1. For a measurable function (or random variable) $X : (\Omega_1, \mathcal{F}) \to (\Omega_2, \mathcal{S})$ with $\mathcal{S} = \sigma(\mathcal{A})$, where $\mathcal{A}$ is a algebra on Ω_2, show that

$$\sigma(X) = \sigma(X^{-1}(\mathcal{A})).$$

Remark: The condition : "$\mathcal{A}$ is a algebra" can be removed, in this case, $X^{-1}(\sigma(\mathcal{A})) = \sigma(X^{-1}(\mathcal{A}))$.

2. Let $X_1, \cdots, X_n$ be a sequence of independent random variables, all defined on $(\Omega, \mathcal{A}, P)$.

   (a) Show that $\{\omega : \sum_{n=1}^{\infty} X_n(\omega) \text{ converges}\}$ and $\left\{\omega : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) \text{ exists}\right\}$ are tail events.

   (b) Show that $\limsup_{n\to\infty} X_n$ and $\liminf_{n\to\infty} X_n$ are a.s. constant.

3. Let $f \in L^1(\Omega, \mathcal{A}, \mu)$, show that if $\mu(A_n) \to 0$,

$$\int_{A_n} |f| \mathrm{d}\mu \to 0 \quad \text{as } n \to \infty.$$

4. (a) Suppose $X_n \xrightarrow{p} X$ and also that $|X_n| \le Y$ for all n, and $Y \in L^p$. Show that $|X| \in L^p$ and $X_n \xrightarrow{L^p} X$. (Theorem 17.4)

   (b) Let $\{X_j\}_{j\ge1}$ be i.i.d. with $X_j \in L^1$. Let $Y_j = e^{X_j}$, show that

$$\left(\prod_{i=1}^{n} Y_j\right)^{1/n}$$

   converges to a constant a.s. (Exercise 20.3)

5. Let $\{X_n\}_{n\ge1}$ be a submartingale with $\sup_n \mathrm{E}|X_n| < \infty$.

   (a) Let $Y_n = \lim_{m\to\infty} \mathrm{E}(X_m|\mathcal{F}_n)$, show that Y_n is $\mathcal{F}_n$-measurable.

   (b) Prove that

$$\mathrm{E}Y_n < \infty.$$

   (c) Show that Y_n is well-defined and is a martingale.

6. (a) State Doob's Optional Sampling Theorem.

   (b) Let $\{X_n\}$ be a martingale, and T is a stopping time bounded by c, show that

$$\mathrm{E}(X_c|\mathcal{F}_T) = X_T \; a.s.$$

(c) Let $\{X_n\}_{n\geq 1}$ be a martingale and let S, T be stopping times bounded by a constant, with $S \leq T$ a.s. Show that

$$\mathrm{E}(X_T|\mathcal{F}_S) = X_S \ a.s.$$

(d) If there is no condition: $S \leq T$ in (c), show that

$$\mathrm{E}(X_T|\mathcal{F}_S) = X_{T\wedge S} \ a.s.$$

1. Solution

$\because S = \sigma(\mathcal{A})$, $\therefore \sigma(X^{-1}(\mathcal{A})) \subseteq \sigma(X^{-1}(S)) = \sigma(X)$

If $B \in \mathcal{A}$. $\therefore \sigma(X^{-1}(B)) \subseteq \sigma(X^{-1}(\mathcal{A}))$

Let $\mathcal{H} = \{B \in S = \sigma(\mathcal{A}) : \sigma(X^{-1}(\mathcal{A})) \supseteq \sigma(X^{-1}(B))\}$, $\therefore \mathcal{A} \in \mathcal{H}$

(i) $\phi \in \mathcal{H}$.

(ii) $B \in \mathcal{H}$, $B^c \in S$, $\sigma(X^{-1}(B^c)) = \sigma((X^{-1}(B))^c) \subseteq \sigma(X^{-1}(\mathcal{A}))$ $\therefore B^c \in \mathcal{H}$.

