Integration and differentiation, integration of vector-valued functions, rectifiable curves, discussion of main problem, uniform convergence

6.20 Theorem *Let $f \in \mathscr{R}$ on $[a, b]$. For $a \le x \le b$, put*

$$F(x) = \int_a^x f(t)\, dt.$$

Then F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0, and

$$F'(x_0) = f(x_0).$$

Proof Since $f \in \mathscr{R}$, f is bounded. Suppose $|f(t)| \le M$ for $a \le t \le b$. If $a \le x < y \le b$, then

$$|F(y) - F(x)| = \left| \int_x^y f(t)\, dt \right| \le M(y - x),$$

by Theorem 6.12(c) and (d). Given $\varepsilon > 0$, we see that

$$|F(y) - F(x)| < \varepsilon,$$

provided that $|y - x| < \varepsilon / M$. This proves continuity (and, in fact, uniform continuity) of F.

Now suppose f is continuous at x_0. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon$$

if $|t - x_0| < \delta$, and $a \le t \le b$. Hence, if

$$x_0 - \delta < s \le x_0 \le t < x_0 + \delta \quad \text{and} \quad a \le s < t \le b,$$

we have,

$$\left| \frac{F(t) - F(s)}{t - s} - f(x_0) \right| = \left| \frac{1}{t - s} \int_s^t [f(u) - f(x_0)]\, du \right| < \varepsilon.$$

It follows that $F'(x_0) = f(x_0)$.

6.21 The fundamental theorem of calculus *If $f \in \mathscr{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof Let $\varepsilon > 0$ be given. Choose a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ so that $U(P,f) - L(P,f) < \varepsilon$. The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(t_i)\,\Delta x_i$$

for $i = 1, \ldots, n$. Thus

$$\sum_{i=1}^{n} f(t_i)\,\Delta x_i = F(b) - F(a).$$

It now follows from Theorem 6.7(*c*) that

$$\left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \varepsilon.$$

Since this holds for every $\varepsilon > 0$, the proof is complete.

6.22 Theorem (integration by parts) *Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in \mathscr{R}$, and $G' = g \in \mathscr{R}$. Then*

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

Proof Put $H(x) = F(x)G(x)$ and apply Theorem 6.21 to H and its derivative. Note that $H' \in \mathscr{R}$, by Theorem 6.13.

6.24 Theorem *If $\mathbf{f}$ and $\mathbf{F}$ map $[a, b]$ into R^k, if $\mathbf{f} \in \mathscr{R}$ on $[a, b]$, and if $\mathbf{F}' = \mathbf{f}$, then*

$$\int_a^b \mathbf{f}(t)\,dt = \mathbf{F}(b) - \mathbf{F}(a).$$

6.25 Theorem *If* $\mathbf{f}$ *maps* $[a, b]$ *into* R^k *and if* $\mathbf{f} \in \mathscr{R}(\alpha)$ *for some monotonically increasing function* α *on* $[a, b]$, *then* $|\mathbf{f}| \in \mathscr{R}(\alpha)$, *and*

$$\left|\int_a^b \mathbf{f}\, d\alpha\right| \le \int_a^b |\mathbf{f}|\, d\alpha. \tag{40}$$

Proof If $f_1, \ldots, f_k$ are the components of $\mathbf{f}$, then

$$|\mathbf{f}| = (f_1^2 + \cdots + f_k^2)^{1/2}. \tag{41}$$

By Theorem 6.11, each of the functions f_i^2 belongs to $\mathscr{R}(\alpha)$; hence so does their sum. Note that the square root function is continuous on $[0, \infty)$. If we apply Theorem 6.11 once more, (41) shows that $|\mathbf{f}| \in \mathscr{R}(\alpha)$.

