Part III Elliptic PDEs
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Contents

Leecture 1

Elliptic PDEs

Lectures 1–12 G. Tajanauskas `tajanauskas@dpmms.cam.ac.uk`
Lectures 13–24 Nushan Wickramasekera

Prerequisites Part II Analysis of PDEs
Reading:

- Gilbarg and Trudinger, *Elliptic PDEs of 2nd order*, [GTGT77]

- (Paper): L. Simon "Schauder estimates by scaling", Calc. Var. PDE **5** (1997), pp. 391–407.
- (Old lecture notes): `winterscompositeness.wordpress.com/lecture-notes/`

Broader reading:

- Folland "Introduction to PDEs"
- Evans and Gariepy "Measure Theory and Fine Properties of Functions"

1 Introduction

We study 2^{nd} order elliptic PDEs on (a domain in) $\mathbb{R}^n$, as e.g. arising from variational problems, and ultimately, non-linear PDEs. But first, it is imperative to understand linear theory.

Setup: Consider for $\Omega \subset \mathbb{R}^n$ open, bounded

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad (x, z, p) \mapsto F(x, z, p)$$

and consider the variational problem of extremising the functional:

$$\mathcal{F}[u] := \int_\Omega F(x, u(x), \nabla u(x)) \, \mathrm{d}x \,, \quad u \in \mathcal{S} \,,$$

where we adopt the notation $\partial = D = \nabla$, assuming F is sufficiently regular and $\mathcal{S}$ a suitable vector space of functions $u : \Omega \to \mathbb{R}$, frequently

$$\mathcal{S} = H^1(\Omega) = \{f \in L^2(\Omega) : \nabla f \in L^2(\Omega)\}$$

or

$$\mathcal{S} = C^{1,\alpha}(\overline{\Omega}), \quad \alpha \in (0, 1) \,,$$

which we will encounter later.

Suppose now that u minimises $\mathcal{F}[u]$ subject to[1] $u|_{\partial\Omega} = g$ for some given $g : \partial\Omega \to \mathbb{R}$. So for all $\varphi \in \mathcal{S}$ $\mathcal{F}[u + t\varphi] \geq \mathcal{F}[u]$. This means that

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}[u + t\varphi] = 0$$

or

$$\left.\frac{d}{dt}\right|_{t=0} \int_\Omega F(x, u + t\varphi, \nabla u + t\nabla\varphi) \, \mathrm{d}x = 0 \,.$$

Assuming enough regularity to exchange d/dt and $\int$, get

$$\int_\Omega (\partial_z F(x, u, \nabla u)\varphi + \mathrm{div}_p(\partial_p F)(x, u, \nabla u)) \, \mathrm{d}x = 0 \,. \tag{1.1}$$

To ensure the perturbed function $u + t\varphi$ has correct boundary conditions (BCs), we need $\varphi|_{\partial\Omega} = 0$. Integrating (1.1) by parts yields

$$\int_\Omega \varphi(x) \left(\partial_z F - \partial_i D_{p_i} F\right)(x, u, \nabla u) \, \mathrm{d}x = 0$$

[1] this tends to be needed for well-posedness.

for all $\varphi \in \mathcal{S}$ and so the Fundamental Lemma of Calc. of Var. gives

$$\frac{\partial F}{\partial z} - \partial_i \left(\frac{\partial F}{\partial p_i} \right) = 0 \quad \text{in } \Omega \, ,$$

that is, the Euler–Lagrange eqns for $F(\mathcal{F})$, which can be re-written as

$$\frac{\partial F}{\partial z} - \partial_i \partial_j u \frac{\partial^2 F}{\partial p_i \partial p_j} = 0 \tag{1.2}$$

– a 2^{nd} order *quasilinear PDE* in u, which means the term in front of $\partial^2 u$ does not depend on $\partial^2 u$.

More generally, consider the PDE

$$\partial_i \Big(a_{ij}(x, u, \nabla u) \partial_j^2 u - b(x, u, \nabla u) \Big) = 0 \, . \tag{1.3}$$

Definition 1.1. *We say (1.3) is* elliptic in Ω *if* $a_{ij}(x, u, \nabla u)$ *is a positive-definite matrix in* Ω. *In the case (1.2), this is then equivalent to "F is convex in p".*

Example. (Dirichlet energy.) When $F(x, z, p) = |p|^2$ one gets

$$\Delta u = 0 \, . \tag{1.4}$$

Extremizers of (1.4) are called *harmonic functions.*

Example. (Minimal surfaces.) When $F(x, z, p) = \sqrt{1 + |p|^2}$ (observe the interpretation of the functional $\mathcal{F}[u]$ [ ]), one gets

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \tag{0.5}$$

– the *minimal surface equation.*

Remark. *Locally* $\nabla u \approx$ *constant, so (0.5) looks similar to (1.4), and so solutions have similar local properties. But the existence theory for (1.4) is "trivial", while the existence theory for (0.5) may fail. (Global properties are important!) For entire solutions (i.e. defined on all of* $\mathbb{R}^n$*), global behaviour very different:*

Theorem 1.1 (Liouville)**.** *Let* $u : \mathbb{R}^n \to \mathbb{R}$, $u \in C^2$, $\Delta u = 0$ *and* u *is bounded, then* $u \equiv$ *const.*

Theorem 1.2 (Bernstein)**.** *The only entire solutions to (0.5) in* $\mathbb{R}^n$ *are planar (u is linear)* $\Leftrightarrow n \leq 7$.

1.1 Harmonic Functions

1.2 Basic Properties

Let $\Omega \subset \mathbb{R}^n$ be a domain (open and connected).

Definition 1.2. *A function* $u \in C^2(\Omega)$ *is* harmonic *if* $\Delta u = 0$, <u>*subharmonic*</u> *if* $\Delta u \geq 0$, <u>*superharmonic*</u> *if* $\Delta u \leq 0$ *in* Ω.

Write $B_\rho(y) = \{x : |x - y| < \rho\}$.

Theorem 1.3 (Mean Value Property (MVP))**.** *If $u \in C^2(\Omega)$ is subharmonic and $B_\rho(y) \subset \Omega$, then*

$$u(y) \leq \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} u(x)\ \mathrm{d}x, \quad \omega_n = |B_1(0)| \tag{1.5}$$

i.e.,

$$u(y) \leq \frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho(y)} u(z)\ \mathrm{d}z\,. \tag{1.6}$$

If u is superharmonic*, then the inequalities are reversed. Finally, if u is* harmonic*, then we have equality.*

Proof. Let $u \in C^2(\Omega)$ be subharmonic. Then, we have

$$\begin{aligned}
0 &\leq \int_{B_\rho(y)} \Delta u\ \mathrm{d}x \\
&\overset{\text{IBP}}{=} \int_{\partial B_\rho(y)} \nabla u \cdot \nu\ \mathrm{d}\sigma, \quad \nu = \text{outward unit normal} \\
&\overset{x=y+\rho\nu}{=} \int_{S^{n-1}} \nabla u \cdot \rho\ \mathrm{d}\sigma \\
&= \rho^{n-1} \int_{S^{n-1}} \frac{\partial}{\partial \rho} \left(u(y+\rho\nu)\right) \mathrm{d}\sigma\,.
\end{aligned}$$

This is true for all $\rho > 0$

$$0 \leq \frac{\partial}{\partial \rho} \int_{S^{n-1}} u(y+\rho\nu)\ \mathrm{d}\sigma$$

i.e. the map $\rho \mapsto \int_{S^{n-1}} u(y+\rho\nu)\ \mathrm{d}\sigma$ is increasing. Hence, for $0 \leq \rho \leq r$

$$\int_{S^{n-1}} u(y+\rho\nu)\ \mathrm{d}\sigma \leq \int_{S^{n-1}} u(y+r\nu)\ \mathrm{d}\sigma\,.$$

By continuity, taking $\rho \to 0$ gives

$$n\omega_n u(y) \leq \int_{\partial B_\rho(y)} u(z)\ \mathrm{d}z.$$

This gives (1.6). To get (1.5), integrate in ρ. The superharmonic case is similar and the harmonic case combines both. $\square$

Remark. *The MVP characterises harmonic functions . Hint: can either proceed via a mollification argument for $u \in L^1_{\boldsymbol{loc}}$ or if $u \in C^0$ an alternative approach is to use the solvability of the Dirichlet problem in balls by elementary means (i.e. using the Poisson kernel), while* only *assuming the u satisfies the* local MVP*, that is for any point in the domain, there is a ball s.t. the MVP holds on that ball.*

Lecture 2

Theorem 1.4 (Strong Maximum Principle)**.** *Suppose $u \in C^2(\Omega)$ is sub-harmonic on Ω ($\Delta u \geq 0$), and suppose there exists $y_0 \in \Omega$ such that*

$$u(y_0) = \sup_{\partial\Omega} u\,.$$

Then $u \equiv const.$

Remark. *If u is superharmonic, then same statement holds with "*sup*" $\to$ "*inf*". If u is harmonic, both work.*

Proof. Let $M = \sup u < \infty$ and let

$$\Sigma := \{y \in \Omega : u(y) = M\}.$$

By assumption, $\Sigma \neq \emptyset$ since $y_0 \in \Sigma$, and Σ is closed as u is continuous. As Ω is connected, it suffices to show that Σ is open. Then $\Sigma = \Omega$.

Pick $y \in \Sigma$. By the MVP for $\rho > 0$ s.t. $B_\rho(y) \subset \Omega$, then

$$M = u(y) \leq \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} u(z) \, \mathrm{d}z$$

so

$$\frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} (M - u(z)) \, \mathrm{d}z \leq 0.$$

But of course, $M - u \geq 0$, so must have $u \equiv M$ on $B_\rho(y)$. So Σ is open. $\square$

Here the SMP is easy given the MVP. For more general PDEs, this is not so, we prove a weaker statement first.

Theorem 1.5 (Weak Maximum Principle (WMP))**.** *Suppose $\Omega \subset \mathbb{R}^n$ is bounded and $u \in C^2(\Omega) \cap C(\overline{\Omega})$. If u is subharmonic on Ω, then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

Proof. Immediate from SMP: since Ω is bounded and $u \in C(\overline{\Omega}) \Rightarrow$ sup and inf are attained. By the SMP, these are not attained in Ω° (unless $u \equiv$ const). $\square$

Remark. *If u is superharmonic $\Rightarrow$ replace "*sup*" with "*inf*". If u harmonic, both hold.*

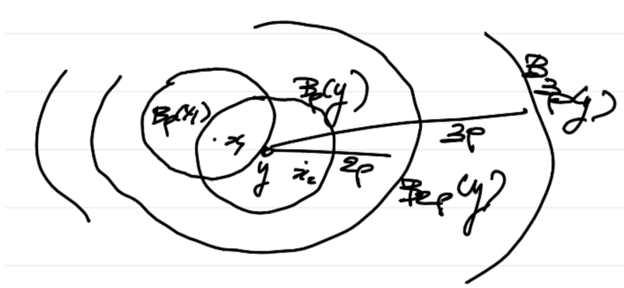
The MVP states that u always an 'average' of itself. Suggests that u cannot vary too much. Can we use this to relate sup and inf? It turns out we can.

Theorem 1.6 (Harnack's Inequality)**.** *Suppose $u \in C^2(\Omega), u \geq 0$ and $\Delta u = 0$ in Ω. Then if $\Omega' \subseteq \Omega$ is any closed subdomain, we have*

$$\sup_{\Omega'} u \leq C \cdot \inf_{\Omega'} u$$

for some $C = C(\Omega', \Omega) > 0$.

Proof. First, choose $y \in \Omega'$ and $\rho > 0$ s.t. $B_{4\rho}(y) \subset\subset \Omega$ and pick $x_1, x_2 \in B_\rho(y)$.



The MVP gives

$$u(x_1) = \frac{1}{\omega_n \rho^n} \int_{B_\rho(x_1)} u \, \mathrm{d}x, \quad B_\rho(x_1) \subset B_{2\rho}(y) \Rightarrow \leq \frac{1}{\omega_n \rho^n} \int_{B_{2\rho}(y)} u$$

$$\leq u(x_2) = \frac{1}{\omega_n (3\rho)^n} \int_{B_{3\rho}(x_2)} u \geq \int_{B_{2\rho}(y)} u \cdot \left(\frac{1}{\omega_n (3\rho)^n} \right), \quad B_{2\rho}(y) \subset B_{3\rho}(x_2)\,.$$

Combining these gives

$$u(x_1) \leq 3^n u(x_2) \quad \forall x_1, x_2 \in B_\rho(y)\,.$$

So Harnack holds locally in balls with constant independent of x_1, x_2, y as long as $B_{4\rho}(y) \subset \Omega$.

Now, choose $x_1, x_2 \in \Omega' \subseteq \Omega$, s.t.

$$\sup_{\Omega'} u = u(x_1), \quad \inf_{\Omega'} u = u(x_2)\,.$$

Then by path connectedness of Ω', there exists a path $\gamma \in \Omega'$ joining x_1 and x_2.

$$4\rho < \operatorname{dist}(\Omega', \partial\Omega) \leq \operatorname{dist}(\gamma, \partial\Omega)$$

Choose $\rho > 0$ s.t. $4\rho < \operatorname{dist}(\gamma, \partial\Omega)$. Now choose $\mathcal{N} = \mathcal{N}(\Omega', \Omega)$ s.t. we can cover γ by $\mathcal{N}$ balls of radius ρ, $\gamma \subset \bigcup B_\rho(y_i)$, $y_i \in \Omega'$.

Then apply the local result along each ball to get

$$u(x_1) \leq 3^n \cdot 3^n \cdot \dots \cdot 3^n u(x_2) \leq 3^{n\mathcal{N}} u(x_2)\,.$$

□

Theorem 1.7 (Derivative Estimates)**.** *Suppose $u \in C^2(\Omega)$ is harmonic in $\Omega \subset \mathbb{R}^n$ a domain. Then if $B_\rho(y) \subset \Omega$, then*

$$|\nabla u(y)| \leq \frac{C}{\rho} \sup_{\partial B_\rho(y)} |u|$$

for some $C = C(n)$.

Proof. Observe that for each $i \leq i \leq n$, $\Delta u = 0$ and so $0 = \partial_i(\Delta u) = \Delta(\partial_i u)$ in Ω. So for all i, $\partial_i u$ is harmonic. By the MVP,

$$\partial_i u(y) = \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} \partial_i u = \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} \nabla u \cdot \vec{e}_i = \frac{1}{\omega_n \rho^n} \int_{\partial B_\rho(y)} u \cdot \vec{n} \cdot \vec{e}_i \, \frac{ds}{\rho}$$

So

$$|\partial_i u(y)| \leq \frac{n}{\omega_n \rho^{n+1}} \sup |u| \int_{\partial B_\rho(y)} ds = \frac{n}{\rho} \sup_{\partial B_\rho(y)} |u|\,.$$

□

Remark. *Can apply this result repeatedly to get that for $\Omega' \subseteq \Omega$ and any multi-index α s.t. $|\alpha| \leq k < \infty$, and $u \in C^{2+k}(\Omega)$, $\Delta u = 0$ in Ω, then*

$$\sup_{\Omega'} |\partial^\alpha u| \leq C \cdot \sup_{\Omega} |u|$$

for some $C = C(C(n), \alpha, \Omega, \Omega') > 0$. Thus, $|\partial^\alpha u|_{C^0(\Omega')} \leq C \cdot |u|_{C^0(\Omega)}$

Also by the MVP for some $y \in B_\rho(x) \subset \Omega$

$$\sup_{\Omega'} |u(y)| = |u(x)| = \left| \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} u(x) \, \mathrm{d}x \right| \leq C \int_\Omega |u|$$

i.e.

$$|\partial^\alpha u|_{C^0(\Omega')} \leq C \cdot |u|_{L^1(\Omega)}.$$

Theorem 1.8 (Uniqueness of solutions to Dirichlet problem)**.** *Suppose that Ω is bounded and $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ and $\Delta u_i = f$ in Ω, $u_1 = u_2$ on $\partial\Omega$. Then $u_1 = u_2$ on Ω.*

Proof. Set $w = u_1 - u_2$. Then $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$.

By applying WMP, get $w = 0$ in Ω. □

Remark. *Can of course integrate by parts where last WMP won't apply for non-divergence form equations.*

Lecture 3

Last time, recall we showed that for $\Omega' \subseteq \Omega$

$$\sup_{\Omega'} |\partial^\alpha u| \leq C \cdot \int_\Omega |u| \, .$$

Theorem 1.9 (Liouville's Theorem for Harmonic Functions)**.** *Let $u \in C^\infty(\mathbb{R}^n)$ be harmonic on $\mathbb{R}^n$ and suppose u grows sub-linearly at ∞, then $u \equiv$ const.*

Remark. *"Growing sublinearly" means*

$$|u(x)| \leq C \cdot (1 + |x|^\alpha), \quad \alpha \in (0, 1).$$

Proof. From derivative estimates (Thm 1.7) we know that for all $y \in \mathbb{R}^n$

$$|\nabla u(y)| \leq \frac{C}{\rho} \sup_{B_\rho(y)} |u| \, .$$

Plugging in the growth assumption gives

$$|\nabla u(y)| \leq \frac{C}{\rho} \sup_{B_\rho(y)} |u| \leq \frac{C}{\rho}(1 + (\rho + |y|)^\alpha) \, .$$

Finally, taking $\rho \to \infty$ gives $\nabla u(y) = 0$, but y was arbitrary, so we are done. □

1.3 Existence Theory for Harmonic Functions

Classical problem: solve the Dirichlet problem for the Laplacian on Ω bounded and

$$\varphi : \overline{\Omega} \to \mathbb{R} \text{ continuous,}$$

we wish to find $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \quad u = \varphi & \text{on } \partial\Omega \end{cases}$$

We will assume for simplicity that $\partial\Omega$ is smooth and $\varphi \in C^\infty(\overline{\Omega}, \mathbb{R})$.

Methods to solve the problem:

(I) Hilbert Space Method: Use the Riesz representation theorem to get $u \in H^1(\Omega)$. Then deal with regularity afterwards. Relies on the equation being linear (cf. Analysis of PDE).

(II) Direct Method of Calculus of Variations: Rephrase $\Delta u = 0$ as a variational problem (i.e. the Euler–Lagrange equation of $\int |\nabla u|^2$), and prove existence using the functional.

(III) Perron's Method: Use the fact that solvability in balls implies solvability in more general domains. More later. Based on maximum principles.

Remark. *In all cases we obtain a regular solution first, and improve regularity later.*

Let us look at (II) in detail. Define

$$\mathcal{L} = \left\{ w \in H^1(\Omega) : w - \varphi \in H_0^1(\Omega) \right\}$$

i.e. H^1 functions which agree with φ on the boundary. Check $\varphi \in \mathcal{L}$, so $\mathcal{L} \neq \emptyset$. Set

$$E[u] = \int_\Omega |\nabla u|^2$$

and define

$$\beta = \inf_{w \in \mathcal{L}} E[w]\,.$$

By definition of inf, there exists $(w_j) \subset \mathcal{L}$ s.t.

$$E[w_j] \overset{j \to \infty}{\longrightarrow} \beta\,.$$

We want to extract a convergent subsequence and show that its limit is a solution. Clearly for j large

$$\int_\Omega |\nabla w_j|^2 \leq \beta + 1\,.$$

Since $w_j - \varphi \in H_0^1(\Omega)$, by the Poincaré inequality,

$$\int_\Omega |w_j - \varphi|^2 \leq C \cdot \int_\Omega |\nabla (w_j - \varphi)|^2$$

$$\Rightarrow \int_\Omega |w_j|^2 \leq C(\Omega, \varphi, \beta) < \infty\,.$$

Indeed,

$$|w_j - \varphi|^2_{L^2(\Omega)} \le C' |\nabla(w_j - \varphi)|^2_{L^2(\Omega)} \le C(\Omega, \varphi, \beta) < \infty$$
$$|w_j|^2 - 2\langle w_j, \varphi \rangle_{L^2(\Omega)} \le C(\Omega, \varphi, \beta)$$
$$\Rightarrow |w_j|^2_{L^2(\Omega)} \le C(\varphi, \Omega, \beta) + \varepsilon |w_j|^2_{L^2} + \frac{1}{\varepsilon} |\varphi|^2_{L^2}$$
$$\Rightarrow |w_j|^2_{L^2(\Omega)} \le C(\varphi, \Omega, \beta)\,.$$

Indeed,

$$|w_j - \varphi|^2_{L^2(\Omega)} \le C' |\nabla(w_j - \varphi)|^2_{L^2(\Omega)} \le C(\Omega, \varphi, \beta) < \infty.$$

$$|w_j|^2 - 2\langle w_j, \varphi \rangle_{L^2(\Omega)} \le C(\Omega, \varphi, \beta)$$

$$\Rightarrow |w_j|^2_{L^2(\Omega)} \le C(\varphi, \Omega, \beta) + \varepsilon |w_j|^2_{L^2} + \frac{1}{\varepsilon} |\varphi|^2_{L^2}$$

$$\Rightarrow |w_j|^2_{L^2(\Omega)} \le C(\varphi, \Omega, \beta)\,.$$

So we have $|w_j|^2_{H^1(\Omega)} \le C$ for j large, so by Banach–Alaoglu

$$w_j \rightharpoonup w \quad \text{in } H^1(\Omega)$$

and by Rellich–Kondrachov (see below)

$$w_{j_k} \to w \quad \text{in } L^p(\Omega)$$

for some $w \in H^1(\Omega)$.

Theorem 1.10 (Rellich–Kondrachov)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $1 \le p < n$*

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \quad \textit{and} \quad W^{1,p}_0(\Omega) \subset\subset L^q(\Omega) \quad \textit{for } 1 \le q < p^*,$$

where $p^* = \frac{np}{n-p}$. When $p = 2$,

$$p^* = \frac{2n}{n-2} > 2 = p \quad \textit{iff } n > 2\,.$$

Proof. Omitted, see the book of Evans, [Eva22]. □

Hence for all $v \in H^1(\Omega)$ we have

$$\int_\Omega \nabla w_j \cdot \nabla v \to \int_\Omega \nabla w \cdot \nabla v\,.$$

Also, it is clear that $w_j - \varphi \rightharpoonup w - \varphi$ in $H^1(\Omega)$ as φ is smooth. But $w_j - \varphi \in H^1_0(\Omega)$ and $H^1_0(\Omega)$ is norm-closed, hence weakly closed. This follows from Hahn–Banach for any convex subset of a Banach space. For completeness, see the lemma below.

Lemma 1.1 (Mazur)**.** *X a Banach space. Then if $C \subset X$ a convex subset then*

$$C \textit{ is norm closed} \iff C \textit{ is weakly closed}.$$

Proof. $\underline{\Longleftarrow :}$ [✍] .

$\underline{\Longrightarrow :}$ We show $X \setminus C$ is weakly open. Let $x \in X \setminus C$. By Hahn–Banach separation, there exists f such that $f(x) = \alpha$ and $f(z) < \alpha$ for all $z \in C$. Then

$$x \in \left\{ z \in X : |f(z) - \alpha| < \frac{1}{2}|f(x)| \right\} \subset X \setminus C \text{ is weakly open}.$$

□

Hence $w \in H^1(\Omega)$ i.e. $w \in \mathcal{L}$. Finally since $\mathcal{E}[\cdot]$ is sequentially weakly lower semicontinuous in $H^1(\Omega)$, we have:

$$\mathcal{E}[w] \leq \liminf_{k\to\infty} \mathcal{E}[w_{j_k}] = \beta$$

so $\mathcal{E}[w] = \beta$. $\mathcal{E}[\cdot]$ is sequentially weakly lower semi-continuous means for all $u_j \rightharpoonup u$ in $H^1(\Omega)$, $\mathcal{E}[u] \leq \liminf_{j\to\infty} \mathcal{E}[u_j]$. To see this, note

$$\int_\Omega Du_j \cdot Dv \to \int_\Omega Du \cdot Dv$$

so with $v = u$

$$\int_\Omega Du_j \cdot Du \to \int_\Omega |Du|^2$$

so

$$\mathcal{E}[u] = \lim_{j\to\infty} \int_\Omega Du_j \cdot Du = \liminf_j \int_\Omega Du_j \cdot Du \leq \liminf \mathcal{E}[u_j]^{1/2}\mathcal{E}[u]^{1/2}$$

We have found a global min w, i.e.

$$\forall v \in H^1_0(\Omega), u + tv \in \mathcal{L}, \quad \mathcal{E}[w + tv] \geq \mathcal{E}[w]$$

i.e. the derivative of $\mathcal{E}[w + tv]$ at $t = 0$ vanishes.

$$f(t) := \mathcal{E}[w + tv] \Rightarrow \lim_{t\to 0} \frac{\mathcal{E}[w + tv] - \mathcal{E}[w]}{t} = 2\int_\Omega Dw \cdot Dv \quad \forall v \in H^1_0(\Omega).$$

This is the *weak formulation* of $\Delta u = 0$.

