Talor Series and Fourier Series.

1^{st} Preliminaries:

Fourier series: $f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos n\theta+b_n\sin n\theta)$, $a_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\cos n\theta d\theta$, $b_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\sin n\theta d\theta$

Talor series: $f(x)=f(0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

eg: Find Fourier series of $\cos^n\theta$, let $z=e^{i\theta}$, $\cos^n\theta=\left(\frac{z+\bar{z}}{2}\right)^n=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}z^{2k-n}$

$\cos^3\theta=\frac{1}{8}[z^3+\frac{1}{z^3}+3(z+\frac{1}{z})]\overset{z=e^{i\theta}}{=}\frac{1}{4}\cos 3\theta+\frac{3}{4}\cos\theta$

2^{nd} For complex function $f(z)=\sum_{i=0}^{\infty}a_i z^i$, $\to$ fix θ, $f(x)=\sum_{i=1}^{\infty}b_n x^i$

$\to$ fix r: $f(re^{i\theta})=a_0+a_1\cos\theta+\cdots=Re[a_0+a_1e^{i\theta}\cdots]=Re[f(e^{i\theta})]$

eg1: $f(z)=\frac{1}{1-z}=\frac{1}{1-re^{i\theta}}=\frac{(1+re^{-i\theta})}{(1-re^{i\theta})(1+re^{-i\theta})}=\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}+i\left(\frac{r\sin\theta}{1-2r\cos\theta+r^2}\right)$

$Re(f)=\frac{1-r\cos\theta}{r^2-2\cos\theta+1}=1+r\cos\theta+r^2\cos 2\theta+\cdots=1+\sum_{k=1}^{\infty}r^n\cos n\theta$

eg2: $f(z)=\ln(1+e^{i\theta})=\ln[2\cos\frac{\theta}{2}]+i\frac{\theta}{2}$

$\ln(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}\cdots$, $Im[\ln(1+e^{i\theta})]=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\theta}{n}=\frac{\theta}{2}$

$\int\theta d\theta=\frac{\theta^2}{2}=\int_{-\pi}^{\pi}\frac{\frac{\theta^2}{2}}{2\pi}d\theta+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\theta$

(For detail: $\frac{\theta^2}{2}=C+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\theta$, $\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\theta^2}{2}d\theta=C+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\langle\cos n\theta,1\rangle$)

$\theta=\pi$, $\frac{\pi^2}{2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(-1)^n}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ $=C$

For Real part, $\ln(\cos\theta)=-\ln 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos 2n\theta$

$\frac{1}{1+z}=\frac{1+r\cos\theta+ir\sin\theta}{1+2r\cos\theta+r^2}$

$P_r(\theta)=\frac{1}{1-z}+\frac{\bar{z}}{1-\bar{z}}$

Poission kernel