Part III Advanced Probability
Based on lectures by P. Sousi

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1 Conditional Expectation

Lecture 1

1.1 Basic definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Remember the following definitions

Definition 1.1 (Sigma algebra). $\mathcal{F}$ is a sigma algebra if and only if: $(\mathcal{F} \in \mathcal{P}\Omega)$

1. $\Omega \in \mathcal{F}$
2. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
3. $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{F} \implies \bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{F}$

Definition 1.2 (Probability measure). $\mathbb{P}$ is a probability measure if

1. $\mathbb{P} : \mathcal{F} \to [0, 1]$ (i.e. a set function)
2. $\mathbb{P}(\Omega) = 1$, and $\mathbb{P}(\varnothing) = 0$
3. $(A_n)_{n\in\mathbb{N}}$ pairwise disjoint $\implies \mathbb{P}\left(\bigcup\limits_{n\in\mathbb{N}} A_n\right) = \sum\limits_{n=1}^{\infty} \mathbb{P}(A_n)$.

Definition 1.3 (Random Variable). $X : \Omega \to \mathbb{R}$ is a random variable if for all B open in $\mathbb{R}$, $X^{-1}(B) \in \mathcal{F}$.

Remark. Observe that the sigma algebra $\mathcal{G} = \{B \subseteq \mathbb{R} : X(B) \in \mathcal{F}\} \supseteq \mathcal{O} \implies \mathcal{G} \supseteq \mathcal{B}(\mathbb{R})$, the former being the collection of open sets in $\mathbb{R}$ and the latter the Borel sigma algebra on $\mathbb{R}$ with the usual topology, namely, $\sigma(\mathcal{O})$ (see below for the notation).

Let $\mathcal{A}$ be a collection of subsets of Ω. We define

$$\begin{aligned} \sigma(\mathcal{A}) &= \text{smallest sigma algebra containing } \mathcal{A} \\ &= \bigcap \{\mathcal{T} : \mathcal{T} \text{ sigma algebra containing } \mathcal{A}\}. \end{aligned}$$

Definition 1.4 (Borel sigma algebra on $\mathbb{R}$). Let $\mathcal{O} = \{\text{open sets}\mathbb{R}\}$. Then, the Borel sigma algebra $\mathcal{B}(\mathbb{R})(:= \mathcal{B})$ is defined as above, namely,

$$\mathcal{B}(\mathbb{R}) := \sigma(\mathcal{O}).$$

Let $(X_i)_{i\in I}$ be a family of random variables, then $\sigma(X_i : i \in I)$ = the smallest sigma algebra that makes them all measurable. We also have the characterisation $\sigma(X_i : i \in I) = \sigma(\{\underbrace{\{\omega \in \Omega : X_i(\omega) \in B\}}_{X_i^{-1}(B)}, i \in I, B \in \mathcal{B}(\mathbb{R})\})$.

1.2 Expectation

Note we use the following for the indicator function on some event A

$$\mathbf{1}(A)(x) = \mathbf{1}(x \in A) := \left.\begin{matrix} 1, & x \in A \\ 0, & x \notin A \end{matrix}\right\}, \quad A \in \mathcal{F}.$$

We now begin the construction of the expectation of generic random variables.

Positive simple random variables: $X = \sum\limits_{i=1}^{n} \mathbf{1}(A_i), c_i \geqslant 0, A_i \in \mathcal{F}$..

$$\mathbb{E}[X] := \sum_{i=1}^{n} c_i \mathbb{P}(A_i).$$

Non-negative random variables: $(X \geqslant 0)$. We proceed by approximation. Namely, let $X_n(\omega) := 2^{-n}\lfloor 2^{-n} \cdot X(\omega)\rfloor \wedge n \uparrow X(\omega), n \to \infty$. Now, by monotone convergence,

$$\mathbb{E}[X] :=\uparrow \lim_{n\to\infty} \mathbb{E}[X_n] = \sup \mathbb{E}[X].$$

General random variables: Have the decomposition $X = X^+ - X^-$, where $X^+ = X \vee 0$, $X^- = -X \wedge 0$. If one of $\mathbb{E}[X^+], \mathbb{E}[X^-] < \infty$ then set

$$\mathbb{E}[X] := \mathbb{E}[X^+] - \mathbb{E}[X^-].$$

Definition 1.5. X is called integrable if $\mathbb{E}[|X|] < \infty$.

Definition 1.6. Let $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$. Then for all $A \in \mathcal{F}$, set

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Now for an integer-valued random variable X, we set:

$$\mathbb{E}[X|B] := \frac{\mathbb{E}[X \cdot \mathbf{1}_B]}{\mathbb{P}(B)}$$

1.3 Conditional expectation with respect to countably generated sigma algebras

Lecture 2 We now extend the definition of the conditional expectation for a countably generated sigma algebra. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We call the sigma algebra $\mathcal{G}$ coutnably generated if there exists a colection $(B_n)_{n\in\mathbb{N}}$ of pairwise disjoint events such that $\bigcup_{n\in I} B_n = \Omega$ with (I countable) and $\mathcal{G} = \sigma(B_i : i \in I)$.

Let X be an integrable random variable. We want to define $\mathbb{E}[X|\mathcal{G}]$.

Define $X'(\omega) = \mathbb{E}[X|B_i]$, whenever $w \in B_i$, i.e.

$$X' = \sum_{i\in I} \mathbf{1}(B_i) \cdot \mathbb{E}[X|B_i].$$

We make the convention that $\mathbb{E}[X|B_i] = 0$ if $\mathbb{P}(B_i) = 0$. It is easy to check that X' is $\mathcal{G}-$measurable. We also have that

$$\mathcal{G} = \left\{ \bigcup_{j\in} B_j : J \subseteq I \right\}$$

and X' satisfies for all $G \in \mathcal{G}$:$\mathbb{E}[X \cdot \mathbf{1}_G] = \mathbb{E}[X' \cdot \mathbf{1}_G]$ and

$$\begin{aligned}
\mathbb{E}[|X'|] &\leqslant \mathbb{E}\left[\sum_{i\in I} |\mathbb{E}[X|B_i]\mathbf{1}(B_i)\right] \\
&= \sum_{i\in I} \mathbb{P}(B_i) \cdot |\mathbb{E}[X|B_i]| \\
&\leqslant \sum_{i\in I} \mathbb{P}(B_i) \cdot \underbrace{\mathbb{E}[X \cdot \mathbf{1}(B_i)]}_{\mathbb{P}(B_i)} \\
&= \mathbb{E}[|X|] < \infty.
\end{aligned}$$

1.4 General case

We say $A \in \mathcal{F}$ happens a.s. if $\mathbb{P}(A) = 1$. Recall (from measure theory and basic functional analysis):

Theorem 1.1 (Monotone Convergence Theorem (MCT)). Let $(X_n)_{n\in\mathbb{N}}$ be such that $X_n \geqslant 0, X$ be random variables such that $X_n \uparrow X$ as $n \to \infty$. Then, $\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$ as $n \to \infty$.

Theorem 1.2 (Dominanted Convergenec Theorem (DCT)). Let $(X_n)_{n\in\mathbb{N}}$ be random variables such that $X_n \to X$ a.s. as $n \to \infty$ and $|X_n| \leqslant Y$ a.s. for all $n \in \mathbb{N}$, where Y is integrable, then $\mathbb{E}[X_n] \to \mathbb{E}[X]$, as $n \to \infty$.

Let $1 \leqslant p < \infty$ and f a measurable function, then set $\|f\|_p := (\mathbb{E}[\|f\|^p])^{\frac{1}{p}}$. If $p = \infty$, then set $\|f\|_\infty := \inf\{\lambda : |f| \leqslant \lambda \text{ a.s.}\}$. Recall for all p, the Lebesgue spaces, $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}) = \{f : \|f\|_p < \infty\}$.

