Summation by parts, absolute convergence, addition and multiplication of series, rearrangement, limits of functions, continuous functions

Theorem $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

Proof Let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!}, \qquad t_n = \left(1+\frac{1}{n}\right)^n.$$

$$t_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$$
$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right).$$

Hence $t_n \le s_n$, so that

$$\limsup_{n\to\infty} t_n \le e.$$

Next, if $n \ge m$, $$t_n \ge 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right).$$

Let $n \to \infty$, keeping m fixed. We get

$$\liminf_{n\to\infty} t_n \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!},$$

so that

$$s_m \le \liminf_{n\to\infty} t_n.$$

Letting $m \to \infty$, we finally get

$$e \le \liminf_{n\to\infty} t_n. \qquad \square$$

Theorem (Root Test) *Given* Σa_n, *put* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

Then

(*a*) *if* $\alpha < 1$, Σa_n *converges;*
(*b*) *if* $\alpha > 1$, Σa_n *diverges;*
(*c*) *if* $\alpha = 1$, *the test gives no information.*

Theorem (Ratio Test) *The series* Σa_n

(*a*) *converges if* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(*b*) *diverges if* $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *for all* $n \geq n_0$, *where* n_0 *is some fixed integer.*

Examples

(*a*) Consider the series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots,$$

for which

$$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(\frac{2}{3}\right)^n=0,$$

$$\liminf_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[2n]{\frac{1}{3^n}}=\frac{1}{\sqrt{3}},$$

$$\limsup_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[2n]{\frac{1}{2^n}}=\frac{1}{\sqrt{2}},$$

$$\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{2}\left(\frac{3}{2}\right)^n=+\infty.$$

The root test indicates convergence; the ratio test does not apply.

(*b*) The same is true for the series

$$\frac{1}{2}+1+\frac{1}{8}+\frac{1}{4}+\frac{1}{32}+\frac{1}{16}+\frac{1}{128}+\frac{1}{64}+\cdots,$$

where

$$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{8},$$

$$\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}=2,$$

but

$$\lim\sqrt[n]{a_n}=\tfrac{1}{2}.$$

Definition Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a *power series*. The numbers c_n are called the *coefficients* of the series; z is a complex number.

Theorem *Given the power series $\Sigma c_n z^n$, put*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \qquad R = \frac{1}{\alpha}.$$

(If $\alpha = 0$, $R = +\infty$; if $\alpha = +\infty$, $R = 0$.) Then $\Sigma c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

Proof Put $a_n = c_n z^n$, and apply the root test:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}.$$

Note: R is called the radius of convergence of $\Sigma c_n z^n$.

Theorem *Given two sequences $\{a_n\}$, $\{b_n\}$, put*

$$A_n = \sum_{k=0}^{n} a_k$$

if $n \geq 0$; put $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Proof

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

$$= \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Theorem *Suppose*

(a) *the partial sums A_n of Σa_n form a bounded sequence;*
(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;
(c) $\lim_{n \to \infty} b_n = 0$.

Then $\Sigma a_n b_n$ converges.

Proof Choose M such that $|A_n| \leq M$ for all n. Given $\varepsilon > 0$, there is an integer N such that $b_N \leq (\varepsilon/2M)$. For $N \leq p \leq q$, we have

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\leq M\left|\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p\right| \\ &= 2Mb_p \leq 2Mb_N \leq \varepsilon.\end{aligned}$$

Convergence now follows from the Cauchy criterion.

Taking $a_n = (-1)^{n+1}$, $b_n = |c_n|$, we get

Theorem *Suppose*

(a) $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$;
(b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ $\quad (m = 1, 2, 3, \ldots)$;
(c) $\lim_{n \to \infty} c_n = 0$.

Then Σc_n converges.

Theorem *Suppose the radius of convergence of $\Sigma c_n z^n$ is 1, and suppose $c_0 \geq c_1 \geq c_2 \geq \cdots$, $\lim_{n\to\infty} c_n = 0$. Then $\Sigma c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.*

Proof Put $a_n = z^n$, $b_n = c_n$. Observe that A_n is bounded, since

$$|A_n| = \left|\sum_{m=0}^{n} z^m\right| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \leq \frac{2}{|1 - z|},$$

if $|z| = 1$, $z \neq 1$.

The theorem follows. $\square$

DEFINITION A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

THEOREM **The Absolute Convergence Test**

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof The assertion follows from the inequality

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k|,$$

plus the Cauchy criterion.

DEFINITION A series that is convergent but not absolutely convergent is called **conditionally convergent**.

Example

(a) For $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$, the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

The original series converges because it converges absolutely.

