$f(x,y)=\begin{cases}\frac{1}{2}e^{-\frac{y}{2}};0<x<1,y>0\\0,\text{otherwise.}\end{cases}.Z=X-Y.$Show the $p.d.f.$ of Z:

Démonstration.

D:={(x,y)|$0<x<1,y>0$}.Set $\begin{cases}U=X,\\Z=X-Y.\end{cases}\Rightarrow\begin{cases}X=U,\\Y=U-Z.\end{cases}\Rightarrow J(X,Y\to U,Z)=\left\|\begin{matrix}1&0\\1&-1\end{matrix}\right\|=1\Rightarrow g(u,z)=f(x,y)*J(X,Y\to U,Z)=\frac{1}{2}\exp\left(\frac{z}{2}-\frac{u}{2}\right)\mathbb{I}(D)$, equivalently, $D=\{(u,z)|0<u<1,z<u\}$.

Then the result can be determined by the value of z:

1.$0\leqslant Z<1$: $f(z)=\int_z^1\frac{1}{2}\exp\left(\frac{z}{2}-\frac{u}{2}\right)\mathrm{du}=1-\exp\left(\frac{z}{2}-\frac{1}{2}\right)$; 2.$Z<0$, $f(z)=\int_0^1\frac{1}{2}\exp\left(\frac{z}{2}-\frac{u}{2}\right)\mathrm{du}=e^{\frac{z}{2}}\left(1-e^{-\frac{1}{2}}\right)$; 3.$Z\geqslant1$, $D=\emptyset\Rightarrow f(z)=0$. □