Finite sets, countable sets, uncountable sets

Definition (Equal cardinality). We say that two sets X and Y have equal cardinality iff there exists a bijection $f : X \to Y$ from X to Y. In this case we say that X and Y are equivalent and write $X \sim Y$.

Example. The sets $\{0, 1, 2\}$ and $\{3, 4, 5\}$ have equal cardinality, since we can find a bijection between the two sets.

Example. If X is the set of natural numbers and Y is the set of even natural numbers, then the map $f : X \to Y$ defined by $f(n) := 2n$ is a bijection from X to Y, and so X and Y have equal cardinality, despite Y being a subset of X and seeming intuitively as if it should only have "half" of the elements of X.

Proposition. Let X, Y, Z be sets. If X has equal cardinality with Y, then Y has equal cardinality with X. If X has equal cardinality with Y and Y has equal cardinality with Z, then X has equal cardinality with Z.

Definition. Let n be a natural number. A set X is said to have cardinality n, iff it has equal cardinality with $\{i \in \mathbb{N} : 1 \leq i \leq n\}$. We also say that X has n elements iff it has cardinality n.

Example. Let a, b, c, d be distinct objects. Then $\{a, b, c, d\}$ has the same cardinality as $\{i \in \mathbb{N} : 1 \leq i \leq 4\} = \{1, 2, 3, 4\}$ and thus has cardinality 4.

Proposition. (Uniqueness of cardinality). Let X be a set with some cardinality n. Then X cannot have any other cardinality, i.e., X cannot have cardinality m for any $m \neq n$.

Definition. (Finite sets). A set is finite iff it has cardinality n for some natural number n; otherwise, the set is called infinite. If X is a finite set, we use $^\sharp(X)$ to denote the cardinality of X.

Definition (Countable sets). A set X is said to be countably infinite (or just countable) iff it has equal cardinality with the natural numbers $\mathbb{N}$. A set X is said to be at most countable iff it is either countable or finite. We say that a set is uncountable if it is infinite but not countable.

Theorem Every infinite subset of a countable set A is countable.

Proof. Suppose $E \subset A$, and E is infinite. Arrange the elements of A in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows: Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen $n_1, \cdots, n_{k-1} (k = 2, 3, 4, \cdots)$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$. Putting $f(k) = x_{n_k} (k = 1, 2, 3, ...)$, we obtain a bijection between E and $\mathbb{N}$.

Theorem The set of real numbers $\mathbb{R}$ is uncountable.

Proof Assume that the set of real numbers can be arranged as

$$a_1, a_2, \cdots,.$$

Take a closed interval I_1 such that $a_1 \notin I_1$. Take a closed interval $I_2 \subset I_1$ such that $a_2 \notin I_2, |I_2| \leq \frac{1}{2}|I_1|$. Continuing in this process, we can construct a sequence of closed intervals $\{I_n\}_{n=1}^{\infty}$ such that

$$I_1 \supset I_2 \supset \cdots, \ |I_{n+1}| \leq \frac{1}{2}|I_n|, n = 1, \cdots,.$$

Let $\{a\} = \cap_{n=1}^{\infty} I_n$; then $a \neq a_i, \ \forall i$. This is a contradiction.

Theorem Let $\{E_k\}_{k \in \mathbb{N}}$ be a countable family of countable sets. Then the set $E = \cup_{k=1}^{\infty} E_k$ is countable.

Proof We arrange the elements of E_k as follows:

$$\begin{array}{l} E_1 : \; x_{11}, x_{12}, x_{13}, \cdots \\ E_2 : \; x_{21}, x_{22}, x_{23}, \cdots \\ E_3 : \; x_{31}, x_{32}, x_{33}, \cdots \\ \cdots \end{array}$$

These elements can be arranged in a sequence

$$(\star) \quad x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{23}, x_{14}, \cdots$$

If any two of the sets E_n have elements in common, these will appear more than once in $(\star)$. Hence there is a subset $T \subset \mathbb{N}$ such that $T \sim E$, which shows that E is at most countable Since $E_1 \subset E$, and E_1 is infinite, E is infinite, and thus countable.

Theorem Let A be a countable set, and let

$$B_n = \{(a_1, \cdots, a_n),\ a_k \in A, k = 1, ..., n\}.$$

Then B_n is countable.

Proof We use induction. It is evident that $B_1 = A$ is countable. Assume that B_{n-1} is countable $(n = 2, 3, 4, \cdots)$. The elements of B_n are of the form

$$(b, a)(b \in B_{n-1}, a \in A).$$

For every fixed b, the set of pairs (b, a) is equivalent to A, and hence countable. Thus B_n is the union of a countable set of countable sets. By the above theorem, B_n is countable.

