Monotonic functions, infinite limits and limit at infinity, the derivative of a real function

Definition (Monotonic functions). Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *monotone increasing* iff $f(y) \geq f(x)$ whenever $x, y \in X$ and $y > x$. We say that f is *strictly monotone increasing* iff $f(y) > f(x)$ whenever $x, y \in X$ and $y > x$. Similarly, we say f is *monotone decreasing* iff $f(y) \leq f(x)$ whenever $x, y \in X$ and $y > x$, and *strictly monotone decreasing* iff $f(y) < f(x)$ whenever $x, y \in X$ and $y > x$. We say that f is *monotone* if it is monotone increasing or monotone decreasing, and *strictly monotone* if it is strictly monotone increasing or strictly monotone decreasing.

Example The function $f(x) := x^2$, when restricted to the domain $[0, \infty)$, is strictly monotone increasing, but when restricted instead to the domain $(-\infty, 0]$, is strictly monotone decreasing. Thus the function is strictly monotone on both $(-\infty, 0]$ and $[0, \infty)$, but is not strictly monotone (or monotone) on the full real line $(-\infty, \infty)$.

Continuous functions are not necessarily monotone (consider for instance the function $f(x) = x^2$ on $\mathbf{R}$), and monotone functions are not necessarily continuous; for instance, consider the function $f : [-1, 1] \to \mathbf{R}$ defined earlier by

$$f(x) := \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Let $f : E(\subset \mathbb{R}) \to \mathbb{R}$. $a \in E$ is a point of discontinuity of f

$\Leftrightarrow$ $\exists \varepsilon > 0 \, \forall \delta > 0 \, \exists x \in E$ Such that $|x - a| < \delta$ & $|f(x) - f(a)| \geq \varepsilon$

Theorem *Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point of x of (a, b). More precisely,*

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t). \tag{25}$$

Furthermore, if $a < x < y < b$, then

$$f(x+) \leq f(y-). \tag{26}$$

Proof By hypothesis, the set of numbers $f(t)$, where $a < t < x$, is bounded above by the number $f(x)$, and therefore has a least upper bound which we shall denote by A. Evidently $A \leq f(x)$. We have to show that $A = f(x-)$.

Let $\varepsilon > 0$ be given. It follows from the definition of A as a least upper bound that there exists $\delta > 0$ such that $a < x - \delta < x$ and

$$\underline{A - \varepsilon < f(x - \delta) \leq A.} \tag{27}$$

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Since f is monotonic, we have

$$f(x - \delta) \leq f(t) \leq A \qquad (x - \delta < t < x). \tag{28}$$

Combining (27) and (28), we see that

$$|f(t) - A| < \varepsilon \qquad (x - \delta < t < x).$$

Hence $f(x-) = A$.

The second half of (25) is proved in precisely the same way.

Next, if $a < x < y < b$, we see from (25) that

$$f(x+) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t). \tag{29}$$

The last equality is obtained by applying (25) to (a, y) in place of (a, b). Similarly,

$$f(y-) = \sup_{a<t<y} f(t) = \sup_{x<t<y} f(t). \tag{30}$$

Comparison of (29) and (30) gives (26).

Corollary *Monotonic functions have no discontinuities of the second kind.*

Theorem *Let f be monotonic on (a, b). Then the set of points of (a, b) at which f is discontinuous is at most countable.*

Proof Suppose, for the sake of definiteness, that f is increasing, and let E be the set of points at which f is discontinuous.

With every point x of E we associate a rational number $r(x)$ such that

$$f(x-) < r(x) < f(x+).$$

Since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$, we see that $r(x_1) \neq r(x_2)$ if $x_1 \neq x_2$.

We have thus established a 1-1 correspondence between the set E and a subset of the set of rational numbers. The latter, as we know, is countable.

Definition (Left and right limits). Let X be a subset of $\mathbf{R}$, $f : X \to \mathbf{R}$ be a function, and let x_0 be a real number. If x_0 is an adherent point of $X \cap (x_0, \infty)$, then we define the *right limit* $f(x_0+)$ of f at x_0 by the formula

$$f(x_0+) := \lim_{x \to x_0; x \in X \cap (x_0, \infty)} f(x),$$

provided of course that this limit exists. Similarly, if x_0 is an adherent point of $X \cap (-\infty, x_0)$, then we define the *left limit* $f(x_0-)$ of f at x_0 by the formula

$$f(x_0-) := \lim_{x \to x_0; x \in X \cap (-\infty, x_0)} f(x),$$

again provided that the limit exists. (Thus in many cases $f(x_0+)$ and $f(x_0-)$ will not be defined.)

For any real number x, a neighborhood of x is any segment $(x-\delta,\ x+\delta)$.

Definition For any real c, the set of real numbers x such that $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly, the set $(-\infty, c)$ is a neighborhood of $-\infty$.

