Theorem If $\{I_n\}$ is a sequence of finite closed intervals in $\mathbb{R}$ such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \cdots)$, then $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof Let $I_n = [a_n, b_n]$ and let E be the set of all a_n. Then $E \neq \emptyset$ and bounded above by b_1. Let $x = \sup E$. If $m, n \in \mathbb{N}$, then

$$a_n \leq a_{m+n} \leq b_{m+n} \leq b_m,$$

so that $x \leq b_m$ for each m. Since $a_m \leq x$, we see that

$$x \in I_m \text{ for } m = l, 2, 3, \cdots .$$

Thus $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \cdots)$, then $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof Let I_n consist of all points $x = (x_1, \cdots, x_k)$ such that

$$a_{n,j} \leq x_j \leq b_{n,j} \ (1 \leq j \leq k; n = 1, 2, 3, ...),$$

and put $I_{n,j} = [a_{n,j}, b_{n,j}]$. For each j, the sequence $\{I_{n,j}\}$ satisfies the hypotheses of the above theorem. Hence there are real numbers $x_j^\star (l \leq j \leq k)$ such that

$$a_{n,j} \leq x_j^\star \leq b_{n,j} \ (1 \leq j \leq k; n = I, 2, 3, ...),$$

Setting $x^\star = (x_1^\star, ..., x_k^\star)$, we see that $x^\star \in I_n$ for $n = 1, 2, 3, \cdots$. The theorem follows.

Theorem Every k-cell is compact.

Proof Let I be a k-cell, consisting of all points $x = (x_1, \cdots, x_k)$ such that $a_i \le x_i \le b_i (1 \le i \le k)$. Observe that there is a $\delta > 0$ such that $|x - y| < \delta$ for all $x, y \in I$.
Suppose $\exists$ an open cover $\{G_a\}$ of I which contains no finite subcover of I. Put $c_j = (a_j + b_j)/2$. The intervals $[a_j, c_j]$ and $[c_j, b_j]$ determine 2^k k-cells Q_i whose union is I. At least one of these sets Q_i, call it I_1, cannot be covered by any finite subcollection of $\{G_a\}$(otherwise I could be so covered). We next subdivide I_1 and continue the process. The sequence $\{I_n\}$ satisfies
(a) $I_1 \supset I_2 \supset \cdots$;
(b) I_n is not covered by any finite subcollection of $\{G_a\}$;
(c) if $x, y \in I_n$, then $|x - y| \le 2^{-n}\delta$.
There is a point $x^\star$ which lies in every I_n. For some b, $x^\star \in G_b$. Since G_b is open, $\exists r > 0$ such that $|y - x^\star| < r \Rightarrow y \in G_b$. If n is so large that $2^{-n}\delta < r$, then (c) implies that $I_n \subset G_b$, which contradicts (b).

Theorem Let E be a subset of $\mathbb{R}^k$. The following conditions are equivalent.
(a) E is closed and bounded.
(b) E is compact.
(c) Every infinite subset of E has a limit point in E.

Proof If (a) holds, then $E \subset I$ for some k-cell I. Since a k-cell is compact and any closed subset of a compact set in a metric space is compact, we know that (b) is true. If (b) is true, then (c) is also true since every infinite subset of a compact set C in a metric space has a limit point in C.
Let us prove that (c) implies (a). If E is not bounded, then E contains infinitely many points x_k with

$$|x_k| > n_k \ (n = 1, 2, 3, ...),$$

where $n_k \in \mathbb{N}, k = 1, 2, 3, \cdots$, and $n_1 < n_2 < \cdots \to \infty$. The set S consisting of these points x_k is infinite and has no limit point in $\mathbb{R}^k$. This is a contraction. Thus (c) implies that E is bounded.

If E is not closed, then there is a point $x_0 \in \mathbb{R}^k$ which is a limit point of E but not a point of E. For $n = 1, 2, 3, ...$, there are points $x_n \in E$ such that $|x_n - x_0| < 1/n$. Let S be the set of these points x_n. Then S is infinite (otherwise there exists $\epsilon_0 > 0$ such that $|x_n - x_0| \geq \epsilon_0$ for any $n \in \mathbb{N}$), S has x_0 as a limit point, and S has no other limit point in $\mathbb{R}^k$. For if $y \in \mathbb{R}^k$, $y \neq x_0$ then

$$
\begin{aligned}
|x_n - y| &\geq |x_0 - y| - |x_n - x_0| \\
&\geq |x_0 - y| - \frac{1}{n} \geq \frac{1}{2}|x_0 - y|
\end{aligned}
$$

for sufficiently large n; this shows that y is not a limit point of S. Thus S has no limit point in E. This is a contradiction.

Theorem (Weierstrass) Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Theorem A nonempty perfect set P in $\mathbb{R}^k$ is uncountable.

Proof Since P has limit points, P is infinite. Assume P is countable and denote the points of P by $x_1, x_2, \cdots$. We construct a sequence $\{V_n\}$ of neighborhoods, as follows.
Let V_1 be any neighborhood of x_1. Since x_1 is a limit point of P, V_1 contains a point of P which is different from x_1. Thus there is a neighborhood V_2 such that $\overline{V_2} \subset V_1$, $x_1 \notin \overline{V_2}$ and $\overline{V_2} \cap P \neq \emptyset$. Continuing in this way, we can find neighborhoods V_{n+1} such that i) $\overline{V}_{n+1} \subset V_n$, ii) $x_n \notin \overline{V}_{n+1}$ and iii) $\overline{V}_{n+1} \cap P \neq \emptyset$. $\overline{V}_n$ is compact since it is bounded and closed. Set $K_n = \overline{V}_n \cap P$. Since $x_n \notin K_{n+1}$, no point of P lies in $\cap_{n=1}^{\infty} K_n$. Since $K_n \subset P$, this implies that $\cap_{n=1}^{\infty} K_n = \emptyset$. But each K_n is compact and nonempty, by (iii), and $K_n \supset K_{n+1}$. Thus $\cap_{n=1}^{\infty} K_n \neq \emptyset$. This is a contradiction.

Corollary Every interval $[a, b] (a < b)$ is uncountable. In particular, the set of all real numbers is uncountable.

Definition Two subsets A and B of a metric space X are said to be separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A. A set $E \subset X$ is connected if E is not a union of two nonempty separated sets.

Remark Separated sets are disjoint, but disjoint sets need not be separated. For example, the interval $[0, 1]$ and the segment $(1, 2)$ are not separated. However, the segments $(0, 1)$ and $(1, 2)$ are separated.

Theorem A subset E of $\mathbb{R}$ is connected if and only if it has the following property: If $x \in E,\ y \in E$, and $x < z < y$, then $z \in E$.