Continuity and compactness, continuity and connectedness, discontinuity

Given two functions $f : X \to Y$ and $g : X \to Z$, one can define their *direct sum* $f \oplus g : X \to Y \times Z$ defined by $f \oplus g(x) := (f(x), g(x))$, i.e., this is the function taking values in the Cartesian product $Y \times Z$ whose first co-ordinate is $f(x)$ and whose second co-ordinate is $g(x)$.

Lemma *Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions, and let $f \oplus g : X \to \mathbf{R}^2$ be their direct sum. We give $\mathbf{R}^2$ the Euclidean metric.*

(a) *If $x_0 \in X$, then f and g are both continuous at x_0 if and only if $f \oplus g$ is continuous at x_0.*

(b) *f and g are both continuous if and only if $f \oplus g$ is continuous.*

Lemma *The addition function $(x, y) \mapsto x + y$, the subtraction function $(x, y) \mapsto x - y$, the multiplication function $(x, y) \mapsto xy$, the maximum function $(x, y) \mapsto \max(x, y)$, and the minimum function $(x, y) \mapsto \min(x, y)$, are all continuous functions from $\mathbf{R}^2$ to $\mathbf{R}$. The division function $(x, y) \mapsto x/y$ is a continuous function from $\mathbf{R} \times (\mathbf{R}\backslash\{0\}) = \{(x, y) \in \mathbf{R}^2 : y \neq 0\}$ to $\mathbf{R}$. For any real number c, the function $x \mapsto cx$ is a continuous function from $\mathbf{R}$ to $\mathbf{R}$.*

Theorem

(a) Let $f_1, \ldots, f_k$ be real functions on a metric space X, and let $\mathbf{f}$ be the mapping of X into R^k defined by

$$\mathbf{f}(x) = (f_1(x), \ldots, f_k(x)) \qquad (x \in X);$$

then $\mathbf{f}$ is continuous if and only if each of the functions $f_1, \ldots, f_k$ is continuous.
(b) If $\mathbf{f}$ and $\mathbf{g}$ are continuous mappings of X into R^k, then $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ are continuous on X.

The functions $f_1, \ldots, f_k$ are called the *components* of $\mathbf{f}$. Note that $\mathbf{f} + \mathbf{g}$ is a mapping into R^k, whereas $\mathbf{f} \cdot \mathbf{g}$ is a *real* function on X.

Lemma *Suppose that $g : D \to \mathbb{R}$ (or $\mathbb{C}$) is continuous at $p \in D$, and that $g(p) \neq 0$. Then there exists $\delta > 0$ with the property that for all $x \in D$ with $|x - p| < \delta$*

$$g(x) \neq 0$$

as well.

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Proof. Let $\varepsilon := \frac{|g(p)|}{2} > 0$. Since g is continuous at p, we may find $\delta > 0$ such that for all $x \in D$ with $|x - p| < \delta$

$$|g(x) - g(p)| < \varepsilon = \frac{|g(p)|}{2}.$$

This implies

$$|g(x)| > \frac{|g(p)|}{2} > 0.$$

□

Definition Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ (or $\mathbb{C}$), and $0 < \alpha < 1$. f is called α-Hölder continuous (for $\alpha = 1$ Lipschitz continuous) if for any closed and bounded interval $I \subset D$ there exists an $m_I \in \mathbb{R}$ with

$$|f(x) - f(y)| \leq m_I |x - y|^\alpha \text{ for all } x, y \in I.$$

One easily checks that if $f, g : D \to \mathbb{R}(\mathbb{C})$ are α-Hölder continuous, then so is their sum $f + g$, and likewise λf, for any $\lambda \in \mathbb{R}(\mathbb{C})$.

Definition The vector space of α-Hölder continuous functions $f : D \to \mathbb{R}$ (resp. $\mathbb{C}$) will be denoted by $C^{0,\alpha}(D, \mathbb{R})$ (resp. $C^{0,\alpha}(D, \mathbb{C})$). We also write $C^{0,\alpha}(D)$ for $C^{0,\alpha}(D, \mathbb{R})$ and, for $0 < \alpha < 1, C^{0,\alpha}(D)$ as $C^\alpha(D)$.