Corollary Any closed interval $[a, b], a < b$ is uncountable.

Proof If $a < b, c < d$, then $[a, b] \sim [c, d]$. Let us show that $[0, 1]$ is uncountable. If $[0, 1]$ is countable, then for any integer n, the interval $[n, n+1] \sim [n+1, n+2]$ is countable. This implies that the set of real numbers $\mathbb{R} = \cup_{n=-\infty}^{\infty}[n, n+1]$ is countable, which is a contradiction.

Corollary The set of all rational numbers is countable.

Proof Note that every rational r is of the form b/a, where a and b are integers. The set of pairs (a, b), and therefore the set of fractions b/a, is countable.

Theorem Let $E = \{(a_1, a_2, \cdots,): \ a_i = 0 \text{ or } 1\}$. Then E is uncountable.

Proof Assume that E is countable and we arrange the elements of E as

$$x_1 = (x_1^1, x_1^2, \cdots), x_2 = (x_2^1, x_2^2, \cdots), \cdots .$$

We define an element $y = (y_1, y_2, \cdots)$ by

$$y_i = \begin{cases} 0, & \text{if } x_i^i = 1, \\ 1, & \text{if } x_i^i = 0. \end{cases}$$

Then $y \in E$ but $y \neq x_i$ for any $i \in \mathbb{N}$. This is a contradiction.

For $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \mathbb{R}^k$, the usual distance between x and y is

$$\|x - y\| = \left(\sum_{j=1}^{k} |x_j - y_j|^2 \right)^{1/2} .$$

If $x \in \mathbb{R}^k$ and $r > 0$, the open (or closed) ball B with center at x and radius r is defined to be the set of all $y \in R^k$ such that $\|y - x\| < r$ (or $\|y - x\| \leq r$).
A set $E \subset \mathbb{R}^k$ is convex if

$$\lambda x + (1 - \lambda) y \in E$$

whenever $x \in E,\ y \in E$, and $0 < \lambda < 1$.

Example. i) Balls are convex. For if $\|y - x\| < r$, $\|z - x\| < r$, and $0 < \lambda < 1$, we have

$$\begin{aligned} \|\lambda y + (1-\lambda) z - x\| &= \|\lambda(y - x) + (1-\lambda)(z - x)\| \\ &\leq \lambda\|(y - x)\| + (1-\lambda)\|z - x\| \\ &< \lambda r + (1-\lambda) r = r. \end{aligned}$$

If $a_i < b_i$; for $i = 1, ..., k$, the set of all points $x = (x_1, \cdots, x_k)$ in $\mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i, \ 1 \leq i \leq k$ is called a k-cell. Thus a 1-cell is an interval, a 2-cell is a rectangle, etc.

Example. k-cells are convex.

Metric Spaces

Definition 1 A metric space is a pair (X, d), where X is a set and $d : X \times X \to [0, \infty)$ for all $x, y, z \in X$ has the following properties:

- (Positivity) $d(x, y) = 0 \iff x = y$,
- (Symmetry) $d(x, y) = d(y, x)$,
- (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$.

A function $d : X \times X \to [0, \infty)$ that satisfies these axioms is called a distance function on X.

Example 1. The set $\mathbb{R}$ of all real numbers endowed with the distance function $d(x, y) = |x - y|$, where $|x|$ is the absolute value of x, is a metric space.

Similarly, the set of all complex numbers $\mathbb{C}$ is a metric space with the distance function $d(z, w) = |z - w|$, where $|z|$ is the modulus of z in $\mathbb{C}$.

Example 2. Let X be a nonempty set. The function

$$d(x, y) = \begin{cases} 0, \text{ if } x = y, \\ 1, \text{ if } x \neq y, \end{cases}$$

is a metric, called the discrete metric (also known as the trivial metric) on X. The space (X, d) is called the discrete metric space.

Example 3. Let $C[a,b] = \{x(t) : x(t) \text{ is continuous on } [a,b]\}$ and define

$$d_1(x,y) := \max_{a \le t \le b} |x(t) - y(t)|, \ d_2(x,y) := \int_a^b |x(t) - y(t)| dt.$$

Then d_1 and d_2 are metrics on $C[a,b]$.