Definition (Infinite adherent points). Let X be a subset of $\mathbf{R}$. We say that $+\infty$ is *adherent* to X iff for every $M \in \mathbf{R}$ there exists an $x \in X$ such that $x > M$; we say that $-\infty$ is *adherent* to X iff for every $M \in \mathbf{R}$ there exists an $x \in X$ such that $x < M$.

In other words, $+\infty$ is adherent to X iff X has no upper bound, or equivalently iff $\sup(X) = +\infty$. Similarly $-\infty$ is adherent to X iff X has no lower bound, or iff $\inf(X) = -\infty$. Thus a set is bounded if and only if $+\infty$ and $-\infty$ are not adherent points.

Definition (Limits at infinity). Let X be a subset of $\mathbf{R}$ with $+\infty$ as an adherent point, and let $f : X \to \mathbf{R}$ be a function. We say that $f(x)$ *converges to* L as $x \to +\infty$ in X, and write $\lim_{x \to +\infty; x \in X} f(x) = L$, iff for every $\varepsilon > 0$ there exists an M such that f is ε-close to L on $X \cap (M, +\infty)$ (i.e., $|f(x) - L| \leq \varepsilon$ for all $x \in X$ such that $x > M$). Similarly we say that $f(x)$ *converges to* L as $x \to -\infty$ iff for every $\varepsilon > 0$ there exists an M such that f is ε-close to L on $X \cap (-\infty, M)$.

Example Let $f : (0, \infty) \to \mathbf{R}$ be the function $f(x) := 1/x$. Then we have $\lim_{x \to +\infty; x \in (0,\infty)} 1/x = 0$.

Definition Let f be a real function defined on $E \subset R$. We say that

$$f(t) \to A \text{ as } t \to x,$$

where A and x are in the extended real number system, if for every neighborhood U of A there is a neighborhood V of x such that $V \cap E$ is not empty, and such that $f(t) \in U$ for all $t \in V \cap E$, $t \neq x$.

Definition Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \qquad (a < t < b, t \neq x),$$

and define

$$f'(x) = \lim_{t \to x} \phi(t),$$

provided this limit exists

If f' is defined at a point x, we say that f is *differentiable* at x. If f' is defined at every point of a set $E \subset [a, b]$, we say that f is differentiable on E.

Theorem *Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x.*

Proof As $t \to x$, we have

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \to f'(x) \cdot 0 = 0.$$

Theorem *Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, fg, and f/g are differentiable at x, and*

(a) $(f+g)'(x) = f'(x) + g'(x)$;

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;

(c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$.

In (c), we assume of course that $g(x) \neq 0$.

Proof

Let $h = fg$. Then

$$h(t) - h(x) = f(t)[g(t) - g(x)] + g(x)[f(t) - f(x)].$$

If we divide this by $t - x$ and note that $f(t) \to f(x)$ as $t \to x$, (b) follows. Next, let $h = f/g$. Then

$$\frac{h(t) - h(x)}{t - x} = \frac{1}{g(t)g(x)}\left[g(x)\frac{f(t) - f(x)}{t - x} - f(x)\frac{g(t) - g(x)}{t - x}\right].$$

Letting $t \to x$, we obtain (c).

Theorem *Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If*

$$h(t) = g(f(t)) \qquad (a \le t \le b),$$

then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x). \tag{3}$$

Proof Let $y = f(x)$. By the definition of the derivative, we have

$$f(t) - f(x) = (t - x)[f'(x) + u(t)], \tag{4}$$

$$g(s) - g(y) = (s - y)[g'(y) + v(s)], \tag{5}$$

where $t \in [a, b]$, $s \in I$, and $u(t) \to 0$ as $t \to x$, $v(s) \to 0$ as $s \to y$. Let $s = f(t)$. Using first (5) and then (4), we obtain

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= [f(t) - f(x)] \cdot [g'(y) + v(s)] \\ &= (t - x) \cdot [f'(x) + u(t)] \cdot [g'(y) + v(s)], \end{aligned}$$

or, if $t \ne x$,

$$\frac{h(t) - h(x)}{t - x} = [g'(y) + v(s)] \cdot [f'(x) + u(t)]. \tag{6}$$

Letting $t \to x$, we see that $s \to y$, by the continuity of f, so that the right side of (6) tends to $g'(y)f'(x)$, which gives (3).

Examples

(*a*) Let f be defined by

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \qquad (x \neq 0).$$

For $t \neq 0$,

$$\frac{f(t) - f(0)}{t - 0} = \sin \frac{1}{t}.$$

As $t \to 0$, this does not tend to any limit, so that $f'(0)$ does not exist.

(*b*) Let f be defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \qquad (x \neq 0).$$

At $x = 0$, we appeal to the definition, and obtain

$$\left| \frac{f(t) - f(0)}{t - 0} \right| = \left| t \sin \frac{1}{t} \right| \le |t| \qquad (t \neq 0);$$

letting $t \to 0$, we see that

$$f'(0) = 0.$$

Thus f is differentiable at all points x, but f' is not a continuous function, since $\cos(1/x)$ does not tend to a limit as $x \to 0$.

