Uniform convergence and continuity, uniform convergence and integration, uniform convergence and differentiation

Theorem *Let $K \subset \mathbb{R}$ or $\mathbb{C}$ and $f_n : K \to \mathbb{R}$ (or $\mathbb{C}$) continuous functions which converge uniformly to $f : K \to \mathbb{R}$ (resp. $\mathbb{C}$). Then the function f is continuous.*

Proof. Let $x \in K, \varepsilon > 0$. By virtue of the uniform convergence of (f_n), there exists a sufficiently large $N \in \mathbb{N}$ so that for all $\xi \in K$ we have

$$|f_N(\xi) - f(\xi)| < \frac{\varepsilon}{3}.$$

Corresponding to x and ε we then determine a $\delta > 0$ so that

$$|f_N(y) - f_N(x)| < \frac{\varepsilon}{3} \quad \text{for all } y \in K \text{ with } |x - y| < \delta.$$

This is possible as the functions f_N are by assumption continuous. We then have for all $y \in K$ with $|x - y| < \delta$

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

whereby f is continuous at x and therefore also in K, as $x \in K$ was arbitrary. $\square$

Using the same method as in the proof of the above theorem, we have

Let E be a metric space.

7.12 Theorem *If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.*

7.13 Theorem *Suppose K is compact, and*

(a) *$\{f_n\}$ is a sequence of continuous functions on K,*
(b) *$\{f_n\}$ converges pointwise to a continuous function f on K,*
(c) *$f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.*

Then $f_n \to f$ uniformly on K.

Proof Put $g_n = f_n - f$. Then g_n is continuous, $g_n \to 0$ pointwise, and $g_n \geq g_{n+1}$. We have to prove that $g_n \to 0$ uniformly on K.

Let $\varepsilon > 0$ be given. Let K_n be the set of all $x \in K$ with $g_n(x) \geq \varepsilon$. Since g_n is continuous, K_n is closed (Theorem 4.8), hence compact (Theorem 2.35). Since $g_n \geq g_{n+1}$, we have $K_n \supset K_{n+1}$. Fix $x \in K$. Since $g_n(x) \to 0$, we see that $x \notin K_n$ if n is sufficiently large. Thus $x \notin \bigcap K_n$. In other words, $\bigcap K_n$ is empty. Hence K_N is empty for some N (Theorem 2.36). It follows that $0 \leq g_n(x) < \varepsilon$ for all $x \in K$ and for all $n \geq N$. This proves the theorem.

$\square$

Remark Let us note that compactness is really needed here. For instance, if

$$f_n(x) = \frac{1}{nx+1} \qquad (0 < x < 1;\ n = 1, 2, 3, \ldots)$$

then $f_n(x) \to 0$ monotonically in $(0, 1)$, but the convergence is not uniform.

7.14 Definition If X is a metric space, $\mathscr{C}(X)$ will denote the set of all complex-valued, continuous, bounded functions with domain X.

We associate with each $f \in \mathscr{C}(X)$ its *supremum norm*

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Since f is assumed to be bounded, $\|f\| < \infty$. It is obvious that $\|f\| = 0$ only if $f(x) = 0$ for every $x \in X$, that is, only if $f = 0$. If $h = f + g$, then

$$|h(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$$

for all $x \in X$; hence

$$\|f + g\| \leq \|f\| + \|g\|.$$

If we define the distance between $f \in \mathscr{C}(X)$ and $g \in \mathscr{C}(X)$ to be $\|f - g\|$, then $\mathscr{C}(X)$ is a metric space.

7.15 Theorem *The above metric makes $\mathscr{C}(X)$ into a complete metric space.*

Proof Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$. This means that to each $\varepsilon > 0$ corresponds an N such that $\|f_n - f_m\| < \varepsilon$ if $n \geq N$ and $m \geq N$. It follows (by Theorem 7.8) that there is a function f with domain X to which $\{f_n\}$ converges uniformly. By Theorem 7.12, f is continuous. Moreover, f is bounded, since there is an n such that $|f(x) - f_n(x)| < 1$ for all $x \in X$, and f_n is bounded.

Thus $f \in \mathscr{C}(X)$, and since $f_n \to f$ uniformly on X, we have $\|f - f_n\| \to 0$ as $n \to \infty$.