Theorem 1.3. $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Hilbert space, with inner product $\langle u, v\rangle_2 = \mathbb{E}[u \cdot v]$. Furthermore, for any closed subspace $\mathcal{H}$, if $f \in \mathcal{L}^2$, there exists a unique $g \in \mathcal{H}$ s.t. $\|f - g\|_{\mathcal{L}^2} = \inf_{h\in\mathcal{H}} \|f - h\|_{\mathcal{L}^2}$ and $\langle f - g, h\rangle = 0$, for all $h \in \mathcal{H}$. We say that g is the orthogonal projection of f in $\mathcal{H}$.

We now construct the conditional expectation in the general case, for any integrably random variable with respect to an arbitrary sigma algebras.

Theorem 1.4 (Conditional Expectation). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subseteq \mathcal{F}$ a sub-sigma algebra, $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists an integrable random variable Y satisfying:

1. Y is $\mathcal{G}-$measurable
2. for all $G \in \mathcal{G}, \mathbb{E}[X \cdot \mathbf{1}(G)] = \mathbb{E}[Y \cdot \mathbf{1}(G)]$.

Moreover, Y unique in the sense that if Y' also satisfies the above $1), 2)$, then $Y = Y'$ a.s.. We call Y a version of the conditional expectation of X given G. We write $Y = \mathbb{E}[X\mathcal{G}]$ a.s. If $\mathcal{G} = \sigma(Z)$, where Z is a random variable, then we write $\mathbb{E}[Z] = \mathbb{E}[X|\mathcal{G}]$.

Remark. 2) could be replaced by $\mathbb{E}[X \cdot Z] = \mathbb{E}[Y \cdot Z]$ for all Z bounded $\mathcal{G}-$measurable random variables.

We now state and prove the main theorem of this section:

Proof. (Theorem 1.4) Uniqueness: Let Y, Y' satisfy $1), 2)$. Let $A = \{Y > Y'\} \in \mathcal{G}$. Then 2)

$$\begin{aligned}
&\implies \mathbb{E}[Y \cdot \mathbf{1}(A)] = \mathbb{E}[Y' \cdot \mathbf{1}(A)] = \mathbb{E}[X \cdot \mathbf{1}(A)] \\
&\implies \mathbb{E}[(Y - Y') \cdot \mathbf{1}(A)] = 0 \\
&\implies \mathbb{P}(A) = \mathbb{P}(Y > Y') = 0 \\
&\implies Y \leqslant Y' \text{ a.s..}
\end{aligned}$$

We similarly obtain $Y \geqslant Y'$ a.s., hence we deduce that $Y = Y'$ a.s.

Existence: three steps.

1. Assume that $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Observe that $\mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P})$ is a closed subspace of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Hence, Theorem 1.3, we have the decomposition $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) = \mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P}) \oplus \mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P})^\perp$. Then, we have the corresponding decomposition $X = Y + Z$, where $Y \in \mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P})$ and $Z \in \mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P})$ respectively. Define $\mathbb{E}[X\mathcal{G}] := Y$, Y is $\mathcal{G}-$measurable and for all $A \in \mathcal{G}$, $\mathbb{E}[X \cdot \mathbf{1}(A)]\mathbb{E}[Y \cdot \mathbf{1}(A)] = \mathbb{E}[Z \cdot \mathbf{1}(A)]$ since $Z \in \mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P})^\perp$.

   Claim: If $X \geqslant 0$, a.s. then $Y \geqslant 0$ a.s. Indeed, let $A = \{Y < 0\} \in \mathcal{G}$. Then observe that $0 \leqslant \mathbb{E}[X \cdot \mathbf{1}(A)] = \mathbb{E}[Y \cdot \mathbf{1}(A)] \leqslant 0$. Hence $\mathbb{E}[Y \cdot \mathbf{1}(A)] = 0$ and so $\mathbb{P}(A) = 0$, gibing $Y = 0$ a.s.

2. Assume $X \geqslant 0$.
   Define $X_n = X \wedge n \leqslant n$, meaning X_n is bounded for all $n \in \mathbb{N}$. So $X_n \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Let $Y_n = \mathbb{E}[X_n]$ a.s.. $(X_n)_{n\in\mathbb{N}}$ is an increasing sequence. By the claim abose, so is $(Y_n)_{n\in\mathbb{N}}$ a.s. Define $Y = \limsup_n Y_n$ meaning Y is $\mathcal{G}-$measurable and $Y =\uparrow \lim_{n\to\infty} Y_n$ a.s. Now, we have that for all $A \in \mathcal{G}$, $\mathbb{E}[X_n \cdot \mathbf{1}(A)] = \mathbb{E}[Y_n \cdot \mathbf{1}(A)]$. Thus, by theorem 1.1 (MCT), $\mathbb{E}[X \cdot \mathbf{1}(A)] = \lim_{n\to\infty} \mathbb{E}[X_n \cdot \mathbf{1}(A)] = \lim_{n\to\infty} \mathbb{E}[Y_n \cdot \mathbf{1}(A)] = \mathbb{E}[Y \cdot \mathbf{1}(A)]$.

3. X general in $\mathcal{L}^1$.
   Decompose as before $X = X^+ - X^-$. Define, $\mathbb{E}[X\mathcal{G}] = \mathbb{E}[X^+|\mathcal{G}] - \mathbb{E}[X^-|\mathcal{G}]$.

□

Lecture 3

Remark. From the second step of the proof of Theorem 1.4 we see that we can define $\mathbb{E}[X|\mathcal{G}]$ for all $X \geqslant 0$, not necessarily integrable. It satisfies all conditions $1), 2)$ except for the integrability one.

Definition 1.7. $\underbrace{\mathcal{G}_1, \mathcal{G}_2, \dots}_{\text{sigma algebras}} \subset \mathcal{F}$. We call them <u>independent</u> if whenever $G_i \in \mathcal{G}_i$ and $i_1 < \dots i_k$ for some $k \in \mathbb{N}$, then $\mathbb{P}(G_{i_1} \cap \dots \cap G_{i_k}) = \prod_{j=1}^{k} \mathbb{P}(G_{i_j})$.

Moreover, let X be a random variable and $\mathcal{G}$ a sigma algebra, then they are said to be int if $\sigma(X)$ is independent of $\mathcal{G}$.

<u>Properties of conditional expectations:</u> Fix $X, y \in \mathcal{L}^1$, $G \in \mathcal{F}$.

1. $\mathbb{E}[\mathbb{E}[X\mathcal{G}]] = \mathbb{E}[X]$ (take $A = \Omega$)
2. If X is $\mathcal{G}-$measurable, then $\mathbb{E}[X\mathcal{G}] = X$ a.s.
3. If X is independent of $\mathcal{G}$, then $\mathbb{E}[X\mathcal{G}] = \mathbb{E}[X]$
4. If $X \geqslant 0$ a.s., then $\mathbb{E}[X\mathcal{G}] \geqslant 0$ a.s.
5. For $\alpha, \beta \in \mathbb{R}$ $\mathbb{E}[\alpha X + \beta Y|\mathcal{G}] = \alpha\mathbb{E}[X] + \beta\mathbb{E}[Y]$
6. $\mathbb{E}[X|\mathcal{G}]| \leqslant \mathbb{E}[|X||\mathcal{G}]$ a.s.