Next time: regularity theory (even if solutions of $\Delta u = 0$ are smooth).

Lecture 4

1.4 Interior Regularity

We wish to prove more regularity for the weak solution. A key result in this direction is the following.

Theorem 1.11 (Weyl's Lemma)**.** *Weakly harmonic functions are smooth. That is, for $\Omega \subset \mathbb{R}^n$ open and $u \in L^1_{loc}(\Omega)$, if*

$$\int_\Omega u\Delta v = 0 \quad \forall v \in C_c^\infty(\Omega)$$

then $u \in C^\infty(\Omega)$ and $\Delta u = 0$.

Proof. Mollify u: take $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \equiv 0$ in $\mathbb{R}^n \setminus B_\epsilon(0)$, $\varphi \geq 0$

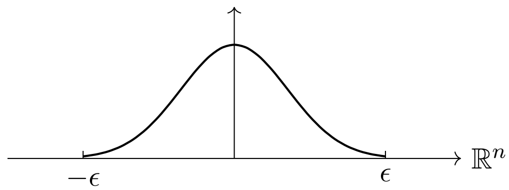


Figure 1: empty image

and

$$\int_{\mathbb{R}^n} \varphi = 1\,.$$

w.l.o.g. take φ to be radially symmetric. For $\epsilon > 0$ put

$$\varphi_\epsilon(x) = \frac{1}{\sigma_n \epsilon^n} \varphi\left(\frac{x}{\epsilon}\right)\,.$$

Then $\varphi_\epsilon \in C_c^\infty(B_\epsilon(0))$, $\varphi_\epsilon \geq 0$ and

$$\int_{\mathbb{R}^n} \varphi_\epsilon = 1.$$

This is the "standard mollifier". Define

$$u_\epsilon(x) = (\varphi_\epsilon * u)(x).$$

Then this is well-defined for $x \in \Omega_\epsilon \coloneqq \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$ (u only defined in Ω). Then u_ϵ is smooth in Ω_ϵ and $u_\epsilon \to u$ in $L^1_{\text{loc}}(\Omega)$. [Eva18, Theorem 4.1] and also $\Delta u_\epsilon = 0$ give

$$\frac{\partial}{\partial x_i} u_\epsilon(x) = \int_\Omega u(y) \frac{\partial}{\partial x_i} \varphi_\epsilon(x-y)\ \mathrm{d}y = -\int_\Omega u(y) \frac{\partial}{\partial y_i} \varphi_\epsilon(x-y)\ \mathrm{d}y$$

$$\Rightarrow \Delta_x u_\epsilon(x) = \int_\Omega u(y) \Delta_y \varphi_\epsilon(x-y)\ \mathrm{d}y = 0.$$

By a priori derivative estimates for harmonic functions, for $\Omega' \subset\subset \Omega$,

$$\sup_{\Omega'} |D^\alpha u_\epsilon| \leq C \int_{\Omega'_{\epsilon_1} \subseteq \Omega} |u_\epsilon|$$

for some $\epsilon_1 = \epsilon_1(\Omega')$ small, where

$$\Omega'_\epsilon \coloneqq \Omega' \cup \{x \in \Omega : \operatorname{dist}(x, \partial\Omega') < \epsilon_1\}.$$

But $u_\epsilon \to u$ in $L^1_{\text{loc}}(\Omega)$, so for small enough ϵ_1,

$$\int_{\Omega'_{\epsilon_1}} |u_\epsilon| \leq C\left(\int_{\Omega'_{\epsilon_1}} |u| + 1\right)$$

Hence

$$\sup_{\Omega'} |D^\alpha u_\epsilon| \leq C\left(\int_{\Omega'_{\epsilon_1}} |u| + 1\right)$$

i.e. $D^\alpha u_\epsilon$ uniformly bounded in $L^\infty(\Omega')$. Hence (as bounded derivatives $\Rightarrow$ equicontinuous) by Arzelà–Ascoli

$$\exists (\epsilon_{j_k})_{k=1}^\infty \subset (\epsilon_j) \text{ s.t. } \exists \tilde{u} \in C^\infty(\Omega') \text{ s.t. } u_{\epsilon_{j_k}} \to \tilde{u} \text{ in } C^k(\Omega') \quad \forall k \in \mathbb{N}.$$

Hence

$$\Delta \tilde{u} = \lim \Delta u_{\epsilon_j} = 0 \text{ in } \Omega$$

as Ω' was arbitrary. But also $u_\epsilon \to u$ a.e. in Ω (properties of mollification) $\Rightarrow u = \tilde{u}$ a.e. □

Remark. *We do not say anything about boundary regularity. But it is possible to get at least $u \in C^\infty(\overline{\Omega})$.*

Let's now improve our $C^\infty(\partial\Omega)$ existence result to $C^0(\partial\Omega)$.

Theorem 1.12 (Existence and Uniqueness for the Dirichlet Problem with $C^0(\partial\Omega)$ data)**.** *Suppose Ω is bounded with sufficiently regular (*) boundary $\partial\Omega$. Then for any $\varphi \in C^0(\partial\Omega)$, there exists*

$$u \in C^\infty(\Omega) \cap C^0(\overline{\Omega}) \quad \textit{solving} \quad \begin{cases} \Delta u = 0 & \textit{in } \Omega \\ u = \varphi & \textit{on } \partial\Omega\,. \end{cases}$$

Remark. *We might have $\int_\Omega |\nabla u|^2 = \infty$ for this solution.*

Proof. Choose a sequence $(\varphi_n)_n \subset C^\infty(\mathbb{R}^n)$ s.t. $\varphi_n \to \varphi$ on $\partial\Omega$ uniformly. Then we know there exists $u_n \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ s.t.

$$\Delta u_n = 0, \quad u_n = \varphi_n \text{ on } \partial\Omega.$$

Then for all $n, m \in \mathbb{N}$,

$$\Delta(u_n - u_m) = 0 \text{ in } \Omega, \quad u_n - u_m = \varphi_n - \varphi_m \text{ on } \partial\Omega.$$

By the WHP,

$$\sup_{\overline{\Omega}} |u_n - u_m| \le \sup_{\partial\Omega} |u_n - u_m| = \sup_{\partial\Omega} |\varphi_n - \varphi_m| \to 0 \quad \text{as } n, m \to \infty\,.$$

So $(u_n)_n$ is Cauchy in $C^0(\overline{\Omega})$. By completeness of $C^0(\overline{\Omega})$, there exists $u \in C^0(\overline{\Omega})$ s.t. $u_n \to u$ uniformly on $\overline{\Omega}$. In particular, $u = \varphi$ on $\partial\Omega$.

Furthermore, by the derivative estimates for $(u_n)_n$, also converges in $C^k(\Omega')$ $\forall \Omega' \subseteq \Omega$ so $u \in C^\infty(\Omega)$ and $\Delta u = 0$. □

Remark. *If $\partial\Omega$ is C^2, then (*) is satisfied, [ ] . More generally, enough to have exterior sphere condition:*

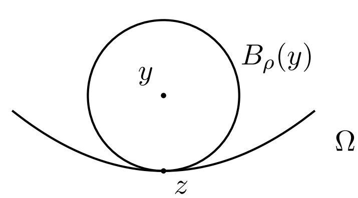


Figure 2: Exterior sphere condition: $B_\rho(y) \cap \Omega = \{z\}$

There exist bounded domains in which this fails (and the conclusion of the Thm fails), e.g. when $\partial\Omega$ has a cusp.

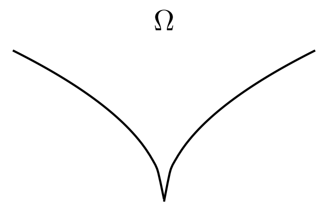


Figure 3: A domain Ω with an upward-opening concave cusp on the boundary

Having established the basic theory and properties of Harmonic functions, we turn out attention to more general 2^{nd} order *elliptic* ddifferential operators.

2 General 2^{nd} Order Elliptic Operators

From now on, write

$$Lu = a^{ij} D^2_{ij} u + b^i \partial_i u + cu.$$

Work on $\Omega \subset \mathbb{R}^n$ open and $u \in C^2(\overline{\Omega})$, with $a^{ij}, b^i, c : \Omega \to \mathbb{R}$ and consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

for given $f : \Omega \to \mathbb{R}$ and $\varphi : \partial\Omega \to \mathbb{R}$. If we can write L in divergence form,

$$Lu = \partial_i (a^{ij} \partial_j u) + b^i \partial_i u + cu,$$

then one can use Hilbert space theory. If not, we have to use Schauder theory. The main idea is to deform L into Δ using a series of rescalings (does not involve Sobolev spaces)! Since $u \in C^2(\Omega)$, let's assume $a^{ij} = a^{ji}$.

Definition 2.1. *(I) L is **elliptic** in Ω if the matrix $a^{ij}(x)$ is positive definite in Ω. That is,*

$$0 < \lambda(x)|\xi|^2 \le a^{ij}(x)\xi_i \xi_j \le \Lambda(x)|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

where $\lambda(x) =$ min. eigenvalue of $a^{ij}(x)$, $\Lambda(x) =$ max.

*(II) L is **strictly elliptic** in Ω if there exists $\lambda_0 > 0$ s.t. $\lambda(x) \ge \lambda_0$ for all $x \in \Omega$.*

*(III) L is **uniformly elliptic** in Ω if L is elliptic and $\Lambda(x)$ is uniformly bounded in Ω.*

Remark. *uniform ellipticity does not necessarily imply strict ellipticity.*

Example: Minimal Surface Equation

$$\nabla \cdot \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0$$

i.e.

$$a^{ij} = \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2}, \quad \frac{1}{\sqrt{1 + |Du|^2}}$$

is elliptic, but not uniformly.

Lecture 5 Goal: to obtain existence and regularity results for general 2^{nd} order elliptic operators with $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$. Note we do not exploit (yet) any divergence structure for L ($a^{ij}, c \in C^1$).

2.1 Basic Properties

Theorem 2.1 (Weak Maximum Principle)**.** *Suppose that L is elliptic and that*

$$\sup_{\Omega}\left|\frac{b^i}{\lambda}\right| < \infty.$$

Suppose Ω is bounded, open and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $Lu \geq 0$ (i.e. u is a subsolution) then

1. *If $c = 0$, then $\sup_{\Omega} u = \sup_{\partial\Omega} u$.*
2. *If $c \leq 0$, then $\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+$ where $u^+ \coloneqq \max(u, 0)$.*

Remark. *The assumption that $c \leq 0$ in Ω is crucial: e.g., $n = 1$, $\Omega = (0, \pi)$, $u'' + u = 0$, $u(x) = \sin x$, with $c = 1$,*

$$\sup u = 1, \quad \sup_{\partial\Omega} u = 0.$$

Or $n = 1$, $\Omega = (0, 2\pi)^2$, $Lu + 2u = 0$, $u(x, y) = \sin(x)\sin(y)$. Here also $u|_{\partial\Omega} \equiv 0$.

Proof. (1) ($c = 0$). If $Lu > 0$ in Ω, then in fact SMP holds. Indeed, if $x_0 \in \Omega$ is a local max, then

$$Du(x_0) = 0 \quad \text{and} \quad \partial_i \partial_j u(x_0) \leq 0.$$

Since $a^{ij}(x_0) \geq 0$, have $a^{ij}\partial_i\partial_j u(x_0) \leq 0 = \mathrm{Tr}(A \cdot \nabla^2 u(x_0)) \leq 0$. To see this briefly, diagonalise both to see that

$$\mathrm{Tr}\left(\underbrace{\Lambda}_{\succ 0} \cdot \underbrace{(\cdots)}_{\preceq 0}\right) \leq 0.$$

Hence

$$0 < Lu(x_0) = a^{ij}D^2_{ij}u(x_0) + b^i\partial_i u(x_0) \leq 0 \qquad \lightning .$$

More generally, if $Lu \geq 0$, consider $v(x) = e^{\gamma x_1}$ for some $\gamma.0$ to be chosen (or more generally any index i, here w.l.o.g. taken to be $i = 1$, for which $\sup\left|\frac{b^i}{\lambda}\right| < \infty$). We thus have

$$\partial_1 v = \gamma e^{\gamma x_1}, \quad \partial_i v = 0 \quad \forall i \neq 1,$$

$$\partial_1\partial_1 v = \gamma^2 e^{\gamma x_1}, \quad \partial_i\partial_j v = 0 \quad \forall i \neq j.$$

Then

$$\begin{aligned} Lv &= e^{\gamma x_1}\left(a^{11}\gamma^2 + b^1\gamma\right) = e^{\gamma x_1}\left(\gamma^2 a^{11} + \gamma b^1\right) \\ &= \lambda e^{\gamma x_1}\left(\gamma^2 + \frac{b^1}{\lambda}\gamma\right) > 0 \text{ in } \Omega \text{ by choosing } \gamma \text{ large.} \end{aligned}$$

Since $Lu \geq 0 \Rightarrow L(u + \varepsilon v) > 0 \quad \forall \varepsilon > 0$. Applying the first case, have

$$\begin{aligned} u(x) \leq \sup_{\partial\Omega}(u + \varepsilon v) &\leq \sup_{\partial\Omega} u + \varepsilon \sup_{\partial\Omega} v \\ &\leq \sup_{\partial\Omega} u + \varepsilon \sup_{\Omega} v. \end{aligned}$$

Take $\varepsilon \to 0$ to get $u(x) \leq \sup_{\partial\Omega} u$. True for all x, so $\sup_\Omega u \leq \sup_{\partial\Omega} u$. The inequality

$$\sup_{\partial\Omega} u \leq \sup_\Omega u$$

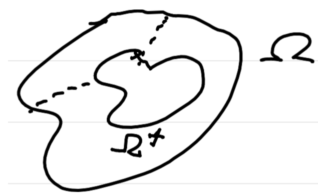
is trivial. (2) ($c \leq 0$). Define

$$L_0 u = a^{ij} D^2_{ij} u + b^i \partial_i u,$$

$$\text{Define } \Omega^+ := \{x \in \Omega : u(x) > 0\}.$$

Since $cu \leq 0$,

$$L_0 u = Lu - cu \geq 0 \text{ on } \Omega^+.$$



Note: if $\Omega^+ = \emptyset$, then $u \leq 0$ on Ω and so $u^+ \equiv 0$, so conclusion is trivial. w.l.o.g. assume $\Omega^+ \neq \emptyset$. Then $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$ and there exists $x_0 \in \partial\Omega^+ \cap \Omega$ such that $u(x_0) = 0$. (If not, then $\partial\Omega^+ \cap \Omega = \emptyset$, then

$$\partial\Omega^+ \subset \Omega^+ \subset \Omega,$$

so

$\partial\Omega^+ \subset \Omega \setminus \Omega^+$, so $u|_{\partial\Omega^+} \leq 0$. But this contradicts (1) for L_0 on Ω^+. Hence

$$\sup_\Omega u = \sup_{\Omega^+} u = \sup_{\partial\Omega^+} u^+ \leq \sup_{\partial\Omega^+} u^+, \quad \text{as } \Omega^+ \neq \emptyset,$$

$$\leq \sup_{\partial\Omega} u^+ \quad \text{as points on } \partial\Omega^+ \text{ are either in } \Omega^\circ \text{ or } \partial\Omega.$$

□

Some corollaries

Corollary 2.1. *Let Ω be bounded, open and $\Omega \in C^2(\overline{\Omega})$. Suppose L is elliptic and*

$$\sup_\Omega \left| \frac{b^i}{\lambda} \right| < \infty, \quad c \leq 0 \textit{ in } \Omega.$$

Then

(1) If $Lu \leq 0$ and then $u \geq \inf_{\partial\Omega} u = \min(u, 0)$,

(2) If $Lu = 0$, then $\sup |u| = \sup_{\partial\Omega} |u|$.

Proof. [ ] . □

Corollary 2.2. *Let L as above. Suppose*

$$u, v \in C^2(\Omega) \cap C(\overline{\Omega}) \textit{ satisfy } Lu \geq 0, \quad Lv = 0, \quad Lw \leq 0.$$

Then

(i) If $u \leq v$ on $\partial\Omega$, then $u \leq v$ on $\overline{\Omega}$.

(ii) If $v \leq w$ on $\partial\Omega$, then $v \leq w$ on $\overline{\Omega}$.

Proof. [ ] . $\square$

We want to work towards a SMP. This is achieved by the following geometric lemma.

Theorem 2.2 (Hopf Boundary Point Lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, C^1, $y \in \partial\Omega$ and suppose $\partial\Omega$ satisfies the **interior sphere condition** at y: there exist $R > 0, z \in \Omega$ s.t. $B_\rho(z) \subset \Omega$, $y \in \partial B_\rho(z)$. Let L be uniformly elliptic in Ω with*

$$\sup_\Omega |b^i| + \sup_\Omega |c| < \infty.$$

Suppose $u \in C^2(\Omega) \cap C^0(\Omega \cup \{y\})$ and satisfies $u(y) > u(x)\ \forall x \in \Omega,\ Lu \geq 0$ in Ω. Finally, assume one of the following:

(i) $c = 0$ in Ω,

(ii) $c \leq 0$ on Ω and $D_\nu u(y) > 0$,

(iii) $D_\nu u(y) = 0$ (and no assumption on c).

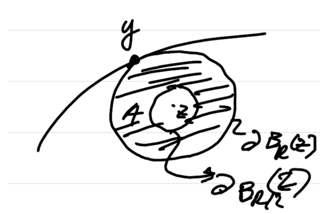
Then

$$\frac{\partial u}{\partial \nu}(y) < 0 \text{ if it exists, where}$$

ν is the outward unit normal at y to $\partial B_\rho(z)$.

Remark. *$\frac{\partial u}{\partial \nu}(y) > 0$ is given by the WMP.*

Proof. Let $A = B_R(z) \setminus \overline{B_r(z)}$ for some $0 < r < R$.



<u>Cases (i)/(ii):</u> on A consider

$$v(x) = e^{-\alpha|x-z|^2}, \quad e^{-\alpha R^2}$$

$$\partial_i v(x) = -2\alpha(x_i - z_i)e^{-\alpha|x-z|^2}$$

$$\partial_i\partial_j v(x) = -2\alpha\delta_{ij}e^{-\alpha|x-z|^2} + 4\alpha^2(x_i-z_i)(x_j-z_j)e^{-\alpha|x-z|^2}\,.$$

So on A

$$\begin{aligned}
Lv &= e^{-\alpha|x-z|^2}\left(a^{ij}(-2\alpha\delta_{ij} + 4\alpha^2(x_i-z_i)(x_j-z_j))\right) \\
&\quad - 2\alpha a^{ii} - 2\alpha b^i(x_i-z_i) - ce^{-\alpha R^2} \\
&\geq e^{-\alpha|x-z|^2}\left(4\alpha^2|x-z|^2\cdot\lambda - 2\alpha\Lambda n\right) \\
&\quad - 2\cdot\sup_\Omega|b|\cdot|x-z| - |c| \\
&\geq e^{-\alpha d(x)^2/2}\left(2\alpha\lambda - 2\alpha n\cdot\sup_\Omega\frac{|b|}{\lambda} - \sup_\Omega\frac{|c|}{\lambda}\right) - \alpha R\sup_\Omega\frac{|b|}{\lambda} - \sup_\Omega\frac{|c|}{\lambda}
\end{aligned} \tag{2.1}$$

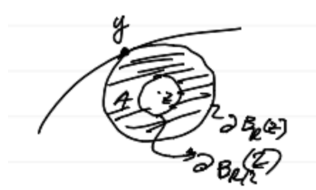
□

Lecture 6

Proof (Cont'd). Thus far, we have constructed

$$v(x) = e^{-\alpha|x-z|^2} - e^{-\alpha R^2}$$

such that $Lv > 0$ on A.



Put

$$w(x) = u(x) - u(y) + \varepsilon v(x), \quad \varepsilon > 0 \text{ TBD}\,.$$

Have

$$Lw = Lu + \varepsilon Lv - cu(y) > 0 \quad \text{in } A \text{ by above.}$$

Also have $v|_{\partial B_R(z)} = 0$ and $u(x) \leq u(y)$ on $\overline{\Omega}$, so $w|_{\partial B_r(z)} \leq 0$. Also, $u(x) < u(y)$ on $\partial B_R(z)$, so we can choose $\varepsilon > 0$ small enough s.t.

$$w|_{\partial B_R(z)} \leq 0.$$

Hence $w \leq 0$. Applying the WMP to w in A, we get

$$u(x) - u(y) + \varepsilon v(x) \leq 0 \quad \text{in } A\,.$$

Choose $t < 0$ so that

$$\frac{u(y+t\nu) - u(y)}{t} \geq -\varepsilon\frac{v(y+t\nu) - v(y)}{t}$$

Sending $t \to 0$:

$$\begin{aligned}
\frac{\partial u}{\partial\nu}(y) \geq -\varepsilon\frac{\partial v}{\partial\nu}(y) &= -\varepsilon\partial_i v(y)\cdot\frac{(y_i - z_i)}{R} \\
&= 2\alpha\varepsilon Re^{-\alpha R^2} > 0.
\end{aligned}$$

Case (iii): $(u(y) = 0)$ Consider

$$L^* = L - c^+ \text{ s.t.}$$

$$L^* u = a^{ij}\partial_i\partial_j u + b^i \partial_i u + \underbrace{(c - c^+)}_{c^- \le 0} u \,.$$

Then

$$\widetilde{L}u = Lu - c^+ u \ge 0$$

So apply the previous case to L. $\square$

Theorem 2.3 (Strong Maximum Principle (SMP))**.** *Suppose $\Omega \subset \mathbb{R}^n$ is a domain (not necessarily bounded) such that $\partial\Omega \neq \emptyset$ satisfies the interior sphere condition for all $y \in \partial\Omega$. Let L be uniformly elliptic,*

$$\sup_{\Omega} \left(\frac{|b| + |c|}{\lambda} \right) < \infty, \quad u \in C^2(\Omega), \quad u \in C(\overline{\Omega}),$$

$$M = \sup_{\Omega} u < \infty, \quad \textit{and} \quad Lu \ge 0 \ \textit{in}\ \Omega.$$

Then:

(i) If $c = 0$ and $u(y) = M$ for some $y \in \Omega$, then $u \equiv M$ in Ω.

(ii) If $c \le 0$, $M \ge 0$, and $u(y) = M$ for some $y \in \Omega$, then $u \equiv M$ in Ω.

(iii) If $M = 0$ and $u(y) = M = 0$ for some $y \in \Omega$, then $u \equiv 0$ in Ω.

Proof. Let $\Sigma = \{x \in \Omega : u(x) = M\}$. By continuity, Σ is closed in Ω. Suppose $\Omega \setminus \Sigma \neq \emptyset$. Pick $z \in \Omega \setminus \Sigma$ such that

$$\operatorname{dist}(z, \Sigma) > \operatorname{dist}(z, \partial\Sigma),$$

see the illustration below.

How?

- first pick $z_1 \in \partial\Sigma \cap \Omega$,
- then pick $r > 0$ such that $B_\rho(z_1) \subset \Omega$,
- then pick any $z \in B_\rho(z_1) \setminus \Sigma$.

Then let

$$R = \sup \{\rho : B_\rho(z) \subset \Omega \setminus \Sigma\} \,.$$

By construction, there exists $y_\rho \in \partial B_\rho(z) \cap \Sigma$. Since $u(y) = M$, this contradicts the Hopf boundary point lemma. So $\Omega \setminus \Sigma = \emptyset$ implies $\Omega = \Sigma$. ases follow directly from Hopf. $\square$

Corollary 2.3 (Comparison Principle). *Let $Lu = a^{ij}\partial_i\partial_j + b^i\partial_i + c$ be uniformly elliptic in $\Omega \subset \mathbb{R}^n$ with*

$$\sup_{\Omega}\left(\frac{|b|+|c|}{\lambda}\right) < \infty.$$

Suppose $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy:

$$Lu \leq Lv \quad \forall\ u \leq v \text{ in } \Omega.$$

Then $u = v$ on Ω, or $u < v$ on Ω.

