Part III Distribution Theory Notes
Based on lectures by A. Ashton

Notes taken by Pantelis Tassopoulos

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Lecture 1

- Hörmander, *Analysis of Partial Differential Operators* Vol. 1
- Reed–Simon, *Methods of Mathematical Physics* Vol. 1
- Distribution Theory: Foundations

1 Motivation

You have probably seen Dirac delta function from previous calculus/differential equations classes:

$$\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = f(x_0), \quad f \text{ “nice”}.$$

Can we define $\delta'(x-x_0)$? One can try as follows

$$\int_0^{\infty} \delta'(x-x_0) f(x)\, dx = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^{\infty} \left[\delta(x-x_0+\varepsilon) - \delta(x-x_0)\right] f(x)\, dx$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left[f(x_0-\varepsilon) - f(x_0)\right] = -f'(x_0).$$

i.e.

$$\int \delta'(x-x_0) f(x)\, dx = -\int \delta(x-x_0) f'(x)\, dx.$$

Is this okay — is it rigorous? We will make this definition rigorous using distribution theory.

Fourier transform of poly x^n

If $f \in L^1(\mathbb{R})$,

$$\left[\int_{\mathbb{R}} |f|\, dx < \infty\right] \quad \text{then} \quad \hat{f}(\lambda) := \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\, dx,$$

How might we take F.T. of $f(x) = x^n$? Might recall identity

$$\delta(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x}\, dx$$

Might then get

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} x^n e^{-i\lambda x}\, dx$$

$$= \left(i \frac{\partial}{\partial \lambda}\right)^n \int_{-\infty}^{\infty} e^{-i\lambda x}\, dx$$

$$= i^n \cdot 2\pi \cdot \delta^{(n)}(\lambda)$$

Recall Parseval's theorem

$$\int_{-\infty}^{\infty} \hat{g}(\lambda) f(\lambda)\, d\lambda = \int_{-\infty}^{\infty} g(x) \hat{f}(x)\, dx$$

Define F.T. of $g(x) = x$ to be the f'n

$$\lambda \mapsto \hat{x}(\lambda) \quad \text{such that}$$

$$\int_{-\infty}^{\infty} \hat{x}(\lambda) f(\lambda)\, d\lambda = \int_{-\infty}^{\infty} x f(x)\, dx \quad \text{for all NICE } f\text{'s.}$$

Again, we will makke this rigorous using Distribution Theory.

Distributional solutions to PDEs

From linear acoustics, air pressure $p = p(t, x)$ satisfies wave eqn:

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = 0. \tag{*}$$

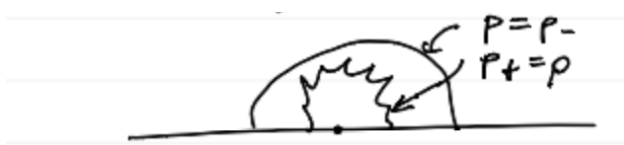


Figure 1: We might be interested for physical solutions to $(*)$ that have discontinuities and thus cannot be treated using classical theory.

Introduce nice $f(t, x)$, $f \in C_c^\infty(\mathbb{R}^2)$. $(*)$ implies

$$\iint \left(\frac{\partial^2 p}{\partial t^2} - \Delta p \right) f\, dxdt = 0 \Rightarrow \iint \left(p \frac{\partial^2 f}{\partial t^2} - p \Delta f \right) dxdt = 0.$$

We say that $p = p(t, x)$ is a weak solution to $(*)$ if

$$\iint \left(p \frac{\partial^2 f}{\partial t^2} - p \Delta f \right) dxdt = 0 \quad \text{for all } f \in C_c^\infty(\mathbb{R}^2).$$

(Try $p = H(t - t_0)$ — Heaviside function,)

In each case, to extend a definition to a larger domain of applicability, we had to introduce a space of "nicer" functions. Thus the theme of distribution theory: functions get replaced by linear maps on some auxiliary space of test functions V.

A distribution is a linear map $u : V \to \mathbb{C}$, i.e., we study the topological dual to V. Let $\langle \cdot, \cdot \rangle$ denote the pairing between V and V^*, i.e., for any $u \in V^*$, $f, g \in V$, and scalars $\alpha, \beta \in \mathbb{C}$,

$$\langle u, \alpha f + \beta g \rangle = \alpha \langle u, f \rangle + \beta \langle u, g \rangle.$$

The topological dual V^* consists of linear maps $u : V \to \mathbb{C}$ such that

$$f_n \to f \text{ in } V \implies \langle u, f_n \rangle \xrightarrow{\mathbb{C}} \langle u, f \rangle.$$

E.g., $V = C_c^\infty(\mathbb{R})$, equipped with the topology of local uniform convergence.

We say $f_n \to f$ in V if, for any compact $K \subset \mathbb{R}$, for all $n \geq 0$,

$$\sup_{x \in K} \left| \left(\frac{d}{dx} \right)^n (f_n(x) - f(x)) \right| \to 0.$$

Define

$$\delta_{x_0} : V \to \mathbb{C}, \quad \langle \delta_{x_0}, f \rangle = f(x_0), \text{ for all } f \in V.$$

Note, $\langle \delta_{x_0}, f_n \rangle \to \langle \delta_{x_0}, f \rangle$ if $f_n \to f$ in V where $V = C_c^\infty(\Omega)$, $\quad \Omega \subset \mathbb{R}^n$ open. $f_n \xrightarrow{V} f$ in the inductive limit topology. One can also treat functions as distribution by defining

$$g \in C_c^\infty(\mathbb{R}), \quad f \mapsto \int f(x)g(x)\,dx, \quad f \in C_c^\infty$$

Lecture 2

2 Distributions

2.1 Notation and Preliminaries

Let X, Y be open subsets of $\mathbb{R}^n$, K compact. Integrals over $X, \mathbb{R}^n$ will then be written:

$$\int_X [\cdot] := \int_X [\cdot]\,dx.$$

2.2 Distributions and Test Functions

First space of test functions

Definition 2.1. *The space $\mathcal{D}(X)$ consists of smooth functions $\varphi : X \to \mathbb{C}$, which have compact support. We say a sequence $\{\varphi_n\}$ in $\mathcal{D}(X)$ tends to zero in $\mathcal{D}(X)$, ($\varphi_n \to 0$ in $\mathcal{D}(X)$) if there exists a compact set $K \subset X$ such that supp(φ_n) $\subset K$, and*

$$\sup |\partial^\alpha \varphi_n| \to 0 \textit{ for each multi-index } \alpha.$$

Functions in $\mathcal{D}(X)$ have nice properties. For example, if $\varphi \in \mathcal{D}(X)$, then $\varphi \equiv 0$ before you reach ∂X. Let $\partial X \supseteq A = \mathbb{R}^n \setminus X$ be closed, $\operatorname{dist}(A, \operatorname{supp}(\varphi)) \geq \delta > 0$. Then $\operatorname{supp}(\varphi)$ is closed and compact.

Figure 2: Support of a test function

This means that integration by parts is easy:

$$\int_X \varphi\, \partial_j \psi\, dx = -\int_X \partial_j \varphi\, \psi\, dx.$$

Since $\varphi \in \mathcal{D}(X)$ is smooth,

$$\varphi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h),$$

and

$$\sum_{|\alpha|\le N} \frac{h^\alpha}{\alpha!} \left(\partial^\alpha\varphi(z) - \partial^\alpha\varphi(\xi)\right) \to 0 \quad \text{uniformly in } x,\, \xi \in B_h(x).$$

Since $\partial^\alpha\varphi$ is smooth and compactly supported, this implies uniform continuity. Therefore,

$$\frac{R_N(x,h)}{|h|^N} \to 0 \quad \text{as } h \to 0 \text{ uniformly in } h.$$

Definition 2.2. *A linear map $u : \mathcal{D}(X) \to \mathbb{C}$ is called a <u>distribution</u> if: for every compact $K \subset X$, there exist constants $C > 0$, $N \in \mathbb{N}$ such that:*

$$|\langle u, \varphi\rangle (=: u(\varphi))| \le C \sum_{|\alpha|\le N} \sup|\partial^\alpha\varphi| \tag{$\bigstar$}$$

for all $\varphi \in \mathcal{D}(X)$ with supp$(\varphi) \subset K$. The space of all such maps is denoted by $D'(X)$, and its elements are called <u>distributions on X</u>.

If the same N can be used in the inequality above for all compact $K \subset X$, we call the least such N the <u>order</u> of u, written ord(u). Note: $N \equiv N_K, \quad C \equiv C_K$

For $x_0 \in X$, define

$$\langle \delta_{x_0}, \varphi\rangle := \varphi(x_0), \quad \varphi \in \mathcal{D}(X).$$

Then $\delta_{x_0} : \mathcal{D}(X) \to \mathbb{C}$ is linear and

$$|\langle \delta_{x_0}, \varphi\rangle| = |\varphi(x_0)| \le \sup|\varphi|.$$

So $C = 1$, $N = 0$ in the inequality. See that $\operatorname{ord}(\delta_{x_0}) = 0$. For a collection $\{f_\alpha\} \subset C(X)$, define $T : \mathcal{D}(X) \to \mathbb{C}$ by

$$\langle T, \varphi\rangle = \sum_{|\alpha|\le M} \int_X f_\alpha\, \partial^\alpha\varphi\, dx.$$

Take $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi) \subset K$. Then

$$|\langle T, \varphi\rangle| \le \sum_{|\alpha|\le M} \int_K |f_\alpha| \cdot |\partial^\alpha\varphi|\, dx$$

$$\le \left(\max_{|\alpha|\le M} \int_K |f_\alpha|\, dx \right) \cdot \sum_{|\alpha|\le M} \sup|\partial^\alpha\varphi|.$$

So $(*)$ holds with

$$C = \max_\alpha \int_K |f_\alpha|\, dx, \quad N = M,$$

and hence $T \in D'(X)$. Note this estimate still holds if the $\{f_\alpha\}$ are only locally integrable, i.e., $f_\alpha \in L^1_{\text{loc}}(X)$. That is, if for all compact $K \subset X$,

$$\int_K |f_\alpha|\, dx < \infty.$$

Henceforth we will make the following abuse of notation: if $f \in L^1_{\text{loc}}(X)$, can define

$$T_f : \mathcal{D}(X) \to \mathbb{C}, \quad \langle T_f, \varphi\rangle = \int_X f\varphi\, dx$$

(the previous example with $M = 0$). We simply write $T_f \equiv f$.

Lemma 2.1. *Let $\varphi_m \to 0$ in $\mathcal{D}(X)$. Then a linear map $u : \mathcal{D}(X) \to \mathbb{C}$ belongs to $D'(X)$ if and only if*

$$\lim_{m\to\infty} \langle u, \varphi_m \rangle = 0 \quad \textit{for all such sequences.}$$

$$[\langle u, \varphi_m \rangle \to \langle u, \varphi \rangle, \quad \varphi_m \to \varphi \quad \varphi_m \to \varphi \Leftrightarrow \varphi_m - \varphi \to 0]$$

Proof. ($\Rightarrow$) Suppose $u \in D'(X)$ and $\varphi_m \to 0$ in $\mathcal{D}(X)$. Then $\operatorname{supp}(\varphi_m) \subset K$, and by seminorm estimate, there exist $C, N > 0$:

$$|\langle u, \varphi_m \rangle| \leq C \cdot \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi_m| \to 0.$$

($\Leftarrow$) Suppose not, i.e. $u : \mathcal{D}(X) \to \mathbb{C}$ linear and $\varphi_m \to 0$ in $\mathcal{D}(X)$, $\Rightarrow \langle u, \varphi_m \rangle \to 0$, but <u>no estimate of form (★)</u> holds, i.e. there exists a compact set $K \subset X$ such that for all choices of C, N, (*) <u>fails on some test function</u> with $\operatorname{supp}(\varphi) \subset K$. In particular, if we take $C = N = m$, there must exist some $\varphi_m \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi_m) \subset K$ and

$$|\langle u, \varphi_m \rangle| > m \cdot \sum_{|\alpha| \leq m} \sup |\partial^\alpha \varphi_m|.$$

Can replace φ_m with $\varphi_m / \langle u, \varphi_m \rangle$. So $\langle u, \varphi_m \rangle = 1$ w.l.o.g. i.e.

$$1 > m \cdot \sum_{|\alpha| \leq m} \sup |\partial^\alpha \varphi_m| \Rightarrow \sup |\partial^\alpha \varphi_m| < \tfrac{1}{m}, \quad m \geq |\alpha|.$$

But the $\varphi_m \to 0$ in $\mathcal{D}(X)$, a contradiction since $\langle u, \varphi_m \rangle = 1 \not\to 0$. □

2.3 Limits in $D'(X)$

Often have sequence $\{u_m\}$ in $D'(X)$. If there is some $u \in D'(X)$ such that

$$\langle u_m, \varphi \rangle \to \langle u, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(X),$$

say $u_m \xrightarrow{D'} u$.

Theorem 2.1 (Non-Examinable)**.** *If $\{u_m\}$ is a sequence in $D'(X)$ and*

$$\langle u, \varphi \rangle := \lim_{m\to\infty} \langle u_m, \varphi \rangle$$

exists for all $\varphi \in \mathcal{D}(X)$, then $u \in D'(X)$.

Take $u_m \in D'(X)$ defined by

$$\langle u_m, \varphi \rangle = \int \sin(mx) \varphi(x) \, dx$$

$$[u_m = \sin(mx)]$$

On IBP (integration by parts),

$$|\langle u_m, \varphi \rangle| = \left| \frac{1}{m} \int \cos(mx) \cdot \varphi'(x) \, dx \right| \to 0 \text{ as } m \to \infty.$$

i.e. $\sin(mx) \to 0$ in $D'(X)$.