Below we provid:we expensions of useful measure theoretic results for the expectation to their corresponding conditional counetparts. First recall:

Lemma 1.1 (Fatou's Lemma). Let $X_n \geqslant 0$ for all $n \in \mathbb{N}$. Then

$$\mathbb{E}[\liminf_n X_n] \leqslant \liminf_n \mathbb{E}[X_n] \quad \text{a.s}$$

Theorem 1.5 (Jensen's Inequality). If X is integrable and $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function, then

$$\phi(\mathbb{E}[X]) \leqslant \mathbb{E}[\phi(X)] \quad \text{a.s.}$$

Now the results themselves:

Theorem 1.6 (Conditional Monotone Convergence theorem (MCT)). Let $\mathcal{G} \subset \mathcal{F}$ be sigma algebras, $X_n \geqslant 0$ a.a. and $X_n \uparrow X$, as $n \to \infty$ a.s. Then

$$\mathbb{E}[X_n|\mathcal{G}] \uparrow \mathbb{E}[X|\mathcal{G}] \quad \text{a.s.}$$

Proof. Theorem 1.6 Set $Y_n = \mathbb{E}[X_n\mathcal{G}]$ a.s. Observe that Y_n is a.s. increasing. Set $Y = \limsup_n Y_n$. Y_n is $\mathcal{G}-$measurable, hence, so is Y (as a lim sup of $\mathcal{G}-$measurable random variables) is also $\mathcal{G}-$measurable. Also, $Y = \lim_{n\to\infty} Y_n$ a.s.

<u>Need to show:</u> $\mathbb{E}[Y \cdot \mathbf{1}(A)]\mathbb{E}[X \cdot \mathbf{1}(A)]$ for all $A \in \mathcal{G}$. Indeed,

$$\begin{aligned}\mathbb{E}[Y \cdot \mathbf{1}(A)] &= \mathbb{E}[\lim_{n\to\infty} Y_n \cdot \mathbf{1}(A)] \overset{\text{MCT}}{=} \lim_{n\to\infty} \mathbb{E}[Y_n \cdot \mathbf{1}(A)] \\ &= \lim_{n\to\infty} \mathbb{E}[X_n \cdot \mathbf{1}(A)] = \mathbb{E}[X \cdot \mathbf{1}(A)].\end{aligned}$$

□

Proof. Theorem 1.1 $\liminf_n X_n = \lim_{n\to\infty}\left(\inf_{k\geqslant n} X_k\right)$, the limit of an increasing sequence. By Theorem 1.1, we have

$$\lim_{n\to\infty}\mathbb{E}[\inf_{k\geqslant n} X_n|\mathcal{G}] = \mathbb{E}[\liminf_n X_n|\mathcal{G}]$$

and

$$\mathbb{E}[\inf_{k\geqslant n} X_k|\mathcal{G}] \overset{\text{a.s.}}{\leqslant} \inf_{k\geqslant n}\mathbb{E}[X_k|\mathcal{G}]^1$$

which gives the result

$$\mathbb{E}[\liminf_n X_n] \leqslant \liminf_n \mathbb{E}[X_n] \quad \text{a.s.}$$

□

Theorem 1.7 (Conditional Dominated Convergence Theorem). SUppose $X_n \to X$ a.s. $n\to\infty$ and $|X_n| \leqslant Y$ a.s. for all $n \in \mathbb{N}$ with Y integrable. Then $\mathbb{E}[X_n\mathcal{G}] \to \mathbb{E}[X\mathcal{G}]$ a.s. as $n\to\infty$.

Proof. From $-Y \leqslant X_n \leqslant Y$, we have $X_n + Y \geqslant 0$ for all $n \in \mathbb{N}$ and $Y - X_n \geqslant 0$a.s. By Theorem 1.1,

$$\begin{aligned}\mathbb{E}[X+Y\mathcal{G}] &= \mathbb{E}[\liminf_n (X_n+Y)|\mathcal{G}] \\ &\leqslant \liminf_n \mathbb{E}[X_n+Y|\mathcal{G}] = \liminf_n \mathbb{E}[X_n\mathcal{G}] + \mathbb{E}[X]\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}[|X-Y||\mathcal{G}] &= \mathbb{E}[Y - \liminf_n X_n|\mathcal{G}] \\ &\leqslant \mathbb{E}[Y] + \liminf_n \mathbb{E}[X_n|\mathcal{G}]\end{aligned}$$

Hence,

$$\limsup_n \mathbb{E}[X_n|\mathcal{G}] \leqslant \mathbb{E}[X|\mathcal{G}]$$

and

$$\liminf_n \mathbb{E}[X_n|\mathcal{G}] \geqslant \mathbb{E}[X|\mathcal{G}]$$

a.s., concluding the proof.

□

Theorem 1.8 (Conditional Jensen). Let $X \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$, $\phi:\mathbb{R}\to\mathbb{R}$ be a convex function s.t. $\phi(X)$ is integrable or $\phi(X)\geqslant 0$

$$\phi(\mathbb{E}[X|\mathcal{G}]) \leqslant \mathbb{E}[\phi(X)|\mathcal{G}] \quad \text{a.s.}$$

Proof. <u>Claim:</u> (true for any convex function, no proof given) $\phi(x) = \sup_{i\in\mathbb{N}}(a_i x + b_i)$, $a_i b_i \in \mathbb{R}$. Thus,

$$\mathbb{E}[\phi(X)|\mathcal{G}] \geqslant a_i\mathbb{E}[X|\mathcal{G}] + b_i \quad \text{for all } i \in \mathbb{N}.$$

Taking the supremum gives [2]

$$\begin{aligned}\mathbb{E}[\phi(X)|\mathcal{G}] &\geqslant \sup_{i\in\mathbb{N}}(a_i\mathbb{E}[X|\mathcal{G}] + b_i) \\ &= \phi(\mathbb{E}[X|\mathcal{G}]) \quad \text{a.s.}\end{aligned}$$

□

Corollary 1.8.1. For all $1 \leqslant p < \infty$ $\|\mathbb{E}[X|\mathcal{G}]\|_p \leqslant \|X\|_p$.

Proof. Apply conditional Jensen. □

[1] can take the infinum due to countability that preserves a.s.
[2] can take the supremum due to countability which again preserves a.s.

Proposition 1.1 (Tower Property). Let X be integrable and $\mathcal{H} \subseteq \mathcal{G}$ sigma algebras. Then

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}] \quad \text{a.s.}$$

Proof. (a) $\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}-$measurable.

(b) For all $A \in \mathcal{H}$ NTS:

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}] \cdot \mathbf{1}(A)] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}] \cdot \mathbf{1}(A)]$$

Indeed, both terms above are equal to $\mathbb{E}[X \cdot \mathbf{1}(A)]$ since $A \in \mathcal{G} \subseteq \mathcal{H}$. □

Proposition 1.2. Let $X \in \mathcal{L}^1$, $\mathcal{G} \subseteq \mathcal{F}$, Y bounded $\mathcal{G}-$measurable. Then

$$\mathbb{E}[X \cdot Y|\mathcal{G}] = Y \cdot \mathbb{E}[X|\mathcal{G}].$$

Proof. (a) RHS is clearly $\mathcal{G}-$measurable.

(b) For all $A \in \mathcal{G}$:

$$\begin{aligned} \mathbb{E}[X \cdot Y \cdot \mathbf{1}(A)] &= \mathbb{E}[Y \cdot \mathbb{E}[X\mathcal{G}] \cdot \mathbf{1}(A)] \\ \mathbb{E}[X \cdot (\underbrace{Y \cdot \mathbf{1}(A)}_{\mathcal{G}\text{-meas. and bounded}})] &= \mathbb{E}[\mathbb{E}[X|\mathcal{G}] \cdot Y \cdot \mathbf{1}(A)] = RHS. \end{aligned}$$

(Also observe that by a monotone class argument, we have for any integrable function $f : \Omega \to \mathbb{R}$, $\mathbb{E}[X \cdot f] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}] \cdot f]$) □

Lecture 4 We are building towards the Theorem

Theorem 1.9. $X \in \mathcal{L}^1, \mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$. Assume $\sigma(\mathcal{G}, \mathcal{H}) \perp \mathcal{H}$, Then

$$\mathbb{E}[X|\sigma(\mathcal{G}, \mathcal{H})] = \mathbb{E}[X|\mathcal{G}] \quad \text{a.s.}$$

We begin with a definition

Definition 1.8. Let $\mathcal{A}$ be a collection of sts. It is called a $\pi-$system if for all $A, B \in \mathcal{A}$, we also have $A \cap B \in \mathcal{A}$.