Proof. Then $(u - v) \geq 0$ in Ω and $Lu - Lv \leq 0$ in Ω. If there exists $x_0 \in \Omega$ such that $u(x_0) = v(x_0)$, then

$$\text{SMP(III)} \Rightarrow u \equiv v \text{ in } \Omega.$$

If not, then $u \neq v$ in Ω and so $u < v$ in Ω. □

Corollary 2.4 (Uniqueness for Neumann Problem). *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\partial\Omega$ satisfies the interior sphere condition at each point. Suppose L is uniformly elliptic with*

$$\frac{|b|+|c|}{\lambda} \in L^\infty(\Omega), \quad c \leq 0.$$

Then if $u_1, u_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$\begin{cases} Lu_i = f & \text{in } \Omega \\ \dfrac{\partial u_i}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

for some $f : \Omega \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$, then

$$u_1 = u_2 + c \quad \text{for some constant } c.$$

Proof. Let $u = u_1 - u_2$ satisfies

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

Let $\mathcal{M} = \sup_{\overline{\Omega}} u \geq 0$ (otherwise take $-u$). By the SMP, if $u \not\equiv \mathcal{M}$ on $\overline{\Omega}$, then there exists $y \in \partial\Omega$ such that $u(y) = \mathcal{M}$ and $u(x) < u(y)$ for all $x \in \Omega$. By Hopf's Lemma,

$$\frac{\partial u}{\partial \nu}(y) \neq 0,$$

a contradiction. □

Remark. *This says that the trivial Neumann problem with zero data has solutions which are constant, i.e. $Lu = 0 \Rightarrow u \equiv$ const. But $Lu = cu \Rightarrow u \equiv cu$ for all $x \in \Omega$, so if $c \neq 0$, $u \equiv 0$. This constant only non-zero when $c = 0$.*

What happens for non-zero RHS?

The following will be critical for Schauder theory.

Theorem 2.4 (Maximum Principle A Priori Estimate (MPAPE))**.** *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain, L elliptic, $c \le 0$, $\beta = \frac{|b|}{\lambda} \in L^\infty(\Omega)$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $f : \Omega \to \mathbb{R}$.*

Then if $Lu \ge f$, then

1. $\sup_\Omega u \le \sup_{\partial\Omega} u + C \cdot \sup_\Omega \left(\frac{|f|}{\lambda}\right)$
2. *if $Lu = f$, then*
$$\sup |u| \le \sup_{\partial\Omega} |u| + C \cdot \sup_\Omega \left(\frac{|f|}{\lambda}\right)$$

$C = C(\beta, \operatorname{diam}(\Omega))$.

Proof. Put $d := \operatorname{diam}(\Omega) = \sup_{x,y\in\Omega} |x-y|$. As Ω bounded, we have

$$\Omega \subset \{x : x \in \mathbb{R}^n, |x| \le \alpha + d\}$$

for some $\alpha \in \mathbb{R}$, w.l.o.g. $\alpha = 0$.

Idea: construct subsolution and use WMP. Let

$$v(x) = \sup_{\partial\Omega} u^+ + Ce^{\alpha d} - e^{\alpha x_1} \sup_\Omega \left(\frac{|f|}{\lambda}\right)$$

where α is to be determined. We now estimate using ellipticity taking $\alpha = \beta + 1$

$$\begin{aligned}
(a^{ij}\partial_i\partial_j + b^i\partial_i)e^{\alpha x_1} &= e^{\alpha x_1}(\alpha^2 a^{11} + \alpha b^1) \\
&\ge e^{\alpha x_1}\lambda\left(\alpha^2 + \frac{b^1}{\lambda}\alpha\right) \\
&\ge e^{\alpha x_1}\lambda\left(\alpha^2 - \beta\alpha\right) \\
&\ge \lambda\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
Lv &= \left(a^{ij}\partial_i\partial_j + b^i\partial_i\right)\left(-e^{\alpha x_1}\sup_\Omega\left(\frac{|f|}{\lambda}\right)\right) + cv \\
&\le cv - \sup_\Omega\left(\frac{|f|}{\lambda}\right) \\
&\le -\frac{1}{C}\sup_\Omega\left(\frac{|f|}{\lambda}\right)
\end{aligned}$$

Now, since $Lu \ge f$, it it follows from the above that

$$L(u-v) \ge f - Lv \ge f + \lambda \sup_\Omega\left(\frac{|f|}{\lambda}\right) \ge 0\,.$$

To Be Continued... $\square$

Lecture 7

Proof (continued). Had

$$v(x) := \sup_{\partial\Omega} u^+ + \left(e^{(\beta+1)d} - e^{(\beta+1)x_2}\right) \sup_{\Omega} \frac{|f|}{\lambda}$$

and showed $L(u-v) \geq 0$. Since $u \leq u^+$, $u|_{\partial\Omega} \leq v|_{\partial\Omega}$ by construction, so by the WMP, $u \leq v$ in Ω, so

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C \cdot \sup_{\Omega} \left(\frac{|f|}{\lambda}\right) ,$$

where $C := \sup_{\Omega} \left(e^{(\beta+1)d} - e^{(\beta+1)x_1}\right)$.

Finally, to obtain (II), namely, when $Lu = f$, apply (I) to $-u$ and combine. □

2.2 Hölder Spaces

Fix $\Omega \subset \mathbb{R}^n$ open, let $\alpha \in (0,1)$.

Definition 2.2. *We say that $u : \Omega \to \mathbb{R}$ is* uniformly Hölder continuous *with exponent α or* uniformly α-Hölder continuous *if*

$$[u]_{\alpha;\Omega} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty$$

This is the Hölder semi-norm.

If $\alpha = 1$, this says u is uniformly Lipschitz. If $\alpha > 1$, that would make $u = \text{const}$ (MVT).

Definition 2.3 (2.19)**.** *We say that u is* locally α-Hölder continuous *in Ω if for all $K \subset\subset \Omega$, $u|_K$ is uniformly α-Hölder continuous.*

Let $k \in \mathbb{N} \cup \{\infty\}$. Recall for a multi-index $\beta \in \mathbb{N}^n$, $|\beta| := \sum \beta_i$ and $C^k(\Omega) := \left\{u : \Omega \to \mathbb{R} : D^\beta u \text{ exists and is continuous } \forall \beta \text{ s.t. } |\beta| \leq k\right\}$

Definition 2.4 (2.19 continued)**.** *Define the Hölder spaces $C^{k,\alpha}(\Omega)$:*

$$C^{k,\alpha}(\Omega) := \left\{u \in C^k(\Omega) : D^\beta u \textit{ is locally } \alpha\textit{-Hölder continuous } \forall \beta, |\beta| = k\right\}$$

and

$$C^{k,\alpha}(\overline{\Omega}) := \left\{u \in C^k(\overline{\Omega}) : D^\beta u \textit{ uniformly } \alpha\textit{-Hölder cts}\right\} .$$

We write for $\alpha \in (0,1)$,

$$C^{0,\alpha}(\Omega) := \text{Hölder} \quad C^{0,\alpha}(\overline{\Omega}) := C^\alpha(\overline{\Omega})$$

$$C^{k,0}(\Omega) := C^k(\Omega), \quad C^{k,0}(\overline{\Omega}) := C^k(\overline{\Omega}), \quad k \in \mathbb{N} \cup \{0\}$$

Remark. *Note $C^{k+1}(\overline{\Omega}) \subset C^{k,\alpha}(\overline{\Omega})$, indeed: Lipschitz $\Rightarrow$ ctsly diff-able. (But Lipschitz $\not\Rightarrow$ diff-able a.e., see [Eva18]!)*

Finally, define

$$C_0^{k,\alpha}(\Omega) \coloneqq C_c^{k,\alpha}(\Omega) \quad \text{i.e. } u \in C^{k,\alpha}(\Omega) \coloneqq \{u \in C^{k,\alpha(\Omega)} : \operatorname{supp}(u) \subset\subset \Omega\}$$

where as usual $\operatorname{supp}(u) = \overline{\{x \in \Omega : u(x) \neq 0\}}$. To get norms on these spaces, put for $k \in \mathbb{N}$, $u \in C^k(\overline{\Omega})$,

$$[u]_{k,\Omega} = \left[D^\beta u\right]_{0,\Omega} \quad \text{(recall for } \alpha = 0 \text{ norm is } C^0\text{-norm)}$$

$$= \sup_{|\beta|=k} \left|D^\beta u\right|_{0,\Omega} = \sup_{|\beta|=k} \sup_{x\in\Omega} \left|D^\beta u(x)\right|$$

For $u \in C^{k,\alpha}(\overline{\Omega})$ put

$$[u]_{k,\alpha;\Omega} \coloneqq \left[D^\beta u\right]_{\alpha,\Omega} = \sup_{|\beta|=k} \left[D^\beta u\right]_{\alpha,\Omega}$$

Note these are semi-norms. To get norms,

$$\|u\|_{C^k(\overline{\Omega})} \equiv |u|_{k,\Omega} \equiv \sum_{j=0}^{k} \left|D^j u\right|_{0,\Omega} \quad \text{and}$$

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} \equiv |u|_{k,\alpha,\Omega} \equiv |u|_{k,\Omega} + \left[D^k u\right]_{\alpha,\Omega}$$

With these norms, C^k and $C^{k,\alpha}$ become Banach spaces.

Theorem 2.5 (Arzelà-Ascoli for Hölder Spaces)**.** *Let $\Omega \subset \mathbb{R}^n$ open, $k \in \mathbb{N}$, $\alpha \in (0,1)$. If $u_j \in C^{k,\alpha}(\Omega)$ satisfies*

$$\sup_j |u_j|_{k,\alpha;\Omega} < \infty$$

for all

$$\Omega' \subset\subset \Omega, \text{ then there exists } u \in C^{k,\alpha}(\Omega) \text{ and a subsequence } (u_{j_k})_j \text{ s.t. } u_{j_k} \to u \text{ in } C^k(\Omega')$$

Remark. *Nothing is said about convergence in $C^{k,\alpha}(\overline{\Omega})$.*

Proof. A bit tedious, but good for the soul! [✍] . □

Two more ingredients before starting Schauder theory.

Interpolation: Consider Banach spaces

$$X \subset\subset Y \subset Z,$$

then we can bound the norm in Y by X and Z norms. Interpolation is the exchange of sizes of X-and$Z$-norms. Here

$$C^{k,\alpha}(\overline{\Omega}) \subset\subset C^k(\overline{\Omega}) \subset C^0(\overline{\Omega})$$

Theorem 2.6 (Interpolation Inequality for Hölder Spaces)**.** *Let $\varepsilon > 0$, $|\ell \in \mathbb{N}$, $\alpha \in (0,1]$. Then there exists $C = C(n,m,\alpha,\varepsilon) \in (0,\infty)$ s.t. if $u \in C^{|\ell,\alpha}(\overline{B_R(x_0)})$, then*

$$R^\ell \left|D^| \ell u\right|_{0,B_R(x_0)} \leq \varepsilon R^{\ell+\alpha} \left[D^\ell u\right]_{\alpha,B_R} + C \cdot |u|_{0,B_R(x_0)}.$$

for all $x_0 \in \mathbb{R}^n$, $R \leq L$.

Sketch Proof (details [✍]). By rescaling and shifting, i.e. considering

$$v(x) := u(x_0 + Rx)$$

suffices to prove for $R = 1$, $x_0 = 0$. Then argue by contradiction and Arzelà-Ascoli. □

Ingredient 2 (Simon's Absorbing Lemma)

Lemma 2.1. *Let $B_\rho(x) \subset \mathbb{R}^n$ be fixed, S a non-negative, sub-additive function on the collection of sub-balls of $B_\rho(x)$, i.e. if*

$$B_\rho(y_j) \subset \bigcup_{j=1}^{N} B_{r_j}(y_j) \subset B_\rho(x)$$

then

$$S(B_\rho(y)) \leq \sum_{j=1}^{N} S(B_{r_j}(y_j)).$$

Let $\lambda \in [1, \infty)$, $\theta \in (0, 1]$. Then there exists $\delta = \delta(n, \lambda, \theta) > 0$ s.t. the following holds: Suppose that for all balls $B_\rho(y) \subset B_R(x)$ we have:

$$\rho^\lambda S(B_{\theta\rho}(y)) \leq \delta^\lambda S(B_\rho(y)) + \gamma$$

for some fixed $\gamma > 0$. Then

$$R^\lambda S(B_{R/2}(x)) \leq C\gamma$$

for some $C = C(n, \lambda, \theta)$.

Remark. *This says that if S is a local bound on S, then we can "absorb" the S-term on the RHS to get a global bound.*

Lecture 8

Proof of Simon's Absorbing Lemma. Put

$$Q = \sup_{B_{\theta\rho}(y) \subset B_R(x)} \rho^\lambda S(B_{\theta\rho}(y)).$$

Recall we had

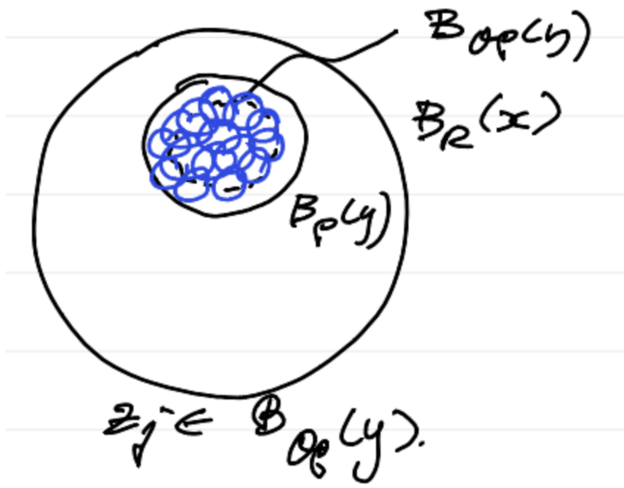
$$(*) \quad \rho^\lambda S(B_{\theta\rho}(y)) \leq \delta^\alpha S(B_\rho(y)) + \gamma.$$

By the subadditivity of S, we have

$$Q \leq R^\lambda S(B_\rho(x)) < \infty.$$

Fix any $B_\rho(y) \subset B_R(x)$. Cover $B_{\theta\rho}(y)$ by a collection of balls

$$\left\{B_{(1-\theta^2)\frac{\rho}{2}}(z_j)\right\}_{j=1}^{N}, \quad N \leq C(\theta, n), \quad z_j \in B_{\theta\rho}(y).$$



How? Choose a maximal, pairwise disjoint collection of balls

$$\left\{B_{(1-\theta^2)\frac{\rho}{2}}(z_j)\right\}_{j=1}^N .$$

We claim that these z_j's work. Indeed, if not, then there exists $z \in B_{\theta\rho}(y) \setminus \bigcup_{j=1}^N B_{(1-\theta^2)\frac{\rho}{2}}(z_j)$ and we have that

$$\operatorname{dist}(z, z_j) \geq (1-\theta)\,\theta^2\rho \quad \forall j.$$

So

$$B_{(1-\theta^2)\frac{\rho}{2}}(z) \cap B_{(1-\theta^2)\frac{\rho}{2}}(z_j) = \emptyset.$$

Which contradicts maximality. Now, to bound N first note (from considering the radii)

$$\bigcup_{j=1}^N B_{(1-\theta^2)\frac{\rho}{2}}(z_j) \subset B_{(1-\theta^2)\frac{\rho}{2}+\theta\rho}(y).$$

Since the LHS is disjoint, by volume bound:

$$N\omega_n\left(\frac{(1-\theta)\theta^2\rho}{2}\right)^n \leq \omega_n\left(\frac{(1-\theta)\theta^2\rho}{2}+\theta\rho\right)^n \Rightarrow N \leq \left(\frac{(1-\theta)\theta^2+2}{(1-\theta)\theta^2}\right)^n .$$

Which is independent of ρ and θ. By sub-additivity, we have

$$\rho^\lambda S(B_{\theta\rho}(y)) \leq \rho^\lambda \sum_{j=1}^N S\left(B_{(1-\theta)\theta^2\rho}(z_j)\right)$$

From $(*)$, we have:

$$\rho^\lambda \Rightarrow (1-\theta)\theta \Rightarrow\leq ((1-\theta)\theta)^{-\lambda}\sum_{j=1}^N\left(\delta((1-\theta)\theta^2\rho)^\lambda S(B_{(1-\theta)\theta^2\rho}(z_j))+\gamma\right) \leq \delta((1-\theta)\theta)^{-\lambda}NQ+N\gamma((1-\theta)\theta)^{-\lambda}$$

Now take the sup over all $B_{\theta\rho}(y) \subset B_R(x)$:

$$Q \leq \delta C_1 Q + C_2\gamma$$

where C_1, C_2 depend on n, θ, λ. Then choose $\delta > 0$ small enough ($= \frac{1}{2C_1}$), then

$$Q \leq 2C_2\gamma.$$

$\square$

3 Schauder Theory

3.1 Interior Schauder Estimates

We will first prove interior estimates in the unit ball, and then extend them to more general domains.

Main point: if coefficients of L are α-Hölder cts, then any $C^{2,\alpha}$ solution of $Lu = f$ can be bounded in $C^{2,\alpha}$ on a smaller ball by $|u|$ and $|f|$.

Theorem 3.1 (Unit Scale Interior Schauder Estimates)**.** *Let $\alpha \in (0,1)$, $\beta > 0$, and suppose*

$$a^{ij}, b^i, c \in C^{0,\alpha}(\overline{B_1(0)})$$

with

$$|a^{ij}|_{0,\alpha;B_1(0)} + |b^i|_{0,\alpha;B_1(0)} + |c|_{0,\alpha;B_1(0)} \leq \beta.$$

Suppose L is strictly elliptic, i.e.

$$\exists \lambda > 0 \text{ s.t. } a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall x \in \overline{B_1(0)},\ \xi \in \mathbb{R}^n.$$

Then if $u \in C^{2,\alpha}(B_1(0)) \cap C^{0,\alpha}(\overline{B_1(0)})$ and $f \in C^{0,\alpha}(B_1(0))$ satisfy $Lu = f$ in $B_1(0)$, then

$$|u|_{2,\alpha;B_{1/2}(0)} \leq C\left(|u|_{0,B_1(0)} + |f|_{0,\alpha;B_1(0)}\right)$$

for some constant $C = C(n, \lambda, \alpha, \beta)$.

Remark.

- *Can never take $\alpha = 0$ or $\alpha = 1$ in these cases (The statement in Thm 3.1 with $\alpha = 0$, $\alpha = 1$ is false!).*
- *Strict ellipticity gives lower bound for λ and upper bound on $|a^{ij}|_{0,\alpha;B_1(0)}$ gives an upper bound on Λ, so $\delta = \frac{\lambda}{\Lambda}$ is bounded above $\Rightarrow$ strict $\Rightarrow$ uniform ellipticity.*
- *Remarkable result:* $\sup|u|$ *controls the derivatives of u!*
- *Will in fact strengthen this to*

$$|u|_{2,\alpha;B_{1/2}(0)} \leq C\left(|u|_{0,B_1(0)} + |f|_{0,\alpha;B_1(0)}\right) \quad \forall\theta \in (0,1), \quad C = C(n, \lambda, \alpha, \beta, \theta).$$

- *There are no assumptions or conclusions about the C^2-norm up to the boundary.*
- *The Schauder estimate gives a compactness property for the space of solutions to $Lu = f$. If $(u_k) \subset C^{2,\alpha}(\overline{B_1(0)}) \cap C^0(\overline{B_1(0)})$ solve $Lu_k = f$ in $B_1(0)$ and*

$$\gamma = \sup_k \sup_{B_1(0)} |u_k| < \infty\,,$$

then estimate $\Rightarrow$

$$|u_k|_{2,\alpha;B_{1/2}(0)} \leq C(\gamma, n, \theta, \beta, \alpha, \delta, f).$$

So by Arzelà-Ascoli there exist $(u_{k_\ell}), u \in C^{2,\alpha}(B_1(0))$ s.t. $u_{k_\ell} \to u$ in $C^2(B_\theta(0))$ for all $\theta \in (0,1)$. Passing to the limit, then $Lu = f$.

Proof. Write for $r > 0$, $B_r \coloneqq B_r(0)$. Working in a slightly smaller ball, we can assume w.l.o.g. that $|u|_{2,B_1} < \infty$. We proceed in three steps:

1. Reduction step
2. Contradiction step
3. Simplified PDE step

Step 1: Reduction Step.

<u>Claim:</u> It suffices to prove the following:

For any given $\delta \in (0,1)$, $C > 0$ s.t.

$$[D^2u]_{\alpha;B_{1/2}} \leq \delta \left([D^2u]_{0,\alpha;(B_1)} + C\left(|u|_{2,B_1} + |f|_{0,\alpha;(B_1)}\right)\right) \tag{3.1}$$

Proof of claim:

Suppose $Lu = f \Rightarrow u$ satisfies 3.1. By the Hölder interpolation inequality, one has

$$[D^2u]_{\alpha;B_{1/2}} \leq 2\delta \left([D^2u]_{\alpha;B_1} + C\left(|u|_{0,B_1} + |f|_{0,\alpha;(B_1)}\right)\right) .$$

Strategy for step 1: take $B_\rho(y) \subseteq B_1(0)$ and shift and scale: $\tilde{u}(x) = u(y + \rho x)$. Then $\tilde{u}$ will satisfy a new PDE and a new inequality, which we will call $\widetilde{(3.1)}$.

Lecture 9 Now, take any sub-ball $B_\rho(y) \subset B_1(0)$ and write

$$\tilde{u}(x) = \rho u(y + \rho x).$$

Then $\tilde{u}$ satisfies

$$\tilde{a}^{ij} D^2_{ij}\tilde{u} + \tilde{b}^i \partial_i \tilde{u} + \tilde{c}\tilde{u} = \tilde{f}\,, \tag{*}$$

where

$$\begin{aligned}
\tilde{a}^{ij}(x) &= a^{ij}(y + \rho x),\\
\tilde{b}^i(x) &= \rho b^i(y + \rho x),\\
\tilde{c}(x) &= \rho^2 c(y + \rho x),\\
\tilde{f}(x) &= \rho^2 f(y + \rho x).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\left|\tilde{a}^{ij}\right|_{0,\alpha;B_1} + \left|\tilde{b}^i\right|_{0,\alpha;B_1} + |\tilde{c}|_{0,\alpha;B_1} &\leq \left|a^{ij}\right|_{0,\alpha;B_\rho(y)} + \rho^\alpha \left[a^{ij}\right]_{\alpha;B_\rho(y)}\\
&\quad + \rho \left|b^i\right|_{0;B_\rho(y)} + \rho^{1+\alpha}\left[b^i\right]_{\alpha;B_\rho(y)}\\
&\quad + \text{similar for } c\\
&\leq \beta \quad \text{as } \rho < 1.
\end{aligned}$$

Since $\tilde{a}^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2$, the PDE $(*)$ is strictly elliptic. So by assumption (3.1) holds for $\tilde{u}$, call it $\widetilde{(3.1)}$. Expressing $\widetilde{(3.1)}$ in terms of u gives:

$$\rho^{2+\alpha}\left[D^2u\right]_{\alpha;B_{\rho/2}} \leq 2\delta\rho^{2+\alpha}\left[D^2u\right]_{\alpha;B_\rho(y)}$$

$$
\begin{aligned}
&+ C\left(|u|_{0;B_\rho(y)} + \rho^{2+\alpha}|f|_{C^{0,\alpha};B_\rho(y)}\right) \\
&\leq 2\delta\rho^{2+\alpha}\left[D^2u\right]_{\alpha;B_\rho(y)} + C\left(|u|_{0;B_1} + |f|_{C^{0,\alpha};B_1}\right).
\end{aligned}
$$

$$
:= \gamma, \text{ indep. of } \rho \text{ and } y.
$$

So by the absorbing lemma, choose δ suitably, have

$$
\left[D^2u\right]_{\alpha;B_{1/2}} \leq C\left(|u|_{0;B_1} + |f|_{C^{0,\alpha};B_1}\right)
$$

where C depends only on $n, \alpha, \lambda, \beta$. This is the conclusion of the theorem (use interpolation again).