To prove (40), put $\mathbf{y} = (y_1, \ldots, y_k)$, where $y_j = \int f_j\, d\alpha$. Then we have $\mathbf{y} = \int \mathbf{f}\, d\alpha$, and

$$|\mathbf{y}|^2 = \sum y_i^2 = \sum y_j \int f_j\, d\alpha = \int \left(\sum y_j f_j\right) d\alpha.$$

By the Schwarz inequality,

$$\sum y_j f_j(t) \le |\mathbf{y}|\,|\mathbf{f}(t)| \qquad (a \le t \le b); \tag{42}$$

hence Theorem 6.12(*b*) implies

$$|\mathbf{y}|^2 \le |\mathbf{y}| \int |\mathbf{f}|\, d\alpha. \tag{43}$$

If $\mathbf{y} = \mathbf{0}$, (40) is trivial. If $\mathbf{y} \ne \mathbf{0}$, division of (43) by $|\mathbf{y}|$ gives (40).

6.26 Definition A continuous mapping γ of an interval $[a, b]$ into R^k is called a *curve* in R^k. To emphasize the parameter interval $[a, b]$, we may also say that γ is a curve on $[a, b]$.

If γ is one-to-one, γ is called an *arc*.
If $\gamma(a) = \gamma(b)$, γ is said to be a *closed curve*.

Remark It should be noted that we define a curve to be a *mapping*, not a point set. Of course, with each curve γ in R^k there is associated a subset of R^k, namely the range of γ, but different curves may have the same range.

We associate to each partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ and to each curve γ on $[a, b]$ the number

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|.$$

We define the length of γ as

$$\Lambda(\gamma) = \sup \Lambda(P, \gamma),$$

where the supremum is taken over all partitions of $[a, b]$.
If $\Lambda(\gamma) < \infty$, we say that γ is *rectifiable*.

6.27 Theorem *If γ' is continuous on $[a, b]$, then γ is rectifiable, and*

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

Proof If $a \le x_{i-1} < x_i \le b$, then

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left|\int_{x_{i-1}}^{x_i} \gamma'(t)\, dt\right| \le \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt.$$

Hence

$$\Lambda(P, \gamma) \le \int_a^b |\gamma'(t)|\, dt$$

for every partition P of $[a, b]$. Consequently,

$$\Lambda(\gamma) \le \int_a^b |\gamma'(t)|\, dt.$$

To prove the opposite inequality, let $\varepsilon > 0$ be given. Since γ' is uniformly continuous on $[a, b]$, there exists $\delta > 0$ such that

$$|\gamma'(s) - \gamma'(t)| < \varepsilon \qquad \text{if } |s - t| < \delta.$$

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, with $\Delta x_i < \delta$ for all i. If $x_{i-1} \le t \le x_i$, it follows that

$$|\gamma'(t)| \le |\gamma'(x_i)| + \varepsilon.$$

Hence

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt &\le |\gamma'(x_i)|\, \Delta x_i + \varepsilon\, \Delta x_i \\
&= \left|\int_{x_{i-1}}^{x_i} [\gamma'(t) + \gamma'(x_i) - \gamma'(t)]\, dt\right| + \varepsilon\, \Delta x_i \\
&\le \left|\int_{x_{i-1}}^{x_i} \gamma'(t)\, dt\right| + \left|\int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(t)]\, dt\right| + \varepsilon\, \Delta x_i \\
&\le |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon\, \Delta x_i.
\end{aligned}$$

If we add these inequalities, we obtain

$$\begin{aligned}
\int_a^b |\gamma'(t)|\, dt &\le \Lambda(P, \gamma) + 2\varepsilon(b - a) \\
&\le \Lambda(\gamma) + 2\varepsilon(b - a).
\end{aligned}$$

Since ε was arbitrary,

$$\int_a^b |\gamma'(t)|\, dt \le \Lambda(\gamma).$$

This completes the proof.