Lecture 3

2.4 Basic Operations

2.4.1 Differentiation + Multiplication by smooth functions

For $u \in C^\infty(X) \subseteq L^1_{\text{loc}}(X)$, $\partial^\alpha u \in D'(X)$

$$\langle \partial^\alpha u, \varphi \rangle = \int_X \varphi \, \partial^\alpha u \, dx, \quad \varphi \in \mathcal{D}(X)$$

$$= (-1)^{|\alpha|} \int_X \partial^\alpha \varphi \, u \, dx$$

Leads to the following definition.

Definition 2.3. *For $u \in D'(X)$, $f \in C^\infty(X)$, define*

$$\langle \partial^\alpha (fu), \varphi \rangle := (-1)^{|\alpha|} \langle u, f \partial^\alpha \varphi \rangle$$

for $\varphi \in \mathcal{D}(X)$. Call $\partial^\alpha u$ the <u>distributional derivatives</u> of u. Note $\partial^\alpha (fu) \in D'(X)$

For δ_x, have

$$\langle \partial^\alpha \delta_x, \varphi \rangle = (-1)^{|\alpha|} \langle \delta_x, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(x) \,.$$

Now define Heaviside function

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Thus, $H \in L^1_{\text{loc}}(\mathbb{R})$ and we can compute its distributional derivative

$$\langle H', \varphi \rangle := -\langle H, \varphi' \rangle = -\int_0^\infty \varphi'(x) \, dx = \varphi(0) = \langle \delta_0, \varphi \rangle$$

So, $H' = \delta_0$. Generally, say $u = v$ in $D'(X)$ if

$$\langle u, \varphi \rangle = \langle v, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(X).$$

Lemma 2.2. *If $u \in D'(\mathbb{R})$ and $u' = 0$ in $D'(X)$, then $u = const.$*

Proof. Fix $\varphi \in D(\mathbb{R})$, $\langle \mathbf{1}, \varphi \rangle = \int \varphi \, dx = 1$. For $\varphi \in D(\mathbb{R})$, write

$$\varphi = (\varphi - \langle \mathbf{1}, \varphi \rangle \cdot \mathbf{1}) + \langle \mathbf{1}, \varphi \rangle \cdot \mathbf{1} = \varphi_A + \varphi_B$$

Note that $\langle \mathbf{1}, \varphi_A \rangle = \int \varphi_A dx = 0$. So we have

$$\psi_A(x) = \int_{-\infty}^x \varphi_A(t) \, dt \,.$$

Thus, $\varphi_A \in D(\mathbb{R})$ with $\varphi_A' = \varphi_A$. Hence,

$$\langle u, \varphi \rangle = \langle u, \varphi_A \rangle + \langle u, \varphi_B \rangle = \langle u, \varphi_A \rangle + \langle \mathbf{1}, \varphi \rangle \langle u, \mathbf{1} \rangle = -\langle u', \varphi_A \rangle + c \langle \mathbf{1}, \varphi \rangle \quad \text{with } c = \text{constant}$$

So $u \equiv$ const. in $D'(\mathbb{R})$. □

2.4.2 Translation + Reflection

If $\varphi \in D(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$, define reflection + translation by

$$\check{\varphi}(x) = \varphi(-x), \qquad (T_h\varphi)(x) = \varphi(x-h)$$

Definition 2.4 (1.4)**.** *For $u \in D'(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$, define*

$$\langle \check{u}, \varphi \rangle = \langle u, \check{\varphi} \rangle, \quad \langle T_h u, \varphi \rangle = \langle u, T_{-h}\varphi \rangle$$

for $\varphi \in D(\mathbb{R}^n)$.

Lemma 2.3. *For $u \in D'(\mathbb{R}^n)$ define*

$$V_h = \frac{T_h u - u}{|h|}$$

If $\frac{h}{|h|} \to m \in \mathbb{S}^{n-1}$ as $|h| \to 0$, then

$$V_h \to m \cdot \nabla u \quad \textit{in } D'(\mathbb{R}^n).$$

Proof. For $\varphi \in D(\mathbb{R}^n)$, by definition of V_h

$$\langle V_h, \varphi \rangle = \left\langle u, \frac{T_h\varphi - \varphi}{|h|} \right\rangle .$$

By Taylor's theorem:

$$(T_h\varphi - \varphi)(x) = \varphi(x+h) - \varphi(x) = -\sum_i h_i \frac{\partial \varphi}{\partial x_i} + R_\varphi(x,h)$$

Know that $R_\varphi = o(|h|)$ in $D(\mathbb{R}^n) \Rightarrow$ so by sequential continuity, Lemma 2.1

$$\begin{aligned}
\langle V_h, \varphi \rangle &= -\sum_i \frac{h_i}{|h|} \langle u, \frac{\partial \varphi}{\partial x_i} \rangle + o(1) \\
&= \left\langle \sum_i \frac{h_i}{|h|} \frac{\partial u}{\partial x_i}, \varphi \right\rangle + o(1) \\
&\to \langle m \cdot \nabla u, \varphi \rangle \quad \text{as } |h| \to 0
\end{aligned}$$

□

2.4.3 Convolution between $D'(\mathbb{R}^n)$ and $D(\mathbb{R}^n)$

For $\varphi \in D(\mathbb{R}^n)$, have

$$(E_x\check{\varphi})(y) = \check{\varphi}(y-x) = \varphi(x-y)$$

If $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, define convolution with $\varphi \in D(\mathbb{R}^n)$:

$$\begin{aligned}
(u * \varphi)(x) &= \int u(x-y)\varphi(y)\, dy \\
&= \int \varphi(x-y) u(y)\, dy \\
&= \langle u, T_x\check{\varphi} \rangle
\end{aligned}$$

Definition 2.5. *For $u \in D'(\mathbb{R}^n)$, $\varphi \in D(\mathbb{R}^n)$ define:*

$$u * \varphi(x) = \langle u, T_x\check{\varphi} \rangle$$

Question: How regular do $u * \varphi(x)$?

Lemma 2.4. *For $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, write $\Phi_x(y) = \varphi(x, y)$. If for each $x \in \mathbb{R}^n$ exists neighbourhood $N_x \subset \mathbb{R}^n$ and compact $K \subset \mathbb{R}^n$ such that*

$$\operatorname{supp}(\Phi|_{N_x \times \mathbb{R}^n}) \subset N_x \times K$$

then

$$\partial_x^\alpha \langle u, \Phi_x \rangle = \langle u, \partial_x^\alpha \Phi \rangle \quad \textit{for } u \in D'(\mathbb{R}^n).$$

(Think of $y \mapsto \Phi_x(y)$ as family of test functions).

Proof. From definition and Taylor's theorem

$$\Phi_{x+h}(y) - \Phi_x(y) = \sum_i h_i \frac{\partial \Phi}{\partial x_i}(x, y) + R_1(x, y, h)$$

For $|h|$ sufficiently small, $x \in N_x$ so

$$\operatorname{supp}(R_1(x, \cdot, h)) \subseteq K, \quad \text{also have} \quad \sup |\partial_y^\alpha R(x, y, h)| = o(|h|).$$

So

$$R_1(x, \cdot, h) = o(|h|) \text{ in } D(\mathbb{R}^n).$$

By sequential continuity:

$$\langle u, \Phi_{x+h} - \Phi_x \rangle = \sum_i h_i \left\langle u, \frac{\partial \Phi}{\partial x_i} \right\rangle + o(|h|).$$

So

$$\frac{\partial}{\partial x_i} \langle u, \Phi_x \rangle = \langle u, \frac{\partial \Phi}{\partial x_i} \rangle.$$

Now for higher order multi-indices, the result follows by induction. □

Corollary 2.1. *If $u \in D'(\mathbb{R}^n)$ and $\varphi \in D(\mathbb{R}^n)$, then*

$$u * \varphi \in \mathcal{C}^\infty(\mathbb{R}^n) \quad \textit{and} \quad \partial^\alpha(u * \varphi) = u * \partial^\alpha \varphi.$$

Remark. *Observe that $(u * \varphi)(x) = \langle u, T_x \check{\varphi} \rangle$ so $\Phi_x = T_x \check{\varphi}$ in previous lemma.*

Lecture 4

2.5 Density of $D(\mathbb{R}^n)$ in $D'(\mathbb{R}^n)$

We can use the previous result to prove an important theorem. We need some preliminary results.

Lemma 2.5. *If $u \in D'(\mathbb{R}^n)$, $\varphi, \psi \in D(\mathbb{R}^n)$, then $(u * \varphi) * \psi = u * (\varphi * \psi)$*

Proof. Fix $x \in \mathbb{R}^n$.

$$\begin{aligned}
((u * \varphi) * \psi)(x) &= \int_{\mathbb{R}^n} (u * \varphi)(x - y) \psi(y)\, dy \\
&= \int \langle u(z), \varphi(x - y - z), \psi \rangle(y)\, dy \quad \text{(abuse)} \\
&= \int \langle u(z), \varphi(x - y - z)\psi(y), \rangle dy \\
&= \lim_{h \to 0} \sum_{m \in \mathbb{Z}^n} \langle u(z), \varphi(x - z - hm)\psi(hm), h \rangle^n \\
&\overset{\text{finite sum} \forall h > 0}{=} \lim_{h \to 0} \langle u(z), \sum_{m \in \mathbb{Z}^n} \varphi(x - z - hm)\psi(hm), h \rangle^n
\end{aligned}$$

$$\stackrel{(\dagger)}{=} \langle u(z), (\varphi * \psi)(x - z) \rangle \quad \text{(by sequential continuity)}$$
$$= (u * (\varphi * \psi))(x)$$

To establish (†) define, for $|h| \leq 1$, family of functions $\{F_h^z\} \quad \zeta \longmapsto \sum_{m \in \mathbb{Z}^n} \varphi(x - z - hm)\psi(hm) \cdot h^n$ It is straightforward to show that $\mathrm{supp}(F_h)$ lies in some fixed, compact $K \subset \mathbb{R}^n$. Also, F_h are smooth. Note that for each α,

$$\sup_z |\partial^\alpha F_h(z)| \leq M_\alpha$$

so for each α, $z \mapsto \partial^\alpha F_h(z)$ is uniformly bounded and equicontinuous:

$$|\partial^\alpha F_h(x) - \partial^\alpha F_h(y)| = \left| \int_0^1 \frac{d}{dt} \partial^\alpha F_h(tx + (1-t)y)\, dt \right| = \left| \int_0^1 (x - y) \cdot \nabla \partial^\alpha F_h(tx + (1-t)y)\, dt \right|$$
$$\lesssim_\alpha |x - y|, \qquad (A \lesssim B \Rightarrow \exists C > 0 : A \subset C \cdot B)\,.$$

Applying Arzelà–Ascoli and diagonal argument, get sequence $\{h_k\}$ s.t.

$$\sup_{z \in K} |\partial^\alpha F_{h_k}(z) - \partial^\alpha (\varphi * \psi)(x - z)| \to 0 \quad \text{for each } \alpha.$$

□

Theorem 2.2. *For $u \in D'(\mathbb{R}^n)$, there exist $\varphi_k \in D(\mathbb{R}^n)$ s.t.*

$$\varphi_k \to u \quad \textit{in } D'(\mathbb{R}^n) \quad k \to \infty.$$

$u_k \to u$ in $D'(\mathbb{R}^n)$ if $\langle u_n, \phi \rangle \to \langle u, \phi \rangle$ for all $\phi \in D(\mathbb{R}^n)$).

Proof. The idea is to set $\psi_k \in D(\mathbb{R}^n)$ with $\int \psi_k\, dx = 1$, $\mathrm{supp}(\psi_k) \to \{0\}$, see the figure below. Informally, these smooth functions approximate the Dirac delta at the origin. Interpreting u as a function, then we see that the distributional convolution with a Dirac delta centred at $x \in \mathbb{R}^n$ is just evaluation at x, so we takes a guess the convolutions of u with the ψ_k and then localise by some appropriate cutoff. That is,

$$\text{‘}u * \delta(x) = \int \delta(x - y) u(y)\, dy \approx [u * \psi_k(x)] \cdot \chi_k(x)\text{’}$$

$$\chi = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 2 \end{cases}, \quad \chi_k(x) = \chi\left(\frac{x}{k}\right), \quad \chi \in C_c^\infty(\mathbb{R}^n)$$

We will make this precise below.

Figure 3: Smooth bump functions ‘converging’ distributionally to the Dirac delta at the origin.