(b) For $\sum_{n=1}^{\infty}\frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, which contains both positive and negative terms, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots,$$

which converges by comparison with $\sum_{n=1}^{\infty}(1/n^2)$ because $|\sin n| \leq 1$ for every n. The original series converges absolutely; therefore it converges.

(c) The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent.

Theorem *If* $\Sigma a_n = A$, *and* $\Sigma b_n = B$, *then* $\Sigma(a_n + b_n) = A + B$, *and* $\Sigma c a_n = cA$, *for any fixed* c.

Proof Let

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k.$$

Then

$$A_n + B_n = \sum_{k=0}^{n} (a_k + b_k).$$

Since $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$, we see that

$$\lim_{n\to\infty} (A_n + B_n) = A + B.$$

Definition Given Σa_n and Σb_n, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad (n = 0, 1, 2, \ldots)$$

and call Σc_n the *product* of the two given series.

This definition may be motivated as follows. If we take two power series $\Sigma a_n z^n$ and $\Sigma b_n z^n$, multiply them term by term, and collect terms containing the same power of z, we get

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \cdots \\ &= c_0 + c_1 z + c_2 z^2 + \cdots .\end{aligned}$$

Setting $z = 1$, we arrive at the above definition.

If
$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k, \qquad C_n = \sum_{k=0}^{n} c_k,$$
and $A_n \to A$, $B_n \to B$, then it is not at all clear that $\{C_n\}$ will converge to AB, since we do not have $C_n = A_n B_n$.

Example The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

converges . We form the product of this series with itself and obtain

$$\sum_{n=0}^{\infty} c_n = 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{3}} + \frac{1}{\sqrt{4}}\right) + \cdots,$$

$$c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}.$$

From
$$(n-k+1)(k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \le \left(\frac{n}{2}+1\right)^2,$$

We get
$$|c_n| \ge \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}.$$
Thus $\sum_{n=0}^{\infty} c_n$ diverges.

Theorem *Suppose* (a) $\sum_{n=0}^{\infty} a_n$ *converges absolutely,* (b) $\sum_{n=0}^{\infty} a_n = A$,
(c) $\sum_{n=0}^{\infty} b_n = B$, (d) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ $(n = 0, 1, 2, \ldots)$.
Then

$$\sum_{n=0}^{\infty} c_n = AB.$$

Proof Put

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k, \qquad C_n = \sum_{k=0}^{n} c_k, \qquad \beta_n = B_n - B.$$

Then

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots + a_n(B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 \end{aligned}$$

Put

$$\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 .$$

We wish to show that $C_n \to AB$. Since $A_n B \to AB$, it suffices to show that

$$\lim_{n\to\infty} \gamma_n = 0.$$

Put

$$\alpha = \sum_{n=0}^{\infty} |a_n| .$$

[It is here that we use (*a*).] Let $\varepsilon > 0$ be given. By (*c*), $\beta_n \to 0$. Hence we can choose N such that $|\beta_n| \leq \varepsilon$ for $n \geq N$, in which case

$$\begin{aligned} |\gamma_n| &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0| \\ &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \varepsilon\alpha. \end{aligned}$$

Keeping N fixed, and letting $n \to \infty$, we get

$$\limsup_{n\to\infty} |\gamma_n| \leq \varepsilon\alpha,$$

since $a_k \to 0$ as $k \to \infty$. Since ε is arbitrary, we get

$$\lim_{n\to\infty} \gamma_n = 0. \qquad \square$$

Theorem *If the series* Σa_n, Σb_n, Σc_n *converge to* A, B, C, *and* $c_n = a_0 b_n + \cdots + a_n b_0$, *then* $C = AB$.

Definition Let $\{k_n\}$, $n = 1, 2, 3, \ldots,$ be a sequence in which every positive integer appears once and only once . Putting

$$a_n' = a_{k_n} \qquad (n = 1, 2, 3, \ldots),$$

we say that $\Sigma a_n'$ is a *rearrangement* of Σa_n .

Let $\mathbf{Z}^+$ be the set of non-negative integers. A bijection from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ is called a permutation of $\mathbf{Z}^+$.

Theorem *Suppose that $\sum_{j=0}^{\infty} z_j$ converges absolutely, and that $\sum_{j=0}^{\infty} z_j = s$. If σ is a permutation of $\mathbf{Z}^+$ then $\sum_{j=0}^{\infty} z_{\sigma(j)}$ converges to s.*

