Part III Functional Analysis
Based on lectures by A. Zsák

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Michaelmas 2023

These notes are produced entirely from the course I took, and my subsequent thoughts. They are not necessarily an accurate representation of what was presented, and may have in places been substantially edited. All errors are almost surely mine. Please send any corrections to pkt28@cam.ac.uk.

This course covers many of the major theorems of abstract Functional Analysis. It is intended to provide a foundation for several areas of pure and applied mathematics. The following topics are covered:

Hahn–Banach Theorems on the extension of linear functionals. Locally convex spaces.

Duals of the spaces $L_p(\mu)$ and $C(K)$. The Radon–Nikodym Theorem and the Riesz Representation Theorem.

Weak and weak-* topologies. Theorems of Mazur, Goldstine, Banach–Alaoglu. Reflexivity and local reflexivity.

Hahn–Banach Theorems on separation of convex sets. Extreme points and the Krein–Milman theorem. Partial converse and the Banach–Stone Theorem.

Banach algebras, elementary spectral theory. Commutative Banach algebras and the Gelfand representation theorem. Holomorphic functional calculus.

Hilbert space operators, C^*-algebras. The Gelfand–Naimark theorem. Spectral theorem for commutative C^*-algebras. Spectral theorem and Borel functional calculus for normal operators.

Prerequisites

Thorough grounding in basic topology and analysis. Some knowledge of basic functional analysis and basic measure theory (most of which was recalled either in lectures or via handouts). In Spectral Theory we will make use of basic complex analysis. For example, Cauchy's Theorem, Cauchy's Integral Formula and the Maximum Modulus Principle.

Literature

1. Allan, Graham R. *Introduction to Banach spaces and algebras (prepared for publication by H. Garth Dales).* Oxford University Press, 2011.
2. Bollobás, Béla *Linear analysis: an introductory course.* Cambridge University Press, 1990.
3. Murphy, Gerard J. *C^*-Algebras and Operator Theory.* Academic Press, Inc., 1990.
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Contents

1 Hahn-Banach Extension Theorems

Lecture 1 We start with setting up some notation.

1. Let X be a normed space. The dual space of X is denoted by X^* and is the space of all bounded linear functionals on X. Observe that X^* is always a Banach space in the operator norm
$$\|f\| = \sup\{|f(x)| : x \in B_X\}, \quad f \in X^*.$$
Recall that $B_X = \{x \in X : \|x\| \leqslant 1\}$ (the unit ball in X), and $S_X = \{x \in X : \|x\| = 1\}$ (the unit sphere in X).

2. Let X, Y be normed spaces. We write $X \sim Y$ if X, Y are *isomorphic*, i.e. there exists a linear bijection $T : X \to Y$ s.t. T, T^{-1} are continuous in the norm topologies.

3. Let X, Y be normed spaces. We write $X \cong Y$ if X, Y are *isometrically isomorphic*, i.e. there exists a surjective linear map $T : X \to Y$ s.t. $\|Tx\| = \|x\|$ for all $x \in X$.

4. For $x \in X$, we write $\langle x, f\rangle = f(x)$. Note that $\langle x, f\rangle = |f(x)| \leqslant \|f\| \cdot \|x\|$.

Examples:

1. For $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $\ell_p^* \cong \ell_q$ (isometrically isomorphic)
2. If H is a Hilbert space, then $H^* \cong H$ (conjugate linear in the complex case).

Definition 1.1. Let X be a real vector space. A functional $p : X \to \mathbb{R}$ is:

(i) positive homogeneous if $p(tx) = tp(x)$ for all $x \in X$ and $t > 0$

(ii) sub-additive if $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$.

Theorem 1.1. Let X be a real vector space and $p : X \to \mathbb{R}$ be positive homogeneous and sub-additive. Let $Y \leqslant X$ and $g : Y \to \mathbb{R}$ be a linear functional s.t. $g(y) \leqslant p(y)$ for all $y \in Y$. Then, there exists linear $f : X \to \mathbb{R}$ s.t. $f\restriction_Y = g$ and $f(x) \leqslant p(x)$ for all $x \in X$.

Recall now Zorn's lemma, which is needed to prove Theorem 1.1 in complete generality. Let $(P, \leqslant)$ be a poset.

- If $A \subset \mathcal{P}$, $x \in P$, then x is an *upper bound* for A if for all $x \in A$, $a \leqslant x$.
- x is a *maximal element* if for all $y \in P$, $y \geqslant x$ implies $y = x$
- A collection of subsets $\mathcal{C}$ of P is called a *chain* if for any two subsets $C, D \in \mathcal{C}$, either $C \subseteq D$ or vice versa.

Lemma 1.1 (Zorn). If $P \neq \varnothing$ and every non-empty chain has an upper bound, then P has a maximal element.

Proof of Theorem 1.1. Let P be the set of pairs (Z, h) where Z is a subspace of X with $Y \subseteq Z$, $h : Z \to \mathbb{R}$ linear, $h\restriction_Y = g$ and for all $z \in Z$, $h(z) \leqslant p(z)$. Observe that P is partially ordered by
$$(Z_1, h_1) \leqslant (Z_2, h_2) \iff Z_1 \subseteq Z_2, \quad h_2\restriction_{Z_1} = h_1.$$
Also, we have $P \neq \varnothing$ since (Y, g) is in P. If $\{(Z_i, h_i)\}_{i\in I}$ is a chain in P with $I \neq \varnothing$, then setting $Z = \bigcup_{i\in I} Z_i$ and $h : Z \to \mathbb{R}$ by requiring that $h\restriction_{Z_i} = h_i$, for $i \in I$, we have that (Z, h) is in P and it is an upper bound for the chain. So by Zorn, P has a maximal element (Z, h).

It suffices to show that $Z = X$. Suppose not, and fix $x \in X/Z$. Let $W = \text{span}(Z \cup \{x\})$ and $f : W \to \mathbb{R}$, $f(z + \lambda x) = h(z) + \lambda\alpha$, for $z \in Z$, $\lambda \in \mathbb{R}$ for some $\alpha \in \mathbb{R}$. We seek $\alpha \in \mathbb{R}$ s.t. for all $w \in W$, $f(w) \leqslant p(w)$. Then, $(W, f) \in P$ and (W, f) is strictly bigger than (Z, h), which would establish a contradiction.

Need: $h(z) + \lambda\alpha \leqslant p(z + \lambda\alpha)$ for all $z, \lambda \in \mathbb{R}$. Since p is positive homogeneous, this is equivalent to:

$$\left\{ \begin{array}{l} h(z) + \alpha \leqslant p(z + x) \\ h(z) - \alpha \leqslant p(z - x) \end{array} \right\} \text{ for all } z \text{ in } Z.$$

That is, $h(y) - p(y - x) \leqslant \alpha \leqslant p(z + x) - h(z)$ for all $y, z \in Z$. This holds since, for $y, z \in Z$:

$$h(y) + h(z) = h(y + z) \leqslant p(y + z) = p(y - x + z + x) \leqslant p(y - x) + p(z + x).$$

□

Definition 1.2. Let X be a real or complex vector space. A semi-norm on X is a functional $p : X \to \mathbb{R}$ s.t.:

- for all $x \in X$, $p(x) \geqslant 0$
- for all $x \in X$ and all $\lambda \in \mathbb{R}$, $p(\lambda x) = |\lambda| \cdot p(x)$
- for all $x, y \in X$, $p(x + y) \leqslant p(x) + p(y)$.

Note: $Norm \implies seminorm \implies (sub-additive\,\&\,positive\ homogeneous)$

Theorem 1.2 (Hahn Banach). Let X be a real of complex vector space and p be a semi-norm on X. Let Y be a subspace of X and $g : Y \to \mathbb{C}$ linear s.t. for all $y \in Y$ $|g(y)| \leqslant p(y)$. Then there exists linear functional f on X s.t. $f\restriction_Y = g$ and for all $x \in X$ $|f(x)| \leqslant p(x)$.

Lecture 2 *Proof.* Real case: for all $y \in Y$ $g(y) \leqslant |g(y)| \leqslant p(y)$. By Theorem 1.1 there exists linear functional $f : X \to \mathbb{R}$ s.t. $f\restriction_Y = g$ and for all $x \in X$ $f(x) = p(x)$. For $x \in X$, we have also $-f(x) = f(-x) \leqslant p(-x) = p(x)$, so $|f(x)| \leqslant p(x)$.

Complex case: $\text{Re}(g) : Y \to \mathbb{R}$, $(\text{Re})(y) = \text{Re}(g(y))$, is real linear. For all $y \in Y$ $|\text{Re}(g)(y)| \leqslant |g(y)| \leqslant p(y)$. By the real case, there exists a real linear map $h : X \to \mathbb{R}$ s.t. $h\restriction_Y = \text{Re}(g)$ and for all $x \in X$ $|h(x)| \leqslant p(x)$.

Claim: there exists unique complex linear map $f : X \to \mathbb{C}$ s.t. $h = \text{Re}(f)$.

Proof of claim: Uniqueness If we have such an f, then for any $x \in X$, $f(x) = h(x) + \text{Im}(f) = h(x) + \text{Im}(-if(ix)) = h(x) - ih(ix)$. Existence *define* $f(x) = h(x) - ih(ix)$, for $x \in X$. Then f is real linear and $f(x) = if(x)$ for all $x \in X$. Hence, f is complex linear and $\text{Re}(f) = h$, by definition.

We have $f : X \to \mathbb{C}$ linear s.t. $\text{Re}(f) = h$. Then $\text{Re}(f)\restriction_Y = h\restriction_Y = \text{Re}(g)$, so by uniqueness $f\restriction_Y = g$. Given $x \in X$, write $|f(x)| = \lambda f(x)$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$; now, $|f(x)| = \lambda f(x) = f(\lambda x) = \text{Re}(f)(\lambda x)$ [1] $= h(\lambda x) \leqslant p(\lambda x) = |\lambda| p(x) = p(x)$. □

Remark. For a complex vector space Y, let $\mathbf{Y}_\mathbb{R}$ be Y viewed as a real vector space. The proof above shows that for a normed space, X, the map $f \mapsto \text{Re}(f) : (X^*) \to (\mathbf{X}_\mathbb{R}^*)$ is an isometric isomorphism.

Corollary 1.2.1. Let X be a real or complex vector space, p a semi-norm on X and $x_0 \in X$. Then there exists linear functional f on X s.t. $f(x_0) = p(x_0)$ and for all $x \in X$ $|f(x)| \leqslant p(x)$.

Proof. Let $Y = \text{span}\{x_0\}$, define $g : Y \to (\mathbb{R}, \mathbb{C})$ $g(\lambda x_0) = \lambda p(x_0)$. Then g is linear and $g(x_0) = p(x_0), |g(\lambda x_0)| = |\lambda| \cdot |p(x_0)| = p(\lambda x_0)$. So for all $y \in Y$ $|g(y)| \leqslant p(y)$. By Theorem 1.2, there exists linear function f on X s.t. $f\restriction_Y = g$ and for all $x \in X$ $|f(x)| \leqslant p(x)$. Note that $f(x_0) = g(x_0) = p(x_0)$. □

[1] $|f(x)| \in \mathbb{R}$.

Theorem 1.3 (Hahn-Banach). Let X be a real or complex normed space.

- It Y is a subspace of X and $g \in Y^*$ then there exists $f \in X^*$ s.t. $f\restriction_Y = g$ and $\|f\| = \|g\|$
- Given $x_0 \in X/\{0\}$, there exists $f \in S_{X^*}$[a] s.t. $f(x_0) = \|x_0\|$.

[a] unit sphere.

Proof. (i) let $p(x) = \|g\| \cdot \|x\|$, for $x \in X$. Then p is a semi-norm on X and for all $y \in Y$, $|g(y)| \leqslant \|g\| \cdot \|y\|$. By Theorem 1.2 there exists linear functional $f : X \to (\mathbb{R}, \mathbb{C})$ s.t. $f\restriction_Y = g$ and for all $x \in X$ $|f(x)| \leqslant p(x) = \|g\| \cdot \|x\|$, which implies $\|f\| \leqslant \|g\|$; since $f\restriction_Y = g$, we also have $\|f\| \geqslant \|g\|$, so we have the desired equality $\|f\| = \|g\|$.

(ii) Apply Corollary 1.2.1 with $p(x) = \|x\|$, to get a linear functional f on X s.t. for all $x \in X$ $|f(x)| \leqslant \|x\|$ and $f(x_0) = \|x_0\|$. It follows that $\|f\| = 1$. $\square$

Remark. 1. part (i) is a sort of linear version of Tietze's extension theorem: given K compact, Hausdorff, $L \subseteq K$ closed, $g : K \to (\mathbb{R}, \mathbb{C})$ continuous, there exists continuous $f : K \to (\mathbb{R}, \mathbb{C})$ s.t. $f\restriction_L = g$ and $\|f\|_\infty = \|g\|_\infty$.

2. part (i) shows that for all $x \neq y \in X$ there exists $f \in X^*$ s.t. $f(x) \neq f(y)$ (use $x_0 = x - y$). X^* separates points of X. (This is a sort of linear version of Uryshon's lemma: $C(K)$ separates points of K, K compact, Hausdorff).

3. The f in part (ii) is called a norming functional for x_0. It shows that $\|x_0\| = \max\{|\langle x_0, g\rangle| : g \in B_{X^*}\}$. Another name for f: support functional at x_0. Assume X is real, $\|x\| = 1$. Then, $B_X \subseteq \{x \in X : f(x) \leqslant 1\}$.

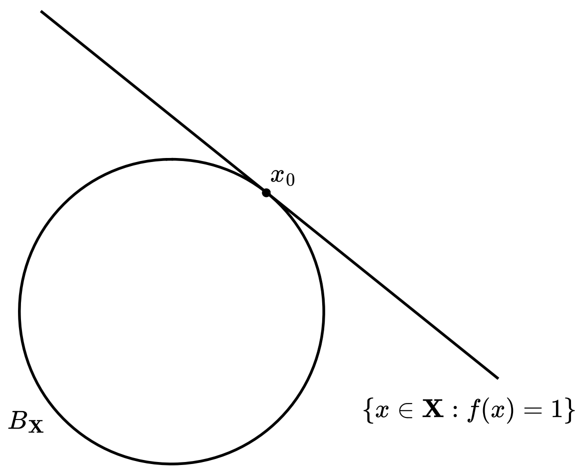

Figure 1: Illustration of support a functional, see the remark above. The pre-image of 1 under f is tangent to B_X at x_0.

Bidual Let X be a normed space. Then $X^{**} = (\mathbf{X}^*)^*$ is called the bidual or second dual of X. For $x \in X$, we define $\hat{x} : X^* \to$ scalar, by $\hat{x}(f) = f(x)$, for all $f \in X^*$ (evaluation at x). Then $\hat{x}$ is linear, and $|(\hat{f})| = |f(x)| \leqslant \|f\| \cdot \|x\|$, so $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leqslant \|x\|$. The map $x \mapsto \hat{x} : X \to X^{**}$ is called the canonical embedding of X into X^{**}.

Theorem 1.4. The canonical embedding of X into X^{**} is an isometric isomorphism into X^{**}.

Proof. Linearty: $(\widehat{\lambda x + \mu y})(f) = f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) = \lambda \hat{x}(f) + \mu \hat{y}(f)$ for all $x, y \in X$, λ, μ scalars and $f \in X^*$

Isometry: if $x \in X/\{0\}$, then there exists norming functional f of x and so $\|\hat{x}\| \geqslant |\hat{x}(f)| = |f(x)| = \|x\|$. $\square$

Remark. 1. In bracket notation: $\langle f, \hat{x}\rangle = \langle x, f\rangle$ (for $x \in X$ and $f \in X^*$).

2. Let $\hat{X} = \{\hat{x} : x \in X\}$-the image of $X \in X^*$. Then, Theorem 1.4 says that $X \cong \hat{X} \subseteq X^{**}$. We often identify $\hat{X}$ with X and think of X isometrically as a subspace of X^{**}. Note that X is complete $\iff$ $\hat{X}$ is closed in X^{**}.

3. More generally, $\overline{\hat{\mathbf{X}}}$ is a Banach space (closed in X^{**}) containing an isometric copy of X as a dense subspace. We thus proved that normed spaces have completions.

Definition 1.3 (Reflexivity). A normed space X is called reflexive if the canonical embedding $X \hookrightarrow X^{**}$ is surjective.

Examples: (Reflexivity)

1. $\ell_p, 1 < p < \infty$
   Hilbert spaces
   finite-dimensional normed spaces
   $L_p(\mu), 1 < p < \infty$ (later!)

2. $c_0, \ell_1, \ell_\infty, L_1[0,1]$ are not reflexive.

Remark. If X is reflexive, then $X \cong X^{**}$. Note however that there exist Banach spaces X s.t. $X \cong X^{**}$ but X is not reflexive.

1.1 Dual Operators

Lecture 3

Let $\mathbf{X}, \mathbf{Y}$ be normed spaces. Recall

$$\mathcal{B}(\mathbf{X}, \mathbf{Y}) = \{T : X \to Y : T \text{ is linear and bounded}\}.$$

This is a normed space in the operator norm:

$$\|T\|_{X\to Y} = \sup_{\|x\|_X \leqslant 1} \|Tx\|_Y .$$

If Y is complete, then so is $(\mathcal{B}(\mathbf{X}, \mathbf{Y}), \|\cdot\|_{X\to Y})$. For $T \in \mathcal{B}(\mathbf{X}, \mathbf{Y})$, the dual operator of T, is the map $T^* : X^* \to Y^*, T^*g = g \circ T$ for $g \in Y^*$[2]. In the bracket notation;

$$\langle x, T^*g\rangle = \langle Tx, g\rangle, \quad \text{for } x \in X,\ g \in Y^*.$$

T^* is linear:

$$\begin{aligned}
\langle x, T^*(\lambda g + \mu h)\rangle &= \langle Tx, \lambda g + \mu h\rangle \\
&= \lambda\langle Tx, g\rangle + \mu\langle Tx, h\rangle \\
&= \lambda\langle x, T^*g\rangle + \mu\langle x, T^*h\rangle \\
&= (\lambda T^*g + \mu T^*)(x) \\
&= \langle x, \lambda T^*g + \mu T^*h\rangle.
\end{aligned}$$

T^* is bounded:

$$\begin{aligned}
\|T^*\| &= \sup_{\|g\|_{Y^*}\leqslant 1} \|T^*g\| \\
&= \sup_{\|g\|_{Y^*}\leqslant 1} \sup_{\|x\|_X\leqslant 1} \|g \circ T(x)\| \\
&= \sup_{\|x\|_X\leqslant 1} \sup_{\|g\|_{Y^*}\leqslant 1} \|g \circ T(x)\| \\
&= \sup_{\|x\|_X\leqslant 1} \|Tx\| = \|T\| .
\end{aligned}$$

Remark. If $\mathbf{X}, \mathbf{Y}$ are Hilbert spaces, and identify $\mathbf{X}, \mathbf{Y}$ with X^* and Y^* respectively, then $T^* : Y \to X$ is the adjoint of T.

[2] well-defined.

Example:
$1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$, $R : \ell_p \to \ell_q$, the right shift $R(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ then $R^* : \ell_q \to \ell_p$ is the left shift.

Properties:

1. $(\mathrm{Id}_X)^* = \mathrm{Id}_X^*$.
2. $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ for $S, T \in \mathcal{B}(\mathbf{X}, \mathbf{Y})$, and λ, μ scalars.
3. $(ST)^* = T^* S^*$ for $T \in \mathcal{B}(\mathbf{X}, \mathbf{Y})$ and $S \in \mathcal{B}(\mathbf{Y}, \mathbf{Z})$
   $(ST)^*(h \in Z^*) = h \circ S \circ T = T^* h \circ S = T^* S^*(h)$
4. $T \mapsto T^* : \mathcal{B}(\mathbf{X}, \mathbf{Y}) \to \mathcal{B}(\mathbf{Y}^*, \mathbf{X}^*)$ is an into isometric isomorphism.
5. Let $x \in X$ then $\langle g, T^* \hat{x} \rangle = \langle T^* g, \hat{x} \rangle = \langle x, T^* g \rangle = \langle Tx, g \rangle = \langle g, \widehat{Tx} \rangle$ for all $g \in Y^* \implies T^{**} \hat{x} \equiv \widehat{Tx}$. In other words, the following diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, hook, "\iota_X"] & Y \arrow[d, hook, "\iota_Y"] \\ X^{**} \arrow[r, "T^{**}"] & Y^{**} \end{tikzcd}$$

   commutes (vertical arrows are canonical embeddings).

Remark. From the (above) properties, if $X \sim Y$ then $X^* \sim Y^*$.

1.2 Quotient Spaces

Let X be a normed space and Y be a closed subspace. Then the quotient space X/Y becomes a normed space in the quotient norm:

$$\|x + Y\|_{X/Y} = d(x, Y) = \inf_{y \in Y} \|x + y\|.$$

The quotient map : $q : X \to X/Y, q(x) = x + Y$ is linear and bounded with $\|q(x)\|_{X/Y} \leqslant \|x\|_X$ for all $x \in X$, so $\|q\| \leqslant 1$. It maps the open unit ball $B_X = \{x \in X : \|x\| < 1\}$ onto $D_{X/Y}$. Indeed, for $x \in D_X$, then $\|q(x)\| \leqslant \|x\| < 1$. Conversely, if $z \in B_{X/Y}$ and $z = q(x)$, then $\|z\| < 1 \implies \inf_{y \in Y} \|x + Y\| < 1 \implies$ there exists $y \in Y$ s.t. $\|x + y\| < 1 \implies x + y \in D_X$ and $q(x + y) = q(x) = z$. It follows that q is an open map and $\|q\| = 1$ (provided $Y \neq X$).

If Z is another normed space, $T \in \mathcal{B}(X, Z)$ and $Y \subseteq \ker(T)$, then there exists a unique map $\tilde{T} : X/Y \to Z$ such that

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "q"] & Z \\ X/Y \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

commutes. Hence, $T = \tilde{T} \circ q$; moreover, $\tilde{T}$ is linear and $\tilde{T}(D_{X/Y}) = \tilde{T}(q(D_X)) = T(D_X)$ and so it follows that $\left\|\tilde{T}\right\| = \|T\|$.

Theorem 1.5. Let X be a normed space. If X^* is separable, then so is X.

Remark. The converse is false in general. For instance, $X = \ell_1, X^* = \ell_\infty$.

Proof. Since X^* is separable, then so is S_{X^*}. Let $\{f_n : n \in \mathbb{N}\}$ be a dense subset of S_{X^*}. For all n there exists $x_n \in B_X$ s.t. $f_n(x_n) > 1/2$. Let $Y = \overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}}$.

Claim: suffices to show $Y = X$.

Suppose not: Then, by Theorem 1.3 we can pick $g \in (X/Y)^*$ with $\|g\| = 1$, that is a norming functional. Let $f = g \circ q$ ($q : X \to X/Y$ is the quotient map). Then $\|f\| = \|g\| = 1 \implies f \in S_{X^*}$. By density, we have that there exists $n \in \mathbb{N}$ s.t. $\|f - f_n\| < \frac{1}{10}$ (something small). So

$$|(f - f_n)(x_n)| \leqslant \|f - f_n\| \cdot \|x_n\| < \frac{1}{10},$$

but

$$|(f - f_n)(x_n)| = |f_n(x_n)| > \frac{1}{2}, \quad n \to \infty,$$

a contradiction. □

Theorem 1.6. Let X be a separable normed space. Then X embeds isometrically into ℓ_∞.

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be dense in X and for all $n \in \mathbb{N}$ let $f_n \in S_{X^*}$ with $f_n(x_n) = \|x_n\|$ (wlog $X \neq \{0\}$). Define $T : X \to \ell_\infty, Tx = (f_n(x_n))$. It is clear that T is linear.

Well-defined: $|f_n(x)| \leqslant \|f_n\| \cdot \|x\| \leqslant \|x\|$, for all $n \in \mathbb{N}$ which implies $\|Tx\|_\infty \leqslant \|x\| < \infty$, hence $Tx \in \ell_\infty$.

T isometric: already $\|Tx\|_\infty \leqslant \|x\|$ for all x. Also, $\|Tx_n\|_\infty = \|x_n\|$ for all n. By density and continuity, $\|Tx\|_\infty = \|x\|$ for all $x \in X$. □

Lecture 4 **Remark.**

1. The result says ℓ_infty is isometrically universal for the class of separable Banach spaces, $\mathcal{SB}$.
2. Dual result: every separable Banach space is a quotient of ℓ_1 (see the Example sheets).

Theorem 1.7 (Vector-valued Liouville). Let X be a complex Banach space and $f : \mathbb{C} \to X$ be holomorphic and bounded, then f is constant.

Proof. We have that there exists $M \geqslant 0$, s.t. for all $z \in \mathbb{C}$, $\|f(z)\| \leqslant M$. Also, for $w \in \mathbb{C}$, $\lim_{z \to w} \frac{f(z) - f(w)}{z - w}$ exists in X and we denote this by $f'(z)$. Fix $\phi \in X^*$ and consider $\phi \circ f : \mathbb{C} \to \mathbb{C}$. This is holomorphic and bounded.

Bounded: $|\phi(f(z))| \leqslant \|\phi\| \cdot \|f(z)\| \leqslant \|\phi\| \cdot \|z\|$ for all $z \in \mathbb{C}$.

Holomorphic:

$$\frac{\phi(f(z)) - \phi(f(w))}{z - w} \overset{[3]}{=} \phi\left(\frac{f(z) - f(w)}{z - w}\right) \to \phi(f'(z)), \quad \text{as } w \to^{[4]} z.$$

Now, by scalar Liouville, $\phi \circ f$ is constant. Hence, $\phi \circ f(z) = \phi(f(0))$ for all $z \in \mathbb{C}$. Fix$z \in \mathbb{C}$, $\phi(f(z)) = \phi(f(0))$, for all $\phi \in X^*$. Since X^* separates points of X, $f(z) = f(0)$ for all $z \in \mathbb{C}$. □

[3] linearity.
[4] ϕ is continuous.

1.3 Locally Convex Spaces

Definition 1.4 (Locally convex space (LCS)). A locally convex space is a pair $(X, \mathcal{P})$, where X is a real/complex vector space and $\mathcal{P}$ is a family of semi-norms on X that separate points of X in the sense that for all $x \in X \neq \{0\}$ there exists semi-norm $P_X \in \mathcal{P}$ s.t. $P_X \neq 0$. The family $\mathcal{P}$ defines a topology on X:

$$\mathcal{U} \subseteq X \text{ is open} \iff \forall x \in \mathcal{U}\ \exists n \in \mathbb{N} \exists p_1, \ldots, p_n \in \mathcal{P}$$
$$\exists \epsilon > 0 \text{ s.t. } \{y \in X : p_k(y - x) < \epsilon, 1 \leqslant k \leqslant n\} \subseteq \mathcal{U}.$$

Remark.

1. Vector addition and scalar multiplication are continuous.
2. This topology is Hausdorff.
3. $x_n \to x \in X \iff$ for all $p \in \mathcal{P}$, $p(x - x_n) \to 0$.
4. Let Y be a subspace of X. Let $\mathcal{P}_Y = \{p\restriction_Y : p \in \mathcal{P}\}$. Then the pair $(Y, \mathcal{P}_Y)$ is a LCS and its topology is the subspace topology induced by $(X, \mathcal{P})$.
5. Let $\mathcal{P}, \mathcal{Q}$ be families of semi-norms on X both separating points of X. We say $\mathcal{P}, \mathcal{Q}$ are equivalent, write $\mathcal{P} \sim \mathcal{Q}$ if they define the same topology on X. Then $(X, \mathcal{P})$ is metrisable iff there exists countable family $\mathcal{Q} \sim \mathcal{P}$.

Definition 1.5 (Fréchect space). AFréchet space is a complete metrisable locally convex space.

Examples:

1. A normed space $(X, \|\cdot\|)$ is a LCS (here $\mathcal{P} = \{\|\cdot\|\}$).
2. Let $\mathcal{U} \subseteq \mathbb{C}$ be non-empty open. Let $\mathcal{O}(\mathcal{U}) = \{f : \mathcal{U} \to \mathbb{C} : f \text{ holomorphic}\}$.
   For $K \subseteq \mathcal{U}$ define $\mathcal{P}_K(f) = \sup_{z \in K} |f(z)|$. Let $\mathcal{P} = \{\mathcal{P}_K : K \subseteq \mathcal{U}, K \text{ compact}\}$. Then $(\mathcal{O}(\mathcal{U}), \mathcal{P})$ is a LCS. Note further that there exists K_n, $n \in \mathbb{N}$, a sequence of compact subsets of $\mathcal{U}$ s.t. $\mathcal{U} = \bigcup_{n \in \mathbb{N}} K_n$ and for all $n \in \mathbb{N}$ $K_n \subset (K_{n+1})^\circ$ (a compact exhaustion of $\mathcal{U}$). Montel's Theorem from complex analysis gives that $(\mathcal{O}(\mathcal{U}), \mathcal{P})$ is not normable: there is no norm on $\mathcal{O}(\mathcal{U})$ that gives the same topology, that is the topology of local uniform convergence. To see this, suppose for a contradiction that there exists norm s.t. $\|\cdot\| \sim \mathcal{P}$, then for all $f \in B_{\mathcal{O}(\mathcal{U})}$, for all $p \in \mathcal{P}$, $p(f) \leqslant C_p \cdot \|f\| := C_p < \infty$ (since $\tau_{\mathcal{P}} = \tau_{\mathcal{O}(\mathcal{U})}$) which implies that that unit ball is compact (by the above and Montel's Theorem), hence sequentially compact due to the metrisability of the norm topology on $\mathcal{O}(\mathcal{U})$. So we conclude that $\mathcal{O}(\mathcal{U})$ is finite-dimensional, a contradition.
3. Fix $d \in \mathbb{N}$ and a non-empty open set $\Omega \subseteq \mathbb{R}^d$. Let $\mathcal{C}^\infty = \{f : \Omega \to \mathbb{R}^d : f \text{ is infinitely differentiable}\}$. Given a multi-index, namely, a $d-$tuple $\alpha \in \mathbb{N}^d$, it defines a differential operator:
$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$
   For a compact set $K \subset \Omega$, $\alpha \in \mathbb{N}^d$, define $p_{k,\alpha}(f) = \sup\{|D^\alpha f(z)| : z \in K\}$. Let $\mathcal{P} = \{p_{k,\alpha} : K \subset \Omega, K \text{ compact}, \alpha \in \mathbb{N}^d\}$. Then $(\mathcal{C}^\infty(\Omega), \mathcal{P})$ is a LCS. It's a Fréchet space and non-normable.

Lemma 1.2. Let $(X, \mathcal{P}), (Y, \mathcal{Q})$ be LCS and $T : X \to Y$ be a linear map. Then the following are equivalent (TFAE):

(i) T is continuous.

(ii) T is continuous at 0.

(iii) For all $q \in \mathcal{Q}$ there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$, $c > 0$ s.t.

$$q(Tx) \leqslant C \cdot \max_{1 \leqslant k \leqslant n} p_k(x) \text{ for all } x \in X.$$

Proof. $\underline{(i) \iff (ii)}$: translation is continuous since vector addition is continuous.

$\underline{(ii) \iff (iii)}$: given $q \in \mathcal{Q}$, let $\mathcal{V} = \{y \in Y : q(y) \leqslant 1\}$. Then $\mathcal{V}$ is a neighbourhood of zero in Y, so there exists a nbhd of zero in X s.t. $T(X) \subseteq \mathcal{V}$. Then there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$, $\epsilon > 0$ s.t. wlog $\mathcal{U} = \{x \in X : p_k(x) \leqslant \epsilon, 1 \leqslant k \leqslant n\}$. Let $p(x) = \max_{1 \leqslant k \leqslant n} p_k(x)$, for $x \in X$. If $p(x) = 1$ then $p(\epsilon x) = \epsilon \implies \epsilon x$ is in $\mathcal{U}$. So $q(Tx) \leqslant 1 \implies q(Tx) \leqslant \frac{1}{\epsilon} p(x)$ by homogeneity for any x s.t. $p(x) > 0$. If $p(x) = 0 \implies p(\lambda x) = 0$ for all λ scalars giving $q(T(\lambda x)) \leqslant 1$ for all λ scalars. Hence, $q(Tx) \leqslant \frac{1}{\epsilon} p(x)$, concluding the proof of this equivalence.