Fix $\psi \in D(\mathbb{R}^n)$ with $\int \psi\, dx = 1$, set

$$\psi_k(x) = k^n \psi(kx), \qquad \int \psi_k\, dx = 1.$$

Fix $\chi \in D(\mathbb{R}^n)$ with $\chi = 1$ on $|x| < 1$ and $\chi = 0$ on $|x| > 2$. For $u \in D'(\mathbb{R}^n)$ and arbitrary $\varphi \in D(\mathbb{R}^n)$, consider

$$\langle \varphi_k, \varphi \rangle \quad \text{where } \chi_k(x) = \chi\left(\frac{x}{k}\right)$$
$$\varphi_k = (u * \psi_k)\chi_k$$

$$\langle \varphi_k, \varphi \rangle = \langle u * \psi_k, \chi_k \varphi \rangle = (u * \psi_k) * (\chi_k \varphi^\vee)(0) = u * (\psi_k * (\chi_k \varphi^\vee))(0) \quad \text{(by Lemma 2.5)}$$

$$\psi_k * (\chi_k \theta)^\vee (x) = \int_{\mathbb{R}^n} k^n \psi(k(x-y)) \cdot \chi\left(\frac{-y}{k}\right) \theta^\vee(-y)\, dy = \int \psi(y) \cdot \chi\left(\frac{y}{k} - \frac{x}{k}\right) \theta\left(\frac{y}{k} - x\right) dy$$

making the change of variables $y' = k(x-y) \iff y = x - \frac{y'}{k}$. Now, we compute

$$\psi_k * (\chi_k \theta)^\vee (x) = \varphi(-x) + R_k(-x), \quad \text{where}$$

$$R_k(z) = \int \varphi(y) \left[\chi\left(\frac{y}{k} + \frac{z}{k}\right) \varphi\left(\frac{y}{k} + x\right) - \varphi(x)\right] dy\,.$$

So

$$\langle \varphi_k, \theta \rangle = u * \theta^\vee(0) + u * R_k^\vee(0) = \langle u, \theta \rangle + \langle u, R_k \rangle.$$

It is straightforward to show that $R_k \to 0$ in $D'(\mathbb{R}^n)$, [ ] . Sequential continuity implies that $\langle \varphi_k, \theta \rangle \to \langle u, \theta \rangle$ as $k \to \infty$, concluding the proof. □

3 Distributions of Compact Support

Let $Y \subset X$ be open. We say $u \in D'(X)$ *vanishes* on Y if $\langle u, \varphi \rangle = 0 \quad \forall \varphi \in D(Y)$.

Definition 3.1. *For $u \in D'(X)$, define the support of u by*

$$\operatorname{supp}(u) = X \setminus \bigcup_{\substack{Y \subset X \\ \text{open} \\ u \text{ vanishes on } Y}} Y$$

Example. For $\delta_x \in D'(\mathbb{R}^n)$, $\operatorname{supp}(\delta_x) = \{x\}$.

Claim: If $u \in D'(X)$ vanishes on a collection $\{U_i\}$ of open sets, then u vanishes on their union. Indeed, suppose

$$\operatorname{supp}(\varphi) \subset \bigcup_i U_i\,.$$

By compactness, there exists $\{U_{i_j}\}_{j=1}^N$ such that $\operatorname{supp}(\varphi) \subset \bigcup_j U_{i_j}$. Now fix a partition of unity $\{\psi_j\}_{j=1}^N$ subordinate to $\{U_{i_j}\}$, i.e.

$$\operatorname{supp}(\psi_j) \subset U_{i_j} \quad \text{and} \quad \sum_{j=1}^N \psi_j = 1.$$

Then,

$$\langle u, \varphi \rangle = \langle u, \sum_{j=1}^N \psi_j \varphi \rangle \sum_{j=1}^N \langle u, \psi_j \varphi \rangle = 0 \quad \text{since } \operatorname{supp}(\psi_j \varphi) \subset U_{i_j}\,.$$

Corollary 3.1.

$$\operatorname{supp}(u) = \textit{Complement of largest open set on which } u \textit{ vanishes}$$

Lecture 5

3.1 More test functions and distributions

Definition 3.2. *Define $\mathcal{E}(X)$ to be the space of smooth functions, $\varphi : X \to \mathbb{C}$, s.t. for any compact $K \subseteq X$ and for each multi-index α,*

$$D^\alpha \varphi \to 0 \textit{ locally uniformly,}$$

i.e.

$$\sup_{x \in K} |D^\alpha \varphi(x)| < \infty \quad \textit{for all compact } K \subseteq X.$$

Definition 3.3. *A linear map $u : \mathcal{E}(X) \to \mathbb{C}$ is a* <u>*distribution in $\mathcal{E}'(X)$*</u> *if for every compact $K \subseteq X$ there exists $c_K > 0$ and N such that*

$$|\langle u, \varphi \rangle| \leq c_K \sup_{|\alpha| \leq N} \sup_{x \in K} |D^\alpha \varphi(x)|$$

for all $\varphi \in \mathcal{E}(X)$ supported in K.

So $\mathcal{E}'(X) =$ distributions with compact support.

Lemma 3.1. *A linear map $u : \mathcal{E}(X) \to \mathbb{C}$ belongs to $\mathcal{E}'(X)$ if and only if*

$$\langle u, \varphi_n \rangle \to 0 \quad \text{for all } \{\varphi_n\} \subseteq \mathcal{E}(X) \text{ such that } \varphi_n \to 0 \text{ in } \mathcal{E}(X).$$

Proof. Almost identical to the analogous result for $\mathcal{D}(X)$. Compare compact exhaustion of X.

($\Rightarrow$): There exists $K \subset X$ compact, c_K, $N \geq 0$ such that for all $\varphi \in \mathcal{E}(X)$,

$$|\langle u, \varphi \rangle| \leq c_K \sum_{|\alpha| \leq N} \sup_K |D^\alpha \varphi|.$$

Thus, $\varphi_n \to 0$ in $\mathcal{E}(X)$ implies $\sup_K |D^\alpha \varphi_n| \to 0$ for all α, so $n \to \infty$ gives

$$\langle u, \varphi_n \rangle \to 0 \quad \text{as required.}$$

($\Leftarrow$): Suppose $u \notin \mathcal{E}'(X)$. Then for all $K \subset X$ compact, for all $N \geq 0$, there exists $\varphi \in \mathcal{E}(X)$ such that

$$|\langle u, \varphi \rangle| > c \sum_{|\alpha| \leq N} \sup_K |D^\alpha \varphi|.$$

Let $\{K_n\}$ be a compact exhaustion of X. Then there exists $\{\varphi_n\}$ such that

$$|\langle u, \varphi_n \rangle| > n \cdot \sum_{|\alpha| \leq n} \sup_{K_n} |D^\alpha \varphi_n| \quad \text{for all } n \in \mathbb{N}.$$

W.l.o.g. suppose $\langle u, \varphi_n \rangle = 1$ for all $n \geq$. Then

$$\sum_{|\alpha| \leq n} \sup_{K_n} |D^\alpha \varphi_n| \to 0 \quad \text{as } u_n \to \infty \Rightarrow \varphi_n \to 0 \text{ in } \mathcal{E}(X), \text{ but } \langle u, \varphi_n \rangle \geq 1 \text{ for all } n\,,$$

a contradiction, completing the proof. □

Lemma 3.2. *If $u \in \mathcal{E}'(X)$, then $u|_{\mathcal{D}(X)}$ defines an element of $D'(X)$ with compact support. Conversely, if $u \in D'(X)$ has compact support, then there exists a unique $\tilde{u} \in \mathcal{E}'(X)$ such that* $\operatorname{supp}(u) = \operatorname{supp}(\tilde{u})$ *and $\tilde{u}|_{\mathcal{D}(X)} = u$.*

Proof. Note that $\mathcal{D}(X) \subset \mathcal{E}(X)$, so if $u \in \mathcal{E}'(X)$ then $u|_{\mathcal{D}(X)}$ is well-defined. There exist a compact $K \subset X$, and constants $C, N \geq 0$ such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi| \quad \forall \varphi \in \mathcal{D}(X).$$

So $u|_{\mathcal{D}(X)} \in D'(X)$ and $\operatorname{supp}(u) \subseteq K$.

If $u \in D'(X)$ has compact support, fix $\rho \in \mathcal{D}(X)$ such that $\rho = 1$ on a neighborhood of $\operatorname{supp}(u)$. Define $\tilde{u} : \mathcal{E}(X) \to \mathbb{C}$ by

$$\langle \tilde{u}, \varphi \rangle := \langle u, \rho\varphi \rangle \quad \forall \varphi \in \mathcal{E}(X).$$

Note that $\operatorname{supp}(\rho\varphi) \subset \operatorname{supp}(\rho) \equiv K$. Since $\rho \in \mathcal{D}(X)$, there exist constants $C, N \geq 0$ such that

$$|\langle \tilde{u}, \varphi \rangle| = |\langle u, \rho\varphi \rangle|$$

$$\begin{aligned}&\le C \sum_{|\alpha|\le N} \sup_K |\partial^\alpha(\rho\varphi)| \\ &\le C \sum_{|\alpha|\le N} \sup_K |\partial^\alpha \varphi| \quad \text{(by Leibniz rule).}\end{aligned}$$

So $\tilde{u} \in \mathcal{E}'(X)$. Suppose there exists $\tilde{v} \in \mathcal{E}'(X)$ with $\tilde{v}|_{\mathcal{D}(X)} = u = \tilde{u}|_{\mathcal{D}(X)}$ and $\operatorname{supp}(\tilde{v}) = \operatorname{supp}(\tilde{u}) = \operatorname{supp}(u)$. With $\rho \in \mathcal{D}(X)$ as before,

$$\begin{aligned}\langle \tilde{v}, \varphi\rangle &= \langle \tilde{v}, \rho\varphi\rangle + \langle \tilde{v}, (1-\rho)\varphi\rangle \\ &= \langle \tilde{u}, \rho\varphi\rangle + \langle \tilde{u}, (1-\rho)\varphi\rangle \\ &= \langle \tilde{u}, \varphi\rangle.\end{aligned}$$

Since $(1-\rho)\varphi$ has support disjoint from $\operatorname{supp}(\tilde{v})$, the second term vanishes:

$$\langle \tilde{v}, \varphi\rangle = \langle \tilde{u}, \varphi\rangle.$$

Hence $\tilde{v} = \tilde{u}$. □

3.2 Convolution between $\mathcal{E}'(\mathbb{R}^n)$ and $D'(\mathbb{R}^n)$

For $\varphi \in \mathcal{E}(\mathbb{R}^n), u \in \mathcal{E}'(\mathbb{R}^n)$ define convolution as before:

$$u * \varphi(x) = \langle u, \tau_x \check{\varphi}\rangle$$

Find $u * \varphi \in \mathcal{E}(\mathbb{R}^n)$. Note that $u * \varphi = 0$ unless $(z - y) \in \operatorname{supp}(\varphi)$ for some $y \in \operatorname{supp}(u)$, i.e.

$$\operatorname{supp}(u * \varphi) \subset \operatorname{supp}(u) + \operatorname{supp}(\varphi).$$

In particular, if $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in D(\mathbb{R}^n)$ then $u * \varphi \in D(\mathbb{R}^n)$.

Definition 3.4. *Let $u, v \in D'(\mathbb{R}^n)$, at least one of which has compact support. Then define*

$$\langle u * v, \varphi\rangle := \langle u, v * \varphi\rangle \quad \forall \varphi \in D(\mathbb{R}^n).$$

Then $u * v \in D'(\mathbb{R}^n)$, [✍] .

Lemma 3.3. *For u, v as in Definition 3.4, $u * v = v * u$.*

Proof. Recall Lemma 2.5, if $u \in D'(\mathbb{R}^n)$ and $\varphi, \psi \in D(\mathbb{R}^n)$ then

$$(u * \varphi) * \psi = u * (\varphi * \psi) \quad (\dagger)$$

Same holds if $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{E}(\mathbb{R}^n)$ with at least one of $\operatorname{supp}(\varphi), \operatorname{supp}(\psi)$ compact.

For $\varphi, \psi \in D(\mathbb{R}^n)$,

$$\begin{aligned}(u * v) * (\varphi * \psi) &= u * (v * (\varphi * \psi)) \\ &= u * ((v * \varphi) * \psi) \\ &= u * [\varphi * (v * \psi)] \\ &= (u * \varphi) * (v * \psi)\end{aligned}$$

Use $\varphi * \psi = \psi * \varphi$, then

$$\begin{aligned}(u * v) * (\varphi * \psi) &= (u * \varphi) * (v * \psi) \\ &= (u * \psi) * (v * \varphi) \\ &= (v * u) * (\varphi * \psi) \quad (\dagger)\end{aligned}$$

So if $E = u * v - v * u$, then $E * (\varphi * \psi) = 0$ for all $\varphi, \psi \in D(\mathbb{R}^n)$ and

$$(E * \varphi) * \psi = 0 \quad \forall \psi \in D(\mathbb{R}^n) \Rightarrow E * \varphi = 0 \quad \forall \varphi \in D(\mathbb{R}^n) \Rightarrow E = 0 \text{ in } D'(\mathbb{R}^n) \text{ i.e.}$$

$$\langle u, v * \varphi \rangle = \langle v, u * \varphi \rangle .$$

$\square$

So for any $u \in D'(\mathbb{R}^n)$ we have

$$\delta_0 * u = u * \delta_0 = u$$

since for $\varphi \in D(\mathbb{R}^n)$

$$\begin{aligned}(u * \delta_0)(\varphi) &= u * (\delta_0 * \varphi) = u * \varphi \\ (\delta_0 * \varphi)(x) &= \langle \delta_0, \tau_x \check{\varphi} \rangle \\ &= \tau_x \check{\varphi}(0) \\ &= \check{\varphi}(-x) = \varphi(x).\end{aligned}$$

Lecture 6

3.3 Tempered Distributions + Fourier Analysis

3.3.1 The test function space $\mathcal{S}(\mathbb{R}^n)$

Definition 3.5. *The test function space, written $\mathcal{S}(\mathbb{R}^n)$ consists of all smooth $\varphi \in C^\infty(\mathbb{R}^n)$ s.t.*

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| < \infty$$

for all multi-indices α, β.

In words: φ rapidly decays to 0 as $|x| \to \infty$ faster than any polynomial growth.

Examples. $x \mapsto \frac{1}{1+|x|^2} \in \mathcal{S}(\mathbb{R}^n), x \mapsto \exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$.

