

# WHAT IS... A DIFFERENTIAL FORM?

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## 1. WHY DIFFERENTIAL FORMS?

Why are differential forms useful? Here are a few reasons:

- They provide a unified language for treating all the important results of vector calculus, such as the generalized fundamental theorem of calculus, Green's theorem, the Divergence (or Gauss-Ostrogradski) theorem, and Stokes's theorem, which can all be stated as

$$\int_{\partial M} \omega = \int_M d\omega.$$

- Differential forms are a natural language for the equations of electromagnetism (Maxwell's equations).
- They are an extremely useful tool in geometry, topology, and differential equations (e.g., de Rham theory, Hodge theory, etc.).

Learning about differential forms requires some effort, but that effort is well worth it!

## 2. DIFFERENTIAL FORMS ON $\mathbb{R}^3$

A differential form on  $\mathbb{R}^3$  is an expression involving symbols like  $dx, dy$ , and  $dz$ . There are four types of forms on  $\mathbb{R}^3$ : 0-forms, 1-forms, 2-forms, and 3-forms. 0-forms are just functions  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ . 1-forms are expressions of the form

$$\omega = adx + bdy + cdz,$$

where  $a, b, c$  are real-valued functions on  $\mathbb{R}^3$ .

A 2-form on  $\mathbb{R}^3$  is an expression of the form

$$\Phi = Adx \wedge dy + Bdy \wedge dz + Cdz \wedge dx,$$

where  $A, B, C$  are again real-valued functions on  $\mathbb{R}^3$  and the meaning of  $dx \wedge dy, dy \wedge dz$ , and  $dz \wedge dx$  will be explained later.

A 3-form on  $\mathbb{R}^3$  is an expression of the form

$$\Omega = f dx \wedge dy \wedge dz,$$

where  $f$  is a real-valued function on  $\mathbb{R}^3$  and the meaning of  $dx \wedge dy \wedge dz$  will be explained below.

For example,  $\omega = yzdx + zxdy + xydz$  is a 1-form and  $\Phi = zdx \wedge dy - xdy \wedge dz + ydz \wedge dx$  is a 2-form.

(It would be more correct to call  $\omega, \Phi$ , and  $\Omega$  form *fields*, since they depend on the point  $(x, y, z)$  in space, but we'll be lazy and keep calling them forms.)

## 3. WHAT DO FORMS DO?

But what does a 1-form do? For example, a function  $f$  can be applied to a number  $x$  to produce another number,  $f(x)$ . What can a form be applied to? The answer is: a  $k$ -form, where  $0 \leq k \leq 3$ , acts on  $k$ -tuples of vectors  $(v_1, \dots, v_k)$  and the output it produces is a real number. For example, if  $\omega$  is a 1-form and  $v \in \mathbb{R}^3$  a vector, then  $\omega(v)$  is defined and is just a real number. Similarly, if  $\Phi$  is a 2-form, then for any two vectors  $v, w$ ,  $\Phi(v, w)$  is a real number, and so on.

Furthermore,  $\omega(v + w) = \omega(v) + \omega(w)$  and  $\omega(tv) = t\omega(v)$ , for any scalar  $t$ . In other words,  $\omega$  is a *linear* function on  $\mathbb{R}^3$ .

It's a little more complicated with 2-forms  $\Phi$ , since they depend on two vectors. Now, if we fix  $w$ , then  $v \mapsto \Phi(v, w)$  is a linear function on  $\mathbb{R}^3$  and if we fix  $v$ , then  $w \mapsto \Phi(v, w)$  is also a linear function. This can be summarized by saying that  $\Phi$  is **bilinear**. Moreover,

$$\Phi(w, v) = -\Phi(v, w),$$

i.e.,  $\Phi$  is anti-symmetric. (Note that  $\Phi(v, v)$  is always zero.)

But *how* do we compute  $\omega(v)$ ? For that, we need to know what  $dx(v)$ ,  $dy(v)$ , and  $dz(v)$  are. The answer is easy:  $dx(v)$  is just the  $x$ - (i.e., first) component of  $v$ ,  $dy(v)$  is the  $y$ -component of  $v$ , etc.

For example if  $\omega = y dx + z dy - \pi dz$  and  $v = (-1, 0, 2)$ , then  $\omega(v) = ydx(v) + zdy(v) - \pi dz(v) = -y - 2\pi$ .

Evaluating 2-forms is a little harder, but here's how it goes. First we need to know how  $(dx \wedge dy)(v, w)$  (and so on) is defined. Given

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3,$$

$(dx \wedge dy)(v, w)$  is the determinant of the submatrix of the matrix  $[v|w]$  (whose columns are  $v$  and  $w$ ) obtained by only using the first two rows. That is,

$$(dx \wedge dy)(v, w) = \det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} = v_1 w_2 - v_2 w_1 = dx(v)dy(w) - dy(v)dx(w).$$

Similarly,

$$(dy \wedge dz)(v, w) = \det \begin{bmatrix} v_2 & w_2 \\ v_3 & w_3 \end{bmatrix} = v_2 w_3 - v_3 w_2 = dy(v)dz(w) - dz(v)dy(w).$$

Note that only the second ( $y$ -) and third ( $z$ -) row of the matrix  $[v|w]$  were used to form the determinant. Similarly for  $dx \wedge dz$ .