Proof By considering real and imaginary parts, it is enough to consider an absolutely convergent real series $\sum_{j=0}^{\infty} a_j$. First consider the case where all the terms are non-negative. If $n \in \mathbf{Z}^+$ and $k = \sup\{\sigma(j) : 1 \leq j \leq n\}$ then $\sum_{j=0}^{n} a_{\sigma(j)} \leq \sum_{i=0}^{k} a_i \leq s$. Thus $\sum_{j=0}^{\infty} a_{\sigma(j)}$ converges, and $\sum_{j=0}^{\infty} a_{\sigma(j)} \leq s$. By the same token,

$$s = \sum_{j=0}^{\infty} a_j = \sum_{j=0}^{\infty} a_{\sigma^{-1}\sigma(j)} \leq \sum_{j=0}^{\infty} a_{\sigma(j)}.$$

In the general case, write $a_j = a_j^+ - a_j^-$. Then $a_{\sigma(j)} = a_{\sigma(j)}^+ - a_{\sigma(j)}^-$, so that $\sum_{j=0}^{\infty} a_{\sigma(j)}$ converges to $\sum_{j=0}^{\infty} a_{\sigma(j)}^+ - \sum_{j=0}^{\infty} a_{\sigma(j)}^-$, and

$$\sum_{j=0}^{\infty} a_{\sigma(j)} = \sum_{j=0}^{\infty} a_{\sigma(j)}^+ - \sum_{j=0}^{\infty} a_{\sigma(j)}^- = \sum_{j=0}^{\infty} a_j^+ - \sum_{j=0}^{\infty} a_j^- = \sum_{j=0}^{\infty} a_j. \qquad \square$$

Definition Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or

$$\lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta.$$

The symbols d_X and d_Y refer to the distances in X and Y, respectively.

Theorem

$$(4) \quad \lim_{x \to p} f(x) = q \quad \Leftrightarrow \quad (5) \quad \lim_{n \to \infty} f(p_n) = q$$

for every sequence $\{p_n\}$ *in* E *such that*

$$(6) \qquad p_n \neq p, \qquad \lim_{n \to \infty} p_n = p.$$

Proof Suppose (4) holds. Choose $\{p_n\}$ in E satisfying (6). Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ if $x \in E$ and $0 < d_X(x, p) < \delta$. Also, there exists N such that $n > N$ implies $0 < d_X(p_n, p) < \delta$. Thus, for $n > N$, we have $d_Y(f(p_n), q) < \varepsilon$, which shows that (5) holds.

Conversely, suppose (4) is false. Then there exists some $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$ (depending on δ), for which $d_Y(f(x), q) \geq \varepsilon$ but $0 < d_X(x, p) < \delta$. Taking $\delta_n = 1/n$ ($n = 1, 2, 3, \ldots$), we thus find a sequence in E satisfying (6) for which (5) is false.

Definition (Limiting value of a function). Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, and let $f : X \to Y$ be a function. If $x_0 \in X$ is an adherent point of E, and $L \in Y$, we say that $f(x)$ *converges to* L *in* Y *as* x *converges to* x_0 *in* E, or write $\lim_{x \to x_0; x \in E} f(x) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), L) < \varepsilon$ for all $x \in E$ such that $d_X(x, x_0) < \delta$.

Remark Some authors exclude the case $x = x_0$ from the above definition, thus requiring $0 < d_X(x, x_0) < \delta$. In our current notation, this would correspond to removing x_0 from E, thus one would consider $\lim_{x \to x_0; x \in E \setminus \{x_0\}} f(x)$ instead of $\lim_{x \in x_0; x \in E} f(x)$.

Remark The requirement that x_0 be an adherent point of E is necessary for the concept of limiting value to be useful, otherwise x_0 will lie in the exterior of E, the notion that $f(x)$ converges to L as x converges to x_0 in E is vacuous (for δ sufficiently small, there are no points $x \in E$ so that $d(x, x_0) < \delta$).

We have two definitions about the limiting value of a map between metric spaces and shall use the second one.

Definition Let $D \subset \mathbb{R}$ (or $\mathbb{C}$) and $f: D \to \mathbb{R}$ (or $\mathbb{C}$) be a function. We say that $\lim_{x \to p} f(x) = y$ if and only if for every sequence $(x_n)_{n \in \mathbb{N}} \subset D$ with $\lim_{n \to \infty} x_n = p$ we have $\lim_{n \to \infty} f(x_n) = y$.

Theorem $\lim_{x \to p} f(x) = y \quad \Leftrightarrow \quad \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D \quad \textit{with } |x - p| < \delta :$

$$|f(x) - y| < \varepsilon . \tag{1}$$

Proof. " $\Leftarrow$ " Let $(x_n)_{n \in \mathbb{N}} \subset D$ be a sequence with $\lim_{n \to \infty} x_n = p$. We have

$$\forall \delta\ \exists N \in \mathbb{N}\ \forall n \geq N : |x_n - p| < \delta . \tag{2}$$

For $\varepsilon > 0$ we determine $\delta > 0$ as in (1) and then N as in (2): It follows that for $n \geq N$:

$$|f(x_n) - y| < \varepsilon.$$

We have therefore shown

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N : |f(x_n) - y| < \varepsilon,$$

so $\lim_{n \to \infty} f(x_n) = y$ and therefore, by definition, $\lim_{x \to p} f(x) = y$.

