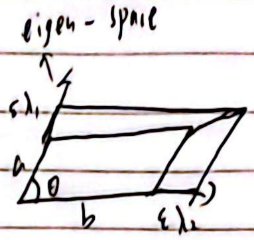
1^{st} $\operatorname{Tr} A$ = divergence.

set vector field $f: \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = Ax = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$

$$A = \nabla f(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}, \quad \operatorname{Tr}(A) = \operatorname{div}(f)$$

Rk: One can see $\text{Divergence} = \dfrac{\text{rate of }\overset{\text{area}}{\vee}\text{change}}{\text{area}}$



2^{nd} (1) $\operatorname{Tr}(A) = \sum_i \lambda_i(A)$: If:

new area: $a(1+\varepsilon\lambda_1) \cdot b(1+\varepsilon\lambda_2)\sin\theta$

$$\text{divergence} = \lim_{\text{area}\to 0} \frac{\text{rate of area change}}{\text{area}} = \frac{ab\sin\theta\,(1+\varepsilon\lambda_1)(1+\varepsilon\lambda_2) - ab\sin\theta}{ab\sin\theta}$$

$$= (1+\varepsilon\lambda_1)(1+\varepsilon\lambda_2) - 1 = \lambda_1 + \lambda_2 + \varepsilon^2\lambda_1\lambda_2 \overset{\varepsilon\to 0}{=} \lambda_1 + \lambda_2$$

(2) $\det(e^{tA}) = e^{t\operatorname{Tr}(A)}$, set $\det(e^{tA}) = S(t)$ be area at time t

$$\operatorname{Tr}(A) = \frac{S'(t)}{S(t)} \Rightarrow c e^{\operatorname{Tr}(A)t} = S(t), \quad t=0,\ c = S(0) = 1$$

$$\Rightarrow \det(e^{tA}) = S(t) = e^{\operatorname{Tr}(A)t}$$

(3) $\operatorname{Tr}(A) = \operatorname{Tr}(P^{-1}AP)$: For vector field: $\vec{x} \to \vec{x} + A\vec{x}$

apply U: $U\vec{x} \to U\vec{x} + UA\vec{x}$

$$\operatorname{Tr}(UAU^{-1}) = \frac{\text{rate Area}\ \det(U)}{\text{Area}\ \det(U)} = \operatorname{Tr}(A) \qquad \vec{y} = U\vec{x} \quad \vec{y} \to \vec{y} + UAU^{-1}\vec{y}$$

(4) $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$: set $B \in GL_n(\mathbb{R})$, $\operatorname{Tr}(AB) = \operatorname{Tr}(B^{-1}(BA)B) = \operatorname{Tr}(BA)$

(5) for $A \in GL_n(\mathbb{R})$, $\dfrac{d\,\det(A(t))}{dt} = \det(A(t))\,\mathrm{Tr}\left[A^{-1}(t)\dfrac{dA(t)}{dt}\right] \cdots ①$

If A may not invertible: $\dfrac{d\,\det(A(t))}{dt} = \mathrm{Tr}\left[\mathrm{adj}(A(t))\dfrac{dA(t)}{dt}\right]$

For ①: let $s(t) = \det(A(t))$, for vector field generated by M:

$\mathrm{Tr}\,M = \dfrac{s'(t)}{s(t)}$. WTF. M: For $A(t)$ acts on basis vector

$(1,0)^T, (0,1)^T, \longrightarrow A(t)(1,0)^T, A(t)(0,1)^T \xrightarrow{M} A'(t)(1,0)^T, A'(t)(0,1)^T$

Since $A(t) \in GL_n(\mathbb{R}) \Rightarrow A'(t)A^{-1}A(1,0)^T = A'(t)(1,0)^T$?

$\Rightarrow \mathrm{Tr}(A^{-1}(t)A'(t)) \cdot \det(A(t)) = \dfrac{d\,\det(A(t))}{dt}$