$\underline{(iii) \iff (ii)}$: Let $\mathcal{V}$ be a nbhd of zero in Y. Then, there exists $n \in \mathbb{N}$, $q_1, \ldots, q_n \in \mathcal{Q}$ and $\epsilon > 0$ s.t. wlog $\mathcal{V} = \{y \in Y : q_j(y) \leqslant \epsilon \text{ for } 1 \leqslant j \leqslant n\}$. For each $1 \leqslant j \leqslant n$, there exists $m_j \in \mathbb{N}$, $p_{j1}, \ldots, p_{jm_j} \in \mathcal{P}$, $C_j > 0$ s.t. $q_j(Tx) \leqslant C_j \cdot \max_{1 \leqslant i \leqslant m_j} p_{ji}(x)$ for all $x \in X$. Finally, let
$\mathcal{U} = \{x \in X : p_{ji}(x) < \frac{\epsilon}{C_j}, 1 \leqslant i \leqslant m_j, 1 \leqslant j \leqslant n\}$ so then $T(\mathcal{U}) \subseteq \mathcal{V}$. □

Definition 1.6. Let $(X, \mathcal{P})$ be a LCS. The <u>dual space</u> of X is the space X^* of all linear functionals which are continuous wrt the topology $(X, \mathcal{P})$.

Lecture 5

Lemma 1.3. Let f be a linear functional on a LCS $(X, \mathcal{P})$. Then f is in $X^* \iff \ker f$ is closed.

Proof. $\underline{\Longleftarrow}$: $\ker f = f^{-1}(\{0\})$ is closed if f is continuous.

$\underline{\Longrightarrow}$: If $\ker f = X$, then $f \equiv 0$ is continuous.
Assume $\ker f \neq X$ and fix $x_0 \in X \backslash \ker f$. Since $X \backslash \ker f$ is open, there exists $n \in \mathbb{N}$,$p_1, \ldots, p_n \in \mathcal{P}$ and $\epsilon > 0$ s.t. $\{x \in X : p_k(x - x_0) < \epsilon, 1 \leqslant k \leqslant n\} \subseteq X \backslash \ker f$. Let $\mathcal{U} = \{x \in X : p_k(x) < \epsilon, 1 \leqslant k \leqslant n\}$. Then $\mathcal{U}$ is a nbhd of zero in X, and $(x_0 + \mathcal{U}) \cap \ker f = \varnothing$.

Note that $\mathcal{U}$ is convex and, in the real case, symmetric ($x \in \mathcal{U}$ implies $-x \in \mathcal{U}$). In the complex case, <u>balanced</u> ($x \in \mathcal{U}$, $|\lambda| \leqslant 1$ implies $\lambda x \in \mathcal{U}$), and hence so is $f(\mathcal{U})$ as f is linear. If $f(\mathcal{U})$ is not bounded, then $f(\mathcal{U})$ is the whole scalar field, and hence so is $f(x_0 + \mathcal{U}) = f(x_0) + f(\mathcal{U})$, a contradiction as zero is not in $f(x_0 + \mathcal{U})$. So there exists $M > 0$ s.t. $|f(x)| < M$ for all $x \in \mathcal{U}$. So given $\delta > 0$, $\frac{\delta}{M}\mathcal{U}$ is a nbhd of zero in $\mathcal{X}$ and $f\left(\frac{\delta}{M}\mathcal{U}\right) \subseteq \{\lambda \text{ scalar}, \lambda < \delta\}$. Thus, f is continuous at zero, hence everywhere. Thus f is in X^*. □

Theorem 1.8. Let $(X, \mathcal{P})$ be a LCS.

(i) Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ s.t. $f\restriction_Y = g$.

(ii) Given a closed subspace Y of X and $x_0 \in X \backslash Y$, there exists $f \in X^*$ s.t. $f\restriction_Y = 0$ and $f(x_0) \neq 0$.

Remark. So X^* separates the points of X.

Proof. (i) by lemma 1.2, there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$ and $C > 0$ s.t. for all $y \in Y$ $|g(y)| \leqslant C \cdot \max_{1 \leqslant k \leqslant n} p_k(y)$. Let $p(x) = C \max_{1 \leqslant k \leqslant n} p_k(x)$, for $x \in X$. Then, p is a semi-norm on X and for all $y \in Y$ $|g(y)| \leqslant p(y)$. By Theorem 1.2, there exists a linear functional f on X s.t. $f\restriction_Y = g$ and for all $x \in X$, $|f(x)| \leqslant p(x)$. Now, finally observe that by lemma 1.2, f is in X^*.

(ii) Let $Z = \operatorname{span}(Y \cup \{x_0\})$ and define a linear functional g on Z by $g(y + \lambda x_0) = \lambda$, for $y \in Y$ and λ scalar. Then $g\restriction_Y = 0, g(x_0) = 1 \neq 0$ and $\ker g = Y$ is closed, so $g \in Z^*$ by lemma 1.3. By part (i), there exists $f \in X^*$ s.t. $f\restriction_Z = g$ and this works.

□

2 Dual Spaces of $L_p(\mu)$ and $\mathcal{C}(K)$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For $\leqslant < p < \infty$,

$$L_p(\mu) = \left\{ f : \Omega \to \text{scalar} : f \text{ is measurable and } \int_\Omega |f|^p d\mu < \infty \right\}$$

This is a normed space in the L_P norm $\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}$.

$p = \infty$: A measurable function $f : \Omega \to$ scalar is essentially bounded if there is $N \in \mathcal{F}$, $\mu(N) = 0$, and $f\restriction_{\Omega\setminus N}$ is bounded.

$L_\infty(\mu) = \{f : \Omega \to \text{scalar} : f \text{ measurable and essentially bounded}\}$. This is again a normed space in the $L^\infty-$ norm:

$$\|f\|_\infty = \text{essup}\,|f| = \inf \left\{ \sup_{\Omega\setminus N} |f| : N \in \mathcal{F}, \mu(N) = 0 \right\}.$$

The inf is attained: there exists $N \in \mathcal{F}$, $\mu(N) = 0$, $\|f\|_\infty = \sup_{\Omega\setminus N} |f|$.
In all the cases, we identify functions f, g if $f = g$ a.e.

Theorem 2.1. $L_p(\mu)$ is complete for $1 \leqslant p \leqslant \infty$.

Proof. Can be found in any standard reference in measure theory, see the literature provided. □

2.1 Complex Measures

Let Ω be a set, $\mathcal{F}$ a $\sigma-$field on Ω. A complex measure on $\mathcal{F}$ is a countably additive function $\nu : \mathcal{F} \to \mathbb{C}$. For $A \in \mathcal{F}$, the total variation measure $|\nu|$ of ν is defined as follows:

$$|\nu(A)| = \sup \left\{ \sum_{k=1}^n A_k : A = \bigcup_{k=1}^n A_k \text{ is a measurable partition of } A \right\}^5.$$

Then, $|\nu| : \mathcal{F} \to [0, \infty]$ is a positive measure. Later we see that $|\nu|$ is a finite measure. The total variation of ν is $\|\nu\|_1 = |\nu|(\Omega)$.

Continuity: if ν is a complex measure on $\mathcal{F}$ and $(A_n) \subseteq \mathcal{F}$, then:

(i) if $A_n \subseteq A_{n+1}$, then $\nu(\bigcup_n A_n) = \lim_{n\to\infty} \nu(A_n)$

(ii) if $A_{n+1} \subseteq A_n$, then $\nu(\bigcap_n A_n) = \lim_{n\to\infty} \nu(A_n)$.

Signed measure: Ω a set, $\mathcal{F}$ a $\sigma-$algebra on Ω.
A signed measure on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{R}$.

Theorem 2.2 (Hahn decomposition). Let Ω be a set, $\mathcal{F}$ a $\sigma-$algebra on Ω, ν a signed measure on $\mathcal{F}$. Then there exists a measurable partition $P \cup N$ of Ω s.t. for all $A \in \mathcal{F}$, $A \subseteq P$ implies $\nu(A) \geqslant 0$ and for all $A \in \mathcal{F}$, $A \subseteq N$ implies $\nu(A) \leqslant 0$.

Remark. 1. The decomposition $\Omega = P \cup N$ is called the Hahn decomposition of ν (or of Ω).

2. Lets us define $\nu^+(A) = \nu(A \cap P), \nu^-(A) = -\nu(A \cap N)$, for $A \in \mathcal{F}$. Then ν^+, ν^- are finite positive measures such that $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$. These determine ν^+, ν^- uniquely and $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν.

[5] $A_k \in \mathcal{F}, A_j \cap A_k = \varnothing\ \forall j \neq k$.

3. If ν is a complex measure on $\mathcal{F}$ then $\mathrm{Re}(\nu), \mathrm{Im}(\nu)$ are signed measures with Jordan decompositions $\nu_1 - \nu_2 + i(\nu_3 - \nu_4)$-the Jordan decomposition of ν. Then $\nu_k \leqslant |\nu|, 1 \leqslant k \leqslant 4$ and $|\nu| \leqslant \nu_1 + \nu_2 + \nu_3 + \nu_4$. So $|\nu|$ is a finite measure.

4. If ν is a signed measure on $\mathcal{F}$ with Jordan decomposition $\nu^+ - \nu^-$, then $\nu^+(A) = \sup\{\nu(B) : B \in \mathcal{F}, B \subseteq A\}$, for $A \in \mathcal{F}$.

Proof of Theorem 2.2. The strategy is to define $\nu^+(A) = \sup\{\nu(B) : B \in \mathcal{F}, B \subseteq A\}$ for $A \in \mathcal{F}$. Then $\nu^+ \geqslant 0$ and ν^+ is finitely additive.

Key step: $\nu^+(\Omega) \geqslant 0$.

By contradiction, assume not; construct sequences $(A_n), (B_n)$ with $A_0 = \Omega$, $\nu^+(A_n) = \infty$, $B_n \subset A_n$ and $\nu(B_n) > n$. Now by the finite additivity of ν^+, pick $A_{n+1} = B_n$ or $A_n \backslash B_n$, to ensure the initial condition ($\nu^+(A_{n+1}) = \infty$) is satisfied.

Claim: this will contradict $\sigma-$additivity.

To see this, note that (A_n) is by construction a decreasing sequence wrt inclusion. By $\sigma-$additivity of ν, $\nu(\cap_n A_n) = \lim_{n\to\infty} \nu(A_n)$. Thus, it cannot be the case that $A_{n+1} = B_n$ infinitely often, since $\nu(\cap_n A_n) < \infty$ (being a signed measure). Thus, there exists $N \in \mathbb{N}$ s.t. for all $n \geqslant N$, $A_{n+1} = A_n \backslash B_n$. Now, $\nu(A_k) = \nu(A_k \backslash B_k) + \nu(B_k) > \nu(A_{k+1}) + k > \nu(A_{k+1})$ for $k \geqslant N$ and so $\nu(A_k) < \nu(A_{k-1}) - k < \nu(A_N) - k, k \to -\infty$, a contradiction.

Claim: there exists $P \in \mathcal{F}$ s.t. $\nu^+(\Omega) = \nu(P)$.

By approximation, take (A_n) s.t. $\nu(A_n) > \nu^+(\Omega) - 2^{-n}$. We will see that the choice $P = \bigcup_n \bigcap_{m \geqslant n} A_m$ works. Let $N = \Omega \backslash P$. By $\sigma-$additivity of ν, have that $\nu(P) = \lim_{n\to\infty} \nu\left(\bigcap_{m\geqslant n} A_m\right)$. Now, for $j \geqslant n$, consider $\bigcap_{n \leqslant m \leqslant j} A_m$, we first see that

$$\nu\left(\bigcap_{n\leqslant m\leqslant n+1} A_m\right) = -\nu(A_n \cup A_{n+1}) + \nu(A_n) + \nu(A_{n+1})$$
$$> -\nu^+(\Omega) + 2\nu^+(\Omega) - 2^{-n} - 2^{-n-1} > \nu^+(\Omega) - 2^{-n-1}.$$

By inducting, we see that:

$$\nu\left(\bigcap_{n\leqslant m\leqslant n+p} A_m\right) > \nu^+(\Omega) - \sum_{m=0}^{p} 2^{-n-m}.$$

and so

$$\nu\left(\bigcap_{n\leqslant m} A_m\right) = \lim_{p\to\infty} \nu\left(\bigcap_{n\leqslant m\leqslant n+p} A_m\right) > \nu^+(\Omega) - \sum_{m=0}^{\infty} 2^{-n-m} = \nu^+(\Omega) - 2^{-n}.$$

which allows us to conclude that $\nu(P) = \nu^+(\Omega)$ upon taking limits.

Now, with $N = \Omega \backslash P$, define the set functions $\overline{\nu}_\pm : \mathcal{F} \to \mathbb{R}$ by $\overline{\nu}_+(E) = \nu(E \cap P)$ and $\overline{\nu}_-(E) = \nu(E \cap N)$ for $E \in \mathcal{F}$.

Observe first that $\overline{\nu}_- \leqslant 0$. Indeed, suppose there exists $E \in \mathcal{F}$ such that $\nu(E \cap N) > 0$. Then, we see that $\nu^+(\Omega) = \nu^+(P) < \nu(E \cap N) + \nu(P) = \nu((E \cap N) \cup P) \leqslant \nu^+(\Omega)$, a contradiction. Thus, $\overline{\nu}_-$ is a negative measure.

Claim: $\nu(N) = \inf\{\nu(E) : E \in \mathcal{F}\}$.

Suppose otherwise, then there exists $E \in \mathcal{F}$ s.t. $\nu(E) < \nu(N)$, which implies $\nu(\Omega \backslash E) = \nu(\Omega) - \nu(E) > -\nu(N) + \nu(\Omega) = \nu(P)$ and so $\nu(\Omega \backslash E) > \nu(P)$, a contradiction.

Now, we can prove $\overline{\nu}_+ \geqslant 0$. Indeed, suppose there exists $E \in \mathcal{F}$ such that $\nu(E \cap P) < 0$. Then, we see that $\nu(N) \leqslant \nu((E \cap P) \cup N) = \nu(N) + \nu(E \cap P) < \nu(N)$, a contradiction. Thus, $\overline{\nu}_+$ is a positive measure.

Claim: $\overline{\nu}_-(E) = \inf\{\nu(A) : A \subseteq E, A \in \mathcal{F}\}$.

Suppose otherwise, then there exists $E \in \mathcal{F}$ s.t. $\nu(A) < \nu(E \cap N)$, which implies $\nu(A \cap N) < \nu(A \cap P) + \nu(A \cap N) = \nu(A) < \nu(E \cap N)$ and so $\nu(A \cap N) < \nu(E \cap N)$ and so $\nu((E \backslash A) \cap N) = \nu(E \cap N \backslash A \cap N) > 0$ a contradiction.

Finally, we observe that

Claim: $\overline{\nu}_+(E) = \sup\{\nu(A) : A \subseteq E, A \in \mathcal{F}\}$.

Suppose otherwise, then there exists $E \in \mathcal{F}$ s.t. $\nu(A) > \nu(E \cap P)$, which implies $\nu(E \cap P) < \nu(A) = \nu(A \cap P) + \nu(A \cap N) < \nu(A \cap P)$ and so $\nu(E \cap P) < \nu(A \cap P)$ and so $\nu((E \backslash A) \cap P) = \nu(E \cap P \backslash A \cap)) < 0$ a contradiction, and we finally obtain the desired decomposition. □

Lecture 6

Definition 2.1 (Absolute Continuity). Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $\nu : \mathcal{F} \to \mathbb{C}$ be a complex measure. ν is absolutely continuous wrt μ, written $\nu << \mu$ if for all $A \in \mathcal{F}$, $\mu(A) = 0 \implies \nu(A) = 0$.

Remark.
1. $\nu << \mu \implies |\nu| << \mu$. So if ν has Jordan decomposition $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ and $\nu << \mu$, then $\nu_k << \mu, 1 \leqslant k \leqslant 4$.
2. If $\nu << \mu$, then for all $\epsilon > 0$, there exists $\delta > 0$ s.t. for all $A \in \mathcal{F}$ $\mu(A) < \delta \implies |\nu(A)| < \epsilon$.

Example:
For f in $L_1(\mu)$ define $\nu(A) = \displaystyle\int_A f d\mu$, $A \in \mathcal{F}$. By the Theorem of Dominated Convergence (DCT), ν is a complex measure and $\mu(A) = 0 \implies \nu(A) = 0$, i.e. $\nu << \mu$.

Definition 2.2. A set in $\mathcal{F}$ is said to be a $\sigma-$finite set (wrt μ) if there is a sequence $(A_n)_{n \in \mathbb{N}} \in \mathcal{F}$ s.t. $A = \displaystyle\bigcup_{n \in \mathbb{N}} A_n$ and for all $n \in \mathbb{N}$, $\mu(A_n) < \infty$. We say μ is $\sigma-$finite if Ω is a $\sigma-$finite set.

Theorem 2.3 (Radon-Nikodym). Let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma-$finite emasure space and $\nu : \mathcal{F} \to \mathbb{C}$ be a complex measure s.t. $\nu << \mu$. Then there exists a unique $f \in L_1(\mu)$ s.t. $\nu(A) = \displaystyle\int_A f d\mu$ for all $A \in \mathcal{F}$. Moreover, f takes values in $\mathbb{C}/\mathbb{R}/\mathbb{R}^+$ according to whether ν is a complex, signed or positive measure respectively.

Proof. Uniqueness: standard.

Existence: wlog ν is a finite positive measure (Jordan decomposition) and wlog μ is a finite measure ($\sigma-$finiteness).

Let $\mathcal{H} = \{h : \Omega \to \mathbb{R}^+ : h \text{ integrable and } \int_A h d\mu \leqslant \nu(A)\ \forall A \in \mathcal{F}\}$. $\mathcal{H} \neq \varnothing$ $(0 \in \mathcal{H})$ and $h_1, h_2 \in \mathcal{H}$ implies $h_1 \vee h_2 = \max\{h_1, h_2\}$ is in $\mathcal{H}$. Also, if (h_n) are in $\mathcal{H}$ s.t. $h_n \uparrow h$, then h is in $\mathcal{H}$. Let $\alpha = \displaystyle\sup_{h \in \mathcal{H}} \int_\Omega f d\mu, 0 \leqslant \alpha \leqslant \nu(\Omega)$.

Claim: there exists $f \in \mathcal{H}$ s.t. $\alpha = \int_\Omega f d\mu$.

We construct such an $f \in \mathcal{H}$. Take $f_n \in \mathcal{H}$ s.t. $\int_A g_n d\mu \leqslant \nu(A)$, for all $A \in \mathcal{F}$ and $\int_\Omega f_n d\mu \to \alpha$. The same holds if we replace f_n by $f_1 \vee \cdots \vee f_n$, and so wlog we can assume the sequence is non-decreasing. Now, by induction, there exists sets $E_1, \ldots, E_n \in \mathcal{F}$ pairwise disjoint s.t. $\bigcup_{k=1}^{n} E_k = \Omega$ and $g_n = \sum_{k=1}^{n} f_j \mathbf{1}_{E_j}$ and $\int_A g_n d\mu = \sum_{j=1}^{n} \int_{E_j \cap A} f_n d\mu \leqslant \sum_{j=1}^{n} \nu(E_j \cap A) = \nu(A)$. Since g_n is non-decreasing, take the pointwise supremum to obtain $f_0 := \sup_n g_n$, which is in $\mathcal{H}$ by the above, and is seen to work by inspection.

Now consider the signed measures $\nu_1, \nu_2, (\lambda_n)_{n\in\mathbb{N}} : \mathcal{F} \to \mathbb{R}$, s.t. $\nu_1 = \nu - \nu_2, \nu_2(A) = \int_A f_0 d\mu$ and $\lambda_n(A) = \nu_1(A) - \frac{1}{n}\mu(A)$ for all $A \in \mathcal{F}$. Then there exist (Hahn decomposition) $(P_n), (N_n)$ in $\mathcal{F}$ s.t. $\Omega = P_n \cup N_n, P_n = \Omega \backslash N_n$, s.t. $\lambda_n(E) \geqslant 0$ for all $E \in \mathcal{F}$ s.t. $E \subseteq P_n$. Now, for such E, we have $\lambda_n(E) = \nu_1(E) - \frac{1}{n}\mu(E) \geqslant 0$ and so $\nu(E) = \nu_1(E) + \nu_2(E) \geqslant \int_E f_0 d\mu + \frac{1}{n}\int_E d\mu$. Let $\tilde{f}_n = f_0 + \frac{1}{n}\mathbf{1}(P_n)$. Observe that for all $E \in \mathcal{F}$, $\int_E \tilde{f}_n d\mu = \int_E f_0 d\mu + \frac{1}{n}\int_{E\cap P_n} d\mu \leqslant \nu(E)$ by the above and the fact that μ is a positive measure. We have by the above that $\tilde{f}_n$ is in $\mathcal{H}$ for all $n \in \mathbb{N}$ and $\alpha \leqslant \int_\Omega \tilde{f}_n d\mu \leqslant \alpha$ and so $\mu(P_n) = 0$ for all $n \in \mathbb{N}$. Thus, by $\sigma-$additivity, $\mu\left(\bigcup_n P_n\right) = 0$.

Let $N = \Omega \backslash \bigcup_n P_n$, then for all $E \in \mathcal{F}$ s.t. $E \subseteq N$, $\lambda_n(E) = \nu_1(E) - \frac{1}{n}\mu(E) \leqslant 0$ for all $n \in \mathbb{N}$ and so $\nu_1(E) \leqslant 0$, i.e. $\nu(E) \leqslant \nu_2(E)$. The reverse inequality is obtained by observing that f_0 is in $\mathcal{H}$ and so we see that $\nu(E) = \nu_2(E)$ for such E. Finally, since $\nu << \mu$, for all $E \in \mathcal{F}$, $\nu(E) = \nu(E \cap N) = \nu_2(E \cap N) = \nu_2(E) = \int_E f_0 d\mu$, which concludes the proof. □

Remark.
1. Without assuming $\nu << \mu$, the proof shows that there exists a decomposition (Lebesgue decomposition) $\nu = \nu_1 + \nu_2$, where $\nu_2(A) = \int_A f d\mu$, and $\nu_2 \perp \mu$ (orthogonal), i.e. there exists a measurable partition $\Omega = P \cup N$, s.t. $\mu(P) = 0$ ($\mu(A) = 0$, for all $A \subseteq P$), $|\nu_2(P)| = 0$ ($\nu_2(A) = 0$, for all $A \subseteq N$).
2. The unique f in Theorem 2.3 is the Radon-Nikodym derivative of ν wrt μ, denoted $\frac{d\nu}{d\mu}$. The result says that $\nu(A) = \int_\Omega \mathbf{1}_A d\nu = \int_A f d\mu = \int_\Omega \mathbf{1}_A \frac{d\nu}{d\mu} d\nu$. Hence a measurable function g is $\nu-$integrable iff $g\frac{d\nu}{d\mu}$ is $\mu-$integrable and then $\int_\Omega g d\nu = \int_\Omega g \frac{d\nu}{d\mu} d\nu$.

2.2 The dual space of L_p

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $1 \leqslant p < \infty$ and $1 < q \leqslant \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L_q = L_q(\mu)$, define $\phi_g : L_p \to$ scalars by $\phi_g(f) = \int_\Omega f g d\mu$, for $f \in L_p$. By Hölder, the product fg is in $L_1(\mu)$ and $|\phi_g(f)| \leqslant \|f\|_p \cdot \|g\|_q$. So ϕ_g is well-defined and clearly linear, also bounded with $\|\phi_g\| \leqslant \|g\|_q$ and so ϕ_g is an element of L_p^*. So we have the map

$$\begin{aligned} \phi : L_q &\to L_p^* \\ g &\mapsto \phi_g. \end{aligned}$$

This map is linear and bounded with $\|\phi\| \leqslant 1$.

Theorem 2.4. Let $(\Omega, \mathcal{F}, \mu), p, q, \phi$ be as above.

(i) If $1 < p < \infty$, then ϕ is an isometric isomorphism. So $L_p^* \cong L_q$.

(ii) If $p = 1$ and μ is a $\sigma-$finite, then $L_1^* \cong L_\infty$.

Proof. Proof of (i): ϕ is isometric. Fix $g \in L_q$. We know $\|\phi_g\| \leqslant \|g\|_q$. Let λ be a measurable function s.t. $|\lambda| = 1$ and $\lambda g = |g|$. Let $f = \lambda |g|^{q-1}$. Then, $\|f\|_p^p = \int_\Omega |f|^p d\mu = \int_\Omega |g|^{p(q-1)} d\mu = \int_\Omega |g|^q d\mu = \|g\|_q^q$. Hence, $\|g\|_q^{\frac{q}{p}} \cdot \|\phi_g\| \geqslant |\phi_g(f)| = \int_\Omega |g|^q d\mu = \|g\|_q^q$, so $\|\phi_g\| \geqslant \|g\|_q^{q-\frac{q}{p}} = \|g\|_q$.

ϕ is onto: Fix $\psi \in L_p^*$. We seek $g \in L_q$ s.t. $\psi = \phi_g$ (Idea: $\psi(\mathbf{1}_A) = \int_A g d\mu$).

Case 1: μ is finite.

Then for $A \in \mathcal{F}$ and $\mathbf{1}_A \in L_p$ so can define $\nu(a) = \psi(\mathbf{1}_A)$. It is an easy check using the DCT that $\nu : \mathcal{F} \to \mathbb{C}$ is indeed a complex measure and $\nu << \mu$. If $A \in \mathcal{F}$, $\mu(A) = 0$, then $\mathbf{1}_A = 0$ almost everywhere (a.e.) in $L_p(\mu)$, so $\nu(A) = \psi(\mathbf{1}_A) = 0$. Then $\nu << \mu$. By Theorem 2.3, there exists $g \in L_1(\mu)$ s.t. $\nu(A) = \int_A g d\mu$ for all $A \in \mathcal{F}$. So $\psi(\mathbf{1}_A) = \int_\Omega \mathbf{1}_A g d\mu$, for $A \in \mathcal{F}$. Hence, $\psi(f) = \int_\Omega fg d\mu$ for all simple functions f. Now given $f \in L_\infty(\mu)$, there exists simple $f_n \to f \in L_\infty(\mu)$ (hence in $L_p(\mu)$ since μ is finite). So $\psi(f_n) \to \psi(f)$ and $f_n g \to fg \in L_1(\mu)$, using Hölder for $p = 1, \infty$. So $\psi(f) = \int_\Omega fg d\mu$ for all $f \in L_\infty(\mu)$. For $n \in \mathbb{N}$, let $A_n = \{|g| \leqslant n\}$ and $f_n = \lambda \cdot \mathbf{1}_{A_n} |g|^{q-1}$, where $|\lambda| = 1, \lambda g = |g|$.
Now, $\int_\Omega f_n g d\mu = \int_{A_n} |g|^q d\mu = \psi(f_n)$ (as f_n is in L_∞). $\psi(f_n) \leqslant \|\psi\| \cdot \|f_n\|_p = \|\psi\| \left(\int_{A_n} |g|^q\right)^{\frac{1}{p}}$. By monotone convergence, we deduce that $\left(\int_{A_n} |g|^q\right)^{\frac{1}{q}} \leqslant \|\psi\|$ and hence that g is in L_q. Given $f \in L_p$, there exists $f_n \to f$ simple in L_p. So $\psi(f_n) \to \psi(f)$ and $f_n g \to fg \in L_1$ (Hölder for the pair (p,q)). Hence, $\psi(f) = \int_\Omega fg d\mu$, concluding the case where μ is finite.

Before we treat the more general case, observe that for $A \in \mathcal{F}$, let $\mathcal{F}_A = \{B \in \mathcal{F} : B \subseteq A\}$ and $\mu_A = \mu\restriction_{\mathcal{F}_A}$, $(A, \mathcal{F}_A, \mu_A)$ is a measure space. Then $L_p(\mu_A) \subseteq L_p(\mu)$ (where we identify $f \in L_p(\mu_A)$ with $f \cdot \mathbf{1}_A \in L_p(\mu)$; this is an isometric embedding). Let $\psi_* = \psi\restriction_{L^p(\mu_A)}$.

Claim: If A, B are in $\mathcal{F}$ s.t. $A \cap B$ is empty, then $\|\psi_{A \cup B}\| = (\|\psi_A\|^q + \|\psi_A\|^q)^{\frac{1}{q}}$.
Lecture 7 Observe that [6]

$$\begin{aligned}(\|\psi_A\|^q + \|\psi_A\|^q)^{\frac{1}{q}} &= \sup\{a\|\psi_A\| + b\|\psi_B\| : a, b, \geqslant 0, a^p + b^p \leqslant 1\} \\ &= \sup\{a|\psi_A(f)| + b|\psi_B(g)| : a, b, \geqslant 0, a^p + b^p \leqslant 1, f \in B_{L_p(\mu_A)}, g \in B_{L_p(\mu_B)}\} \\ &= \sup\{|a\psi_A(f) + b\psi_B(g)| : a, b, \geqslant 0, a^p + b^p \leqslant 1, f \in B_{L_p(\mu_A)}, g \in B_{L_p(\mu_B)}\}.\end{aligned}$$

Now, $a\psi_A(f) + b\psi_B(g) = \psi_{A \cup B}(af + bg)$ (embed $f, g \in L_p(\mu)$ be extending f, g to zero outside A, B respectively). Now, continuing the above we obtain

$$= \sup\{|\psi_{A \cup B}(h)| : h \in B_{L_p(\mu_{A \cup B})}\} = \|\psi_{A \cup B}\|$$

as required, concluding the proof of the finite case.

Case 2: μ is $\sigma-$finite.

There exists a measurable partition $\Omega = \bigcup_{n \in \mathbb{N}} A_n$, of Ω, s.t. $\mu(A_n) < \infty$ for all n. By Case 1, for all $n \in \mathbb{N}$, there exists $g_n \in L_q(\mu_A)$ s.t. $\psi_{A_n} = \psi_{g_n}$, i.e. $\psi(f) = \int_{A_n} f g_n d\mu$, for all $f \in L_p(\mu_{A_n})$. By Claim 2, $\sum_{k=1}^n \|g_k\|_q^q = \sum_{k=1}^n \|\psi_{A_n}\|^q = \|\psi_{\cup_{k=1}^n A_k}\|^q \leqslant \|\psi\|^q$. If we define g on Ω by setting $g = g_n$ on A_n, then g is in L_q. Thus, $\psi(f) = \psi_g(f)$ for all $f \in L_p(\mu_n)$, for all n. Hence, $\psi(f) = \phi_g(f)$ on $\text{span}\{\bigcup_{n \in \mathbb{N}} L_p(\mu_n)\} = L_p(\mu)$.

Case 3: general μ.

First assume that for $f \in L_p(\mu)$, $\{f \neq 0\}$ is $\sigma-$ finite. Indeed, $\{f \neq 0\} = \bigcup_{n \in \mathbb{N}} \{|f| > \frac{1}{n}\}$ and $\mu(\{|f| > \frac{1}{n}\}) \leqslant n^p \cdot \|f\|_p^p < \infty$ by Markov's inequality.
Chose $(f_n) \in B_{L_p}$ s.t. $\psi(f_n) \to \|\psi\|$. Then $A = \bigcup_{n \in \mathbb{N}} \{f_n \neq 0\}$ is $\sigma-$finite and $\|\psi_A\| = \|\psi\|$. By the claim previously established, $\|\psi\| = \left(\|\psi_A\|^q + \|\psi_{\Omega \setminus A}\|^q\right)^{\frac{1}{q}}$. By case 2, there exists a $g \in L_q(\mu_A) \subseteq L_q(\mu)$ s.t. $\phi_A = \phi_g$. So for all $f \in L_p(\mu)$, $\psi(f) = \psi_A(f\restriction_A) + \psi_{\Omega \setminus A}(f\restriction_{\Omega \setminus A}) = \int_A f\restriction_A g d\mu = \int_\Omega fg d\mu$

[6] using the fact that $(\ell_q^2)^* \equiv \ell_p^2$.

