Definition. A point x in a metric space X is said to be a limit of a sequence of points $(x_n) \subset X$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$, for all $n \geq N$.

If x is a limit of the sequence (x_n), we say that (x_n) converges to x and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

If a sequence has a limit, it is called convergent. Otherwise, it is called divergent. Observe that $x_n \to x \Leftrightarrow d(x_n, x) \to 0$.

A subset Y of a metric space (X, d) is bounded if there exists $x \in X$ and $r > 0$, such that $Y \subset B(r, x)$. Otherwise, Y is unbounded.

Any convergent sequence is bounded, since if $x_n \to x$, then there exists $N \in \mathbb{N}$ such that $d(x_n, x) < 1$ for all $n \geq N$ and so

$$d(x_n, x) \leq \max\left(1, \max_{j=1,\cdots,N-1} d(x_j, x)\right), \ \forall n \in \mathbb{N}.$$

Lemma 1 Let X be a metric space.
i) If $p \in X,\ p' \in X$, and if $\{p_n\}$ converges to p and to p', then $p' = p$.
ii) If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p_n \to p$.

Proof i) We have

$$d(p, p') \leq d(p_n, p) + d(p_n, p').$$

Taking $n \to \infty$, we get $d(p, p') = 0$. Thus $p = p'$.
ii) For each positive integer n, there is a point $p_n \in E$ such that $d(p_n, p) < 1/n$. Hence $p_n \to p$.

Theorem Suppose $x_n \in \mathbb{R}^k$ $(n = 1, 2, 3, ...)$ and $x_n = (a_{1,n}, \cdots, a_{k,n})$. Then $\{x_n\}$ converges to $x = (a_1, \cdots, a_k)$ if and only if

$$\lim_{n\to\infty} a_{j,n} = a_j \ (1 \le j \le n).$$

Proof " $\Rightarrow$ " follows from $|a_{j,n} - a_j| \le |x_n - x|$.

" $\Leftarrow$ ": We have for each $\epsilon > 0$ there corresponds an integer N such that $n \ge N$ implies

$$|a_{j,n} - a_j| < \frac{\epsilon}{\sqrt{k}} \ (1 \le j \le k).$$

Hence $n \ge N$ implies

$$|x_n - x| = \left\{ \sum_{j=1}^{k} |a_{j,n} - a_j|^2 \right\}^{1/2} < \epsilon,$$

so that $x_n \to x$.

Lemma 2. $x_n \to x \Leftrightarrow \forall$ open set U that contains x there exists an N such that $x_n \in U$ for every $n \geq N$.

Proof $\Rightarrow$: Given any open set U that contains x there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$, and so $\exists\ N$ such that $x_n \in B(x, \epsilon) \subset U$ for all $n \geq N$.
$\Leftarrow$: For any $\epsilon > 0$, since the set $B(x, \epsilon)$ is open and contains x, there exists an N such that $x_n \in B(x, \epsilon)$ for every $n \geq N$. Thus $x_n \to x$. □

Lemma 3. A subset A of (X, d) is closed $\Leftrightarrow$ whenever $(x_n) \subset A$ with $x_n \to x$ it follows that $x \in A$.

Proof $\Rightarrow$: Let $(x_n) \subset A$ with $x_n \to x$. If $x \notin A$, i.e. $x \in A^c$, there would exist $B(x, r) \subset A^c$ since A^c is open. It follows that $B(x, r) \cap A = \emptyset$, which contradicts to $x_n \to x$.
$\Leftarrow$: Take $x \notin A$. If for any $\epsilon > 0$, $B(x, \epsilon) \cap A \neq \emptyset$, by taking $\epsilon = \frac{1}{n}$, we would get $(x_n) \subset A$ such that $d(x, x_n) < \frac{1}{n}$, which implies that $x_n \to x$ and so $x \in A$. This is a contradiction. Thus there exists an $r > 0$ such that $B(x, r) \cap A = \emptyset$, that is $B(x, r) \subset A^c$. Consequently, A^c is open. □

Definition Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \cdots$. The sequence $\{p_{n_k}\}$ is called a subsequence of $\{p_n\}$. If $\{p_{n_k}\}$ converges, its limit is called a subsequential limit of $\{p_n\}$.

Observe that $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p.

Theorem (a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.
(b) Any bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof (a) If $\{p_n\}$ is a finite set then there is a p and a sequence $\{n_k\}$ with $n_1 < n_2 < n_3 < \cdots$, such that $p_{n_1} = p_{n_2} = \cdots = p$. The subsequence $\{p_{n_k}\}$ converges to p.
If $\{p_n\}$ is an infinite set, then $\{p_n\}$ has a limit point $p \in X$. Choose n_1 so that $d(p, p_{n_1}) < 1$. Having chosen $n_1, \cdots, n_{n_{k-1}}$, we can choose an integer $n_k > n_{k-1}$ such that $d(p, p_{n_k}) < 1/k$. Then $\{p_{n_k}\}$ converges to p.