Example 4. For any integer $k \geq 1$, the function $d : \mathbb{R}^k \times \mathbb{R}^k \to [0,\infty)$ defined by

$$d(x,y) = \left(\sum_{j=1}^{k} |x_j - y_j|^2 \right)^{1/2},$$

is a metric on the set $\mathbb{R}^k$, called the standard metric on $\mathbb{R}^k$.

Example 5. More generally, take $X = \mathbb{R}^n$ with any one of the metrics

$$d_{l^p}(x,y) = \begin{cases} \left(\sum_{j=1}^{n} |x_j - y_j|^p\right)^{1/p}, & 1 \le p < \infty, \\ \max_{j=1,\cdots,n} |x_j - y_j|, & p = \infty. \end{cases}$$

It is easy to see that d_{l^∞} is a metric. For the case $1 \le p < \infty$, we need only to use the

(Minkowski's inequality.) For arbitrary complex numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ and a real number $p \ge 1$,

$$\left(\sum_{j=1}^{n} |x_j + y_j|^p\right)^{1/p} \le \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{1/p}.$$

Proof. We may assume that both real numbers

$$u = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \text{ and } v = \left(\sum_{j=1}^{n} |y_j|^p\right)^{1/p}$$

are positive. By the triangle inequality, we have

$$|x_k + y_k|^p \le (|x_k| + |y_k|)^p = (u+v)^p \left(\frac{u}{u+v} \frac{|x_k|}{u} + \frac{v}{u+v} \frac{|y_k|}{v} \right)^p.$$

Since $\frac{u}{u+v} + \frac{v}{u+v} = 1$ and x^p is convex for $p \ge 1, x \ge 0$, we have

$$\left(\frac{u}{u+v} \frac{|x_k|}{u} + \frac{v}{u+v} \frac{|y_k|}{v} \right)^p \le \frac{u}{u+v} \frac{|x_k|^p}{u^p} + \frac{v}{u+v} \frac{|y_k|^p}{v^p}.$$

Hence

$$|x_k + y_k|^p \le (u+v)^p \left(\frac{u}{u+v} \frac{|x_k|^p}{u^p} + \frac{v}{u+v} \frac{|y_k|^p}{v^p} \right).$$

By summing both sides of the above inequality, we obtain

$$\sum_{j=1}^{n} |x_j + y_j|^p \le (u+v)^p.$$

Example 6. If d is a metric on X and $A \subset X$, then $d|_{(A \times A)}$ is a metric on A.

Example 7. If (X_1, d_1) and (X_2, d_2) are metric spaces, the product metric d on $X_1 \times X_2$ is given by

$$d((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1), d(x_2, y_2)\}.$$

Other metrics are sometimes used on $X_1 \times X_2$, for instance,

$$d(x_1, y_1) + d(x_2, y_2) \text{ or } \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}.$$

Definition. A subset $U \subset X$ of a metric space (X, d) is called open if, for every $x \in U$, there exists a constant $\epsilon > 0$ such that the open ball

$$B(x, \epsilon) := \{y \in X | d(x, y) < \epsilon\}$$

(centered at x with radius ϵ) is contained in U.

We shall also call an open ball with center x a neighborhood of x.

A subset $E \subset X$ of a metric space (X, d) is called closed if $E^c = X \setminus E$ is open.

Some basic facts about open sets

• Every ball $B(x, r)$ is open, for if $y \in B(x, r)$ and $d(x, y) = s$ then $B(y, r - s) \subset B(x, r)$.

• X and $\emptyset$ are both open.

• The union of any family of open sets is open.

• The intersection (resp. union) of any finite family of open (resp. closed) sets is open (resp. closed). Indeed, if $U_1, \cdots, U_n$ are open and $x \in \cap_{i=1}^n U_i$, for each j there exists $r_j > 0$ such that $B(x, r_j) \subset U_j$, and then $B(x, r) \subset \cap_{i=1}^n U_i$ where $r = \min(r_1, \cdots, r_n)$, so $\cap_{i=1}^n U_i$ is open.

Definition 3. A point $x \in E$ is said to be an interior point of E if

$$\exists r > 0, \; s.t. \; B(x, r) \subset A.$$

The interior of E is the set of all its interior points and is denoted by E^o. A point x (not in E) is an exterior point of E when

$$\exists r > 0, \; s.t. B(x, r) \subset X \setminus E.$$

All other points are called boundary points of E.
The set of interior and boundary points of E is called the closure of E and denoted by $\overline{E} = E^o \cup \partial E$. Note that $\overline{E}$ is also the intersection of all closed sets containing E.