Theorem 1.10 (Uniquenes of extension). Let (E, ξ)be a measurable space and let $\mathcal{A}$ be a $\pi-$system generating the sigma algebra ξ. Let μ, ν be two measures on (E, ξ) with $\mu(E) = \nu(E) < \infty$. If $\mu = \nu$ on $\mathcal{A}$, then $\mu = \nu$ on ξ.

Proof. (Theorem 1.9) NTS: for all $F \in \sigma(\mathcal{G}, \mathcal{H})$

$$\mathbb{E}[X \cdot \mathbf{1}_F] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}] \cdot \mathbf{1}_F]$$

Now, set $\mathcal{A} = \{A \cap B : A \in \mathcal{G}, B \in \mathcal{H}\}$. It is easy to check that $\mathcal{A}$ is a $\pi-$system generating $\sigma(\mathcal{G}, \mathcal{H})$. If $F = A \cap B$ for some $A \in \mathcal{G}$ and $B \in \mathcal{H}$, Then

$$\begin{aligned} \mathbb{E}[X \cdot \mathbf{1}(A \cap B)] &= \mathbb{E}[X \cdot \mathbf{1}(A) \cdot \mathbf{1}(B)] \\ &= \mathbb{E}[X \cdot \mathbf{1}(A)] \cdot \mathbb{E}[\mathbf{1}(B)] \overset{H \perp \sigma(\mathcal{G}, \mathcal{H})}{=} \mathbb{E}[\mathbb{E}[X|\mathcal{G}] \cdot \mathbf{1}(A \cap B)]. \end{aligned}$$

Now assume $X \geqslant 0$; in the general case, decompose $X = X^+ - X^-$ and apply same argument to both $X^\pm$. Define the measures $\mu(F) = \mathbb{E}[X \cdot \mathbf{1}(F)]$ and $\nu(F) = \mathbb{E}[X \cdot \mathbf{1}(F)]$ for all $F \in \sigma(\mathcal{G}, \mathcal{H})$. Observe that $\mu(\Omega) = \nu(\Omega) = \mathbb{E}[X] < \infty$ and we have shown that $\mu = \nu$ on $\mathcal{A}$. Thus, $\mu = \nu$ on $\sigma(\mathcal{G}, \mathcal{H})$ which finally implies the result

$$\mathbb{E}[X|\sigma(\mathcal{G}, \mathcal{H})] = \mathbb{E}[X|\mathcal{G}] \quad \text{a.s.}$$

□

Examples:

Definition 1.9 (Gaussian). $(X_1, X_2, \cdots, X_n) \in \mathbb{R}^n$ has the Gaussian distribution if and only if for all scalars $a_1, a_2, \cdots, a_n \in \mathbb{R}$, $a_1X_1 + \cdots a_nX_n$ has the Gaussian distrubition in $\mathbb{R}$.

A stochastic process (more on that later) $(X_t)_{t\geqslant 0}$ is a Gaussian process if for all $t_1 < t_2 < \cdots t_n$ the vector $(X_{t_1}, X_{t_2}, \cdots, X_{t_n})$ is Gaussian.

Let (X, Y) be a Gaussian vector in $\mathbb{R}^2$. We compute $\mathbb{E}[X|Y]$.
Let $X' = \mathbb{E}[X|Y]$. Looking for f a Borel measurable function s.t. $\mathbb{E}[X|Y] = f(Y)$ a.s. Let $f(y) = ay + b$ for some $a, b \in \mathbb{R}$ to be determined. Now, $X' = aY + b$, $\mathbb{E}[X'] = \mathbb{E}[X] = a\mathbb{E}[Y] + b$ and $\mathbb{E}[X' \cdot Y] = \mathbb{E}[X \cdot Y] \implies \mathbb{E}[(X - X') \cdot Y] = 0$. Thus $\text{Cov}(X - X', Y) = 0 \implies \text{Cov}(X, Y) = a^2\text{Var}(Y)$.

Need to check: that for all Z bounded $\sigma(Y)-$measurable, $\mathbb{E}[(X - X') \cdot Z] = 0$.
Indeed, observe that $(X - X', Y)$ is a Gaussian vector and since $\text{Cov}(X - X', Y) = 0 \implies X - X' \perp Y \implies (X - X') \perp Z$.

2. Let (X, Y) be a random vector with density in $\mathbb{R}^2$ with joint density function $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$. Let $h : \mathbb{R} \to \mathbb{R}$ be a Borel function such that $h(X)$ is integrable. We now compute $\mathbb{E}[h(X)|Y]$.
We have for all g bounded $\sigma Y-$measurable functions.

$$\begin{aligned}
\int_{\mathbb{R}^2} h(x)g(y)f_{X,Y}(x, y)\,\mathrm{d}x\,\mathrm{d}y &= \mathbb{E}[h(X)g(Y)] \\
&= \mathbb{E}[\mathbb{E}[h(X)|Y]g(Y)] = \mathbb{E}[\phi(Y)g(Y)] \\
&= \int_{\mathbb{R}^2} \phi(y)g(y)f_{Y(y)}\,\mathrm{d}y
\end{aligned}$$

where $f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x, y)\,\mathrm{d}x$ and $\phi : \mathbb{R} \to \mathbb{R}$ is some Borel measurable function. Hence,

$$\phi(y) = \begin{cases} \int_{\mathbb{R}} h(x)\dfrac{f_{X,Y}(x, y)}{f_Y(y)}\,\mathrm{d}\mathrm{d}x, & f_Y(y) > 0 \\ 0, & \text{otherwise} \end{cases}$$

can be seen to work. Thus, we obtain

$$\mathbb{E}[h(X)|Y] = \phi(Y) \quad \text{a.s.}$$

2 Discrete Time Martingales

Definition 2.1 (Filtration). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a pobability space. A filtration is a sequences of increasing sigma sub-algebras of $\mathcal{F}$, $(\mathcal{F}_n)_{n\in\mathbb{N}}$, $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all $n \in \mathbb{N}$. We call $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}})$ a filtered probability space.

Let $X = (X_n)_{n\in\mathbb{N}}$ be a sequence of random variables/a stochastic process. Then it induces $(\mathcal{F}_n^X)_{n\in N}$, where $\mathcal{F}_n^X := \sigma(X_{:k\leqslant n})$ for all $n \in \mathbb{N}$: the canonical filtration associated to X. We call X adapted to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if $X$is $\mathcal{F}_n-$measurable for all $n \in \mathbb{N}$.X is called integrable if X_n is integrable for all $n \in \mathbb{N}$.

Definition 2.2 (Martingale discrete time). Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space. Let $X = (X_n)_{n\in\mathbb{N}}$be an integrabl and adapted process.

- X is called a martingale if $\mathbb{E}[X_n|\mathcal{F}_m] = X_m$ a.s. for all $n \geqslant m$.

- X is called a super-martingale if $\mathbb{E}[X_n|\mathcal{F}_m] \leqslant X_m$ a.s. for all $n \geqslant m$.
- X is called a sub-martingale if $\mathbb{E}[X_n|\mathcal{F}_m] \geqslant X_m$ a.s. for all $n \geqslant m$.