(extend g in the usual sense.).

Proof of (ii) (μ is $\sigma-$ finite).

ϕ is isometric: Let $g \in L_\infty$. We know already that $\|\phi_g\| \leqslant \|g\|_\infty$ (Hölder). Fix $s < \|g\|_\infty$. Then $\mu(\{|g| > s\}) > 0$. Since μ is $\sigma-$finite, there exists $A \subseteq \{|g| > s\}$ s.t. $0 < \mu(A) < \infty$. Choose a measurable function λ s.t. $|\lambda| = 1$ and $\lambda g = |g|$. Then λg is in $L_1(\mu)$, $\|\lambda g\|_1 = \mu(A)$. Now, $\mu(A) \cdot \|\phi_g\| \geqslant |\phi_g(\lambda \mathbf{1}_A)| = \int_A |g| \geqslant s\mu(A)$. We deduce $\|\phi_g\| > s$ and so $\|\phi_g\| \geqslant s$ and hence $\|\phi_g\| \geqslant \|g\|_\infty$.

ϕ is onto: Fix $\psi \in L_1^*$. Seek $g \in L_\infty$ s.t. $\psi = \phi_g$.

Case 1: μ is finite. Define $\nu(A) = \psi(\mathbf{1}_A)$ for all $A \in \mathcal{F}$ and proceed in the same way as for $p > 1$.

Case 2: μ is $\sigma-$finite. This time we prove

Claim: If A, B are in $\mathcal{F}$ s.t. $A \cap B$ is empty, then $\|\psi_{A \cup B}\| = \max\{\|\psi_A\|, \|\psi_B\|\}$.
Observe like before that [7]

$$\begin{aligned}
\max\{\|\psi_A\|, \|\psi_B\|\} &= \sup\{a\|\psi_A\| + b\|\psi_B\| : a, b, \geqslant 0, a^p + b^p \leqslant 1\} \\
&= \sup\{a|\psi_A(f)| + b|\psi_B(g)| : a, b, \geqslant 0, a + b \leqslant 1, f \in B_{L_1(\mu_A)}, g \in B_{L_1(\mu_B)}\} \\
&= \sup\{|a\psi_A(f) + b\psi_B(g)| : a, b, \geqslant 0, a + b \leqslant 1, f \in B_{L_1(\mu_A)}, g \in B_{L_1(\mu_B)}\} \\
&= \sup\{|\psi_{A \cup B}(h)| : h \in B_{L_1(\mu_{A \cup B})}\} \\
&= \|\psi_{A \cup B}\|
\end{aligned}$$

as required.

To conclude, proceed in an entirely analogous way using a measurable partition of $\Omega = \bigcup_{n \in \mathbb{N}} A_n$, s.t. μ_{A_n} is a finite measure for all $n \in \mathbb{N}$. By Case 1, for all $n \in \mathbb{N}$, there exists $g_n \in L_\infty(\mu_{A_n})$ s.t. $\psi_{A_n} = \psi_{g_n}$, i.e. $\psi(f) = \int_{A_n} f g_n d\mu$, for all $f \in L_1(\mu_{A_n})$. Now, by the previous claim,

$$\left\| \sum_{k=1}^n g_k \mathbf{1}_{A_k} \right\|_\infty = \left\|\psi_{\cup_{k=1}^n A_k}\right\| = \max_{1 \leqslant k \leqslant n} \|\psi_{A_k}\| \leqslant \|\psi\|.$$

If we define g on Ω by setting $g = g_n$ on A_n, then g is in L_∞ with $\|g\|_\infty \leqslant \|\psi\|$. Thus, $\psi(f) = \psi_g(f)$ for all $f \in L_1(\mu_{A_n})$, for all n. Hence, $\psi(f) = \phi_g(f)$ on $\text{span}\{\bigcup_{n \in \mathbb{N}} L_1(\mu_n)\} = L_1(\mu)$.

□

Corollary 2.4.1. For $1 < p < \infty$, for a measure space $(\Omega, \mathcal{F}, \mu)$ $L_p(\mu)$ is reflexive.

Proof. Let ψ be in L_p^{**}. then $g \mapsto \psi(\phi_g) : L_q \to$ scalars is in L_q^* ($\frac{1}{p} + \frac{1}{q} = 1$). By Theorem 2.4($i$), there exists $f \in L_p$ s.t. $\langle \phi_g, \psi \rangle = \int_\Omega f g d\mu = \langle f, \phi_g \rangle = \langle \phi_g, \hat{f} \rangle$ for all $g \in L_q$. Then $\psi = \hat{f}$, since $L_p^* = \{\phi_g : g \in L_q\}$. □

2.3 $\mathcal{C}(K)$ spaces

Throughout, K is a compact, Hausdorff topological space. Define

$$\mathcal{C}(K) = \{f : K \to \mathbb{C} : f \text{ continuous}\},$$

a complex Banach space in the sup-norm: $\|f\|_\infty = \sup_K |f|$.

$$\mathcal{C}^{\mathbb{R}}(K) = \{f : K \to \mathbb{R} : f \text{ continuous}, \}$$

is a real Banach space with norm $\|f\|_\infty = \sup_K |f|$.

$$\mathcal{C}^+(K) = \{f \in \mathcal{C}(K) : f \geqslant 0\}.$$

[7] using the fact that $(\ell_1^2)^* \equiv \ell_\infty^2$.

Moreover,

$$\mathcal{M}(K) = \mathcal{C}(K)^*,$$

is a complex Banach space in the operator norm.

$$\mathcal{M}^{\mathbb{R}}(K) = \{\phi \in \mathcal{M}(K) : \phi(f) \in \mathbb{R}, \forall f \in \mathcal{C}^R(K)\},$$

is a closed, real-linear subspace of $\mathcal{M}(K)$.

$$\mathcal{M}^+(K) = \{\phi : \mathcal{C}(K) \to \mathbb{C} : \phi \text{ is linear} \phi(f) \geqslant 0, \forall f \in \mathcal{C}^+(K)\}.$$

Elements of $\mathcal{M}^+(K)$ are called positive linear functionals.

Aim: identify $\mathcal{M}(K), \mathcal{M}^{\mathbb{R}}(K)$.

Lemma 2.1. (i) For all $\phi \in \mathcal{M}(K)$, there exist unique $\phi_1, \phi_2 \in \mathcal{M}^R(K)$, $\phi = \phi_1 + i\phi_2$

(ii) $\phi \mapsto \phi\restriction_{\mathcal{C}^{\mathbb{R}}(K)} : \mathcal{M}^{\mathbb{R}}(K) \to (\mathcal{C}^{\mathbb{R}}(K))^*$ is an isometric isomorphism.

(iii) $\mathcal{M}^+(K) \subset \mathcal{M}(K)$ and $\mathcal{M}^+(K) = \{\phi \in \mathcal{M}(K) : \|\phi\| = \phi(\mathbf{1}_K)\}$.

(iv) For all $\phi \in \mathcal{M}^{\mathbb{R}}(K)$, there exist unique $\phi^+, \phi^- \in \mathcal{M}^+(K)$ s.t. $\phi = \phi^+ - \phi^-$ and $\|\phi\| = \|\phi^+\| + \|\phi^-\|$.

Proof. (i) Let ϕ be in $\mathcal{M}(K)$. Define $\overline{\phi} : \mathcal{C}(K) \to \mathbb{C}$, by $\overline{\phi}(f) = \phi(\overline{f})$. Then, $\overline{\phi}$ is in $\mathcal{M}(K)$ and ϕ is in $\mathcal{M}^{\mathbb{R}}(K) \iff \phi = \overline{\phi}$[8].

Uniqueness: assume $\phi = \phi_1 + i\phi_2$ where $\phi_1, \phi_2 \in \mathcal{M}^{\mathbb{R}}(K)$. Then $\overline{\phi} = \phi_1 - i\phi_2$ so $\phi_1 = \frac{\phi+\overline{\phi}}{2}, \phi_2 = \frac{\phi-\overline{\phi}}{2}$.

Existence: check that the above works.

(ii) Let ϕ be in $\mathcal{M}^{\mathbb{R}}(K)$. The fact that $\left\|\phi\restriction_{\mathcal{C}^{\mathbb{R}}(K)}\right\| \leqslant \|\phi\|_{\mathcal{C}(K)}$ is clear. Let f be in $B_{\mathcal{C}(K)}$. Choose $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and $\lambda\phi(f) = |\phi(f)|$. So $|\phi(f)| = \phi(\lambda f) = \phi(\mathrm{Re}(\lambda f)) + i\phi(\mathrm{Im}(\lambda f)) = \phi(\mathrm{Re}(\lambda f)) \leqslant \left\|\phi\restriction_{\mathcal{C}^{\mathbb{R}}(K)}\right\| \cdot \|\mathrm{Re}(\lambda f)\|_\infty \leqslant \left\|\phi\restriction_{\mathcal{C}^{\mathbb{R}}(K)}\right\|$.
Hence, $\left\|\phi\restriction_{\mathcal{C}^{\mathbb{R}}(K)}\right\| \geqslant \|\phi\|$. Finally, given $\psi \in (\mathcal{C}^{\mathbb{R}}(K))^*$, define $\phi(f) = \psi(\mathrm{Re}(f)) + i\psi(\mathrm{Im}(f))$, for $f \in \mathcal{C}(k)$. Then ϕ is in $\mathcal{M}(K)$ and $\phi\restriction_{\mathcal{C}^{\mathbb{R}}(K)} = \psi$.

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(iii) $\mathcal{M}^+(K) \subset \mathcal{M}(K)$: let ϕ be in $\mathcal{M}^+(K)$.
For $f \in \mathcal{C}^{\mathbb{R}}(K)$, $\|f\|_\infty \leqslant 1$ we have $\mathbf{1}_K \pm f \geqslant 0$, so $\phi(\mathbf{1}_K \pm f) \geqslant 0$. So $\phi(f)$ is in $\mathbb{R}$ and $|\phi(f)| \leqslant \phi(\mathbf{1}_K)$. So $\phi\restriction_{\mathbf{C}^{\mathbb{R}}(K)}$ is in $(\mathcal{C}^{\mathbb{R}}(K))^*$ and $\left\|\phi\restriction_{\mathbf{C}^{\mathbb{R}}(K)}\right\| = \phi(\mathbf{1}_K)$. By (ii), ϕ is in $\mathcal{M}(K)$, $\|\phi\| = \phi(\mathbf{1}_K)$.

$\mathcal{M}^+(K) = \{\phi \in \mathcal{M}(K) : \|\phi\| = \phi(\mathbf{1}_K)\}$ ("$\supseteq$"): let ϕ be in $\mathcal{M}(K)$ with $\|\phi\| = \phi(\mathbf{1}_K)$. Wlog, $\|\phi\| = \phi(\mathbf{1}_K) = 1$. Fix $f \in B_{\mathcal{C}^{\mathbb{R}}(K)}$, let $\phi(f) = \alpha + i\beta$, with $\alpha, \beta \in \mathbb{R}$.

Need: $\beta = 0$. For $t \in \mathbb{R}$, $|\phi(f + it\mathbf{1}_K)|^2 = \alpha^2 + (\beta+t)^2 = \alpha^2 + \beta^2 + 2\beta t \leqslant \|f + it\mathbf{1}_K\|_\infty^2 \leqslant 1 + t^2$, so $\beta = 0$. Given $f \in \mathcal{C}^+(K)$, with $0 \leqslant f \leqslant 1$ on K it follows that $|2f - \mathbf{1}_K| \leqslant 1$, so $\|2f - \mathbf{1}_K\|_\infty \leqslant 1$. So $|\phi(2f - \mathbf{1}_K)| \leqslant 1$, i.e. $-1 \leqslant 2\phi(f) \leqslant 1$, which implies $\phi(f) \geqslant 0$.

(iv) Let ϕ be in $\mathcal{M}^{\mathbb{R}}(K)$. Assume $\phi = \psi_1 - \psi_2$, where $\psi_1, \psi_2 \in \mathcal{M}^+(K)$. For $f, g \in \mathcal{C}^+(K)$ with $0 \leqslant g \leqslant f$, $\psi_1(f) \geqslant \psi_1(g) = \phi(g) + \psi_2(g) \geqslant \phi(g)$. So $\psi_1(f) \geqslant \sup\{\phi(g) : 0 \leqslant g \leqslant f\}$. Define for $f$in $\mathcal{C}(K)$

$$\phi^+(K) = \sup\{\phi(g) : 0 \leqslant g \leqslant f\}$$

Note that $\phi^+(f) \geqslant 0$, $\phi^+(f) \leqslant \|\phi\| \cdot \|f\|_\infty$, $\phi^+(f) \geqslant \phi(f)$. Furthermore, it is easy to check that $\phi^+(t_1 f_1 + t_2 f_2) = t_1\phi^+(f_1) + t_2\phi^+(f_2)$ for all $f_1, f_2 \in \mathcal{C}^+(K)$, $t_1, t_2 \in \mathbb{R}^+$. Next, for $f \in \mathcal{C}^{\mathbb{R}}(K)$, write $f = f_1 - f_2$, both in $\mathcal{C}^+(K)$[9] and define $\phi^+(f) = \phi^+(f_1) - \phi^+(f_2)$. This is

[8] check!

[9] e.g. $f_1 = f \vee 0, f_2 = (-f) \vee 0$.

well-defined and $\mathbb{R}-$linear (check). Finally, for f in $\mathcal{C}(K)$, let $\phi^+(f) = \phi^+(\operatorname{Re} f) + i\phi^+(\operatorname{Im} f)$. Then ϕ^+ is $\mathbb{C}-$linear and since $\phi^+(f) \geqslant 0$ for all $f \in \mathcal{C}^+(K)$, we have ϕ^+ is in $\mathcal{M}^+(K)$. Define $\phi^- = \phi^+ - \phi$. For $f \in \mathcal{C}^+(K)$, $\phi^+(f) \geqslant \phi(f)$ implies that ϕ^- is in $\mathcal{M}^+(K)$ and $\phi = \phi^+ - \phi^-$. $\|\phi\| \leqslant \|\phi^+\| + \|\phi^-\| = \phi^+(\mathbf{1}_K) + \phi^-(\mathbf{1}_K) = 2\phi^+(\mathbf{1}_K) - \phi(\mathbf{1}_K)$. Given $f \in \mathcal{C}^+(K)$ with $0 \leqslant f \leqslant 1$, $-1 \leqslant 2f - 1 \leqslant 1$, so $2\phi(f) - \phi(\mathbf{1}_K) = \phi(2f - \mathbf{1}_K) \leqslant \|\phi\|$. Taking the supremum over f, we deduce that $2\phi^+(\mathbf{1}_K) - \phi(\mathbf{1}_K) = \phi(2f - \mathbf{1}_K)$. So $\|\phi\| = \|\phi^+\| + \|\phi^-\|$.

Uniqueness: Assume $\phi = \psi_1 - \psi_2$, where ψ_1, ψ_2 are in $\mathcal{M}^+(K)$ and $\|\phi\| = \|\psi_1\| + \|\psi_2\|$. From initial observation, $\psi_1 \geqslant \phi^+$ on $\mathcal{C}^+(K)$ and so $\psi_2 = \psi_1 - \phi \geqslant \phi^+ - \phi = \phi^-$ on $\mathcal{C}^+(K)$. Hence, $\psi_1 - \psi^+ = \psi_2 - \phi^-$ is in $\mathcal{M}^+(K)$. By (iii), $\|\psi_1 - \psi^+\| + \|\psi_2 - \psi^-\| = \psi_1(\mathbf{1}_K) - \phi^+(\mathbf{1}_K) + \psi_2(\mathbf{1}_K) - \phi^-(\mathbf{1}_K) = \|\psi_1\| + \|\psi_2\| - \|\phi^+\| - \|\phi^-\| = \|\phi\| - \|\phi\| = 0$. Thus, $\psi_1 = \phi^+, \psi_2 = \phi^-$.

□

2.4 Topological Preliminaries

We begin with some definitions and key topological results that will be useful in obtaining the characterisation of the dual spaces $(\mathcal{C}(K))^*$.

1. K being compact, Hausdorff is normal: given disjoint closed sets E, F there exists disjoint open sets $\mathcal{U}, \mathcal{V} \in K$ s.t. $E \subset \mathcal{U}$, $F \subset \mathcal{V}$. Equivalently, given $E \subset \mathcal{U} \subseteq K$, E closed, $\mathcal{U}$ open, there exists $\mathcal{V}$ open s.t. $E \subset \mathcal{V} \subseteq \mathcal{U}$ (use normality in $E, K \backslash \mathcal{U}$).
2. Urysohn Lemma: given disjoint closed sets $E, F \in K$, there exists a continuous function $f : K \to [0, 1]$ s.t. $f \restriction_E = 0$ and $f \restriction_F = 1$.
3. Notation: $f \prec \mathcal{U}$ means $\mathcal{U} \subseteq K$ open $f : K \to [0, 1]$ is continuous and the support of f $\operatorname{supp}(f) = \overline{\{x \in K : f(x) \neq 0\}} \subseteq \mathcal{U}$. $E \prec \mathcal{U}$ means E is a closed subset of K, $f : K \to [0, 1]$ continuous and $f \restriction_E = 1$.
   Urysohn says: $E \subseteq \mathcal{U} \subseteq K$, E closed, $\mathcal{U}$ open, then there exists a continuous function f s.t. $E \prec f \prec \mathcal{U}$ ($E \subseteq \mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq \mathcal{U}$, $\mathcal{V}$ open and apply Urysohn to $E, F = K \backslash \mathcal{V}$).

Lemma 2.2. Let $E, \mathcal{U}_1 \dots \mathcal{U}_n$ be subsets of K ($n \in \mathbb{N}$), E closed, $\mathcal{U}_j$ open for $1 \leqslant j \leqslant n$ s.t. $E \subseteq \cup_{j=1}^n \mathcal{U}_j$. Then

(i) there exist open sets $\mathcal{V}_j, 1 \leqslant j \leqslant n$, s.t. $\overline{\mathcal{V}}_j \subseteq \mathcal{U}_j$ for all j and $E \subseteq \cup_{j=1}^n \mathcal{V}_j$.

(ii) there exist $f_j \prec \mathcal{U}_j, 1 \leqslant j \leqslant n$, s.t. $0 \leqslant \sum_{j=1}^n f_j \leqslant 1$ on K and $\sum_{j=1}^n f_j = 1$ on E.

Proof. (i) We proceed by induction on n.

$n = 1$: is just a restatement of normality of K.

$n > 1$: $E \backslash \mathcal{U} \backslash \subseteq \bigcup_{j<n} \mathcal{U}_j$, so by induction there exist open sets $\mathcal{V}_j, j < n$, s.t. $\overline{\mathcal{V}}_j \subseteq \mathcal{U}_j$ and $E \backslash \mathcal{U}_n \subseteq \cup_{j<n} \mathcal{V}_j$. So $E \backslash \cup_{j<n} \mathcal{V}_j \subseteq \mathcal{U}_n$ and so by normality, there exists open $\mathcal{V}_n$ s.t. $E \backslash \cup_{j<n} \mathcal{V}_j \subseteq \mathcal{V}_n \subseteq \overline{\mathcal{V}}_n \subseteq \mathcal{U}_n$.

(ii) Let $\mathcal{V}_j$ be as in part (i). By Urysohn, there exists h_j s.t. $\overline{\mathcal{V}}_j \prec h_j \prec \mathcal{U}_j$ for $1 \leqslant j \leqslant n$, and there exists h_0 s.t. $K \backslash \bigcup_{j=1}^n \mathcal{V}_j \prec h_0 \prec K \backslash E$.

Let $h = h_0 + \sum_{j=1}^n h_j$. Then $h \geqslant 1$ on K. Let $f_j = \frac{h_j}{h}$ for all j. Then $0 \leqslant \sum_{j=1}^n f_j \leqslant 1$ on K and $\sum_{j=1}^n f_j = 1$ on E where $f_j \prec \mathcal{U}_j$ for all j.

□

2.5 Borel Measures

Let X be a Hausdorff space. Let $\mathcal{G}$ be the family of open sets in X. The Borel $\sigma-$algebra of X is $\mathcal{B} = \sigma(\mathcal{G})$, the $\sigma-$algebra generated by $\mathcal{G}$. members of $\mathcal{B}$ are called Borel sets. A Borel measure on X is a (positive) measure μ on $\mathcal{B}$. we say μ is regular if

(i) $\mu(E) < \infty$ for all $E \subseteq X$, E compact.

(ii) $\mu(A) = \inf\{\mu(\mathcal{U}) : A \subseteq \mathcal{U} \in \mathcal{G}\}$ for all $A \in \mathcal{B}$.

(iii) $\mu(\mathcal{U}) = \sup\{\mu(E) : E \subseteq \mathcal{U}, E \text{ compact}\}$.

A complex Borel measure ν is regular if $|\nu|$ is regular. If X is compact, Hausdorff, then a Borel measure μ on X is regular

$$\begin{aligned}&\iff \mu(X) < \infty \text{ and } \mu(A) = \inf\{\mu(\mathcal{U}) : A \subseteq \mathcal{U} \in \mathcal{G}\} \text{ for all } A \in \mathcal{B}.\\ &\iff \mu(X) < \infty \text{ and } \mu(A) = \sup\{\mu(E) : E \subseteq A, E \text{ closed}\} \text{ for all } A \in \mathcal{B}.\end{aligned}$$

2.6 Integration with respect to complex measures

Let Ω be a set, $\mathcal{F}$ a $\sigma-$algebra on Ω and ν a complex measure on $\mathcal{F}$. Then ν has Jordan decomposition $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$. Say a measurable function $f : \Omega \to \mathbb{C}$ is $\nu-$integrable if f is $|\nu|-$integrable (i.e. $\int_\Omega |f| d|\nu| < \infty$) iff f is ν_k-integrable for all k. So we define

$$\int_\Omega f d\nu = \int_\Omega f d\nu_1 - \int_\Omega f d\nu_2 + i \int_\Omega f d\nu_3 - i \int_\Omega f d\nu_4.$$

Lecture 9 Properties:

1. $\int_\Omega \mathbf{1}_A d\nu = \nu(A)$, for all $A \in \mathcal{F}$.
2. Linearity: if $f, g : \Omega \to \mathbb{C}$ are $\nu-$integrable, $a, b \in \mathbb{C}$, then $af + bg$ is $\nu-$integrable and $\int_\Omega (af + bg) d\nu = a \int_\Omega f d\nu + b \int_\Omega g d\nu$.
3. Dominated Convergence (DC): let $(f_n)_{n\in\mathbb{N}}$, f, g, be emasurable functions s.t. $f_n \to f$ a.e. (wrt $|\nu|$) and g is in $L_1(|\nu|)$ and for all n $|f_n| \leqslant g$ then f is $\nu-$integrable and $\int_\Omega f_n d\nu \to \int_\Omega f d\nu$ (True for ν_k for all k, so true for ν).
4. $\left|\int_\Omega f d\nu\right| \leqslant \int_\Omega |f| d|\nu|$ for all $f \in L_1(\nu)$ (True for simple functions by 1&2 and for general f, use DCT).

Let ν be a complex Borel measure on K (compact, hausdorff). Then for f continuous, then

$$\int_K |f| d|\nu| \leqslant \|f\| \cdot |\nu|(K).$$

So, f is $\nu-$integrable. Define $\phi : \mathcal{C}(K) \to \mathbb{C}$ by $\phi(f) = \int_K f d\nu$. Then ϕ is in $\mathcal{M}(K)$ and $\|\phi\| \leqslant |\nu|(K) = \|\nu\|_1$ (TV norm). If ν is a signed measure, then ϕ is a member of $\mathcal{M}^{\mathbb{R}}(K)$. If ν is a positive measure, then ϕ is in $\mathcal{M}^+(K)$.

Theorem 2.5 (Riesz Representation Theorem). For every $\phi \in \mathcal{M}^+(K)$, there exists a unique regular Borel measure μ on K that represents ϕ, i.e. $\phi(f) = \int_K f d\mu$ for all continuous f. Moreover,

$$\|\phi\| = \mu(K) = \|\mu\|_1 \text{ TV norm of } \mu.$$

Proof. Uniqueness: Assume μ_1, μ_2 both represent ϕ. Let $E \subseteq \mathcal{U} \subseteq K$, where E is closed and $\mathcal{U}$ is open, then by Urysohn, there exists f continuous s.t. $E \prec f \prec \mathcal{U}$. Now, $\mu_1(E) \leqslant \int_K f d\mu_1 = \phi(f) = \int_K f d\mu_2 \leqslant \mu_2(\mathcal{U})$. Take infemum over $\mathcal{U}$ open and use regularity to deduce that $\mu_1(E) \leqslant \mu_2(E)$, and by symmetry $\mu_1(E) = \mu_2(E)$ agree on closed sets, and we conclude that $\mu_1 = \mu_2$ for all $A \in \mathcal{F}$ by regularity from below.

Existence: Define for $\mathcal{U} \in \mathcal{G}$ (i.e. $\mathcal{U}$ open), $\mu^*(\mathcal{U}) = \sup\{\phi(f) : f \prec \mathcal{U}\}$. Note that $\mu^*(\mathcal{U}) \geqslant 0$, and for $\mathcal{V} \supseteq \mathcal{U}$, $\mathcal{U}, \mathcal{V} \in \mathcal{G}$, then $\mu^*(\mathcal{V}) \geqslant \mu^*(\mathcal{U})$ and hence $\mu^*(\mathcal{U}) \leqslant \mu^*(\mathcal{K})$ but $\mu^*(K) = \phi(\mathbf{1}_K)$ $(f \prec K$

implies $f \leqslant \mathbf{1}_K$ and ϕ is in $\mathcal{M}^+(K)$). It follows that for $\mathcal{U} \in \mathcal{G}$, $\mu^*(\mathcal{U}) = \inf\{\mu^*(\mathcal{U}) : A \subseteq \mathcal{U} \in \mathcal{G}\}$. Extend the definition of μ^*: for $A \subseteq K$ let $\mu^*(A) = \inf\{\mu^*(\mathcal{U}) : A \subseteq \mathcal{U} \in \mathcal{G}\}$.

Claim:μ^* is an outer measure.

We easily have that $\mu^*(\varnothing) = 0$ and for all $A \subseteq B \subseteq K$ $\mu^*(A) \leqslant \mu^*(B)$. It remains to show that if for all n in $\mathbb{N}$

$$(A_n) \subseteq K, \text{ then } \mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leqslant \sum_n \mu^*(A_n).$$

To see this, first fix $\mathcal{U}_n \in \mathcal{G}$ for $n \in \mathbb{N}$ and let $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$. Fix $f \prec \mathcal{U}$ and let $E = \operatorname{supp} f$. Then $E \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, by compactness, $E \subseteq \bigcup_{k=1}^{n} \mathcal{U}_k$ for some $n \in \mathbb{N}$. By lemma 2.2, there exist $h_j \prec \mathbf{U}_j$, $1 \leqslant j \leqslant n$, $\sum_{j=1}^{n} h_j \leqslant 1$ on K and is equal to 1 on E. So $f = \sum_{j=1}^{n} fh_j$ and hence $\phi(f) = \sum_{j=1}^{n} \phi(fh_j) \leqslant \sum_{j=1}^{n} \mu^*(\mathcal{U}_j) \leqslant \sum_{j=1}^{\infty} \mu^*(\mathcal{U}_j)$ as $fh_j \prec \mu^*(\mathcal{U}_j)$ for all j.

Taking the supremum of f, we deduce $\mu^*(\mathcal{U}) \leqslant \sum_{j=1}^{\infty} \mu^*(\mathcal{U}_j)$. It follows easily that $\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leqslant \sum_n \mu^*(A_n)$ for arbitrary sets (just approximate using an $\frac{\epsilon}{2^n}$ argument). We now let $\mathcal{M}$ be the set of μ^*-measurable subsets of K, then $\mathcal{M}$ is a $\sigma-$algebra and $\mu^*\restriction_{\mathcal{M}}$ is a measure on $\mathcal{M}$.
Next we show that $\mathcal{B} \subseteq \mathcal{U}$. Enough to show that $\mathcal{G} \subseteq \mathcal{M}$. Let $\mathcal{U}$ be in $\mathcal{G}$. We need to show: $\mu^*(A) \geqslant \mu^*(A \cap \mathcal{U}) + \mu^*(A\backslash\mathcal{U})$ for all $A \subseteq K$. First let $A = \mathcal{V} \in \mathcal{G}$. Fix $f \prec \mathcal{V} \cap \mathcal{U}$, fix $g \prec \mathcal{V}\backslash \operatorname{supp} f$. Then $f + g \prec \mathcal{V}$, and thus $\mu^*(\mathcal{V}) \geqslant \phi(f+g) = \phi(f) + \phi(g)$. Taking the supremum over g, we get $\mu^*(\mathcal{V}) \geqslant \phi(f) + \mu^*(\mathcal{V}\backslash \operatorname{supp} f) \geqslant \phi(f) + \mu^*(\mathcal{V} \cap \mathcal{U})$. Now let $A \subset K$ be arbitrary. Fix $\mathcal{V} \in \mathcal{G}$ s.t. $A \subseteq \mathcal{V}$, then $\mu^*(\mathcal{V}) \geqslant \mu^*(\mathcal{V} \cap \mathcal{U}) + \mu^*(\mathcal{V} \cap \mathcal{U}) \geqslant \mu^*(A \cap \mathcal{U}) + \mu^*(A\backslash\mathcal{U})$. Taking the infinum over all such $\mathcal{V}$, we have that $\mu^*(A) \geqslant \mu^*(A \cap \mathcal{U}) + \mu^*(A\backslash\mathcal{U})$.

Now, $\mu := \mu^*\restriction_{\mathcal{B}}$ is a Borel measure on K. We have that $\mu(K) = \phi(\mathbf{1}_K) = \|\phi\| < \infty$ and by definition, μ is regular. It remains to show that

$$\phi(f) = \int_K f d\mu$$

for all continuous f. It is enough to check that for all $f \in \mathcal{C}^{\mathbb{R}}(K)$ and then to show that $\phi(f) \leqslant \int_K f d\mu$ (by applying the it to $-f$).

Fix $a < b \in \mathbb{R}$ s.t. $f(K) \subseteq [a,b]$. Wlog, $a > 0$, since $\phi(\mathbf{1}_K) = \int_K \mathbf{1}_K d\mu$. Let $\epsilon > 0$; choose $0 \leqslant y_0 < a \leqslant y_1 < \cdots < y_n = b$ s.t. $y_j < y_{j+1} + \epsilon$ for all $1 \leqslant j \leqslant n$. Let $A_j = f^{-1}((y_{j-1}, y_j])$. Then, $K = \bigcup_{j=1}^{n} A_j$ and this is a measurable partition. Choose closed sets E_j and open sets $\mathcal{U}_j$ s.t. $E_j \subseteq A_j \subseteq \mathcal{U}_j$ and $\mu(\mathcal{U}_j\backslash E_j) < \frac{\epsilon}{n}$ (by regularity) and $f(\mathcal{U}_j) \subseteq (y_{j-1}, y_j + \epsilon)$. By lemma 2.2 there exist $h_j \prec \mathcal{U}_j$, $1 \leqslant j \leqslant n$, $\sum_{j=1}^{n} h_j \leqslant 1$ on K. Now [10]

$$
\begin{aligned}
\phi(f) = \sum_{j=1}^{n} \phi(fh_j) \leqslant \sum_{j=1}^{n} (y_j + \epsilon)\phi(h_j) \\
&\leqslant \sum_{j=1}^{n} (y_j + \epsilon)\mu(\mathcal{U}_j) \leqslant \sum_{j=1}^{n} (y_{j-1} + 2\epsilon)\left(\mu(\mathcal{U}_j) + \frac{\epsilon}{n}\right) \\
&\leqslant \sum_{j=1}^{n} y_{j-1}\mu(\mathcal{U}_j) + \epsilon(b+\epsilon) + 2\epsilon\mu(K) + 2\epsilon^2 \\
&= \int_K \sum_{j=1}^{n} y_{j-1}\mathbf{1}_{E_j} d\mu + \mathcal{O}(\epsilon) \\
&\leqslant \textstyle\int_K f d\mu + \mathcal{O}(\epsilon).
\end{aligned}
$$

Hence, $\phi(f) \leqslant \int_K f d\mu$, since $\epsilon > 0$ was arbitrary.