The set X is partitioned into three parts: its interior E^o, its exterior $(\overline{E})^c$, and its boundary ∂E.
We call E is dense in X if $\overline{E} = X$, and nowhere dense if $\overline{E}$ has empty interior.

Definition Let X be a metric space. Let $E \subset X$, $x_0 \in X$.

i) We say that x_0 is an adherent point of E if for every $r > 0$, the ball $B(x_0, r)$ has a non-empty intersection with E. The set of all adherent points of E is just the closure of E.

ii) A point p is an accumulation point (or limit point) of a set E if every open ball around it contains other points of E,

$$\forall \epsilon > 0, \ \exists q \neq p, q \in E \cap B(p, \epsilon).$$

Note that p is not necessarily an element of E. If $q \in E$ and q is not an accumulation point of E, then q is an isolated point of E.

iii) E is closed if every limit point of E is a point of E.

iv) E is perfect if E is closed and if every point of E is a limit point of E.

Theorem. If p is a limit point of a set E, then for any $r > 0$, $B(p, r)$ contains infinitely many points of E.

Proof. Suppose $\exists$ a ball $B(p, r)$ which contains only a finite number of points of E. Let $q_1, \cdots, q_n$ be those points of $B(x, r) \cap E$, which are distinct from p, and put

$$s = \min_{1 \leq i \leq n} d(p, q_i).$$

Clearly, $s > 0$ and $B(p, s)$ contains no point q of E such that $q \neq p$, so that p is not a limit point of E. This is a contradiction.

Corollary A finite point set has no limit points.

Theorem Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets. Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c.$$

Proof Set

$$A = \left(\bigcup_\alpha E_\alpha\right)^c, \; B = \bigcap_\alpha E_\alpha^c.$$

If $x \in A$, then $x \notin \bigcup_\alpha E_\alpha$, hence $x \notin E_\alpha$ for every α, so that $x \in \bigcap_\alpha E_\alpha^c$. Thus $A \subset B$.

Conversely, if $x \in B$, then $x \in E_\alpha^c$, for every α, hence $x \notin E_\alpha$ for any α, hence $x \notin \bigcup_\alpha E_\alpha$, so that $x \in \left(\bigcup_\alpha E_\alpha\right)^c$. Thus $B \subset A$. It then follows that $A = B$.

Theorem A set E is open if and only if its complement is closed.

Proof Let E^c be closed. Choose $x \in E$. Then $x \notin E^c$, and x is not a limit point of E^c. Hence there exists an open ball $B(x, r)$ such that $E^c \cap B(x, r) = \emptyset$, that is, $B(x, r) \subset E$. Thus x is an interior point of E, and E is open.
Next, suppose E is open. Let x be a limit point of E^c. Then every open ball $B(x, r)$ contains a point of E^c, so that x is not an interior point of E. Since E is open, this means that $x \in E^c$. It follows that E^c is closed.

Corollary i) A set F is closed if and only if its complement is open.
ii) The intersection of any family of closed sets is closed.
iii) The union of finitely many closed sets is closed.

Definition. If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the closure of E is the set $\overline{E} = E \cup E'$.

Theorem If X is a metric space and $E \subset X$, then
(a) $\overline{E}$ is closed,
(b) $E = \overline{E}$ if and only if E is closed,
(c) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

By (a), (c), $\overline{E}$ is the smallest closed subset of X that contains E.

Proof (a) If $p \in X$ and $p \notin \overline{E}$ then p is neither a point of E nor a limit point of E. Hence p has a neighborhood which does not intersect E. The complement of E is therefore open. Hence E is closed.
(b) If $E = \overline{E}$, (a) implies that E is closed. If E is closed, then $E' \subset E$, hence $E = \overline{E}$.
(c) If F is closed and $F \supset E$, then $F \supset F'$, hence $F \supset E'$. Thus $F \supset \overline{E}$.

Theorem Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

Proof If $y \in E$ then $y \in \overline{E}$. Assume $y \notin E$. For every $h > 0$ there exists a point $x \in E$ such that $y - h < x < y$, for otherwise $y - h$ would be an upper bound of E. Thus y is a limit point of E. Hence $y \in E'$.

Definition. Suppose $E \subset Y \subset X$, where X is a metric space. We say that E is open relative to Y if to each $p \in E$ there is an $r > 0$ such that $B(p, r) \cap Y \subset E$.

