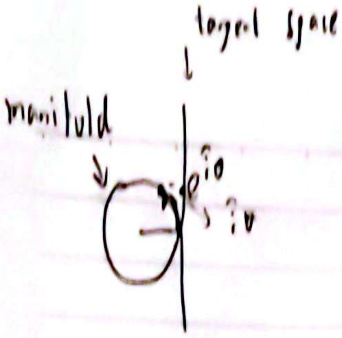

Start with $\begin{cases} \text{velocity } \vec{v} : \\ \text{location } p \end{cases}$ on manifold $\exp_p(\vec{v})$?

Back to e^x. consider $e^x = g(1)$. g satisfies: $\begin{cases} g(0) = 1 \\ g'(t) = x\,g(t) \end{cases} \Rightarrow g(t) = e^{tx}$

in diagram: start $1 \xrightarrow[\text{initial velocity}]{x} \underset{\text{intermediate point}}{\bullet} \xrightarrow{x g(t)} \bullet\, e^x$

$g(0) \qquad g(t) \qquad g(1)$

g is used to describe point on manifold.

$g : [0,1] \to M$

Generally. Start. $p \xrightarrow{\vec{v}}$ Intermediate point $\xrightarrow{\text{parallel transport } v}$ end $\exp_p(\vec{v})$

$g(0) \qquad g(t) \qquad g(1)$

consider $f(x) \xrightarrow[\text{velocity}]{\frac{df}{dx}} \underset{\text{inter-M}}{\bullet} \xrightarrow{\frac{\partial g}{\partial x}} \bullet\, \exp\left(\frac{d}{dx}\right) f(x)$

$g(0) \qquad g(t,x) \qquad g(1,x)$

g satisfies:

$$\begin{cases} g(0,x) = f(x) \\ \frac{\partial g}{\partial t} = \frac{\partial g}{\partial x} \end{cases}$$

$\Downarrow$

$$g(x,t) = f(x+t)$$

$$\Rightarrow \underbrace{\exp\left(t \frac{d}{dx}\right)}_{\text{shift operator}} f(x) = f(x+t)$$

generally: $\exp(\vec{a} \cdot \nabla) f(\vec{x}) = f(\vec{x} + \vec{a})$

Furthermore: $f(x) \xrightarrow[\text{(in phase space)}]{\text{"velocity" } \frac{d^2 f}{dx^2}} \bullet \xrightarrow{\frac{\partial^2 g}{\partial x^2}} \exp\left(\frac{d^2}{dx^2}\right) f(x)$

$g(0) \qquad g(t,x) \qquad g(1,x)$

$$\Rightarrow \begin{cases} g(0,x) = f(x) & \text{if } x \in \mathbb{R} \\ \frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2} \end{cases} \quad g(t,x) = \int_{\mathbb{R}} f(s)\, G(x,t;s)\, ds = \exp\left(t \frac{d^2}{dx^2}\right) f(x)$$

where $G(x,t;s) = \frac{1}{2\sqrt{\pi t}} \exp\left[-\frac{(x-s)^2}{4t}\right]$

So $p \xrightarrow{\text{vector attach top}} \underset{TM}{\bullet} \xrightarrow{\text{vector to } g(t)}$ end $\exp(\text{vector field})(p)$

$g(0) \qquad g(t) \qquad g(1)$

Deduce: $g(t) = \underbrace{\exp[(\text{vector field})\, t]}_{\text{flow}}\, (p) \xrightarrow[p = \vec{v} \text{ on manifold}]{\text{vector field is linear} \to A} \Rightarrow \exp(tA)\, \vec{v}$ which is the trajectory of $\vec{v}$