SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Course Name: Probability **Dept.:** Department of Mathematics

Exam Duration: 180 mins **Venue:** Depart. of Math.

Question No.	1	2	3	4	5	6	7
Score	16 marks	16 marks	12 marks	16 marks	12 marks	12 marks	16 marks

This exam paper contains **seven** questions and the score is **100** in total. (Please hand in your exam paper, answer sheet, and your scrap paper to the proctor when the exam ends.)

Notations:

- $\mathbb{R}$: the real number set, i.e., $\mathbb{R} = (-\infty, \infty)$.
- $\mathcal{B}$: the Borel σ-algebra of $\mathbb{R}$.
- $\mathrm{Re}\,(\phi_X(t))$: real part of the characteristic function $\phi_X(t)$ of random variable X.
- a.e.: almost everywhere.
- a.s.: almost surely.
- i.i.d.: independent and identically distributed.
- $Z \sim N(0,1)$: random variable Z has standard normal distribution.

ANSWER THE FOLLOWING SEVEN QUESTIONS:

1. Answer the following two questions:

(a) Let f be a measurable function on a measure space $(\Omega, \mathcal{A}, \mu)$. Suppose $\int_\Omega f \,\mathrm{d}\mu$ exists and $A_1, A_2, \ldots$ form a partition of Ω. Show that

$$\int_\Omega f \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{A_n} f \,\mathrm{d}\mu.$$

(b) Assume that X is a random variable defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that

$$\mathbb{E}|X| < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \mathbb{P}(|X| > n) < \infty.$$

2. Answer the following two questions:

(a) Let f be a measurable function on a measure space $(\Omega, \mathcal{A}, \mu)$. Show that if f is integrable, then $\mu(A_n) \to 0$ as $n \to \infty$ implies

$$\int_{A_n} |f| \mathrm{d}\mu \to 0.$$

(b) Consider the integral

$$I_n = \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-3x} \, \mathrm{d}x.$$

(i) Guess the value of $\lim_{n\to\infty} I_n$.

(ii) Prove that your guess is correct.

4.16 设 $\alpha > 0$, 求极限 $\lim\limits_{n\to\infty} (\mathrm{R}) \int_0^n \left(1+\frac{x}{n}\right)^n e^{-(\alpha+1)x} dx$. 令 $f_n(x) = \left(1+\frac{x}{n}\right)^n e^{-(\alpha+1)x} \varphi_{[0,n]}(x)$, 则 $\lim\limits_{n\to\infty} f_n(x) = e^{-\alpha x}$.

3. Let X_1 and X_2 be two independent $\mathbb{R}$-valued random variables with distributions F_1 and F_2, respectively. Show that the distribution function of the sum $X_1 + X_2$ is given by

$$\mathbb{P}(X_1 + X_2 \le x) = \int_{\mathbb{R}} F_1(x - x_2) \, \mathrm{d}F_2(x_2).$$

4. Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let X be a $\mathbb{R}$-valued random variable, and for $t \in \mathbb{R}$, define the characteristic function of X by

$$\phi_X(t) := \int_{\mathbb{R}} e^{itx} \mathbb{P}^X(dx) = \mathbb{E}\left[e^{itX}\right] = \mathbb{E}\left[\cos(tX) + i\sin(tX)\right].$$

where $\mathbb{P}^X$ is the distribution measure of X.

(a) Show that, for any $t \in \mathbb{R}$,

$$1 - \mathrm{Re}\left(\phi_X(2t)\right) \le 4\left[1 - \mathrm{Re}\left(\phi_X(t)\right)\right].$$

(b) Let $\{X_j\}_{j\ge1}$ be a sequence of independent and identically distributed random variables with $\mathbb{E}(X_j) = \mu$ and $\mathrm{Var}(X_j) = \sigma^2 \in (0, \infty)$. Let

$$S_n = \sum_{j=1}^n X_j \quad \text{and} \quad Y_n = \frac{S_n - n\mu}{\sqrt{n}\sigma}.$$

Use characteristic function to show that Y_n converges in distribution to Z, where $Z \sim N(0, 1)$.

5. Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, assume that $\{X_n\}_{n\ge1}$ is a sequence of random variables, and X is a random variable, all of which are from $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$. If $X_n \xrightarrow{P} X$, show that there exists a subsequence $\{n_k\}_{k\ge1}$ such that $X_{n_k} \xrightarrow{\text{a.s.}} X$.

6. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, and let f_0 and f_1 be two probability density functions. Assume $f_0(x) > 0$ for all x and X_k has f_0 as its density function. Define

$$\xi_n = \frac{f_1(X_1) \cdots f_1(X_n)}{f_0(X_1) \cdots f_0(X_n)}, \quad n = 1, 2, \ldots$$

Show that $\{\xi_n\}_{n \geq 1}$ is a martingale with respect to $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$.

7. Answer the following two questions:

(i) State the Doob's Optional Sampling Theorem.

(ii) Let $\{X_n\}_{n \geq 0}$ be an integrable and adapted process. Show that $\{X_n\}_{n \geq 0}$ is a martingale if and only if $\mathbb{E}[X_S] = \mathbb{E}[X_T]$ for all stopping times S and T that take at most two real-valued values.

1. Solution

(a) $\int_\Omega f\,d\mu = \int_{\bigcup_{n=1}^{\infty} A_n} f\,d\mu = \int_\Omega \lim_{m\to\infty}\sum_{n=1}^{m} f I_{A_n}\,d\mu \overset{MCT}{=} \lim_{n\to\infty}\int_\Omega \sum_{n=1}^{m} f I_{A_n}\,d\mu = \lim_{m\to\infty}\sum_{n=1}^{m}\int_{A_n} f\,d\mu =$

(b) "$\Rightarrow$" $\sum_{n=1}^{\infty} P\{|X|>n\} = \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} P\{|X|=m\} = \sum_{n=2}^{\infty} n P\{|X|=n\} \le \sum_{n=1}^{\infty} n P\{|X|=n\} = E|X| < \infty$

"$\Leftarrow$" $E|X| = \sum_{n=1}^{\infty} n P\{|X|=n\} = \sum_{n=2}^{\infty} n P\{|X|=n\} + P\{|X|=1\} \le \sum_{n=1}^{\infty} P\{|X|>n\} + 1 < \infty$

GPT:

(a) 由于 $A_1, A_2, \cdots$ 构成 Ω 的一个分割，我们有 $A_i \cap A_j = \phi$ 对 $i \ne j$，且 $\bigcup_{n=1}^{\infty} A_n = \Omega$，于是

$\int_\Omega f\,d\mu = \int_{\bigcup_{n=1}^{\infty} A_n} f\,d\mu = \cdots\cdots$

(b) "$\Rightarrow$" $E[|X|] = \int_0^\infty P\{|X|>x\}\,dx$

$\sum_{n=1}^{\infty} P\{|X|>n\} \le \sum_{n=1}^{\infty}\int_n^{n+1} P\{|X|>x\}\,dx$ [$P\{|X|>x\}$ 是一个减函数 $\therefore P\{|X|>n\} \ge \int_n^{n+1} P\{|X|>x\}\,dx$]

$\therefore \sum_{n=1}^{\infty} P\{|X|>n\} \le \int_1^\infty P\{|X|>x\}\,dx \le \int_0^\infty P\{|X|>x\}\,dx = E[|X|] < \infty$

"$\Leftarrow$" $E[|X|] = \int_0^1 P\{|X|>x\}\,dx + \int_1^\infty P\{|X|>x\}\,dx = \int_0^1 P\{|X|>x\}\,dx + \sum_{n=1}^{\infty}\int_n^{n+1} P\{|X|>x\}\,dx$

$\le \int_0^1 P\{|X|>x\}\,dx + \sum_{n=1}^{\infty}\int_n^{n+1} P\{|X|>n\}\,dx = \int_0^1 P\{|X|>x\}\,dx + \sum_{n=1}^{\infty} P\{|X|>n\} < \infty$

2. Solution

(a) For a sequence of numbers $\{a_n\}_{n\ge1}$ increasing to ∞, we decompose

$\mu|fI_{A_n}| = \mu(|f| I_{A_n\cap\{|f|\le a_n\}}) + \mu(|f| I_{A_n\cap\{|f|>a_n\}}) \le a_n\mu(A_n) + \mu(|f| I_{\{|f|>a_n\}})$

$\because f$ is integrable, $|f I_{\{|f|>a_n\}}| \le |f|$, By DCT $\mu(|f| I_{\{|f|>a_n\}}) \to \mu(|f| I_{\{|f|=\infty\}}) = 0$

We set $a_n = (\mu(A_n))^{-1/2}$. $\therefore \mu(|fI_{A_n}|) \le (\mu(A_n))^{1/2} + \mu(|f| I_{\{|f|>a_n\}}) \to 0$

(b) (i) $I_n = \frac{1}{2}$

(ii) Let $f_n = I_{[0,n]}(1+\frac{x}{n})^n e^{-3x}$, then $f_n \uparrow f = I_{(0,\infty)} e^{-2x}$ $((1+\frac{x}{n})^n \uparrow)$