Step 2: Contradiction via Arzelà-Ascoli Suppose there exists $\delta > 0$ such that for all $k \in \mathbb{N}$, there exists a_k^{ij}, b_k^i, c_k such that

$$
|a_k^{ij}|, |b_k^i|, |c_k| \leq \beta \quad (\beta \text{ independent of } k), \quad a_k^{ij}\xi_i\xi_j \geq \lambda|\xi|^2,
$$

and

$$
u_k \in C^{2,\alpha}(B_1) \cap C^\alpha(\overline{B}_1) \text{ solving } L_k u_k = f_k, \quad \text{for } f_k \in C^{0,\alpha}(\overline{B}_1)
$$

<u>but</u>

$$
\left[D^2u_k\right]_{\alpha;B_{1/2}} > \delta\left[D^2u_k\right]_{\alpha;B_1} + k\left(|u_k|_{2;B_1} + |f_k|_{C^{0,\alpha};B_1}\right) \tag{3.2}
$$

By definition of $\left[D^2u_k\right]_{\alpha;B_{1/2}}$ and by passing to a subsequence, we may assume that there exist $x_k, y_k \in B_{1/2}$ and fixed limit s.t.

$$
\frac{|D_{\ell m}u_k(x_k) - D_{\ell m}u_k(y_k)|}{|x_k - y_k|^\alpha} \geq \tfrac{1}{2}\left[D^2u_k\right]_{\alpha;B_{1/2}}.
$$

By taking an appropriate subsequence in $\{x_k, y_k\}$, for all k, let $\rho_k := |x_k - y_k|$. Then

$$
\tfrac{1}{2}\left[D^2u_k\right]_{\alpha;B_{1/2}} < \frac{|D_{\ell m}u_k(x_k)| + |D_{\ell m}u_k(y_k)|}{\rho_k^\alpha} \leq \frac{2|u_k|_{2;B_1}}{\rho_k^\alpha}.
$$

Inequality (3.2) thus implies

$$
2\left[D^2u_k\right]_{\alpha;B_{1/2}} < \frac{2\left[D^2u_k\right]_{\alpha;B_1}}{k\rho_k^\alpha}.
$$

So in particular,

$$
\rho_k^\alpha < \frac{4}{k} \to 0.
$$

(Note $\alpha = 0$ does *not* imply $\rho_k \to 0$.) Next, rescale appropriately and take the limit, set

$$
\tilde{u}_k(x) := \frac{u_k(x_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha}\left[D^2u_k\right]_{\alpha;B_1}},
$$

where

$$
q_k(x) := u_k(x_k) + \rho_k x^i \partial_i u_k(x_k) + \frac{\rho_k^2}{2}x^i x^j D_{ij}^2 u_k(x_k).
$$

Note. x has nothing to do with x_k.

By construction, $\tilde{u}_k(0) = 0$, $D\tilde{u}_k(0) = 0$, $D^2\tilde{u}_k(0) = 0$, and $\tilde{u}_k$ is defined on B_1 and

$$\tilde{u}_k \text{ defined on } B_{1/\rho_k}\left(-\frac{x_k}{\rho_k}\right) \supset B_{1/\rho_k}(0), \quad x_k \in B_{1/2}(0)\,.$$

So by direct calculation

$$\left[D^2\tilde{u}_k\right]_{\alpha;B_{1/(2\rho_k)}(0)} \leq 1. \quad \text{(Taylor with remainder)}$$

Hence for any $R \geq 1$ (using (3.2) to control $\tilde{u}_k|_{B_\rho}$) we have

$$|\tilde{u}_k|_{2,\alpha;B_\rho} \leq C \cdot \mathbb{R}^{2+\alpha}.$$

Now, by Arzela–Ascoli, passing to a subsequence, there exists $v \in C^{2,\alpha}_{\text{loc}}(\mathbb{R}^n)$ s.t. $\tilde{u}_k \to v$ in $C^2(B_\rho(0))$ for all $R > 0$ and s.t.

$$\left[D^2 v\right]_{\alpha;\mathbb{R}^n} \leq 1.$$

Step 3: Find a PDE for v.

First put $w_k(x) = u_k(x_k + \rho_k x)$. This satisfies $\tilde{L}_k w_k = \tilde{f}_k$ in $B_{1/(2\rho_k)}(0)$, where

$$\tilde{L}_k = \tilde{a}^{ij}_k D^2_{ij} + \tilde{b}^i_k \partial_i + \tilde{c}_k,$$

where $\tilde{a}^{ij}_k$ are as before with $\rho \mapsto \rho_k$ and $x_k \mapsto y$, i.e. $\tilde{a}^{ij}_k = a^{ij}_k(x_k + \rho_k x)$ etc. We now have

$$w_k(x) = \rho_k^{2+\alpha}\left[D^2 u_k\right]_{\alpha;B_1} \tilde{u}_k(x) + q_k(x),$$

and also

$$\tilde{L}_k \tilde{u}_k = g_k = \frac{\tilde{f}_k - \tilde{L}_k q_k}{\rho_k^{2+\alpha}\left[D^2 u_k\right]_{\alpha;B_1}}.$$

We will find that L_k tends uniformly to a constant coefficient operator, and $g_k \to$ constant. Indeed,

$$\left[\tilde{a}^{ij}_k\right]_{\alpha;B_{1/(2\rho_k)}} \lesssim \rho_k^\alpha \left[a^{ij}_k\right]_{\alpha;B_1} \leq \rho_k^\alpha \to 0 \quad \text{as } k \to \infty \quad (\alpha > 0 \text{ as above: scaling and containment}).$$

! $\alpha = 0$ fails here!

By Arzelà–Ascoli on $\tilde{a}^{ij}_k$, we get (up to subsequence)

$$\tilde{a}^{ij}_k \to \tilde{a}^{ij} \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n) \text{ locally uniformly, and the limit has } [\tilde{a}^{ij}]_{\alpha;\mathbb{R}^n} = 0 \quad \forall \alpha < 0.$$

Hence, $\tilde{a}^{ij}$ is constant.

$$[\tilde{b}^i_k]_{0;B_{1/\rho_k}} \leq \rho_k^\beta \to 0,$$

$$[\tilde{c}_k]_{0;B_{1/\rho_k}} \leq \rho_k^\beta \to 0,$$

so $\tilde{b}^i_k \to 0$, and $\tilde{c}_k \to 0$ locally uniformly on $\mathbb{R}^n$. Finally, $g_k \to 0$ locally uniformly. Taking the limit in $\tilde{a}^{ij}_k D^2_{ij}\tilde{v}_k = g_k$, we obtain

$$\tilde{a}^{ij} D^2_{ij}\tilde{v} = 0 \quad \text{on } \mathbb{R}^n \tag{3.3}$$

where $\tilde{a}^{ij}$ are constants. Also, in the limit $\tilde{a}^{ij}\xi^i\xi^j \geq \lambda|\xi|^2$, so we are still strictly elliptic. Diagonalise $A = (\tilde{a}^{ij})$,

$$PAP^T = Q = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \quad \lambda_i \geq \lambda > 0 \quad \forall i.$$

Let $w(x) = v(P^T x)$. Then

$$D^2 w(x) = P^T D^2 \tilde{v}(P^T x) P,$$

so (3.3) becomes

$$0 = \operatorname{tr}(AD^2\tilde{v}) = \operatorname{tr}(QD^2 w(P^T x)) = \sum_{i=1}^n \lambda_i D^2_{ii} w(P^T x)$$

and we deduce

$$\sum_{i=1}^n \lambda_i D^2_{ii} w(x) = 0.$$

Rescaling $\tilde{w}(x) = w(\sqrt{\lambda_1}x_1, \ldots, \sqrt{\lambda_n}x_n)$, gives

$$\Delta\tilde{w} = 0 \quad \text{on } \mathbb{R}^n \quad \text{and} \quad [D^2\tilde{w}]_{\alpha;\mathbb{R}^n} < \infty$$

and in particular, that $\tilde{w}$ is smooth and so $\Delta(D^2_{ij}\tilde{w}) = 0$ on $\mathbb{R}^n$. But by Hölder continuity,

$$|D^2_{ij}\tilde{w}(x)| \leq |D^2_{ij}\tilde{w}(0)| + [D^2\tilde{w}]_{\alpha;\mathbb{R}^n}|x|^\alpha,$$

Liouville's Theorem gives $D^2_{ij}\tilde{w} \equiv$ const(cannot use Liouville for $\alpha = 1$).

Lecture 10

Recall, we found $\tilde{w}$ s.t. $D^2\tilde{w} \equiv$ constant or a limit, of $u_k \to \tilde{v}$ in C^2. But $D^2 v(0) = 0$, so $D^2 v \equiv 0$. On the other hand, consider $\zeta_k = \frac{x_k - y_k}{\rho_k}$ with $|\zeta_k| = 1$, and

$$u_k(x_k + \rho_k\zeta_k) = u_k(y_k)$$

so

$$|D^2_{\ell m}\tilde{u}_k(\zeta_k)| = \left|\frac{\rho_k^2 D^2_{\ell m}u_k(y_k) - D^2_{\ell m}u_k(x_k)}{\rho_k^{2+\alpha}[D^2 u_k]_{\alpha;B_1}}\right| \geq \frac{1}{2}\frac{[D^2 u_k]_{\alpha;B_{1/2}}}{\rho_k^{2+\alpha}[D^2 u_k]_{\alpha;B_1}}.$$

By choice of x_k, y_k

$$|D^2_{\ell m}\tilde{u}_k(\zeta_k)| \geq \frac{1}{2}\frac{[D^2 u_k]_{\alpha;B_{1/2}}}{[D^2 u_k]_{\alpha;B_1}} \overset{3.2}{>} \frac{\delta}{2}. \tag{3.3}$$

Since ζ_k is bounded and have, up to a subsequence, $\zeta_k \to \zeta$, then in the limit

$$|D^2_{\ell m}v(\zeta)| \leq \delta/2 \overset{!}{>} \frac{\delta}{2}.$$

This contradicts $D^2\tilde{v} \equiv 0$. $\square$

So we have proven

$$|u|_{2,\alpha;B_{1/2}} \leq C\left(|u|_{0;B_1} + |f|_{0,\alpha;B_1}\right).$$

Corollary 3.1 (Scale-invariant interior Schauder Estimate)**.** *Suppose $B_\rho(x_0) \subset \mathbb{R}^n$ and $a^{ij}, f^i, c \in C^{0,\alpha}(B_\rho(x_0))$ are strictly elliptic,*

$$a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2, \quad \lambda > 0, \quad \forall x \in B_\rho(x_0), \quad \forall \xi \in \mathbb{R}^n.$$

Suppose also that

$$|a^{ij}|_{0;B_\rho(x_0)} + R^\alpha[a^{ij}]_{\alpha;B_\rho(x_0)} + R(|b^i|_{0;B_\rho(x_0)} + R^\alpha[b^i]_{\alpha;B_\rho(x_0)}) + R^2(|c|_{0;B_\rho(x_0)} + R^\alpha[c]_{\alpha;B_\rho(x_0)}) \leq \beta,$$

for some $\beta \geq 0$. Suppose $u \in C^{2,\alpha}(B_\rho(x_0)) \cap C^0(\overline{B_\rho(x_0)})$ satisfies $Lu = f \in C^{0,\alpha}(B_\rho(x_0))$.

Then,

$$|u|'_{2,\alpha;B_{R/2}(x_0)} \leq C\left(|u|_{0;B_\rho(x_0)} + R|f|_{0;B_\rho(x_0)} + R^{2+\alpha}[f]_{\alpha;B_\rho(x_0)}\right),$$

where

$$|u|'_{k,\alpha;B_\rho(y)} = \sum_{j=0}^{k} \rho^j |D^j u|_{0;B_\rho(y)} + r^{k+\alpha}[D^k u]_{\alpha;B_\rho(y)},$$

and $C = C(\alpha, n, \lambda, \beta)$ (indep. of u and R).

Proof. Apply Theorem 3.1 with $x \coloneqq x_0 + Rx$. □

Corollary 3.2 (Exterior Schauder Estimates in General Domains)**.** *Let $\alpha \in (0,1)$ and let $\Omega \subset \mathbb{R}^n$ open, bounded, and suppose that $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$ where*

$$|a^{ij}|_{0;\Omega} + |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta,$$

with $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2 > 0$ for all $x \in \Omega$, $\xi \in \mathbb{R}^n$.

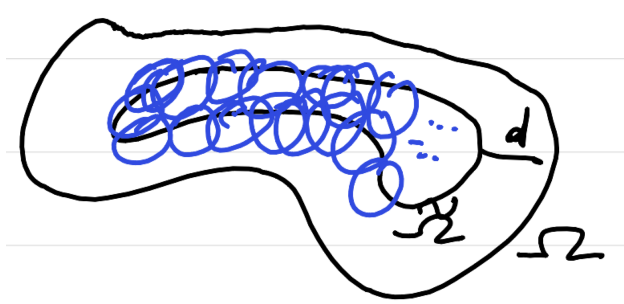
Suppose $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ solves $Lu = f \in C^{0,\alpha}(\Omega)$. Then for all $\tilde{\Omega} \subset\subset \Omega$,

$$|u|_{2,\alpha;\tilde{\Omega}} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right),$$

where $C = C(\alpha, n, \beta, \lambda, \operatorname{dist}(\tilde{\Omega}, \partial\Omega))$.

Proof. Let $d \coloneqq \operatorname{dist}(\tilde{\Omega}, \partial\Omega) = \sup\{r > 0 : (\tilde{\Omega})_r \subset \Omega\}$, where $(\tilde{\Omega})_r = \bigcup_{x\in\Omega} B_r(x)$ is the r-neighbourhood of $\tilde{\Omega}$. Then $x \in \tilde{\Omega} \Rightarrow \forall x \in \tilde{\Omega}, B_{d/2}(x) \subset \Omega$, so

$$|a^{ij}|'_{0,\alpha;B_{d/2}(x)} + d|b^i|'_{0,\alpha;B_{d/2}(x)} + d^2|c|'_{0,\alpha;B_{d/2}(x)} \leq C(d) \cdot \beta.$$



Then by Corollary (3.1), we get

$$|u|_{0;B_{d/2}(x)} + d|Du|_{0;B_{d/2}(x)} + d^2|D^2u|_{0;B_{d/2}(x)}$$

$$
\begin{aligned}
&+ d^{2+\alpha}[D^2u]_{\alpha;B_{d/2}(x)} \\
&\le C\left(|u|_{0;B_d(x)} + d^2|f|_{0;B_d(x)} + d^{2+\alpha}[f]_{\alpha;B_d(x)}\right) \\
&\le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right) \\
&\Rightarrow C = C(n,\lambda,\alpha,\beta,d).
\end{aligned} \tag{3.4}
$$

In particular,

$$|u(x)| + |Du(x)| + |D^2u(x)| \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right) \quad \forall x \in \tilde{\Omega}.$$

Hence,

$$|u|_{2,\alpha;\tilde{\Omega}} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right). \tag{*}$$

But also by (3.4)

$$\sup_{x\neq y\in\tilde{\Omega}, |x-y|<\frac{d}{2}} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \le C\cdot \mathrm{RHS}.$$

On the other hand, if $|x-y| \ge d/2$, then

$$\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \le \left(\frac{d}{2}\right)^{-\alpha} \cdot 2 \cdot |u|_{2,\alpha;\tilde{\Omega}}.$$

Hence,

$$[D^2u]_{\alpha;\tilde{\Omega}} \le C\cdot \mathrm{RHS}. \tag{**}$$

Finally, combining (*) and (**) allows us to conclude the proof. □

4 Boundary Schauder Estimates

Let us establish some notation used throughout this section. Write

$$\mathbb{R}^n_\pm = \{(x', x^n) : x' \in \mathbb{R}^{n-1}, x^n \gtrless 0\},$$

$$B^\pm_\rho(y) = B_\rho(y) \cap \mathbb{R}^n_\pm, \quad B^\pm_\rho := B^\pm_\rho(0),$$
$$S_\rho(y) = B_\rho(y) \cap \{x^n = 0\}, \quad S_R := S_R(0).$$

Theorem 4.1 (3.4 (Boundary Schauder Estimates on Unit Ball))**.** *As before $\alpha \in (0,1)$, $a^{ij}, b^i, c \in C^{0,\alpha}(B_1^+)$ and*

$$|a^{ij}|_{0,\alpha;B_1^+} + |b^i|_{0,\alpha;B_1^+} + |c|_{0,\alpha;B_1^+} \le \beta,$$

and $a^{ij}\xi_i\xi_j \ge \lambda|\xi|^2$ for all $x \in B_1^+$, for all $\xi \in \mathbb{R}^n$. Suppose $u \in C^{2,\alpha}(B_1^+)$ solves:

$$\begin{cases} Lu = f \in C^{0,\alpha} \text{ in } B_1^+, \\ u = \varphi \in C^{2,\alpha}(B_1^+) \text{ on } S_1. \end{cases}$$

Then

$$|u|_{2,\alpha;B^+_{1/2}} \le C\left(|u|_{0;B_1^+} + |f|_{0,\alpha;B_1^+} + |\varphi|_{2,\alpha;B_1^+}\right).$$

Proof. By considering $v := u - \varphi$, it suffices to consider the case $\varphi \equiv 0$ ($\varphi \in C^{2,\alpha}(B_1^+)$). Proceed as in Thm 3.1. The reduction step is exactly the same (note a boundary version of the absorbing lemma is needed, and upon making the natural modifications to the statement one argues as before). Steps 2 and 3 are key, which we now detail.

Step 2:

Claim: for all $\delta > 0$, there exists $C = C(n, \lambda, \alpha, \beta, \delta)$ s.t.

$$[D^2 u]_{\alpha; B_{1/2}^+} \leq \delta [D^2 u]_{\alpha; B_1^+} + C \left(|u|_{2; B_1^+} + |f|_{0,\alpha; B_1^+} \right).$$

proof of claim: Argue by contraditction. As before, up to a subsequence there exist $x_k, y_k \in B_1^+$, $u_k \in C^{2,\alpha}(B_1^+)$ that solve $L_k = f_k \in C^{,\alpha}(B_1^+)$ and

$$[D^2 u_k]_{\alpha; B_{1/2}^+} > \delta [D^2 u_k]_{\alpha; B_1^+} + k \left(|u_k|_{2; B_1^+} + |f_k|_{0,\alpha; B_1^+} \right),$$

as well as

$$\left| \frac{D^2_{\ell m} u_k(x_k) - D^2_{\ell m} u_k(y_k)}{|x_k - y_k|^\alpha} \right| > \tfrac{1}{2} [D^2 u_k]_{\alpha; B_{1/2}^+}.$$

Then, as before, $\rho_k := |x_k - y_k| \to 0$ as $k \to \infty$. We now have two cases to consider, either

(I)

$$\limsup_{k \to \infty} \frac{\operatorname{dist}(x_k, S_1)}{\rho_k} = \infty,$$

or (II)

$$\limsup_{k \to \infty} \frac{\operatorname{dist}(x_k, S_1)}{\rho_k} = \mu < \infty.$$

Lecture 11 **Claim:** for all $\delta > 0$, there exists C s.t.

$$[D^2 u]_{\alpha; B_{1/2}^+} \leq \delta [D^2 u]_{\alpha; B_1^+} + C \cdot \left(|u|_{2; B_1^+} + |f|_{0,\alpha; B_1^+} \right).$$

Two cases:

(1) either $\limsup_{k \to \infty} \frac{\operatorname{dist}(x_k, S_1)}{\rho_k} = \infty$,

(2) or $\limsup_{k \to \infty} \frac{\operatorname{dist}(x_k, S_1)}{\rho_k} = \mu < \infty$.

Case 1: Here for all $R > 0$ and k sufficiently large

$$\frac{1}{2} \geq \operatorname{dist}(x_k, S_1) \geq R \cdot \rho_k \quad (\text{as } x_k \in B_{1/2}^+),$$

so we have

$$B_{R\rho_k}(x_k) \subset B_1^+.$$

Set (as before)

$$\tilde{u}_k(x) = \frac{u_k(x_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1^+}},$$

where

$$q_k(x) = u_k(x_k) + (\rho_k x)^i D_i u_k(x_k) + \frac{1}{2} \rho_k^2 x^i x^j D^2_{ij} u_k(x_k).$$

Then $\tilde{u}_k$ is defined in $B_R(0)$, and

$$|\tilde{u}_k|_{2,\alpha;B_\rho(0)} \le C(R)$$

using Arzelà–Ascoli, proof goes through as in Thm 3.1.

Case 2: Here $\limsup_{k\to\infty} \frac{\operatorname{dist}(x_k,S_1)}{\rho_k} = \mu < \infty$.

Let $z_k = \operatorname{proj}_{\{x^n=0\}}(x_k)$, i.e.

$$z_k = (x_k^1, \dots, x_k^{n-1}, 0).$$

As before, look at $C^{2,\alpha}$, i.e. define

$$\tilde{u}_k(x) = \frac{u_k(z_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha}[D^2 u_k]_{\alpha;B_1^+}},$$

where

$$q_k(x) = u_k(z_k) + (\rho_k x)^i D_i u_k(z_k) + \frac{\rho_k^2}{2} x^i x^j D_{ij}^2 u_k(z_k).$$

$$= (\rho_k \partial_n u_k(z_k))x^n + \frac{\rho_k^2}{2} D_{nj}^2 u_k(z_k) x^i x^j$$

because $u|_{S_1} = 0$ and $\partial_i u|_{S_1} = 0$ for all $i \ne n$. In particular, as before, have

$$\left[D^2 \tilde{u}_k\right]_{\alpha;B_+^R(0)} \le 1$$

and for any $R > 0$,

$$|\tilde{u}_k|_{2,\alpha;B_+^R(0)} \le C(R) \quad \text{for } k \text{ suff. large.}$$

Also,

$$\tilde{u}_k|_{S_R} = 0 \quad \text{since on } \{x^n = 0\}, \quad q_k(x) = 0.$$

Set

$$\xi_k = \frac{x_k - z_k}{\rho_k}, \quad \eta_k = \frac{y_k - z_k}{\rho_k}.$$

Then for k sufficiently large,

$$|\xi_k| \le 2\mu \quad \text{and}$$

$$|\eta_k| \le \frac{|x_k - y_k| + |x_k - z_k|}{\rho_k} \le 1 + 2\mu.$$

So both sequences are bounded (and lie in compact subsets of $\mathbb{R}^n$), so can find convergent subsequences $\xi_k \to \xi$, $\eta_k \to \eta$. Then

$$D_{\ell m}^2 \tilde{u}_k(\xi_k) = \frac{D_{\ell m}^2 u_k(x_k) - D_{\ell m}^2 u_k(z_k)}{\rho_k^\alpha [D^2 u_k]_{\alpha;B_1^+}},$$

and

$$D_{\ell m}^2 \tilde{u}_k(\eta_k) = \frac{D_{\ell m}^2 u_k(y_k) - D_{\ell m}^2 u_k(z_k)}{\rho_k^\alpha [D^2 u_k]_{\alpha;B_1^+}}.$$

So,

$$\left|D_{\ell m}^2 \tilde{u}_k(\xi_k) - D_{\ell m}^2 \tilde{u}_k(\eta_k)\right| = \frac{|D_{\ell m}^2 u_k(x_k) - D_{\ell m}^2 u_k(y_k)|}{\rho_k^\alpha [D^2 u_k]_{\alpha;B_1^+}} \ge \frac{1}{2}[D^2 u_k]_{\alpha;B_{1/2}^+} \ge \frac{\delta}{2} > 0 \qquad (*)$$

(using contradiction assumption of proof of claim for the first inequality).

Then by Arzelà–Ascoli we obtain

$$v \in C^{2,\alpha}(\mathbb{R}^n_+ \cap \{x^n = 0\}) \quad \text{s.t. } \tilde{u}_k \to v$$

in C^2 on compact subsets of $\mathbb{R}^n_+ \cup \{x^n = 0\}$. As before, v satisfies

$$a_k^{ij} D^2_{ij} v = 0 \quad \text{– elliptic,}$$

and $\tilde{a}_k^{ij}$ is constant in k, and also

$$v|_{x^n=0} = 0.$$

Then again as before, by rotation and scaling, we get that there exists $w \in C^2(\overline{H})$, ($H = \{x^n > 0\}$), such that

$$\begin{cases} \Delta w = 0 & \text{on } H \\ w = 0 & \text{on } \partial H \end{cases}$$

By making an odd reflection in ∂H (see below), we can extend w to a harmonic $\tilde{w}$ on all of $\mathbb{R}^n$ with $[D^2\tilde{w}]_{\alpha;\mathbb{R}^n} < \infty$. But then this implies that $\partial^2\tilde{w}$ is harmonic and grows sublinearly, hence $\tilde{w}$ is constant (by Liouville). But then this contradicts ($*$) after taking it to the limit and so we are done with the claim.