Remark. If X is a (super/sub)martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$, then it is also a martingale with respect to $(\mathcal{F}_n^X)_{n\in\mathbb{N}}$. To see this, one has to use the tower property 1.1: $\mathcal{F}_n^X \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}$ implies $\mathbb{E}[X_n|\mathcal{F}_m^X] = \mathbb{E}[\mathbb{E}[X_n|\mathcal{F}_m]|\mathcal{F}_m^X]$ (since $\mathbb{E}[X_n|\mathcal{F}_m]$ a.s.).

Examples:

1. Let $(\xi_i)_{i\in\mathbb{N}}$ be iid. s.t. $\mathbb{E}[\xi_i] = 0$ for all $i \in \mathbb{N}$ and define $X = (X_n)_{n\in\mathbb{N}}$ by $X_n = \xi_1 + \cdots + \xi_n$ for all $n \in \mathbb{N}$, $X_0 = 0$. X is a martingales with respect to $(\mathcal{F}_n^\xi)_{n\in\mathbb{N}}$.
2. Let $(\xi_i)_{i\in\mathbb{N}}$ be iid. s.t. $\mathbb{E}[\xi_i] = 1$ for all $i \in \mathbb{N}$ and define $X = (X_n)_{n\in\mathbb{N}}$ by $X_n = \prod_{i=1}^n \xi_i$ for all $n \in \mathbb{N}$, $X_0 = 1$. X is again a martingales with respect to $(\mathcal{F}_n^\xi)_{n\in\mathbb{N}}$.

Lecture 5 Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space.

Definition 2.3 (Stopping time discrete time). A stopping time T is a random variable $T : \Omega \to \mathbb{Z}_+ \cup \{\infty\}$ s.t. $\{T \leqslant n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Equivalently, if $\{f = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ since

$$\{T = n\} = \underbrace{\{T \leqslant n\}}_{\mathcal{F}_n} \setminus \underbrace{\{T \leqslant n-1\}}_{\mathcal{F}_{n-1}\subset\mathcal{F}_n} \in \mathcal{F}_n.$$

and

$$\{T \leqslant n\} = \bigcup_{k=1}^n \{T = k\} \in \mathcal{F}_k \subset \mathcal{F}_n.$$

Examples:

1. Constant time are trivially stopping times.
2. Let $X = (X_n)_{n\in\mathbb{N}}$ be a stochastic process taking values in $\mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$ (X adapted). Define
$$T_A = \{n \geqslant 0 : X_{n\in A}\}.$$
Then $\{T_A \leqslant n\} = \bigcup_{k=0}^n \{X_{k\in A}\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ (with convention $\inf \varnothing = \infty$).
3. $L_A = \sup\{n \geqslant 0 : X_{n\in A}\}$ is NOT a stopping time.

Properties: $S, T, (T_n)_{n\in\mathbb{N}}$ stopping times. Then $S \wedge T, S \vee T, \inf_n T_n, \sup_n T_n, \liminf_n T_n, \limsup_n T_n$ are also stopping times.

Definition 2.4 (Stopping time sigma algerbra). It $T$is a stopping time, define

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leqslant t\} \in \mathcal{F}_t\}$$

Let $(X_n)_{n\geqslant 0}$ be a process. Write $X_T(\omega) = X_{T(\omega)}(\omega)$ for $\omega \in \Omega$ whenever $T(\omega) < \infty$. Define the stopped process: $X_t^T := X_{T\wedge t}$.

Proposition 2.1. Let S and T be stopping times, and let X be an adapted process, then:

1. If $S \leqslant T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.
2. $X_T\cdot$ is $\mathcal{F}_T$−measurable.
3. X^T is adapted.

4. If X is integrable, then the stopped process iss integrable.

Proof. 1. Immediate from definition.

2. Let $A \in \mathcal{B}(\mathbb{R})$. Need to show:

$$\{X_T \mathbf{1}(T < \infty)\} \cap \{T \leqslant t\} \in A, \quad \text{for all } t \geqslant 0.$$

Indeed, we have that

$$\{X_T \mathbf{1}(T < \infty)\} = \bigcup_{s=0}^{t} \underbrace{\{X_s \in A\}}_{\mathcal{F}_s \subseteq \mathcal{F}_t} \cap \underbrace{\{T = s\}}_{\mathcal{F}_s} \in \mathcal{F}_t.$$

3. $X_t^T = X_{T \wedge t}$, this being $\mathcal{F}_{T \wedge t}-$measurable $\subseteq \mathcal{F}_t-$measurable by 1), so we conclude it is $\mathcal{F}_t-$measurable.

4.

$$\begin{aligned}
\mathbb{E}[|X_t^T|] &= \mathbb{E}[|X_{T \wedge t}|] \\
&= \sum_{s=0}^{t-1} \mathbb{E}[|X_s| \cdot \mathbf{1}(T = s)] + \mathbb{E}[|X_t| \cdot \mathbf{1}(T \geqslant t)] \\
&\leqslant \sum_{s=0}^{t} \mathbb{E}[|X_s|] \quad \underbrace{< \infty}_{X_t \text{ is integrable}} .
\end{aligned}$$

□

We now state and prove a fundamental theorem in Martingale theory:

Theorem 2.1 (Optional Stopping Theorem discrete time). Let $(X_n$ be a martingale.

1. If T is a stopping time, then the stopped process X^T is also a martingale. In particular for all $t \geqslant 0$:
$$\mathbb{E}[X_{T \wedge t}] = \mathbb{E}[X_0].$$

2. It $S \leqslant T$ are bounded stopping times, then
$$\mathbb{E}[X_T | \mathcal{F}_S] = X_T, \quad \text{a.s.}$$
and hence $\mathbb{E}[X_T] \mathbb{E}[X_S]$.

3. It there exists an integrable random variable Y such that $|X_n \leqslant Y|$ for all $n \geqslant 0$ and T is finite, then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

4. If there exists $M \geqslant 0$ such that $|X_{n+1} - X_n| \leqslant M$ for all $n \in \mathbb{N}$ and T is a stopping time with $\mathbb{E}[T] < \infty$, then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Proof. 1. NTS: for all $t \geqslant 0$, $\mathbb{E}[X_{T \wedge t} | \mathcal{F}_{t-1}] = X_{T \wedge t}$ a.s. Indeed,

$$\begin{aligned}
\mathbb{E}[X_{T \wedge t} | \mathcal{F}_{t-1}] &= \sum_{s=0}^{t-1} \mathbb{E}[X_s \cdot \mathbf{1}(T = s) | \mathcal{F}_{t-1}] \mathbb{E}[X - -t] \cdot \mathbf{1}(T \geqslant t) | \mathcal{F}_{t-1}] \\
&= \sum_{s=0}^{t-1} \mathbf{1}(T = s) \cdot X_s + X_{t-1} \cdot \mathbf{1}(T \geqslant t) \quad \text{a.s.} \\
&= \sum_{s=0}^{t-2} \mathbf{1}(T = s) \cdot X_s + X_{t-1} \cdot \mathbf{1}(T \geqslant t - 1) \quad \text{a.s.} \\
&= X_{T \wedge t-1} \quad \text{a.s.}
\end{aligned}$$

2. $S \leqslant T \leqslant n, n \in \mathbb{N}$ fixed. Let $A \in \mathcal{F}_S$. <u>NTS:</u> $\mathbb{E}[X_T \cdot \mathbf{1}(A)] = [X_s \cdot \mathbf{1}(A)]$. We compute

$$\begin{aligned}
X_T - X_S &= (X_T - X_{T-1}) + \cdots + (X_{S+1} - X_S) \\
&= \sum_{k=0}^{n-1} (X_{k+1} - X_k) \cdot \mathbf{1}(S \leqslant k < T).
\end{aligned}$$