#### 4. GEOMETRIC MEANING OF FORMS

You may say, okay, but what does it all mean? To explain that, let us first descend into two dimensional space and look at vectors  $v = (v_1, v_2)^T$  ( $\cdot^T$  stands for transpose – we like to think of vectors as columns) and  $w = (w_1, w_2)^T$ . Consider the oriented **parallelogram**  $P(v, w)$  determined by  $v$  and  $w$ , in that order. “Oriented” means that we care about the order of  $v$  and  $w$ . What is the signed area of  $P(v, w)$ ? (“Signed” just means that we think of the “area” of  $P(w, v)$  as being  $-P(v, w)$ .) The answer is

$$\text{signed area}(P(v, w)) = \det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} = v_1 w_2 - v_2 w_1.$$

I encourage you to check this formula!

In view of this, we now see that:

*$(dx \wedge dy)(v, w)$  is the (signed) area of the (oriented) parallelogram defined by the projection of the vectors  $v$  and  $w$  onto the  $xy$ -plane.*

Similarly,  $dx(v)$  is just the (signed) length of the projection of  $v$  to the  $x$ -axis, and so on.

#### 5. DIFFERENTIATION OF FORMS

Differential forms can be differentiated. Differentiation of a  $k$ -form produces a  $(k+1)$ -form. All you have to know to differentiate a form on  $\mathbb{R}^3$  are the following rules:

- (a) if  $f$  is a 0-form, i.e., a (smooth) function, its differential is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

- (b) if  $\omega = f dx$  or  $\omega = g dy$  or  $\omega = h dz$  is a 1-form, then

$$d(f dx) = df \wedge dx, \quad d(g dy) = dg \wedge dy, \quad d(h dz) = dh \wedge dz.$$

- (c) if  $\Phi = A dx \wedge dy$  or  $B dy \wedge dz$ , or  $C dz \wedge dx$  is a 2-form, then

$$d(A dx \wedge dy) = dA \wedge dx \wedge dy, \quad d(B dy \wedge dz) = dB \wedge dy \wedge dz, \quad \text{etc.}$$

- (d) differentiation is linear, i.e.,  $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$  and  $d(c\omega) = cd\omega$ , for any constant  $c$ .

- (e) For any form  $\omega$ ,  $d(d\omega) = 0$ .

You may ask: does this have any relation to the operation of differentiation of functions as we know it? The answer is yes, if you look at it the right way. To see what I mean, let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a (smooth) function. We know that

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Let us think of  $h$  as a 1-dimensional vector and the interval  $[x, x+h]$  as the 1-dimensional “parallelogram” (i.e. segment)  $P_x(h)$  determined by the vector  $h$  anchored at the point  $x$ . We orient  $P_x(h) = [x, x+h]$  as usual from left to right. The boundary  $\partial P_x(h)$  is just the set  $\{x, x+h\}$ . We can think of  $f(x+h) - f(x)$  as the integral of  $f$  over  $\partial P_x(h)$ , taking into account the orientation of  $P_x(h)$ . Then we can think of the derivative of  $f$  as

$$f'(x) = \lim_{h \rightarrow 0} \frac{1}{h} \int_{\partial P_x(h)} f.$$

In analogy with this, we can define the differential of a 1-form  $\omega$  in the following way: at the point  $p = (x, y, z) \in \mathbb{R}^3$ ,  $d\omega$  is the 2-form which to the pair of vectors  $v, w \in \mathbb{R}^3$  assigns the following number:

$$d\omega_p(v, w) = \lim_{h \rightarrow 0} \frac{1}{h^2} \int_{\partial P_p(v, w)} \omega,$$

where  $P_p(v, w)$  is the parallelogram defined by  $v$  and  $w$  anchored at the point  $p$ . Note that the boundary  $\partial P_p(v, w)$  of this parallelogram consists of four line segments. So now the question is: what is the meaning of the integral of  $\omega$  over  $\partial P_p(v, w)$ ? The answer is not that hard and is given in the next section.

## 6. INTEGRATION OF FORMS

A  $k$ -form can be integrated over a (piecewise smooth)  $k$ -dimensional object. For instance, 1-forms are integrated over curves, 2-forms over surfaces, and 3-forms over 3-dimensional solids.

We will only define the integral of a 1-form  $\omega$  over a curve  $C$ . If  $C$  is parametrized by  $\gamma : [a, b] \rightarrow \mathbb{R}^3$ , i.e.,  $C = \{\gamma(t) : a \leq t \leq b\}$ , then

$$\int_C \omega = \int_a^b \omega(\dot{\gamma}(t)) dt.$$

Observe that  $\omega(\dot{\gamma}(t))$  is just a scalar function of  $t$  and the integral on the right-hand side is just the ordinary Riemann integral. It can be shown that  $\int_C \omega$  does not depend on the choice of the parametrization  $\gamma$  of  $C$ .