Sequences and series of functions

7.1 Definition Suppose $\{f_n\}$, $n = 1, 2, 3, \ldots$, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n \to \infty} f_n(x) \qquad (x \in E). \tag{1}$$

Under these circumstances we say that $\{f_n\}$ converges on E and that f is the *limit*, or the *limit function*, of $\{f_n\}$.

Similarly, if $\Sigma f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in E), \tag{2}$$

the function f is called the *sum* of the series Σf_n.

The main problem which arises is to determine whether important properties of functions are preserved under the limit operations (1) and (2). For instance, if the functions f_n are continuous, or differentiable, or integrable, is the same true of the limit function? What are the relations between f_n' and f', say, or between the integrals of f_n and that of f ?

To say that f is continuous at a limit point x means

$$\lim_{t \to x} f(t) = f(x).$$

Hence, to ask whether the limit of a sequence of continuous functions is continuous is the same as to ask whether

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t). \tag{3}$$

Example Let

$$f_n(x) = \frac{x^2}{(1 + x^2)^n} \qquad (x \text{ real}; n = 0, 1, 2, \ldots),$$

and consider

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1 + x^2)^n}. \tag{6}$$

Since $f_n(0) = 0$, we have $f(0) = 0$. For $x \neq 0$, the last series in (6) is a convergent geometric series with sum $1 + x^2$

$$f(x) = \begin{cases} 0 & (x = 0), \\ 1 + x^2 & (x \neq 0), \end{cases} \tag{7}$$

so that a convergent series of continuous functions may have a discontinuous sum.

Example

Let $f_n : [0,1] \to \mathbb{R}$ be the function $f_n(x) = x^n$. The sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the function f defined by:

$$f(x) := \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1. \end{cases}$$

For $x = 1$ we always have $f_n(x) = 1$, whereas for $0 \le x < 1$, given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$, e.g. the smallest natural number greater than $\frac{\log \varepsilon}{\log x}$, such that

$$|f_n(x) - 0| = |f_n(x)| = x^n < \varepsilon \quad \text{for all } n \ge N.$$

We observe that the limit function f is not continuous, although all the f_n are continuous. The concept of pointwise convergence is therefore too weak to allow for continuity properties to carry over to limit functions.

Example Suppose that $f^{(n)} : [a,b] \to \mathbf{R}$ a sequence of Riemann-integrable functions on the interval $[a,b]$. If $\int_{[a,b]} f^{(n)} = L$ for every n, and $f^{(n)}$ converges pointwise to some new function f, this does not mean that $\int_{[a,b]} f = L$. An example comes by setting $[a,b] := [0,1]$, and letting $f^{(n)}$ be the function $f^{(n)}(x) := 2n$ when $x \in [1/2n, 1/n]$, and $f^{(n)}(x) := 0$ for all other values of x. Then $f^{(n)}$ converges pointwise to the zero function $f(x) := 0$. On the other hand, $\int_{[0,1]} f^{(n)} = 1$ for every n, while $\int_{[0,1]} f = 0$. In particular, we have an example where

$$\lim_{n\to\infty} \int_{[a,b]} f^{(n)} \neq \int_{[a,b]} \lim_{n\to\infty} f^{(n)}.$$

Example Let

$$f_n(x) = \frac{\sin nx}{\sqrt{n}} \qquad (x \text{ real}, n = 1, 2, 3, \ldots),$$

and

$$f(x) = \lim_{n \to \infty} f_n(x) = 0.$$

Then $f'(x) = 0$, and

$$f_n'(x) = \sqrt{n} \cos nx,$$

so that $\{f_n'\}$ does not converge to f'. For instance,

$$f_n'(0) = \sqrt{n} \to +\infty$$

as $n \to \infty$, whereas $f'(0) = 0$.

