$x^3+1=(x+1)(x+w)(x+w^2)$ where $w=e^{\frac{2\pi}{3}i}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$\Rightarrow f^3+g^3=(f+g)(f+wg)(f+w^2g)=(f+g)(f+wg)\overline{(f+wg)}$, $f,g\in\mathbb{Q}$

claim $\exists\, h, q, x$ s.t. $\begin{cases} f+g=q^3+kx \\ (f+wg)(f+w^2g)=f^2-fg+g^2=h^3 \end{cases}$ — ①

By ①: $\frac{f^3}{h^3}+\frac{g^3}{h^3}+(-q)^3=kx$

RK: ~~Since~~ 1. Set $u^3=f+wg$, $\bar{u}^3=f+w^2g$, $h=u\bar{u}\in\mathbb{R}$.

2. $-(wu^3+\overline{wu^3})=-(wf+w^2g+w^2f+wg)=-(w+\bar{w})(f+g)=f+g$

WTF: $-(wu^3+\overline{wu^3})=q^3+kx$

Set $u=w-x$, $u^3=1-3w^2x+3wx^2-x^3$

$-(wu^3+\overline{wu^3})=\underbrace{(1-x)}_{q}{}^3+\underbrace{9}_{k}x$

Then: $x=\frac{f^3}{9h^3}+\frac{g^3}{9h^3}+\frac{(x-1)^3}{9}$ $\begin{cases} f=\frac{u^3+\bar{u}^3}{2}=\frac{(w-x)^3+(\bar{w}-x)^3}{2} \\ g=\frac{u^3-\bar{u}^3}{2w}=\frac{(w-x)^3-(\bar{w}-x)^3}{2w} \\ h^3=(w-x)(\bar{w}-x) \end{cases}$

Th: $\exists\, a\in\mathbb{Q}$: $a=\left(\frac{a-3^2}{3^2}\right)^3+\left(\frac{-a^3+3^5a+3^6}{3^3a^2+3^4a+3^6}\right)^3+\left(\frac{3^3a+3^5a^2}{3^2a^2+3^4a+3^6}\right)^3$