□

Corollary 2.5.1. For every $\phi \in \mathcal{M}(K)$, there exists a unique regular complex Borel measure ν on K that represents ϕ, namely, $\phi(f) = \int_K f d\nu$ for all continuous f. Moreover, $\|\phi\| = \|\nu\|_1$ and if ϕ is in $\mathcal{M}^{\mathbb{R}}(K)$, then ν is a signed measure.

Proof. Existence: Apply lemma 2.1 and theorem 2.5 to obtain a regular complex Borel measure ν that represents ϕ.

Need: $\|\nu\|_1 = \|\phi\|$.

Lecture 10 This will give uniqueness, if ν_1, ν_2 represent ϕ, then $\nu_1 - \nu_2$ represents $\phi - \phi = 0$, then $\|\nu_1 - \nu_2\|_1 = 0$, hence $\nu_1 = \nu_2$. $\|\phi\| \leqslant \|\nu\|_1$, was already done before Theorem 2.5. Take a measurable partition $K = \bigcup_{j=1}^{n} A_j$. Fix $\epsilon > 0$ and closed sets E_j, open sets $\mathcal{U}_j$ s.t. $E_j \subseteq A_j \subseteq \mathcal{U}_j$, $|\nu|(\mathcal{U}_j \backslash E_j) < \frac{\epsilon}{n}$ ($|\nu|$ is regular). Can also assume that $\mathcal{U}_j \subseteq K \backslash \bigcup_{i \neq j} E_i$, for all $1 \leqslant j \leqslant n$. Fix $\lambda_j \in \mathbb{C}$ s.t. $|\lambda_j| = 1$, $\lambda_j \nu(E_j) = |\nu(E_j)|$, $1 \leqslant j \leqslant n$. By lemma 2.2, there exist $h_j \prec \mathcal{U}_j$, $1 \leqslant j \leqslant n$, $\sum_{j=1}^{n} h_j \leqslant 1$ on K. then for all j $E_j \prec h_j$. Hence,

$$
\begin{aligned}
\left| \int_K \left(\sum_{j=1}^{n} \lambda_j \mathbf{1}_{E_j} - \sum_{j=1}^{n} \lambda_j h_j \right) d\nu \right| &\leqslant \sum_{j=1}^{n} \int_K |\mathbf{1}_{E_j} - \sum_{j=1}^{n} h_j| d|\nu| \\
&\leqslant \sum_{j=1}^{n} |\nu|(\mathcal{U}_j \backslash E_j) < \epsilon.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\sum_{j=1}^{n} |\nu(A_j)| &\leqslant \sum_{j=1}^{n} |\nu(E_j)| + \epsilon = \sum_{j=1}^{n} \lambda_j \nu(E_j) + \epsilon \\
&= \int_K \sum_{j=1}^{n} \lambda_j \mathbf{1}_{E_j} d\nu + \epsilon \leqslant \left| \int_K \left(\sum_{j=1}^{n} \lambda_j h_j \right) d\nu \right| + 2\epsilon \\
&= \left| \phi\left(\sum_{j=1}^{n} \lambda_j h_j \right) \right| + 2\epsilon \\
&= \|\phi\| \cdot \left\| \textstyle\sum_{j=1}^{n} \lambda_j h_j \right\|_\infty + 2\epsilon \leqslant \|\phi\| + 2\epsilon
\end{aligned}
$$

using the fact the the expression in the second to last line is a convex combination of function with sup norm equal to one. Hence, it follows that $\|\nu\|_1 \leqslant \|\phi\|$. □

[10]using that $f \leqslant y_j \leqslant \epsilon$ and $h_j \prec \mathcal{U}_j$ and $\phi \in \mathcal{M}^+(K)$.

Corollary 2.5.2. The space of regular complex Borel measures is a complex Banach space in the $\|\nu\|_1$ (total variation norm) and is isometrically isomorphic $\mathcal{M}(K)$.
The space of regular real Borel measures is a real Banach space in the $\|\nu\|_1$ (total variation norm) and is isometrically isomorphic $\mathcal{M}^{\mathbb{R}}(K)$.

3 Weak Topologies

Let X be a set and $\mathcal{F}$ be a family of function s.t. each $f \in \mathcal{F}$ is a function $f : X \to Y_f$, where Y_f is a topological space.

The weak topology $\sigma(X, \mathcal{F})$ on X generated by $\mathcal{F}$ is the smallest topology on X s.t. each $f \in \mathcal{F}$ is continuous (is easily see to exist).

Remark.

1. $\mathcal{S} = \{f^{-1}(\mathcal{U}) : f \in \mathcal{F}, \mathcal{U} \subseteq Y_f open\}$ is a sub-base of $\sigma(X, \mathcal{F})$. So $\mathcal{V} \subseteq X$ is open, i.e. it is in $\sigma(X, \mathcal{F})$ iff for all $x \in \mathcal{V}$, there exist $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathcal{F}$ and open sets $\mathcal{U}_j \subseteq Y_{f_j}$ (open nbhds of $f_j(x)$) for $1 \leqslant j \leqslant n$ s.t. x is in $\bigcap_{j=1}^{n} f^{-1}(\mathcal{U}_j) \subseteq \mathcal{V}$.
2. If $\mathcal{S}_f$ is a sub-base in Y_f, then $\{f^{-1(\mathcal{U})} : f \in \mathcal{F}, \mathcal{U} \in \mathcal{S}_f\}$, is a sub-base for $\sigma(X, \mathcal{F})$.
3. If Y_f is Hausdorff for all $f \in \mathcal{F}$ and $\mathcal{F}$ separates points in X (i.e., for all $x \neq y$, there exists $f \in \mathcal{F}$ s.t. $f(x) \neq f(y)$). Then $\sigma(X, \mathcal{F})$ is Hausdorff (easy to check).
4. $Y \subseteq X$, let $\mathcal{F}_Y = \{f\restriction_Y : f \in \mathcal{F}\}$. Then $\sigma(Y, \mathcal{F}_Y) = \sigma(X, \mathcal{F})\restriction_Y$ (check!).
5. Universal property: let Z be a topological space and $q : Z \to X$ be a function. Then g is continuous iff $f \circ g : Z \to Y_f$ is continuous for all $f \in \mathcal{F}$.

Examples:

1. Let X be a topological space, let $Y \subseteq X$ and $\iota : Y \to X$ be the inclusion map. Then, $\sigma(Y, \{\iota\})$ is the subspace topology of Y.
2. let Γ be a set, X_γ a topological space for all $\gamma \in \Gamma$ and $X = \prod_{\gamma \in \Gamma} X_\gamma = \{X : X$ is a function on Γ s.t. $\forall \gamma \in \Gamma, x(\gamma) \in X_\gamma\}$. For $x \in X$, $\gamma \in \Gamma$ we often write x_γ for $x(\gamma)$. We think of x as the "$\Gamma-$tuple", $(x_\gamma)_{\gamma \in \Gamma}$. For each γ we have $\pi_\gamma : X \to X_\gamma, x \mapsto x_\gamma$ $((x_\delta)_{\delta \in \Gamma})$ the evaluation at γ, or projection onto X_γ. The weak topology $\sigma(X, \{\pi_\gamma : \gamma \in \Gamma\})$ is called the product topology on X. $\mathcal{V}$ is open iff for all $x = (x_\gamma)_{\gamma \in \Gamma} \in \mathcal{V}$, there exist $n \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_n \in \Gamma$ and open neighbourhoods $\mathcal{U}_j$ of $x_{\gamma_j} in X_{\gamma_j}$ s.t.
$$\{y = (y_\gamma)_{\gamma \in \Gamma} \in X : y_{\gamma_j} \in \mathcal{U}_j, 1 \leqslant j \leqslant n\} \subseteq \mathcal{V}$$
.

Proposition 3.1. Let X be a set. For each $n \in \mathbb{N}$, let (Y_n, d_W) be a metric space and $f_n : X \to Y_n$ be a function s.t. $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ separates points of X. Then $\sigma(X, \mathcal{F})$ is metrisable.

Proof. Define
$$d(x, y) = \sum_{n=1}^{\infty} \min(|f_n(x) - f_n(y)|, 1) \cdot 2^{-n}, \text{ for } x, y \text{ in } X.$$
This is a metric on X (easy to check) ($\mathcal{F}$ separating points implies that for $x \neq y$, $d(x, y) > 0$). Fiven $\epsilon \in (0, 1)$ and $d(x, y) < \frac{\epsilon}{2^n}$, then $|f_n(x) - f_n(y)| < \epsilon$. So each f_n is continuous wrt the topology τ induced by d. So $\sigma = \sigma(X, \mathcal{F}) \subseteq \tau$. Fix $x \in X$, then $y \mapsto \min(|f_n(x) - f_n(y)|, 1) \cdot 2^{-n}$ is $\sigma-$continuous. By the Weierstrass M-test, $\sum_{n=1}^{\infty} \min(|f_n(x) - f_n(y)|, 1) \cdot 2^{-n}$ is univormly convergent, hence $\sigma-$continuous. So, $\{y \in X : d(y, x) < \epsilon\}$ is $\sigma-$open. Hence, $\tau \subseteq \sigma$ and $\tau = \sigma$. □

Theorem 3.1 (Tychonov). The product of compact topological spaces is compact in the product topology.

Proof. We have $X = \prod_{\gamma\in\Gamma} X_\gamma$ as in examples 3. Assume each X_γ is compact. Let $\mathcal{F}$ be a family of closed subsets of X with the finite intersection property (FIP). We need to show that $\bigcap_{F\in\mathcal{F}} F \neq \varnothing$ (equivalent to compactness).
By Zorn, there exists a maximal family $\mathcal{A}$ of subsets of X s.t. $\mathcal{F} \subseteq \mathcal{A}$ and $\mathcal{A}$ has the FIP ($\mathcal{M} = \{\mathcal{A} \subseteq \mathcal{P}(X) : \mathcal{A} \supseteq \mathcal{F}$ & $\mathcal{A}$ has the FIP$\}$, and every chain has a maximal element. Check!).
We will show that $\bigcap_{A\in\mathcal{A}} \overline{A} \neq \varnothing$.

<u>Note</u>:

1. $A_1, \ldots, A_n \in \mathcal{A}$ implies that $A = \bigcap_{i=1}^{n} A_i$ is in $\mathcal{A}$.
   Indeed, for all $B_1, \ldots, B_m \in \mathcal{A}$, s.t. $A \cap B_1 \cap \cdots \cap B_m \neq \varnothing$ so $\mathcal{A} \cup \{A\}$ has the FIP. Hence, A is in $\mathcal{A}$.

2. $B \subseteq X$, $B \cap A \neq \varnothing$ for all $A \in \mathcal{A}$ implies B is in $\mathcal{A}$. Indeed, for $A_1, \ldots, A_n \in \mathcal{A}$ s.t. $\bigcup_{i=1}^{n} A_i \neq \varnothing$ and $B \cap \bigcup_{i=1}^{n} A_i \neq \varnothing$, then $A \cup \{B\}$ has the FIP and using maximality, we conclude that B is in $\mathcal{A}$.

Let $\gamma \in \Gamma$. Then $\{\pi_\gamma(A) : A \in \mathcal{A}\}$ has the FIP. Since X_γ is compact, $\bigcap_{A\in\mathcal{A}} \overline{\pi_\gamma(A)} \neq \varnothing$. Fix $x_\gamma \in \bigcap_{A\in\mathcal{A}} \overline{\pi_\gamma(A)} \neq$. Let $x = (x_\gamma)_{\gamma\in\Gamma}$ and $\mathcal{U}$ be an open neighbourhood of $\underline{x}$. We show that $\mathbf{U} \cap A \neq \varnothing$ for all $A \in \mathcal{A}$. Then $x \in A$, for all $A \in \mathcal{A}$. Wlog, $\mathcal{U} = \bigcup_{j=1}^{n} \pi_{\gamma_j}^{-1}(\mathcal{U}_j)$ for $n \in \mathbb{N}, \gamma_1, \ldots, \gamma_n \in \mathcal{F}$, $\mathcal{U}_j$ is an open neighbourhood of $x_{\gamma_j} \in X_{\gamma_j}$. So $\mathcal{U}_j \cap \bigcup_{j=1}^{n} \pi_{\gamma_j}^{-1}(A_j) \neq \varnothing$ for all $A \in \mathcal{A}$, so $\pi_{\gamma_j}^{-1}(\mathcal{U}_j) \in \mathcal{A}$ by note 2 above. By 1 above, $\mathcal{U} \in \mathcal{A}$ and hence, $\mathcal{U} \cap A \neq \varnothing$ for all $A \in \mathcal{A}$.
We have thus demonstrated that for all $A \in \mathcal{A}$, $x \in \overline{A}$, which concludes the proof. ∎

Lecture 11

3.1 Weak topologies on vector spaces

Let E be a real or complex vector space. Let F be a subspace of the space of all linear functionals on E that separates points, i.e. for all $x \in E, x \neq 0$, then there exists $f \in F, f(x) \neq 0$. Consider the weak topology $\sigma(E, F)$. So $\mathcal{U} \subseteq E$ is open iff for all $x \in \mathcal{U}$, there exists $n \in \mathbb{N}, f_1, \ldots, f_n \in F, \epsilon > 0$ s.t. $\{y \in E : |f_j(y - x)| < \epsilon, 1 \leqslant j \leqslant n\} \subseteq \mathcal{U}$. For $f \in F, x \in E, p_f(x) = |f(x)|$. Let $\mathcal{P} = \{p_f : f \in F\}$. Then $(E, \mathcal{P})$ is a locally convex space (LCS) whose topology is $\sigma(E, F)$. So $\sigma(E, F)$ is Hausdorff and vector addition and scalar multiplication are continuous.

Lemma 3.1. Let E be as above, ler $f, g_1, \ldots, g_n$ be linear functionals on E s.t. $\bigcup_{j=1}^{n} \ker g_j \subseteq \ker f$. Then $f \in \operatorname{span}\{g_1, \ldots, g_n\}$.

Proof. Let $\mathbb{K}$ be the scalar field. Define $T : E \to \mathbb{K}^n$ by $Tx = (g_j(x))_{j=1}^n$. Then $\ker(T) = \bigcup_{j=1}^{n} \ker g_j \subseteq \ker f$ and hence we have a factorisation

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "T"] & \mathbb{K} \\ \mathbb{K}^n \arrow[ur, "h"'] & \end{tikzcd}$$

with h linear, $f = h \circ T$. Then there exists $(a_j(x))_{j=1}^n \in \mathbb{K}^n$ s.t. $h(y) = \sum_{j=1}^n a_j y_j$ for all $y \in \mathbb{K}^n$. So for all $x \in E, f(x) = h(Tx) = \sum_{j=1}^n a_j g_j(x)$. So $f = \sum_{j=1}^n a_j g_j$ as required. □

Proposition 3.2. Let E, F be as above, let f be a linear function on E. Then f is $\sigma(E,F)-$continuous iff $f \in F$. So, $(E, \sigma(E,F))^* = F$.

Proof. $\Longleftarrow$: holds by definition.

$\Longrightarrow$: there exists an open neighbourhood $\mathcal{U}$ of 0 in E s.t. for all $x \in \mathcal{U}$, $|f(x)| < 1$. Wlog, (shrink $\mathcal{U}$ if necessary) $\mathcal{U} = \{x \in E : |g_j(x)| < \epsilon, 1 \leqslant j \leqslant n\}$ for some $n \in \mathbb{N}, g_1, \ldots, g_n \in F, \epsilon > 0$. If $x \in \bigcup_{j=1}^n \ker g_j$, then $ambx \in \mathcal{U}$ for all scalars λ and hence $|f(x)| = |\lambda| \cdot |f(x)| < 1$ for all λ. So $f(x) = 0$. By lemma 3.1, $f \in \operatorname{span}\{g_1, \ldots, g_n\}$. □

Examples:

1. Let X be a normed space. The weak topology on X is the topology $\sigma(X, X^*)$ on X. (X^* annihilates points of X by Hahn-Banach). We sometimes write, (X, w) for $(X, \sigma(X, X^*))$. Open sets in $\sigma(X, X^*)$ are called weak open, or $w-$open. $\mathcal{U} \subseteq X$ is $w-$open $\iff$ for all $x \in \mathcal{U}$, there exists $n \in \mathbb{N}, f_1, \ldots, f_n \in X^*, \epsilon > 0$ s.t. $\{y \in X : |f_j(y-x)| < \epsilon, 1 \leqslant j \leqslant n\}$.

2. Let X be a normed space. The weak star topology or w^*-topology on X^* is the topology $\sigma(X^*, X)$ on X^*. Here, we are identifying X with its image in X^{**} under the canonical embedding. Open sets in $\sigma(X^*, X)$ are called w^*-open and $\mathcal{U} \subseteq X^*$ is weak-* open iff for all $f \in \mathcal{U}$, there exist $n \in \mathbb{N}, x_1, \ldots, x_n \in X, \epsilon > 0$ s.t. $\{y \in X^* : |g(x_j) - f(x_j)| < \epsilon, 1 \leqslant j \leqslant n\} \subseteq \mathcal{U}$.

Properties:

1. (W, w) and $(\mathbf{X}^*, \mathbf{w}^*)$ (this is $(X^*, \sigma(X^*, X))$) are LCS and hence Hausdorff with continuous vector space operations.

2. $\sigma(X, X^*) \subseteq \|\cdot\|-$topology with equality iff $\dim X < \infty$.

3. $\sigma(X, X^*) \subseteq \sigma(X^*, X) \subseteq \|\cdot\|$, where equality in the first inclusion is achieved iff X is reflexive, and for the latter iff $\dim X^* = \dim X < \infty$.

4. Let Y be a subspace of X. Then, $\sigma(X, X^*)\restriction_Y = \sigma(Y, \{f \in X^*\}) = \sigma(Y, Y^*)$ by Hahn-Banach. Similarly, $\sigma(X^{**}, X^*)\restriction_X = \sigma(X, X^*)$. So in other words, the canonical embedding $X \to X^{**}$ is also a weak-to-weak-* homeomorphism between X and $\hat{X}$.

Proposition 3.3. Let X be a normed space.

(i) A linear functional f on X is continuous in the weak topology iff $f \in X^*$. So $(X, w)^* = X^*$.

(ii) A linear functional f on X^* is w^*- continuous iff $f \in X$, i.e. $f = \hat{x}$ for some $x \in X$. So $(X^*, w^*)^* = X$. It follows that $\sigma(X^*, X) = \sigma(X^*, X^{**})$ iff X is reflexive.

Definition 3.1 (Weak Boundedness). Let X be a normed space, then a subset $A \subseteq X$ is weakly bounded if $\{f(x) : x \in A\}$ is bounded for all $f \in X^*$ (iff for all w-neighbourhood of 0 in X, there exists $\lambda > 0$ s.t. $A \subseteq \lambda \mathcal{U}$).
A subset $B \subseteq X^*$ is weak-* bounded if $\{f(x) : x \in B\}$ is bounded for all $x \in X$ (iff iff for all

w^*-neighbourhood of 0 in X^*, there exists $\lambda > 0$ s.t. $B \subseteq \lambda\mathcal{U}$).

3.2 Principle of Uniform Boundedness (PUB)

Let X be a Banach space, Y be a normed space and $\mathcal{T} \subseteq \mathcal{B}(,\mathcal{Y})$. If $\mathcal{T}$ is pointwise bounded $\left(\sup_{T\in\mathcal{T}} \|Tx\| \text{ for all } x \in X\right)$, then T is uniformly bounded $\left(\sup_{T\in\mathcal{T}} \|T\| < \infty\right)$.

Proposition 3.4. (i) $A \subseteq X$ is weakly bounded implies that A is $\|\cdot\|$ −bounded.

(ii) $B \subseteq X^*$ is weak-* bounded and X is complete implies that B is $\|\cdot\|$ −bounded.

Proof. (ii) $B \subseteq X^* = \mathcal{B}(X, \text{scalars})$, B weak-* bounded says B is pointwise bounded. So done by PUB.
(i) $\hat{A} = \{\hat{x} : x \in A\} \subseteq X^{**} = \mathcal{B}(X^*, \text{scalars})$. A weakly bounded iff $\hat{A}$ is pointwise bounded and so can conclude again by PUB. □

Notation: We write $x_n \xrightarrow{w} x$ if $(x_n)_{n\in\mathbb{N}}$ converges to x in the weak topology (in some normed space). Note that $x_n \xrightarrow{w} x$ in X iff $\langle x_n, f\rangle \to \langle x, f\rangle$ for all $f \in X^*$. We write $f_n \xrightarrow{w^*} f$ in X^* if $(f_n)_{n\in\mathbb{N}}$ converges to f in the weak-* topology (in some dual space) iff $\langle x, f_n\rangle \to \langle x, f\rangle$ for all $x \in X$.

Consequences of PUB: Let X be a Banach space, Y a normed space, (T_n) a sequence in $\mathcal{B}(X, Y)$. If $T : X \to Y$ is a function s.t. $T_n \to T$ pointwise on X (i.e. $T_n x \to Tx$ for all $x \in X$), then $T \in \mathcal{B}(X, X)$, $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$ and $\|T\| \leqslant \liminf_n \|T_n\|$.

Proposition 3.5. Let X be a normed space.

(i) If $x_n \xrightarrow{w} x$ in X, then $\sup_n \|x_n\| < \infty$ and $\|x\| \leqslant \liminf_n \|x_n\|$.

(ii) If $f_n \xrightarrow{w^*} f$ in X^* and X is complete, then $\sup_n \|f_n\| < \infty$ and $\|f\| \leqslant \liminf_n \|f_n\|$.

Proof. (ii) We have that $f_n \to f$ pointwise in $X^* = \mathcal{B}(X, \text{scalars})$. Result follows by PUB.
(i) Since $x_n \xrightarrow{w} x$, $\hat{x}_n \to \hat{x}$ pointwise in $X^{**} = \mathcal{B}(X^*, \text{scalars})$ and we conclude by PUB again. □

Lecture 12 For the above, the converse is not true. We can find a sequence that converges weakly but not in the norm topology. For instance,

Example:
In $\ell_p, 1 < p < \infty$, $e_n = (0, \ldots, 0, \underset{n^{th}\text{-entry}}{1}, 0, \ldots, 0)$, $e_n \xrightarrow{w} 0$, but clearly $e_n \overset{\|\cdot\|}{\nrightarrow} 0$.

3.3 Hahn-Banach Separation Theorems

Let $(X, \mathcal{P})$ be a LCS. Let $\mathcal{C}$ be a convex subspace of X, s.t. $0 \in \operatorname{int}\mathcal{C}$. Then define $\mu_{\mathcal{C}} : X \to \mathbb{R}, \mu_{\mathcal{C}} = \inf\{t > 0 : x \in t\mathcal{C}$.

Well-defined: $\frac{1}{n}x \to 0$ as $n \to \infty$, so there exists $n \in \mathbb{N}$ s.t. $\frac{1}{n}x \in \mathcal{C}$. $\mu_{\mathcal{C}}$ is the Minkowski functional (gauge functional) of $\mathcal{C}$.

Example:
If X is a normed space and $\mathcal{C} = B_X$, then $\mu_{\mathcal{C}} = \|\cdot\|$.

Lemma 3.2. $\mu_{\mathcal{C}}$ is positive homogeneous and sub-additive. Moreover, $\{x : \mu_{\mathcal{C}} < 1\} \subset \mathcal{C} \subset \{x : \mu_{\mathcal{C}} \leqslant 1\}$. The first inclusion is an equality if $\mathcal{C}$ open.

Proof. Positive homogeneous: for $x \in X, s, t, > 0$ we have $sx \in st\mathcal{C} \iff x \in t\mathcal{C}$. Hence, $\mu_{\mathcal{C}}(sx) = s\mu_{\mathcal{C}}(x)$. A;so holds for $s = 0$, since $\mu_{\mathcal{C}}(0) = 0$.

Subadditivity: First an observation: $\mu_{\mathcal{C}} < t$ implies $x \in t\mathcal{C}$. Indeed, there exists $t' < t$ s.t. $x \in t'\mathcal{C}$. Then, $\frac{x}{t} = (1 - \frac{t'}{t}) \cdot 0 + \frac{t'}{t} \cdot \frac{x}{t'} \in \mathcal{C}$ by the convexity of $\mathcal{C}$.

Now, let $x, y \in X$. Fix $s > \mu_{\mathcal{C}}(x)$, $t > \mu_{\mathcal{C}}(y)$. Then $x \in s\mathcal{C}$, $y \in t\mathcal{C}$. So, $x + y = \left(\frac{s}{s+t} \cdot \frac{x}{s} + \frac{t}{s+t} \cdot \frac{x}{t}\right)(s+t) \in (s+t)\mathcal{C}$ by convexity. So $\mu_{\mathcal{C}}(x+y) \leqslant s+t$, and hence $\mu_{\mathcal{C}}(x+y) \leqslant \mu_{\mathcal{C}}(x) + \mu_{\mathcal{C}}(y)$.

Next, if $\mu_{\mathcal{C}}(x) < 1$, then $x \in \mathcal{C}$ by above. If $\mathcal{C}$ is open and $x \in \mathcal{C}$, then there exists $n \in \mathbb{N}$ s.t. $(1 + \frac{1}{n})x \in \mathcal{C}$, since $(1 + \frac{1}{n})x \xrightarrow{n \to \infty} x$ and $\mathcal{C}$ open. Hence, $\mu_{\mathcal{C}}(x) \leqslant \frac{1}{1+\frac{1}{n}} < 1$.

Finally, $x \in \mathcal{C}$ implies that $\mu_{\mathcal{C}}(x) \leqslant 1$. Then, by homogeneity, $\mu_{\mathcal{C}}((1 - \frac{1}{n})x) < 1$ for all n, so $(1 - \frac{1}{n})x \in \mathcal{C}$ for all n, since $(1 - \frac{1}{n})x \to x$, the in case $\mathcal{C}$ is closed $x \in \mathcal{C}$. □

Remark. If $\mathcal{C}$ is symmetric (in real case) or balanced (in complex case)m then $\mu_{\mathcal{C}}$ is a semi-norm. If, in addition $\mathcal{C}$ is bounded, then $\mu_{\mathcal{C}}$ is a norm.

Theorem 3.2. Hahn-Banach Separation Theorem Let $(X, \mathbb{P})$ be a LCS and $\mathcal{C}$ be an open convex subset of X with $0 \in \operatorname{int} \mathcal{C}$. let $x_0 \in X \backslash \mathcal{C}$. Then there exists $f \in X^*$ s.t. $f(x_0) > f(x)$ for all $x \in \mathcal{C}$.(In complex case: $\operatorname{Re}(f(x_0))) > \operatorname{Re}(f(x))$ for all $x \in \mathcal{C}$).

Remark. From now on we work with real scalars and the complex case will follow, since

$$f \mapsto \operatorname{Re} f : X^* \to X^*_{\mathbb{R}}$$

is a real linear injection.

Proof. Consider $\mu_{\mathcal{C}}$. By lemma 3.2, $\mathcal{C} = \{s : \mu_{\mathcal{C}}(x) < 1\}$ and so $\mu_{\mathcal{C}}(x_0) \geqslant 1$. Let $Y = \operatorname{span}\{x_0\}$ and $g : Y \to \mathbb{R}$, $g(\lambda x_0) = 1 \leqslant \mu_{\mathcal{C}}(x_0)$. Hence, $g \leqslant \mu_{\mathcal{C}}$ on Y.
By Theorem 1.1, there exists linear $f : X \to \mathbb{R}$ s.t. $f\restriction_Y = g$ and $f \leqslant \mu_{\mathcal{C}}$ on X. For all $x \in \mathcal{C}$, $f(x) \leqslant \mu_{\mathcal{C}}(x) < 1 = f(x_0)$. We also gave $f(x) < 1$ on $\mathcal{C}$ and so $|f(x)| < 1$ on $\mathcal{C} \cap (-\mathcal{C})$. Since $\mathcal{C} \cap (-\mathcal{C})$ is an open neighbourhood of 0, we have that $f \in X^*$. □

Theorem 3.3. Let $(X, \mathcal{P})$ be a LCS. Let $A, B \neq \varnothing$, disjoint convex subsets of X.

(i) If A is open, there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ s.t. $f(x) < \alpha \leqslant f(y)$ for all $x \in A, y \in B$.

(ii) If A is compact, and B is closed, then there exists $f \in X^*$ s.t. $\sup_A f < \inf_B f$.

Proof. (i) Fix $\alpha \in A, b \in B$. Let $\mathcal{C} = A - B + b - \alpha$ and $x_0 = b - \alpha$. Then $\mathcal{C}$ is open, convex, $0 \in \mathcal{C}$ and $x_0 \notin \mathcal{C}$ $(A \cap B = \varnothing)$. By Theorem 3.2, there exists $f \in X^*$ s.t. $f(z) < f(x_0)$ for all $z \in \mathcal{C}$. So for all $x \in A, y \in B$ $f(x - y + x_0) < f(x_0)$, i.e $f(x) < f(y)$. In particular, $f \neq 0$. Let $\alpha = \inf f$. Then $\alpha \leqslant f(y)$ for all $y \in B$. Since $f \neq 0$, there exists $u \in X$ s.t. $f(u) > 0$.Now, given $x \in A$, $x + \frac{1}{n}u \to x$ and since A is open, there exists $n \in \mathbb{N}$ s.t. $x + \frac{1}{n}u \in A$. Then $f(x) < f(x + \frac{1}{n}u) \leqslant \alpha$.

(ii) Claim: there exists open, convex neighbourhood of 0 in X,$\mathcal{U}$ s.t. $(A + \mathcal{U}) \cap B = \varnothing$

Indeed, for $x \in A$, there exists open neighbourhood $\mathcal{U}_x$ of 0 s.t. $(x + \mathcal{U}_x) \cap B = \varnothing$ (B is closed). Since $0 + 0 = 0$ and "+" is continuous, there exists open neighbourhood $\mathcal{V}_x$ of 0 s.t. $\mathcal{V}_x + \mathcal{V}_x \subseteq \mathcal{U}_x$. Wlog, $\mathcal{V}_x$ is convex and symmetric. By compactness, there exist $x_1, \ldots, x_n \in A$ s.t. $A \subseteq \bigcup_{i=1}^{n}(x_i + \mathcal{V}_{x_i})$. Let $\mathcal{U} = \bigcap_{i=1}^{n} \mathcal{V}_{x_i}$. Given $x \in A$, there exists i s.t. $x \in x_i + \mathcal{V}_{x_i}$. So, $x + \mathcal{U} \subseteq x \in x_i + \mathcal{V}_{x_i} + \mathcal{U} \subseteq x \in x_i + \mathcal{V}_{x_i} + \mathcal{V}_{x_i} \subseteq x_i + \mathcal{U}_{x_i}$ is disjoint from B. So, $A + \mathcal{U}$ is disjoint from B.

Now, apply part (i) with $A+\mathcal{U}, B$ to show that there exists $f \in X^*$ s.t. $f(x+u) < f(y)$ for all $x \in A, y \in B, u \in \mathcal{U}$. In particular, $f \neq 0$ so there exists $z \in X$ s.t. $f(z) > 0$. Also, $\frac{1}{n}z \xrightarrow{n\to\infty} 0$, so there exists $n \in \mathbb{N}$ s.t. $\frac{1}{n}z \in \mathcal{U}$. So $f(x) + \frac{1}{n}f(z) < f(y)$ for all $x \in A, y \in B$. It follows that $\sup\limits_{A} f < \inf\limits_{B} f$.

$\square$

Theorem 3.4 (Mazur). Let $\mathcal{C}$ be a convex subset of a normed space X. Then $\overline{\mathcal{C}}^{\|\cdot\|} = \overline{\mathcal{C}}^{w}$. In particular, $\mathcal{C}$ is $\|\cdot\|$ −closed iff $\mathcal{C}$ is weakly closed.

Proof. Wlog, $\mathcal{C} \neq \emptyset$.