Thus,

$$\mathbb{E}[X_T \cdot \mathbf{1}(A)] \overset{(A \in \mathcal{F}_S)}{=} \mathbb{E}[X_S \cdot \mathbf{1}(A)] + \sum_{k=0}^{n-1} \mathbb{E}[(X_{k+1} - X_k) \cdot \mathbf{1}(S \leqslant k < T) \cdot \mathbf{1}(A)]$$

Have, $A \cap \{S \leqslant k\} \in \mathcal{F}_k$ and $A \cap \{T > k\} \in F_k$. Thus, $\mathbf{1}(S \leqslant k < T) \cdot \mathbf{1}(A)$ is $\mathcal{F}_k-$measurable. Using $\mathbb{E}[X_{k+1}|\mathcal{F}_k] = X_k$ a.s. we deduce that

$$\begin{aligned} \mathbb{E}[(X_{k+1} - X_k) \cdot \mathbf{1}(S \leqslant k < T] \cdot \mathbf{1}(A)] &= \mathbb{E}[\overbrace{\mathbb{E}[(X_{k+1} - X_k)|\mathcal{F}_k]}^{0} \cdot \mathbf{1}(S \leqslant k < T] \cdot \mathbf{1}(A)] \\ &= 0 \end{aligned}$$

Thus, $\mathbb{E}[X_T|\mathcal{F}_S] = X_S$ a.s.

3. By the Optional Stopping Theorem applied to $(X_{T \wedge n})_{n \geqslant 0}$, we have

$$\mathbb{E}[X_{T \wedge n}] = \mathbb{E}[X_0] \quad \text{for all } n \geqslant 0.$$

Now, T being finite a.s. implies that $X_T = \lim_{n \to \infty} X_{T \wedge n}$ a.s. By assumption, have $|X_{T \wedge n}| \leqslant Y$ a.s. for all $n \in \mathbb{N}$ and so can apply DCT to conclude.

4. Observe that for all $n \geqslant 1$

$$X_{T \wedge n} - X_0 = \sum_{k=0}^{n-1} (X_k - X_0) \cdot \mathbf{1}(T = k) + (X_n - X_0)\mathbf{1}(T \geqslant n)$$

Hence, we have the bound (using that $|X_{k+1} - X_k| \leqslant M$ a.s. for all $k \geqslant 0$)

$$\begin{aligned} |X_{T \wedge n} - X_0| &\leqslant M \sum_{k=0}^{n-1} k\mathbf{1}(T = k) + n\mathbf{1}(T \geqslant n) \\ &\leqslant \mathbb{E}[T] < \infty \quad \text{a.s.} \end{aligned}$$

Now, $\mathbb{E}[T] < \infty$ gives $T < \infty$ a.s. and so can deduce as before that $X_T = \lim_{n \to \infty} X_{T \wedge n}$ and use the DCT to conclude the argument.

□

Corollary 2.1.1. Let X be a positive superartingale, T a stopping time such that $T < \infty$ a.s., then

$$\mathbb{E}[X_T] \leqslant \mathbb{E}[X_0].$$

Proof. Use Fatou 1.1: $\mathbb{E}[\liminf_{t \uparrow \infty} X_{T \wedge t}] \leqslant \liminf_{t \uparrow \infty} \mathbb{E}[X_{T \wedge t}] \leqslant \mathbb{E}[X_0]$. □

Simple random walk on $\mathbb{Z}$

Let $(\xi_i)_{i \geqslant 0}$ be iid Bernoulli random variables with success probability $1/2$. Define the process $(X_n)_{n \geqslant 0}$ by setting $X_n = \xi_1 + \cdots + \xi_n$ for all $n \geqslant 1$ and $X_0 = 0$. Furthermore, let $T = \inf\{n \geqslant 0 : X_n = 1\}$. Using the analysis below, we will see that $\mathbb{P}(T < \infty) = 1$. The Optional Stopping Theorem gives $\mathbb{E}[X_{T \wedge t}] = 0$ for all $t \geqslant 0$. Yet, $1 = \mathbb{E}[X)_T] \neq 0$. We thus see that the condition $\mathbb{E}[T] < \infty$ in 4) is necessary, since $\mathbb{E}[T] = \infty$.

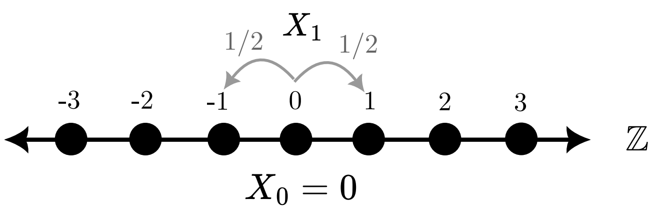

Figure 1: Illustration of simple random walk (first step) on $\mathbb{Z}$.

Lecture 6

We consider again the example of the simple random walk 2 $(X_n)_{n\in\mathbb{N}}$ and define the stopping times

$$T_c = \inf n \geqslant 0 : X_{n=c}, \quad c \in \mathbb{Z}$$

Set $T = T_{-a} \wedge T_b$ for $ab \in \mathbb{Z}$. We now ask what is $\mathbb{P}(T_{-a} \wedge T_b)$?

To answer this, note first that $X_n^T = X_{T\wedge n}$ is a martingale by the Optional Stopping Theorem and we also have the bounded differences $|X_{n+1} - X_n| \leqslant 1$ for all $n \geqslant 1$.

Claim: $\mathbb{E}[T] < \infty$.
To show this, we will *stochastically dominate* T be a geometric random variable, which automatically has a finite expectation and then conclude using the non-negativity of T. Now we have that for the sequence $\xi_1, \xi_2, \cdots, \xi_{a+b}$ the probability that they all are either $+1$ or -1 is $2 \cdot 2^{-(a+b)}$ by independence, call this event A_1. The same is true for the shifted sequence $\xi_{k(a+b)+1} \cdots \xi_{(k+1)(a+b)}$ for all $k \in \mathbb{N}$, where we call the corresponding event A_k.

Thus, we can bound T by the the random variable

$$Z(\omega) = \inf\{n \geqslant 0 : \omega \in A_n\}$$

which has the distribution $Z \sim \mathit{Geom}(2 \cdot 2^{-(a+b)})$. Thus, $\mathbb{E}[T] < \mathbb{E}[Z] \leqslant (a+b) \cdot 2^{a+b-1} < \infty$. Thus, we conclude by the OST that $\mathbb{E}[X_T] = \mathbb{E}[X_0] = 0$. Hence, $-a\mathbb{P}(T_a < T_b) + b\mathbb{P}(T_b < T_{-a}) = 0$ and so a quick computation yields that $\mathbb{P}(T_{-a} < T_b) = \frac{b}{a+b}$.

3 Martingale Convergence Theorem

Theorem 3.1 (Almost sure martingale convergence theorem). Let X be a supermartingale bounded in $\mathcal{L}^1$, i.e. satisfying $\sup_n \mathbb{E}[|X_n|] < \infty$. Then, there exists $X_\infty \in \mathcal{L}^1(\mathcal{F}_\infty), \mathcal{F}_\infty = \sigma(\mathcal{F}_n : n \geqslant 0)$ such that $X_n \overset{n\to\infty}{\longrightarrow} X_\infty$, a.s.