To finish the proof of the theorem, by interpolation and scaling. Just as in Thm 3.1, we have for any $B_\rho(y) \subset B_1^+$ with $y \in \{x^n = 0\}$, we have

$$\rho^{2+\alpha}[D^2u]_{\alpha;B^+_{\rho/2}(y)} \leq \delta \cdot \rho^{2+\alpha}[D^2u]_{\alpha;B^+_\rho(y)} + C \cdot \left(|u|_{0;B_1^+} + |f|_{0,\alpha;B_1^+}\right).$$

Also, by the interior estimate, for any $B_\rho(y)$ s.t. $\overline{B_\rho(y)} \subset B_1^+$, we have

$$\rho^{2+\alpha}[D^2u]_{\alpha;B_{\rho/2}(y)} \leq C \cdot \left(|u|_{0;B_1^+} + |f|_{0,\alpha;B_1^+}\right).$$

Then the conclusion now follows from the *boundary absorbing lemma.* □

Proposition 4.1 (Reflection Principle for Harmonic Functions)**.** *Let $\Omega^+ \subset \mathbb{R}^n_+$ be an open subset of $\mathbb{R}^n_+$ and let $T = \partial\Omega^+ \cap \{x^n = 0\}$. Let Ω^- be the reflection of Ω^+ in $\{x^n = 0\}$, i.e.*

$$\Omega^- = \{(x', -x^n) : (x', x^n) \in \Omega^+\}.$$

Let $v \in C^2(\Omega^+) \cap C^0(\Omega^+ \cap T)$ and $\bar{v}$ be the odd reflection of v in T, i.e.

$$\bar{v} : \Omega^+ \cup T \cup \Omega^- \to \mathbb{R}, \qquad \bar{v}(x', x^n) = \begin{cases} v(x', x^n), & (x', x^n) \in \Omega^+ \\ -v(x', -x^n), & (x', -x^n) \in \Omega^-. \end{cases}$$

Then if $\Delta v = 0$ in Ω^+ and $v|_T = 0$, then

$$\bar{v} \in C^2((\Omega^+ \cup T \cup \Omega^-)^\circ) \quad \textit{and} \quad \Delta\bar{v} = 0.$$

Proof. (Use that the even-odd extension is continuous on $\Omega^+ \cup T \cup \Omega^-$ and recall that the local MVP for continuous functions implies Harmonicity, using the maximum principle to conclude). □

Remark. *This is trivial if $T = \emptyset$, as then $\Omega^+ \cup \Omega^-$ disjoint. Important part is C^2 across T.*

Proposition 4.2 (Absorbing Lemma, Boundary Version)**.** *Given $\theta \in (0,1)$, $\mu \in \mathbb{R}$, then there exists $\delta = \delta(n, \delta, \mu)$ and $C = C(n, \delta, \mu)$ s.t.: if $R > 0$,*

$$\mathcal{B} = \left\{ B_\rho(y) \subset B_R^+(0) \right\}, \quad \mathcal{B}^+ = \left\{ B_\rho^+(y) : y^n = 0, B_\rho(y) \subset B_R^+(0) \right\},$$

and $S : \mathcal{B} \cup \mathcal{B}^+ \to \mathbb{R}_{\geq 0}$ sub-additive function satisfying:

$$\rho^\mu S(B_{\theta\rho}^+(y)) \leq \delta \cdot \rho^\mu S(B_\rho^+(y)) + \gamma \quad \text{for all } B_\rho^+(y) \in \mathcal{B}^+,$$

and

$$\rho^\mu S(B_{\theta\rho}(y)) \leq \delta \cdot \rho^\mu S(B_\rho(y)) + \gamma \quad \text{for } B_\rho(y) \in \mathcal{B},$$

then

$$R^\mu S(B_{\theta\rho}^+(0)) \leq C\gamma.$$

Proof. . □

Lecture 12 **Shorthand:** write "hypotheses (H)" for: "Suppose $\Omega \subset \mathbb{R}^n$ is a *bounded* domain and $a \in (0,1)$. Suppose $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$ are such that

$$|a^{ij}|_{0,\alpha;\Omega} + |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta,$$

and suppose that there exists $\lambda > 0$ such that

$$a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$

As always, $L = a^{ij}\partial_i\partial_j + b^i\partial_i + c$."

Theorem 4.2 (Boundary Schauder Estimates in General Domains)**.** *Suppose (H) holds, Ω is a $C^{2,\alpha}$ domain. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that if $u \in C^{2,\alpha}(\overline{\Omega})$, $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^{2,\alpha}(\Omega)$ solve*

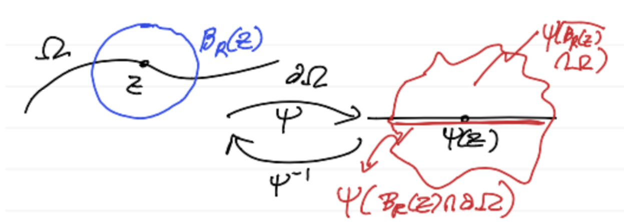
$$\begin{cases} Lu = f, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases}$$

then for all $x \in \partial\Omega$,

$$\rho^{2+\alpha}[D^2u]_{2,\alpha;B_\epsilon(x)\cap\Omega} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right).$$

Remark. *Need Ω to be $C^{2,\alpha}$ to have any chance of u being $C^{2,\alpha}$ on $\partial\Omega$.*

Proof. Pick $z \in \partial\Omega$. By definition there exists $R > 0$ and $\Psi : B_R(z) \to D \subset \mathbb{R}^n$ a $C^{2,\alpha}$-diffeomorphism such that:



In other words, $\Psi(B_R(z)\cap\Omega)\subset\{y^n>0\}$ and $\Psi(B_\rho(z)\cap\partial\Omega)\subset\{y^n=0\}$, i.e. Ψ rectifies $\partial\Omega$ near z. Let $x=(x^1,x^2,\dots,x^n)$ be coordinates in Ω and let $y=(y^1,y^2,\dots,y^n)$ be coordinates in D. Let $\tilde u(y)=u(\Psi^{-1}(y))$ be the pullback of u along Ψ^{-1}. Then

$$\tilde u|_{\{y^n=0\}\cap D}=(\varphi\circ\Psi^{-1})|_{\{y^n=0\}\cap D}=:\tilde\varphi\,.$$

To apply Theorem 4.2 (Uniq. boundary Schauder) need to find PDE satisfied by $\tilde u$ and show it satisfies the hypotheses. Note $u(x)=\tilde u(\Psi(x))$, so

$$D_{x^i}u=D_{y^k}\tilde u\,\frac{\partial\psi^k}{\partial x^i},\quad\text{so}$$

$$\partial^2_{x^ix^j}u=\partial^2_{y^\ell y^k}\tilde u(\Psi(x))\,\frac{\partial\psi^\ell}{\partial x^i}\,\frac{\partial\psi^k}{\partial x^j}+D_{y^k}\tilde u(\Psi(x))\,\frac{\partial^2\psi^k}{\partial x^i\partial x^j}$$

Hence can find the coefficients of the new PDE explicitly:

$$\begin{gathered}A^{\ell k}\partial^2_{y^\ell y^k}\tilde u+B^\ell D_{y^\ell}\tilde u+C\tilde u=\tilde f\quad\text{on }D\\ \tilde u=\tilde\varphi\quad\text{on }D\cap\{y^n=0\}\end{gathered}$$

where

$$\begin{gathered}A^{\ell k}=a^{ij}\frac{\partial\psi^k}{\partial x^i}\frac{\partial\psi^\ell}{\partial x^j},\quad B^\ell=\frac{\partial\psi^\ell}{\partial x^i}b^i+a^{ij}\frac{\partial^2\psi^\ell}{\partial x^i\partial x^j},\\ C=c\circ\psi^{-1},\quad\tilde f=f\circ\psi^{-1}\end{gathered}$$

Rescale: choose $\sigma>0$ s.t. $B_\sigma(\Psi(z))\subset D$ (as D open). Want to apply Thm 4.2 with $\bar u(y)=\tilde u(\Psi(z)+\sigma y)$, we will get

$$|\tilde u|'_{2,\alpha;B^+_{\sigma/2}(\Psi(z))}\le C\Big(|\tilde u|_{0;B^+_\sigma(\Psi(z))}+\sigma^2|\tilde f|_{\alpha;B^+_\sigma(\Psi(z))}+|\tilde\varphi|'_{2,\alpha;B^+_\sigma(\Psi(z))}\Big)\tag{†}$$

for some $C=C(n,\lambda,\alpha,\beta,\Psi)$. To apply Thm 4.2, we need to check that the assumptions are satisfied. That is,

(a) Coefficients are bounded, and strictly ellipticity.

(b) $|A^{\ell k}|'_{0,\alpha;B^+_\sigma(\Psi(z))}+|B^\ell|'_{0,\alpha;B^+_\sigma(\Psi(z))}+|C|'_{0,\alpha;B^+_\sigma(\Psi(z))}\le\mu(\Psi)\beta$

(b) For this, note

$$A^{\ell k}(y)\xi^\ell\xi^k=a^{ij}\partial_i(\xi\cdot\Psi)\partial_j(\xi\cdot\Psi)$$

and a^{ij} elliptic gives the lower bound

$$\ge\lambda|D(\xi\cdot\Psi)|_|^2\Psi^{-1}(y)\ge\lambda c(\Psi)|\xi|^2\,.$$

The last inequality follows from

$$\xi\cdot y=\xi\cdot\Psi(\Psi^{-1}(y))$$

and by the chain rule,

$$\begin{gathered}\xi=D(\xi\cdot\Psi)|_{\Psi^{-1}(y)}\cdot D\Psi^{-1}|_y\\ |\xi|\le|D(\xi\cdot\Psi)||_{\Psi^{-1}(y)}\cdot\underbrace{\left\|D\Psi^{-1}\right\|_0}_{C(\Psi)^{-1/2}\in(0,\infty)}\,.\end{gathered}$$

Can check (a) similarly [✍]. So transforming RHS of (†) for $\tilde{u} \to u, \tilde{f} \to f, \tilde{\varphi} \to \varphi$ have

$$|\tilde{u}|'_{2,\alpha;B^+_{\sigma/2}(\Psi(z))} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$

where $C = C(n, \lambda, \alpha, \beta, \Psi, \sigma)$. Pick $\sigma > 0, r = r(z)$ s.t.

$$B_r(z) \subset \Psi^{-1}(B_{\sigma/2}(\Psi(z)))\,.$$

Then, by the above, we have

$$|u|_{2,\alpha;B_r(z)\cap\Omega} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right) \tag{††}$$

All this was done for a fixed $z \in \partial\Omega$ so have $\Psi = \Psi_z$, $\sigma = \sigma_z$, $C = C_z$.

We finish with a compactness argument. Clearly $\partial\Omega \subset \bigcup_{z\in\partial\Omega} B_{r(z)/2}(z)$, by compactness there is a finite subcover $\partial\Omega \subset \bigcup_{j=1}^N B_{r(z_j)/2}(z_j)$. Then let $\epsilon = \min_{j\in\{1,\dots,N\}}\left\{\frac{r(z_j)}{2}\right\}$
$C = \max_{1\le j\le N}\left\{C_{z_j}\right\}$. Then for any $x \in \partial\Omega$

$$|u|_{2,\alpha;B_\epsilon(z)\cap\Omega} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)\,.$$

□

5 Global Schauder Estimates

Can combine *interior* and *boundary* estimates to deduce the following result.

Theorem 5.1 (Global Schauder Estimates)**.** *Suppose hypothesis (H) holds. Let Ω be a $C^{2,\alpha}$ domain. Then if $u \in C^{2,\alpha}(\Omega)$, $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^{2,\alpha}(\Omega)$ satisfy*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

then

$$|u|_{2,\alpha;\Omega} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\partial\Omega}\right)$$

where $C = C(n, \lambda, \alpha, \beta, \Omega)$.

Proof. Let $\epsilon = \epsilon(\Omega)$ be as in the proof of Theorem 4.2. Then let
$\Omega_\epsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \epsilon/4\}$. By interior estimates, we have

$$|u|_{2,\alpha;\Omega_\epsilon} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right)\,.$$

Then note that $\Omega \setminus \Omega_\epsilon \subset \bigcup_{x\in\partial\Omega} B_{\epsilon/4}(x)$. Therefore for all $x \in \Omega$,

- either $x \in \Omega_\epsilon$ giving

$$|u(x)| + |Du(x)| + |D^2u(x)| \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right)$$

- or $x \in \Omega \setminus \Omega_\epsilon$, and some $B_{\epsilon/4}(y)$ contains x for some $y \in \partial\Omega$. By Theorem 4.2

$$|u(x)| + |Du(x)| + |D^2u(x)| \le |u|_{2,\alpha;B_{\epsilon/4}(y)\cap\Omega} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\partial\Omega}\right)$$

So in both cases we obtain

$$|u|_{2;\Omega} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\partial\Omega}\right)$$

To be continued... □

Lecture 13

- ⊛ Global Schauder Estimates
- ⊛ Solvability of the "Dirichlet problem"

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

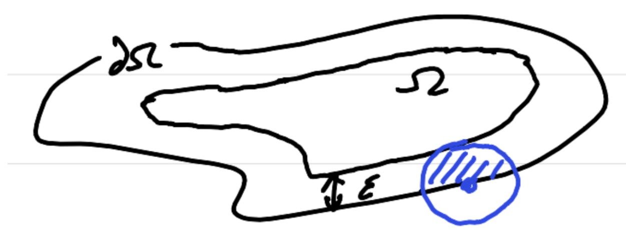
Quasilinear Theory (2$^{\text{nd}}$ order)

De Giorgi–Nash–Moser (*a priori* estimates)
Application to minimal surface equation

Proof (continued). We have just established

$$|u|_{2;\Omega} \le C_1\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right) \qquad (1)$$
$$C_1 = C_1(n, \lambda, \alpha, \beta, \Omega)\,.$$

It remains to bound $\left[D^2u\right]_{\alpha;\Omega}$. Let $x, y \in \Omega$, $x \neq y$.



Let ε be as in Thm 4.2. Suppose $|x-y| < \varepsilon/4$. There are two cases to consider. First suppose both

$$x, y \in \Omega_\varepsilon := \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > \varepsilon/4\}\,.$$

In this case, interior Schauder estimate gives:

$$\frac{|D^2_{ij}u(x) - D^2_{ij}u(y)|}{|x-y|^\alpha} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right)\,. \qquad (\circledast)$$

Alternatively, if either $x \in \Omega \setminus \Omega_\varepsilon$ or $y \in \Omega \setminus \Omega_\varepsilon$, then $x, y \in B_{\varepsilon/2}(z)$, for some $z \in \partial\Omega$. Then Thm 4.2 gives ⊛.

Finally, suppose that $|x-y| \ge \varepsilon/4$:

$$\begin{aligned} \frac{|D^2_{ij}u(x) - D^2_{ij}u(y)|}{|x-y|^\alpha} &\le (\varepsilon/4)^{-\alpha}\left(|D^2_{ij}u(x)| + |D^2_{ij}u(y)|\right) \\ &\le 2(\varepsilon/4)^{-\alpha}|u|_{2;\Omega} \\ &\le (\varepsilon/4)^{-\alpha} \cdot C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right) \end{aligned}$$

by 1, concluding the proof. □

6 Solvability of the Dirichlet problem

Having established a priori estimates of solutions to the aforementioned Dirichlet Problem, we now turn our attention to another set of important questions. Given $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$, the Dirichlet problem for L is: given $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$, does there exist a solution $u \in C^{2,\alpha}(\overline{\Omega})$ to:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{DP}$$

If exists, is it unique?

Theorem 6.1. *Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ a bounded $C^{2,\alpha}$ domain. Suppose $a^{ij}, b^i, c \in C^{2,\alpha}(\overline{\Omega})$, $c \leq 0$ in Ω (las condition turns out to be necessary,) and a^{ij} satisfy the strict ellipticity condition*

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \lambda > 0 \text{ constant}, \forall x \in \Omega, \xi \in \mathbb{R}^n .$$

Then,

1. *for any given $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$, the Dirichlet problem*

$$Lu = f \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega$$

has a solution $u \in C^{2,\alpha}(\overline{\Omega})$.

2. $\iff$ *For any given $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$, the (DP)*

$$\Delta u = f \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega$$

has a solution $u \in C^{2,\alpha}(\overline{\Omega})$.

Proof. By considering $u - \varphi$ in place of u, it suffices to establish the equivalence for the case $\varphi \equiv 0$, since

$$\begin{bmatrix} Lu = f \text{ in } \Omega \\ u = \varphi \text{ on } \partial\Omega \end{bmatrix} \iff \begin{bmatrix} Lv = f - L\varphi \text{ in } \Omega \\ v = 0 \text{ on } \partial\Omega, u = v + \varphi \end{bmatrix}$$

So assume $\varphi = 0$. Note that the subspace

$$C_0^{2,\alpha}(\overline{\Omega}) := \left\{ v \in C^{2,\alpha}(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega \right\}$$

is a closed subspace of $C^{2,\alpha}(\overline{\Omega})$, with respect to the usual norm, hence Banach.

Consider one-parameter family of operators:

$$L_t : C_0^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega}), \quad t \in [0,1] .$$

$$L_t = tL + (1-t)\Delta, \quad \text{so } L_0 = \Delta,\, L_1 = L .$$

Let

$$Lu \equiv a^{ij}D^2_{ij}u + b^iD_iu + cu$$

$$a_t^{ij} = ta^{ij} + (1-t)\delta^{ij}, \quad b_t^i = tb^i, \quad c_t = tc .$$

Let

$$\beta = \sum |a_t^{ij}|_{0,\alpha;\Omega} + \sum |b_t^i|_{0,\alpha;\overline{\Omega}\Omega} + |c_t|_{0,\alpha;\Omega} .$$

Then,

$$\sum |a_t^{ij}|_{0,\alpha;\Omega} + \sum |b_t^i|_{0,\alpha;\Omega} + |c_t|_{0,\alpha;\Omega} \leq \max\{1,\beta\} \quad \forall t \in [0,1]$$

and similarly

$$a_t^{ij}\xi_i\xi_j \geq \min\{1,\lambda\}|\xi|^2 \quad \forall t \in [0,1]\,.$$

The Global Schauder Estumates, Thm 5.1) now gives

$$|u|_{2,\alpha;\Omega} \leq C\left(|u|_{0;\Omega} + |Lu|_{0,\alpha;\Omega}\right) \quad \forall u \in C_0^{2,\alpha}(\overline{\Omega})\,,$$

for some

$$C = C(n,\lambda,\gamma,\beta,\Omega)\,.$$

□

Lecture 14

Proof of Thm 6.1 (continued). $L_t\colon C_0^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$,

$$L_t = tL + (1-t)\Delta.$$

Global Schauder give $|u|_{2,\alpha;\Omega} \leq C_1(|u|_{0;\Omega} + |L_t u|_{0,\alpha;\overline{\Omega}})$, for all $u \in C_0^{2,\alpha}(\overline{\Omega})$, $C_1 = C_1(n,\lambda,\alpha,\beta,\Omega)$ (indep. of t and u).

Since $c \leq 0$, by the max. principle, a priori estimate, Thm 2.4

$$|u|_{0;\Omega} \leq C_2|L_t u|_{0;\overline{\Omega}}, \quad C_2 = C_2(n,\lambda,\alpha,\beta,\Omega).$$

So

$$|u|_{2,\alpha;\overline{\Omega}} \leq C|L_t u|_{0,\alpha;\overline{\Omega}} \quad \forall u \in C_0^{2,\alpha}(\overline{\Omega}),$$

for all $u \in C_0^{2,\alpha}(\overline{\Omega})$, $C = C(n,\lambda,\kappa,\beta,\Omega)$. This says L_t is injective. Solvability of $L_t u = f$ in $C_0^{2,\alpha}(\overline{\Omega})$ is equivalent to surjectivity of L_t (by the injectivity of L_t). We will show if L_t is surjective for some $t \in [0,1]$, then it is surjective for all $t \in [0,1]$.

Let $s \in [0,1]$ and suppose L_s is bijective. The estimate above can be written as

$$|L_s^{-1}g|_{2,\alpha;\overline{\Omega}} \leq C|g|_{0,\alpha;\overline{\Omega}}, \quad \forall g \in C^{0,\alpha}(\overline{\Omega}).$$

Now, fix $f \in C^{0,\alpha}(\Omega)$. The following chain of equivalences holds,

$$L_t u = f \iff L_s u + (L_t - L_s)u = f \iff u + L_s^{-1}((L_t - L_s)u) = L_s^{-1}f \iff u = L_s^{-1}f + \underbrace{L_s^{-1}((L_s - L_t)u)}_{T_t u}$$

<u>Claim:</u> $T_t\colon C_0^{2,\alpha}(\overline{\Omega}) \to C_0^{2,\alpha}(\overline{\Omega})$ is a contraction mapping provided $|t-s| \leq \gamma$, where

$$\gamma = \gamma(C(n,\alpha,\beta,\lambda),\Omega).$$

Indeed, for $u, v \in C_0^{2,\alpha}(\overline{\Omega})$,

$$\begin{aligned}
|T_t u - T_t v|_{2,\alpha;\overline{\Omega}} &= \left|L_s^{-1}(L_s - L_t)(u-v)\right|_{2,\alpha;\overline{\Omega}} \\
&= |s-t| \cdot \left|L_s^{-1}(L-\Delta)(u-v)\right|_{2,\alpha;\overline{\Omega}} \\
&\leq C \cdot |s-t| \cdot |(L-\Delta)(u-v)|_{2,\alpha;\overline{\Omega}} \quad \text{(direct computation)}
\end{aligned}$$

$$\leq \tilde{C} \cdot |s-t| \cdot |(u-v)|_{2,\alpha;\overline{\Omega}}.$$

So if $|s-t| \leq \frac{1}{2\tilde{C}}$, then T_t is a contraction.

By the contraction mapping principle, T_t has a unique fixed point $u \in C_0^{2,\alpha}(\overline{\Omega})$. If solvability of $L_s u = f$ for $u \in C_0^{2,\alpha}(\overline{\Omega})$ holds for some $s \in [0,1]$, then solvability of $L_t u = f$ for $u \in C_0^{2,\alpha}(\overline{\Omega})$ holds for all $t \in [s-\gamma, s+\gamma]$. By breaking $[0,1]$ into intervals of length 2γ and applying this conclusion on each subinterval, we arrive at the conclusion of the Theorem. □

Remark. *The method of proof is called the* continuity method. *The exact main steps of solvability of L:*

(i) Use Thm 6.1 to prove solvability when $\Omega = B$ a ball

(ii) Perron's method: "solvability in balls $\Rightarrow$ solvability in general domains".

Proposition 6.1. *Let $B = B_R(y) \subset \mathbb{R}^n$ be any open ball. If $f \in C^\infty(\overline{B})$, $\varphi \in C^\infty(\overline{B})$, then there is a unique function $u \in C^\infty(\overline{B})$ s.t.*

$$\Delta u = f \text{ in } B, \quad u = \varphi \text{ on } \partial B.$$

Proof (Sketch). By considering $v = u - \varphi$, one can simply reduce the proof to showing the analogous statement for $\Delta v = f - \Delta\varphi$, $v = 0$ on ∂B. Now, by Riesz rep. theorem, there exists a weak solution $u \in W_0^{1,2}(B)$. Regularity theory (difference quotient arguments) give $v \in C^\infty(\overline{B})$, see the book of Evans. □

One can generalise this to the case $f \in C^{0,\alpha}(\overline{B})$, $\varphi \in C^0(\overline{B})$ or $(\varphi \in C^{2,\alpha}(\partial B))$.