Definition (Local maxima and minima)**.** Let $f : X \to \mathbf{R}$ be a function, and let $x_0 \in X$. We say that f attains a *local maximum* at x_0 iff there exists a $\delta > 0$ such that the restriction $f|_{X \cap (x_0-\delta, x_0+\delta)}$ of f to $X \cap (x_0 - \delta, x_0 + \delta)$ attains a maximum at x_0. We say that f attains a *local minimum* at x_0 iff there exists a $\delta > 0$ such that the restriction $f|_{X \cap (x_0-\delta, x_0+\delta)}$ of f to $X \cap (x_0 - \delta, x_0 + \delta)$ attains a minimum at x_0.

Remark If f attains a maximum at x_0, we sometimes say that f attains a *global* maximum at x_0, in order to distinguish it from the local maxima defined here. Note that if f attains a global maximum at x_0, then it certainly also attains a local maximum at this x_0, and similarly for minima.

Example Let $f : \mathbf{R} \to \mathbf{R}$ denote the function $f(x) := x^2 - x^4$. This function does not attain a global minimum at 0, since for example $f(2) = -12 < 0 = f(0)$, however it does attain a local minimum, for if we choose $\delta := 1$ and restrict f to the interval $(-1, 1)$, then for all $x \in (-1, 1)$ we have $x^4 \leq x^2$ and thus $f(x) = x^2 - x^4 \geq 0 = f(0)$, and so $f|_{(-1,1)}$ has a local minimum at 0.

Definition Let f be a real function defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

Theorem *Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$.*

Proof Let $\delta > 0$ so that

$$a < x - \delta < x < x + \delta < b.$$

If $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

Letting $t \to x$, we see that $f'(x) \geq 0$.

If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \leq 0,$$

which shows that $f'(x) \leq 0$. Hence $f'(x) = 0$.

Remark The analogous statement for local minima is also true.

Remark The above theorem does not work if the open interval (a, b) is replaced by a closed interval $[a, b]$. For instance, the function $f : [1, 2] \to \mathbf{R}$ defined by $f(x) := x$ has a local maximum at $x_0 = 2$ and a local minimum $x_0 = 1$ (in fact, these local extrema are global extrema), but at both points the derivative is $f'(x_0) = 1$, not $f'(x_0) = 0$. Thus the endpoints of an interval can be local maxima or minima even if the derivative is not zero there. Finally, the converse of it is false.

Examples.

1) $f(x) = x^2, f : \mathbb{R} \to \mathbb{R}$, has a global minimum at $x_0 = 0$.

2) $f(x) = x^3, f : \mathbb{R} \to \mathbb{R}$, satisfies $f'(0) = 0$ although it neither has a local minimum nor a local maximum at $x_0 = 0$. Thus, $f'(x_0) = 0$ is not a sufficient condition for a local minimum or maximum.

3) $f(x) = x, f : [0, 1] \to \mathbb{R}$, has a local minimum at $x_0 = 0$ and a local maximum at $x_0 = 1$. However, at neither point $f'(x_0) = 0$. The reason is that the domain of definition of $f, [0, 1]$, does not contain an open interval around $x_0 = 0$ or $x_0 = 1$.

4) $f(x) = 2x^3 - 3x^2, f : \mathbb{R} \to \mathbb{R}$, satisfies $f'(x_0) = 0$ for $x_0 = 0, 1$, and it has a local maximum at $x_0 = 0$, and a local minimum at $x_0 = 1$.

Theorem (Rolle's theorem) *Suppose that f is a real-valued function, defined on a closed interval $[a, b]$, which is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Suppose that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof If $f(x) = f(a)$ for all $x \in (a, b)$ then $f'(x) = 0$ for all $x \in (a, b)$. Otherwise f is not monotonic on $[a, b]$, and therefore has a local maximum or local minimum at an interior point c of $[a, b]$. Then $f'(c) = 0$, by the above theorem. $\square$

Proposition *If f is a continuous function on an interval I then f is injective if and only if f is strictly monotonic.*

Proof If f is strictly monotonic, then certainly f is injective. Suppose that f is not strictly monotonic, and suppose for example that $a < b < c$ while $f(a) < f(c) < f(b)$. Then there exists $d \in [a, b]$ such that $f(d) = f(c)$, contradicting the fact that f is injective. Other possibilities are dealt with in the same way. $\square$

Corollary *Suppose that $f'(x) \neq 0$, for each $x \in (a,b)$. Then f is strictly monotonic.*

Proof

If not, f is not injective and so there exists $a \leq a' < b' \leq b$

for which $f(a') = f(b')$. But then there exists $a' < c < b'$ with $f(c') = 0$, giving a contradiction.