Before we embark on the proof of this theorem, we need so me supporting results. First we have a result from analysis and we set up some notation. Let $x - (x_{n n\in\mathbb{N}})$ be a real sequence and let $a < b$ be reals. We proceed to define the *number of upcrossings of the sequence* before time $n \in \mathbb{N}$. Wec constructrecursively the sequence of times:

$$\begin{aligned} T_0(x) &= 0 \\ S_{k+1}(x) &= \inf\{n \geqslant T_k(x) : x_n \leqslant a\} \\ T_{k+1}(x) &= \inf\{n \geqslant S_{k+1}(x) : x_n \geqslant b\} \end{aligned}$$

and

$$N_n([a,b], X) = \sup\{k \geqslant 0 : T_k(x) \leqslant n\}$$

Observe that as $n \to \infty$, $N_n([a,b],x) \uparrow N([a,b],x) = \sup\{kgeq0 : T_k(x) < \infty\}$ (see figure 2 for an illustration).

Lemma 3.1. Let $X = (X_n)$ be a real sequence. Then X converges in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ if and on ly if for all $a < b$, $a, b \in \mathbb{Q}$, $N([a,b],X) < \infty$.

Proof. $\underline{\Longrightarrow:}$ Suppose x converges, if $a < b$ such that $N([a,b],x) = \infty$, then $\liminf_n x_n \leqslant a < b \leqslant \limsup_n x_n$, a contradiction.
$\underline{\Longleftarrow:}$ if not, then $\liminf_n x_n < \limsup_n x_n$ which implies that there exists $a < b$ in $\mathbb{Q}$ with $\liminf_n x_n < a < b < \limsup_n x_n$, and hence $N([a,n],x) = \infty$, a contradiction. $\square$

Now we state it Doob's upcrossing Inequality

Lemma 3.2 (Doob's upcrossing inequality). Let X be a supermartingale, then for all $n \in \mathbb{N}$:

$$(b-a)\cdot \mathbb{E}[N_n([a,b],X)] \leqslant \mathbb{E}[(X_n-a)^-]$$

Proof. It is not hard to check that the sequences of times in 3 are stopping times. Now we have:

$$\begin{aligned}&\sum_{k=1}^{n}(X_{T_k\wedge n}-X_{S_k\wedge n})\\ &=\underbrace{\sum_{k=1}^{N_n}(X_{T_k}-X_{S_k})}_{\geqslant N_n\cdot(b-a)}+(X_n-X_{S_{N_n+1}})\mathbf{1}(S_{N_n+1}\leqslant n)\end{aligned}$$

Since $T_{k\wedge n} \geqslant S_{k\wedge n}$, the OST gives $\mathbb{E}[X_{T_k\wedge n}] \leqslant \mathbb{E}[X_{S_k\wedge n}]$. Note:

$$\underbrace{X_n - X_{S_{N_n}+1}}_{\geqslant (X_n-a)\wedge 0=-(X_n-a)^-} \quad \mathbf{1}(S_{N_n+1}\leqslant n).$$

Thus, taking expectations on both sides gives:

$$0 \geqslant (b-a)\cdot\mathbb{E}[N_n]-\mathbb{E}[(X_n-a)^-].$$

thus concluding the proof. □

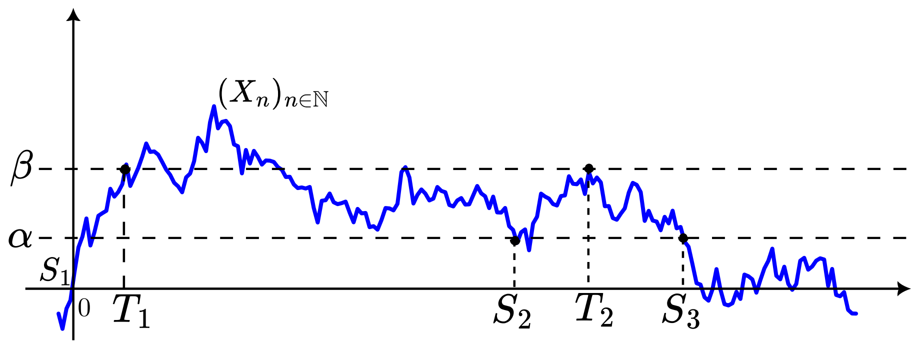

Figure 2: Illustration of upcrossings for the process $(X_n)_{n\in\mathbb{N}}$.

Now we proceed to the proof of the martingale convergence theorem:

Proof. (Theorem 3.1) Fix $a < b$, in $\mathbb{Q}$. Have

$$\begin{aligned}\mathbb{E}[N_{n([a,b],X)}] &\leqslant (b-a)^-\underbrace{\mathbb{E}[(X_n-a)^-]}_{\leqslant \mathbb{E}[|X_n|+a]}\\ &\leqslant (b-a)^-\left(\sup_{n\geqslant 0}\underbrace{\mathbb{E}[|X_n|]}_{<\infty}+a\right)\end{aligned}$$

Also have $N_n([a,b],X)\uparrow N([a,b],X)$ as $n\to\infty$. By monotone convergence: $\mathbb{E}[N([a,b],X)]<\infty$. Set

$$\Omega_0=\bigcap_{a<ba,b,\in\mathbb{Q}}\{N([a,b],X)<\infty\}\in\mathcal{F}_\infty$$

and $\mathbb{P}(\Omega_0)=1$. On Ω_0, X converges. set

$$X_\infty=\begin{cases}\lim\limits_{n\to\infty}X_n & \text{on } \Omega_0\\ 0, & \text{on } \Omega\backslash\Omega_0.\end{cases}$$

So, X_∞ is $\mathcal{F}_\infty-$measurable, $X_n \overset{n\to\infty}{\longrightarrow} X_\infty$ a.s. and

$$\mathbb{E}[|X_\infty|]=\mathbb{E}[\liminf_n |X_n|]\leqslant \liminf_{\mathbb{E}[X_n]}<\infty.$$

□

Corollary 3.1.1. Let B be a upermaartingale. Then, X converges a.s.

Proof. $\mathbb{E}[|X_n|] = \mathbb{E}[X_n] \leqslant \mathbb{E}[X_0]$. Apply the martingale convergence theorem to conclude. $\square$

Lecture 7

4 Doob's inequalities

Theorem 4.1 (Doob's maximal inequality). Let X be a non-negative submartingale and set $X_n^* = \sup_{0 \leqslant k \leqslant n} X_k$. Then for all $\lambda \geqslant 0$,

$$\begin{aligned} \lambda \cdot \mathbb{P}(X_n^* \geqslant \lambda) &\leqslant \mathbb{E}[X_n \cdot \mathbf{1}(X_n^* \geqslant \lambda)] \\ &\leqslant \mathbb{E}[X_n]. \end{aligned}$$

Proof. Let $T = \inf\{k \geqslant 0 : X_k \geqslant \lambda\}$ (it is a stopping time). Then $\{X_n^* \geqslant \lambda\} = \{T \leqslant n\}$. Also have that $X_{T \wedge n}$ is a submartingale by the OST. Then $\mathbb{E}[X_{T\wedge n}] \leqslant \mathbb{E}[X_n]$. Now,

$$\begin{aligned} \mathbb{E}[X_{T\wedge n}] &= \mathbb{E}[X_T \cdot \mathbf{1}(T \leqslant n)] \\ &+ \mathbb{E}[X_n \cdot \mathbf{1}(T > n)] \\ &\geqslant \lambda \cdot \mathbb{P}(T \leqslant n) + \mathbb{E}[X_n \cdot \mathbf{1}(T > n)] \\ &\implies \lambda \cdot \mathbb{P}(T \leqslant n) \leqslant \mathbb{E}\left[X_n \cdot \mathbf{1}(\underbrace{T \leqslant n}_{=\{X_n^* \geqslant \lambda\}}) \right] \\ &\leqslant \mathbb{E}\left[X_n\right] \end{aligned}$$