Proposition 6.2. *Let $B = B_R(y) \subset \mathbb{R}^n$. If $f \in C^{0,\alpha}(\overline{B})$ and $\varphi \in C^0(\overline{B})$ then there exists a unique $u \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ s.t.*

$$\Delta u = f \text{ in } B, \quad u = \varphi \text{ on } \partial B.$$

If $\varphi \in C^{2,\alpha}(\overline{B})$, then $u \in C^{2,\alpha}(\overline{B})$.

Proof. Idea is to mollify f, φ to get smooth data, use Prop 6.1 to solve for these smooth approximations and then take a limit. More precisely, consider the standard mollifier

$$\eta(x) = \begin{cases} \eta = c \cdot e^{\frac{1}{|x|^2-1}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}, \ c \text{ chosen s.t. } \int_{\mathbb{R}^n} \eta = 1$$

Define for $\sigma > 0$, $\eta_\sigma(x) = \sigma^{-n}\eta\left(\frac{x}{\sigma}\right)$. Choose $\sigma_k \to 0^+$. Extend f to $\tilde{f} \in C_c^{0,\alpha}(\mathbb{R}^n)$ and $\varphi \to \tilde{\varphi} \in C_c^0(\mathbb{R}^n)$. Mollify $\tilde{f}, \tilde{\varphi}$ and set

$$f_k(x) = \int_{\mathbb{R}^n} f(y)\eta_{\sigma_k}(x-y) \, \mathrm{d}y = \int_{\mathbb{R}^n} f(x-y)\eta_{\sigma_k}(y) \, \mathrm{d}y$$

$$\varphi_k(x) = \int_{\mathbb{R}^n} \tilde{\varphi}(y)\eta_{\sigma_k}(x-y) \, \mathrm{d}y = \int_{\mathbb{R}^n} \tilde{\varphi}(x-y)\eta_{\sigma_k}(y) \, \mathrm{d}y \, .$$

Note that $f_k \to f$, $\varphi_k \to \varphi$ uniformly in $\overline{B}$. We in fact have $|f_k|_{0,\alpha;\mathbb{R}^n} \leq |\tilde{f}|_{0,\alpha;\mathbb{R}^n}$ and $|\varphi_k|_{0;\mathbb{R}^n} \leq |\tilde{\varphi}|_{0,;\mathbb{R}^n}$ (direct computation).

By Prop 6.1, get $u_\sigma \in C^\infty(\overline{B})$ s.t.

$$\Delta u_\sigma = f_\sigma \text{ in } B, \quad u_\sigma = \varphi_\sigma \text{ on } \partial B.$$

□

Lecture 15

Proof of Prop 6.2 (Cont'd). To solve

$$\Delta u = f \text{ in } B, \quad u = \varphi \text{ on } \partial B \text{ for } f \in C^{0,\alpha}(\overline{B}), \varphi \in C^0(\overline{B}),$$

$$\Delta u_k = f_k \text{ in } B, \quad u_k = \varphi_k \text{ on } \partial B,$$

$$\Delta(u_k - u_l) = f_k - f_l \text{ in } B, \quad u_k - u_l = \varphi_k - \varphi_l \text{ on } \partial B.$$

By the max. principle a priori estimate, Theorem 2.4, we have

$$|u_k - u_l|_{0;\overline{B}} \leq |\varphi_k - \varphi_l|_{0;\partial B} + C \cdot |f_k - f_l|_{0;\overline{B}} \to 0 \quad \text{as } k, l \to \infty\,.$$

So u_k is Cauchy and hence converges uniformly to some $u \in C^0(\overline{B})$. In particular, $u = \varphi$ on ∂B.

Now applying interior Schauder estimates gives for all $B' \subset\subset B$,

$$|u_k|_{2,\alpha;B'} \leq C\left(|u_k|_{0;\overline{B}} + |f_k|_{0,\alpha;\overline{B}}\right) \leq C\left(|u|_{0;\overline{B}} + |\tilde{f}|_{0,\alpha;\mathbb{R}^n} + 1\right)$$

from the mollification bounds. Passing to a subsequence (without relabelling), there exists $u \in C^{2,\alpha}(B')$ s.t. $u_k \to u$ in $C^2(B')$ (by Arzelà-Ascoli). Since $u_k \to u$ pointwise, we have $v = u$ in B' and so in particular, $u \in C^{2,\alpha}(B')$. By passing to limit in $\Delta u_k = f_k$ in B, get $\Delta u = f$ in B. $B' \subset\subset B$ is arbitrary, so $u \in C^{2,\alpha}(B)$ and $\Delta u = f$ in B.

For the 2nd part when $\varphi \in C^{2,\alpha}(\overline{B})$, repeat the argument (after extending φ to $\tilde{\varphi} \in C^{2,\alpha}_c(\mathbb{R}^n)$, see general extension theorem, [GTGT77, Lemma 6.37], and use *global Schauder estimates* in place of interior estimates. ☐

Proposition 6.3. *Let $B \subset \mathbb{R}^n$ be a ball, $a \in C^{0,1}$, $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{B})$, $c \leq 0$, $Lu \equiv a^{ij} D^2_{ij} + b^i \partial_i + c$ be strictly elliptic. Then for any $f \in C^{0,\alpha}(\overline{B})$ and $\varphi \in C^0(\overline{B})$, there exists unique $u \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ s.t. $Lu = f$ in B, $u = \varphi$ on ∂B.*

Proof. Combine Thm 6.1 with the proof of Prop. 6.2. ☐

Perron's method: We'll assume the following for the rest of the section. Hypothesis (H): $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ bounded,

$$\sum |a^{ij}|_{0,\alpha;\Omega} + \sum |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta,$$

$$a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2, \quad \lambda > 0 \text{ constant}\,, \underline{c \leq 0}\,.$$

Observation 1: Fix $f \in C^{0,\alpha}(\Omega)$, and suppose that $u \in C^2(\Omega)$. Then u is a subsolution to $Lu = f$ in Ω (i.e. $Lu \geq f$ in Ω): if for every ball $B \subset\subset \Omega$ we have that $u \leq u_B$, where $u_B \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ is the unique function satisfying $Lu_B = f$ in B, $u_B = u$ on ∂B (such u_B exists by Proposition 6.3). This follows from the weak maximum principle (Existence).

Observation 2: $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^0(\overline{\Omega})$. Define

$$S_\varphi \equiv \left\{v \in C^2(\Omega) \cap C^0(\overline{\Omega}) : Lv \geq f \text{ in } \Omega,\ v \leq \varphi \text{ on } \partial\Omega\right\}$$

Then if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solves $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$, then

$$u(x) = \sup_{v \in S_\varphi} v(x)\,.$$

Definition 6.1. *Let $f \in C^{0,\alpha}(\Omega)$. A function $u \in C^0(\Omega)$ is a* subsolution *(resp.* supersolution*) to $Lu = f$ in Ωif, for every ball $B \subset\subset \Omega$, we have $u \leq u_B$ (resp. $u \geq u_B$) in B, (where u_B is the unique function in $C^{2,\alpha}(B) \cap C^0(\overline{B})$ such that $Lu_B = f$ in B, $u_B = u$ on ∂B).*

Definition 6.2. *Let $u \in C^0(\Omega)$ be a subsolution to $Lu = f$ in Ω. Let $B \subset\subset \Omega$ be a ball, then the* L-lift *of u wrt B is the function U_B defined by*

$$U_B(x) = \begin{cases} u_B(x), & x \in B, \\ u(x), & x \in \Omega \setminus B. \end{cases}$$

Lemma 6.1. *We have the following:*

(i) Let $u, v \in C^0(\overline{\Omega})$, if u is a subsolution and v is a supersolution to $Lu = f$ in Ω, and if $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.

(ii) If $u_1, u_2 \in C^0(\overline{\Omega})$ are subsolutions to $Lu = f$ in Ω, then $u(x) = \max\{u_1(x), u_2(x)\}$ is again (continuous and) a subsolution to $Lu = f$.

(iii) If $u \in C^0(\Omega)$ is a subsolution, and $B \subset\subset \Omega$, then the L-lift of u is again a (cts) subsolution.

Proof. , application of max principle. $\square$

With the above in mind, **define** for $\varphi \in C^0(\overline{\Omega})$, $f \in C^{0,\alpha}(\Omega)$ fixed

$$S_\varphi \equiv \left\{ v \in C^0(\overline{\Omega}) : v \text{ is a subsolution in } \Omega,\ v \leq \varphi \text{ on } \partial\Omega \right\},$$

and set $u(x) = \sup_{v \in S_\varphi} v(x)$.

Lecture 16

Theorem 6.2. *Let **hyp**(H) hold. Let $f \in C^{0,\alpha}(\Omega)$ and $\varphi \in C^0(\overline{\Omega})$. Define*

$$S_\varphi = \left\{ v \in C^0(\overline{\Omega}) : v \textit{ is a subsolution to } Lu = f \textit{ in } \Omega,\ v \leq \varphi \textit{ on } \partial\Omega \right\}.$$

Set

$$u(x) = \sup_{v \in S_\varphi} v(x) \quad \forall x \in \Omega\,.$$

Then, the function u defined as above is well-defined (i.e. $S_\varphi \neq \emptyset$ and $u(x) \in \mathbb{R}$) and we have $u \in C^{2,\alpha}(\Omega)$ and solves $Lu = f$ in Ω.

Remark. *1. Even though we use the function φ to get the solution u as above, in this theorem, there is no claim about the behaviour of u on approach to $\partial\Omega$.*

2. Once we know $u \in C^2(\Omega)$, we of course have that

$$u(x) = \sup_{v(x)} \quad \textit{for } v \in C^2(\Omega),$$

$$\begin{cases} \textit{subsolution to} \\ Lu = f, v \leq \varphi \end{cases}$$

However, the proof of the theorem (including $u \in C^2(\Omega)$) will crucially depend on Lemma 6.1(ii), (iii):

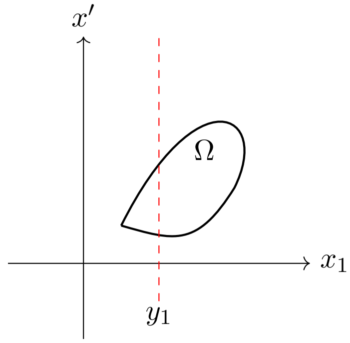
$$i.e., u_1, u_2 \text{ subsolutions } \Rightarrow \max\{u_1, u_2\} \text{ is a subsolution}]$$

L-lift of a subsolution is a subsolution.

Note they are not valid for the smaller class of C^2 subsolutions. In this sense, the philosophy of the proof is similar to the Hilbert space (variational) approaches to solving PDEs.

(Proof of Theorem 6.2). First we check that $S_\varphi \neq \emptyset$. Pick $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ such that

$$\Omega \subseteq \{x \in \mathbb{R}^n : x_1 \geq y_1\}.$$



Let $d = \sup_{x\in\Omega} |x - y| < \infty$ (Ω is bounded). Set

$$s(x) = -\sup_{\partial\Omega} |\varphi| - (e^{\gamma d} - e^{\gamma(x_1 - y_1)}) \cdot \sup_{\Omega} |f|,$$

for some sufficiently large constant γ. By direct calculation, we have

$$Ls = e^{\gamma(x_1-y_1)} \cdot \sup_{\Omega} |\varphi| \cdot \left(a^{11}\gamma^2 + b^1\gamma + c\right) - c \cdot (\sup_{\Omega} |\varphi| + e^{\gamma d}\gamma^d \sup_{\Omega} |f|).$$

Now, $c \leq 0$ gives the lower bound

$$Ls \geq e^{\gamma(x_1-y_1)} \cdot \sup_{\Omega} |f|(a^{11}\gamma^2 + b^1\gamma + c) \geq \sup_{\Omega} |f| \geq f \quad \text{if } \gamma \text{ is sufficiently large.}$$

Also,

$$s \leq -\sup_{\partial\Omega} |\varphi| \leq \varphi \quad \text{on } \partial\Omega.$$

Thus, $s \in S_\varphi$, so $S_\varphi \neq \emptyset$. Moreover, if $s_1 = -s$, then

$$Ls_1 = -Ls \leq -\sup_{\Omega} |f| \leq f \quad \text{in } \Omega, \quad s_1 \geq \varphi \text{ on } \partial\Omega \quad \text{by } (**).$$

So by Lemma 6.1(i), $v \leq s_1$, for all $v \in S_\varphi$. In particular, $u(x) \leq s_1$, for all $x \in S_\varphi$, hence u is well-defined.

Fix $z \in \Omega$, and choose $R > 0$ such that $\overline{B_R(z)} \subset \Omega$. By definition of $u(z)$, there exist $v_j \in S_\varphi$ s.t. $v_j(z) \to u(z)$. Let $\tilde{v}_j = \max\{v_j, s\}$, $s \in S_\varphi$ (by Lemma 6.1). So, $u(z) \geq \tilde{v}_j(z) \geq v_j(z)$ giving $\tilde{v}_j(z) \to u(z)$, 1 and $s \leq \tilde{v}_j \leq s_1(= -s)$ implies $\sup_\Omega |\tilde{v}_j| \leq \sup_\Omega |s|$.

Now, let V_j be the L-lift of $\tilde{v}_j$ w.r.t. the ball $B_R(z)$. So we have $LV_j = f$ in $B_R(z)$, $V_j = \tilde{v}_j$ on $\partial B_R(z)$. $V_j \in \tilde{S}_\varphi$ (Lemma 6.1iii)), and $V_j \geq \tilde{v}_j$ gives

$$u(z) \geq V_j(z) \geq \tilde{v}_j(z).$$

By interior Schauder estimates:

$$|V_j|_{C^{2,\alpha}(B_{R/2}(z))} \leq C\left(|V_j|_{0;B_R(z)} + |f|_{0,\alpha;B_R(z)}\right)$$
$$\overset{\text{max. principle estimate}}{\leq} C\left(|\tilde{v}_j|_{0;\partial B_R(z)} + |f|_{0,\alpha;B_R(z)}\right) \leq C \cdot C\left(\sup_\Omega |s| + |f|_{0,\alpha;B_R(z)}\right).$$

Now, Arzelà–Ascoli implies that there exists $V \in C^{2,\alpha}(B_{R/2}(z))$ s.t. up to a subsequence, $V_j \to V$ in $C^2(\overline{B_{R/2}(z)})$. In particular $LV = f$ (passing to limit in $LV_j = f$). By 1, $V(z) = u(z)$.

Claim: $u \equiv V$ in $B_{R/16}(z)$. This will complete the proof, since $V \in C^{2,\alpha}(B_{R/16}(z))$ and solves $Lv = f$, and $z \in \Omega$ is arbitrary.

Proof of claim: Since $u \geq V_j$ (since $V_j \in S_\varphi$), we also have $u \geq V$ in $B_{R/2}(z)$. If the claim is false, then there exists $z_1 \in B_{R/16}(z)$ s.t. $V(z_1) < u(z_1)$ and so there exists $w \in S_\varphi$ s.t. $V(z_1) < w(z_1) \leq u(z_1)$, 2. Let $w_j = \max\{w, V_j\} \in S_\varphi$. Note that $u \geq w_j \geq V_j$. Let W_j be the L-lift of w_j w.r.t. $B_{R/4}(z_1)$. By interior Schauder estimate as before there exists $w \in C^{2,\alpha}(B_{R/8}(z_1))$ s.t. up to a subsequence, $w_j \to w$ in $C^2(B_{R/8}(z_1))$. Have $Lw = f$ in $B_R(z_1)$.

Now, $W_j \geq w_j \geq V_j$ in $B_{R/4}(z_1)$, 3 which gives $W \geq V$ in $B_{R/8}(z_1)$. ($c \leq 0$, and the strong max. principle) implies $L(W - V) = 0$. By 3 and the fact that $V_j(z) \to u(z)$, we have that $W(z) = V(z)$, but since $z \in B_{R/8}(z_1)$, we have by the SMP that $W \equiv V$ in $B_R(z_1)$.

By 2, $V(z_1) < w(z_1) \underbrace{\leq}_{\text{defn of } w_j} w_j(z_1) \underbrace{\leq}_{\text{L-lift of } w_j} W_j(z_1)$ (, L-lift of w_j)

$$V(z_1) < W(z_1) \qquad \text{as } j \to \infty\,,$$

a contradiction. ☐

Lecture 17

Next goal: Discuss the behaviour of u on approach to $\partial\Omega$. We'll show that under a mild regularity condition on Ω (i.e., if Ω satisfies the *exterior sphere condition* at every point on $\partial\Omega$), the Perron solution extends to a C^0 function on $\overline{\Omega}$ and satisfies $u(x) = \varphi(x)$ on $\partial\Omega$.

To do this, we need the notion of barriers.

Definition 6.3. *Let* hyp(H) *hold, and let* $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^{0,\alpha}(\overline{\Omega})$. *Let* $x_0 \in \partial\Omega$.

(i) A sequence of functions $w_i^+ \in C^0(\overline{\Omega})$ *is an* <u>*upper barrier*</u> *at* x_0, *w.r.t.* L, f, φ *if:*

- $\circledast$ w_i^+ *is a super-solution to* $Lu = f$ *in* Ω *with* $w_i^+ \geq \varphi$ *on* $\partial\Omega$, *for each* i;
- $\circledast$ $w_i^+(x_0) \to \varphi(x_0)$ *as* $i \to \infty$.

(ii) A sequence $w_i^- \in C^0(\overline{\Omega})$ *is a* <u>*lower barrier*</u> *at* x_0, *w.r.t.* L, f, φ *if:*

- $\circledast$ w_i^- *is a subsolution to* $Lu = f$ *in* Ω *with* $w_i^- \leq \varphi$ *on* $\partial\Omega$, *for all* i;
- $\circledast$ $w_i^-(x_0) \to \varphi(x_0)$ *as* $i \to \infty$.

Proposition 6.4. *Suppose* hyp(H) *holds,* $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^0(\overline{\Omega})$. *Let* $x_0 \in \partial\Omega$. *Suppose upper and lower barriers at* x_0, *w.r.t.* L, f, φ *exist. Then the Perron solution* u *given by Thm 6.2 has the property that* $u(x) \to \varphi(x_0)$ *as* $x \to x_0$, $x \in \Omega$.

Proof. Let $(w_i^\pm)$ be upper and lower barriers at x_0. Since w_i^+ is a supersolution with $w_i^+ \geq \varphi$ on $\partial\Omega$, we have by Lemma 6.1(i) that $v \leq w_i^+ \in \overline{\Omega}$. Thus for all $v \in \mathrm{S}_\varphi$, $u \leq w_i^+$ in Ω for all i. Also, $w_i^- \leq u$ for all i, since $w_i^- \in \mathrm{S}_\varphi$. Since $w_i^\pm(x_0) \to \varphi(x_0)$ and $w_i^\pm \in C^0(\overline{\Omega})$, we get the conclusion. $\square$

Proposition 6.5. *Suppose* hyp(H) *holds,* $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^0(\overline{\Omega})$. *Let* $x_0 \in \partial\Omega$. *If there exists* $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *s.t.:*

(i) $Lw \leq -1$ *in* Ω,

(ii) $w(x_0) = 0$,

(iii) $w(x) > 0 \quad \forall x \in \overline{\Omega} \setminus \{x_0\}$,

then upper and lower barriers exist at x_0 *w.r.t.* L, f, φ. *In fact, for any sequence* $\varepsilon_i \to 0$, *there exist constants* k_i *s.t.* $w_i^\pm(x) = \varphi(x_0) \pm \varepsilon_i \pm k_i w(x)$ *define upper and lower barriers.*

Proof. Let $\varepsilon > 0$ and choose $r > 0$ s.t. $|\varphi(x) - \varphi(x_0)| < \varepsilon$ for all $x \in B_\delta(x_0) \cap \partial\Omega$. Since $\partial\Omega \setminus B_\delta(x_0)$ is compact, we can find constant ℓ_ε large enough s.t.

$$\begin{cases} \ell_\varepsilon w(x) \geq \varphi(x) - \varphi(x_0) - \varepsilon \\ \ell_\varepsilon w(x) \geq -\varphi(x) + \varphi(x_0) - \varepsilon \end{cases} \qquad \forall x \in \partial\Omega \setminus B_\delta(x_0).$$

Set

$$k_\varepsilon = \max\left\{ \ell_\varepsilon, \sup_{x \in \overline{\Omega}} |f(x) - c(x)\varphi(x_0)| \right\}.$$

Then we compute $Lw_\varepsilon \leq f$ in Ω where

$$w_\varepsilon(x) = \varphi(x_0) + \varepsilon + k_\varepsilon w(x).$$

So take $\varepsilon_n \searrow 0$ we get $w_{\varepsilon_n}^+ := w_{\varepsilon_n}$ is an upper barrier.

Similarly, $w_{\varepsilon_n}^- = \varphi(x_0) - \varepsilon_n - k_{\varepsilon_n} w(x)$. $\square$

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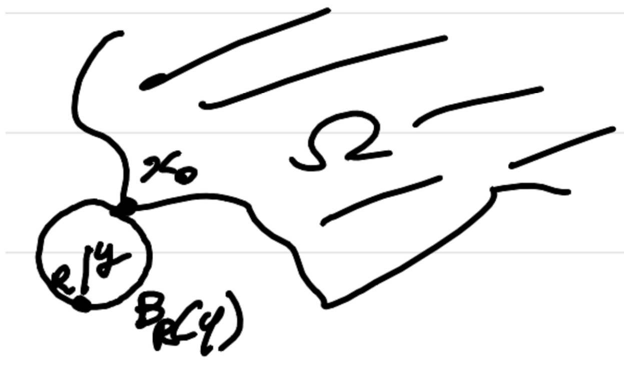
Proposition 6.6. *Suppose* hyp(H) *holds, $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^0(\overline{\Omega})$. Then, if Ω satisfies the exterior sphere condition at $x_0 \in \partial\Omega$, then upper and lower barriers exist at x_0 w.r.t. L, f, φ.*

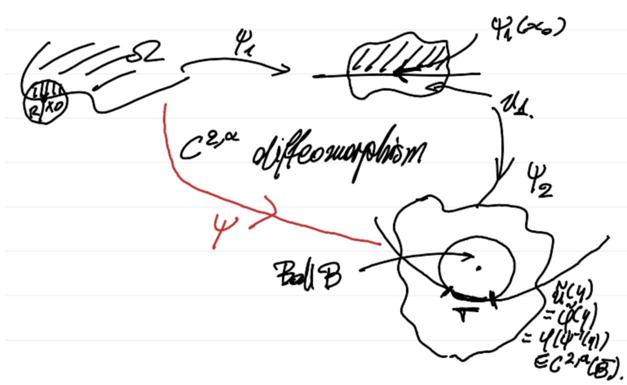


Proof. By assumption, there exists $B_R(y) \subset B^n$ s.t. $\overline{B}_R(y) \cap \overline{\Omega} = \{x_0\}$. Let $w(x) = \mu(R^{-\sigma} - |x-y|^{-\sigma})$ for $x \in \overline{\Omega}$, where $\mu, \sigma > 0$. Then $w \in C^2(\overline{\Omega})$, $w(x_0) = 0$. By direct calculation, , we see that $Lw(x) = -1$ for all $x \in \Omega$, provided $\mu, \sigma > 0$ are chosen appropriately. $\square$

Theorem 6.3. *Let* hyp(H) *hold, $f \in C^{0,\alpha}(\overline{\Omega})$, and $\varphi \in C^0(\overline{\Omega})$. Then there is a unique function $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ s.t. $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$, provided Ω satisfies the exterior sphere condition (e.g. if Ω is a C^2 domain).*

Proof. Let u be given by Thm 6.2. Then $u \in C^{2,\alpha}(\Omega)$ and satisfies $Lu = f$ in Ω. Then extend u to $\overline{\Omega}$ by setting $u(x) = \varphi(x)$ for all $x \in \partial\Omega$. Propositions 6.2-6.4 implies $u \in C^0(\overline{\Omega})$. $\square$

Theorem 6.4. *Suppose that* hyp(H) *holds, and Ω is a bounded $C^{2,\alpha}$ domain. Then for any $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$, there is a unique function $u \in C^{2,\alpha}(\overline{\Omega})$ s.t. $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$.*



Proof. Set $\psi_1 \colon B_R(x_0) \to \mathcal{U}_1$, $\quad \psi_2 \colon \mathcal{U}_1 \to \mathcal{U}_2$ both $C^{2,\alpha}$-diffeomorphisms. Let $\Psi = \psi_2 \circ \psi_1$, then

$$\Psi \colon B_R(y) \to \mathcal{U}_2 \text{ is a } C^{2,\alpha} \text{ diffeom.}$$

s.t. there exist ball B with $B \subset \Psi(\Omega \cap B_R(x_0))$ and

$$T := \Psi\left(\partial\Omega \cap B_\rho(x_0)\right) \subset \partial B \quad \text{for some } \rho > 0.$$

Letting $y = \Psi(x)$ be coordinates in $\mathcal{U}_2$, then the equation $Lu = f$ becomes

$$\tilde{L}\tilde{u}(y) = \tilde{f}(y), \quad \tilde{u}(y) = u(\Psi^{-1}(y)), \quad \tilde{f}(y) = f(\Psi^{-1}(y)),$$
$$\tilde{\varphi}(y) = \varphi(\Psi^{-1}(y)) \quad \text{for } y \in T.$$

Now solve the problem

$$\tilde{L}\tilde{v} = \tilde{f} \text{ in } B, \quad \tilde{v} = \tilde{u} \text{ on } \partial B.$$

and get $\tilde{v} \in C^{2,\alpha}(B) \cap C^0(\overline{B})$. By adapting the same mollification + compactness argument (used to prove Theorem 1.12), we also get that $\underline{v \in C^{2,\alpha}(B \cup T)}$. How?