7.16 Theorem *Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \ldots$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and*

(23)
$$\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha.$$

Proof It suffices to prove this for real f_n. Put

(24)
$$\varepsilon_n = \sup |f_n(x) - f(x)|,$$

the supremum being taken over $a \le x \le b$. Then

$$f_n - \varepsilon_n \le f \le f_n + \varepsilon_n,$$

so that the upper and lower integrals of f (see Definition 6.2) satisfy

(25)
$$\int_a^b (f_n - \varepsilon_n)\,d\alpha \le \underline{\int} f\,d\alpha \le \overline{\int} f\,d\alpha \le \int_a^b (f_n + \varepsilon_n)\,d\alpha.$$

Hence

$$0 \le \overline{\int} f\,d\alpha - \underline{\int} f\,d\alpha \le 2\varepsilon_n[\alpha(b) - \alpha(a)].$$

Since $\varepsilon_n \to 0$ as $n \to \infty$ (Theorem 7.9), the upper and lower integrals of f are equal.

Thus $f \in \mathscr{R}(\alpha)$. Another application of (25) now yields

(26)
$$\left| \int_a^b f\,d\alpha - \int_a^b f_n\,d\alpha \right| \le \varepsilon_n[\alpha(b) - \alpha(a)].$$

Corollary If $f_n \in \mathscr{R}(\alpha)$ *on* $[a, b]$ *and if*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (a \le x \le b),$$

the series converging uniformly on $[a, b]$, *then*

$$\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\,d\alpha.$$

Theorem *Let $I = [a, b]$ be a bounded interval in $\mathbb{R}$. Let $f_n : I \to \mathbb{R}$ be differentiable functions. Assume that*

(i) *there exists a $z \in I$ for which $f_n(z)$ converges*

(ii) *the sequence of derivatives (f'_n) converges uniformly on I.*

Then the sequence (f_n) converges uniformly on I to a differentiable *function f and we have*

$$f'(x) = \lim_{n\to\infty} f'_n(x) \quad \textit{for all } x \in I.$$

Proof. Let $g(x)$ be the limit of $f'_n(x)$. Let $\eta > 0$. Because of uniform convergence of f'_n we can find an $N \in \mathbb{N}$ with the following property

$$\forall\ n, m \geq N : \sup\{|f'_n(x) - f'_m(x)| : x \in I\} < \eta \tag{1}$$
$$\text{and so } \sup\{|f'_n(x) - g(x)| : x \in I\} < \eta \quad \text{for } n \geq N \tag{2}$$

Furthermore, for all $x \in I, n, m \in \mathbb{N}$, we have, by the mean value theorem

$$|f_n(x) - f_m(x) - (f_n(z) - f_m(z))| \leq |x - z| \sup_{\xi \in I} |f'_n(\xi) - f'_m(\xi)|, \tag{3}$$

and therefore

$$|f_n(x) - f_m(x)| \leq |f_n(z) - f_m(z)| + |x - z| \sup_{\xi \in I} |f'_n(\xi) - f'_m(\xi)|,$$

wherefrom, on account of (i) and (1) it follows easily that (f_n) is a Cauchy sequence in $\mathscr{C}(I)$. Therefore, the sequence (f_n) converges to a continuous limit function f.

In particular (i), and thereby the above considerations, hold for every $z \in I$.

In (3) we let m tend to ∞ and obtain from (2)

$$|f_N(x) - f(x) - (f_N(z) - f(z))| \leq |x - z|\eta. \tag{4}$$

For N, which depends only on η, and x we find a $\delta > 0$ with

$$|f_N(x) - f_N(z) - (x - z)f'_N(x)| \leq \eta|x - z| \text{ for } |x - z| < \delta. \tag{5}$$

This follows from our characterization of differentiability.

It follows from (2), (4) and (5) that

$$|f(x) - f(z) - g(x)(x - z)| \leq 3\eta|x - z|, \text{ if } |x - z| < \delta.$$

Since this holds for every $x \in I$ and for all z with $|x - z| < \delta$, it follows from our characterization of differentiability, that $f'(x)$ exists and

$$f'(x) = g(x).$$

$\square$