Definition 3.6. *A linear map $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions, if there exist $C, N \geq 0$ such that*

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha|, |\beta| \leq N} \|\varphi\|_{\alpha,\beta}.$$

Remark. *Arguing as before, one can show that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, that $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ linear belongs to $\mathcal{S}'(\mathbb{R}^n) \Leftrightarrow \langle u, \varphi_m \rangle \to 0 \quad \forall \varphi_m \to 0$ in $\mathcal{S}$.*

Note the inclusions

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$$

$$(\varphi_m \to 0 \text{ in } D \Rightarrow \varphi_m \to 0 \text{ in } \mathcal{S} \Rightarrow \varphi_m \to 0 \text{ in } \mathcal{E})$$

and

$$\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset D'(\mathbb{R}^n).$$

So in a sense, $\mathcal{S}, \mathcal{S}'$, are the Goldilocks pair if we want to do Fourier Analysis. As it will turn out, the Fourier transform is extremely well adapted to $\mathcal{S}$, and by extension through duality, to $\mathcal{S}'$.

3.4 Fourier Transform on $\mathcal{S}(\mathbb{R}^n)$

Definition 3.7. *For $f \in L^1(\mathbb{R}^n)$ define* <u>*Fourier Transform*</u>*:*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \quad \xi \in \mathbb{R}^n$$

Use $\mathcal{F}$ to denote linear map $\mathcal{F} : f \mapsto \hat{f}$.

Note that $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, since for $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}\int_{\mathbb{R}^n} |\varphi|\, dx &= \int_{\mathbb{R}^n} (1+|x|)^{-N}(1+|x|)^N |\varphi|\, dx \\ &\leq C \sum_{|\alpha|\leq N} \|\varphi\|_{\alpha,0} \int_{\mathbb{R}^n} (1+|x|)^{-N} dx < \infty\end{aligned}$$

for $N \geq n+1$.

Lemma 3.4. *If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in \mathcal{C}_0(\mathbb{R}^n)$.*

Proof. Suppose $\lambda_k \to \lambda$ in $\mathbb{R}^n$. Then

$$\lim_{k\to\infty} \hat{f}(\lambda_k) = \lim_{k\to\infty} \int_{\mathbb{R}^n} e^{-ix\cdot\lambda_k} f(x)\, dx = \int_{\mathbb{R}^n} \underbrace{\lim_{k\to\infty} e^{-ix\cdot\lambda_k}}_{g_\lambda(x)} f(x)\, dx = \hat{f}(\lambda)$$

Since $|g_\lambda| \leq 1$, $f \in L^1(\mathbb{R}^n)$,

$$\hat{f}(\lambda) = \int_{\mathbb{R}^n} e^{-ix\cdot\lambda} f(x)\, dx = \hat{f}(\lambda).$$

$\square$

IMPORTANT: Fourier transform interchanges smoothness and decay.

Lemma 3.5. *For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$(D^\alpha \varphi)^\wedge(\lambda) = \lambda^\alpha \hat{\varphi}(\lambda), \quad D = -i\partial$$

$$(x^\beta \varphi)^\wedge(\lambda) = C(-D_\lambda)^\beta \hat{\varphi}(\lambda), \quad D_x := -i\frac{\partial}{\partial x}$$

Proof. By IBP:

$$\begin{aligned}(D^\alpha \varphi)^\wedge(\lambda) &= \int_{\mathbb{R}^n} e^{-i\lambda\cdot x} D^\alpha \varphi(x)\, dx \\ &= C(-1)^{|\alpha|} \int_{\mathbb{R}^n} D_x^\alpha \left[e^{-i\lambda\cdot x}\right] \varphi(x)\, dx \\ &= C \int_{\mathbb{R}^n} \lambda^\alpha e^{-i\lambda\cdot x} \varphi(x)\, dx \\ &= \lambda^\alpha \hat{\varphi}(\lambda)\end{aligned}$$

$$\begin{aligned}(-D_\lambda)^\beta \hat{\varphi}(\lambda) &= (-D_\lambda)^\beta \int_{\mathbb{R}^n} e^{-i\lambda\cdot x} \varphi(x)\, dx \\ &\overset{\text{DCT}}{=} \int_{\mathbb{R}^n} x^\beta e^{-i\lambda\cdot x} \varphi(x)\, dx \\ &= (x^\beta \varphi)^\wedge(\lambda)\end{aligned}$$

$\square$

$$\text{Note Lemmas 3.4, 3.5} \Rightarrow \hat{f}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$$

Might have seen "Fourier inversion" Theorem

$$\varphi(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\lambda} \hat{\varphi}(\lambda) d\lambda$$

Theorem 3.1. *The Fourier Transform is a continuous isomorphism on $\mathcal{S}(\mathbb{R}^n)$, is bijective and $\varphi_m \xrightarrow{\mathcal{S}} 0 \Leftrightarrow \hat{\varphi}_m \xrightarrow{\mathcal{S}} 0$.*

Proof. Know that $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$. Lemma 3.5

$$|\lambda^\alpha D^\beta \hat{\varphi}(\lambda)| = \left| \int D^\alpha (x^\beta \varphi) e^{-i\lambda\cdot x} dx \right| \le \int |D^\alpha (x^\beta \varphi)| dx < \infty. \tag{$\dagger$}$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have $D^\alpha(x^\beta\varphi) \in \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. Hence,

$$\|\hat{\varphi}\|_{\alpha,\beta} < \infty \text{ for all } \alpha, \beta, \text{ i.e., } \hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$$

which means the Fourier transform is a map

$$\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

By suitably applying ($\dagger$) to sequence $\varphi_m \to 0$, find $\hat{\varphi}_m \to 0 \Rightarrow \varphi_m \to 0$.

Want to establish inversion. Consider

$$\int e^{i\lambda\cdot x} \hat{\varphi}(\lambda) d\lambda = \lim_{\varepsilon \to 0} \int e^{i\lambda\cdot x} e^{-\varepsilon|\lambda|^2} \hat{\varphi}(\lambda) d\lambda \tag{PCT}$$

For all $\varepsilon > 0$, by Fubini

$$= \int \varphi(y) \left[\int e^{i\lambda\cdot(x-y)} e^{-\varepsilon|\lambda|^2} d\lambda \right] dy \tag{$\bigstar$}$$

$$= \int \varphi(y) \left[\prod_{j=1}^n \int e^{i\lambda_j(x_j-y_j)} e^{-\varepsilon\lambda_j^2} d\lambda_j \right]$$

$$= \int \varphi(y) \left[\prod_{j=1}^n \left(\frac{\pi}{\varepsilon}\right)^{1/2} e^{-(x_j-y_j)^2/4\varepsilon} \right] dy$$

$$= \int \varphi(y) \left(\frac{\pi}{\varepsilon}\right)^{n/2} e^{-|x-y|^2/4\varepsilon} dy, \quad (\text{ making the change of variables } y^* := \frac{x-y}{2\sqrt{\varepsilon}}).$$

$$\xrightarrow[\text{(DCT)}]{\varepsilon\to 0} \varphi(x)(2\pi)^n \left(\sqrt{\tfrac{1}{\pi}}\right)^n \underbrace{\int e^{-|y|^2} dy}_{=1}$$

$$= (2\pi)^n \varphi(x).$$

$$\textit{i.e.} \quad \varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d} e^{i\lambda\cdot x} \hat{\varphi}(\lambda)\, d\lambda$$

which implies

$$\varphi(x-z) = \mathcal{F}^{-1}\left[\frac{\hat{\varphi}}{(2\pi)^n}\right]$$

So we get a bijection $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and by previous results, if $\varphi_{\varepsilon_k} \to 0$, then $\varphi_k \to 0$.

$$(\bigstar) \qquad \int_{\mathbb{R}} e^{-i\sigma\lambda} e^{-\varepsilon\lambda^2}\, d\lambda = \int_{\mathbb{R}} e^{-\varepsilon\left(\lambda - \frac{i\sigma}{2\varepsilon}\right)^2 - \frac{\sigma^2}{4\varepsilon}}\, d\lambda = e^{-\sigma^2/4\varepsilon} \int_{\mathbb{R}} e^{-\varepsilon\left(\lambda-\frac{i\sigma}{2\varepsilon}\right)^2}\, d\lambda$$

This last integral is a standard Gaussian integral. We can differentiate with respect to σ

$$\frac{\partial}{\partial\sigma}[\cdots] = 0 \quad \text{implies } \sigma = 0 \text{ without loss of generality.}$$

Alternatively, by the Cauchy integral theorem one can compute

$$\int_{\mathbb{R}+\frac{i\sigma}{2\varepsilon}} e^{-\varepsilon\left(\lambda-\frac{i\sigma}{2\varepsilon}\right)^2}\, d\lambda = \int_{\mathbb{R}} e^{-\varepsilon x^2}\, dx = \sqrt{\frac{\pi}{\varepsilon}}.$$

□

Lecture 7

3.5 Fourier Transform on $\mathcal{S}'(\mathbb{R}^n)$

To define F.T. on $\mathcal{S}'(\mathbb{R}^n)$ need Parseval's Theorem.

Lemma 3.6. *If $u, \varphi \in \mathcal{S}(\mathbb{R}^n)$ then*

$$\int \varphi(x)\widehat{\psi}(x)dx = \int \widehat{\varphi}(x)\psi(x)dx$$

Proof. By Fubini:

$$\text{LHS} = \int \varphi(x)\left[\int e^{-i\lambda\cdot x}\psi(\lambda)d\lambda\right]dx = \int \psi(\lambda)\left[\int e^{-i\lambda\cdot x}\varphi(x)dx\right]d\lambda = \int \psi(\lambda)\widehat{\varphi}(\lambda)d\lambda$$ □

If $u \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, then previous lemma states

$$\langle \hat{u}, \varphi\rangle = \langle u, \hat{\varphi}\rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$

Since $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, the RHS is well-defined for <u>any</u> $u \in \mathcal{S}'(\mathbb{R}^n)$.

Definition 3.8. *For $u \in \mathcal{S}'(\mathbb{R}^n)$ define $\hat{u}$ by*

$$\langle \hat{u}, \varphi\rangle := \langle u, \hat{\varphi}\rangle \quad \textit{for all } \varphi \in \mathcal{S}(\mathbb{R}^n)$$

As an example take $u = \delta_0$.

$$\langle \hat{\delta}_0, \varphi\rangle = \langle \delta_0, \hat{\varphi}\rangle = \hat{\varphi}(0) = \int \varphi(x)dx = \langle 1, \varphi\rangle$$

i.e. $\hat{\delta}_0 = 1$ in $\mathcal{S}'(\mathbb{R}^n)$. Also, $u = 1$ compute

$$\langle \hat{1}, \varphi\rangle = \langle 1, \hat{\varphi}\rangle = \int \hat{\varphi}d\lambda = (2\pi)^n\varphi(0) = \langle (2\pi)^n\delta_0, \varphi\rangle$$

In "old" language,

$$\text{"}\delta_0(x) = \frac{1}{(2\pi)^n}\int e^{-i\lambda\cdot x}\, d\lambda\text{"}$$

Straightforward to extend Lemma 3.5 to $\mathcal{S}'(\mathbb{R}^n)$, , i.e.

$$(D_x^\alpha \hat{u}) = \lambda^\alpha \hat{u}$$
$$(x^\beta u)\hat{} = (-D)^\beta \hat{u}\,.$$

Theorem 3.2. *The Fourier Transform defines a continuous injection*

$$\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

Proof. To get injection, note

$$\check{u} = \frac{1}{(2\pi)^n}\,(\hat{u})\hat{}$$

since

$$\langle \check{u}, \varphi\rangle = \langle u, \check{\varphi}\rangle \overset{(*)}{=} \langle u, ((2\pi)^{-n}(\hat{\varphi})\hat{})\rangle = \langle (2\pi)^{-n}(\hat{u})\hat{}, \varphi\rangle$$

$$(*) \quad \varphi(-x) = \frac{1}{(2\pi)^n}\int e^{-i\lambda\cdot x}\hat{\varphi}(\lambda)\,d\lambda = (2\pi)^{-n}(\hat{\varphi})\hat{}.$$

To see that $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, note that

$$\varphi_m \to 0 \text{ in } \mathcal{S}(\mathbb{R}^n) \iff \hat{\varphi}_m \to 0 \text{ in } \mathcal{S}(\mathbb{R}^n)$$

so

$$\langle \hat{u}, \varphi_m\rangle = \langle u, \hat{\varphi}_m\rangle \to 0 \text{ as } m \to \infty,$$

so $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$. For continuity, suppose $u_m \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$, i.e.

$$\langle u_m, \varphi, \to\rangle 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$

$$\iff \quad \langle u_m, \hat{\psi}, \to\rangle 0 \quad \forall \psi \in \mathcal{S}(\mathbb{R}^n)$$

$$\iff \quad \langle \hat{u}_m, \psi, \to\rangle 0 \quad \forall \psi \in \mathcal{S}(\mathbb{R}^n)$$

i.e. $\quad u_m \to 0 \quad \text{in } \mathcal{S}' \iff \hat{u}_m \to 0 \quad \text{in } \mathcal{S}'$. ☐

4 Sobolev Space

Definition 4.1. *For $s \in \mathbb{R}$ define Sobolev space $H^s(\mathbb{R}^n)$ to be the $u \in \mathcal{S}'(\mathbb{R}^n)$ for which $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ can be identified with a (measurable) $\lambda \mapsto \hat{u}(\lambda)$ that satisfies*

$$\|u\|_{H^s}^2 = \int (1+|\lambda|^2)^s |\hat{u}(\lambda)|^2\,d\lambda < \infty.$$

We will use notation

$$\langle\lambda\rangle = (1+|\lambda|^2)^{1/2}$$

so $\langle\lambda\rangle \sim |\lambda|$ as $|\lambda| \to \infty$. Observe that $u \in H^s(\mathbb{R}^n)$ then $\langle\cdot\rangle^s \hat{u} \in L^2(\mathbb{R}^n)$.