"$\overline{\mathcal{C}}^{\|\cdot\|} \subseteq \overline{\mathcal{C}}^{w}$": is true since the weak topology is weaker than the $\|\cdot\|$ −topology.

"$\overline{\mathcal{C}}^{\|\cdot\|} \supseteq \overline{\mathcal{C}}^{w}$": If $x \notin \overline{\mathcal{C}}^{\|\cdot\|}$, then apply Theorem 3.2 (*ii*) to $A = \{x\}$, $B = \overline{\mathcal{C}}^{\|\cdot\|}$ to obtain $f \in X^*$ s.t. $f(x) < \inf_B f := \alpha$. Then, $\{y : f(y) < \alpha\}$ is a weakly open neighbourhood of X, disjoint from B (and hence from $\mathcal{C}$). So $x \notin \mathcal{C}$. □

Corollary 3.4.1 (Mazur). If $x_n \xrightarrow{w} 0$ in a normed space X, then for all $\epsilon > 0$, there exists $x \in \operatorname{conv}\{x_n : n \in \mathbb{N}\}$ s.t. $\|x\| \leqslant \epsilon$.

Proof. $0 \in \overline{\operatorname{conv}\{x_n : n \in \mathbb{N}\}}^{w} = \overline{\operatorname{conv}\{x_n : n \in \mathbb{N}\}}^{\|\cdot\|}$ by Mazur. □

Remark. It follows from this that there exist $p_1 < q_1 < p_2 < q_2 \ldots$ and convex combinations $z_n = \sum_{i=p_n}^{q_n} t_i x_i$ s.t. $z_n \to 0$ in $\|\cdot\|$.

Lecture 13

Theorem 3.5 (Banach-Alaoglu). For any normed space X, (B_{X^*}, w^*) is compact.

Proof. For $x \in X$, let $K_x = \{\lambda : \lambda \text{ scalar }, |\lambda| \leqslant \|x\|\}$. Let $K = \prod_{x \in X} K_x$ in he product topology. Let $\pi_x : K \to K_x$ be the projection $(\lambda_y)_{y \in X} \mapsto \lambda_x$.

Note $K = \{\lambda : X \to \text{scalars} : |\lambda(x)| \leqslant \|x\|\}$, so $B_{X^*} \subseteq K$.

The subspace topology on B_{X^*} is $\sigma(K, \{\pi_x : x \in X\})\restriction_{B_{X^*}} = \sigma(B_{X^*}, \{\pi_x\restriction_{B_{X^*}} : x \in X\}) = \sigma(B_{X^*}, \{\hat{x}\restriction_{B_{X^*}} : x \in X\}) = \sigma(X^*, X)\restriction_{B_{X^*}}$, the weak-* topology. By Theorem 3.1, K is compact. So all we need to show is that B_{X^*} is closed in K. Now,

$$\begin{aligned} B_{X^*} &= \{\lambda \in K : \lambda_{ax+by} = a\lambda_x + b\lambda_y \forall x, y \in X, \forall a, b \in \text{scalars}\} \\ &= \bigcap_{x,y,a,b} \{\lambda \in K : \pi_{ax+by}(\lambda) = a\pi_x(\lambda) + b\pi_y(\lambda)\} \\ &= \bigcap_{x,y,a,b} \{\lambda \in K : \pi_{ax+by}(\lambda) - a\pi_x(\lambda) - b\pi_y(\lambda)^{-1}(\{0\})\} \end{aligned}$$

closed in K as each π_x is continuous. □

Proposition 3.6. Let X be a normed space and K be a compact, Hausdorff space.

(i) X is separable (in the $\|\cdot\|$ −top) iff (B_{X^*}, w^*) is metrisable.

(ii) $\mathcal{C}(K)$ is separable iff K is metrisable.

Proof. (*i*)"$\Longrightarrow$": Fix a dense sequence (x_n) in X. Let $\mathcal{F} = \{\hat{x_n}\restriction_{B_{X^*}} : n \in \mathbb{N}\}$. Then $\mathcal{F}$ separates the points of X, so $\sigma(B_{X^*}, \mathcal{F})$ is Hausdorff and is contained in the weak-* topology. So

$$\mathrm{Id} : (B_{X^*}, w^*) \to (B_{X^*}, \sigma(B_{X^*}, \mathcal{F}))$$

is a continuous bijection from a compact space to a Hausdorff space, and hence a homeomorphism. So $\sigma(B_{X^*}, \mathcal{F})$ is the weak-* topology on B_{X^*}. This is metrisable by proposition 3.1.

(i)"$\Longrightarrow$": By above, $(B_{\mathcal{C}(K)^*}, w^*)$ is metrisable. For $k \in K$, define $\delta_k : \mathcal{C}(K) \to$ scalars by $\delta_k(f) = f(k)$ for all $f \in \mathcal{C}(K)$. Then $\delta_k \in B_{\mathcal{C}(K)^*}$. Hence

$$\begin{aligned} \delta :&\to (B_{\mathcal{C}(K)^*}, w^*) \\ k &\mapsto \delta_k \end{aligned}$$

δ is continuous: let $f \in \mathcal{C}(K)$. Is $\hat{f} \circ \delta$ continuous? For $k \in K$, $(\hat{f} \circ \delta)(k) = \delta_k(f) = f(k)$. Then, $\hat{f} \circ \delta = f$. This is continuous on K. By the universal property of the weak topology, δ is continuous.

δ is injective: $\mathcal{C}(K)$ separates points of K by Urysohn.

Now, $\delta : K \to (\delta(K), w^*)$ is a continuous bijection from compact to Hausdorff, and hence a homeomorphism. Hence K is metrisable.

(ii)"$\Longleftarrow$": K compact metrisable, so K is separable. Fix a dense sequence (x_n) in K. Let $(f_n) = d(x, x_n)$ (d is a metric inducing the topology of K). Let A be the sub-algebra of $\mathcal{C}(K)$ generated by $f_n, n \in \mathbb{N}$ and $\mathbf{1}_K$. The A is separable, A separates points of K, $\mathbf{1}_K \in A$ and in complex case, closed under complex conjugate. By Stone Weierstrass, $\overline{A} = \mathcal{C}(K)$, so $\mathcal{C}(K)$ is separable.

(i)"$\Longleftarrow$": let $K = (B_{X^*}, w^*)$. This is compact, by Theorem 3.5. Since K is metrisable, $\mathcal{C}(K)$ is separable. We prove that $X \hookrightarrow \mathcal{C}(K)$ isometrically. Then done. Let $T : X \to \mathcal{C}(K)$ be $Tx = \hat{x}\restriction_{B_{X^*}}$. then T is linear and $\|Tx\|_\infty = \|\hat{x}\| = \|x\|$. □

Remark.

1. If X is separable, then (B_{X^*}, w^*) is compact, metrisable and hence weak-* sequentially compact(+separable).
2. X is separable implies that X^* is weak-* separable ($X^* = \bigcup_{n\in\mathbb{N}} nB_{X^*}$).
   By mazur, X is separable iff X is weakly separable (weak closure of span of some (x_n) weakly dense in X is $\|\cdot\|$ −closure by Mazur, since it is convex).
   So X weakly separable implies X^* is weak-* separable. The converse is not true in general (e.g. ℓ_∞).
3. The proof shows $(B_{\mathcal{C}(K)^*}, w^*)$ contains a homeomorphic copy of K.
4. Proof also shows that for every normed space X there exists compact, hausdorff K s.t. $X \hookrightarrow \mathcal{C}(K)$ isometically ($K = (B_{X^*}, w^*)$).

Proposition 3.7. Let X be a normed space. Then X^* is separable iff (B_X, w) is metrisable.

Proof. "$\Longrightarrow$": By proposition 3.6 (i), $(B_{X^{**}}, w^*)$ is metrisable. Hence, $(B_X, w) = (B_{X^{**}}, w^*)\restriction_{B_X}$ is metrisable.

"$\Longleftarrow$": let d metrise (B_X, w). Then for all $n \in \mathbb{N}$, there exists finite $F_n \subseteq X^*$ and $\epsilon_n > 0$ s.t. $\mathcal{U}_n = \{x \in B_X : |f(x)| < \epsilon_n \forall f \in F_n\} \subseteq \{x : d(x, 0) < \frac{1}{n}\}$. Let $Z = \operatorname{span} \bigcup_{n\in\mathbb{N}} F_n$.

Claim: $\overline{Z} = X^*$, then done.

Indeed, let $g \in X^*$ and fix $\epsilon > 0$. Then $\{x \in B_X : |g(x)| < \epsilon\}$ is a weak neighbourhood of 0 in B_X and hence contains $\mathcal{U}_n$ for some $n \in \mathbb{N}$. Let $Y = \bigcap_{f\in F_n} \ker f$, then for $x \in B_Y, x \in \mathcal{U}_n$, so, $g(x) < \epsilon$. So $\|g\restriction_{Y^*}\| \leqslant \epsilon$. Now $Y = \bigcap_{f\in F_n} \ker f \subseteq \ker(g - h)$, so by lemma 3.1 $g - h \in \operatorname{span} F_n \subseteq Z$ implies $d(g, z) \leqslant \epsilon$ which gives $g \in \overline{Z}$. □

Theorem 3.6 (Goldstine). For any normed space X, $\overline{B_X}^{w^*} = B_{X^{**}}$ ($\overline{B_X}^{w^*}$ is the closure in $(\mathbf{X}^{**}, \mathbf{w}^*)$ of B_X).

Proof. $B_{X^{**}}$ is weak-* closed (follows from Theorem 3.5) and $B_X \subseteq B_{X^{**}}$ so $\overline{B_X}^{w^*} \subseteq B_{X^{**}}$. Now let $\phi \in X^{**} \backslash \overline{B_X}^{w^*}$. Apply Theorem 3.3 (ii) to (X^{**}, w^*), $A = \{\phi\}, B = \overline{B_X}^{w^*}$ (show weak-* closure of convex set is closed). Now, there exists $f \in X^*$ s.t. $\phi(f) > \sup_B \hat{f}$ (real case), $[\mathrm{Re}(\phi(f))] > \sup_B \mathrm{Re}(\hat{f})$, $\|\phi\| \cdot \|f\| > \sup_{B_X} f$. So $\|\phi\| > 1$. ☐

Example:
Note that $\overline{X}^{w^*} = X^{**}$. So X separable implies X^* is weak-* separable. For instance, $\ell_\infty^* = \ell_1^{**}$ is weak-* separable, but ℓ_∞ is <u>NOT</u> separable.

Indeed, we have that the map

$$\psi : \ell_\infty \to \ell_1^*$$
$$x \mapsto \left(f_x : \ell_1 \to \text{ scalars} : y \mapsto \sum_{n \in \mathbb{N}} x_n y_n \right)$$

is an isometric isomorphism (in the norm topologies). It suffices to show that

$$(\ell_\infty^*, \sigma(\ell_\infty^*, \ell_\infty)) \xrightarrow{\phi} (\ell_1^{**}, \sigma(\ell_1^{**}, \ell_1^*))$$

is a homeomorphism. Observe that $\phi = (\psi^{-1})^*, \phi^{-1} = (\psi)^*$, both dual maps. ψ being an isometric isomorphism in the norm topology implies that the same holds for ϕ. By the previous observation, it suffices to show that for all $y \in \ell_1^*$, $\hat{y} \circ \phi : (\ell_\infty^*, \sigma(\ell_\infty^*, \ell_\infty)) \to$ scalars is continuous. Indeed, observe that for $f \in \ell_\infty^*$, $\hat{y} \circ \phi(f) = \phi(f)(y) = (\psi^{-1})^*(f)(y) = f(\psi^{-1}(y)) = \widehat{\psi^{-1}(y)}(f)$, and so $\hat{y} \circ \phi = \widehat{\psi^{-1}(y)}$, which is weak-* continuous by the universal property of the weak topology, hence we are done.

Lecture 14

Theorem 3.7. Let X be a Banach space. Then TFAE:

(i) X is reflexive.

(ii) (B_X, w) is compact.

(iii) X^* is reflexive.

Proof. <u>$(i) \implies (ii)$</u>: using the canonical embedding (a w–w^* homeomorphism), $(B_X, w) = (B_{X^{**}}, w^*)$ B_X is compact by Banach-Alaoglu (Theorem 3.5).

<u>$(ii) \implies (i)$</u>: $(B_X, w) = (B_{X^{**}}, w^*)$, so B_X is compact in the weak-* topology of X^{**}. So B_X is weak-* closed in X^{**}. By Goldstine, $B_{X^{**}} \supseteq \overline{B_X}^{w^*} = B_X$.

<u>$(i) \implies (iii)$</u>: $(B_{X^*}, w) = (B_{X^*}, w^*)$ by reflexivity and is compact by Theorem 3.5. By $(ii) \implies (i)$, X^* is reflexive.

<u>$(iii) \implies (i)$</u>: By what we have just proved, X^{**} is reflexive. By the implication $(i) \implies (ii)$, <u>$(B_{X^{**}}, w)$</u> is compact. Since, X is complete, X is closed in X^{**}, and hence weakly closed in X^{**} (by Mazur). Hence, $B_X = X \cap B_{X^{**}}$ is a weakly. closed subset of $B_{X^{**}}$ and thus weakly compact[11]. By $(ii) \implies (i)$, X is reflexive. ☐

Remark. If X is separable and reflexive, then (B_X, w) is compact, metrisable. Hence, B_X is weakly sequentially compact.

[11] $B_{X^{**}}$ is weak-* compact by Banach-Alaoglu and the map $\iota : (B_X, w) \to (\hat{B}_X, w^*)$ is a homeomorphism.

Lemma 3.3. Let (K,d) be a non-empty compact metric space. Then there exists a continuous surjection $\phi : \{0,1\}^{\mathbb{N}} \to K$, where $\{0,1\}^{\mathbb{N}}$ is given the product topology.

Proof. Since compact and metric imply totally bounded, that is if $A \subseteq K$ is non-empty, closed and $\epsilon > 0$, then there exist non-empty closed sets $B_1, \ldots, B_n$ s.t. $A = \bigcup_{j=1}^{n} B_j$ and $\operatorname{diam}(B_j) < \epsilon$ for all j.

Applying this[12], there exists a non-empty closed subset K_ϵ of K for all $\epsilon \in \Sigma = \bigcup_{n=1}^{\infty} \{0,1\}^n$ s.t. $K_\varnothing = K$, $K_\epsilon = K_{\epsilon,0} \cup K_{\epsilon,1}$ and $\max_{\epsilon\{0,1\}^n} \operatorname{diam} K_\epsilon \to 0$ as $n \to \infty$. Imagine some picture like the one below:

$$\begin{tikzcd}[column sep=small]
 & & & K \arrow[dll, no head] \arrow[drr, no head] & & & \\
 & K_0 \arrow[dl, no head] \arrow[dr, no head] & & & & K_1 \arrow[dl, no head] \arrow[dr, no head] & \\
K_{00} & & K_{01} & & K_{10} & & K_{11} \\
\iddots & & \vdots & & \vdots & & \ddots
\end{tikzcd}$$

Define $\phi : \{0,1\}^{\mathbb{N}} \to K$, $\phi((\epsilon_i)_{i=1}^{\infty}) =$ the unique point in $\bigcap_{n=1}^{\infty} K_{\epsilon_1,\ldots,\epsilon_n}$ (is well-defined by compactness and nestedness of K_ϵ's).

$\underline{\phi \text{ is onto}}$: given $x \in K$, inductively construct $\epsilon_1, \ldots, \epsilon_n$ s.t. for all n $x \in K_{\epsilon_1,\ldots,\epsilon_n}$.

$\underline{\phi \text{ is continuous}}$: for $\epsilon = (\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$, let $n \in \mathbb{N}$, then for all $\delta = (\delta_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ if $\delta_i = \epsilon_i$ for all $1 \leqslant i \leqslant$, then $d(\phi(d), \phi(\epsilon)) \leqslant \operatorname{diam} K_{\epsilon_1,\ldots,\epsilon_n} \to 0$ as $n \to \infty$. $\square$

Remark. $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the middle third Cantor set Δ via the map

$$(\epsilon_i)_{i=1}^{\infty} \mapsto \sum_{i=1}^{\infty} (2\epsilon_i) \cdot 3^{-i}.$$

Theorem 3.8. Every separable Banach space X embeds isometrically into $\mathcal{C}[0,1]$. So $\mathcal{C}[0,1]$ is isometrically universal for the class of separable Banach spaces ($\mathcal{SB}$).

Proof. From the proof of proposition 3.6 that $X \hookrightarrow \mathcal{C}(K)$ isometrically where $K = (B_{X*}, w^*)$. Since X is separable, K is metrisable. By lemma 3.3, there exists a continuous surjection $\phi : \Delta \to K$. Hence, $\mathcal{C}(K) \hookrightarrow \mathcal{C}(\Delta)$ isometrically via $f \mapsto f \circ \phi$. Also have $\mathcal{C}(\Delta) \hookrightarrow \mathcal{C}([0,1])$ isometrically via $f \mapsto \tilde{f}_1$.

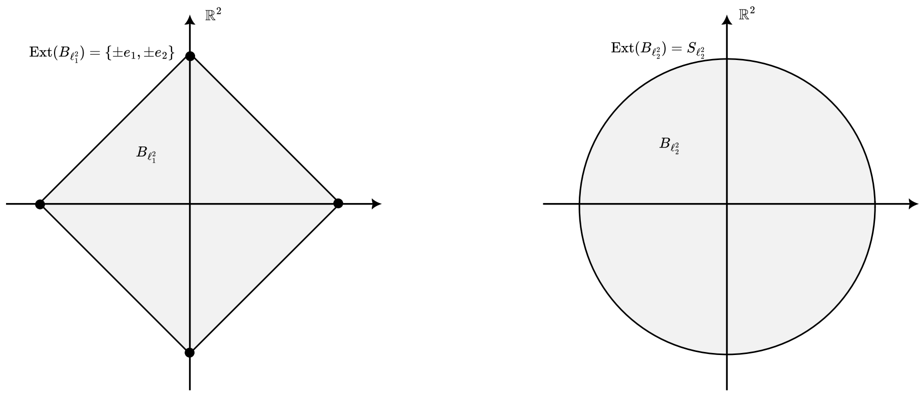
Write $[0,1]\backslash\Delta$ as a disjoint union $\bigcup_{n=1}^{\infty} (a_n, b_n)$. Then $\tilde{f}\restriction_\Delta = f$ for all n, $\tilde{f}$ is linear on $[a_n, b_n]$ with $\tilde{f}(a_n) = f(a_n), \tilde{f}(b_n) = f(b_n)$. $\square$

[12] at each branching point $\epsilon \in \Sigma$, can cover K_ϵ by balls of diameter $\operatorname{diam} K_\epsilon/2$, 'shedding balls' until only the intersection with one remains, hence halving the diameter in a finite depth and proceed like so recursively.

4 Convexity

Let X be a real or complex vector space and $K \subseteq X$ be a convex set. A point $x \in K$ is an extreme point of K if whenever $x = (1-t)y + tz$ for $t \in (0,1),\ y, z \in K$, we have $y = z = x$. Let $\operatorname{Ext} K$ be the set of extreme points of K.

Examples:



(a) $B_{\ell_1^2}$ (b) $B_{\ell_2^2}$

Figure 2: Above are displayed balls and their extreme points in ℓ_1^2, ℓ_1^2 respectively.

Furthermore, for the sequence space c_0, have that $\operatorname{Ext}(B_{c_0}) = \varnothing$.

Indeed, given $x = (x_n) \in B_{c_0}$. Fix $N \in \mathbb{N}$ s.t. $|x_N| < \frac{1}{2}$. Let $y_n = z_n = x_n$ for all $n \neq N \in \mathbb{N}$ and $y_N = x_N + \frac{1}{2}, z_N = x_N - \frac{1}{2}$. Then $y = (y_n)_{n\in\mathbb{N}}, z = (z_n)_{n\in\mathbb{N}} \in B_{c_0}$ and $x = \frac{1}{2}y + \frac{1}{2}z, y \neq x, z \neq x$.

Theorem 4.1 (Krein-Milman). Let $(X, \mathcal{P})$ be a LCS. Let K be a compact, convex subset of X. Then $K = \overline{\operatorname{conv}}(\operatorname{Ext} K)$. In particular, $\operatorname{Ext} K \neq \varnothing$ provided $K \neq \varnothing$.

Corollary 4.1.1. If X is a normed space, then $B_{X*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{Ext} K)$ and $\operatorname{Ext} B_{X*} \neq \varnothing$. Note c_0 is not a dual spce isometrically, i.e. there exists no normed space X s.t. $c_0 \cong X^*$.

Definition 4.1. Let K be a compact convex set in a LCS $(X, \mathcal{P})$. A face of K is a non-empty, compact convex set $E \subseteq K$ s.t. if $y, z \in K$, $t \in (0,1)$, $(1-t)y + tz \in E$, then $y, z \in E$.

Examples:

1. K is a face of K. For $x \in K$, $x \in \operatorname{Ext} K \iff \{x\}$ is a face of K.
2. let $f \in X^*$, $\alpha = \sup\limits_K f$, $E = \{x \in K : f(x) = \alpha\}$ is a face.
   ($E \neq \varnothing$, convex, compact and if $y, z \in K$, $t \in (0,1)$ and $(1-t)y + tz \in E$, then $\alpha = f((1-t)y + tz) = (1-t)f(y) + tf(z) \geqslant \alpha$ giving equality, hence $f(y) = f(z) = \alpha$, hence $y, z \in E$).

   [In the complex case, use $\operatorname{Re} f$. From now on, we only use real scalars.]
3. Let E be a face of K. If F is a face of E, then F is a face of K. So if $x \in \operatorname{Ext} E$, then $x \in \operatorname{Ext} K$.

Proof. Proof of Theorem 4.1 Let E be a face of K. We show $\operatorname{Ext} E \neq \varnothing$.

By Zorn, lemma 1.1, there exists a minimal (wrt inclusion) face F of E. If $|F| > 1$, then pick $x \neq y \in F$ and $f \in X^*$ s.t. $f(x) > f(y)$ (by Hahn-Banach). Then $\mathbb{G} = \{z \in F : f(z) = \sup_F f\}$ is a face of $F, y \notin \mathcal{G}$ so $\mathcal{G} \subsetneq F$, a contradiction. So F is a singleton which means $\operatorname{Ext} E \neq \varnothing$.

Now, let $L = \overline{\operatorname{conv}} \operatorname{Ext} K$. then $L \neq \varnothing$, convex, compact, $L \subseteq K$. Assume $x_0 \in K \backslash L$. By Theorem 3.2, there exists $f \in X^*$ s.t. $f(x_0) > \sup_L f$. Let $\alpha = \sup_K f$, then $E = \{x \in K : f(x) = \alpha\}$ is a face of K. So there's an extreme point z of K with $X \in E$. Since $\alpha \geqslant f(x_0)$, $E \cap L \neq \varnothing$, a contradiction. So $z \notin L$. □

Lecture 15

Lemma 4.1. let $(X, \mathcal{P})$ be a LCS, let $K \subseteq X$ be compact and $x_0 \in K$. Then for a neighbourhood $\mathcal{V}$ of x_0 in X, there exist $f_1, \ldots, f_n \in X^*$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ s.t. $x_0 \in \{x \in X : f_i(x) < \alpha_i, 1 \leqslant i \leqslant n\} \cap K \subseteq \mathcal{V}$.

Proof. let τ be the topology of X defined by $\mathcal{P}$ let $\sigma = \sigma(X, X^*)$. Then $\operatorname{Id} : (K, \tau) \to (K, \sigma)$ is a continuous bijection ($\sigma \subseteq \tau$) from compact to Hausdorff (as X^* separates points of X by Hahn-Banach), so it is a homeomorphism, i.e. $\sigma = \tau$ on K.

□

Lemma 4.2. let $(X, \mathcal{P})$ be a LCS, let $K \subseteq X$ be compact and convex. $x_0 \in \operatorname{Ext} K$. Then for a neighbourhood $\mathcal{V}$ of x_0 in X, there exists $f \in X^*$, $\alpha \in \mathbb{R}$ s.t. $x_0 \in \{x \in X : f(x) < \alpha\} \cap K \subseteq \mathcal{V}$.

Proof. Let $n, f_1, \ldots, f_n \in X^*$, $\alpha_1, \ldots, \alpha_n$ be as in lemma 4.1 and $K_1 = \{x \in K : f_i(x) \geqslant \alpha_i\}$. This is compact and convex. Observe $\bigcup_{i=1}^n K_i \supseteq K \backslash \mathcal{V}$ and $x_0 \notin \bigcup_{i=1}^n K_i$. Also,

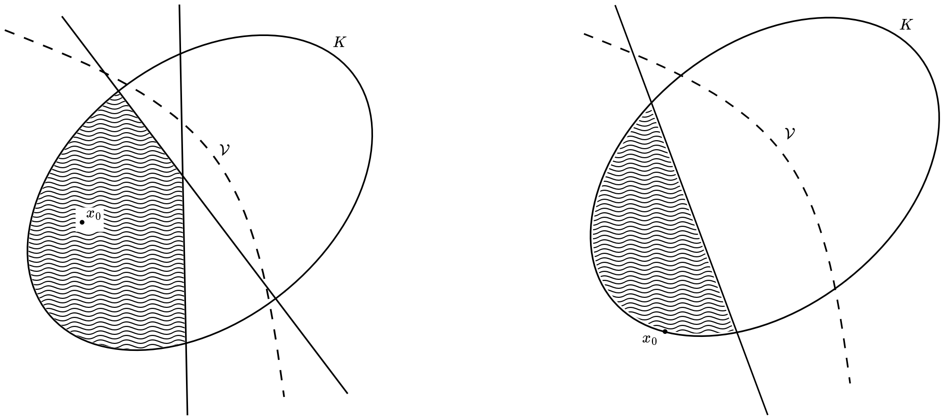
$$\operatorname{conv} \bigcup_{i=1}^n K_i = \left\{ \sum_{i=1}^n t_i x_i : x_i \in K_i, t_i \geqslant 0, \sum_{i=1}^n t_i = 1 \right\}.$$

Since x_0 is an extreme point of K, $x_0 \notin \operatorname{conv} \bigcup_{i=1}^n K_i$ (the case $n = 2$ is true by definition, and use induction to arrive at the general case).

Furthermore,

$$K_1 \times \ldots K_n \times \left\{ (t_i) \in \mathbb{R}^n : t_i \geqslant 0 \forall i, \sum_{i=1}^n t_i = 1 \right\}$$

is compact and $(x_1, \ldots, x_n, (t_i)_{i=1}^n) \mapsto \sum_{i=1}^n t_i x_i$ is continuous (algebraic operaitons "$+, \times$" are continuous in LCS), so the image $B = \operatorname{conv} \bigcup_{i=1}^n K_i$ is compact. By Theorem 3.2, there exists $f \in X^*$ s.t. $f(x_0) < \inf_B f$. Choose $\alpha \in \mathbb{R}$ with $f(x_0) < \alpha < \inf_B f$. Then $x_0 \in \{x \in X : f(x) < \alpha\} \cap K$, which is disjoint from B and hence from $\bigcup_{i=1}^n K_i$ and so is contained in $\mathcal{V}$. □



(a) Illustration of lemma 4.1.

(b) Illustration of lemma 4.2.

Figure 3

Theorem 4.2. Let $(X, \mathcal{P})$ be a locally convex space, $K \subseteq X$ compact, convex and $S \subseteq K$. If $K = \overline{\text{conv}} S$, then $\overline{S} \supseteq \text{Ext}\, K$.

Remark. The closure is necessary. For instance, let S be a dense subset of $S_{\ell_2^2}$. Then $\overline{\text{conv}} S_{\ell_2^2} = B_{\ell_2^2}$ and $\text{Ext}\, B_{\ell_2^2} = S_{\ell_2^2}$. Also, $\text{Ext}\, K$ need not be closed. E.g. in $\mathbb{R}^3$,

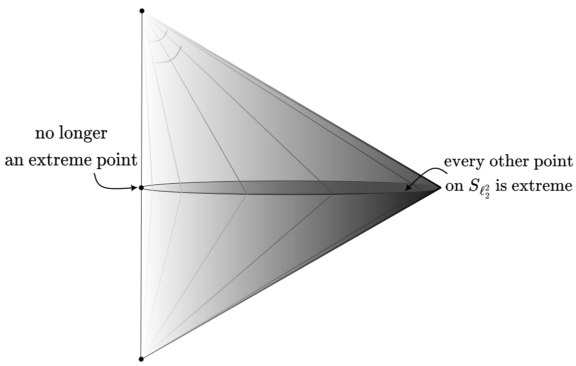


Figure 4: Illustration of extreme points of a double cone in $\mathbb{R}^3$ (which include top and bottom vertices).

Proof. Proof of Theorem 4.2 Assume $x_0 \in \text{Ext}\, K \backslash \overline{S}$. Apply lemma 4.2 with $\mathcal{V} = X \backslash \overline{S}$. So, $f \in X^*$, $\alpha \in \mathbb{R}$ s.t. $x_0 \in \{x \in X : f(x) < \alpha\} \cap K \subseteq \mathcal{V}$. Then, $L = \{x \in K : f(x) \geqslant \alpha\}$ is compact, convex with $L \supseteq S$. Hence, $L \supseteq \overline{\text{conv}} S = K$, a contradiction since $x_0 \notin L$. Thus, $x_0 \in \overline{S}$. $\square$

Remark. One can show that $\text{Ext}\, B_{\mathcal{C}(K)^*} = \{\lambda \delta_k : |\lambda| = 1, k \in K\}$ $(\delta_k(f) = f(k))$, where K is compact, Hausdorff. Can use Theorem 4.2 for "$\subseteq$".

Theorem 4.3 (Banach-Stone). Let K, L be compact, Hausdorff spaces, then $\mathcal{C}(K) \cong \mathcal{C}(L)$ $\iff$ L and K are homeomorphic.

Proof. "$\underline{\Longleftarrow}$": If $\phi : K \to L$ is a homeomorphism then

$$\begin{aligned} \phi^* : \mathcal{C}(L) &\cong \mathcal{C}(K) \\ f &\mapsto f \circ \phi \end{aligned}$$

is an isometric isomorphism.

<u>" $\implies$ "</u>: let $T : \mathcal{C}(L) \cong \mathcal{C}(K)$ be an isometric isomorphism. Then so is its dual $T^* : \mathcal{C}(K)^* \cong \mathcal{C}(L)^*$. So $T^*(B_{\mathcal{C}(K)^*}) = B_{\mathcal{C}(L)^*}$ and $T^*(\operatorname{Ext} B_{\mathcal{C}(K)^*}) = \operatorname{Ext} B_{\mathcal{C}(L)^*}$. Thus, for each $k \in K$, $T^*(\delta_k) = \lambda(k) \cdot \delta_{\phi(k)}$ for some scalar $\lambda(k), |\lambda(k)| = 1$ and some $\phi(k) \in L$. So we have functions

$$\lambda : K \to \text{ scalars}$$
$$\phi : K \to L$$

Now, for all $k \in K$, $\lambda(k) = T^*(\delta_k)(\mathbf{1}_L) = T(\mathbf{1}_L)(k)$, which means $\lambda = T(\mathbf{1}_L) \in \mathcal{C}(K)$, so λ is continuous. Recall, $\delta : K \to (\mathcal{C}(L)^*, w^*)$ is continuous (indeed, it is a homeomorphism between K and $\delta(K)$). Also, $T^* : \mathcal{C}(K)^* \to \mathcal{C}(L)^*$ is $w * -w^*$ continuous. hence, $h \mapsto \overline{\lambda(k)} \cdot T^*(\delta_k) = \delta_{\phi(k)} : K \to (\mathcal{C}(L)^*, w^*)$ is continuous. Since $\phi : K \xrightarrow{T^*} (\delta(L), w^*) \xrightarrow{\delta^{-1}} L$ is a composition of continuous maps, hence continuous.

<u>ϕ is into</u>: Assume $\phi(k_1) = \phi(k_2)$. So $\overline{\lambda(k)} \cdot T^*(\delta_{k_1}) = \overline{\lambda(k)} \cdot T^*(\delta_{k_1})$. Evaluate at $T^{-1}(\mathbf{1}_K)$ to get $\lambda(k_1) = \lambda(k_2)$ and so $\delta_{k_1} = \delta_{k_2}$ (as T^* is injective) which finally gives $k_1 = k_2$.

