(ii) F inte

let $S_k = \sum_{n=1}^{k} a_n$, $S = \sum_{n=1}^{\infty} a_n$

$$\sum_{n=1}^{\infty} r^n a_n = \lim_{n\to\infty} \sum_{k=1}^{n} r^k a_k = \lim_{n\to\infty} \sum_{k=1}^{n} r^k (S_k - S_{k-1})$$

$$= \lim_{n\to\infty}\left[r^n S_n + \sum_{k=1}^{n-1} (r^k S_k - r^{k+1} S_k) \right] = \lim_{n\to\infty}\left[r^n S_n + (1-r)\sum_{k=1}^{n-1} r^k S_k \right]$$

$$= (1-r)\sum_{k=1}^{\infty} r^k S_k \quad \cdots\cdots \text{①}$$

$\forall \varepsilon > 0$, take $N \in \mathbb{N}$, s.t. $|S - S_N| < \frac{\varepsilon}{2}$

$\Rightarrow \exists \delta > 0$ s.t. $0 < 1-r < \delta$, $\left|(1-r)\sum_{k=1}^{N} r^k (S_k - S)\right| < \frac{\varepsilon}{2}$ ... ②

$$\sum_{k=1}^{\infty} r^k = \frac{1}{1-r} \quad \cdots\cdots \text{③}$$

$$\left|\sum_{n=1}^{\infty} r^n a_n - \sum_{n=1}^{\infty} a_n\right| \overset{\text{①}}{=} \left|(1-r)\sum_{k=1}^{\infty} r^k S_k - S\right|$$

$$\overset{\text{③}}{=} \left|(1-r)\sum_{k=1}^{\infty} r^k S_k - (1-r)\sum_{k=1}^{\infty} r^k S\right|$$

$$= \left|(1-r)\sum_{k=1}^{\infty} r^k (S_k - S)\right| \le \left|(1-r)\sum_{k=1}^{N} r^k (S - S_k)\right| + \left|(1-r)\sum_{k=N+1}^{\infty} r^k (S - S_N)\right|$$

$$\overset{\text{②}}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

(b) $\mathrm{Im}(f)$ is constant;

(c) $|f|$ is constant;

one can conclude that f is constant.

14. Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0 = 0$. Prove the **summation by parts** formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

$k \to \infty$

15. Abel's theorem. Suppose $\sum_{n=1}^\infty a_n$ converges. Prove that

$|S_k - S| < \varepsilon$

$$\lim_{r \to 1,\ r<1} \sum_{n=1}^{\infty} r^n a_n = \sum_{n=1}^{\infty} a_n.$$

[Hint: Sum by parts.] In other words, if a series converges, then it is Abel summable with the same limit. For the precise definition of these terms, and more information on summability methods, we refer the reader to Book I, Chapter 2.

16. Determine the radius of convergence of the series $\sum_{n=1}^\infty a_n z^n$ when:

(a) $a_n = (\log n)^2$

(b) $a_n = n!$

(c) $a_n = \frac{n^2}{4^n + 3n}$