Lemma 4.1. *If $u \in H^s(\mathbb{R}^n)$ and $s > n/2$, then $u \in C(\mathbb{R}^n)$.*

Proof. We establish that $\hat{u} \in L^1(\mathbb{R}^n)$.

$$\int |\hat{u}(\lambda)|\,d\lambda = \int \langle\lambda\rangle^{-s}\langle\lambda\rangle^{s}|\hat{u}(\lambda)|\,d\lambda$$

$$\leq \left(\int \langle\lambda\rangle^{-2s}\,d\lambda\right)^{1/2}\left(\int \langle\lambda\rangle^{2s}|\hat{u}(\lambda)|^2\,d\lambda\right)^{1/2}$$

$$= \left[\int_{S^{n-1}} d\sigma \underbrace{\int_0^\infty (1+r^2)^{-s} r^{n-1}\,dr}_{(\dagger)}\right]^{1/2} \cdot \|u\|_{H^s(\mathbb{R}^n)} \qquad (\dagger)$$

Note, $(\dagger) = \mathcal{O}\left(r^{-2s+n-1}\right)$ as $r \to \infty$, so the integral is finite if $s > n/2$. **CANNOT** invoke inverse F.T. — only proved that inversion works on $\mathcal{S}(\mathbb{R}^n)$! We can only write

$$\langle u, \varphi \rangle = \langle \hat{u}, \hat{\varphi} \rangle = \int \hat{u}(\lambda)\overline{\hat{\varphi}(\lambda)}\, d\lambda$$

$$= \int \hat{u}(\lambda) \left[\frac{1}{(2\pi)^n} \int e^{-i\lambda \cdot x} \varphi(x)\, dx \right] d\lambda\, .$$

Since $\hat{u} \in L^1(\mathbb{R}^n)$, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, can apply Fubini to obtain

$$\langle u, \varphi \rangle = \int \varphi(x) \left[\frac{1}{(2\pi)^n} \int e^{i\lambda \cdot x} \hat{u}(\lambda)\, d\lambda \right] dx$$

$$= \int u(x) \hat{\varphi}(x)\, dx$$

where

$$u(x) = \frac{1}{(2\pi)^n} \int e^{i\lambda \cdot x} \hat{u}(\lambda)\, d\lambda.$$

Since $\hat{u} \in L^1(\mathbb{R}^n)$, by Dominated Convergence Theorem we get $u \in C(\mathbb{R}^n)$. $\square$

Corollary 4.1. *If $u \in H^s(\mathbb{R}^n)$ for <u>all</u> $s > n/2$, then $u \in C^\infty(\mathbb{R}^n)$.*

Proof (sketch). Replace u with $D^\alpha u (= \lambda^\alpha \hat{u})$, show $(D^\alpha u)\hat{} \in L^1(\mathbb{R}^n)$etc., conclude $D^\alpha u \in C(\mathbb{R}^n)$ $\square$

Remark. *When understanding regularity, which is a **LOCAL** concept, can confine attention to $\varphi u, \varphi \in \mathcal{D}(\mathbb{R}^n)$.*

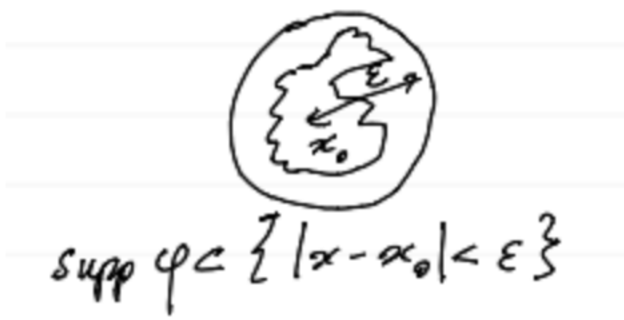


Figure 4: Illustration of support of test function φ in ball or radius ε.

So very rarely need to study u in isolation, φu, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ will do. Thus, if $u \in D'(X)$, can consider $u \in D'(X)$, $\varphi \in \mathcal{D}(X)$, and make extension

$$(\varphi u)_{\text{ext}} \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$$

This leads us to the introduction to the notion of local Sobolev spaces $H^s_{\text{loc}}(X)$, which we turn our attention to next.

Lecture 8

Definition 4.2. *Say $u \in D'(X)$ belongs to the <u>local Sobolev space</u> $H^s_{loc}(X)$ if $u\varphi$ (extends to) an element of $H^s(\mathbb{R}^n)$ for each $\varphi \in \mathcal{D}(X)$.*

Remark. *Note we interpret $\varphi u \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ by*

$$\langle \varphi u, \psi \rangle := \langle u, \varphi \psi \rangle \quad \forall \psi \in \mathcal{E}(\mathbb{R}^n).$$

Well-defined since $\operatorname{supp}(\varphi\psi) \subset X$.

5 § 4: Applications of Fourier Transform

5.1 § 4.1: Elliptic Regularity

Interested in problems of form

$$P(D)u = f \tag{$\bigstar$}$$

where $u, f \in D'(X)$, where P is a polynomial in λ, e.g.

$$P(\lambda) = \lambda_1^2 + \cdots + \lambda_n^2$$

$$P(D) = -\left(\frac{\partial}{\partial x_1}\right)^2 - \cdots - \left(\frac{\partial}{\partial x_n}\right)^2 = -\Delta.$$

Interested in the following question: if $f \in H^s_{\mathrm{loc}}(X)$, can we say that $u \in H^t_{\mathrm{loc}}(X)$ for some $t = t(s, P)$? We will give a positive answer to this when $P(D)$ is elliptic.

Definition 5.1. *An N^{th} order P.D.O.*

$$P(D) = \sum_{|\alpha|=N} c_\alpha D^\alpha$$

(constant coeffs.) has principal symbol defined by

$$\sigma_P(\lambda) = \sum_{|\alpha|=N} c_\alpha \lambda^\alpha.$$

Say $P(D)$ is elliptic if $\sigma_P(\lambda) \neq 0$ on $\mathbb{R}^n \setminus \{0\}$.

Remark. *One guess for constructing solutions to such P.D.O.s would be to take Fourier transforms of ($\bigstar$) and write*

$$\hat{u} = \frac{\hat{f}}{P(\lambda)}.$$

The ellipticity condition will mean that such a guess is largely correct, since , $P(\lambda) \asymp \sigma_P(\lambda), |\lambda| \to \infty$, though this needs to be made precise.

Lemma 5.1. *If $P(D)$ is N^{th} order elliptic then for all sufficiently large $|\lambda| \gg 1$,*

$$|P(\lambda)| \gtrsim |\lambda|^N.$$

Proof. By continuity & compactness, since $\sigma_P(\lambda) \neq 0$ on S^{n-1},

$$\min_{|\lambda|=1} |\sigma_P(\lambda)| = C > 0.$$

Then for $\lambda \in \mathbb{R}^n \setminus \{0\}$,

$$|\sigma_P(\lambda)| = |\lambda|^N \left| \sum_{|\alpha|=N} c_\alpha \left(\frac{1}{|\lambda|}\right)^\alpha \right| \geq C|\lambda|^N.$$

By the triangle inequality,

$$\begin{aligned} |P(\lambda)| &\geq |\sigma_P(\lambda)| - |P(\lambda) - \sigma_P(\lambda)| \\ &\geq \left[C - \frac{|P(\lambda) - \sigma_P(\lambda)|}{|\lambda|^N}\right] |\lambda|^N \end{aligned}$$

Since $|P(\lambda) - \sigma_P(\lambda)| = O(|\lambda|^{N-1})$, choosing $|\lambda|$ sufficiently large so

$$\frac{|P(\lambda) - \sigma_P(\lambda)|}{|\lambda|^N} \leq \frac{C}{2}$$

Hence, for $|\lambda|$ sufficiently large

$$|P(\lambda)| \geq \frac{C}{2}|\lambda|^N \Rightarrow C_1|\lambda|^N \gtrsim \langle\lambda\rangle^N.$$

□

Theorem 5.1. *Suppose $P(D)u$ is N^{th} order elliptic. Then,*

$$P(D)u \in H^s_{loc}(X) \Rightarrow u \in H^{s+N}_{loc}(X).$$

We will first tackle, more "easy" version, relevant if $u \in \mathcal{E}'(\mathbb{R}^n)$. We will crucially use the fact that if $u \in \mathcal{E}'(\mathbb{R}^n)$, then

$$\hat{u} \in \mathcal{E}(\mathbb{R}^n), \quad |\hat{u}(\lambda)| \lesssim \langle\lambda\rangle^M \text{ some } M > 0.$$

(which we will establish using Payley-Wiener theory, see later). When $u \in \mathcal{E}'(\mathbb{R}^n)$, can use the <u>parametrix</u> to prove version of Thm 5.1, which we define below.

Definition 5.2. *Say that $E \in D'(\mathbb{R}^n)$ is a <u>parametrix</u> for $P(D)$ if there exists $\omega \in \mathcal{E}(\mathbb{R}^n)$ such that*

$$P(D)E = \delta_0 + \omega\,.$$

Remark. *This is almost like saying that we can invert the P.D.O., that is having some Green's function G satisfying $P(D)G = \delta_0$, which would allow us to express solutions to find $P(D)u = f$ simply by taking the convolution $G * f$.*

Lemma 5.2. *Every (non-zero), <u>elliptic</u> $P(D)$ admits a <u>parametrix</u> $E \in \mathcal{E}(\mathbb{R}^n \setminus \{0\})$.*

Proof. Fix $R \geq 0$ such that $|P(\lambda)| \gtrsim \langle\lambda\rangle^N$ on $|\lambda| > R$, $\chi \in \mathcal{D}(\mathbb{R}^n)$ such that $\chi = 1$ on $|\lambda| \leq R$ and $\chi = 0$ on $|\lambda| \geq R+1$. Define $E \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\hat{E}(\lambda) = \frac{(1-\chi(\lambda))}{P(\lambda)}$$

Then $\hat{E}$ is smooth and $|\hat{E}| \lesssim \langle\lambda\rangle^{-N}$ for $|\lambda|$ so $\hat{E} \in \mathcal{S}(\mathbb{R}^n) \Rightarrow E \in \mathcal{S}'(\mathbb{R}^n)$. Inverting the Fourier transform, we obtain

$$P(D)E = \delta_0 + \omega$$

where $\hat{\omega} = -\chi \in \mathcal{D}(\mathbb{R}^n) \Rightarrow \omega \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$. For $|\lambda| > R+1$ we have,

$$|(\chi^\beta E)^\wedge(\lambda)| = \left|D^\beta \hat{E}(\lambda)\right| = \left|D^\beta\left(\frac{1}{P(\lambda)}\right)\right| \lesssim \langle\lambda\rangle^{-N-|\beta|}$$

where the last estimate follows from an induction argument, [ ] . So for <u>every</u> $|\beta|$ (particularly $s \in \mathbb{Z}_{\geq 0}$), there is a β multi-index s.t. $\chi^\beta E \in H^s(\mathbb{R}^n)$. So for each α, $D^\alpha(\chi^\beta E)$ is continuous for $|\beta|$ sufficiently large (by the Sobolev embedding lemma). That is, E is smooth away from $x = 0$, $E \in \mathcal{E}'(\mathbb{R}^n \setminus \{0\})$. $\square$

Proof of easy version of Theorem 5.1. If $u \in \mathcal{E}'(\mathbb{R}^n)$, then $\hat{u} \in \mathcal{E}(\mathbb{R}^n)$, using

$$P(D)\hat{E}(\lambda) = 1 + \hat{\omega}$$

i.e.

$$1 = P(\lambda)\hat{E} - \hat{\omega} \in \mathcal{D}(\mathbb{R}^n) \quad \{ \ = 0 \ (\langle\lambda\rangle^{-k}) \ \forall R$$

and so multiplying both sides by $\hat{u}$

$$\hat{u} = \left[P(\lambda)\hat{\omega}\hat{E} - \hat{\omega}\right]\hat{u}\,.$$

Now, since $\hat{w} \in \mathcal{D}(\mathbb{R}^n)$, we have that $\hat{\omega} = o(\langle\lambda\rangle^{-}k)$ for all $k \geq 1$. Thus we have, $P(D)u \in H^s(\mathbb{R}^n)$ and $P(\lambda)\hat{u}\langle\lambda\rangle^s \in L^2(\mathbb{R}^n)$. We now estimate

$$\hat{u}\langle\lambda\rangle^{s+N} = [P(\lambda)\hat{u}\langle\lambda\rangle^s]\underbrace{\hat{E}(\lambda)\langle\lambda\rangle^N}_{\leq 1} - \underbrace{\hat{\omega}\hat{u}\langle\lambda\rangle^{s+N}}_{o(\langle\lambda\rangle^{-k})\forall k\geq 1}$$

and $\left\|\hat{u}\langle\lambda\rangle^{s+N}\right\|_{L^2} \lesssim 1$ which gives $u \in H^{s+N}(\mathbb{R}^n)$. $\square$

The above gives us some idea as how one could proceed to prove the general case of 5.1, i.e. by localising u by multiplying by some test function $\varphi \in \mathcal{D}(X)$. Consider briefly the Elliptic P.D.O. $P(D)$ and suppose $u \in D'(X)$ satisfies in the sense of distributions $P(D)u = f$. Then, after this localisation step (also notice $f \in H^s_{loc}(X)$ means $\varphi f \in H^s(\mathbb{R}^n)$) we can express

$$\text{LHS} = P(D)[\underbrace{\varphi u}_{\in \mathcal{E}'(X)}] + \underbrace{[\varphi, P(D)]}_{\text{ord } N-1}(u)$$

where the second term involves a *commutator* term which we will try to control. One can proceed by iterating this scheme finitely many times and hope this will improve regularity locally, which indeed it will! This is how we will proceed in proving Theorem 5.1.

