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How to find periodic function

Consider $g(x) = \sum_{n=1}^{+\infty} x-n + x + \sum_{n=1}^{+\infty} x+n$, $g(x+1) = g(x)$

for $f(x) = \sum_{n=1}^{+\infty} \frac{1}{x-n} + \frac{1}{x} + \sum_{n=1}^{+\infty} \frac{1}{x+n}$, $f(x+1) = f(x)$

1° $f(x)$ continuous on $x \notin \mathbb{Z}$, since $2n-1 > x^2$, $\frac{1}{n^2-x^2} < \frac{1}{(n-1)^2} \Rightarrow f(x)$ uniformly converge.

2° Herglotz trick: set $f_N(x) = \sum_{n=-N}^{N} \frac{1}{x+n}$, $f(x) \leftarrow f_N$

find $f_N(\frac{x}{2}) + f_N(\frac{x+1}{2}) = 2f_{2N}(x) + \frac{2}{x+2N+1}$

$\Rightarrow f(\frac{x}{2}) + f(\frac{x+1}{2}) = 2f(x)$ $\Rightarrow$ since for $h(x) = \pi\cot(\pi x)$,
$h(\frac{x}{2}) + h(\frac{x+1}{2}) = 2h(x)$

WTS: $f = h$: set $m(x) \overset{\Delta}{=} f(x) - h(x)$, 1° $m(x+1) = m(x)$

2° $\lim_{x\to 0} m(x) = \lim_{x\to 0} \frac{1}{x} - \pi\cot(\pi x) \underset{\text{Talor}}{=} 0 \overset{\text{by } 1°}{\Rightarrow} m(n) = 0$ for $n \in \mathbb{Z}$

3° let $x_m = \underset{x\in[0,1]}{\arg\max}\, m(x) \Rightarrow m(\frac{x_m}{2}) + m(\frac{x_m+1}{2}) = 2m(x_m)$

$\Rightarrow m(\frac{x_m}{2}) = m(\frac{x_m+1}{2}) = m(x_m) \Rightarrow m(x_m) = m(\frac{x_m}{2}) = \cdots = \lim_{n\to+\infty} m(\frac{x_m}{2^n}) = m(0) = 0$

$\Rightarrow \max_x m(x) = 0 \Rightarrow m(x) = 0$

$\Rightarrow \boxed{\pi\cot\pi x = \frac{1}{x} + \sum_{n=1}^{+\infty} \frac{1}{x+n} + \frac{1}{x-n}}$ $\ln\frac{1}{n}$, integral on x for both sides.

$\ln\sin\pi x = \ln x + \sum_{n=1}^{+\infty} \ln(\frac{x^2}{n^2} - 1) + C$

$\Rightarrow \sin\pi x = \pi x \prod_{n=1}^{+\infty} (1 - \frac{x^2}{n^2})$, $(\forall$ for C: $\frac{\sin\pi x}{kx} = \prod_{n=1}^{+\infty} (1 - \frac{x^2}{n^2}) \Rightarrow k = \pi$

So $\sin\pi x$ is a periodic function with $T=1$

$\Rightarrow \cot\pi x$ is $T=1$ $\Rightarrow$ By Fourier Series, we can find periodic function.

RK1: $x\cot x = 1 - 2\sum_{n=1}^{+\infty} \underbrace{\frac{x^2}{\pi^2 n^2 - x^2}}_{\frac{x^2}{\pi^2 n^2} \cdot \frac{1}{1-(\frac{x}{\pi n})^2}} = 1 - 2\sum_{n=1}^{+\infty}\sum_{k=1}^{+\infty} (\frac{x}{\pi n})^{2k}$

$= 1 - 2\sum_{k=1}^{+\infty} (\frac{1}{\pi^{2k}} \sum_{n=1}^{+\infty} \frac{1}{n^{2k}}) x^{2k}$

By Talor's Series of $x\cot x$ can we find $\sum \frac{1}{n^{2k}}$, eg:

$x\cot x = 1 - \frac{x^2}{3} - \frac{1}{45}x^4 - \frac{2}{945}x^6 \cdots$ $0 < |x| < \pi$. $\Rightarrow -\frac{1}{3} = -\frac{2}{\pi^2} \sum_{n=1}^{+\infty} \frac{1}{n^2}$

$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$\Rightarrow$

$-\frac{1}{45} = -\frac{2}{\pi^4} \sum_{n=1}^{+\infty} \frac{1}{n^4} \Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

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RK2: Jacobi's Theorem: A complex function has 2 T_i at most, $\frac{T_1}{T_2} \notin \mathbb{R}$.

$$p(x) = \frac{1}{x^2} + \sum_{m,m'} \left[\frac{1}{(x-2m\omega-2m'\omega')^2} - \frac{1}{(2m\omega+2m'\omega')^2} \right]$$

$$\begin{cases} p(x+2\omega) = p(x) \\ p(x+2\omega') = p(x) \end{cases}$$