(\alpha)$, let $\varepsilon > 0$ be given. There are partitions P_j $(j = 1, 2)$ such that

$$U(P_j, f_j, \alpha) - L(P_j, f_j, \alpha) < \varepsilon.$$

These inequalities persist if P_1 and P_2 are replaced by their common refinement P. Then (20) implies

$$U(P, f, \alpha) - L(P, f, \alpha) < 2\varepsilon,$$

which proves that $f \in \mathscr{R}(\alpha)$.

With this same P we have

$$U(P, f_j, \alpha) < \int f_j \, d\alpha + \varepsilon \qquad (j = 1, 2);$$

hence (20) implies

$$\int f \, d\alpha \le U(P, f, \alpha) < \int f_1 \, d\alpha + \int f_2 \, d\alpha + 2\varepsilon.$$

Since ε was arbitrary, we conclude that

$$(21) \qquad \int f \, d\alpha \le \int f_1 \, d\alpha + \int f_2 \, d\alpha.$$

If we replace f_1 and f_2 in (21) by $-f_1$ and $-f_2$, the inequality is reversed, and the equality is proved.

The proofs of the other assertions of Theorem 6.12 are so similar.

6.13 Theorem *If $f \in \mathscr{R}(\alpha)$ and $g \in \mathscr{R}(\alpha)$ on $[a, b]$, then*

(*a*) $fg \in \mathscr{R}(\alpha)$;

(*b*) $|f| \in \mathscr{R}(\alpha)$ *and* $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$.

Proof If we take $\phi(t) = t^2$, Theorem 6.11 shows that $f^2 \in \mathscr{R}(\alpha)$ if $f \in \mathscr{R}(\alpha)$. The identity

$$4fg = (f+g)^2 - (f-g)^2$$

completes the proof of (*a*).

If we take $\phi(t) = |t|$, Theorem 6.11 shows similarly that $|f| \in \mathscr{R}(\alpha)$. Choose $c = \pm 1$, so that

$$c \int f\,d\alpha \ge 0.$$

Then

$$\left| \int f\,d\alpha \right| = c \int f\,d\alpha = \int cf\,d\alpha \le \int |f|\,d\alpha,$$

since $cf \le |f|$.

6.17 Theorem *Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$.*

Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. In that case

$$\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx. \tag{27}$$

Proof Let $\varepsilon > 0$ be given and apply Theorem 6.6 to α': There is a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that

$$U(P, \alpha') - L(P, \alpha') < \varepsilon. \tag{28}$$

The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$\Delta\alpha_i = \alpha'(t_i)\,\Delta x_i$$

for $i = 1, \ldots, n$. If $s_i \in [x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\,\Delta x_i < \varepsilon, \tag{29}$$

by (28) and Theorem 6.7(*b*). Put $M = \sup|f(x)|$. Since

$$\sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\,\Delta x_i$$

it follows from (29) that

$$\left|\sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\,\Delta x_i\right| \le M\varepsilon. \tag{30}$$

$$\sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i \le U(P, f\alpha') + M\varepsilon,$$

for all choices of $s_i \in [x_{i-1}, x_i]$, so that

$$U(P, f, \alpha) \le U(P, f\alpha') + M\varepsilon.$$

The same argument leads from (30) to

$$U(P, f\alpha') \le U(P, f, \alpha) + M\varepsilon.$$

Thus

$$|U(P, f, \alpha) - U(P, f\alpha')| \le M\varepsilon. \tag{31}$$

Now note that (28) remains true if P is replaced by any refinement. Hence (31) also remains true. We conclude that

$$\left|\overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,dx\right| \le M\varepsilon.$$

But ε is arbitrary. Hence

$$\overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f(x)\alpha'(x)\,dx,$$

for *any* bounded f. The equality of the lower integrals follows from (30) in exactly the same way. The theorem follows.

6.19 Theorem (change of variable) *Suppose φ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathscr{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by*

$$\beta(y) = \alpha(\varphi(y)), \qquad g(y) = f(\varphi(y)).$$

Then $g \in \mathscr{R}(\beta)$ and

(37) $$\int_A^B g\,d\beta = \int_a^b f\,d\alpha.$$

Proof To each partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ corresponds a partition $Q = \{y_0, \ldots, y_n\}$ of $[A, B]$, so that $x_i = \varphi(y_i)$. All partitions of $[A, B]$ are obtained in this way. Since the values taken by f on $[x_{i-1}, x_i]$ are exactly the same as those taken by g on $[y_{i-1}, y_i]$, we see that

(38) $$U(Q, g, \beta) = U(P, f, \alpha), \qquad L(Q, g, \beta) = L(P, f, \alpha).$$

Since $f \in \mathscr{R}(\alpha)$, P can be chosen so that both $U(P, f, \alpha)$ and $L(P, f, \alpha)$ are close to $\int f\,d\alpha$. Hence (38), combined with Theorem 6.6, shows that $g \in \mathscr{R}(\beta)$ and that (37) holds. This completes the proof.

Let us note the following special case:

Take $\alpha(x) = x$. Then $\beta = \varphi$. Assume $\varphi' \in \mathscr{R}$ on $[A, B]$. If Theorem 6.17 is applied to the left side of (37), we obtain

(39) $$\int_a^b f(x)\,dx = \int_A^B f(\varphi(y))\varphi'(y)\,dy.$$