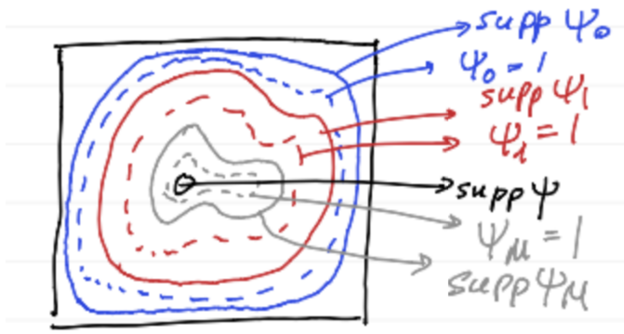


Figure 5: Iteration of localisation argument.

Proof (Thm 5.1). We will use the following facts, :

- $u \in \mathcal{E}'(\mathbb{R}^n)$ then $\exists t \in \mathbb{R}$ s.t. $u \in H^t(\mathbb{R}^n)$
- If $u \in H^t(\mathbb{R}^n) \Rightarrow D^\alpha u \in H^{t-|\alpha|}(\mathbb{R}^n)$
- If $s > t \Rightarrow H^t(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$
- If $\psi \in C^\infty(\mathbb{R}^n)$ and $u \in H^s(\mathbb{R}^n) \Rightarrow \psi u \in H^s(\mathbb{R}^n)$

Fix $\varphi \in \mathcal{D}(X)$. Introduce test functions

$$\{\psi_0, \psi_1, \dots, \psi_M\} \subset \mathcal{D}(\mathbb{R}^n) \text{ s.t. } \psi_{i-1} = 1 \text{ on } \operatorname{supp}\psi_i \text{ and } \operatorname{supp}\varphi \subset \operatorname{supp}(\psi\mu) \subset \cdots \subset \operatorname{supp}\psi_0.$$

(see the figure above). Note that $\psi_0 u \in \mathcal{E}'(\mathbb{R}^n)$, so $\psi u \in H^t(\mathbb{R}^n)$ (for some t). Then

$$P(D)[\psi_1 u] = \psi_1 P(D)u + [P(D), \psi_1](u)$$

$$= \psi_1 P(D)u + [P(D), \psi_1](\psi_0 u) \quad \text{ since } \psi_0 = 1 \text{ on } \operatorname{supp}\psi_1$$

$$\in H^s(\mathbb{R}^n) \oplus H^{t-N+1}(\mathbb{R}^n) \quad (\text{same } t).$$

I.e. $P(D)[\psi_1 u] \in H^{\tilde{A}_1}(\mathbb{R}^n)$ where

$$\tilde{A}_\mu = \min\{s, t - N + 1\} \quad \text{i.e.}$$

$$\int \langle\lambda\rangle^{2\tilde{A}_1} |P(\lambda)[\psi_1 u]^\wedge(\lambda)|^2 \, d\lambda < \infty \tag{†}$$

since $|P(\lambda)| \gtrsim \langle\lambda\rangle^N$ for $|\lambda|$ sufficiently large.

$$(\dagger) \Rightarrow \int \langle\lambda\rangle^{2(\tilde{A}_1+N)} |[\psi_1 u]^\wedge(\lambda)|^2 \, d\lambda \lesssim \int \langle\lambda\rangle^{2\tilde{A}_1} |P(\lambda)[\psi_1 u]^\wedge(\lambda)|^2 \, d\lambda < \infty$$

i.e. $\psi_1 u \in H^{A_1}(\mathbb{R}^n)$, $A_1 = \tilde{A}_1 + N = \min\{s+N, t+1\}$. Similarly,

$$P(D)[\psi_2, u] = \psi_2 P(D)u + [P(D), \psi_2](u) = \psi_2 P(D)u + [P(D), \psi_2](\psi_1 u)$$

since $\psi_1 = 1$ on $\operatorname{supp}\psi_2$, which gives that

$$\psi_2 u \in H^{A_2}(\mathbb{R}^n), \quad A_2 = \min\{s, t-N+1\} + N = \min\{s+N, t+2\}.$$

Proceeding inductively,

$$\psi_\mu u \in H^{A_\mu}(\mathbb{R}^n), \quad A_\mu = \min\{s+N, t+\mu\}.$$

Now, choose M s.t. $M \geq s+N-t$, so $A_M = s+N$. Since $\psi_M = 1$ on $\operatorname{supp}\varphi$ get $\varphi u \in H^{s+N}(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(X)$ arbitrary, we conclude $u \in H^{s+N}_{\text{loc}}(X)$. $\square$

5.2 Fundamental Solutions

To solve PDEs of form $P(D)u = f$, can use fundamental solutions.

Definition 5.3. *Say $E \in D'(\mathbb{R}^n)$ is a fundamental sol^n for $P(D)$ if*

$$P(D)E = \delta.$$

Lemma 5.3. *Fundamental sol^n for*

$$P(D) = \frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$$

is given by $E = \frac{1}{\pi z}$. ($z = x_1 + ix_2 \in \mathbb{C} \simeq \mathbb{R}^2$)

Proof. $E \in L^1_{\text{loc}}(\mathbb{R}^2)$, so $E \in D'(\mathbb{R}^2)$. For $\varphi \in D(\mathbb{R}^2)$

$$\left\langle \frac{\partial}{\partial \bar{z}} E, \varphi \right\rangle = -\left\langle E, \frac{\partial \varphi}{\partial \bar{z}} \right\rangle = -\lim_{\varepsilon \to 0} \int_{|z|>\varepsilon} \frac{1}{\pi z} \frac{\partial \varphi}{\partial \bar{z}}\, dx \quad \text{(Dominated Convergence)}.$$

$$= -\lim_{\varepsilon \to 0} \int_{|z|>\varepsilon} \frac{\partial}{\partial \bar{z}} \left(\frac{\varphi}{\pi z}\right) dx$$

$$\left(\int_{\partial D} \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\right) dx_1 dx_2 = \int_{\partial D} P dx_1 + Q dx_2, \quad \text{Green's Theorem}\right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{\varphi}{z}\, dz \quad (dz = dx_1 + i dx_2).$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_0^{2\pi} \varphi(\varepsilon\cos\theta, \varepsilon\sin\theta) \frac{i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}} d\theta = \frac{1}{2\pi} \cdot 2\pi\varphi(0) = \langle \delta_0, \varphi \rangle \quad \text{(by DCT again)}.$$

$\square$

Lemma 5.4. *The fundamental solution for the heat operator*

$$P(D) = \partial_t - \Delta_x \quad \text{on } \mathbb{R} \times \mathbb{R}^n$$

is

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{2t}\right), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Proof. Note that $P(D)E = 0$ on $t \geq \varepsilon > 0$. Now, for $\varphi \in \mathcal{D}(\mathbb{R}^n)$:

$$\begin{aligned}
\langle(\partial_t - \Delta_x)E, \varphi\rangle &= -\langle E, (\partial_t + \Delta_x)\varphi\rangle \\
&= \lim_{\varepsilon\to 0} -\int_\varepsilon^\infty dt \int_{\mathbb{R}^n} dx\, E(x,t)\,(\partial_t + \Delta_x)\,\varphi \\
&= \lim_{\varepsilon\to 0}\left(-\int_{\mathbb{R}^n} dx\, E\varphi\big|_{t=\varepsilon}^\infty + \int_\varepsilon^\infty dt \int_{\mathbb{R}^n} dx\,\varphi\,[\partial_t E - \Delta_x E]\right) \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} dx\,(4\pi\varepsilon)^{-n/2}\exp\left(-\frac{|x|^2}{2\varepsilon}\right)\varphi(x,\varepsilon), \qquad \frac{x}{\sqrt{2\varepsilon}} = y \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} dy\,(2\pi)^{-n/2}\left(\frac{1}{(4\pi\varepsilon)^{n/2}}\right)\exp(-|y|^2)\varphi(\sqrt{2\varepsilon}y) \\
&\overset{\text{(DCT)}}{=} \varphi(0)\cdot\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-|y|^2}dy = \varphi(0) = \langle\delta_0, \varphi\rangle
\end{aligned}$$

□

To approach more general P.D.O.s, one can attempt to construct fundamental solutions by considering the following

$$\langle E, \varphi\rangle = \frac{1}{(2\pi)^n}\int \frac{\hat{\varphi}(-\lambda)}{P(\lambda)}d\lambda$$

$$\langle P(D)E, \varphi\rangle = \langle E, P(-D)\varphi\rangle = \frac{1}{(2\pi)^n}\int \frac{P(\lambda)\hat{\varphi}(-\lambda)}{P(\lambda)}d\lambda = \varphi(0).$$

One would need to check that this is well defined. The main difficulty comes in finding a domain of integration (which was purposefully left blank), so as to avoin any 'singularities' of $1/P(\lambda)$. We will see the following construction due to Hörmander, namely, the Hörmander's Staircase which gives the construction of fundamental sol to general P.D.Os.

Lecture 10

In other words, we will try to construct a surface $\Sigma \subset \mathbb{C}^n$ s.t. $\Sigma \simeq \mathbb{R}^n$ (homotopic) and for which

$$\langle E, \varphi\rangle = \frac{1}{(2\pi)^n}\int_\Sigma \frac{\hat{\varphi}(-\lambda)}{P(\lambda)}\,d\lambda, \quad \varphi \in \mathcal{D}(\mathbb{R}^n)$$

defines an element of $D'(\mathbb{R}^n)$. Note then

$$\langle P(D)E, \varphi\rangle = \langle E, P(-D)\varphi\rangle$$

$$= \frac{1}{(2\pi)^n}\int_\Sigma \frac{P(\lambda)\,\hat{\varphi}(-\lambda)}{P(\lambda)}\,d\lambda = \frac{1}{(2\pi)^n}\int_\Sigma \hat{\varphi}(-\lambda)\,d\lambda = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{\varphi}(-\lambda)\,d\lambda = \varphi(0))$$

We willl use complex analysis, and the fact that $\Sigma \simeq \mathbb{R}^n$. Will call Σ "Hörmander's staircase."

Lemma 5.5. *For $\lambda \in \mathbb{R}^n$, write $\lambda = (\lambda', \lambda_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$ etc. For each $\lambda' \in \mathbb{R}^{n-1}$, if $\varphi \in \mathcal{E}(\mathbb{R}^n)$ then*

$$\mathbb{C} \ni z \mapsto \hat{\varphi}(\lambda', z) \text{ is holomorphic and}$$

there exists

$$\delta > 0 \text{ such that } |\hat{\varphi}(\lambda', z)| \lesssim_m (1+|\lambda|)^{-m}e^{\delta|\operatorname{Im} z|}, \quad m = 0, 1, 2, \ldots$$

Remark. *This establishes fast decay at horizontal infinity, so*

$$\int_{\mathbb{R}+i\eta}\hat{\varphi}(\lambda', z)\,dz = \int_{\mathbb{R}}\hat{\varphi}(\lambda', \lambda_n)\,d\lambda_n \quad \forall\eta \in \mathbb{R}. \quad \textit{(Cauchy's Theorem)}$$

Theorem 5.2. *For every non-zero $P(D)$ there exists a fundamental solution.*

Proof. By scaling and rotating coordinate axes, can fully assume $\varphi(\lambda)$ does the form:

$$P(\lambda', \lambda_n) = \lambda_n^M + \sum_{m=0}^{M-1} a_m(\lambda')\lambda_n^m$$

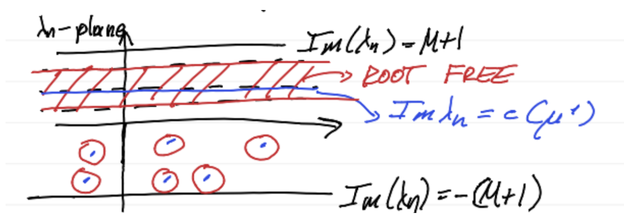
Let us <u>fix</u> $\mu' \in \mathbb{R}^{n-1}$. Then

$$P(\mu', \lambda_n) = \prod_{i=1}^{M} (\lambda_n - \tau_i(\mu')) \quad \text{where } \sum_i \tau_i(\mu') = 0$$

are the zeros of the polynomial $\lambda_n \mapsto P(\mu', \lambda_n)$.