(b) follows from (a), since every bounded subset of $\mathbb{R}^k$ lies in a compact subset of $\mathbb{R}^k$.

Definition A sequence (x_n) in a metric space (X, d) is called a Cauchy sequence if, for every $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for any two integers $n, m \geq n_0$, we have $d(x_n, x_m) < \epsilon$. A metric space (X, d) is called complete if every Cauchy sequence in X converges.

For any $n \in \mathbb{N}$, $\mathbb{R}^n$, equipped with the Euclidean metric, is complete, because a Cauchy sequence in $\mathbb{R}^n$ is Cauchy in each coordinate.

Definition Let A be a non-empty set of the metric space (X, d). The diameter of A is defined as

$$\operatorname{diam}(A) = \sup_{x,y \in A} d(x, y).$$

Remark If $\{p_n\}$ is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, p_{N+2}, \cdots$, then $\{p_n\}$ is a Cauchy sequence if and only if

$$\lim_{N \to \infty} \operatorname{diam}(E_N) = 0.$$

Theorem. Let E be a in a metric space X, then

$$\text{diam } \overline{E} = \text{diam } E.$$

Proof. Let $p, q \in \overline{E}$ and $\epsilon > 0$. By the definition of $\overline{E}$, there are points p', q', in E such that $d(p,p') < \epsilon/2,\ d(q,q') < \epsilon/2$. Hence

$$d(p,q) \le d(p,p') + d(p',q') + d(q,q') < \epsilon + d(p',q') \le \epsilon + \text{diam}(E).$$

It follows that

$$\text{diam } \overline{E} \le \epsilon + \text{diam } E.$$

Since ϵ is arbitrary, this proves the theorem.

Theorem. (Nested sets theorem). *Let* $A_1 \supset A_2 \supset \cdots$ *be a decreasing chain of non-empty closed subsets of a complete metric space* (X, d) *and let* $\operatorname{diam}(A_n) \to 0$ *as* $n \to \infty$*. Then* $\cap_{n=1}^{\infty} A_n$ *consists of exactly one point.*

Proof. Pick in each A_n a point a_n. If $N \in \mathbb{N}$ and $k, j > N$, then since $A_n \downarrow$, the points a_k and a_j belong to A_N. Thus, $d(a_j, a_k) \le \operatorname{diam}(A_N) \to 0$ as $N \to \infty$, i.e., (a_n) is Cauchy. Let $a = \lim a_n$. For any N and any $k > N$, $a_k \in A_N$. Hence, $a = \lim a_k \in A_N$, i.e., $a \in \cap_{N=1}^{\infty} A_N$. Note that $\cap_{n=1}^{\infty} A_n \subset A_N$ for all N, and so

$$\operatorname{diam}\left(\cap_{n=1}^{\infty} A_n\right) \le \operatorname{diam}(A_N) \to 0, \ \ N \to \infty.$$

But a set of diameter zero reduces to a single point. □

Theorem. (a) In any metric space X, every convergent sequence is a Cauchy sequence.
(b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.

Proof. (a) follows from the definition.
(b) Let $\{p_n\}$ be a Cauchy sequence in X. Since X is compact, $\{p_n\}$ contains a convergent subsequence, say $p_{n_k} \to p$. Given $\epsilon > 0$, since $\{p_n\}$ is Cauchy, there exists an $N_1 \in \mathbb{N}$ such that

$$d(p_l, p_m) < \epsilon/2, \ l, m \geq N_1.$$

From $p_{n_k} \to p$, we can find an $N_2 \in \mathbb{N}$ such that if $k \geq N_2$, then

$$d(p_{n_k}, p) < \epsilon/2.$$

Setting $K = \max\{N_1, N_2\}$, we have for $n \geq K$ that

$$d(p_n, p) \leq d(p_n, p_{n_K}) + d(p_{n_K}, p) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $p_n \to p$.

Remark The above proof shows that a Cauchy sequence converges if it contains a convergent subsequence.

Corollary All compact metric spaces and all Euclidean spaces are complete.

Corollary Every closed subset E of a complete metric space X is complete.

Proof. Let $\{p_n\}$ be a Cauchy sequence in E; then $p_n \to p$ for some $p \in X$ since X is complete, and actually $p \in E$ since E is closed.

We know that convergent sequences in a metric space are bounded. However, bounded sequences in $\mathbb{R}^k$ need not converge. On the other hand, any bounded monotone sequence in $\mathbb{R}$ converges.