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KKT:

1° lead in : Lagrange Multiplier Method

$$\min f(x) \quad \text{s.t } y = g(x) \Rightarrow L(x,y) = f(x,y) + \lambda[y - g(x)]$$

Calculate $\nabla L = 0$

generally: $\min_{x \in \mathbb{R}^n} f(x)$ $\quad L(x,\lambda) = f(x) + \sum \lambda_i g_i(x)$

s.t. $g_i(x) = a_i^T \cdot x + b_i \le 0$ $\Rightarrow$ If at (x^*, λ^*) take max $\Rightarrow$

$i \in [\![1, m]\!]$, $a_i \in \mathbb{R}^n$, $b \in \mathbb{R}$ $\quad \nabla L(x^*, \lambda^*) = 0$

Remark: $\forall \lambda_i \ge 0$ $\begin{cases} \lambda_i = 0, & g_i(x) \text{ is slack condition} \\ \lambda_i > 0, & g_i(x) \text{ is compact condition} \end{cases}$

2° Dual problem:

(P): $\min_{x \in \mathbb{R}^n} f_0(x)$ $\quad L(x,\lambda,\nu) = f_0(x) + \sum \lambda_i f_i(x) + \sum \nu_i h_i(x)$

s.t. $g_i(x) \le 0$, $i \in [\![1, m]\!]$ $\Longleftrightarrow$ (P) $\min_x \max_{\lambda,\nu} L(x,\lambda,\nu)$

s.t. $\lambda \succeq 0 \Rightarrow \forall \lambda_i \ge 0$.

$h_i(x) = 0$, $j \in [\![1, q]\!]$

proof for equivalence: $D := \{\text{feasible set}\}$

1° $x \notin D$, $\max_{\lambda,\nu} L(x,\lambda,\nu) = f_0(x) + \underset{g_i(x) > 0,\ \lambda_i > 0}{\infty} = \infty$

2° $x \in D$, $\max_{\lambda,\nu} L(x,\lambda,\nu) = f_0(x)$

$\square$
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concave (can be proved)

↑

$x^* = \arg\min L$

Dual function: $g(\lambda,\nu) = \min_x L(x,\lambda,\nu) = f(x^*) + \sum \lambda_i g_i(x^*) + \sum \nu_i h_i(x^*)$

(D): $\max_{\lambda,\nu} g(\lambda,\nu)$

s.t. $\nabla_x L(x,\lambda,\nu) = 0$

$\lambda \succeq 0$

Compare:

$p^* \triangleq$ (P): $\min_x \max_{\lambda,\nu} L(x,\lambda,\nu)$ $\geq$ (D) $\max_{\lambda,\nu} \min_x L(x,\lambda,\nu) \triangleq D^*$

s.t $\lambda \succeq 0$

↓

weak dual "=" strong dual.

3rd Lagrange-dual-function:

$L(x,\lambda,\nu) = f_0(x) + \sum \lambda_i g_i(x) \triangleq t + \lambda^T u$ ($t = f_0(x)$, $u = (f_1(x), \dots, f_m(x))$)

(P) feasible set: (D):

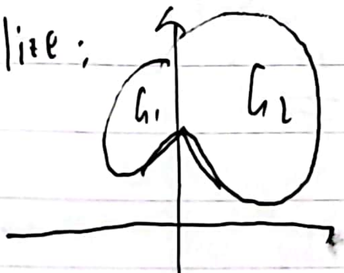
$\{(t,u) \mid t = f_0(x), u_i = g_i(x), x \in D\} = G_1$ $\{(t,u) \mid t = f_0(x), u_i = g_i(x), x \in \mathbb{R}^n\} = G_2$

$D = \left\{ x \,\middle|\, \begin{array}{l} f_i(x) \le 0 \\ h_i(x) = 0 \end{array} \right\}$

⇓ ⇓

problem $\min_x \{t \mid (t,u) \in G_1, u \le 0\}$ $\max_\lambda \min_x \{t + \lambda^T u \mid (t,u) \in G_2, \lambda \succeq 0\}$

Visualize:



f_0 convex

constraint convex

↗

Th. (get strong dual): If a convex problem satisfies

~~strong du~~ slater condition ⇒ strong dual.

has at least
1 interior point. recall

Slater condition:

$\exists\, x \in \text{relint}(D)$ s.t. $g_i(x) < 0$, $Ax = b$.

The strong dual and KKT: If convex problem is strong dual $\Rightarrow$ satisfies KKT.

4th KKT (def): ① primal: $g_i(x) \le 0$, $h_i(x) = 0$ ② dual: $\nabla_x L(x, \lambda, \nu) = 0$, $\lambda \succeq 0$

③ achieve op-sol both primal and dual (strong dual)

④ $\lambda_i g_i(x^*) = 0$
$$\lambda_i \ge 0 \begin{cases} \lambda_i = 0 \Rightarrow \lambda_i g_i(x^*) = 0 \\ \lambda_i > 0 \Rightarrow g_i(x^*) \ne 0 \Rightarrow \lambda_i g_i(x^*) = 0 \end{cases}$$