Example Let

$$f_n(x) = n^2 x(1-x^2)^n \qquad (0 \le x \le 1,\ n = 1, 2, 3, \ldots). \tag{10}$$

We have

$$\lim_{n\to\infty} f_n(x) = 0 \qquad (0 \le x \le 1). \tag{11}$$

$$\int_0^1 x(1-x^2)^n \, dx = \frac{1}{2n+2}.$$

$$\int_0^1 f_n(x)\, dx = \frac{n^2}{2n+2} \to +\infty \qquad \text{as } n \to \infty.$$

Thus the limit of the integral need not be equal to the integral of the limit.

7.7 Definition We say that a sequence of functions $\{f_n\}$, $n = 1, 2, 3, \ldots$, converges *uniformly* on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon \tag{12}$$

for all $x \in E$.

We say that the series $\Sigma f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of partial sums defined by

$$\sum_{i=1}^{n} f_i(x) = s_n(x)$$

converges uniformly on E.

Remark If a sequence f_n: X → Y of functions converges pointwise (or uniformly) to a function f : X → Y, then the restrictions $f_n|_E$: $E \rightarrow$ Y of f to some subset E of X will also converge pointwise (or uniformly) to $f|_E$.

Example The functions $f_n\colon (0,1) \to \mathbb{R}$ defined by $f_n(x) = x^n$, $n = 2,3,4, \ldots$, converge pointwise, but do not converge uniformly to the function $f = 0$.

Example Let $f^{(n)} : [0,1] \to \mathbf{R}$ be the functions $f^{(n)}(x) := x/n$, and let $f : [0,1] \to \mathbf{R}$ be the zero function $f(x) := 0$. Then it is clear that $f^{(n)}$ converges to f pointwise. Now we show that in fact $f^{(n)}$ converges to f uniformly. We have to show that for every $\varepsilon > 0$, there exists an N such that $|f^{(n)}(x) - f(x)| < \varepsilon$ for every $x \in [0,1]$ and every $n \geq N$. To show this, let us fix an $\varepsilon > 0$. Then for any $x \in [0,1]$ and $n \geq N$, we have

$$|f^{(n)}(x) - f(x)| = |x/n - 0| = x/n \leq 1/n \leq 1/N.$$

Thus if we choose N such that $N > 1/\varepsilon$ (note that this choice of N does not depend on what x is), then we have $|f^{(n)}(x) - f(x)| < \varepsilon$ for all $n \geq N$ and $x \in [0,1]$, as desired.

7.8 Theorem *The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies*

$$|f_n(x) - f_m(x)| \leq \varepsilon. \tag{13}$$

Proof Suppose $\{f_n\}$ converges uniformly on E, and let f be the limit function. Then there is an integer N such that $n \geq N$, $x \in E$ implies

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2},$$

so that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \varepsilon$$

if $n \geq N$, $m \geq N$, $x \in E$. Conversely, suppose the Cauchy condition holds. Then, the sequence $\{f_n(x)\}$ converges, for every x, to a limit which we may call $f(x)$. Thus the sequence $\{f_n\}$ converges on E, to f. We have to prove that the convergence is uniform.

Let $\varepsilon > 0$ be given, and choose N such that (13) holds. Fix n, and let $m \to \infty$ in (13). Since $f_m(x) \to f(x)$ as $m \to \infty$, this gives

$$|f_n(x) - f(x)| \leq \varepsilon \tag{14}$$

for every $n \geq N$ and every $x \in E$, which completes the proof.

7.9 Theorem *Suppose*

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in E).$$

Put

$$M_n = \sup_{x\in E} |f_n(x) - f(x)|.$$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

7.10 Theorem *Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose*

$$|f_n(x)| \le M_n \qquad (x \in E, n = 1, 2, 3, \ldots).$$

Then Σf_n converges uniformly on E if ΣM_n converges.

Proof If ΣM_n converges, then, for arbitrary $\varepsilon > 0$,

$$\left|\sum_{i=n}^{m} f_i(x)\right| \le \sum_{i=n}^{m} M_i \le \varepsilon \qquad (x \in E),$$

provided m and n are large enough. Uniform convergence now follows from Theorem 7.8.