" $\Rightarrow$ " If (1) is not fulfilled then

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in D \quad \text{with } |x - p| < \delta \quad \text{but } |f(x) - y| > \varepsilon . \tag{3}$$

For $n \in \mathbb{N}$ we set $\delta = \frac{1}{n}$ and determine $x_n = x$ corresponding to δ as in (3). Then $|x_n - p| < \frac{1}{n}$ and so $\lim_{n \to \infty} x_n = p$, but for ε as in (3) we have

$$|f(x_n) - y| > \varepsilon,$$

and therefore $\lim_{n \to \infty} f(x_n) \neq y$. $\square$

Def. $(X, d), (Y, d_1)$ are metric spaces, the map

$$T: \ X \ni x \to y = Tx \in Y.$$

For $x_0 \in X, T \ is \ continuous \ at \ x_0$ iff $\forall \epsilon > 0, \exists \delta > 0$, s.t.

$$d(x, x_0) < \delta \Rightarrow d_1(Tx, Tx_0) < \epsilon.$$

T is continuous on $D \subset X$ iff $\forall x \in D$, it is continuous at x. T is called uniformly continuous on D if δ can be the same for all $x_0 \in D$, i.e., δ is independent of $x_0 \in D$.

T is a homeomorphism iff it is bijective, and T and T^{-1} are continuous.

Remark If $f : X \to Y$ is continuous, and K is any subset of X, then the restriction $f|_K : K \to Y$ of f to K is also continuous

Proposition (Continuity preserved by composition). *Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces.*

(a) *If $f : X \to Y$ is continuous at a point $x_0 \in X$, and $g : Y \to Z$ is continuous at $f(x_0)$, then the composition $g \circ f : X \to Z$, defined by $g \circ f(x) := g(f(x))$, is continuous at x_0.*

(b) *If $f : X \to Y$ is continuous, and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is also continuous.*

Example If $f : X \to \mathbf{R}$ is a continuous function, then the function $f^2 : X \to \mathbf{R}$ defined by $f^2(x) := f(x)^2$ is automatically continuous also. This is because we have $f^2 = g \circ f$, where $g : \mathbf{R} \to \mathbf{R}$ is the squaring function $g(x) := x^2$, and g is a continuous function.

Theorem T is continuous at $x_0 \Leftrightarrow$
$\forall \{x_n\} \subset X, x_n \to x_0 \Rightarrow Tx_n \to Tx_0$.

Proof. $\Rightarrow$ It is a consequence of the the definition

$\Leftarrow$: if T is not continuous at x_0, then $\exists \epsilon_0 > 0$, s.t.

$$\forall n \in \mathbb{N}, \exists x_n \in X, d(x_n, x_0) < \frac{1}{n},\ d_1(Tx_n, Tx_0) \geq \epsilon_0 .$$

Hence, $x_n \to x_0$, $Tx_n \nrightarrow Tx_0$.

$X \xrightarrow{f} Y$

Theorem T is continuous
$\Leftrightarrow \forall$open $G \subset Y, T^{-1}(G)$ is open in X.

Proof. $\Rightarrow$: Let $G \subset Y$ be open, $x_0 \in T^{-1}(G)$, then $T(x_0) \in G$, so $\exists\ B(Tx_0, \epsilon) \subset G$. By continuity, $\exists \delta > 0$ s.t. $T(B(x_0, \delta)) \subset B(Tx_0, \epsilon)$. Thus, $B(x_0, \delta) \subset T^{-1}(B(Tx_0, \epsilon)) \subset G^{-1}(G)$. Hence, $T^{-1}(G)$ is open.

$\Leftarrow$: Let $\{x_n\} \subset X, x_n \to x_0$. Fix $\epsilon > 0$. Since $T^{-1}(B(Tx_0, \epsilon))$ is open, $\exists \delta > 0$ s.t. $B(x_0, \delta) \subset T^{-1}(B(Tx_0, \epsilon))$. Thus, $T(B(x_0, \delta)) \subset B(Tx_0, \epsilon)$. From $x_n \to x_0$, $\exists N_0 \in \mathbb{N}$ s.t. $d(x_n, x_0) < \delta, \forall n \geq N_0$. Hence, $d(Tx_n, Tx_0) < \epsilon$, $\forall n \geq N_0$.

Corollary *A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.*

This follows from the theorem, since a set is closed if and only if its complement is open, and since $f^{-1}(E^c) = [f^{-1}(E)]^c$ for every $E \subset Y$.