<u>**Claim:**</u> There exists a horizontal line (see figure below) $\operatorname{Im}\lambda_n = c(\mu')$ in the complex λ_n-plane inside the strip $|\operatorname{Im}\lambda_n| \leq M+1$ such that

$$|\operatorname{Im}(\lambda_n - \zeta_i(\mu'))| > 1, \quad i = 1, 2, \ldots, M.$$



Indeed, $|\operatorname{Im}(\lambda_n)| \leq M+1$ consists of $M+1$ strips of width 2. By the pigeonhole principle, there exists a strip with <u>no roots</u> $\zeta_i(\mu')$ inside it. Choose horizontal line $\operatorname{Im}(\lambda_n) = c(\mu')$ to disect strip. Consequently, $|P(\mu', \lambda_n)| > 1$ on $\operatorname{Im}(\lambda_n) = c(\mu')$. Since the set of roots varies continuously with the coefficients of polynomial, deduce that same statement holds for μ' in a sufficiently small neighborhood of μ', call it $N(\mu')$. We thus get open $N(\mu')$ such that

$$|P(\lambda', \lambda_n)| > 1 \quad \text{for } \operatorname{Im}(\lambda_n) = c(\mu') \text{ and } \lambda' \in N(\mu').$$

Can do this for every $\mu' \in \mathbb{R}^{n-1}$, can generate an open cover of $\mathbb{R}^{n-1}$ with open sets of the form $N(\mu')$. By Heine–Borel can extract locally finite subcover

$$N_1 = N(\mu^1), \quad N_2 = N(\mu^2), \quad \ldots$$

We have that

$$|P(\lambda', \lambda_n)| > 1 \quad \text{on } \left\{\operatorname{Im}(\lambda_n) = c_i = c(\mu^i),\ \lambda' \in N_i\right\}.$$

Define open sets

$$\Delta_1 := N_1, \quad \Delta_i := N_i \setminus \left(\overline{N_1} \cup \cdots \cup \overline{N_{i-1}}\right).$$

Have $(\Delta_i)_i$ are open, disjoint,

$$\bigcup_{j=1}^{\infty} \overline{\Delta}_j = \mathbb{R}^{n-1} \quad \text{and} \quad |P(\lambda', \lambda_n)| > 1 \text{ on } \{\operatorname{Im}\lambda_n = c_i,\ \lambda' \in \Delta_i\}.$$

Define, for $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle E, \varphi \rangle := \frac{1}{(2\pi)^n} \sum_{i=1}^{\infty} \int_{\Delta_i} d\lambda' \left[\int_{\operatorname{Im} \lambda_n = c_i} d\lambda_n \; \frac{\hat{\varphi}(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)} \right]$$

Then

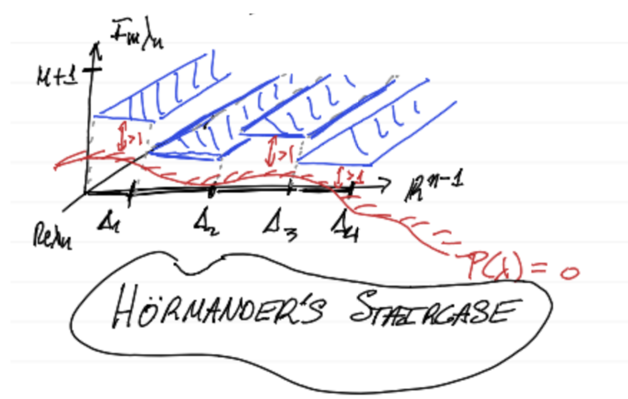
$$\begin{aligned} \langle P(D)E, \varphi \rangle &= \frac{1}{(2\pi)^n} \sum_{i=1}^{\infty} \int_{\Delta_i} d\lambda' \int_{\operatorname{Im} \lambda_n = c_i} \hat{\varphi}(\lambda', \lambda_n) d\lambda_n \\ &= (\text{by Cauchy} + \text{Lemma 5.5}) \\ &= \frac{1}{(2\pi)^n} \sum_{i=1}^{\infty} \int_{\Delta_i} d\lambda' \int_{\mathbb{R}} d\lambda_n \; \hat{\varphi}(-\lambda', -\lambda_n) \end{aligned}$$

$$\overset{(\text{no } i \geq 1 \text{ dependence})}{=} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} d\lambda' \int_{\mathbb{R}} \hat{\varphi}(-\lambda', -\lambda_n) \hat{\varphi}(-\lambda', -\lambda_n) \, d\lambda_n = \varphi(0) = \langle \delta_0, \varphi \rangle \, .$$

Still need to show that $E \in D'(\mathbb{R}^n)$, . This implies

$$P(D)E = \delta_0, \quad E \in D'(\mathbb{R}^n).$$

$\square$



Remark. *To obtain the continuity of zeros, let $\mathcal{U}_\epsilon$ are neighbourhoods, balls of radius ϵ centered around zeros of the polynomial $\lambda_n \mapsto P(\mu', \lambda_n)$, μ' fixed (see the figure below).*

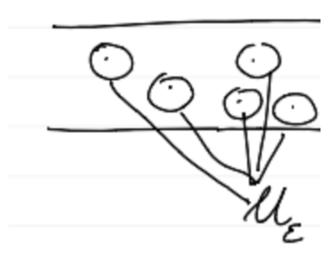


Figure 6: An illustration of ϵ ball nbhds centred around zeros of polynomial $\lambda_n \mapsto P(\mu', \lambda_n)$, μ' fixed.

Can count

$$\#\text{ of zeros inside } \mathcal{U}_\epsilon \text{ (arg-principle)} = \frac{1}{(2\pi i)} \int_{\partial \mathcal{U}_\epsilon} \frac{\partial P / \partial \lambda_n (\mu', \lambda_n)}{P(\mu', \lambda_n)} \, d\lambda_n$$

is continuous in a neighborhood of μ' and integer-valued, hence constant.

Lecture 11

5.3 *Structure Theorem for* $\mathcal{E}'(X)$

We know that if $f \in C(X)$, then $\partial^\alpha f \in D'(X)$ with

$$\langle \partial^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \int f \, \partial^\alpha \varphi \, dx \quad \forall \varphi \in \mathcal{D}(X).$$

Also, note that $\delta_0 = (xH)''$ in $D'(\mathbb{R})$. It is natural to ask: can all distributions be written in the form

$$u = \sum_\alpha \partial^\alpha f_\alpha \quad \text{in } D'(X) \text{ where } f_\alpha \in C(X)?$$

We will prove in case $\mathcal{E}'(X)$, but the result is true more generally.

Lemma 5.6. *If $u \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ then $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ can be identified with the smooth (analytic) function*

$$\lambda \mapsto \hat{u}(\lambda) = \langle u(x), e^{-i\lambda\cdot x} \rangle.$$

Also, there exists $M \geq 0$ such that $|\hat{u}(\lambda)| \lesssim \langle \lambda \rangle^M$.

Proof. Fix $\chi \in \mathcal{D}(\mathbb{R}^n)$ with

$$\chi = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| \geq 2 \end{cases}$$

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ set $\varphi_m = \chi\left(\frac{x}{m}\right)\varphi(x) \in \mathcal{D}(X)$.

<u>Claim:</u> $\varphi_m \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$. For arbitrary α, β:

$$\begin{aligned} \|\varphi_m - \varphi\|_{\alpha,\beta} &= \|x^\alpha D^\beta \left[\varphi(x)(1-\chi(x/m))\right]\|_\infty \\ &= \left\| x^\alpha \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D^\gamma \varphi \cdot D^{\beta-\gamma}(1-\chi(x/m)) \right\|_\infty . \end{aligned}$$

All derivatives of $x \mapsto (1-\chi(x/m))$ tend to zero uniformly, and

$$\left\| x^\alpha D^\beta \varphi \cdot (1-\chi(x/m)) \right\|_\infty \lesssim \sup_{|x|\geq m} |x^\alpha D^\beta \varphi| \cdot \left| \frac{x}{m} \right| \lesssim \frac{\|\varphi\|_{\alpha+1,\beta}}{m} \to 0$$

So, by sequential continuity of $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$,

$$\langle \hat{u}, \varphi \rangle = \lim_{m\to\infty} \langle \hat{u}, \varphi_m \rangle = \lim_{m\to\infty} \langle u, \hat{\varphi}_m \rangle = \lim_{m\to\infty} \left\langle u(x), \int e^{-i\lambda\cdot x} \varphi_m(\lambda) d\lambda \right\rangle$$

By Riemann sum argument in Lemma 2.5, (note each φ_m has compact support)

$$= \lim_{m\to\infty} \int \langle u(x), e^{-i\lambda\cdot x} \rangle \, \varphi_m(\lambda) \, d\lambda$$

Since power series for $x \mapsto e^{-i\lambda\cdot x}$ converges locally uniformly, can interchange $\langle \cdot, \cdot \rangle$ with infinite sum by sequential continuity. So $\hat{u}(\lambda) = \langle u(x), e^{-i\lambda\cdot x} \rangle$ is smooth, and by semi-norm estimate of $u \in \mathcal{E}'(\mathbb{R}^n)$, there exists $N \geq 0$ and compact $K \subset \mathbb{R}^n$ such that

$$|\hat{u}(\lambda)| = |\langle u(x), e^{-i\lambda\cdot x} \rangle| \leq C \sum_{|\alpha|\leq N} \sup_K |D_x^\alpha (e^{-i\lambda\cdot x})| \lesssim \langle \lambda \rangle^N, \quad \lambda \in \mathbb{R}^n$$

So by DCT,

$$\langle \hat{u}, \varphi \rangle = \int \hat{u}(\lambda) \varphi(\lambda) \, d\lambda,$$

i.e. $\hat{u}$ can be identified with $\lambda \mapsto \hat{u}(\lambda)$. $\square$

Theorem 5.3. *For each $u \in \mathcal{E}'(X)$, there exists a finite collection $\{f_\alpha\}$, $f_\alpha \in \mathcal{C}(X)$ and* $\operatorname{supp}(f_\alpha) \subset X$, *such that*

$$u = \sum_\alpha \partial^\alpha f_\alpha \quad \text{in } \mathcal{E}'(CX).$$

Proof. Fix $\rho \in \mathcal{D}(X)$ such that $\rho = 1$ on $\operatorname{supp}(u)$. Then for $\varphi \in \mathcal{E}(X)$, we have

$$\langle u, \varphi \rangle = \langle u, \rho\varphi \rangle + \langle u, (1-\rho)\varphi \rangle = \langle u, \rho\varphi \rangle.$$

By extending by zero, we can treat $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\rho\varphi \in \mathcal{D}(\mathbb{R}^n)$. For some $\psi \in \mathcal{S}(\mathbb{R}^n)$, can write $(\rho\varphi) = (\psi^\wedge)^\wedge$. In fact,

$$(2\pi)^n \psi^\vee = \rho\varphi \tag{$*$}$$

so we have

$$\langle u, \varphi \rangle = \langle u, (\psi^\wedge)^\wedge \rangle = \langle \hat{u}, \hat{\psi} \rangle.$$

Note that

$$\left((1-\Delta)^m \psi\right)^\wedge (\lambda) = \langle \lambda \rangle^{2m} \hat{\psi}(\lambda)$$

where $\Delta \equiv \sum_j \partial^2/\partial x_j^2$. Also,

$$\langle u, \varphi \rangle = \langle \langle \lambda \rangle^{-2m} \hat{u}, [(1-\Delta)^m \psi]^\wedge \rangle.$$

By choosing m sufficiently large and defining

$$f(x) = \frac{1}{(2\pi)^n} \int e^{i\lambda \cdot x} \underbrace{\langle \lambda \rangle^{2m} \hat{u}(\lambda)}_{\in L^1(\mathbb{R}^n)} d\lambda$$

we see that by DCT, $f \in C(\mathbb{R}^n)$. Also,

$$(2\pi)^n f^\vee(x) = (\langle \lambda \rangle^{-2m} \hat{u})^\wedge.$$

Hence

$$\begin{aligned} \langle u, \varphi \rangle = \langle \langle \lambda \rangle^{-2m} \hat{u}, [(1-\Delta)^m \psi]^\wedge \rangle &= \langle (2\pi)^n f^\vee, (1-\Delta)^m \psi \rangle \\ &= \langle f, (1-\Delta)^m ((2\pi)^n \psi) \rangle \end{aligned}$$

and from $(*)$, $= \langle f, (1-\Delta)^m (\rho\varphi) \rangle$. Can expand derivatives, by Leibniz

$$\langle u, \varphi \rangle = \langle f, \sum_\alpha (-1)^{|\alpha|} \partial^\alpha (\rho \partial^\alpha \varphi) \rangle$$

where $\rho \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}(\rho\varphi) \subset X$. So

$$\langle u, \varphi \rangle = \langle \sum_\alpha \partial^\alpha (f_\alpha \rho), \varphi \rangle$$

$$\equiv \langle \sum_\alpha \partial^\alpha f_\alpha, \varphi \rangle$$

where $f_\alpha \in C(X)$ and $\operatorname{supp}(f_\alpha) \subset X$. $\square$

Example. Know that $\delta_0 = (xH)''$. Now, if

- ($\varphi \equiv 1$ on some neighbourhood of 0) $\varphi(0) = 1$, and φ is in $\mathcal{D}(\mathbb{R})$, then $\delta_0 = \varphi\delta_0$.
- $\langle \delta_0, f \rangle = \langle (xH)'', f \rangle$, for f in $\mathcal{D}(\mathbb{R})$.

Therefore, using

$$\varphi'(xH)' = (\varphi' xH)' - \varphi'' xH = -\varphi''(xH) + 2(\varphi' xH)' + \varphi(xH)''$$

which implies

$$(\varphi xH)'' = \varphi''(xH) + 2\varphi'(xH)' + \varphi(xH)'' ,$$

we can compute for any $f \in \mathcal{D}(\mathbb{R})$

$$\langle \delta_0, f\rangle = \langle (\varphi xH)'' + \varphi''(xH) - 2(\varphi'(xH))', f\rangle.$$

Thus,

$$\delta_0 = \underbrace{(\varphi xH)''}_{\in C_c(\mathbb{R})} + \underbrace{\varphi''(xH)}_{\in C_c(\mathbb{R})} - \underbrace{2(\varphi'(xH))'}_{\in C_c(\mathbb{R})}$$

Since $\varphi' = 0$ on some neighbourhood of 0 and $(xH)' = xH' + H = H$ as a distribution.