Can extend $\tilde{u} \in C^0(\overline{B}) \cap C^{2,\alpha}(T')$, $T' \subset\subset T$ to $\tilde{u} \in C^0(\overline{B}) \cap C^{2,\alpha}(G)$, G is some open nbhd of T'. ([GTGT77, p.137]) Then proceed by mollifying extensions of $\tilde{f}, u^+, \tilde{\varphi}$, namely
$f_n, u_n \in C^\infty(\overline{B})$. Have $f_n \to f$ and $u_n \to u$ uniformly in $\overline{B}$, and

$$\begin{cases} |f_n|_{2,\alpha;\overline{B}} \le |f|_{2,\alpha;\overline{B}} \\ |u_n|_{\overline{B}} \le |u|_{0;\overline{B}} \\ |u|_{2,\alpha;G\cap B} \le |u|_{2,\alpha;G\cap B} \quad \forall n \gg 1 \end{cases} \qquad \text{(shrinking } G \text{ if necessary)}$$

Consider now $v_n \in C^\infty(\overline{B})$:

$$\begin{cases} \tilde{L}v_n = f_n & \text{in } \overline{B} \\ v_n = u_n & \text{on } \partial B \end{cases}$$

The usual hyp. (H) holds for $L \Rightarrow$ apply boundary Schauder estimates near the boundary to get that there exists $\epsilon > 0$ s.t.

$$|v_n|_{2,\alpha;\underbrace{B_\epsilon(x_0) \cap B}_{G := G \cap B}} \le C\left(|u_n|_{0;B} + |f_n|_{0;\overline{B}} + |u_n|_{2,\alpha;G}\right).$$

The WMP, a priori bound gives the estimate

$$\le C\left(|u|_{0;B} + |f|_{0;\overline{B}} + |u|_{2,\alpha;G}\right)$$

and the mollification bound

$$\le C\left(|u|_{0;B} + |f|_{0;\overline{B}} + |u|_{2,\alpha;G}\right)$$

and we thus obtain a bound independent of n. Thus, $v_n \in C^{2,\alpha}(\overline{B})$ and are uniformly bounded, thus by Arzelà–Ascoli, we have that there exist subsequence (not relabelled) s.t.

$$v_n \to v^* \in C^{2,\alpha}(\overline{G}), \quad \text{convergence happens} \quad \text{in } C^2(\overline{G}).$$

Thus v^* satisfies $Lv^* = f$ in G. Now, **NTS:** $v^* = \tilde{v}|_G$, $\tilde{v} \in C^{2,\alpha}(B) \cap C^0(\overline{B})$. By interior Schauder estimates, as in Thm. 5.1, we have that up to a subsequence, $v_n \to v \in C^{2,\alpha}_{\text{loc}}(B)$ in $C^2(B)$.
Additionally, $v \in C^0(\overline{B})$ and $v_n \to v$ uniformly in B (apply WMP + a priori estimate). Thus
$v \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ and $\tilde{L}\tilde{v} = \tilde{f}$ in B, $v = \tilde{u}$ on ∂B. By uniqueness, $\tilde{v} = v$ on B, and in particular $\tilde{v}|_G = v|_G = v^* \in C^{2,\alpha}(\overline{G \cap B})$. Hence, pulling back to Ω, there exist nbhd U of
$x_0 \in \partial\Omega$ (chosen arbitrarily) s.t. $u \in C^{2,\alpha}(\overline{U \cup \Omega})$.

On the other hand,

$$\tilde{L}(\tilde{v} - \tilde{u}) = 0 \text{ in } B, \quad \tilde{v} - \tilde{u} = 0 \text{ on } \partial B, \quad \tilde{v} - \tilde{u} \in C^0(\overline{B}).$$

By the weak max. principle, $\tilde{v} \equiv \tilde{u}$ on $\overline{B}$. So in particular $\tilde{u} \in C^{2,\alpha}(B \cup T)$, so $u \in C^{2,\alpha}(\overline{\Omega})$. $\square$

Fredholm Alternative:

Let V a normed space and $T\colon V \to V$ a compact linear map. Then either

(i) the equation $x + Tx = 0$ has a non-zero solution $x \in V$, **or**

(ii) for any given $y \in V$, there is a unique $x \in V$ s.t. $x + Tx = y$.

Proof. Omitted, see [GTGT77, Chapert 5]. □

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Theorem 6.5 (Fredholm Alternative). *Let $\alpha \in (0,1)$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain. Let*

$$a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega}) \quad \text{and} \quad Lu \equiv a^{ij} D^2_{ij} u + b^i D_i u + cu$$

be strictly elliptic. Then either:

(i) The homogeneous problem

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

*has a non-trivial solution $u \in C^{2,\alpha}(\overline{\Omega})$, **or***

(ii) For any given $f \in C^{0,\alpha}(\Omega)$ and $\varphi \in C^{2,\alpha}(\Omega)$, the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.

Remark. *(1) This says that the sharp condition under which (ii) holds is <u>not</u> that $c \le 0$, but that the homog. problem has only the zero solution.*

(2) Failure of (i) is equivalent to the statement that uniqueness holds for solutions to the DP as in (ii), so the theorem can be seen as saying that if uniqueness holds (i.e. DP as in (ii) can have at most one solution), then there exists a solution.

Proof. It suffices to prove the theorem in the special case $\varphi \equiv 0$. (Unique solvability of $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$ for $u \in C^{2,\alpha}(\overline{\Omega})$ is equivalent to unique $v \in C^{2,\alpha}(\overline{\Omega})$ solving $Lu = f - L\varphi$ in Ω, $v = 0$ on $\partial\Omega$.)

So assume $\varphi \equiv 0$. Choose constant $\sigma > \sup_{\overline{\Omega}} c$ and let

$$L_\sigma u \coloneqq Lu - \sigma u \equiv a^{ij} D^2_{ij} u + b^i D_i u + \overbrace{(c - \sigma)}^{\le 0} u$$

so that $c - \sigma \le 0$.

By Theorem 6.1, we know that

$$L_\sigma : C^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$$

is a bijection. By Global Schauder estimates + max. principle estimate,

$$|u|_{2,\alpha;\overline{\Omega}} \le C |L_\sigma u|_{\alpha;\overline{\Omega}}, \quad \forall u \in C^{2,\alpha}_0(\Omega).$$

Equivalently,

$$|L_\sigma^{-1} f|_{2,\alpha;\overline{\Omega}} \le C|L_\sigma u|_{\alpha;\overline{\Omega}}, \quad \forall f \in C^{2,\alpha}(\overline{\Omega}).$$

$L_\sigma^{-1} : C^{0,\alpha}(\overline{\Omega}) \to C_0^{2,\alpha}(\overline{\Omega})$ is a bounded linear operator.

The inclusion $I : C_0^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$ is compact by Arzelà–Ascoli. That is, compactness follows by exploiting boundary regularity (need at least C^2) and proceeding similarly to the proof of the Global Schauder estimates (Theorem 5.1). So $T_\sigma \equiv I \circ L_\sigma^{-1} : C^{0,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$ is compact, i.e. if (u_i) is bdd in $C^{0,\alpha}(\overline{\Omega})$, then $(T_\sigma(u_i))$ has a convergent subseq. in $C^{0,\alpha}(\overline{\Omega})$. Hence so is σT_σ.

By the abstract Fredholm Alternative 6.5, we have either

(i) $u + \sigma T_\sigma u = 0$ has a non-zero solution $u \in C^{0,\alpha}(\overline{\Omega})$, as does $Lu = 0$ **or**,

(ii) for any $f \in C^{0,\alpha}(\overline{\Omega})$, there is a unique function $u \in C^{2,\alpha}(\overline{\Omega})$ s.t. $u + \sigma T_\sigma u = L_\sigma^{-1} f$.

Note that in either case, since $T_\sigma u, L_\sigma^{-1} f \in C_0^{2,\alpha}(\overline{\Omega})$ and $u = -T_\sigma u$ in case (i), and $u = L_\sigma^{-1} f - T_\sigma u$ in case (ii), we have that $u \in C_0^{2,\alpha}(\overline{\Omega})$ automatically. Now just apply L_σ to both sides of the equation in both cases (i), (ii). $\square$

7 Quasilinear second order elliptic theory and the De Giorgi–Nash–Moser theory

Fix $\alpha \in (0,1)$, Ω a bdd $C^{2,\alpha}$ domain,

$$a^{ij}, b^i \in C^{0,\alpha}(\overset{x}{\overline{\Omega}} \times \overset{z}{\mathbb{R}} \times \overset{p}{\mathbb{R}^n}; \mathbb{R}).$$

Suppose

$$a^{ij}(x,z,p)_{ij} \text{ is positive definite for all } (x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \quad \text{(second-order)}\,.$$

Consider the *quasilinear* operator

$$Qu \equiv a^{ij}(x,u,Du) D^2_{ij} u + b(x,u,Du).$$

We are interested in the Dirichlet problem for Q, i.e. the question of solvability for $u \in C^{2,\alpha}(\overline{\Omega})$ of

$$\text{(DP)} \quad \begin{cases} Qu = 0, & \text{in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \end{cases} \quad \text{for given } \varphi \in C^{2,\alpha}(\Omega).$$

To do this, we will rely on the following fixed point theorem.

Theorem 7.1 (Leray–Schauder fixed pt. thm.)**.** *Let X be a Banach space, and $T : X \to X$ a continuous, compact operator (not assumed linear). Suppose there is a constant $M > 0$ such that for any $\sigma \in [0,1]$ and any $x \in X$ satisfying $x = \sigma T(x)$, we have that $|x| \le M$ (indep. of σ). Then there exists $x_0 \in X$ s.t. $x_0 = Tx_0$, i.e. T has a fixed point.*

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To apply this result to the (DP) above: we will take $X = C^{1,\beta}(\overline{\Omega})$ for some fixed $\beta \in (0,1)$. Now, define $T : C^{1,\beta}(\overline{\Omega}) \to C^{1,\beta}(\overline{\Omega})$ by setting

$$T(v) := u \quad \text{for any given } v \in C^{1,\beta}(\overline{\Omega}),$$

where u solves the *linear Dirichlet problem*:

$$\begin{cases} a^{ij}(x,v,Dv)D^2_{ij}u + b(x,v,Dv) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Note for given $v \in C^{1,\beta}(\overline{\Omega})$, $x \mapsto a^{ij}(x,v(x),Dv(x))$, $x \mapsto b(x,v(x),Dv(x))$ are in $C^{0,\alpha}(\overline{\Omega})$.

Also, $a^{ij}(x,v(x),Dv(x)) \geq \lambda_\nu \cdot |\xi|^2$ for some constant $\lambda_0 > 0$.

By Thm 6.4, there is a unique $u \in C^{2,\alpha\beta}(\overline{\Omega}) \subset C^{1,\beta}(\overline{\Omega}) \coloneqq X$ solving $(\mathrm{DP})_{\mathrm{v}}$, so T is well defined. By the Schauder estimates, one can check that T is continuous and compact. Note also that (DP) is equivalent to T having a fixed point, i.e. the existence of $v \in C^{1,\beta}(\overline{\Omega})$ satisfying $v = T(v)$. Such v automatically will be in $C^{2,\alpha\beta}(\overline{\Omega})$.

More generally: $v \in C^{1,\beta}(\overline{\Omega})$ satisfies $v = \sigma T(v)$ for some $\sigma \in [0,1]$

$$\iff v \in C^{2,\alpha\beta}(\overline{\Omega}) \text{ and solves } \begin{cases} a^{ij}(x,v,Dv)D^2_{ij}v + \sigma b(x,v,Dv) = 0 & \text{in } \Omega \\ v = \sigma\varphi & \text{on } \partial\Omega \end{cases}$$

or equivalently,

$$v \in C^{2,\alpha}(\overline{\Omega}) \text{ and solves } \begin{cases} a^{ij}(x,v,Dv)D^2_{ij}v + \sigma b(x,v,Dv) = 0, & \text{in } \Omega \\ v = \sigma\varphi & \text{on } \partial\Omega \end{cases}$$

So by the abstract (Leray–Schauder f.p. Thm 7.1), if there exists

$$\mathcal{M} = \mathcal{M}(n,\varphi,a^{ij},b,\Omega) > 0 \quad \text{and some fixed } \beta = \beta(n,\varphi,a^{ij},b,\Omega) \in (0,1)$$

s.t.

$$|u|_{1,\beta;\overline{\Omega}} \leq \mathcal{M} \quad \text{whenever } u \in C^{2,\alpha}(\overline{\Omega})$$

solves

$$a^{ij}(x,u,Du)D^2_{ij}u + \sigma b(x,u,Du) = 0 \text{ in } \Omega, \quad u = \sigma\varphi \text{ on } \partial\Omega \text{ for some } \sigma \in [0,1],$$

then (DP) has a solution in $C^{2,\alpha}(\overline{\Omega})$.

Remark. *Proving such a bound $|u|_{1,\beta;\Omega} \leq \mathcal{M}$ generally requires additional hypotheses, e.g. in the case of minimal surface equation (see below). This requires the geometric condition that the domain Ω is* "mean convex" *(and $C^{2,\alpha}$).*

Now let's consider operators Q arising as Euler–Lagrange operators associated with functionals of the form

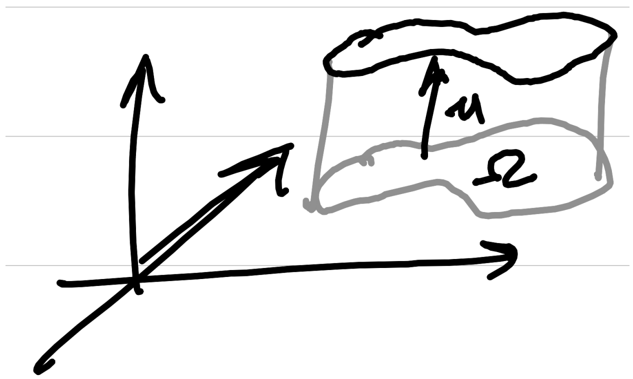
$$\mathcal{F}(u) = \int_\Omega F(x,u,Du)\ \mathrm{d}x,$$

$$F : \overset{x}{\Omega} \times \overset{z}{\mathbb{R}} \times \overset{p}{\mathbb{R}^n} \to \mathbb{R}.$$

with Euler-Lagrange equations, ,

$$0 = \left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(u + t\eta),\ \eta \in C^1_c(\Omega)$$

has divergence structure, which in non-divergence form is a Q as above.



Assume as a further simplification that $F(x,u,p) = F(p)$, i.e. F depends only on the gradient variable $p \in \mathbb{R}^n$. A very important specific example (the prototypical quasilinear 2^{nd} order elliptic operator), is the case when $F(p) = \sqrt{1+|p|^2}$. So we have the area functional

$$\mathcal{F}(u) = \int_\Omega \sqrt{1+|Du|^2}$$

whose E–L eqn is the minimal surface equation:

$$D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right) = 0, \qquad \text{div. form}$$

$$\left[\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}\right] D^2_{ij} u = 0 \qquad \text{non-div. form}$$

In the generality of $F = F(p)$, the E–L equation is

$$D_i(F_{p_i}(Du)) = 0, \qquad \text{div. form}$$

where it is understood that

$$F_{p_i}(p) = D_{p_i} F(p),$$

or, in non-divergence form

$$F_{p_i} F_{p_j}(Du) D^2_{ij} u = 0.$$

If the integrand is convex in p, then $a^{ij}(p) = F_{p_i p_j}(p)$ is elliptic.

In the case of the Minimal Surface Equation, one has

$$a^{ij}(p) = \underbrace{\left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2}\right)}_{\tilde{a}^{ij}(p)} \frac{1}{\sqrt{1+|p|^2}}\,.$$

$[\tilde{a}^{ij}(p)]_{ij}$ has eigenvalues

$$\underbrace{1,\ldots,1}_{(n-1)}, \quad \frac{1}{1+|p|^2}$$

and so is elliptic, but strictly elliptic only if $|Du|$ bounded in $\overline{\Omega}$.

Recall we want a $C^{1,\beta}(\overline{\Omega})$ bound in $C^{2,\alpha}(\overline{\Omega})$ solutions to

$$\begin{cases} F_{p_i}(Du)\, D_i u = 0 & \text{in } \Omega, \\ u = \sigma\varphi & \text{on } \partial\Omega\,. \end{cases}$$

By the WMP, $|u|_{0;\overline{\Omega}} \le |\sigma\varphi|_{0;\Omega} \le |\varphi|_{0;\Omega}$ (first easiest step). Then we need

(i) $|u|_{1;\overline{\Omega}} \le M_1 = M_1(\varphi, F, \Omega)$,

(ii) $[Du]_{\beta;\overline{\Omega}} \le M_2 = M_2(\varphi, F, \Omega)$.

For both of these, we derive and use the PDE satisfied by partial derivatives $w = D_k u$, $k \in \{1, \dots, n\}$. Typically, (i) will come from applying the max principle and (ii) requires De Giorgi–Nash–Moser.

Lecture 21 In particular, considering functionals of the type

$$\mathcal{F}(u) = \int_\Omega F(Du)\, \mathrm{d}x\,, \quad \text{e.g. } F(p) = \sqrt{1+|p|^2}$$

for the Minimal surface equation.

$$\textit{E-L} \text{ equation } \int_\Omega F_{p_i}(Du) D_i \eta = 0\,, (1) \quad \forall \eta \in C_c^1(\Omega) \quad (\Leftrightarrow D_i(F_{p_i}(Du)) = 0 \text{ weakly in } \Omega).$$

We now seek to bound

(i) $|Du|_{0;\beta;\overline{\Omega}} \le M_1 = M_1(\varphi, F, \Omega)$,

(ii) $|Du|_{0;\overline{\Omega}} \le M_2$ ——VERY PROBLEM DEPENDENT.

(iii) $[Du]_{\beta;\overline{\Omega}} \le M_3$.

To do both (ii), (iii), we need the equation for partial derivatives $w = D_k u$, $k \in \{1, \dots, n\}$. Now, replace η with $D_k \eta$ in (1) to deduce

$$\int_\Omega F_{p_i}(Du) D_i D_k \eta = 0 \quad \forall \eta \in C_c^2(\Omega)$$

IBP w.r.t. x_k gieves

$$\int_\Omega D_k(F_{p_i}(Du)) D_i \eta = 0$$

which implies

$$\int_\Omega F_{p_i p_j}(Du) D_j w D_i \eta = 0\,.$$

So w is a *weak solution* to

$$D_i\left(F_{p_i p_j}(Du) \cdot D_j w\right) = 0 \quad \text{in } \Omega. \quad (2)$$

So if $|Du| \le M_2$ on $\overline{\Omega}$, then this is a *uniformly elliptic equation in* Ω.

Step (iii) follows (once we have step (ii)), by applying **De Giorgi–Nash–Moser (DGNM) theory in the interior** to Equation (2). This theory says that if

$$w \in W^{1,2}$$

is a weak solution to a divergence structure equation

$$D_i(a^{ij}(x) D_j w) = 0 \quad \text{in } \Omega,$$

with a^{ij} bounded measurable and strictly elliptic, then there exists $\beta \in (0,1)$ depending only on a bound on $\sum_{i,j} |a^{ij}|_{L^\infty(\Omega)}$ and the ellipticity constant, and $\dim n$, s.t. the solution

$$w \in C^{0,\beta}_{\mathrm{loc}}(\Omega) \quad \text{(with an estimate).}$$

Remarks:

(1) DGNM theory extends to more general div. structure equations which have lower order terms, as well as inhomogeneous terms on the r.h.s. under approp. assumptions. We will only present the theory for pure divergence form, homogeneous eqns as above.

(2) There are also global estimates, giving bounds on $|w|_{0;\beta;\overline{\Omega}}$ subject to appropriate boundary assumptions, and that's what is really needed for the quasilinear applications. We will omit the bdry theory.

De Giorgi–Nash–Moser Theory

We consider operators of the form

$$Lu = D_i(a^{ij} D_j u) \quad \text{in some open set } \Omega \subseteq \mathbb{R}^n.$$

Hypothesis (H):

(i) $a^{ij} \in L^\infty(\Omega)$ with $|a^{ij}|_{L^\infty(\Omega)} \leq \Lambda$, $\Lambda =$ a fixed constant,

(ii) $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ a.e. $x \in \Omega$, for some fixed constant $\lambda > 0$.

Definition 7.1. *A function $u \in W^{1,2}(\Omega)$ is a weak sub(super)solution to $Lu = 0$ in Ω if*

$$\int_\Omega a^{ij} D_j u D_i \varphi \leq 0 \ (\geq 0) \quad \forall \varphi \in W_0^{1,2}(\Omega) \text{ and } \varphi \geq 0.$$

Remark. *A $u \in W^{1,2}(\Omega)$ is a weak solution to $Lu = 0$ in Ω if and only if u is both a weak subsolution and a weak supersolution.*

Theorem 7.2 (Local boundedness of subsolutions)**.** *Suppose hyp(H) holds. If $u \in W^{1,2}(\Omega)$ is a weak subsolution to $Lu = 0$ in Ω, then for any ball $B_{2R}(y) \subset \Omega$ and any $p > 1$,*

$$\sup_{B_R(y)} u \leq C \cdot R^{-n/p} \left\| u^+ \right\|_{L^p(B_{2R}(y))}, \quad \text{where } C = C(n, \lambda, \Lambda, p).$$

Theorem 7.3 (Weak Harnack inequality for non-neg. supersolutions)**.** *Suppose hyp(H) holds. If $u \in W^{1,2}(\Omega)$ is a weak supersolution to $Lu = 0$ in Ω, non-negative in $B_{4R}(y) \subset \Omega$, and if $p \in \left(1, \frac{n}{n-2}\right)$, then*

$$\inf_{B_R(y)} u \geq C \cdot R^{-n/p} \left\| u \right\|_{L^p(B_{2R}(y))}, \quad C = C(n, \lambda, \Lambda, p).$$

Corollary 7.1 (Harnack inequality for non-negative solutions)**.** *Suppose hyp(H) holds. If $u \in W^{1,2}(\Omega)$ is a non-neg. weak solution to $Lu = 0$ in Ω, then for any subdomain $\Omega_1 \subset\subset \Omega$, we have*

$$\sup_{\Omega_1} u \leq C \cdot \inf_{\Omega_1} u,$$

where

$$C = C(n, \lambda, \Lambda, \Omega_1, \Omega).$$

(Proof of Corollary 7.1). Just pick some $p \in (1, \frac{n}{n-2})$, and apply Thms. 7.2 and 7.3 to get the Harnack inequality for balls. Then use the same argument as in the case of non-negative harmonic functions to extend it to domains $\Omega_1 \subset\subset \Omega$. □

Theorem 7.4 (Hölder continuity)**.** *Suppose hyp(H) holds, and suppose that $u \in W^{1,2}(\Omega)$ is a weak solution to $Lu = 0$ in Ω. Then u is (a.e.) locally Hölder continuous in Ω. Moreover, we have the estimates for any ball $B_\rho(y) \subset \Omega$,*

(i) For any $r \in (0, R]$, we have

$$\operatorname{osc}_{B_R(y)} u \le C \cdot \left(\frac{r}{R}\right)^\mu \operatorname{osc}_{B_R(y)} u, \quad (\operatorname{osc} = \sup - \inf).$$

(ii) $u \in C^{0,\mu}(\Omega)$, and

$$R^\mu [u]_{\mu, B_{R/4}(y)} \le C \cdot \sup_{B_R(y)} |u|.$$

$$C = C(n, \lambda, \Lambda), \quad \mu = \mu(n, \lambda, \Lambda) \in (0, 1).$$

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Theorem 7.5. *$Lu = 0$, $u \in W^{1,2}(\Omega)$, $B_R(y) \subseteq \Omega$,*

(i) for all $r \in (0, R]$, $\operatorname{osc}_{B_r(y)} u \le C \left(\frac{r}{R}\right)^\mu \operatorname{osc}_{B_R(y)} u$

(ii) $R^\mu [u]_{\mu; B_{R/4}(y)} \le C \sup_{B_R(y)} |u|$, $\quad C = C(n, \lambda, \Lambda)$

Proof. First check (ii) $\Rightarrow$ (i): $x, z \in B_{R/4}(y)$, $x \ne z$. Let $d = \frac{5}{4}|x - z|$. Observe

$$d \le \frac{5}{4} \cdot \frac{R}{2} = \frac{5}{8} R, \quad B_{\frac{5R}{8}}(x) \subset B_R(y)$$

$$|u(x) - u(z)| \le \operatorname{osc}_{B_d(x)} u \le C \left(\frac{d}{5R/8}\right)^\mu \overbrace{\operatorname{osc}_{B_{5R/8}(z)} u}^{\le \operatorname{osc}_{B_R(y)} u}$$

(by (i)), we have $R^\mu \dfrac{|u(x) - u(z)|}{|x - z|^\mu} \le C \sup\limits_{B_R(y)} |u|$ which gives

$$R^\mu [u]_{\mu; B_{R/4}(y)} \le C \sup_{B_R(y)} |u|\,.$$

To see (ii), we'll use Theorem 7.3. (Note also that $|u|$ is locally subharmonic by Theorem 7.2 applied to u and $-u$.) Suppose now that $r \le R/4$. Set $M_1 = \sup_{B_r(y)} u$, $m_1 = \inf_{B_r(y)} u$, $M_4 = \sup_{B_{4r}(y)} u$, $\quad m_4 = \inf_{B_{4r}(y)} u$ (want to establish $(M_1 - m_1) \le \gamma(M_4 - m_4)$, $\gamma < 1$, and iterate).