(iii) $B_1, \dots, B_n, \dots \in \mathcal{H}$ ..- — - -

$\therefore \sigma(\mathcal{A}) \subseteq \mathcal{H} \subseteq S$ $\therefore \mathcal{H} = S$ $\therefore \sigma(X^{-1}(S)) \subseteq \sigma(X^{-1}(\mathcal{A}))$

$\therefore \sigma(X) = \sigma(X^{-1}(\mathcal{A}))$

2. Solution

(a) For any k, $\{w : \sum_{n=1}^{\infty} X_n(w) \text{ converges}\} \Longleftrightarrow \{w : \sum_{n=k}^{\infty} X_n(w) \text{ converges}\} \in \bigcap_{n=1}^{\infty} \sigma(X_k, k \geq n) = \mathcal{C}_\infty$

For any k, $\{w : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i(w) \text{ exists}\} \Longleftrightarrow \{w : \lim_{n\to\infty} \frac{1}{n} \sum_{i=k}^{n} X_i \text{ exists}\} \in \bigcap_{n=1}^{\infty} \sigma(X_k, k \geq n) = \mathcal{C}_\infty$

(b) $\because \limsup_{n\to\infty} X_n$ and $\liminf_{n\to\infty} X_n$ is $\mathcal{C}_\infty$-measurable, $\therefore P\{\overline{\lim_{n\to\infty}} X_n \leq c\} = 0$ or 1

(For any c), Let $c_0 = \inf\{c : F_{\overline{\lim} X_n}(c) = 1\}$

$\therefore P\{\overline{\lim_{n\to\infty}} X_n \leq c\} = \begin{cases} 1, & c \geq c_0 \\ 0, & c < c_0 \end{cases}$ $\therefore P\{\overline{\lim_{n\to\infty}} X_n = c_0\} = F(c) - F(c^-) = 1$

$\therefore \overline{\lim_{n\to\infty}} X_n = c_0$ a.s.

Similarly, let $b_0 = \inf\{b : F_{\underline{\lim}_{n\to\infty} X_n}(b) = 1\}$, $\underline{\lim}_{n\to\infty} X_n = b_0$ a.s.

3. Solution

For a sequence of numbers $\{a_n\}_{n\geq 1}$ increasing to ∞, we decompose

$\mu|f I_{A_n}| = \mu(|f| I_{A_n \cap \{|f| \leq a_n\}}) + \mu(|f| I_{A_n \cap \{|f| > a_n\}}) \leq a_n \mu(A_n) + \mu(|f| I_{\{|f| > a_n\}})$

$\because f$ is integrable, $||f| I_{\{|f|>a_n\}}| \leq |f|$, By DCT $\mu(|f| I_{\{|f|>a_n\}}) \to \mu(|f| I_{\{|f|=\infty\}}) = 0$

We set $a_n = (\mu(A_n))^{-1/2}$. $\therefore \mu(|f I_{A_n}|) \leq (\mu(A_n))^{1/2} + \mu(|f| I_{\{|f|>a_n\}}) \to 0$

4. Solution

(a)

Proof. Because $E(|X_n|^p) \le E(Y^p) < \infty$, then $X_n \in L^p$. Now

$$\{|X| > Y + \epsilon\} \subset \{|X| > |X_n| + \epsilon\} = \{|X| - |X_n| > \epsilon\} \subset \{|X - X_n| > \epsilon\}.$$

Hence for any $\epsilon > 0$,

$$P\{|X| > Y + \epsilon\} \le P\{|X - X_n| > \epsilon\} \to 0, \quad n \to \infty.$$

Note that

$$A = \{\omega : |X(\omega)| > Y(\omega)\} = \cup_{m=1}^{\infty}\{\omega : |X(\omega)| > Y(\omega) + \frac{1}{m}\} =: \cup_{m=1}^{\infty} A_m.$$

Clearly, $A_m = \{\omega : |X(\omega)| > Y(\omega) + \frac{1}{m}\} \uparrow A$. Then

$$P(A) = P(|X| > Y) = \lim_{m\to\infty} P(A_m) = \lim_{m\to\infty} P(|X| > Y + \frac{1}{m}) = 0.$$

