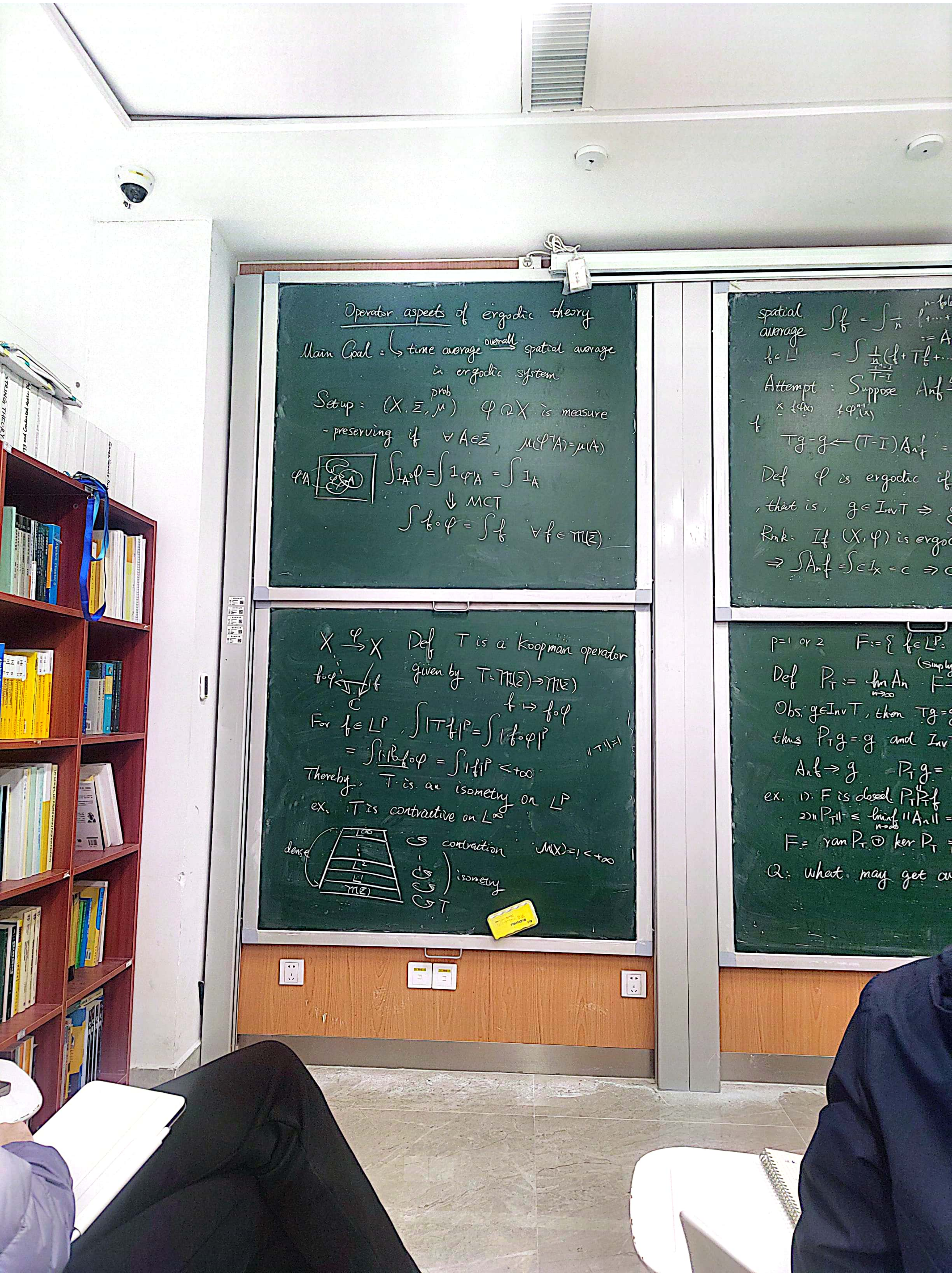
Operator aspects of ergodic theory
Main Goal: time average $\xrightarrow{\text{overall}}$ spatial average in ergodic system
Setup: $(X, \Sigma, \overset{\text{prob}}{\mu})$ $\varphi \circlearrowright X$ is measure-preserving if $\forall A \in \Sigma$, $\mu(\varphi^{-1}A) = \mu(A)$
$\varphi^{-1}A$
$\int 1_A \circ \varphi = \int 1_{\varphi^{-1}A} = \int 1_A$
$\Downarrow$ MCT
$\int f \circ \varphi = \int f \quad \forall f \in \mathcal{M}(\Sigma)$.
$X \xrightarrow{\varphi} X$
$f \circ \varphi$, f, $\mathbb{C}$
Def T is a Koopman operator given by $T: \mathcal{M}(\Sigma) \to \mathcal{M}(\Sigma)$, $f \mapsto f \circ \varphi$
For $f \in L^p$, $\int |Tf|^p = \int |f \circ \varphi|^p = \int |\cdot|^p \circ f \circ \varphi = \int |f|^p < +\infty$
$\|T\| = 1$
Thereby, T is an isometry on L^p
ex. T is contractive on L^∞
dense
L^∞
L^2
L^1
$\mathcal{M}(\Sigma)$
contraction
$\mu(X) = 1 < +\infty$
isometry
T

