Riemann Lemma: $f \in \mathcal{L}^1(\mathbb{R}^n)$, i.e. $f: \mathbb{R}^n \to \mathbb{C}$ measurable and $\|f\|_{\mathcal{L}^1} := \int_{\mathbb{R}^n} |f(x)|\,dx < +\infty$. $\hat{f} := \mathcal{L}(f) = \int_{\mathbb{R}^n} f(x) e^{-ixs} dx$

prove $|\hat{f}(s)| \to 0$ as $s \to +\infty$

proof: $n=1$ For $s \neq 0$, use $x \to x + \frac{\pi}{s}$, $\hat{f}(s) = \int_{\mathbb{R}^n} f(x) e^{-ixs} dx = \int_{\mathbb{R}} f(x + \frac{\pi}{s}) e^{-ixs} e^{-i\pi} dx$

$= -\int_{\mathbb{R}} f(x + \frac{\pi}{s}) e^{-ixs} dx$

$|\hat{f}(s)| \le \frac{1}{2} \int_{\mathbb{R}} |f(x) - f(x + \frac{\pi}{s})|\,dx$, $s \to 0$ by dominated convergence theorem.

$|\hat{f}(s)| \to 0$. $\square$

intuitive: $s \uparrow$ consider $\sin(sx)$ with large frequency in each neibouring interval, $f(x)$ is nearly constant,

and $\int f(x) \sin x\,dx = 0$

Set $T_n(x) = \frac{1}{\pi} \int_0^{\pi} \frac{\sin(n+\frac{1}{2})u}{2\sin\frac{u}{2}} [f(x+u) + f(x-u)]\,du$

Dini: f $\exists\, \delta \in (0, \pi]$, s.t. $\frac{\varphi(u)}{u}$ is absolutely intergrable on $[0, \delta] \Rightarrow \lim_{n\to\infty} T_n(x) = S$

Dirichlet-Jordan test: If periodic function $f(x)$ is of bounded variation on a period; Set $T = 2\pi$. And

a): $\exists f(x_0^+), f(x_0^-)$ b) $\exists\, a > 0$ s.t. $\int_0^a \frac{|f(x_0+t) - f(x_0^+)|}{t} dt, \int_0^a \frac{|f(x_0-t) - f(x_0^-)|}{t} dt$ converge.

then $\lim_{n\to\infty} [S_n f(x_0)] = \frac{1}{2} \cdot (f(x_0^+) + f(x_0^-))$, Since

$[\, D_n(x) = \sum_{k=-n}^{n} e^{ikx} = \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})}$, $S_n(f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t)\,dt$

let $\hat{f}(x) = \frac{f(x^+) + f(x^-)}{2}$

By property of convolution: $|S_n(f)(x) - \hat{f}(x)| = \left| \frac{1}{2\pi} \int_0^{\pi} D_n(t) f(x-t)\,dt - \hat{f}(x) \right|$

Since property of $D_n(x)$: $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t)\,dt = 1 \Rightarrow \frac{1}{2} = \frac{1}{2\pi} \int_0^{\pi} \frac{\sin[(n+\frac{1}{2})t]}{\sin(\frac{t}{2})} dt$

$\Rightarrow |S_n(f)(x) - \hat{f}(x)| = \frac{1}{2\pi} \int_0^{\pi} \sin[(n+\frac{1}{2})t] \cdot \frac{f(x+t) + f(x-t) - f(x^+) - f(x^-)}{\sin\frac{t}{2}} dt$

$n \to \infty$, $\lim_{n\to\infty} S_n(f)(x) = \hat{f}(x)$. note $|f * D_n - f| = \int_{\mathbb{R}} (f(x-y) - f(x)) D_n(y)\,dy \to 0$