Lemma: $f(x), g(x) \in \mathbb{F}[x]$, $f(x), g(x) \neq$ Constant

$\deg(f,g) = (m,n)$ then f, g are coprime

$\Leftrightarrow \exists \deg u < n, \deg v < m$, s.t. $u(x) f(x) = v(x) g(x)$

Proof: intuition

$$\begin{cases} a_0 u_0 = b_0 v_0 \\ a_1 u_0 + a_0 u_1 = b_1 v_0 + b_0 v_1 \\ \quad \cdots \\ a_m u_{n-2} + a_{m-1} u_{n-1} = b_n v_{m-2} + b_{n-1} v_{m-1} \\ a_m u_{n-1} = b_n v_{m-1} \end{cases}$$

$$\Rightarrow \quad \mathrm{res}(f,g) = \begin{vmatrix} a_0 & a_1 & \cdots & a_m & & \\ & a_0 & \cdots & a_{m-1} & a_m & \\ & & \ddots & & \vdots & a_m \\ & & a_0 & & & \\ b_0 & b_1 & \cdots & b_n & & \\ & b_0 & b_1 & \cdots & b_n & \\ & & \ddots & b_0 & b_1 & \cdots & b_n \end{vmatrix}$$

: 结式,
Sylvester行列式

$\Rightarrow f, g$ not coprime $\Leftrightarrow \mathrm{res}(f,g) = 0$

e.g. $\begin{cases} 5x^2 - 6xy + 5y^2 - 16 = 0 \\ 2x^2 - (1+y)x + y^2 + y - 4 = 0 \end{cases}$

$f(x) = 5x^2 - 6xy + 5y^2 - 16$

$g(x) = 2x^2 - (1+y)x + y^2 - y - 4$

Treat y as constant, what $\exists x$ satisfies 2 equation

i.e. $\mathrm{res}(f,g) = 0$

$$\mathrm{res}(f,g) = \begin{vmatrix} 5 & -6y & 5y^2-16 & 0 \\ 0 & 5 & -6y & 5y^2-16 \\ 2 & -(1+y) & y^2-y-4 & 0 \\ & 2 & -(1+y) & y^2-y-4 \end{vmatrix} = 32(y-2)(y-1)(y+1)^2 = 0$$

1° $y = 2$

$\begin{cases} 5x^2 - 12x + 4 = 0 & x_1 = \frac{2}{5}, x_2 = 2 \\ 2x^2 - 3x - 2 = 0 & x_1 = -\frac{1}{2}, x_2 = 2 \end{cases}$

2° $y = 1$ $\Rightarrow x = -1$

3° $y = -1$ $\Rightarrow x = 1$

4°