<u>ϕ is onto</u>: Given $l \in L$, since T^* is onto, there exists a scalar μ, $|\mu| = 1, k \in K$ s.t. $T^*(\mu\delta_k) = \delta_l$. So $\mu\lambda(k)\delta_{\phi(k)} = \delta_l$. Evaluate at $\mathbf{1}_L$ to get $\mu\lambda(k) = 1$ and so $\phi(k) = l$. $\square$

5 Banach Algebras

A real or complex algebra is a real or resp. complex vector space A with multiplication $A \times A :\rightarrow A$, $(a, b) \mapsto a \cdot b$ s.t.

(i) $a(bc) = (ab)c$

(ii) $a(b + c) = ab + ac$, $(a + b) \cdot c = ac + bc$

(iii) $\lambda(ab) = (\lambda a)b = a(\lambda b)$

for all $a, b, c \in A, b$ scalar.

A is unital if there exists $\mathbf{1} \in A$ s.t. $1 \neq 0$ and for all $x \in A$ $\mathbf{1}a = a\mathbf{1} = a$. This element is unique, called the unit of A.

An algebra norm on A is a norm on A s.t. for all $a, b \in A$, $\|ab\| \leqslant \|a\| \cdot \|b\|$. A normed algebra is an algebra with an algebra norm. note that multiplication is continuous (as well as addition and scalar multiplication). A Banach algebra (BA) is a complete normed algebra.

A unital normed algebra is a normed algebra, A with an element $\mathbf{1} \in A$ s.t. for all $x \in A$, $\mathbf{1}a = a\mathbf{1} = a$ and s.t. $\|\mathbf{1}\| = 1$ ($\|\mathbf{1}\| \leqslant \|\mathbf{1}\| \cdot \|\mathbf{1}\|$ and $1 \leqslant \|\mathbf{1}\|$). If A is a normed algebra which is also a unital algebra (but not assuming $\|\mathbf{1}\| = 1$), then $|||a||| = \sup\{\|ab\| : \|b\| \leqslant 1\}$ defines an equivalent norm on A that makes A a unital normed algebra.

A unital Banach algebra is a complete unital normed algebra. A linear map $\theta : A \to B$ between algebras is a homomorphism if for all $a, b \in A$ $\theta(ab) = \theta(a) \cdot \theta(b)$. If in addition A and B are unital with units $\mathbf{1}_A$ and $\mathbf{1}_B$ and $\theta(\mathbf{1}_A) = \mathbf{1}_B$, then θ is a unital homomorphism. In the category of normed algebras, an isomorpism will mean a continuous homomorphism with continuous inverse. BUT, homomorphisms are not assumed continuous.

Lecture 16 Note: from now on, the scalar field is $\mathbb{C}$.

Examples:

1. $\mathcal{C}(K)$, K compact Hausdorff, is a commutative, unital BA with pointwise multiplication in the uniform norm.

2. Let K be compact, Hausdorff, A uniform algebra on K is a closed sub-algebra of $\mathcal{C}(K)$ that separates points of K and contains the constant functions.

3. The disk algebra $A(\Delta) = \{f \in \mathcal{C}(\Delta) : f$ holomorphic on the interior of $\Delta\}$, $\Delta = \{z \in \mathbb{C} : |z| \leqslant 1\}$.

   More generally, let $K \subseteq \mathbb{C}, K \neq \varnothing$ compact. We have the following uniform algebras on $K : \mathcal{P}(K) \subseteq \mathcal{R}(K) \subseteq \mathcal{O}(K) \subseteq A(K) \subseteq \mathcal{C}(K)$, where $\mathcal{P}(K), \mathcal{R}(K), \mathcal{O}(K)$ are the closures in $\mathcal{C}(K)$ of respectively, polynomials, rational functions with no pole in K, functions holomorphic on some open neighbourhood of K. $A(K) = \{f \in \mathcal{C}(K) : f$ holomorphic on $\text{int}(K)\}$. Later, $\mathcal{R}(K) = \mathcal{O}(K)$ say, $\mathcal{R}(K) = \mathcal{R}(K)$ if and only of $\mathbb{C}\backslash K$ is connected. In general $A(K) \neq \mathcal{O}(K)$, $A(K) = \mathcal{C}(K) \iff \text{int}(K) = \varnothing$.

4. $L_1(\mathbb{R})$ with the L_1-norm and convolution $f * g(x) = \displaystyle\int_{\mathbb{R}} f(y)g(x-y)dy$ is a commutative Banach algebra without a unit (Riemann-Lebesgue lemma).

5. If X is a Banach space, then $\mathcal{B}(X)$ with composition an operator norm is a unital Banach algebra. It is not commutative if $dimX > 1$.

   special case: if X is a Hilbert space, then $\mathcal{B}(X)$ is a C^*-algebra (see later).

5.0.1 Elementary constructions

1. If A is a unital algebra with unit $\mathbf{1}$, then a unital sub-algebra is a sub-algebra B of A s.t. $\mathbf{1} \in B$. If A is a normed algebra, then the closure of a sub-algebra of A is a sub-algebra of A.

2. Unitisation: The unitisation of an algebra A is the vector space direct sum $A_+ = A \bigoplus \mathbb{C}$ with multiplication $(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda, \mu)$. Then A_+ is a unital algebra with unit $\mathbf{1} = (0, 1)$.

   The ideal $\{(a, 0) : a \in A\}$ is isomorphic to A and will always be identified with $A/$ We can weite $A = \{a + \lambda\mathbf{1} : a \in A, \lambda \in \mathbb{C}\}$. If A is a normed algebra, then A_+ becomes a unital normed algebra with $\|a + \lambda\mathbf{1}\| = \|a\| + |\lambda|$. Then A is a closed ideal of A_+. If A is a Banach algebra, then A_+ is a unital Banach algebra.

3. The closure of an ideal of a normed algebra is an ideal. If $\mathcal{J}$ is a closed ideal of the normed algebra of A, then $A\backslash\mathcal{J}$ is a normed algebra in the quotient norm. If A is a unital normed algebra and $\mathcal{J}$ is a proper closed ideal of $A(\mathcal{J} \neq A)$, then $A\backslash\mathcal{J}$ is a unital normed algebra with $\mathbf{1} + \mathcal{J}$ ($\|\mathbf{1} + \mathcal{J}\| \leqslant \|\mathbf{1}\| = 1$ and $\|\mathbf{1} + \mathcal{J}\| \geqslant 1$ from an earlier observation).

4. let $\tilde{A}$ be the Banach space completion of a normed algebra. Then $\tilde{A}$ is a Banach algebra with the following multiplication: given $a, b \in \tilde{A}$, choose sequences $(a_n), (b_n)$ in A s.t. $a_n \to a, b_n \to b$ and define $a \cdot b = \lim_{n\to\infty} a_n \cdot b_n$.

5. Let A be a unital Banach algebra. Let $X = A$ thought of as a Banach space. For $a \in A$, define $L_a : X \to X$, $L_a(x) = a \cdot x$. Then $L_a \in \mathcal{B}(X)$ and $\|L_a\| = \|a\|$. The map $L : A \to \mathcal{B}(X)$, $a \mapsto L_a$, is an isometric unital HM (homomorphism).

Lemma 5.1. Let A be a unital Banach algebra and $a \in A$. Ift $\|\mathbf{1} - a\| < 1$, then a is invertible (there exists $b \in A$ s.t. $ab = ba = \mathbf{1}$) and $\|a^{-1}\| \leqslant \frac{1}{1-\|\mathbf{1}-a\|}$.

Proof. For all $n \in \mathbb{N}$, $\|(\mathbf{1} - a)^n\| \leqslant \|\mathbf{1} - a\|^n$, so $\sum_{n=0}^{\infty} \|(\mathbf{1} - a)^n\| < \infty$. Hence, $\sum_{n=0}^{\infty} (1 - a)^n$ converges ($(1 - a)^0 = 1$).

Let $b = \sum_{n=0}^{\infty} (\mathbf{1} - a)^n$. Then $(\mathbf{1} - a)b = b(\mathbf{1} - a) = \sum_{n=1}^{\infty} (\mathbf{1} - a)^n = b - 1$, and so $ab = ba = \mathbf{1}$. So, $b = a^{-1}$ and $\|a^{-1}\| = \|b\| \leqslant$ Let $b = \sum_{n=0}^{\infty} (\mathbf{1}-a)^n$. Then $(\mathbf{1}-a)b = b(\mathbf{1}-a) = \sum_{n=1}^{\infty} (\mathbf{1}-a)^n = b-1$, and so $ab = ba = \mathbf{1}$. So, $b = a^{-1}$ and $\|a^{-1}\| = \|b\| \leqslant \sum_{n=1}^{\infty} \|(\mathbf{1} - a)^n\| \leqslant \sum_{n=1}^{\infty} \|\mathbf{1} - a\|^n = \dfrac{1}{1 - \|\mathbf{1} - a\|}$. □

Notation: we let $\mathcal{G}(A)$ denote the group of invertibles of a unital algebra A.

Corollary 5.0.1. Let A be a unital Banach algebra.

(i) $\mathcal{G}(A)$ is open in A.

(ii) $x \mapsto x^{-1}$ is a continuous function on $\mathcal{G}(A)$.

(iii) Assume $(x_n) \subseteq \mathcal{G}(A)$, $x_n \to x \in A\backslash\mathcal{G}(A)$. Then $\|x_n^{-1}\| \to \infty$ as $n \to \infty$.

(iv) If $x \in \partial\mathcal{G}(A) = \overline{\mathcal{G}(A)}\backslash\mathcal{G}(A)$, then there exists (z_n) in A s.t. $\|z_n\| = 1$ for all n and $z_n \cdot x \to 0$ and $x \cdot z_n$ as $n \to \infty$. It follows that x has no left or right inverse in A, not even in any unital algebra B containing A as a (not necessarily unital) sub-algebra.

Proof. (i) Let $x \in \mathcal{G}(A)$. If $y \in A$ and $\|y - x\| \leqslant \frac{1}{\|x^{-1}\|}$, then $\|\mathbf{1} - x^{-1}y\| = \|x^{-1}(x - y)\| \leqslant \|x^{-1}\| \cdot \|x - y\| < 1$. Hence, by lemma 5.1, $x^{-1}y \in \mathcal{G}(A)$, which implies that $y = x \cdot x^{-1}y \in \mathcal{G}(A)$.

(ii) Let us fix $x \in \mathcal{G}(A)$. For $y \in \mathcal{G}(A)$ $y^{-1} - x^{-1} = y^{-1}(x-y)x^{-1}$ so $\|y^{-1} - x^{-1}\| \leqslant \|y^{-1}\| \cdot \|x^{-1}\| \cdot \|x-y\|$. If $\|x-y\| < \frac{1}{2\|x^{-1}\|}$, then $|\, \|y^{-1}\| - \|x^{-1}\| \,| \leqslant 2 \cdot \|x^{-1}\|^2 \cdot \|x-y\| \to 0$ as $y \to x$.

(iii) From proof of (i), if $\|x - x_n\| < \frac{1}{x_n^{-1}}$, then $x \in \mathcal{G}(A)$, a contradiction. So $\|x - x_n\| \geqslant \frac{1}{x_n}$. Since, $\|x - x_n\| \to 0$, the result follows.

(iv) Given $x \in \partial\mathcal{G}(A)$, there exists a sequence $(x_n) \subseteq \mathcal{G}(A)$, $x_n \to x$. By part (iii) $\|x_n\| \to \infty$, let $z_n = \frac{x_n^{-1}}{\|x_n^{-1}\|}$, for all $n \in \mathbb{N}$. Then $z_n x = z_n x_n + z_n(x - x_n) = \frac{1}{\|x_n^{-1}\|} + z_n(x - x_n) \to 0$, by the above and since $\|z_n(x - x_n)\| \leqslant \|z_n\| \cdot \|x - x_n\| \to 0$. Similarly, $xz_n \to 0$.

Assume that B is a unital BA and A is a sub-algebra of B. If $y \in A$ and $yx = \mathbf{1}_B$, then $yxz_n = z_n$. So $\|z_n\| = 1 = \|yxz_n\| \leqslant \|y\| \cdot \|xz_n\|$, $n \to \infty$, a contradiction. Similarly, there is no $y \in B$ s.t. $xy = \mathbf{1}_B$.

□

Lecture 17

Definition 5.1. Let A be an algebra (always complex) and let $x \in A$. The spectrum $\sigma_A(x)$ of x in A is defined as follows: if A is unital, then $\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin \mathcal{G}(A)\}$ and if A is non-unital then $\sigma_A(x) := \sigma_{A_+}(x)$.

Examples:

1. $A = M_n(\mathbb{C})$, $x \in A$, $\sigma_A(x)$ is the set of eigenvalues (evals) of x.
2. $A = \mathcal{C}(K)$, K compact Hausdorff, $f \in A$, $\sigma_A(f) = f(K)$.
3. X a Banach space, $A = \mathcal{B}(X)$, $T \in A$, then $\sigma_A(T) = \{\lambda \in \mathbb{C} : \lambda \operatorname{Id} - T \text{ not an isomorphism}\}$.

Theorem 5.1. Let A be a Banach algebra, $x \in A$. Then $\sigma_A(x)$ is a non-empty, compact subset of $\{\lambda \in \mathbb{C} : |\lambda| \leqslant \|x\|\}$.

Proof. Wlog, A is a unital Banach algebra. If $|\lambda| > \|x\|$, then $\|x\| < 1$, so by lemma 5.1, $\mathbf{1} - \frac{x}{\lambda} \in \mathcal{G}(A)$ and so $\lambda\mathbf{1} - x = \lambda(\mathbf{1} - \frac{x}{\lambda}) \in \mathcal{G}(A)$. Hence, $\sigma_A(x) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leqslant \|x\|\}$. Also, $\sigma_A(x)$ is the inverse image of the closed set $A \backslash \mathcal{G}(A)$ (corollary 5.0.1(i)) under the continuous function $\lambda \mapsto \mathbb{C} \to A : \lambda\mathbf{1} - x$ and hence $\sigma_A(x)$ is closed. It follows that $\sigma_A(x)$ is compact.

$\sigma_A(x)$ is non-empty: consider $f : \mathbb{C} \backslash \sigma_A(x) \to A$, $f(\lambda) = (\lambda\mathbf{1} - x)$. By corollary 5.0.1$(ii)$ f is continuous and for $\lambda \neq \mu$:

$$
\begin{aligned}
f(\lambda) - f(\mu) = f(\lambda)((\mu\mathbf{1} - x) - (\lambda\mathbf{1} - x))f(\mu) \\
&= f(\lambda)(\mu - \lambda)f(\mu) \\
&= (\mu - \lambda)f(\lambda)f(\mu).
\end{aligned}
$$

So $\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -f(\lambda)f(\mu) \to -f(\mu)^2$ as $\lambda \to \mu$ because f is continuous. Thus, f is holomorphic. If $|\lambda| > \|x\|$ then $\lambda\mathbf{1} - x \in \mathcal{G}(A)$ and $\|(\lambda\mathbf{1} - x)^{-1}\| = \frac{1}{|\lambda|}\left\|\left(\mathbf{1} - \frac{x}{\lambda}\right)^{-1}\right\| \leqslant \frac{1}{|\lambda|}\frac{1}{1 - \|\frac{x}{\lambda}\|} = \frac{1}{|\lambda| - \|x\|} \to 0$ as $|\lambda| \to \infty$. If $\sigma_A(x)$ where empty, then f is a bounded entire function, so by vector-valued Liouville, f is constant, and since $f(\lambda) \to 0$ as $|\lambda| \to \infty$, $f \equiv 0$, a contradiction. □

Corollary 5.1.1 (Gelfand-Mazur). A complex unital normed division ($\mathcal{G}(A) = A \backslash \{0\}$) algebra is isometrically isomorphic to $\mathbb{C}$.

Proof. Let us define the map $\theta : \mathbb{C} \to A$, $\theta(\lambda). = \lambda \cdot \mathbf{1}$. then θ is an isomtric homomorphism. To show that it is onto, fix any $x \in A$. Let B be the completion of A. Then B is a unital Banach algebra. Then by Theorem 5.1, $\sigma_B(x)$ is non-empty which implies that there exists $\lambda \in \mathbb{C}$ s.t. $\lambda\mathbf{1} - x$ is NOT invertible in B, hence $\lambda\mathbf{1} - x$ is not in $\mathcal{G}(A)$ which means that $\lambda\mathbf{1} - x = 0$ and so $\theta(\lambda) = x$. □

Definition 5.2 (Spectral radius). Let A be a Banach algebra and $x \in A$. The spectral radius $r_A(x)$ of x in A is $r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$. From Theorem 5.1, $r_A(x)$ is well-defined and $r_A(x) \leqslant \|x\|$.

Note: let x, y be comuting elements of a unital algebra A. Then $x \cdot y \in \mathcal{G}(A) \iff x \in \mathcal{G}(A)$ and $y \in \mathcal{G}(A)$ (use the fact that $z(xy) = (xy)z = 1$ gives $yzx = yzx \cdot yxz = yxz = 1$).

Lemma 5.2 (Spectral Mapping Theorem for polynomials). Let A be a unital Banach algebra and $x \in A$. Then for a complex polynomial $p = \sum_{k=0}^{n} a_k z^k$ we have

$$\sigma_A(p(x)) = \{p(\lambda) : \lambda \in \sigma_A(x)\} = p(\sigma_A(x))$$

where $p(x) = \sum_{k=0}^{n} a_k z^k$ and $x^0 = \mathbf{1}_A$.

Proof. Wlog $n \neq 1$ and $a_n \neq 0$ ($\sigma_A(\lambda\mathbf{1}) = \{\lambda\}$). Fix $\mu \in \mathbb{C}$. Write $\mu - p(z) = c \cdot \prod_{k=1}^{n}(\lambda_k - z)$ for some $c, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $c \neq 0$. note that $\{\lambda : p(\lambda) = \mu\} = \{\lambda_1, \ldots, \lambda_n\}$. Now $\mu \notin \sigma_A(p(x)) \iff \mu\mathbf{1} - p(x) = \prod_{k=1}^{n}(\lambda_k\mathbf{1} - x)$ is invertible $\iff \lambda_k - x\lambda_k\mathbf{1} - x$ is invertible (use previous note on commutativity and invertibility) $\iff$ there exists no $\lambda \in \sigma_A(x)$ s.t. $p(\lambda) = \mu$. The result now follows. $\square$

Theorem 5.2 (Beurling-Gelfand Spectral Radius Formula (SRF)). Let A be a unital Banach algebra, $x \in A$. Then

$$r_A(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}}.$$

Proof. Wlog A is unital. By lemma 5.2, if $\lambda \in \sigma_A(x)$ and $n \in \mathbb{N}$, then $\lambda^n \in \sigma(x^n)$. By Theorem 5.1, $|\lambda^n| \leqslant \|x^n\|$ and $|\lambda| \leqslant \|x^n\|^{\frac{1}{n}}$. It thus follows that $r_A(x) \leqslant \inf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}}$.

Consider $f : \mathbb{C}\backslash\sigma_A(x) \to A, f(\lambda) = (\lambda\mathbf{1} - x)^{-1}$, by the proof of Theorem 5.1, f is holomorphic. Note that $\mathbb{C}\backslash\sigma_A(x) \supseteq \{|\lambda| > r_A(x)\} \supseteq \{\lambda : |\lambda| > \|x\|\}$. If $|\lambda| > \|x\|$, then $f(\lambda) = \frac{1}{\lambda}\left(\mathbf{1} - \frac{x}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{x^n}{\lambda^n}$ (by the proof of lemma 5.1).

Fix $\phi \in A^*$ (Banach space dual). Then $\phi \circ f$ is holomorphic on $\mathbb{C}\backslash\sigma_A(x)$ and if $|\lambda| > \|x\|$, then $\phi \circ f(\lambda) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{\phi(x^n)}{\lambda^n}$. This Laurent expansion must also be valid on $\{|\lambda| > r_A(x)\}$. So for $|\lambda| > r_A(x)$ and for $\phi \in A^*$, $\phi\left(\frac{x^n}{\lambda^n}\right) \to 0$ as $n \to \infty$. So for $|\lambda| > r_A(x)$, $\frac{x^n}{\lambda^n} \xrightarrow{w} 0$. By proposition 3.5, there exists $M > 0$ s.t. for all $n \in \mathbb{N}$, $\left\|\frac{x^n}{\lambda^n}\right\| \leqslant M^{\frac{1}{n}}$ and so $\limsup \|x^n\|^{\frac{1}{n}} \leqslant |\lambda|$. We have thus proved that $r_A(x) \leqslant \inf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leqslant \liminf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leqslant \limsup_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leqslant r_A(x)$. $\square$

Theorem 5.3. Let A be a unital Banach algebra and B be a closed, unital sub-algebra of A. Let $x \in B$. Then, $\sigma_B(x) \supseteq \sigma_A(x)$ and $\partial\sigma_B(x) \subseteq \sigma_A(x)$. It follows that $\sigma_B(x)$ is the union of $\sigma_A(x)$ and some of the bounded components of $\mathbb{C}\backslash\sigma_A(x)$.

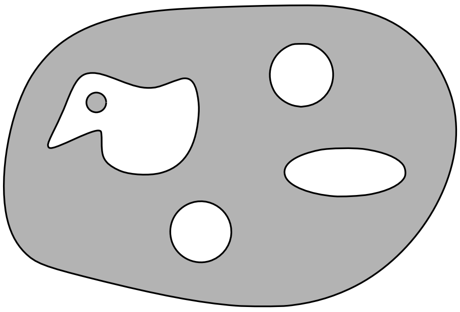
Before we proceed with the proof of the above, we prove a topological lemma.

Lemma 5.3. Suppose $\mathcal{V}$ and $\mathcal{W}$ are open sets in some topological space X s.t. $\mathcal{V} \subseteq \mathcal{W}$ and $\mathcal{W}$ contains non boundary points of $\mathcal{V}$. Then $\mathcal{V}$ is a union of components of $\mathcal{W}$.

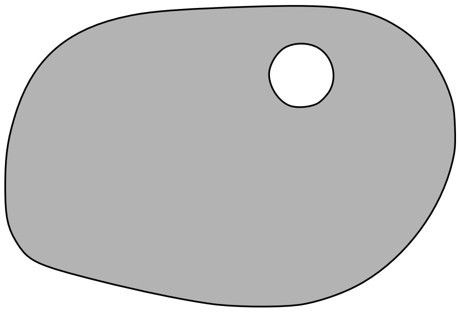
Proof. Let Ω be a component of W that intersects $\mathcal{V}$. Let $\mathcal{U}$ be the complement of $\overline{\mathcal{V}}$. Since $\mathcal{W}$ contains no boundary point of $\mathcal{V}$, Ω is the union of two disjoint open sets $\Omega \cap \mathcal{V}$ and $\Omega \cap \mathcal{U}$. Since Ω is connected, $\Omega \cap \mathcal{U}$ is empty and so it follows that $\Omega \subseteq \mathcal{V}$. □

Proof of Theorem 5.3. $\sigma_B(x) \supseteq \sigma_A(x)$ holds since an element invertible in B is also invertible in A. Let $\lambda \in \partial\sigma_B(x)$. then, there exist $(\lambda_n) \subseteq \mathbb{C}\backslash\sigma_B(x)$ s.t. $\lambda_n \to \lambda$. So $\lambda_n\mathbf{1} - x \in \mathcal{G}(B)$ and $\lambda_n\mathbf{1} - x \to \lambda\mathbf{1} - x \in B\backslash\mathcal{G}(B)$, which means $\lambda\mathbf{1} - x \in \partial\mathcal{G}(B)$. By corollary 5.0.1($iv$), $\lambda\mathbf{1} - x$ is not invertible in A, that is $\lambda \in \sigma_A(x)$.

To conclude, let Ω_A, Ω_B be the complements in $\mathbb{C}$ of $\sigma_A(x), \sigma_B(x)$ respectively. The preceding discussion implies that $\partial\Omega_B \subseteq \sigma_A(x)$ and so can use the topological lemma with $\mathcal{V} = \Omega_B, \mathcal{W} = \Omega_A$. Thus, Ω_B is the union of components of Ω_A. This means that $\sigma_B(x)$ is the union of $\sigma_A(x)$ and some bounded components of $\Omega_A = \sigma_A(x)\backslash\sigma_A(x)$. □



(a) $\sigma_A(x)$



(b) $\sigma_B(x)$

Figure 5: Illustration of Thheorem 5.3 for a sub-algebra $B \subseteq A$, $x \in B$.

Lecture 18

Proposition 5.1. Let A be a unital Banach algebra and C a maximal commutative sub-algebra of A (wrt inclusion). Then C is a unital closed sub-algebra of A. Moreover, for all $x \in C$, $\sigma_C(x) = \sigma_A(x)$.

Proof. $\overline{C}$ is a commutative sub-algebra of A. $\overline{C} \supseteq C$ and by maximality $\overline{C} = C$ is closed.
$C + \mathbb{C}\cdot\mathbf{1}$ is a commutative sub-algebra of A contains C, so by maximality $C = C + \mathbb{C}\cdot\mathbf{1}$, i.e. $\mathbf{1} \in C$. Fix $x \in C$. We know that $\sigma_C(x) \supseteq \sigma_A(x)$. Assume $\lambda \in \mathbb{C}\backslash\sigma_A(x)$/ Let $y = (\lambda\mathbf{1} - x)^{-1}$ (in A). Have for all $z \in C$, $z(\lambda\mathbf{1} - x) = (\lambda\mathbf{1} - x)z$ as C is commutative and hence $yz = zy$. It follows that the sub-algebra generated by $C \cup \{y\}$ is commutative, so by maximality it is in C and so $y \in C$ and $\lambda \notin \sigma_C(x)$. Hence, $\sigma_C(x) \subseteq \sigma_A(x)$. □

Definition 5.3. A non-zero homomorphism $\phi : A \to \mathbb{C}$ on an algebra A is called a character on A. Let Φ_A be the set of all characters on A. If A is unital, then $\phi(\mathbf{1}_A) = 1$ for all characters ϕ.

Lemma 5.4. Let A be a Banach algebra and $\phi \in \Phi_A$. Then ϕ is continuous and $\|\phi\| \leqslant 1$. Moreover, if A is a unital Banach algebra, then $\|\phi\| = 1$.

Proof. Wlog, A is a unital Banach algebra: can define $\phi_+ : A_+ \to \mathbb{C}$ by $\phi_+(a + \lambda\mathbf{1}) = \phi(a) + \lambda$. Then $\phi_+ \in \Phi_{A_+}$ and $\phi_+\restriction_A = \phi$. Now assume that A is a unital Banach algebra and $\phi \in \Phi_A$. Let $x \in A$ and assume $\phi(x) > \|x\|$. By Theorem 5.1, $\phi(x) \notin \sigma_A(x)$. So $\phi(x)\mathbf{1} - x \in \mathcal{G}(A)$. So $1 = \phi(x) = \phi((\phi(x)\mathbf{1} - x)\cdot(\phi(x)\mathbf{1} - x)^{-1}) = (\phi(x)\mathbf{1} - x) = 0$, a contradiction. So $|\phi(x)| \leqslant \|X\|$, giving $\|\phi\| \leqslant 1$. In fact $\|\phi\| = 1$ since $\phi(\mathbf{1}) = 1$. □

Lemma 5.5. Let A be a unital Banach algebra and $\mathcal{J}$ be a proper ideal of A. Then $\overline{\mathcal{J}}$ is also a proper ideal. In particular, maximal ideals are closed.

Proof. Since $\mathcal{J}$ is proper, $\mathcal{J} \cap \mathcal{G}(A)$ is empty. By corollary 5.0.1, $\mathcal{G}(A)$ is open giving that $\overline{\mathcal{J}} \cap \mathcal{G}(A)$ is empty, hence $\overline{\mathcal{J}}$ is proper. We have shown that if $\mathcal{M}$ is a maximal ideal of A, then $\mathcal{M}$ is proper and hence so is $\overline{\mathcal{M}}$. By maximality, $\mathcal{M} = \overline{\mathcal{J}}$ is closed. □

Notation: For an algebra A, we let $\mathcal{M}_A$ be the set of all maximal ideals of A.

Theorem 5.4. Let A be a commutative unital Banach algebra. Then the map

$$\begin{aligned}\Phi_A &\to \mathcal{M}_A \\ \phi &\mapsto \ker \phi\end{aligned}$$

is a bijection.

Proof. Well-defined: let $\phi \in \Phi_A$. Since ϕ is a homomorphism, $\ker \phi$ is an ideal of A. Since ϕ is a non-zero linear functional, $\ker \phi$ is a 1−codimensional sub-space. So $\ker \phi$ is a maximal ideal.

Injective: assume $\phi, \psi \in \Phi_A$ and $\ker \phi = \ker \psi$. For $x \in A$, $\phi(x)\mathbf{1} - x \in \ker \phi = \ker \psi$, which implies $\psi(\phi(x)\mathbf{1} - x) = 0$ giving $\phi(x) \cdot \psi(\mathbf{1}) = \psi(x) = \phi(x)$.

Surjective: let $\mathcal{M} \in \mathcal{M}_A$. By lemma 5.5, $\mathcal{M}$ is closed, so $A\backslash\mathcal{M}$ is a unital Banach algebra in the quotient norm. From algebra, $A\backslash\mathcal{M}$ is a field, so a division algebra. By corollary 5.1.1 (Galfand-Mazur), $A\backslash\mathcal{M} \cong \mathbb{C}$. So the quotient map $q : A \to A\backslash\mathcal{M}$ "is" a character and $\ker q = \mathcal{M}$. □

Corollary 5.4.1. Let A be a commutative unital Banach algebra and $x \in A$. Then

(i) $x \in \mathcal{G}(A) \iff$ for all $\phi \in \Phi_A$, $\phi(x) \neq 0$.

(ii) $\sigma_A(x) = \{\phi(x) : \phi \in \Phi_A\}$.

(iii) $r_A(x) = \sup\{|\phi(x)| : \phi \in \Phi_A\}$

Proof. (i) If $x \in \mathcal{G}(A)$, then for all characters ϕ, $1 = \phi(\mathbf{1}) = \phi(x \cdot x^{-1}) = \phi(x) \cdot (\phi(x))^{-1}$ implying that $\phi(x) \neq 0$.
Assume that $x \notin \mathcal{G}(A)$, then $\mathcal{J} = xA = \{xa : a \in A\}$ is a proper ideal of A, and so is contained in a maximal ideal which is $\ker \phi$ for some character ϕ by Theorem 5.4. So $\phi(x) = 0$ since $x \in \mathcal{J} \subseteq \ker \phi$.

(ii) $\lambda \in \sigma_A(x) \iff (\lambda\mathbf{1} - x) \notin \mathcal{G}(A) \iff$ (by (i)) there exists $\phi \in \Phi_A$ s.t. $\phi(\lambda\mathbf{1} - x) = 0$, i.e. $\lambda = \phi(x)$.

(iii) Is immediate from (ii).

□

Corollary 5.4.2. Let x, y be commuting elements of a Banach algebra A. Then

$$\begin{aligned}r_A(x + y) &\leqslant r_A(x) + r_A(y) \\ r_A(x \cdot y) &\leqslant r_A(x) \cdot r_A(y).\end{aligned}$$

Proof. Wlog, A is a commutative unital Banach Algebra. ($A \to A_+$ if necesary and then replace A by a maximal commutative sub-algebra containing x, y and use proposition 5.1). Then for all characters ϕ, $|\phi(x + y)| \leqslant |\phi(x)| + |\phi(y)| \leqslant r_A(x) + r_A(y)$ by corollary 5.4.1. Taking supremum over all characters ϕ gives $r_A(x + y) \leqslant r_A(x) + r_A(y)$. Argue analogously for the remaining inequality. □

Examples:

1. $A = \mathcal{C}(K)$, K compact, Hausdorff. $\Phi_A = \{\delta_k : k \in K\}$ $(\delta_k(f) = f(k))$. "$\supseteq$" is easy to check.
"$\subseteq$": let $\mathcal{M} \in \mathcal{M}_A$. Seek $k \in K$ s.t. $\mathcal{M} = \ker \delta_k$. Assume there is non such A. Then for all $k \in K$, there exist $f_k \in \mathcal{M}$ s.t. $f_k(k) \neq 0$. By continuity, there exists open

neighbourhoods $\mathcal{U}_k$ of k s.t. $f_k\restriction_{\mathcal{U}_k} \neq 0$. By compactness, there exist $k_1, \ldots, k_n \in K$ s.t. $\bigcup^n \mathcal{U}_{k_j} = K$. Then $g = \sum_{j=1}^n |f_{k_j}|^2 > 0$ on K. So $\frac{1}{g} \in \mathcal{C}(K)$. Also, $g = \sum_{j=1}^n f_{k_j} \cdot \overline{f}_{k_j} \in \mathcal{M}$, a contradiction.

