矩阵作为表示的便利，
也有线性变换的证明。
来源是微分算子和二项式定理。

$f(x) = \frac{1}{2}x^T A x$, $\nabla^2 f(x) = ?$

$f: V \to W$. V, W are vector-space

$f(x+\Delta x) = f(x) + Df(x)\cdot \Delta x + O(\Delta x^2)$

Where $Df: V \to \mathcal{L}(V, W)$: linear mapping set from $V \to W$.

$Df(x)\cdot \Delta x \in W$

Remark: $Df(x) \in \mathcal{L}(V,W) \cong M_{V\to W}$

$D f_i \circ f_j \longleftrightarrow A_i A_j$

For $f(x) = \frac{1}{2}x^T A x$

$f(x+\Delta x) = \cancel{\frac{1}{2}x^T A x +} \frac{1}{2}(x+\Delta x)^T A (x+\Delta x)$

$= \frac{1}{2}x^T A x + \frac{1}{2}\Delta x^T A x + \frac{1}{2}x^T A \Delta x + \frac{1}{2}\Delta x^T A \Delta x$

By Talor expension: $f(x+\Delta x) = f(x) + Df(x)\Delta x + O(\Delta x^2)$

Then $Df(x)\cdot \Delta x = \frac{1}{2}\Delta x^T A x + \frac{1}{2}x^T A \Delta x$

Since $\Delta x^T A x \in \mathbb{R}$

$= \frac{1}{2}x^T A^T \Delta x + \frac{1}{2}x^T A \Delta x$

$Df(x) = \frac{1}{2}x^T A^T + \frac{1}{2}x^T A$

$Df: \mathbb{R}^{1\times n} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, $Df(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^{n\times 1} \to \nabla f(x)$

$Df(x)\cdot \Delta x \longleftrightarrow (\nabla f(x))^T \Delta x$

$\nabla f(x) = (\frac{1}{2}A + A^T) x$

$\nabla f = \frac{1}{2}(A + A^T)$. When A is symmetry , $\nabla f = A$

more precisely

For $f(x+\Delta x) = f(x) + Df(x)\Delta x + D^2 f(x)\cdot(\Delta x, \Delta x) + o(\Delta x^3)$

$D^2 f: \mathbb{R}^{1\times n} \to \mathcal{L}(V \to \mathcal{L}(V, W)) \cong B\mathcal{L}(V, W)$

$D^2 f(x) \in B\mathcal{L}(V, W)$

Then $\cancel{\frac{1}{2}\Delta x^T f(x) \Delta x}$ $\frac{1}{2}\Delta x^T A \Delta x = D^2 f(x)\cdot(\Delta x, \Delta x)$

Since $D^2 f(x) \in \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R})) \cong \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \cong M_{n\times n}(\mathbb{R})$

$\Delta x^T M \Delta x = \frac{1}{2}\Delta x^T A \Delta x$

$M = \frac{A}{2}$