Integration of 2- and 3-forms is defined analogously, but is a little more cumbersome to write, so we won't.

## 7. GRAD, CURL, DIV IN THE LANGUAGE OF FORMS

Let  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a vector field with components  $F_1, F_2, F_3$ . Thinking of  $F$  as a force field, we can associate to it the **work form**,  $\omega_F$ , defined by

$$\omega_F(v) = F \cdot v.$$

Note that  $\omega_F$  is a 1-form. It is not hard to see that

$$\omega_F = F_1 dx + F_2 dy + F_3 dz.$$

We can also associate to  $F$  a 2-form, called the **flux form**  $\Phi_F$ , in the following way:

$$\Phi_F(v, w) = \det [F|v|w],$$

i.e.,  $\Phi_F(v, w)$  is the determinant of the  $3 \times 3$  matrix whose columns are  $F, v$ , and  $w$ . It can be shown that, if  $F = (F_1, F_2, F_3)$ , then

$$\Phi_F = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy.$$

To a function  $f$ , we associate the **density form**, which is a 3-form defined by

$$\varrho_f = f dx \wedge dy \wedge dz.$$

If  $C$  is a curve and  $S$  is a surface, then the work of the force field along  $C$  is just  $\int_C \omega_F$ , whereas the flux of  $F$  over  $S$  is  $\int_S \Phi_F$ .

On the other hand, in vector calculus we define the gradient of a function  $f$  by  $\nabla f = (f_x, f_y, f_z)^T$ , the curl of a vector field  $F = (F_1, F_2, F_3)$  by

$$\text{curl } F = \nabla \times F,$$

and the divergence of  $F$  by

$$\operatorname{div} F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

How do these notions relate to the work form, flux form, and density form? Answer: if  $f$  is a function and  $F$  a vector field, both on  $\mathbb{R}^3$ , then

$$df = \omega_{\nabla f}, \quad d\omega_F = \Phi_{\operatorname{curl} F}, \quad d\Phi_F = \varrho_{\operatorname{div} F}.$$

This can be summarized as follows:

Vector calculus in $\mathbb{R}^3$		Forms in $\mathbb{R}^3$
functions	=	0-forms
$\downarrow \nabla$		$\downarrow d$
vector fields	$\xrightarrow{\text{work } \omega}$	1-forms
$\downarrow \operatorname{curl}$		$\downarrow d$
vector fields	$\xrightarrow{\text{flux } \Phi}$	2-forms
$\downarrow \operatorname{div}$		$\downarrow d$
functions	$\xrightarrow{\text{density } \varrho}$	3-forms

Note that the property  $d \circ d = 0$  on forms, is expressed by  $\operatorname{curl} \circ \nabla = 0$  and  $\operatorname{div} \circ \operatorname{curl} = 0$  in vector calculus.

## 8. GENERAL STORY

**Differential forms naturally “live” on smooth manifolds.** These are geometric objects which locally look like  $\mathbb{R}^n$ , for some  $n \geq 1$ . For instance,  $\mathbb{R}^3$ , circles, spheres, tori (= toruses), ellipsoids, the Möbius band, etc. are examples of smooth manifolds. If  $M$  is a smooth manifold of dimension  $n$ , then a differential  $k$ -form (where  $0 \leq k \leq n$ ) is an assignment  $\alpha$  to each point  $p \in M$  of a  $k$ -linear alternating map  $\alpha_p$  on the tangent space  $T_p M$  of  $M$  at  $p$ . To say that  $\alpha_p$  is  $k$ -linear means that

$$\alpha_p : \overbrace{T_p M \times \cdots \times T_p M}^{k \text{ times}} \rightarrow \mathbb{R}$$

and  $\alpha_p$  is linear in each variable (with all other variables fixed). By alternating we mean that whenever two variables are flipped, the sign of  $\alpha_p$  flips, e.g.,

$$\alpha_p(v_1, \dots, v_j, \dots, v_i, \dots, v_k) = -\alpha_p(v_1, \dots, v_i, \dots, v_j, \dots, v_k).$$

Differential forms on smooth manifolds can also be differentiated and integrated, and we have a general Stokes's theorem in the same form as in  $\mathbb{R}^3$ :

$$\int_{\partial M} \omega = \int_M d\omega,$$

for any  $k$ -form  $\omega$  and  $(k+1)$ -dimensional manifold  $M$ .

If you want to learn more about differential forms, I refer you to Hubbard & Hubbard [2] and Flanders [1]. There are of course many other excellent references.

## REFERENCES

1. Harley Flanders, *Differential forms with applications to the physical sciences*, Dover, 1989.
2. John Hubbard and Barbara Burke Hubbard, *Vector calculus, linear algebra, and differential forms: A unified approach*, Matrix Editions, 2009.