2. Let $K \subseteq \mathbb{C}$, K compact and non-empty. Then $\Phi_{\mathcal{R}(K)} = \{\delta_w : w \in K\}$.

3. $\Phi_{A(\Delta)} = \{\delta_w : w \in \Delta\}$ where $A(\Delta)$ is the disc algebra.

4. Wiener algebra: $W = \{f \in \mathcal{C}(S^1) : \sum_{n\in\mathbb{Z}} |\hat{f}_n| < \infty\}$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, $\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$. W is a commutative unital Banach algebra with pointwise operations in the norm $\|f\|_1 = \sum_{n\in\mathbb{Z}} |\hat{f}_n|$. [It is isometrically isomorphic to $\ell_1(\mathbb{Z})$ which is a Banach algebra in the ℓ_1-norm and convolution product. That is for $a = (a_n), b = (b_n)$, $(a * b)_n = \sum_{j+k=n} a_k b_j, n \in \mathbb{Z}$. The isomorphism is given by $f \mapsto (\hat{f}_n)_n \in \mathbb{Z}$]. Have $\Phi_W = \{\delta_w : w \in S^1\}$, so $\sigma_W(f) = f(S^1)$. So if $f \in \mathcal{C}(S^1)$ has absolutely convergence Fourier series and is nowhere zero, then $\frac{1}{f} \in W$ and so has an absolutely convergence Fourier series and is nowhere zero (Wiener's Theorem).

Lecture 19

Definition 5.4. Let A be a commutative unital Banach algebra. Then

$$\begin{aligned}\Phi_A &= \{\phi \in B_{A*} : \phi(ab) = \phi(a)\phi(b)\forall a, b \in A, \phi(\mathbf{1}_A = 1\}\\ &= B_{A*} \cap (\hat{ab} - \hat{a}\cdot\hat{b})^{-1}(\{0\}) \cap \mathbf{1}_A^{-1}(\{1\})\end{aligned}$$

is weak-* closed. (Here for $x \in A$, $\hat{x} \in A^{**}$ is its canonical image in A^{**}). Hence, Φ_A is w^*-compact. The $w^* - topology$ on Φ_A is called the Gelfand topology. Φ_A with the Gelfand topology is the spectrum of A OR the character space of A OR the maximal ideal space of A. For $x \in A$, $\hat{x}\restriction_{\Phi_A}$ is continuous on Φ_A wrt the Gelfand topology; we denote $\hat{x}\restriction_{\Phi_A}$ by $\hat{x}$. So $\hat{x} \in \mathcal{C}(\Phi_A)$-called the Gelfand transform of x. The map

$$\begin{aligned}A &\to \mathcal{C}(\Phi_A)\\ x &\mapsto \hat{x}\end{aligned}$$

is the Gelfand map.

Theorem 5.5 (Gelfand Representation Theorem). Let A be a commutative unital Banach algebra, then the Gelfand map is a continuous unital homomorphism $A \to \mathcal{C}(\Phi_A)$. For $x \in A$

(i) $\|\hat{x}\|_\infty = r_A(x) \leqslant \|x\|$.

(ii) $\sigma_{\mathcal{C}(\Phi_A)}(\hat{x}) = \sigma(x)$.

(iii) $x \in \mathcal{G}(A) \iff \hat{x} \in \mathcal{G}(\mathcal{C}(\Phi_A))$.

Proof. The Gelfand map is linear since $x \to \hat{x} : A \to A^{**}$ is linear.

Homomorphism: for $x, y \in A$ $\widehat{xy}(\phi) = \phi(xy) = \phi(x)\phi(y) = \hat{x}\cdot\hat{y}$ for all $\phi \in \Phi_A$, so $\widehat{xy} = \hat{x}\hat{y}$.

Unital: $\hat{\mathbf{1}}_A(\phi) = \phi(\mathbf{1}_A) = 1$ for all $\phi \in \Phi_A$, so $\hat{\mathbf{1}}_A = \hat{\mathbf{1}}_{\Phi_A}$.

Continuity: follows once we prove (i).

(i) $\|\hat{x}\|_\infty = \sup\{|\hat{x}(\phi)| : \phi \in \Phi_A\} \overset{Cor5.4.1(iii)}{=} r_A(x) \overset{Thm5.1}{\leqslant} \|x\|$.

(ii) $\sigma_{\mathcal{C}(\Phi_A)}(\hat{x}) = \{|\overbrace{\hat{x}(\phi)}^{=\phi(x)}|: \phi \in \Phi_A\} \overset{Cor5.4.1(ii)}{=} \sigma_A(x)$.

(iii) Immediate.

□

Note: the Gelfand map need not be injective or surjective. Using Theorem, 5.2 its kernel is

$$\begin{aligned} \{x \in A : \sigma_A(x) = \{0\}\} &= \{x \in A : \overbrace{\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}}^{\text{quasi-nilpotent}=0} \} \\ &= \bigcap_{\phi\in\Phi_A} \ker\phi \\ &= \underbrace{\bigcap_{\mathcal{M}\in\mathcal{M}_A} \mathcal{M}}_{\text{Jacobson radical of } A, \mathcal{J}(A)} . \end{aligned}$$

Say A is semi-simple if $\mathcal{J}(A) = \{0\}$.

6 Holomorphic Functional Calculus (HFC)

Recall For a non-empty open set $\mathcal{U} \subseteq \mathbb{C}$, $\mathcal{O}(\mathcal{U}) = \{f : \mathcal{U} \to \mathbb{C} : f \text{ holomorphic}\}$ is a LCS with the topology of local uniform convergence induce by the family of semi-nnorms: $f \mapsto \|f\|_K = \sup_K |f|$ for non-empty compact $K \subseteq \mathcal{U}$. $\mathcal{O}(\mathcal{U})$ is also an algebra with pointwise multiplication which is cotinuous wrt the topology of $\mathcal{O}(\mathcal{U})$ [a Fréchet algebra].

Notation: Define $e, u \in \mathcal{O}(\mathcal{U})$ by $e(z) = 1$ and $u(z) = z$ for all $z \in \mathbb{C}$. $\mathcal{O}(\mathcal{U})$ is a untial algebra with unit e.

Theorem 6.1 (Holomorphic Function Calculus). Let A be a commutative unital Banach algebra, $x \in A, \mathcal{U} \subseteq \mathbb{C}$ open and $\sigma_A(x) \subseteq \mathcal{U}$. Then there exists a unique unital homomorphism $\Theta_x : \mathcal{O}(\mathcal{U}) \to A$ s.t. $\Theta_x(u) = x$. Moreover, $\phi(\Theta_x(f)) = f(\phi(x))$ for all $\phi \in \Phi_A, f \in \mathcal{O}(\mathcal{U})$ and $\sigma_A(\Theta_x(f)) = \{f(\lambda) : \lambda \in \sigma_A(x)\}$.

Note: Think of Θ_x as "evaluation as x"-write $f(x)$ for $\Theta_x(f)$. Then $e(x) = 1$, $u(x) = x$. If p is a polynomial, there exist $n \in \mathbb{N}, a_0, \ldots, a_n \in \mathbb{C}$ s.t. for all $z \in \mathbb{C}$, $p(z) = \sum_{k=0}^{n} a_k z^k$, then $p = \sum_{k=0}^{n} a_k u^k$. So $\Theta_x(p) = p(z) = \sum_{k=0}^{n} a_k (\Theta_x(u))^k = \sum_{k=0}^{n} a_k x^k = p(x)$ as defined in lemma 5.2.
Also, $\phi(f(x)) = f(\phi(x))$ for all $f \in \mathcal{O}(\mathcal{U})$, $\phi \in \Phi_A$ and $\sigma_A(f) = \{f(\lambda) : \lambda \in \sigma_A(x)\} = f(\sigma_A(x))$.

Theorem 6.2 (Runge's Approximation Theorem). Let K be non-empty and compact. Then $\mathcal{O}(K) = \mathcal{R}(K)$, i.e. if f is a function holomorphic on some open neighbourhood of K then for all $\epsilon > 0$, there exists ration function r with no poles in K s.t. $\|f - r\|_K < \epsilon$.

More precisely, given a set Λ consisting of one point from each bounded component of $\mathbb{C}\backslash K$, r can be chosen s.t. all its poles are in Λ. If $\mathbb{C}\backslash K$ is connected, then Λ is empty so in fact we get $\mathcal{O}(K) = \mathcal{P}(K)$.

6.1 Vector-valued integration

Let $a < b$ in $\mathbb{R}$, X be a Banach space and $f : [a, b] \to X$ continuous. We define "$\int_a^b f(t)dt$". We choose dissections $\mathcal{D}_n := a = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n}^{(n)} = b$ s.t. $|\mathcal{D}_n| = \max_{1\leqslant j\leqslant k_n} (t_j^{(n)} - t_{j-1}^{(n)}) \to 0$ as $n \to \infty$.

Since f is uniformly continuous, the limit of

$$\sum_{j=0}^{k_n} f(t_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})$$

exists and is independent of $(\mathcal{D}_n)$. We define $\displaystyle\int_a^b f(t)dt$ to be this limit. It follows that for all $\phi \in X^*$

$$\phi\left(\int_a^b f(t)dt\right) = \int_a^b \phi(f(t))dt.$$

Taking ϕ to be a norming functional for $\displaystyle\int_a^b f(t)dt$, we get

$$\left\|\int_a^b f(t)dt\right\| \leqslant \int_a^b \|f(t)\|\, dt, \quad (\|\phi\| \leqslant 1).$$

Let γ be a path in $\mathbb{C}$ (continuously differentiable), $f : [\gamma] \to X$ be continuous[13]. Define

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Given a chain $\Gamma = (\gamma_1, \ldots, \gamma_n)$[14] and continuous $f : [\gamma] \to X$ define

$$\int_\Gamma f(z)dz = \sum_{j=1}^{n} \int_a^b f(\gamma(t))\gamma'(t)dt$$

and have for all $\phi \in X^*$

$$\phi\left(\int_\Gamma f(z)dz\right) = \int_\Gamma \phi(f(z))dz.$$

and

$$\left\|\int_\Gamma f(z)dz\right\| \leqslant \ell(\Gamma) \cdot \sup_{z\in[\gamma]} \|f(z)\|.$$

Theorem 6.3 (Vector-valued Cauchy's Theorem). Let $\mathcal{U} \subseteq \mathbb{C}$ be open, Γ a cycle[a] in $\mathcal{U}$, s.t. $n(\Gamma, w) = \frac{1}{2\pi i}\displaystyle\int_\gamma \frac{1}{z-w}dz = 0$ for all $w \notin \mathcal{U}$ and $f : \mathcal{U} \to X$ holomorphic. Then

$$\int_\Gamma f(z)dz = 0.$$

[a] a cycle is a chain $\Gamma = (\gamma_1, \ldots, \gamma_n), n \in \mathbb{N}$ of paths $\gamma_j : [a_j, b_j] \to \mathbb{C}$ s.t. there exists a permutation $\rho \in S_n$ s.t. $\gamma_j(b_j) = \gamma_{\rho(j)(a_{\rho(j)})}$ for all $j = 1, \ldots, n$.

Proof. For $\phi \in X^*$, apply the scalar version of Cauchy's Theorem to deduce

$$\phi\left(\int_\Gamma f(z)dz\right) = 0, \quad \text{for all } \phi \in X^*$$

and then apply Hahn-Banach to conclude. $\square$

Lemma 6.1. Let K be a non-empty compact s.t. $K \subseteq \mathcal{U}, \mathcal{U} \subseteq \mathbb{C}$ open. Then there is a cycle Γ such that

$$n(\Gamma, w) = \begin{cases} 1, & w \in K \\ 0, & w \notin \mathcal{U}. \end{cases}$$

[13] $[\gamma]$ denotes the path itself in $\mathbb{C}$.

[14] any finite collection of paths defined as above.

Proof. Note that K being compact means that $\text{dist}(K, \mathbb{C}\backslash\mathcal{U}) = \delta > 0$. Thus, there exists an $n \in \mathbb{N}$[15], s.t. K is covered by finitely many (by compactness) boxes in the dyadic lattice $2^{-n}\mathbf{Z}^2$ where any adjacent to them boxes are also $\subseteq \mathcal{U}$, see figure 6. More precisely, $\mathcal{A} = \{(x,y) \in \mathbb{Z}^2 : [x \cdot 2^{-n}, x \cdot 2^{-n} + 2^{-n}] \times [y \cdot 2^{-n}, y \cdot 2^{-n} + 2^{-n}] \cap K \neq \varnothing\}$. Have $|\mathcal{A}| < \infty$. Now, define $\mathcal{B} = \mathcal{A}\{(x \pm 1, y \pm 1) \in \mathbb{Z}^2 : (x,y) \in A)\}$. Let Γ be the boundary of the boxes above, that is

$$\Gamma = \partial \bigcup_{(x,y)\in\mathcal{B}} [x \cdot 2^{-n}, x \cdot 2^{-n} + 2^{-n}] \times [y \cdot 2^{-n}, y \cdot 2^{-n} + 2^{-n}]$$

oriented counter-clockwise (black curve in figure 6), and note that $\Gamma \subseteq \mathcal{U}\backslash K$.

Now, for any $w \in K$, w is either in the interior of a box or the interior of the union of boxes adjacent to it. Regardless, one computes the winding number around such a curve $\tilde{\Gamma}$ (red in figure6), which is seen to be the same as the winding number of Γ around w, by homotopy invariance (Cauchy's Theorem). One argues similarly for $w \in \mathbb{C}\backslash\mathcal{U}$ to obtain $n(\Gamma, w) = 0$. □

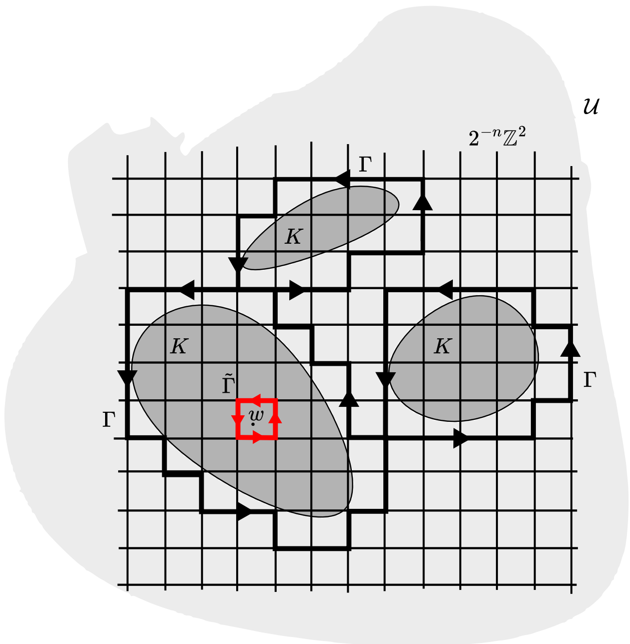


Figure 6: Illustration of proof of Lemma 6.1, where $n \in \mathbb{N}, w \in K$ and $K, \mathcal{U}, \Gamma$ (in black) as in in the lemma.

Lecture 20

Lemma 6.2. Let $A, x, \mathcal{U}$ be as in Theorem 6.1. $K = \sigma_A(x)$ and fix a cycle (guaranteed to exists by Lemma 6.1) Γ in $\mathcal{U}\backslash K$ s.t.

$$n(\Gamma, w) = \begin{cases} 1, & w \in K \\ 0, & w \notin \mathcal{U}. \end{cases}$$

Define the map

Then,

Θ_x is well-defined, linear, continuous.

(i)(ii) For a rational function r with no poles in $\mathcal{U}$, $\Theta_x(r) = r(x)$ in the usual sense.

[15]for instance, take $n \in \mathbb{N}$ s.t. $2\sqrt{2} \cdot 2^{-n} < \frac{\delta}{2}$.

(iii) $\phi(\Theta_x(f)) = f(\phi(x))$ for all $\phi \in \Phi_A, f \in \mathcal{O}(\mathcal{U})$ and $\sigma_A(\Theta_x(f)) = \{f(\lambda) : \lambda \in \sigma_A(x)\}$.

Remark. So we can think of the HFC as a Banach algebra valued Cauchy integral formula. Lemma 6.2 almost proves the theorem (6.1). It remains to show that Θ_x is a homomorphism and it is unique.

Proof. (i) If $z \in [\Gamma]$ then $z \notin K = \sigma_A(x)$. So $z\mathbf{1} - x \in \mathcal{G}(A)$. By the proof of Theorem 5.1, the map $z \mapsto (z\mathbf{1} - x)^{-1}$ is continuous (indeed, holomorphic). So, Θ_x is well-defined. It's also linear by linearity of integration. We also have the estimate

$$\|\Theta_x(f)\| \leqslant \frac{1}{2\pi}\ell(\Gamma) \cdot \sup_{z\in[\gamma]} |f(z)| \cdot \left\|(z\mathbf{1} - x)^{-1}\right\|.$$

Since the map $z \mapsto \left\|(z\mathbf{1} - x)^{-1}\right\|$ is continuous on the compact set $[\Gamma]$, it is bounded. So there exists $M > 0$ s.t. for all $f \in \mathcal{O}(\mathcal{U})$ $\|\Theta_x(f)\| \leqslant M \cdot \|f\|_{[\Gamma]}$.

By Lemma 1.3, Θ_x is continuous.

(ii) First we show $\Theta_x(e) = 1$.

Fix $R > \|x\|$ and let γ be the anticlockwise boundary of $D(0, R)$. Then γ and Γ are homologous in $\mathbb{C}\backslash K$. So, by Cauchy's Theorem and the proof of Lemma 5.1,

$$\begin{aligned}
\Theta_x(e) &= \tfrac{1}{2\pi i}\int_\gamma (z\mathbf{1} - x)^{-1} dz \\
&= \tfrac{1}{2\pi i}\int_\gamma \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{x}{z}\right)^n dz \\
&= \underbrace{\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\frac{x^n}{z^{n+1}}dz}_{\text{sum conv. absolutely and uniformly on } \gamma} \\
&= x^0 = 1.
\end{aligned}$$

Let $r \in \mathcal{O}(K)$ be a rational function. So $r \underset{=K}{=} \frac{p}{q}$, for polynomials p, q s.t. for all $z \in \mathcal{U}$, $q(z) \neq 0$. By Lemma 5.2, $\sigma_A(q(x)) = \{q(\lambda) : \lambda \in \sigma_A(x)\}$ and so $0 \notin \sigma_A(q(x))$. We define $r(x) = p(x) \cdot q(x)^{-1}$ ("usual sense"). For $z, w \in \mathbb{C}$, $r(z) - r(w) = q(z)^{-1}q(w)^{-1}(q(w)p(z) - q(z)p(w)) = q(z)^{-1}q(w)^{-1}(z - w)s(z, w)$, where s is a polynomial in z, w. Hence, $r(z)\mathbf{1} - r(x) = q(z)^{-1}q(w)^{-1}(z\mathbf{1} - w)s(z, w)$ and

$$\begin{aligned}
\Theta_x(r) &= \tfrac{1}{2\pi i}\int_\gamma \underbrace{r(z)}_{r(z)\mathbf{1}-r(x)+r(x)} (z\mathbf{1} - x)^{-1} dz \\
&= \underbrace{\tfrac{1}{2\pi i}\int_\gamma q(z)^{-1}q(w)^{-1}s(z, w)dz}_{=0 \text{ by Cauchy}} + \tfrac{1}{2\pi i}\int_\gamma (z\mathbf{1} - x)^{-1})dz \cdot r(x) \\
&= r(x) \cdot \Theta_x(e) = r(x).
\end{aligned}$$

(iii) $\phi(\Theta_x(f)) = \frac{1}{2\pi i}\int_\gamma f(z)(z\mathbf{1} - x)^{-1}dz = f(\phi(x))$ by Cauchy's integral formula. and so

$$\sigma_A(\Theta_x(f)) \overset{\text{Cor } 5.4.1}{=} \{\overset{=f(\phi(x))}{\phi(\Theta_x(f))} : \phi \in \Phi_A\} = \{f(\lambda) : \lambda \in \sigma_A(x)\}.$$

□

Proof. Proof of Theorem 6.2 Let $A = \mathcal{R}(K)$. Let $x \in A$ be the element $x(z) = z$ for all $z \in K$. $\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda \in \sigma_A(x)\} = K$ (for $\lambda \notin K$, $\frac{1}{\lambda - z}$ is the inverse to $\lambda\mathbf{1} - x$).

Let f be holomorphic on some open set $\mathcal{U} \supseteq K$. Let $\Theta_x : \mathcal{O}(\mathcal{U}) \to A$ be given by Lemma 6.2. $\Theta_x(f)(w) = \delta_w(\Theta_x(f)) = f(\delta_w(x)) = f(w)$ for all $w \in K$. So $\Theta_x(f) = f\restriction_K \in \mathcal{R}(K)$. So

$\mathcal{O}(K) = \mathcal{R}(K)$.

Let us now fix Λ as in the statement of Theorem 6.2. Let B be the closed sub-algebra of A generated by $1, x, (\lambda\mathbf{1} - x)^{-1}, \lambda \in \Lambda$. So B = closure in $\mathcal{C}(K)$ of rational functions with poles in Λ. By Theorem 5.3, $\sigma_B(x)$ is the union of $\sigma_A(x)$ and some of the bounded components of $\mathbb{C}\backslash K$. Since for any such component D there exists $\lambda \in \Lambda \cap D$, so $\lambda \cdot \mathbf{1} - x \in \mathcal{G}(A)$. So $\sigma_B(x) = \sigma_A(x)$. So $\Theta_x(f)$ takes values in B, i.e. $f\restriction_K \in B$. □

Corollary 6.3.1. Let Let $\mathcal{U} \subseteq \mathbb{C}$ be non-empty and open. Then the algebra $\mathcal{R}(\mathcal{U})$ of rational functions with no poles in $\mathcal{U}$ is dense in $\mathcal{O}(\mathcal{U})$.

Proof. Let $f \in \mathcal{O}(\mathcal{U})$ and $\mathcal{V}$ be a neighbourhood of f in $\mathcal{O}(\mathcal{U})$. We need $\mathcal{V} \cap \mathcal{R}(\mathcal{U}) \neq \varnothing$.

Wlog, $\mathcal{V} = \{g \in \mathcal{O}(\mathcal{U}) : \|g - f\|_K < \epsilon\}$ for some non-empty, compact $K \subseteq \mathcal{U}$ and $\epsilon > 0$. Let $\hat{K}$ be the union of K and those bounded components $\mathcal{D}$ of $\mathbb{C}\backslash K$ that are combined in $\mathcal{U}$.

If D is a bounded component of $\mathbb{C}\backslash\hat{K}$, then D is a bounded component of $\mathbb{C}\backslash K$ s.t. $D\backslash\mathcal{U} \neq \varnothing$ so we can fix $\lambda_0 \in D\backslash\mathcal{U}$. Let Λ be the set of all these λ_0's. By Theorem 6.2, there exists rational function r s.t. $\|r - f\|_{\hat{K}} < \epsilon$ and the poles of r are in Λ. Hence, $r \in \mathcal{V} \cap \mathcal{R}(K)$. □

Combining the above results, we can now embark on a proof of Theorem 6.1, which we started this section with.

Proof. Let Θ_x be as in lemma 6.2. Then for all $f, g \in \mathcal{R}(\mathcal{U})$, $\Theta_x(fg) \overset{Lemma 6.2\text{(ii)}}{=} (f \cdot g)(x)$ $= f(x)g(x) = \Theta_x(f) \cdot \Theta_x(g)$ and conclude by density of $\mathcal{R}(\mathcal{U})$ in $\mathcal{O}(\mathcal{U})$ and continuity that Θ_x is a homomorphism.

For uniqueness, assume $\Psi : \mathcal{O}(\mathcal{U}) \to A$ is a continuous unital homomorphism and $\psi(x) = x$. Then for all polynomials p, $\Psi(p) = p(x) = \Theta_x(p)$ and so for all rational $r \in \mathcal{R}(\mathcal{U})$ $\Psi(r) = r(x) = \Theta_x(r)$ and hence $\Psi \equiv \Theta_x$ by density and continuity. □

7 C^*−algebras

A C^*- algebra is a complex algebra A with an involution: a map $A \to A, x \mapsto x^*$ s.t.

(i) $(\lambda x + \mu y)^* = \overline{\lambda} x^* + \overline{\mu} y^*$

(ii) $(xy)^* = y^* x^*$

(iii) $x^{**} = x$

for all $x, y \in A, \lambda, \mu \in \mathbb{C}$. If A is unital, then $\mathbf{1}^* = \mathbf{1}$. A C^*-algebra is a Banach algebra with an involution s.t. the C^*-equation holds:

$$\|x^* x\| = \|x\|^2, \quad \text{for all } x \in A.$$

A complete algebra norm on a *-algebra that satisfies the C^*-equation is a C^*-norm. So a C^*-algebra is a *-algebra with a C^*-norm on it.

Lecture 21 **Remark.**

1. If A is a C^*-algebra, and $x \in A$, then $\|x^*\| = \|x\|$ ($\|x\|^2 = \|x^* x\| \leqslant \|x^*\| \cdot \|x\|$ so $\|x\| \leqslant \|x^*\|$ and hence $\|x^*\| \leqslant \|x^{**}\| = \|x\|$). So the involution is continuous.

   A Banach algebra with an involution s.t. $\|x^*\| = \|x\|$ for all x.

2. If A is a C^*-algebra and if A has a multiplicative identity $\mathbf{1} \neq 0$, then automatically A is a unital C^*-agebra, $\|\mathbf{1}\| = 1$ ($\|\mathbf{1}\|^2 = \|\mathbf{1}^*\mathbf{1}\| = \|\mathbf{1}\|$).

Definition 7.1. A *-sub-algebra of a *-algebra A is a sub-algebra B of A s.t. for all $x \in B$, $x^* \in B$. A C^*-sub-algebra of a C^*-algebra is a closed *-algebra. So a C^*-sub-algebra of a C^*-algebra is a C^*-algebra. The closure of a *-algebra of a C^*-algebra is a *-sub-algebra, so a C^*-algebra.

A *-homomorphism between *-algebras is a homomorphism $\theta : A \to B$ s.t. $\theta(x^*) = \theta(x)^*$ for all $x \in A$. A *-isomorphism is a bijective *-homomorphism.

Examples:

1. $\mathcal{C}(K)$, K compact Hausdorff, is a commutative, unital C^*-algebra with involution $f \mapsto f^*$, where $f^*(k) = \overline{f(k)}$ for all $k \in K$, $f \in \mathcal{C}(K)$.
2. $\mathcal{B}(H)$, H Hilbert space is a unital C^*-algebra with involution $T \mapsto T^*$ where T^* is the adjoint.
3. Any C^*-sub-algebra of $\mathcal{B}(H)$, (H any Hilbert space) is a C^*-algebra.

. . . And that's all folks!

Remark. the Gelfand-Naimark Theorem says that if A is a C^*-algebra then there exists a Hilbert space H s.t. A is isometrically *-isomorphic to some C^*-sub-algebra of $\mathcal{B}(H)$. We will prove the commutative version.

Definition 7.2. Let A be a C^*-algebra and $x \in A$. We say x is

(i) hermitian or self-adjoint if $x^* = x$

(ii) unitary if (A is unital and) $x^* x = x x^* = \mathbf{1}$

(iii) normal if $x^* x = x x^*$

Examples:

1. $\mathbf{1}$ is both hermitian and unitary. In general, hermitian and unitary are normal.
2. $f \in \mathcal{C}(K)$ is Hermitian $\iff$ $f(K) \subseteq \mathbb{R}$ and unitary $\iff$ $f(K)^1$. (Recall: $f(K) = \sigma_{\mathcal{C}(K)}(f)$).

Remark. 1. If A is a C^*-algebra and $x \in A$. Then there exist unique hermitian $h, k \in A$ s.t. $x = h + ik$. [If $x = h + ik$ then $x^* = h - ik$, so $h = \frac{x+x^*}{2}, k = \frac{x-x^*}{2i}$ and conversely, this choice for h, k works].

2. If A is a unital C^*-algebra and $x \in A$, then $x \in \mathcal{G}(A) \iff x^* \in \mathcal{G}(A)$ and in this case $(x^*)^{-1} = (x^{-1})*$.
It follows that $\sigma_A(x^*) = \{\overline{\lambda} : \lambda \in \sigma_A(x)\}$ $(\lambda\mathbf{1} - x \in \mathcal{G}(A) \iff (\lambda\mathbf{1} - x)^* = \overline{\lambda} \cdot \mathbf{1} - x^* \in \mathcal{G}(A))$ so $\sigma_A(x^*) = \sigma_A(x)$.

Lemma 7.1. Let A be a C^*-algebra and $x \in A$. Then $r_A(x) = \|x\|$ provided x is normal.

Proof. Assume x is hermitian. Then $\|x\|^2 = \|x^2\|$ and inductively, $\|x\|^{2^n} = \left\|x^{2^n}\right\|$ for all n. By the spectral radius formula (Theorem 5.2), $r_A(x) = \lim_{n\to\infty} \left\|x^{2^n}\right\|^{\frac{1}{2^n}} = \|x\|$.

If x is normal, then $\|x^*x\| = r_A(x^*x)$ because x^*x is hermitian.

Now, $r_A(x^*x) \leqslant r_A(x*)r_A(x) \overset{(Cor5.4.2)}{\leqslant} \|x^*\| \cdot \|x\|$. But $\|x\|^2 = \|x^*x\|$. So we have equality throughout and so $\|x\| = r_A(x)$. $\square$

Lemma 7.2. Let A be a unital C^*-algebra and $x \in A$. Then $\phi(x^*) = \overline{\phi(x)}$ for all $\phi \in \Phi_A$.

Proof. Wlog we can assume that x is hermitian. [For general x, write $x = h + ik$, h, k hermitian. Then $\phi(x^*) = \phi(h - ik) = \phi(h) - i\phi(k) = \overline{\phi(x)}$ $(\phi(h), \phi(k)$ real)]. Now assume x is hermitian $\phi \in \Phi_A$ and write $\phi(x) = a + ib$, $a, b \in \mathbb{R}$.

<u>Need</u>: For $t \in \mathbb{R}$,

$$\begin{aligned}
|\phi(x + it\mathbf{1})|^2 &= |a + i(b + t)|^2 \\
&= a^2 + (b + t)^2 = a^2 + b^2 + 2bt + t^2 \\
&\leqslant \|x + it\mathbf{1}\|^2 = \|(x + it)^*(x + it)\| \\
&= \|(x - it)^*(x + it)\| = \left\|x^2 + t^2\mathbf{1}\right\| \leqslant \left\|x^2\right\| + t^2.
\end{aligned}$$

Hence, $b = 0$. $\square$

Corollary 7.0.1. Let A be a unital C^*-algebra.

(i) If $x \in A$ is hermitian, then $\sigma_A(x) \subseteq \mathbb{R}$.

(ii) If $x \in A$ is unitary, then $\sigma_A(x)^1$.

(iii) If B is a unital C^*-sub-algebra of A and $x \in B$ is normal then $\sigma_B(x) = \sigma_A(x)$.