$\square$

Theorem 4.2 (Doob's $\mathcal{L}^1$ inequality). Lte $p > 1$ and let X be a martingale or a non-negative submartingale. Set $X_n^* = \sup_{0 \leqslant k \leqslant n} |X_k|$. Then

$$\left\|X_n^8\right\|_p \leqslant \frac{p}{p-1} \left\|X_n\right\|_p .$$

Proof. By Jensen, it is enough to prove 4.2 for a non-negative submartingale. Now, observe that

$$\begin{aligned} &= b \\ (y \wedge k)^p &= \int_k^0 px^{p-1}\mathbf{1}(\mathbf{y} \geqslant \mathbf{x})\,\mathrm{d}x = \mathbb{E}[\int_0^k [x^{p-1}\mathbf{1}(X_n^8)\,\mathrm{d}x] \\ &\overset{\text{Fubini}}{=} \int_0^k px^{p-1} \underbrace{\mathbb{P}(X_n^* \geqslant x)}_{\leqslant \frac{1}{x}\mathbb{E}\left[X_n \cdot \mathbf{1}(X_n^* \geqslant x)\right]} \mathrm{d}x \\ &\leqslant \mathbb{E}\left[\int_0^k px^{p-2} \cdot \mathbf{1}(X_n^* \geqslant x)\,\mathrm{d}x \cdot X_n \right] \\ &= \mathbb{E}\left[\tfrac{p}{p-1} (X_n^* \wedge k)^{p-1} \cdot X_n \right] \\ &\overset{\text{Hölder}}{\leqslant} \tfrac{p}{p-1} \cdot \left\|X_n\right\|_p \cdot \left\|X_n^* \wedge k\right\|_p^{p-1} . \end{aligned}$$

So we proved $\left\|X_n^* \wedge k\right\|_p^p \leqslant \frac{p}{p-1} \left\|X_n\right\|_p \cdot \left\|X_n^* \wedge k\right\|_p^{p-1}$, which implies $\left\|X_n^* \wedge k\right\|_p \leqslant \frac{p}{p-1} \cdot \left\|X_n\right\|_p$. Now take $k \to \infty$ and use monotone convergence to conclude the argument. $\square$

Theorem 4.3 ($\mathcal{L}^p$-convergence theorem). Let X be a martingale and $1 < p < \infty$, then the following are equivalent:

1. X is bounded in $\mathcal{L}\checkmark$, i.e. $\sup_{n \geqslant 0} \left\|X_n\right\|_p < \infty$.

2. X converges 'underlinealmost surely and in $\mathcal{L}^p$ to a limit $X_\infty \in \mathcal{L}^p$.

3. There exists $Z \in \mathcal{L}^p$ s.t. $X_n = \mathbb{E}[Z|\mathcal{F}_n]$ a.s.

Proof. 1) $\implies$ 2): X bounded in $\mathcal{L}^p$ implies X is bounded in $\mathcal{L}^1$. So there exists X_∞ such that $X_n \xrightarrow{n\to\infty} X_\infty$ a.s.
Also, $\mathbb{E}[|X_\infty|^p] = \mathbb{E}\left[\liminf_n |X_n|^p\right] \overset{\text{Fatou}}{\leqslant} \liminf_{\mathbb{E}[|X_n|^p]} < \infty$. Thus, $X_\infty \in \mathcal{L}^p$.

Now, let $X_n^* = \sup_{0\leqslant k\leqslant n} |X_k|$, $X_\infty^* = \sup_{k\in\mathbb{N}} |X_k|$. Thus,

$$|X_n - X_\infty| \leqslant 2X_\infty^*$$

for all $n \in \mathbb{N}$. Thus, it is enough to show by DCT that $X_\infty^* \in \mathcal{L}^p$. By Doob's $\mathcal{L}^p-$inequality, $\|X_n^*\|_p = \frac{p}{p-1} \cdot \sup_{n\in\mathbb{N}} \|X_n\|_p < \infty$ By MCT ($X_n^* \uparrow X_\infty^*$): $\|X_\infty^*\|_p \leqslant \frac{p}{p-1} \sup_{n\in\mathbb{N}} \|X_n\|_p < \infty$ Thus, $X_\infty^* \in \mathcal{L}^p$.

2) $\implies$ 3): $X_n \xrightarrow{n\to\infty} X_\infty$ a.s. and in $\mathcal{L}^p$. Set $Z = X_\infty$. Need to show: $X_n = \mathbb{E}[X_\infty|\mathcal{F}_n]$ for all $n \in \mathbb{N}$.

$$\begin{aligned}\|X_n - \mathbb{E}[X_{oo}|\mathcal{F}_n]\|_p &\overset{m\geqq n}{=} \|\mathbb{E}[X_m - X_\infty|\mathcal{F}_n]\|_p \\ &\overset{\text{contraction (Jensen)}}{\leqslant} \|X_m - X_\infty\|_p \to 0, \quad m\to\infty.\end{aligned}$$

3) $\implies$ 1) : By conditional Jensen, we can conclude. □

Definition 4.1. A martingale of the form $X_n = \mathbb{E}[Z|\mathcal{F}_n]$, $Z \in \mathcal{L}^p$ is called a martingale closed in $\mathcal{L}^p$.

Corollary 4.3.1. Let $Z \in \mathcal{L}^p$, $X_n = \mathbb{E}[Z|\mathcal{F}_n]$ a.s. Then $X_n \xrightarrow{n\to\infty} \mathbb{E}[Z|\mathcal{F}_\infty]$ a.s. and in $\mathcal{L}^p$ where $F_\infty = \sigma(X_n, n \geqslant 0)$.

Proof. By theorem 4.3, we have $X_n \overset{n\to\infty\longrightarrow}{X}{}_\infty$ a.s. and in $\mathcal{L}^p$. Now, we need to show:

$$X_\infty = \mathbb{E}[Z|\mathcal{F}_\infty] \quad \text{a.s.}$$

Now, we have that X_∞ is $\mathcal{F}_\infty-$measurable (being the pointwise limit of $X_n, n \geqslant 0$) and for all $A \in \mathcal{F}_\infty$, $\mathbb{E}[Z\cdot\mathbf{1}(A)] = \mathbb{E}[X_\infty \cdot \mathbf{1}(A)]$. Fix $A \in \bigcup_{n\geqslant 0} \mathcal{F}_n$, a $\pi-$system generating $\mathcal{F}_\infty$. There exists $N \in \mathbb{N}$ such that $A \in \mathcal{F}_N$. Let $n \geqslant N$, now

$$\mathbb{E}[Z\cdot\mathbf{1}(A)] = \mathbb{E}[X_n \cdot \mathbf{1}(A)] \xrightarrow{n\to\infty} \mathbb{E}[X_\infty \cdot \mathbf{1}(A)].$$

□

Definition 4.2 (Uniform integrability). A collection of variables $(X_i)_{i\in I}$ is called uniformly integrable (UI) if

$$\sup_{i\in I} \mathbb{E}[|X_i| \cdot \mathbf{1}(|X_i| > M)] \xrightarrow{M\to\infty} 0.$$

Equivalently, $(X_i)_{i\in I}$ is UI if (X_i) is bounded in $\mathcal{L}^1$ and for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $A \in \mathcal{F}$ with $\mathbb{P}(A) < \delta$,

$$\sup_{i\in I} \mathbb{E}[|X_i| \cdot \mathbf{1}(A_i)] < \epsilon.$$

label = () A UI family is bounded in $\mathcal{L}^1$.

lbbel = () If a family (X_i) is bounded in $\mathcal{L}^p$, $p > 1$, then it is also UI.

Lemma 4.1. Let $(X_n)_{n\in\mathbb{N}}$, X be in $\mathcal{L}^1$ and $X_n \overset{n\to\infty}{\longrightarrow} X$ a.s. Then $X_n \overset{n\to\infty}{\longrightarrow}$ in $\mathcal{L}^1$ if and only if $(X_n)_{n\in\mathbb{N}}$ is UI.