Thus $|X| \le Y$ a.s. and $X \in L^p$. If $\{X_n\}$ does not converge to X in L^p, there exists a subsequence $\{X_{n_k}\}$ s.t. for all k and some $\epsilon > 0$,

$$E\{|X_{n_k} - X|^p\} \ge \epsilon. \tag{6.7}$$

As $X_{n_k} \xrightarrow{P} X$, Theorem $\boxed{6.2.7}$ shows that a further subsequence $\{X_{n_{k_j}}\}$ s.t. $X_{n_{k_j}} - X \xrightarrow{a.s.} 0$ and $|X_{n_{k_j}} - X| \le 2Y$, by Lebesgue DCT

$$E\{|X_{n_{k_j}} - X|^p\} \to 0 \quad \text{as } j \to \infty, \tag{6.8}$$

which contracts ($\boxed{6.7}$). This completes the proof. ■

(a) $\forall m$, $P\{|X| > Y + \frac{1}{m}\} \le P\{|X_n| + |X - X_n| \ge Y + \frac{1}{m}\}$

$\le P\{|X - X_n| \ge \frac{1}{m}\} \to 0 \ (n \to \infty)$

$\therefore P\{|X| > Y\} = \bigcup_{m=1}^{\infty} P\{|X| > Y + \frac{1}{m}\} = \lim_{m\to\infty} P\{|X| > Y + \frac{1}{m}\}$

$= 0$

$\therefore |X| \le Y$, a.s. $\therefore E[|X|^p] \le E[Y^p] < \infty$

Suppose $X_n \overset{L^p}{\not\to} X$, $\forall \varepsilon > 0$, $\exists n_k$, $E[|X_{n_k} - X|^p] \ge \varepsilon$

$\because X_{n_k} \xrightarrow{P} X$ $\therefore \exists X_{n_{k_j}} \xrightarrow{a.s.} X$ $\therefore |X_{n_{k_j}} - X| \le 2Y$

$\therefore E[|X_{n_{k_j}} - X|^p] \to 0$ (DCT), contradiction $E[|X_{n_{k_j}} - X|^p] \ge \varepsilon$

(b) $\because (X_j)_{j\ge1}$ is i.i.d. with $X_j \in L^1$. By SLLN, $\frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s.} E(X_1)$

$(\prod_{j=1}^{n} Y_j)^{\frac{1}{n}} = \exp\{\frac{1}{n}\sum_{j=1}^{n} \ln Y_j\} = \exp\{\frac{1}{n}\sum_{j=1}^{n} X_j\} \xrightarrow{a.s.} \exp\{E(X_1)\} = C$ [$f(x) = \exp\{x\}$ is continuous]

5. Solution

(a) For fixed m, $E[X_m|\mathcal{F}_n]$ is $\mathcal{F}_n$-measurable ($m \ge n$), $\mathcal{F}_n$ is a σ-algebra.

$\therefore Y_n = \lim_{m\to\infty} E(X_m|\mathcal{F}_n)$ is $\mathcal{F}_n$ ....

(b) $E Y_n = E \lim_{m\to\infty} E(X_m|\mathcal{F}_n) \le \liminf_{m\to\infty} E[E(X_m|\mathcal{F}_n)] = \liminf_{m\to\infty} E[X_m] \le \sup_n E|X_n| < \infty$

(c) By Second MCT, $\because \sup_n E|X_n| < \infty$ $\therefore \{Y_n\}_{n\ge1}$ is UI. $\therefore \lim Y_n$ exist

$E(Y_{n+1}|\mathcal{F}_n) = E[\lim_{m\to\infty} E(X_m|\mathcal{F}_{n+1})|\mathcal{F}_n] \overset{DCT}{=} \lim_{m\to\infty} E(X_m|\mathcal{F}_n)$

6. Solution

(a) $\{X_n\}$ ...., T, S, c $\quad S \le T \quad E(X_T|\mathcal{F}_S) = X_S$

(b) For any $A \in \mathcal{F}_T$, $E(X_T) = E(X_0) = E[X_R]$ $\quad R(\omega) = I_A C + I_{A^c} T(\omega)$

$\therefore E[X_T I_A] + E[X_T I_{A^c}] = E[X_T I_{A^c}] + E[X_c I_A]$ $\quad \therefore \int_A X_T \, dP = \int_A X_c \, dP$

(c) ....

(d) ....