$\lim_{n\to\infty}\int_0^n (1+\frac{x}{n})^n e^{-3x}\,dx = \lim_{n\to\infty}\int_0^\infty f_n\,dx = \int_0^\infty \lim_{n\to\infty} f_n\,dx = \int_0^\infty f(x)\,dx = \frac{1}{2}$

3. Solution

$P\{X_1+X_2\le x\} = \int_R P\{X_1 \le x-s\}\,dF(s) = \int_R F(x-s)\,dF(s)$

4. Solution

(a) $1-\mathrm{Re}(\phi_X(2t)) = \int_{\mathbb{R}} (1-\cos(2tx))\, p^X(dx) = \int_{\mathbb{R}} 2(1-\cos^2 tx)\, p^X(dx) = \int 2(1+\cos tx)(1-\cos tx)\, p^X(dx)$

$\leq \int 4(1-\cos tx)\, p^X(dx) = 4[1-\mathrm{Re}(\phi_X(t))]$

(b) $Y_n = \dfrac{S_n - n\mu}{\sqrt{n}\sigma} = \dfrac{1}{\sqrt{n}} \sum_{i=1}^{n} \dfrac{X_i-\mu}{\sigma}$ $\quad M_i = \dfrac{X_i-\mu}{\sigma}$, $E(M_i)=0$, $Var(M_i)=1$.

$\therefore \varphi_{Y_n}(t) = \varphi_{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} M_i}(t) = \varphi_{M_1}^{n}\left(\frac{t}{\sqrt{n}}\right) = \left[\varphi_{M_1}(0) + \frac{t}{\sqrt{n}}\varphi_{M_1}'(0) + \frac{t^2}{2n}\varphi_{M_1}''(0) + o\left(\frac{1}{n}\right)\right]^n$

$= \left[1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right]^n$

$\therefore \lim_{n\to\infty} \varphi_{Y_n}(t) = \lim_{n\to\infty} \left[1-\frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right]^n = \exp\left\{\lim_{n\to\infty} n \ln\left(1-\frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)\right\} = \exp\left\{-\frac{t^2}{2}\right\} = \varphi_Z(t)$

$\therefore Y_n \xrightarrow{D} Z$

5. Solution.

$\because X_n \xrightarrow{P} X$ $\quad \therefore \lim_{n\to\infty} E\left[\dfrac{|X_n - X|}{1+|X_n-X|}\right] = 0$ $\quad \therefore \exists \{n_k\}$, s.t.

$E\left[\dfrac{|X_{n_k}-X|}{1+|X_{n_k}-X|}\right] < \dfrac{1}{2^k}$. By MCT

$E\left[\sum_{k=1}^{\infty} \dfrac{|X_{n_k}-X|}{1+|X_{n_k}-X|}\right] = \sum_{k=1}^{\infty} E\left[\dfrac{|X_{n_k}-X|}{1+|X_{n_k}-X|}\right] = \sum_{k=1}^{\infty} \dfrac{1}{2^k} = 1$

$\therefore \sum_{k=1}^{\infty} \dfrac{|X_{n_k}-X|}{1+|X_{n_k}-X|} < \infty$ a.s. $\quad \therefore |X_{n_k}-X| = 0$ a.s. as $k\to\infty$

$\therefore X_{n_k} \xrightarrow{a.s.} X$

GPT:

$X_n \xrightarrow{P} X$, $\therefore$ For any $\varepsilon > 0$, $\lim_{n\to\infty} P\{|X_n - X| \geq \varepsilon\} = 0$, $\therefore \exists \{n_k\}$, $P\{|X_{n_k}-X| \geq \varepsilon\} < \frac{1}{2^k}$

$\therefore \sum_{k=1}^{\infty} P\{|X_{n_k} - X| \geq \varepsilon\} < \infty$ $\quad \therefore P\{|X_{n_k} - X| \geq \varepsilon, \text{ i.o.}\} = 0$ [Borel-Cantelli Lemma]

$\therefore P\{\exists N, k>N, |X_{n_k} - X| < \varepsilon\} = 1$, i.e. $P\{\lim_{k\to\infty} X_{n_k} = X\} = 1$

6. Solution

(i) For any n, $E[|\xi_n|] = E[\xi_n] = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \dfrac{f_1(x_1)\cdots f_1(x_n)}{f_0(x_1)\cdots f_0(x_n)} f_0(x_1)\cdots f_0(x_n)\, dx_1 \cdots dx_n$

$= \int_{\mathbb{R}} f_1(x_1)\, dx_1 \cdots \int_{\mathbb{R}} f_1(x_n)\, dx_n = 1 < \infty$

(ii) For any n, ξ_n is $\mathcal{F}_n$-measurable.