扫描全能王 创建

$(\ldots\Sigma,\mu)$ $\varphi \circlearrowright X$ is measure preserving if $\forall A \in \Sigma$ $\mu(\varphi^{-1}A) = \mu(A)$

$$\int 1_A \circ \varphi = \int 1_{\varphi^{-1}A} = \int 1_A$$

$\Downarrow$ MCT

$$\int f \circ \varphi = \int f \quad \forall f \in \mathcal{M}(\Sigma)$$

T is an isometry on L^p

T is contractive on L^∞

$L^\infty \circlearrowright$ contraction

$\vdots$

$L^2 \circlearrowright$) isometry

$L^1 \circlearrowright$

$\mathcal{M}(\Sigma) \circlearrowright T$

p=1 or 2 $\quad F := \{ f \in L^p : L^p\text{-}\lim_{n\to\infty} A_n f \text{ exists} \}$

Def $P_T := \lim_{n\to\infty} A_n \quad F \to L^p \cap \mathrm{Inv}\,T \quad f \mapsto \lim_{n\to\infty} A_n f$ (simply $\mathrm{Inv}\,T$)

Obs. $g \in \mathrm{Inv}\,T$, then $Tg = g \quad T^n g = g \quad A_n g = g$

thus $P_T g = g$ and $\mathrm{Inv}\,T \subset F$ Moreover

$A_n f \to g \quad P_T g = g = P_T f \quad P_T^2 = P_T$ on F

条件期望

ex. 1): F is closed. $P_T P_T f$

2): $\|P_T\| \le \liminf_{n\to\infty} \|A_n\| = 1 \quad \ker(I-T)$

$F = \mathrm{ran}\,P_T \oplus \ker P_T = \mathrm{Inv}\,T \oplus \ker P_T$?

Q: What may get averaged out by time?

spatial average $\int f = \int \frac{1}{n}(\overbrace{f + \cdots + f}^{n\text{-folds}}) = \int \frac{1}{n}(f + f\circ\varphi + \cdots + f\circ\varphi^{n-1})$

time-average

$f \in L^1 \quad = \int \frac{1}{n}(f + Tf + \cdots + T^{n-1}f) := \int A_n f \quad \left(:= A_n f,\ \frac{T^n - I}{T - I}\right)$

Attempt: Suppose $A_n f \xrightarrow{L^1} g$. Obs. $g \in \mathrm{Inv}\,T$

$x \quad f(x) \quad f\circ\varphi^{n-1}(x)$ $\qquad$ i.e. $Tg = g$

$$Tg - g \leftarrow (T-I)A_n f = \underbrace{\frac{T^n - I}{n}}_{\le \frac{2}{n}} f \to 0$$

Def φ is ergodic if $\dim_{\mathbb{C}} \mathrm{Inv}\,T = 1$

, that is, $g \in \mathrm{Inv}\,T \Rightarrow g = c 1_X$ for some $c \in \mathbb{C}$.

Rmk: If (X,φ) is ergodic and $A_n f \xrightarrow{L^1} \int f 1_X$

$\Rightarrow \int A_n f = \int c 1_X = c \Rightarrow c = \int f$

When p=2

Sketch:

$L^2 = \mathrm{In}\ldots$

When p=1 $L\ldots$

$L^1 = L\ldots$

Thm. If $T\ldots$

then $L\ldots$

Cor. If $(X\ldots$

$A_n f\ldots$

Obs. $x \ldots$

$Tg = g \quad T\ldots$

Obs. $\ker P\ldots$

$g \in \mathrm{ran}(I - P_T)$

$I - A_n = \frac{1}{n}(\ldots$

Can Koopman operator be adjusted to project I to any sub-1D space of L^1 ?

扫描全能王 创建