Definition A mapping $\mathbf{f}$ of a set E into R^k is said to be *bounded* if there is a real number M such that $|\mathbf{f}(x)| \leq M$ for all $x \in E$.

Theorem *Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.*

Proof Let $\{V_\alpha\}$ be an open cover of $f(X)$. Since f is continuous, each of the sets $f^{-1}(V_\alpha)$ is open. Since X is compact, there are finitely many indices, say $\alpha_1, \ldots, \alpha_n$, such that

$$X \subset f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n}).$$

Since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$, it follows that

$$f(X) \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

Note: We have used the relation $f(f^{-1}(E)) \subset E$, valid for $E \subset Y$. If $E \subset X$, then $f^{-1}(f(E)) \supset E$; equality need not hold in either case.

Corollary *If* $\mathbf{f}$ *is a continuous mapping of a compact metric space* X *into* R^k, *then* $\mathbf{f}(X)$ *is closed and bounded. Thus,* $\mathbf{f}$ *is bounded.*

Proposition Let K be a compact subset of (X, d). Then any continuous function $f : K \to \mathbb{R}$ is bounded and attains its bounds.

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Proof $f(K) \subset \mathbb{R}$ is compact and so is bounded and closed. Let $l = \sup\{f(x), x \in K\}$; then $l < \infty$. Take $\{f(x_n)\} \subset f(K)$ so that $f(x_n) \to l$; then $l \in f(K)$ since $f(K)$ is closed. Hence, there is a $z \in K$ such that $l = f(z)$. Similarly, there is a $y \in K$ such that $f(y) = \inf\{f(x), x \in K\}$.

$\square$

Theorem *Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by*

$$f^{-1}(f(x)) = x \qquad (x \in X)$$

is a continuous mapping of Y onto X.

Proof. Let $g = f^{-1}$ and C be a closed subset of X. We need only to show that $g^{-1}(C) = f(C)$ is a closed subset of Y. Since X is compact and C is a closed subset of X, we know that C is a compact subset of X. Since f is a continuous map from a compact metric space to another metric space, it maps compact sets to compact sets. Thus $f(C)$ is a compact subset of Y and so is a closed subset of Y. $\square$

Definition (Uniform continuity). Let $f : X \to Y$ be a map from one metric space (X, d_X) to another (Y, d_Y). We say that f is *uniformly continuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ whenever $x, x' \in X$ are such that $d_X(x, x') < \delta$.

Definition A function $f : E \to \mathbb{R}$ is *uniformly continuous* on a set $E \subset \mathbb{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ for all points $x_1, x_2 \in E$ such that $|x_1 - x_2| < \delta$.

$$(f : E \to \mathbb{R} \text{ is uniformly continuous }) :=$$
$$= \left(\forall \varepsilon > 0\, \exists \delta > 0\, \forall x_1 \in E\, \forall x_2 \in E \left(|x_1 - x_2| < \delta \Rightarrow \right.\right.$$
$$\left.\left. \Rightarrow |f(x_1) - f(x_2)| < \varepsilon\right)\right).$$

$$(f : E \to \mathbb{R} \text{ is not uniformly continuous}) :=$$
$$\left(\exists \varepsilon > 0\, \forall \delta > 0\, \exists x_1 \in E\, \exists x_2 \in E \left(|x_1 - x_2| < \delta \;\&\; |f(x_1) - f(x_2)| \geq \varepsilon\right)\right).$$

$$(f : E \to \mathbb{R} \text{ is continuous on } E :=$$
$$= \left(\forall a \in E\, \forall \varepsilon > 0\, \exists \delta > 0\, \forall x \in E \left(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\right)\right).$$

Example i) The function $f(x) = \dfrac{1}{x}$

is continuous but not uniformly continuous on (0, 1).

ii) For any $a > 0$, the function $f(x) = \dfrac{1}{x}$

is uniformly continuous on $[a, \infty)$.

iii) The function $g(x) = x^2$ is continuous but not uniformly continuous on $(-\infty, \infty)$.