(iii) For any n, $E[\xi_n \mid \mathcal{F}_{n-1}] = E\left[\frac{f_1(X_1)\cdots f_1(X_{n-1})}{f_0(X_1)\cdots f_0(X_{n-1})} \frac{f_1(X_n)}{f_0(X_n)} \,\Big|\, \mathcal{F}_{n-1}\right] = \xi_{n-1} E\left[\frac{f_1(X_n)}{f_0(X_n)}\right]$

$$= \xi_{n-1} \int_{\mathbb{R}} \frac{f_1(X_n)}{f_0(X_n)} f_0(X_n)\, dX_n = \xi_{n-1} \int_{\mathbb{R}} f_1(X_n)\, dX_n = \xi_{n-1}$$

$\therefore \{\xi_n\}_{n\geq 1}$ is a martingale with respect to $\mathcal{F}_n = \sigma\{X_1, \cdots, X_n\}$

7. Solution

(i) Let $(X_n)_{n\geq 0}$ be a martingale (resp. supermartingale) and let S, T be stopping times bounded by a constant c with $S \leq T$ a.s. Then

$$E[X_T \mid \mathcal{F}_S] = X_S \quad (\text{resp. } \leq) \quad \text{a.s.}$$

proof (Martingale)

Proof. First note that

$$E|X_T| = E\left\{\left|\sum_{n=0}^{c} X_n I_{\{T=n\}}\right|\right\}$$
$$\leq \sum_{n=0}^{c} E\left|X_n I_{\{T=n\}}\right| \leq \sum_{n=0}^{c} E|X_n| < \infty$$

Similarly, $E|X_S| < \infty$. Hence X_S and X_T are integrable.

Second, X_S is $\mathcal{F}_S$-measurable, and $E(X_T|\mathcal{F}_S)$ is also $\mathcal{F}_S$-measurable. It remains to prove that for all $A \in \mathcal{F}_S$, we have

$$\int_A X_T dP = \int_A X_S dP$$

i.e.

$$E(X_T I_A) = E(X_S I_A) \qquad (8.3)$$

To prove (8.3), define a r.v.

$$R(\omega) = S(\omega) I_A(\omega) + T(\omega) I_{A^c}(\omega)$$

$R(\omega)$ is a stopping time. Indeed,

$$\{R \leq n\} = (A \cap \{S \leq n\}) \cup (A^c \cap \{T \leq n\})$$

Because $A \in \mathcal{F}_S$, then $A \cap \{S \leq n\} \in \mathcal{F}_n$. Also $A^c \in \mathcal{F}_S \subset \mathcal{F}_T$, then $A^c \cap \{T \leq$

$n\} \in \mathcal{F}_n$. Thus $\{R \leq n\} \in \mathcal{F}_n$. Thus R is a stopping time.

Now $R \leq T \leq c$, then

$$E(X_R) = E(X_S I_A + X_T I_{A^c}) \qquad (8.4)$$
$$E(X_T) = E(X_T I_A + X_T I_{A^c}) \qquad (8.5)$$

By Theorem 8.2.2, we have

$$E(X_R) = E(X_T) = E(X_0)$$

Then (8.4) and (8.5) yield that

$$E(X_S I_A) - E(X_T I_A) = 0$$

By the definition of conditional expectation, (8.2) holds. ■

proof (supermartingale) Q E 12

(ii) "$\Rightarrow$" By (i)

① $S = a$, $T = b$, $a < b$, then $E[X_T \mid \mathcal{F}_S] = X_S$ a.s.

$\therefore E[E(X_T \mid \mathcal{F}_S)] = E[X_T] = E[X_S]$

② $a > b$ ......

"$\Leftarrow$" $E[X_n|\mathcal{F}_m]$, X_m is $\mathcal{F}_m$ .... $(n>m)$, $R(w) = \begin{cases} n , & w \notin A \\ m , & w \in A \end{cases}$, $\forall A \in \mathcal{F}_m$

R is a stopping time . $E[X_R] = E[I_{A^c}X_n + I_A X_m] = E[X_n]$ $(S=n , T=R)$

$\therefore$ $E[I_A X_m] = E[I_A X_n] \Longleftrightarrow \int_A X_m \, dp = \int_A E(X_n|\mathcal{F}_m) \, dp$ $\therefore$ $X_m = E(X_n|\mathcal{F}_m)$ a.s.

$\therefore$ $\{X_n\}_{n\geq 0}$ is a martingale .