Theorem Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

Proof Suppose E is open relative to Y. To each $p \in E$ there is a positive number r_p such that the conditions $d(p,q) < r_p,\ q \in Y$ imply that $q \in E$. Define

$$G = \bigcup_{p \in E} B(p, r_p).$$

Then G is an open subset of X and $E \subset G \cap Y$. By our choice of $B(p, r_p)$, we have $B(p, r_p) \cap Y \subset E$ for every $p \in E$, so that $G \cap Y \subset E$. Thus $E = G \cap Y$.

Conversely, if G is open in X and $E = G \cap Y$, for every $p \in E$ there is a ball $B(p, r_p) \subset G$. Then $B(p, r_p) \cap Y \subset E$, so that E is open relative to Y.

Definition. By an open cover of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$.

Definition. A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

The requirement is that if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \cdots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}.$$

Observe that if $E \subset Y \subset X$, then E may be open relative to Y without being open relative to X. The property of being open thus depends on the space in which E is embedded.

Theorem Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.

Proof Suppose K is compact relative to X, and let $\{V_\alpha\}$ be open sets of Y, such that $K \subset \cup_\alpha V_\alpha$. There open sets U_α of X, such that $V_\alpha = Y \cap U_\alpha$, for all α; and since K is a compact set of X, there are indices $\alpha_1, \cdots, \alpha_n$ such that

$$K \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_n},$$

which, combining with $K \subset Y$, implies that $K \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. Thus K is compact relative to Y.

If K is a compact subset of Y and U_α is collection of open sets of X which covers K, and put $V_\alpha = Y \cap U_\alpha$. There are indices $\alpha_1, \cdots, \alpha_n$ such that $K \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$ and so $K \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. Thus K is compact relative to X.

Theorem Compact subsets of metric spaces are closed.

Proof Let K be a compact subset of a metric space X. We shall prove that K^c is an open subset of X.
Suppose $p \in X$, $p \notin K$. For any $q \in K$, consider the open balls $B(q, r_q)$ and $B(p, r_q)$, where $r_q < \frac{1}{2}d(p, q)$. Then $B(q, r_q) \cap B(p, r_q) = \emptyset$ and

$$K \subset \cup_{q \in K} B(q, r_q).$$

Since K is compact, there are finitely many points $q_1, \cdots, q_n$ in K such that

$$K \subset B(q, r_{q_1}) \cup \cdots \cup B(q, r_{q_n}).$$

Let $r = \min\{r_{q_1}, \cdots, r_{q_n}\}$; then

$$B(p, r) \cap (B(q, r_{q_1}) \cup \cdots \cup B(q, r_{q_n})) = \emptyset.$$

Thus $B(p, r) \cap K = \emptyset$ and so $B(p, r) \subset K^c$, which shows that K^c is open.

Theorem Closed subsets of compact sets are compact.

Proof Suppose $F \subset K \subset X$, F is closed subset of X, and K is compact. Let $\{V_a\}$ be an open cover of F. If F^c is adjoined to $\{V_a\}$, we obtain an open cover Ω of K. Since K is compact, there is a finite subcollection Φ of Ω which covers K, and hence F. If F^c is a member of Φ, we may remove it from Φ and still retain an open cover of F. Thus there is a finite subcollection of $\{V_a\}$ that covers F.

Corollary If F is closed and K is compact, then $F \cap K$ is compact.

Theorem If $\{K_a\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_a\}$ is nonempty, then $\cap_a K_a \neq \emptyset$.

Proof Fix a member K_1 of $\{K_a\}$ and put $G_a = K_a^c$. Assume that

$$\cap_a K_a = K_1 \cap (\cap_{a \neq 1} K_a) = \emptyset.$$

Then

$$K_1 \subset (\cap_{a \neq 1} K_a)^c = \cup_{a \neq 1} K_a^c = \cup_{a \neq 1} G_a.$$

Thus $\{G_a\}_{a \neq 1}$ is an open cover of K_1. Since K_1 is compact, there are finitely many indices $a_1, \cdots, a_n$ such that

$$K_1 \subset G_{a_1} \cup \cdots \cup G_{a_n},$$

which gives

$$\emptyset = K_1 \cap (G_{a_1} \cup \cdots \cup G_{a_n})^c = K_1 \cap (K_{a_1} \cap \cdots \cap K_{a_n}).$$

This is a contradiction.

Corollary If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$, $n = 1, 2, \cdots$, then $\cap_n K_n \neq \emptyset$.

Theorem If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof If no point of K were a limit point of E, then each $q \in K$ would have a neighborhood V_q which contains at most one point of E (namely, q, if $q \in E$). It is clear that no finite subcollection of $\{V_q\}$ can cover E; and the same is true of K, since $E \subset K$. This contradicts the compactness of K.