Definition Let $D \subset \mathbb{R}$ (or $\mathbb{C}$) and $f : D \to \mathbb{R}$ (or $\mathbb{C}$) a function. The function f is said to be uniformly continuous in D

$$\begin{aligned} \iff \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x_1, x_2 \in D \quad & \text{with } |x_1 - x_2| < \delta : \\ & |f(x_1) - f(x_2)| < \varepsilon \, . \end{aligned} \tag{5}$$

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$. We show that f is continuous at every $p \in \mathbb{R}$. Let $\varepsilon > 0$. We set

$$\delta = \min\left(1, \frac{\varepsilon}{2|p| + 1}\right).$$

If $|x - p| < \delta$ then

$$|x^2 - p^2| = |x - p||x + p| \le |x - p|(|x| + p) < |x - p|(2|p| + 1) < \varepsilon.$$

This shows that f is continuous. We now show that f is not uniformly continuous on $\mathbb{R}$.

For this we prove the negation of (5), namely

$$\begin{aligned} \exists \varepsilon > 0 \, \forall \delta > 0 \, \exists x_1, x_2 \in \mathbb{R} \quad & \text{with } |x_1 - x_2| < \delta \\ & \text{but } |f(x_1) - f(x_2)| > \varepsilon \, . \end{aligned} \tag{6}$$

We choose $\varepsilon = 1$. For $\delta > 0$ there exist $x_1, x_2 \in \mathbb{R}$ with $|x_1 - x_2| = \frac{\delta}{2}, |x_1 + x_2| > \frac{2}{\delta}$. Therefore

$$|x_1^2 - x_2^2| = |x_1 - x_2||x_1 + x_2| > 1$$

which proves (6).

Proposition Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces, and let $f : X \to Y$ and $g : Y \to Z$ be two uniformly continuous functions. Then $g \circ f : X \to Z$ is also uniformly continuous.

Theorem *Let $I = [a, b]$ be a closed and bounded interval and $f : I \to \mathbb{R}$ (or $\mathbb{C}$) a continuous function. Then f is uniformly continuous on I.*

Lemma If $f : (X, d_X) \to (Y, d_Y)$ is continuous and X is compact, then f is uniformly continuous on X : $\forall \epsilon > 0 \; \exists \delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon, \; x, y \in X. \tag{0.3}$$

Proof If f is not uniformly continuous then $\exists \epsilon > 0$ s.t. $\forall \delta > 0$, $\exists x, y \in X$ with $d_X(x, y) < \delta$ and $d_Y(f(x), f(y)) \geq \epsilon$. Taking $\delta = 1/n$, we can find $x_n, y_n \in X$ such that

$$d_X(x_n, y_n) < 1/n \text{ and } d_Y(f(x_n), f(y_n)) \geq \epsilon. \tag{0.4}$$

Since X is compact, $\exists \{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \to x$. It follows that $y_{n_k} \to x$ also. Since f is continuous at x, we can find $\delta > 0$ such that $d_X(z, x) < \delta$ ensures that $d_Y(f(z), f(x)) < \epsilon/2$. Thus for j sufficiently large we have $d_X(x_{n_j}, x) < \delta$, $d_X(y_{n_k}, x) < \delta$. Hence

$$d_Y(f((x_{n_j}), f((y_{n_j}) \leq d_Y(f((x_{n_j}), f(x)) + d_Y(f((y_{n_j}), f(x)) < \epsilon,$$

contradicting (0.4). $\square$

Theorem *Let E be a noncompact set in R^1. Then*

(a) there exists a continuous function on E which is not bounded;
(b) there exists a continuous and bounded function on E which has no maximum. If, in addition, E is bounded, then

(c) there exists a continuous function on E which is not uniformly continuous.