Now, $M_4 - u$ and $u - m_4$ are both non-negative in $B_{4r}(y)$, and satisfy $L(M_4 - u) = 0$, $\quad L(u - m_4) = 0$. So by Theorem 7.3 with $\varphi = 1$,

$$r^{-n} \int_{B_{2r}} (M_4 - u) \le C \cdot \inf_{B_r(y)} (M_4 - u) = C \cdot (M_4 - M_1) \tag{1}$$

and

$$r^{-n} \int_{B_{2r}} (u - m_4) \le C \cdot \inf_{B_r(y)} (u - m_4) = C \cdot (m_1 - m_4)\,. \tag{2}$$

Now, (1) + (2) give $(M_4 - m_4) \leq C \cdot ((M_4 - M_1) + (m_1 - m_4))$, and $(M_1 - m_1) \leq \left(\frac{C-1}{C}\right) \cdot (M_4 - m_4)$ for $\gamma = \frac{C-1}{C} < 1$ which implies

$$\operatorname{osc}_{B_r(y)} u \leq \gamma \operatorname{osc}_{B_{4r}(y)} u \, , \gamma \in (0,1).$$

Iterating this, starting with $r = R/4$ we deduce

$$\operatorname{osc}_{B_{R/4^{j+1}}(y)} u \leq \gamma^j \operatorname{osc}_{B_{R/4}(y)} u, \quad j = 0, 1, 2, \ldots$$

Given any $r \in (0, R/4]$, there exist (unique) j s.t. $4^{j+1}R \leq r \leq 4^j R$, hence

$$\operatorname{osc}_{B_r(y)} u \leq \operatorname{osc}_{B_{R/4^j}(y)} u \leq \gamma^{j-1} \operatorname{osc}_{B_{R/4}(y)} u$$

$$= \gamma^{-1} \cdot 4^{-(\log_4 \gamma) j} \operatorname{osc}_{B_{R/4}(y)} u, \quad \text{let } \mu = \frac{\log \gamma}{\log(1/4)} \in (0,1)$$

$$= \gamma^{-1} \cdot 4^{-j\mu} \operatorname{osc}_{B_{R/4}(y)} u$$

which finally gives the estimate

$$\operatorname{osc}_{B_r(y)} u \leq C \cdot \left(\frac{r}{R}\right)^{\mu} \operatorname{osc}_{B_{R/4}(y)} u \, .$$

□

Proof of Thm 7.2. We are assuming $u \in W^{1,2}(\Omega)$ is a weak subsolution, i.e.

$$\int_\Omega a^{ij} D_i u D_j v \leq 0$$

for all $v \in W_0^{1,2}(\Omega)$, $v \geq 0$. Then $u^+ := \max\{u, 0\}$ is also a subsolution, . So w.l.o.g. we can assume that u is non-negative.

It suffices to prove the theorem assuming in fact that $u \geq \varepsilon$ for arbitrary $\varepsilon > 0$. The general case $u \geq 0$ then follows by applying the conclusion to $u + \varepsilon$ and letting $\varepsilon \to 0$.

By considering $\tilde{u}(x) := u(y + Rx)$, we may also assume $y = 0$, $R = 1$. Let $\beta \geq 0$ and set

$$\nu_k := \min\left\{\varepsilon, u^k, ku\right\} \quad \text{for suff. large } k.$$

<u>Claim:</u> $\nu_k \in W^{1,2}(\overline{B})$, with

$$D\nu_k(x) = \begin{cases} \beta u^{\beta-1} Du(x) & \text{if } x \in \Omega_k := \left\{x \in B : u^\beta(x) < ku(x)\right\}, \\ kDu & \text{if } x \in B \setminus \Omega_k. \end{cases}$$

To see this, note that if $\beta \leq 1$, $\Omega_k = B$ and $v_k = u^\beta$ for sufficiently large k (since $u \geq \varepsilon$).

When $\beta > 1$, then

$$v_k = u \min\left\{u^{\beta-1}, k\right\} = u \cdot g(\omega_k), \quad \text{where } \omega_k = \min\left\{u, k^{\frac{1}{\beta-1}}\right\}, \quad g(t) = t^{\beta-1}.$$

Since $\varepsilon \leq \omega_k \leq k^{\frac{1}{\beta-1}}$ and g and g' are bounded on $[\varepsilon, k^{\frac{1}{\beta-1}}]$, we have that

$$g(\omega_k) \in W^{1,2}(B) \text{ with } D_i(g(\omega_k)) = g'(\omega_k) D_i \omega_k.$$

Thus, by standard facts about Sobolev functions, the claim follows. ($\omega_k \in W^{1,2}$ being min of Sobolev functions.)

Now, fix $\eta \in C_c^1(B)$ and take $v = v_k\eta^2$ in the inequality $\int_B a^{ij} D_i u D_j v \le 0$. This gives

$$\beta \int_{\Omega_k} a^{ij} D_i u \cdot u^{\beta-1} D_j \eta^2 + k \int_{B\setminus\Omega_k} a^{ij} D_i u D_j \omega_k \eta^2$$
$$\le -2\int_{\Omega_k} a^{ij} D_i u \cdot u^\beta \eta D_j\eta - 2k\int_{B\setminus\Omega_k} a^{ij} D_i u \omega_k \eta D_j \eta.$$

By ellipticity and bounds $|a^{ij}|_{L^\infty(B)} \le \Lambda$, this gives:

$$\beta\int_{\Omega_k} |Du|^2 u^{\beta-1}\eta^2 + \lambda k \int_{B\setminus\Omega_k} |Du|^2\eta^2 \le \frac{2\Lambda}{\lambda}\int_{\Omega_k} |Du| u^\beta |\eta||D\eta| + \frac{2\Lambda k}{\lambda}\int_{B\setminus\Omega_k} |Du|\omega_k|\eta||D\eta|.$$

Young's inequality with ε:

$$\int_{\Omega_k} |Du| u^{\frac{\beta}{2}} u^{\frac{\beta}{2}-1}\eta|D\eta| \le \frac{\beta}{2}\int_{\Omega_k}|Du|^2 u^{\beta-1}\eta^2 + \frac{C}{\beta}\int_{\Omega_k} u^{\beta+1}|D\eta|^2.$$

Hence,

$$\beta/2\int_{\Omega_k}|Du|^2u^{\beta-1}\eta^2 + k\varepsilon\int_{B\setminus\Omega_k}|Du|^2\eta^2 \le C/\beta\int_{\Omega_k} u^{\beta+1}|D\eta|^2 + ck\int_{B\setminus\Omega_k} u^2|D\eta|^2, \quad \text{(since } ku \le u^\beta\text{)}.$$

□

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Proof of Thm 7.2 (Cont'd). We have just established that for all $\beta > 0$ and $k \ge 1$ sufficiently large that

$$\beta/2\int_{\Omega_R}|Du|^2u^{\beta-1}\eta^2 + \frac{\lambda}{2}\int_{B\setminus\Omega_k}|Du|^2\eta^2 \le C/\beta\int_{\Omega_R} u^{\beta+1}|D\eta|^2 + C/2\int_{B\setminus\Omega_k} u^2|D\eta|^2,$$

where $C = C\left(\frac{\Lambda}{\lambda}\right)$, $\Omega_k = \{z \in B \colon u(z) \le ku(x)\}$.

$$\Rightarrow \beta/2\int_{\Omega_R}|Du|^2u^{\beta-1}\eta^2 \le C/\beta\int_{\Omega_R}u^{\beta+1}|D\eta|^2 + C\int_{B\setminus\Omega_k}u^2|D\eta|^2.$$

Note that $\mathbf{1}_{\Omega_R} \to \mathbf{1}_B$ pointwise in B ($u > \varepsilon$).

Assuming $\int_B u^{\beta+1}|D\eta|^2 < \infty$, we can let $k \to \infty$ to deduce that

$$\beta/2\int_B |Du|^2u^{\beta-1}\eta^2 \le C/\beta\int_B u^{\beta+1}|D\eta|^2.$$

Let $\alpha = \beta+1$, then

$$\int_B |Du|^2 u^{\alpha-2} \overset{\circledast}{\le} \frac{C}{(\alpha-1)^2}\int_B u^\alpha|D\eta|^2$$

holds for any $\alpha \ge 1$, where $C = C\left(\frac{1}{\lambda}\right)$ provided $\int_B u^\alpha|D\eta|^2 < \infty$. Now compute

$$D\left(u^{\frac{\alpha}{2}}\eta\right) = \frac{\alpha}{2}u^{\frac{\alpha}{2}-1}Du\cdot\eta + u^{\frac{\alpha}{2}}D\eta$$

and estimate

$$\int_B \left|D\left(u^{\frac{\alpha}{2}}\eta\right)\right|^2 \le \frac{C\cdot\alpha^2}{(\alpha-1)^2}\int_B u^{\alpha-1}|Du|^{\frac{1}{2}}\eta^2 + 2\int_B u^\alpha|D\eta|^2.$$

$$\circledast \le \frac{C\cdot\alpha^2}{(\alpha-1)^2}\int_B u^\alpha|D\eta|^2.$$

Recall the Sobolev inequality (for $f\in W_0^{1,2}(B)$)

$$\|f\|_{L^{2\sigma}(B)} \le C\,\|Df\|_{L^2(B)}\,,$$

$$\sigma = \begin{cases} \frac{n}{n-2} & \text{if } n\ge 3,\\ \text{any fixed } \# & \text{if } n=2.\end{cases}$$

for some $C=C(n)$.

Using this with $f=u^{\alpha/2}\eta$, we get from the previous line that,

$$\left(\int_B (u^{\alpha\sigma}\eta^{2\sigma})\right)^{\frac{1}{\alpha\sigma}} \le \left(C\cdot\frac{\alpha^2}{(\alpha-1)^2}\int_B u^\alpha|D\eta|^2\right)^{1/\alpha},$$

subject to

$$\int_B u^\alpha|D\eta|^2<\infty\,.$$

Given $0<r'<r<1$, choose η so that $\eta\in C_c^1(B)$, $\eta\equiv 1$ on $B_{r'}$, and $\eta\equiv 0$ in $B\setminus B_r$, and $|D\eta|\le\frac{2}{r-r'}$ (), so

$$\left(\int_{B_{r'}} u^{\alpha\sigma}\right)^{1/\alpha\sigma} \le \left(\frac{C\alpha^2}{(\alpha-1)^2}\right)^{1/\alpha}\frac{1}{(r-r')^{2/\alpha}}\left(\int_{B_r}u^\alpha\right)^{1/\alpha}.$$

Let $r_j=\frac{1}{2}+\frac{1}{2^{j+1}}$ and take $r=r_{j-1}$, $r'=r_j$, as well as $\alpha=\rho\sigma^{j-1}$ for any $\rho>1$, for $j\ge 0$.

Since $g_\alpha=\frac{\alpha}{\alpha-1}=1+\frac{1}{\alpha-1}$ is decreasing in α, we have

$$g(\rho\sigma^j)\le g(\rho)=\frac{\rho}{\rho-1}$$

$$\left(\int_{B_{r_j}} u^{\rho\sigma^j}\right)^{1/\rho\sigma^j} \le C^{\frac{1}{\rho\sigma^{j-1}}}2^{\frac{2(j+1)}{\rho\sigma^{j-1}}}\left(\int_{B_{r_{j-1}}}u^{\rho\sigma^{j-1}}\right)^{1/\rho\sigma^{j-1}},\quad C=C(\rho,\tfrac{1}{\lambda})$$

$$j=1,2,\ldots$$

Iterating this gives:

$$\left(\int_{B_{r_j}} u^{\rho\sigma^j}\right)^{1/\rho\sigma^j} \le C^{\sum_{j=1}^\infty\frac{j}{\rho\sigma^{j-1}}}\cdot 2^{\frac{2}{p}\sum_{j=1}^\infty\frac{j}{\rho\sigma^{j-1}}}\left(\int_{B_1}u^p\right)^{1/p} = C\cdot|u|_{L^p(B)}$$

Taking $j\to\infty$ gives

$$\sup_{B_{1/2}} u\le C\cdot|u|_{L^p(B)},$$

for some

$$C=C(n,\tfrac{\Lambda}{\lambda},\rho),\quad p>1$$

$\square$

The iteration technique used above to prove the theorem is called the Moser iteration.

Remark. *Using the case $p > 1$ (in fact the case $p = 2$) of the theorem (proved), it is possible to extend to all $p > 0$ (with $C > 0$ depending on p).*

It remains to prove Thm 7.3 (Weak Harnack inequality). For this we need the following lemma first.

Lemma 7.1 (John–Nirenberg)**.** *Let $B = B_1(0) \subset \mathbb{R}^n$, and let $u \in W^{1,1}_{loc}(B)$. Suppose that there is a $M > 0$ such that*

$$\rho^{1-n} \int_{B_\rho(y) \cap B} |Du| \le M < \infty$$

for any ball $B_\rho(y)$.

Then, there exists $p_0 = p_0(n)$ and $c = c(n)$ s.t.

$$\int_B e^{p_0/M \frac{|u-u_B|}{\mu}} \le c,$$

where

$$u_B = \frac{1}{|B|} \int_B u.$$

Proof. Omitted, see [GTGT77, Chapter 7]. $\square$

Proof of Thm 7.3. W.l.o.g., assume $R = 1$, $y = 0$, also $u > \varepsilon$ (else can replace u with $u + \varepsilon$, then let $\varepsilon \downarrow 0$). Thus, have

$$\int_{B_4} a^{ij} D_i u D_j v \ge 0\,, \qquad \text{for all } v \in W^{1,2}_0(B_4)\,. \tag{$\dagger$}$$

One can check, using ellipticity that with $\omega = 1/u$,

$$\int_{B_4} a^{ij} D_i \omega D_j v \le -2 \int_{B_4} \left| \frac{Du}{u^2} \right|^2 v \le 0\,, \qquad \text{for all } v \in W^{1,2}_0(B_4).$$

(putting $\omega^2 \nu$ for test function in ($\dagger$)). So by Thm 7.2,

$$\sup_{B_1} \omega \le C \left(\int_{B_2} \omega^p \right)^{1/p}$$

giving the lower bound

$$\inf_{B_1} u \ge C \left(\int_{B_2} u^{-p} \right)^{-1/p}$$

$$= C \left(\int_{B_2} u^p \right)^{1/p} \left[\left(\int_{B_2} u^p \right) \left(\int_{B_2} u^{-p} \right) \right]^{-1/p}$$

$\square$

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Proof of Thm 7.3 (continued). So ω is a non-negative weak sub-solution. So by Thm 7.2,

$$\sup_{B_1} \omega \leq C \left(\int_{B_3} \omega^p \right)^{1/p} < \infty \quad \forall p \in (0,2]$$

and the remark at end of Thm 7.2 gives

$$\inf_{B_1} u \geq C \left(\int_{B_3} u^{-p} \right)^{-1/p} = C \left(\int_{B_3} u^p \right)^{1/p} \left[\left(\int_{B_3} u^p \right) \left(\int_{B_3} u^{-p} \right) \right]^{-1/p}$$

$$C = C(n, \Lambda/\lambda, p)$$

<u>Claim:</u> there exists

$$p_0 = p_0(n, \Lambda/\lambda) > 0 \quad \text{and} \quad C = C(n, \Lambda/\lambda) \text{ s.t.}$$

$$\left(\int_{B_3} u^{-p_0} \right) \left(\int_{B_3} u^{p_0} \right) \leq C$$

This will prove the theorem for $p = p_0$, and hence (by Hölder inequality) for any $p \in (0, p_0]$. Indeed, we rely on John–Nirenberg lemma. Let

$$\omega = \log u - \frac{1}{|B_3|} \int_{B_3} \log u \quad (\text{since } u \geq \varepsilon, \omega \in W^{1,2}(B_3)).$$

$$D_i \omega = D_i u / u \,, \quad \text{using } \dagger \text{ with } u^{-1} v \text{ in place of } v$$

$$\int a^{ij} \chi D_i \omega \, g^i D_j v - \int a^{ij} \chi D_i \omega \, \frac{1}{u^2} D_j u \cdot v \geq 0$$

which gives

$$\int a^{ij} D_i \omega D_j v \geq \int a^{ij} D_i \omega D_j \omega v \geq \lambda \int |\nabla \omega|^2 v$$

using $|a^{ij}| \leq \Lambda$, replacing v with u^2,

$$\int |\nabla \omega|^2 v^2 \leq \frac{2\Lambda}{\lambda} \int |\nabla \omega| \, |\nabla v| \, v \quad (\text{replace } v \text{ with } v^2) \quad \forall v \in C_c^1(B_4), v \geq 0.$$

$$\overset{ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2}{\leq} \quad \frac{1}{2} \int |\nabla \omega|^2 v^2 + \frac{\Lambda}{2\lambda} \int |\nabla v|^2$$

and

$$\int |D\omega|^2 v^2 \leq \frac{1}{\lambda} \int |Dv|^2 \quad v \in C_c^1(B_{4/5}), v \geq 0. \qquad (2)$$

If $B_{7\rho/6}(y) \subset B_4$, then we can choose in (2) $v \in C_c^1(B_{7\rho/6}(y))$, $v \equiv 1$ on $B_\rho(y)$ and $|Dv| \leq \frac{12}{\rho}$, $v \geq 0$. Hence,

$$\int_{B_\rho(y)} |D\omega|^2 \leq C \rho^{n-2} \Rightarrow \int_{B_\rho(y)} |D\omega| \leq \left(\int_{B_\rho(y)} |D\omega|^2 \right)^{1/2} |B_\rho(y)|^{1/2} \leq C \cdot \rho^{n-1}.$$

We need to check that for any ball $B_\rho(y)$,

$$\int_{B_\rho(y) \cap B_3} |D\omega| \leq C \cdot \rho^{n-1} \qquad (**)$$

If $\rho \geq 1/4$, then

$$\int_{B_\rho(y)\cap B_3} |D\omega| \leq \int_{B_3} |D\omega| \leq C \cdot \rho^{n-1}$$

For $\rho \in (0, 1/4)$, if $B_\rho(y) \cap B_3 \neq \emptyset$, then

$$B_{7\rho/6}(y) \subset B_4, \text{ so again,} \quad \int_{B_\rho(y)\cap B_3} |D\omega| \leq \int_{B_\rho(y)} |D\omega| \leq C \cdot \rho^{n-1}$$

So by John–Nirenberg

$$\int e^{\rho_0|\omega|} \leq C, \quad \text{for } \rho_0 = \rho_0\left(n, \frac{\Lambda}{\lambda}\right), \quad C = C(n)$$

Note $\int_{B_3} \omega = 0$. So

$$\left(\int_{B_3} u^{-\rho_0}\right)\left(\int_{B_3} u^{\rho_0}\right) \leq \left(\int_{B_3} e^{-\rho_0\omega}\right)\left(\int_{B_3} e^{\rho_0\omega}\right) \leq C^2,$$

so the claim holds, and hence the theorem for $\rho \in (0, \rho_0]$. To prove the theorem for $\rho \in (\rho_0, \frac{n}{n-2})$, it suffices to show that given such ρ,

$$\left(\int_{B_3} u^p\right)^{1/p} \leq C \cdot \left(\int_{B_3} u^{p'}\right)^{1/p'}, \quad \text{for some} \quad 0 < p' \leq \rho_0, \quad p' = p'\left(p, n, \frac{\Lambda}{\lambda}\right), \quad C = C\left(n, \frac{\Lambda}{\lambda}\right).$$

To see this, let $\beta > 0$, take $v = u^{-\beta}\eta^2$, so

$$\int a^{ij}\partial_i u\, \partial_j v \geq 0.$$

By the exact same steps as in Thm 7.2:

$$\beta/2 \int u^{-\beta-1}|Du|^2\eta^2 \leq \frac{C}{\beta} \int u^{1-\beta}|D\eta|^2 .$$

Letting $\gamma = 1 - \beta$, and assuming $\gamma \in (\alpha, 1)$ (i.e. $\beta \in (0, 1)$), this gives us

$$\int u^{\gamma-2}|Du|^2\eta^2 \leq \frac{C}{(1-\gamma)^2} \int u^\gamma |D\eta|^2 .$$

So,

$$\left(\int_{B_{r'}} u^{\sigma\gamma}\right)^{1/\sigma\gamma} \leq \left(\frac{C\gamma^2}{(1-\gamma)^2(r-r')^2}\right)^{1/\gamma} \left(\int_{B_r} u^\gamma\right)^{1/\gamma} \tag{$***$}$$

$0 < r' < r < 4$, $\sigma = \frac{n}{n-2}$. Given $p \in (p_0, \sigma)$, choose $N = N(n, \frac{\Lambda}{\lambda}, \rho) \in \mathbb{N}$ such that $\sigma^{-N}\rho \leq \rho_0$. In $(***)$ take $\gamma = \sigma^{-j-1}\rho$, $r = 2 + \frac{j}{N}$, $r' = 2 + \frac{j+1}{N}$, $j = 0, 1, 2, \ldots, N-1$ giving

$$\left(\int_{B_{2+\frac{j}{N}}} u^{\sigma^{-j}\rho}\right)^{1/\sigma^{-j}\rho} \leq \left(\frac{C\sigma^{-j}\rho}{1-\sigma^{-j-1}}\right)^{\frac{2}{\sigma^{-j-1}}} \cdot N^{\frac{2}{\sigma^{-j-1}p}} \cdot \left(\int_{B_{2+\frac{j+1}{N}}} u^{\sigma^{-j-1}\rho}\right)^{1/\sigma^{-j-1}p}$$

for $j = 0, 1, 2, \ldots, N-1$ giving finally

$$\left(\int_{B_2} u^p\right)^{1/p} \leq C \cdot \left(\int_{B_3} u^{\sigma^{-N}p}\right)^{\frac{1}{\sigma^{-N}p}} .$$

$\square$

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