Proof. (i) Let $B = \underline{C^*-\text{algebra generated by } \mathbf{1}, x}$ (check *-sub-alg) $\overline{p(x) : p \text{ poly}}$. B is commutative, so $\sigma)B(x) = \{\phi(x) : \phi \in \Phi_B\}$. By Lemma 7.2, $\sigma_A(x) \subseteq \sigma_B(x) \subseteq \mathbb{R}$.

(ii) Let $B = \underline{C^*-\text{algebra generated by by } \mathbf{1}, x, x^*} = \overline{\{p(x, x^*) : p \text{ poly in two variables }\}}$. B is commutative, so $\sigma_B(x) = \{\phi(x) : \phi \in \Phi_B\}$. By Lemma 7.2, $1 = \phi(\mathbf{1}) = \phi(x^*x) = \overline{\phi(x)}\phi(x)$, hence $|\phi(x)|^2 = 1$. So $\sigma_A(x) \subseteq \sigma_B(x)^1$.

(iii) For the last part, assume $x \in B$ is hermitian. Then $\sigma_A(x) \subseteq \mathbb{R}$, so $\mathbb{C} \backslash \sigma_A(x)$ is connected. So it follows by Theorem 5.3 that $\sigma_A(x) = \sigma_B(x)$.

Now assume $x \in B$ is normal. Then for $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} \lambda\mathbf{1} - x \in \mathcal{G}(A) &\iff \lambda\mathbf{1} - x \in \mathcal{G}(A) \;\&\; (\lambda\mathbf{1} - x)^* \in \mathcal{G}(A) \\ &\overset{\text{commuting elements}}{\iff} (\lambda\mathbf{1} - x)(\lambda\mathbf{1} - x)^* \in \mathcal{G}(A) \\ &\overset{\text{hermitian}}{\iff} (\lambda\mathbf{1} - x)(\lambda\mathbf{1} - x)^* \in \mathcal{G}(B) \\ &\overset{\text{commuting elements}}{\iff} \lambda\mathbf{1} - x \in \mathcal{G}(B) \;\&\; (\lambda\mathbf{1} - x)^* \in \mathcal{G}(B) \\ &\iff \lambda\mathbf{1} - x \in \mathcal{G}(B). \end{aligned}$$

□

Remark. $T \in \mathcal{B}(H)$, T hermitian or unitary, then $\sigma(T) = \partial\sigma(T) \subseteq \sigma_{\text{ap}}(T) =$ set of approximate evals. So $\sigma(T) = \sigma_{\text{ap}}(T)$ (also holds for normal operators).

Lecture 22

Theorem 7.1. Let A be a commutative unital C^*-algebra. Then there exists compact, Hausdorff K s.t. A is isometrically isomorphic to $\mathcal{C}(K)$. In particular, the Gelfand map

$$\begin{aligned} A &\to \mathcal{C}(\Phi_A) \\ x &\mapsto \hat{x} \end{aligned}$$

is an isometric *-isomorphism.

Proof. By Theorem 5.5, the Gelfand map $G : A \to \mathcal{C}(\Phi_A)$ where $\mathcal{G}(x) = \hat{x}\restriction_{\Phi_A}$, is a unital homomorphism. It remains to check the following three properties:

<u>G is a *-homomorphism</u>: $\hat{x^*}(\phi) = \phi(x^*) \overset{Lemma 7.2}{=} \overline{\phi(x)} = \overline{\hat{x}(\phi)} = (\hat{x})^*(\phi)$ for all $\phi \in Phi_A$.

<u>G is isometric</u>: $\|G(x)\| = \|\hat{x}\|_\infty \overset{Thm 5.5(i)}{=} r_A(x) \overset{A \text{ commutative Lemma 7.1}}{=}$ for all $x \in A$.

<u>G is surjective</u>: let $\hat{A}$ be the image of G. So $\hat{A} = \{\hat{x} : x \in A\}$. SInce G is an isometric unital *-homomorphism, it follows that $\hat{A}$ is a closed sub-algebra of $\mathcal{C}(\Phi_A)$ containing the constant functions and closed under conjugation. Also $\hat{A}$ separates points of Φ_A: if $\phi \neq \psi$ in Φ_A, then there exists $x \in A$ s.t. $\phi(x) \neq \psi(x)$, i.e. $\hat{x}(\phi) \neq \hat{x}(\psi)$. By Stone-Weierstrass, $\hat{A} = \mathcal{C}(\Phi_A)$. □

<u>Applications:</u>

1. Let A be a unital C^*-algebra and let $x \in A$. Say x is <u>positive</u> if x is hermitain and $\sigma_A(x) \subseteq [0, \infty)$. We show there exists a unique positive $y \in A$ s.t. $y^2 = x$, called the square root of x, denoted $x^{\frac{1}{2}}$.

   <u>Existence</u>: $B =$ <u>C^*-sub-algebra generated by by $\mathbf{1}, x$</u> $= \{p(x) : p \text{ poly }\}$. B is a commutative unital C^*-algebra. By Theorem 7.1, the Gelfand map

   $$\begin{aligned} B &\to \mathcal{C}(\Phi_B) \\ w &\mapsto \hat{w} \end{aligned}$$

   is an *-isomorphism. Now, we compute $\sigma_{\mathcal{C}(\Phi_B)}(\hat{x}) \overset{Cor 5.4.1(ii)}{=} \sigma_B(x) \overset{Cor 7.0.1}{=} \sigma_A(x) \subseteq [0, \infty)$.

   The map $\phi \in \Phi_B$, $\phi \mapsto \sqrt{\hat{x}(\phi)} \in \mathcal{C}(\Phi_B)$, so there exists a $y \in B$ s.t. $\hat{y}(\phi) = \sqrt{\hat{x}(\phi)}$ for all $\phi \in \Phi_B$. $\hat{y^*} = (\hat{y})^* = \overline{\sqrt{\hat{x}}} = \sqrt{\hat{x}} = \hat{y}$.The Gelfand map is injective, so $y^* = y$, i.e. y is hermitian. Now, $\sigma_A(y) = \sigma_B(y) = \sigma_{\mathcal{C}(\Phi_B)}(\hat{y}) \subseteq [0, \infty)$, so y is positive. Finally, $\hat{y^2} = (\hat{y})^2 = \hat{x}$, so $y^2 = x$. Note that y is a limit of sequence of polynomials in x.

Uniqueness: Assume $z \in A$ is positive and $z^2 = x$. Have $zx = xz = z^3$, so $zp(x) = p(x)z$ for all polynomials p, so $yz = zy$. Let $\tilde{B} = \underline{C^*\text{−sub-algebra generated by } \mathbf{1}, y, z}$. Then $\tilde{B}$ is a commutative unital C^*−algebra containing $y, z, x = y^2 = z^2$. Theorem 7.1 gives that the

$$\begin{aligned} \tilde{B} &\to \mathcal{C}(\Phi_{\tilde{B}}) \\ w &\mapsto \hat{w} \end{aligned}$$

is an isometric *-isomorphism. $\sigma_{\mathcal{C}(\Phi_{\tilde{B}})}(\hat{y}) = \sigma_{\tilde{B}}(y) = \sigma_A(y) \subseteq [0, \infty)$. Also, $\hat{z}^2 = \widehat{z^2} = \hat{x} = \widehat{y^2} = \hat{y}^2$ and hence $\hat{y} = \hat{z}$ and thus $y = z$.

This applies to a positive operator $T \in \mathcal{B}(H)$, where H is a Hilbert space (T is positive $\iff$ for all $x \in H \langle Tx, x\rangle \geqslant 0$).

2. Polar decomposition: let H be a Hilbert space, and $T \in \mathcal{B}(H)$ invertible. Then there exists unique operators $\mathcal{R}, \mathcal{U}$ s.t. $\mathcal{R}$ is positive, $\mathcal{U}$ is unitary and $T = \mathcal{R}\mathcal{U}$.

   Existence: TT^* is positive ($\langle TT^*x, x\rangle = \|T^*x\|^2 \geqslant 0$). Let $\mathcal{R} = (TT^*)^{\frac{1}{2}}$. So $\mathcal{R}^2 = TT^*$ is invertible, and hence so is $\mathcal{R}$ (being the product of $\mathcal{R}, \mathcal{R}$, commuting elements is invertible $\iff$ $\mathcal{R}$). Let $\mathcal{U} = \mathcal{R}^{-1}T$. Then $\mathcal{U}$ is invertible and $\mathcal{U}\mathcal{U}^* = \mathcal{R}^{-1}TT^*(\mathcal{R}^{-1})^* \; \mathcal{R}^{-1} \overset{R^2}{TT^*} \mathcal{R}^{-1} = \text{Id}$.

   Uniqueness: if $T = \mathcal{R}\mathcal{U}$, $\mathcal{R}$ positive, $\mathcal{U}$ unitary, then $TT^* = \mathcal{R}\mathcal{U}\mathcal{U}^*\mathcal{R} = \mathcal{R}^2$ so $\mathcal{R} = \sqrt{TT^*}$ and $\mathcal{U} = \mathcal{R}^{-1}T$.

8 Borel Functional Calculus and Spectral Theory

Throughout we fix:

H non-zero, complex Hilbert space.

$\mathcal{B}(H)$ a bounded linear operator on H.

K compact, Hausdorff.

$\mathcal{B}$ Borel $\sigma-$field on K.

8.1 Operator-valued measures

Definition 8.1 (A resolution of the identity of H over K). A resolution of the identity of H over K (roti of H over K) is a map $P : \mathcal{B} \to \mathcal{B}(H)$ s.t.

(i) $P(\varnothing) = 0$ and $P(K) = \mathrm{Id}$.

(ii) For all $E \in \mathcal{B}$ $P(E)$ is an orthogonal profection.

(iii) For all $E, F \in \mathcal{B}$ $P(E \cap F) = P(E) \circ P(F) = P(F) \circ P(E)$.

(iv) For all $E, F = \varnothing$. Then, $P(E \cup F) = P(E) + P(F)$.

(v) For all $x, y \in H$ the map $P_{x,y} : \mathcal{B} \to \mathbb{C}$ defined by $P_{x,y}(E) = \langle P(E)x, y\rangle$, $E \in \mathcal{B}$, is a regular complex Borel measure.

Example:
$H = L_2[0,1]$, $K = [0,1]$, $P(E)f = \mathbf{1}_E f$.

Simple Properties:

(i) For all $E, F \in \mathcal{B}$ $P(E \cap E), P(F)$ commute (directly follows from (ii) above).

(ii) If $E \cap F = \varnothing$, then $P(E)(H) \perp P(F)(H)$. That is for all $x, y \in H$ $\langle P(E)x, P(F)y\rangle = \langle P(F) \cdot P(E)x, y\rangle\langle P\ (\underbrace{E \cap F}_{=\varnothing})\ x, y\rangle = 0$.

(iii) For $x \in H$, $P_{x,x}$ is a positive measure of total mass $P_{x,x}(K) = \|x\|^2$. ($P_{x,x}(E) = \langle P(E)x, x\rangle = \langle P(E)^2 x, x\rangle\ \langle P(E)x, P(E)x\rangle = \|P(E)x\|^2 \geqslant 0$, which equals $\|x\|^2$ if $E = K$).

(iv) p is finitely additive and for $x \in H$, $E \mapsto P(E)x : \mathcal{B} \to H$ is countably additive. That is, for $E_n \in \mathcal{B}, n \in \mathbb{N}$, $E_n \cap E_m = \varnothing$ for all $m \neq n$,

$$\begin{aligned}\left\langle \sum_{n\in\mathbb{N}} P(E_n)x, y \right\rangle &= \sum_{n\in\mathbb{N}} \langle P(E_n)x, y\rangle = \sum_{n\in\mathbb{N}} P_{x,y}(E_n) \\ &= P_{x,y}\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \left\langle P\left(\bigcup_{n\in\mathbb{N}} E_n\right) x, y \right\rangle\end{aligned}$$

for all $y \in H$ so

$$\sum_{n\in\mathbb{N}} P(E_n)x = P\left(\bigcup_{n\in\mathbb{N}} E_n\right) x.$$

Note that $\sum_{n\in\mathbb{N}} \|P(E_n)\|^2 \leqslant \|x\|^2$ be Bessel's inequality since $\left(P\left(\bigcup_{n\in\mathbb{N}} E_n\right) x\right)_{n\in\mathbb{N}}$ are pairwise orthogonal.

(v) P need not be countably additive, but if $P(E_n) = 0$ for all $n \in \mathbb{N}$ then $P\left(\bigcup_{n\in\mathbb{N}} E_n\right) = 0$.

(vi) For $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}$, consider the sequence $F_1 = E_1$, $F_n = E_n \backslash \bigcup_{i=1}^{n-1} E_i$, for $n > 1$, then

$$P\left(\bigcup_{n\in\mathbb{N}} E_n\right)x = P\left(\bigcup_{n\in\mathbb{N}} F_n\right)x = \sum_{n\in\mathbb{N}} P(F_n)x = 0, \qquad \text{for all } x \in H.$$

Lecture 23

Definition 8.2 (The algebra $L_\infty(P)$). Let P be a resolution of H over K. Say a Borel function $f : K \to \mathbb{C}$ is called $P-$essentially bounded if there exists $E \in \mathcal{B}$ s.t. $P(E) = 0$ and f bounded on $K\backslash E$.
Then define

$$\|f\|_\infty = \inf\{\|f\|_{K\backslash E} : E \in \mathcal{B}, P(E) = 0\},$$

which is attained (check!).
Let $L_\infty(P)$ be the set of all $P-$essentially bounded Borel functions $f : K \to \mathbb{C}$. This is a commutative, unital C^*-algebra with pointwise operations and $\|\cdot\|_\infty$ [As usual, we identify $f, g \in L_\infty(P)$ $P-$a.e., if there exists $E \in \mathcal{B}$ s.t. $P(E) = 0$, $f = g$ on $K\backslash E$].

Lemma 8.1. Let P be as above. Then there exists an isometric, unital *-homomorphism $\Phi : L_\infty(P) \to \mathcal{B}(H)$ s.t.

(i) $\langle \Phi(f)x, y\rangle = \int_K f dP_{x,y}$ for all $f \in L_\infty(P)$ for all $x, y \in H$.

(ii) $\|\Phi(f)x\|^2 = \int_K |f|^2 dP_{x,x}$ for all $f \in L_\infty(P)$ for all $x, y \in H$.

(iii) For $\mathcal{S} \in \mathcal{B}(H)$, $\mathcal{S}$ commutes with all $\Phi(f)$, $f \in L_\infty(P)$ $\iff$ $\mathcal{S}$ commutes with all $P(E)$, $E \in \mathcal{B}$.

Note: $\Phi(f)$ is uniquely determined by (i). We denote $\Phi(f)$ by $\int_K f dP$. So it says

$$\left\langle \int_K f dPx, y \right\rangle = \int_K f dP_{x,y}.$$

Proof. Sketch Define $\Phi(\mathbf{1}_E) = \int_K \mathbf{1}_E dP = P(E)$.

For simple functions $s = \sum_{j=1}^n a_j \mathbf{1}_{E_j}$, $\Phi(s) = \int_K s dP = \sum_{j=1}^n a_j P(E_j)$.

Φ is an isometric *-isomorphism, unital, on simple functions. Extend by density. □

Definition 8.3. let $L_\infty(K)$ be the set of all bounded Borel functions $f : K \to \mathbb{C}$. This is a commutative unital C^*-algebra with pointwise operations and the sup-norm $\|f\|_K = \sup_{z\in K} |f(z)|$. If P is a resolution of the identity of H over K, then $L_\infty(K) \subseteq L_\infty(P)$ and the inclusion is a norm decreasing unital *-homomorphism.

Theorem 8.1 (Spectral Theorem for commutative C^*-algebras). Let $A \subseteq \mathcal{B}(H)$ be a commutative unital C^*-algebra of $\mathcal{B}(H)$. Let $K = \Phi_A$. Then there exists a unique resolution of the identity of H over K, s.t.

$$\int_K \hat{T} dP = T, \qquad \text{for all } T \in A.$$

Moreover,

(i) $P(\mathcal{U}) \neq \varnothing$ for any $\neq \varnothing$, open $\mathcal{U} \subseteq K$.

(ii) $\mathcal{S} \in \mathcal{B}(H)$ commutes with all $T \in A \iff \mathcal{S}$ commutes with all $P(E)$, $E \in \mathcal{B}$.

Proof. By Theorem 7.1 the Gelfand map

$$\begin{aligned} A &\to \mathcal{C}(K) \\ x &\mapsto \hat{x} \end{aligned}$$

is an isometric *-isomorphism and hence so is its inverse

$$\begin{aligned} \mathcal{G}^{-1} : \mathcal{C}(K) &\to A \\ \hat{T} &\mapsto T. \end{aligned}$$

We see a roti P over K which represents $\mathcal{G}^{-1} : \mathcal{G}^{-1}(\hat{T}) = \displaystyle\int_K \hat{T} dP$.

This is an operator version of the Riesz Representation Theorem, Theorem 2.5.

Uniqueness: $T = \displaystyle\int_K \hat{T} dP$ for all T means

$$\langle Tx, y \rangle = \int_K \hat{T} dP_{x,y}, \quad \text{for all } T \in A, x, y \in H.$$

By uniqueness in the Riesz Representation Theorem (RRT), $P_{x,y}$ is uniquely determined for all $x, y \in H$. Since $P_{x,y}(E) = \langle P(E)x, y \rangle$, $P(E)$ is uniquely determined for all $E \in \mathcal{B}$.

Existence: For $x, y \in H$, $\hat{T} \mapsto \langle Tx, y \rangle : \mathcal{C}(K) \to \mathbb{C}$ is in $\mathcal{M}(K) = \mathcal{C}(K)^*$ with norm at most $\|x\| \cdot \|y\|$. By RRT, there exists a unique $\mu_{x,y} \in \mathcal{M}(K)$ s.t.

$$\langle Tx, y \rangle = \int_K \hat{T} d\mu_{x,y}, \quad \text{for all } T \in A.$$

$\|\mu_{x,y}\|_1 \leqslant \|x\| \cdot \|y\|$. Now, by linearity

$$= \lambda \int_K \hat{T} d\mu_{x,z} + \int_K \hat{T} d\mu_{y,z}.$$

By uniqueness in the RRT, $\mu_{\lambda x + y, z} = \lambda \mu_{x,z} + \mu_{y,z}$. If $\hat{T}$ is real-valued, then T is hermitian, so

$$\int_K \hat{T} d\mu_{x,y} = \langle Ty, x \rangle = \overline{\langle Tx, y \rangle} = \int_K \hat{T} d\overline{\mu_{x,y}}.$$

By uniqueness in the RRT, $\mu_{y,x} = \overline{\mu_{x,y}}$.

Fix $f \in L_\infty(K)$. Then $\Theta : H \times H \times \mathbb{C}$, $\Theta(x, y) = \displaystyle\int_K f d\mu_{x,y}$ is a sesquilinear form and $|\Theta(x, y)| \leqslant \|f\|_\infty \cdot \|\mu_{x,y}\|_1 \leqslant \|f\|_\infty \cdot \|x\| \cdot \|y\|$. So there exists $\Psi(f) \in \mathcal{B}(H)$ s.t. $\langle \psi(f)x, y \rangle = \Theta(x, y) = \displaystyle\int_K f d\mu_{x,y}$ and $\|\Psi(f)\| = \|\Theta\| \leqslant \|f\|_K$.

We now have a map $\Psi : L_\infty(K) \to \mathcal{B}(H)$ s.t.

Ψ is linear: clear by the linearity of $\displaystyle\int_K f d\mu_{x,y}$.

Ψ is bounded: $\|\Psi(f)\| \leqslant \|f\|_K$.

Ψ is a *-map:

$$\begin{aligned} \langle \Psi(\overline{f})x, y \rangle &= \int_K \overline{f} d\mu_{x,y} = \overline{\int_K f d\mu_{y,x}} \\ &= \overline{\langle \Psi(f)y, x \rangle} = \langle x, \Psi(f)y \rangle \\ &= \langle \Psi^*(f)x, y \rangle, \quad \text{for all} x, y \in H. \end{aligned}$$

So $\Psi(\overline{(f)}) = \Psi(f)^*$.

$\underline{\Psi\restriction_{\mathcal{C}(K)} = \mathcal{G}^{-1}}$: have $\langle \Psi(\hat{T})x, y\rangle = \int_K \hat{T} d\mu_{x,y} = \langle Tx, y\rangle$ for all x, y. So $\Psi(\hat{T}) = T = \mathcal{G}^{-1}$.

<u>Ψ is multiplicative</u>: for $S, T \in A$.

$$\begin{aligned}\int_K \hat{S}\cdot\hat{T} d\mu_{x,y} &= \int_K \widehat{ST} d\mu_{x,y}\\ &= \langle STx, y\rangle\\ &= \int_K \hat{S} d\mu_{Tx,y}, \quad \text{S} \in A.\end{aligned}$$

By uniqueness in RRT, $\hat{T} d\mu_{x,y} = d\mu_{Tx,y}$ as measures. Hence,

$$\begin{aligned}\int_K f\hat{T} d\mu_{x,y} &= \int_K f d\mu_{Tx,y} = \langle \Psi(f)Tx, y\rangle\\ &= \langle Tx, \Psi(f)^* y\rangle = \int_K \hat{T} d\mu_{x,\Psi(f)^* y}, \quad \text{for all } T \in A, f \in L_\infty(K).\end{aligned}$$

By uniqueness in RRT, $f d\mu_{x,y} = d\mu_{x,\Psi(f)^* y}$. Finally, for $g \in L_\infty(K)$,

$$\begin{aligned}\int_K gf d\mu_{x,y} & \\ &= \int_K g d\mu_{x,\Psi(f)^* y}\\ &= \langle \Psi(gf)x, y\rangle\\ &= \langle \Psi(g)x, \Psi(f)^* y\rangle\\ &= \langle \Psi(f)\Psi(g)x, y\rangle, \quad \text{for all } x, y \in H.\end{aligned}$$

So $\Psi(fg) = \Psi(f)\cdot\Psi(g)$.

Define $P(E) = \Psi(\mathbf{1}_E)$. Easy to check P is a roti of H over K. $P_{x,x}(E) = \langle P(E)x, y\rangle = \int_K \mathbf{1}_E d\mu_{x,y} = \mu_{x,y}(E)$ for all $E \in \mathcal{B}$. So $P_{x,y} = \mu_{x,y}$. Also,

$$\begin{aligned}\left\langle \int_K \hat{T} dP_{x,y} \right\rangle &= \int_K \hat{T} dP_{x,y}\\ &= \langle \Psi(\hat{T})x, y\rangle\\ &= \langle Tx, y\rangle.\end{aligned}$$

So $\int_K \hat{T} dP = T$. (Without Lemma 8.1, could define $\int_K f dP = \Psi(f)$ for $f \in L_\infty(K)$).

(i) Fix $\mathcal{U} \subseteq K$, $\mathcal{U}$ open. By Urysohn, there exists $f : K \to [0,1]$ continuous, s.t. $\operatorname{supp} f \subseteq \mathcal{U}$, $f \neq 0$.
There exists $T \in A$, $\sqrt{f} = \hat{T}$. Then $T \neq 0$ so there exists $x \in H$ s.t. $Tx \neq 0$. $0 < \|Tx\|^2 = \langle T^2 x, x\rangle = \int_K \widehat{T^2} dP_{x,x} = \int_K f dP_{x,x} \leqslant P_{x,x}(\mathcal{U}) = \langle P(\mathcal{U}x), x\rangle$. So $P(\mathcal{U}) \neq 0$.

(ii) Let $\mathcal{S} \in \mathcal{B}(H)$. $\langle \mathcal{S}Tx, y\rangle = \langle Tx, \mathcal{S}^* y\rangle = \int_K \hat{T} dP_{x,\mathcal{S}^* y}$ and $\langle T\mathcal{S}x, y\rangle = \int_K \hat{T} dP_{\mathcal{S}x,y}$.

So

$$\begin{aligned}\mathcal{S}T = T\mathcal{S} \text{ for all } T \in A &\iff P_{x,\mathcal{S}^* y} = P_{\mathcal{S}x,y} \text{ for all } x, y \in H.\\ &\iff \langle P(E)x, \mathcal{S}^* y\rangle = \langle P(E)\mathcal{S}x, y\rangle \text{ for all } x, y \in H, E \in \mathcal{B}.\\ &\iff \mathcal{S}P(E) = P(E)\mathcal{S} \text{ for all } E \in \mathcal{B}.\end{aligned}$$

□

Lecture 24

Let A be a united Banach algebra and $x \in A$. We define $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ($x^0 = 1$) (converges absolutely, so converges in A). If $xy = yx$ in A, then $e^{x+y} = e^x \cdot e^y$.

Lemma 8.2 (Fugledo-Putman-Rosenblum). Let A be a unital C^*-algebra, $x, y, z \in A$ with x, y normal. If $xz = zy$, then $x^*z = zy^*$.

Proof. None given. □

Theorem 8.2 (Spectral Theorem for normal operators). Let $T \in \mathcal{B}(H)$ be normal. Then there exists a unique resolution of the identity of H over $\sigma(T) = \sigma_{\mathcal{B}(H)}(T)$, P s.t. $T = \int_{\sigma(T)} \lambda dP$ (i.e. the spectral decomposition of T). Moreover, $\mathcal{S} \in \mathcal{B}(H)$ commutes with T $\iff$ $\mathcal{S}$ commutes with all $\overset{\text{spectral projections}}{P(E)}$, $E \in \mathcal{B}$.

Proof. Let A be the unital C^*-sub-algebra of $\mathcal{B}(H)$ generated by T.
So $A = \overline{\{p(T, T^*) : p \text{ poly in two variables}\}}$. T normal implies that A is a commutative C^*-sub-algebra. $\sigma_A(T) = \sigma(T)$ by Corollary 7.0.1. By Lemma 7.2, every $\phi \in \Phi_A$ is uniquely determined by $\phi(T)$. $[\phi(T^*) = \overline{\phi(T)}, \phi(p(T, T^*)) = p(\phi(T), \phi(T^*))]$. Thus, the map

$$\begin{aligned} \Phi_A &\to \sigma(T) \\ \phi &\mapsto \phi(T) \end{aligned}$$

is a continuous bijection (Corollary 5.4.1) and thus a homeomorphism. $\hat{T}, \widehat{T^*}$ in $\mathcal{C}(\Phi_A)$ correspond to $\lambda \mapsto \lambda$, $\lambda \mapsto \overline{\lambda}$ in $\mathcal{C}(\sigma(T))$ respectively.

Existence of P: follows from Theorem 8.1.

Uniqueness: if $T = \int_{\sigma(T)} \lambda dP$, then $p(T, T^*) = \int_{\sigma(T)} p(\lambda, \overline{\lambda}) dP$ (Lemma 8.1). So $\langle p(T, T^*)x, y\rangle = \int_{\sigma(T)} p(\lambda, \overline{\lambda}) dP_{x,y}$. Since, $\lambda \mapsto p(\lambda, \overline{\lambda})$ are dense in $\mathcal{C}(\sigma(T))$, by uniqueness in RRT, $P_{x,y}$ are uniquely determined and hence so is P.

If $\mathcal{S}T = T\mathcal{S}$, then $\mathcal{S}T^* = T^*\mathcal{S}$ by Lemma 8.2. Finally, $\mathcal{S}T = T\mathcal{S} \iff \mathcal{S}$ commutes with all elements of A, $\iff$ $\mathcal{S}$ commutes with $P(E)$, for all in $E \in \mathcal{B}$ (Theorem 8.1). □

Theorem 8.3 (Borel Functional Calculus). Let T be a normal operator, let $K = \sigma(T)$ and P be the roti of H over K given by Theorem 8.2. The map

$$\begin{aligned} L_\infty(K) &\to \mathcal{B}(H) \\ f \mapsto f(T) &:= \int_{\sigma(T)} f(\lambda) dP \end{aligned}$$

has the following properties:

(i) it is a unital *-homomorphism s.t. $z(T) = T$ (where $z(\lambda) = \lambda$ for all $\lambda \in K$).

(ii) $\|f(T)\| \leqslant \|f\|_K$ for all $f \in L_\infty(K)$ with equality if $f \in \mathcal{C}(K)$.

(iii) For $\mathcal{S} \in \mathcal{B}(H)$, $\mathcal{S}T = T\mathcal{S} \iff \mathcal{S}f(T) = f(T)\mathcal{S}$ for all $f \in L_\infty(K)$.

(iv) $\sigma(f(T)) \subseteq \overline{f(K)}$ for all $f \in L_\infty(K)$.

Proof. Everything follows from Lemma 8.1, Theorems 8.1 and 8.3. (Note that $f(T) = \Psi(f)$ from Theorem8.1). For (iv), $\sigma(f(T)) \subseteq \sigma_{L_\infty(K)}(f) = \overline{f(K)}$. □

Theorem 8.4 (Polar Decomposition). Let $T \in \mathcal{B}(H)$ be normal. Then, there exists a positive operator $\mathcal{R}$, unitary $\mathcal{U}$ s.t. $T = \mathcal{R}\mathcal{U}$. Also, $T, \mathcal{R}, \mathcal{U}$ pointwise commute.

Proof. Define r, u on $\sigma(T)$:

$$r(\lambda) = |\lambda|,\ u(\lambda) = \begin{cases} \frac{\lambda}{|\lambda|}, & \lambda \neq 0 \\ 1, & \text{if } \lambda = 0 \in \sigma(T). \end{cases}$$

Then, $r, u \in L_\infty(\sigma(T))$ and $ru = z$ ($z(\lambda) = \lambda$ for all $\lambda \in \sigma(T)$) let $\mathcal{R} = r(T), \mathcal{U}$. Then $T = Z(T) = \mathcal{R}\mathcal{U}$. r is positive, u is unitary in $L_\infty(\sigma(T))$ and hence $\mathcal{R}$ is positive, $\mathcal{U}$ is unitary in $\mathcal{B}(H)$. Since $L_\infty(K)$ is commutative, $\mathcal{R}, \mathcal{U}, T$ must commute. □

Theorem 8.5 (Unitaries as exponentials). Let $\mathcal{U} \in \mathcal{B}(H)$ be unitary. Then there exists hermitian Q s.t. $\mathcal{U} = e^{iQ}$.

Proof. By Corollary 7.0.1, $\sigma(u)^1$. Let $f : S^1 \to \mathbb{R}$ be in $L_\infty(S^1)$ s.t. $e^{if(t)} = t$ for all $t \in S^1$. Let $Q = f(\mathcal{U})$. Then Q is hermitian since f is hermitian in $L_\infty(K)$.

$$\sum_{k=0}^{n} \frac{(if(t))^k}{k!} \to t, \qquad \text{uniformly on } S^1.$$
$$\sum_{k=0}^{n} \frac{(iQ^)k}{k!} \to \mathcal{U},$$

i.e. $\mathcal{U} = e^{iQ}$. □

Theorem 8.6 (Connectedness of $\mathcal{G}(\mathcal{B}(H))$). Fix $T \in \mathcal{G}(\mathcal{B}(H))$. $T = \mathcal{R}\mathcal{U}$, $\mathcal{R}$ positive, $\mathcal{U}$ unitary (Theorem 8.4) where $\mathcal{R}, \mathcal{U} \in \mathcal{G}(\mathcal{B}(H))$.

Proof. Since $\mathcal{R}$ is invertible, $\sigma(\mathcal{R}) \subseteq (0, \infty)$. Let $\mathcal{S} = \log(\mathcal{R}) = \displaystyle\int_{\sigma(\mathcal{R})} \log \lambda dP$ (P is a roti of H over K).

$$e^{\mathcal{S}} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\mathcal{S})^k}{k!} = \lim_{n\to\infty} \overbrace{\sum_{k=0}^{n} \frac{(\log\lambda)^k}{k!}}^{\to \text{ uniformly on } \sigma(\mathcal{R})} (\mathcal{R}) = z(\mathcal{R}) = \mathcal{R}.$$

So $T = e^{\mathcal{S} \cdot e^{iQ}}$. The map $[0,1] \to \mathcal{G}(\mathcal{B}(H)) : t \mapsto e^{t\mathcal{S}} \cdot e^{itQ}$ is a continuous path from Id to T. Hence $\mathcal{G}(\mathcal{B}(H))$ is connected. □

End of lecture course.