Proof Suppose first that E is bounded, so that there exists a limit point x_0 of E which is not a point of E. Consider

$$f(x) = \frac{1}{x - x_0} \qquad (x \in E).$$

This is continuous on E, but unbounded. To see that it is not uniformly continuous, let $\varepsilon > 0$ and $\delta > 0$ be arbitrary, and choose a point $x \in E$ such that $|x - x_0| < \delta$. Taking t close enough to x_0, we can then make the difference $|f(t) - f(x)|$ greater than ε, although $|t - x| < \delta$. Since this is true for every $\delta > 0$, f is not uniformly continuous on E.

The function g given by

$$g(x) = \frac{1}{1 + (x - x_0)^2} \qquad (x \in E)$$

is continuous on E, and is bounded, since $0 < g(x) < 1$. It is clear that

$$\sup_{x \in E} g(x) = 1,$$

whereas $g(x) < 1$ for all $x \in E$. Thus g has no maximum on E.

Having proved the theorem for bounded sets E, let us now suppose that E is unbounded. Then $f(x) = x$ establishes (a), whereas

$$h(x) = \frac{x^2}{1 + x^2} \qquad (x \in E)$$

establishes (b), since

$$\sup_{x \in E} h(x) = 1$$

and $h(x) < 1$ for all $x \in E$.

Assertion (c) would be false if boundedness were omitted from the hypotheses. For, let E be the set of all integers. Then every function defined on E is uniformly continuous on E.

Theorem *If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.*

Proof Assume, on the contrary, that $f(E) = A \cup B$, where A and B are nonempty separated subsets of Y. Put $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$.

Then $E = G \cup H$, and neither G nor H is empty.

Since $A \subset \bar{A}$ (the closure of A), we have $G \subset f^{-1}(\bar{A})$; the latter set is closed, since f is continuous; hence $\bar{G} \subset f^{-1}(\bar{A})$. It follows that $f(\bar{G}) \subset \bar{A}$. Since $f(H) = B$ and $\bar{A} \cap B$ is empty, we conclude that $\bar{G} \cap H$ is empty.

The same argument shows that $G \cap \bar{H}$ is empty. Thus G and H are separated. This is impossible if E is connected.

Theorem *Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.*

Proof Since [a, b] is connected and f is continuous, f([a, b]) is connected. Therefore, for any c satisfying f(a) < c < f(b), we have $c \in f([a, b])$ and so there exists an $x \in [a, b]$ such that f(x)=c.

If x is a point in the domain of definition of the function f at which f is not continuous, we say that f is *discontinuous* at x, or that f *has a discontinuity* at x.

4.25 Definition Let f be defined on (a, b). Consider any point x such that $a \leq x < b$. We write

$$f(x+) = q$$

if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$. To obtain the definition of $f(x-)$, for $a < x \leq b$, we restrict ourselves to sequences $\{t_n\}$ in (a, x).

$\lim_{t \to x} f(t)$ exists if and only if $f(x+) = f(x-) = \lim_{t \to x} f(t)$.

Definition Let f be defined on (a, b). If f is discontinuous at a point x, and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the *first kind*, or a *simple discontinuity*, at x. Otherwise the discontinuity is said to be of the *second kind*.

There are two ways in which a function can have a simple discontinuity:

$f(x+) \neq f(x-)$ or $f(x+) = f(x-) \neq f(x)$

Examples

(*a*) Define

$$f(x) = \begin{cases} 1 & (x \text{ rational}), \\ 0 & (x \text{ irrational}). \end{cases}$$

Then f has a discontinuity of the second kind at every point x, since neither $f(x+)$ nor $f(x-)$ exists.

(*b*) Define

$$f(x) = \begin{cases} x & (x \text{ rational}), \\ 0 & (x \text{ irrational}). \end{cases}$$

Then f is continuous at $x = 0$ and has a discontinuity of the second kind at every other point.

(*c*) Define

$$f(x) = \begin{cases} x + 2 & (-3 < x < -2), \\ -x - 2 & (-2 \le x < 0), \\ x + 2 & (0 \le x < 1). \end{cases}$$

Then f has a simple discontinuity at $x = 0$ and is continuous at every other point of $(-3, 1)$.

(*d*) Define

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Since neither $f(0+)$ nor $f(0-)$ exists, f has a discontinuity of the second kind at $x = 0$.