Lecture 12

Note that the existence of fundamental solution theorem (Hörmander Staircase) is called "Malgrange-Ehrenpreis Theorem."

5.4 Paley-Wiener-Schwartz Theorem:

Have seen that if $u \in \mathcal{E}'(\mathbb{R}^n)$, then $\hat{u}$ can be identified with $\lambda \mapsto \hat{u}(\lambda) = \langle u(x), e^{-i\lambda\cdot x}\rangle$. Take complex analytic extension to $z \in \mathbb{C}^n$, call

$$\hat{u}(z) := \langle u(x), e^{-iz\cdot x}\rangle$$

the Fourier-Laplace transform of $u \in \mathcal{E}'(\mathbb{R}^n)$. Know that there exist $C, N > 0$, $K \subset \mathbb{R}^n$ compact

$$|\hat{u}(z)| = |\langle u(x), e^{-iz\cdot x}\rangle| \le C \cdot \sum_{|\alpha|\le N} \sup_K |\partial_x^\alpha (e^{-iz\cdot x})|.$$

Also $z \mapsto \hat{u}(z)$ is entire (power series of $x \mapsto e^{-iz\cdot x}$ converges locally uniformly, so can apply n-termwise (sequential continuity of u) to get power series for $\hat{u}(z)$).

Lemma 5.7. *If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\operatorname{supp}(u) \subset \overline{B}_\delta = \{x \in \mathbb{R}^n : |x| \le \delta\}$, then $\exists C, N \ge 0$ such that*

$$|\hat{u}(z)| \le C \cdot (1 + |z|)^N e^{\delta\cdot|\operatorname{Im} z|}, \quad z \in \mathbb{C}.$$

Proof. Fix $\psi \in C^\infty(\mathbb{R})$ such that $\psi(\tau) = 1$ on $\tau = -\frac{1}{2}$, $\psi(\tau) = 0$ on $\tau \le -1$. For $\varepsilon > 0$, define $\psi_\varepsilon(x) = \psi(\varepsilon(|x| - \delta))$, $x \in \mathbb{R}^n$. Then $\psi_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ and

$$\begin{cases} \psi_\varepsilon = 1 & \text{on } |x| \le \delta + \frac{1}{2\varepsilon} \\ \psi_\varepsilon = 0 & \text{on } |x| \ge \delta + \frac{1}{\varepsilon}\,. \end{cases}$$

Note that $\psi_\varepsilon = 1$ on $\operatorname{supp}(u)$. Since $u \in \mathcal{E}'(\mathbb{R}^n)$, $\exists C, N \ge 0$ such that

$$|\hat{u}(z)| = |\langle u(x), \psi_\varepsilon(x)\cdot e^{-iz\cdot x}\rangle| \le C \cdot \sum_{|\alpha|\le N} \sup_{|x|\le\delta+\frac{1}{\varepsilon}} \left|\partial_x^\alpha \left[\psi_\varepsilon(x) e^{-iz\cdot x}\right]\right|.$$

Also observe that $\partial^\beta \psi_\varepsilon \lesssim \varepsilon^{|\beta|}$ and

$$\partial^\gamma e^{-iz\cdot x} \lesssim |z|^{|\gamma|} e^{(\delta+\frac{1}{\varepsilon})\cdot \operatorname{Im} z|}, \quad \text{on } \operatorname{supp}\psi_\varepsilon$$

which implies

$$|\hat{u}(z)| \lesssim \sum_{|\beta|+|\gamma|\le N} \varepsilon^{|\beta|} |z|^{|\gamma|} e^{(\delta+\frac{1}{\varepsilon})\cdot \operatorname{Im} z|}$$

Taking $\varepsilon = \frac{1}{|z|+1}$ gives the result. ∎

Paley-Wiener-Schwartz is about converse in the sense that if $z \mapsto U(z)$ is entire function of $z \in \mathbb{C}^n$ and $|U(z)| \lesssim C(1+|z|)^N e^{\delta \cdot |\operatorname{Im} z|}$, is it the case that $U = \hat{u}$ where $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\operatorname{supp}(u) \subset \overline{B}_\delta$? Yes.

Theorem 5.4 (P-W-S). *(A) If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp}(\varphi) \subset \overline{B}_\delta$, then $z \mapsto \hat{\varphi}(z)$ is entire and*

$$(\dagger) \quad |\hat{\varphi}(z)| \lesssim_N (1+|z|)^{-N} e^{\delta |\operatorname{Im}(z)|}, \quad z \in \mathbb{C}, N = 0, 1, 2, \ldots$$

Conversely, if $z \mapsto \Phi(z)$ is entire and satisfies (†), then $\Phi = \hat{\varphi}$ for some $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\operatorname{supp}(\varphi) \subset \overline{B}_\delta$.

(B) If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\operatorname{supp}(u) \subset \overline{B}_\delta$, then $z \mapsto \hat{u}(z)$ is entire and $\exists C, N \geq 0$ such that

$$(\ddagger) \quad |\hat{u}(z)| \lesssim (1+|z|)^N e^{\delta |\operatorname{Im}(z)|}, \quad z \in \mathbb{C}^n.$$

Conversely, if $z \mapsto U(z)$ is entire and satisfies (‡), then $U = \hat{u}$ for some $u \in \mathcal{E}'(\mathbb{R}^n)$ with $\operatorname{supp}(u) \subset \overline{B}_\delta$.

Proof. (A: Clear that $z \mapsto \hat{\varphi}(z) = \int e^{-iz\cdot x}\varphi(x)\,dx$ is entire (e.g., $\partial^\alpha \hat{\varphi}/\partial z^\alpha = 0\ \forall \alpha, \forall z \in \mathbb{C}^n$, or apply Morera + Fubini, or expand $z \mapsto e^{-iz\cdot x}$ and integrate termwise). For the estimate (†) (c.f. Lemma 5.5), for arbitrary

$$|\hat{z}^\alpha \hat{\varphi}(z)| = \left| \int z^\alpha e^{-iz\cdot x} \varphi(x)\,dx \right| = \left| \int \left(e^{-iz\cdot x} D^\alpha \varphi(x)\right) dx \right|.$$

Since $\left|e^{-iz\cdot x}\right| = \left|e^{\operatorname{Im} z \cdot x}\right| \leq C e^{\delta |\operatorname{Im} z|}$ on $\operatorname{supp}\varphi$, estimate (†) now follows.

For converse, given entire $z \mapsto \Phi(z)$ obeying (†), define

$$\varphi(x) = \frac{1}{(2\pi)^n} \int e^{i\lambda\cdot x} \Phi(\lambda)\,d\lambda.$$

Follows ((DCT) + (†)) that $\varphi \in C^\infty(\mathbb{R}^n)$. By Cauchy's theorem, entirety of $z \mapsto \Phi(z)$ and estimate (†), have for arbitrary $\alpha \in \mathbb{R}^n$.

$$|\varphi(x)| = \frac{1}{(2\pi)^n} \left| \int e^{i(\lambda + i\eta)\cdot x} \Phi(\lambda + i\eta)\,d\lambda \right|.$$

The above is justified because of rapid horizontal decay of Φ. So by (†):

$$|\varphi(x)| \lesssim_N \int e^{-\eta\cdot x} (1 + |\lambda + i\eta|)^{-N} e^{\delta|\lambda|}\,d\lambda$$

$$\lesssim e^{\delta|\eta \cdot x|}.$$

Take $\eta = \frac{x}{|x|}t$, $t > 0$,

$$= e^{-t(C|x| - \delta)}.$$

If $|x| > \delta$, take $t \to \infty$ to get $\varphi = 0$, i.e. $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp}(\varphi) \subset \overline{B}_\delta$. Taking Fourier transform shows that $\Phi = \hat{\varphi}$.

(B): (⇒): already established (Lemma 5.7).

(⇐): Let $z \mapsto U(z)$ be an entire function satisfying (‡). Then $|_{\mathbb{R}^n} \in \mathcal{S}'(\mathbb{R}^n)$ since $|U(\lambda)| \lesssim \langle \lambda \rangle^N$. Since $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is an isomorphism, $\exists u \in \mathcal{S}'(\mathbb{R}^n)$ s.t. $\hat{u} = U$. For $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\int \varphi\,dx = 1 \quad \text{and} \quad \operatorname{supp}(\varphi) \subset B_1.$$

Set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then $\varphi_\varepsilon \to \delta$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{supp}(\varphi_\varepsilon) \subset B_\varepsilon$. Hence $\hat{\varphi}_\varepsilon \to 1$ in $\mathcal{S}'(\mathbb{R}^n)$.

Define $\hat{u}_\varepsilon(z) = \hat{\varphi}_\varepsilon(z) U(z)$. By (†) for $\hat{\varphi}_\varepsilon$ and (‡) for u, have $\hat{u}_\varepsilon(z) \lesssim_N (1+|z|)^{-N} e^{(\delta+\varepsilon)|\operatorname{Im} z|}$, $N = 0, 1, 2, 3, \ldots$. Hence, $\hat{u}_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp}(\hat{u}_\varepsilon) \subset \overline{B}_{\delta+\varepsilon}$. As $\varepsilon \to 0$, get $\hat{u}_\varepsilon \to \hat{u}$ in $\mathcal{S}'(\mathbb{R}^n)$. ∎

Lecture 13

6 Oscillatory Integrals.

In this section, we would like to make sense of

$$\int e^{i\lambda x}\,dx$$

and more generally, objects of the form

$$\int e^{i\Phi(x,\theta)}a(x,\theta)\,d\theta$$

where $x \in X$, $\theta \in \mathbb{R}^k$. Call real-valued $\Phi \in C^\infty(X \times \mathbb{R}^k \setminus \{0\})$ the phase function and a will belong to a class of functions called symbols. **NOTE:** latter integral will not be well-defined classically since we will allow symbols that get large as $|\theta| \to \infty$.

Lemma 6.1 (Riemann–Lebesgue Lemma). *If $f \in L^1(\mathbb{R})$, then $|\hat{f}(\lambda)| \to 0$ as $|\lambda| \to \infty$.*

Proof. Assume $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$.

$$\begin{aligned}\hat{f}(\lambda) &= \frac{1}{2}\int \left[e^{-i\lambda x}f(x) + e^{-i\lambda x}f(x)\right]dx \quad \text{where } x = x + \pi/\lambda \\ &= \frac{1}{2}\int e^{-i\lambda x}f(x) + e^{-i\lambda x}e^{-i\pi}f(x+\pi/\lambda)\,dx \\ &= \frac{1}{2}\int e^{-i\lambda x}\left[f(x) - f(x+\pi/\lambda)\right]dx\end{aligned}$$

Since $f \in L^1(\mathbb{R})$, given $\varepsilon > 0$, there exists R s.t.

$$\frac{1}{2}\int_{|x|>R}|f(x) - f(x+\pi/\lambda)|\,dx < \varepsilon/4$$

Since $f \in C(\mathbb{R})$, choose R sufficiently large so that

$$\left|\int_{|x|<R} e^{-i\lambda x}\left[f(x) - f(x+\pi/\lambda)\right]dx\right| < \varepsilon/4 \quad \text{(by DCT)}$$

i.e., $|\hat{f}(\lambda)| < \varepsilon/2$ for $|\lambda|$ sufficiently large. Note that $L^1(\mathbb{R}) \cap C(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, so given $g \in L^1(\mathbb{R})$, fix $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ such that $\|f - g\|_{L^1} \leq \varepsilon/2$ So,

$$|\hat{g}(\lambda)| = |\hat{g}(\lambda) - \hat{f}(\lambda) + \hat{f}(\lambda)| \leq |\hat{g} - \hat{f}| + |\hat{f}(\lambda)| < \varepsilon \text{ for } |\lambda| \text{ sufficiently large.}$$

$\square$

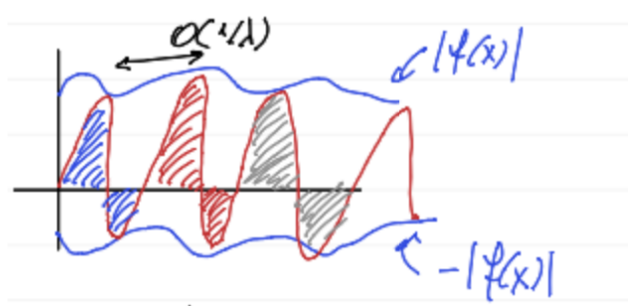


Figure 7: "More oscillation in integrand $\Rightarrow$ more decay of integral"($\int f(x)e^{-i\lambda x}\,dx$) because *oscillation*gives *cancellation*.

More generally, if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\Phi \in C^\infty(\mathbb{R})$, expect

$$\int \varphi(\theta) \cdot e^{i\lambda\Phi(\theta)}\, d\theta$$

to decay as $|\lambda| \to \infty$. E.g., if $\Phi' \neq 0$, then the operator

$$L = \frac{1}{i\lambda\Phi'(\theta)} \frac{d}{d\theta}$$

is well-defined since $|\Phi'(\theta)| \gtrsim 1$ on $\operatorname{supp}\varphi$. Note that

$$Le^{i\lambda\Phi} = e^{i\lambda\Phi}, \quad \text{so}$$

$$\int \varphi(\theta) e^{i\lambda\Phi(\theta)}\, d\theta = \int \varphi(\theta) L e^{i\lambda\Phi(\theta)}\, d\theta = \int \left[L^t \varphi(\theta)\right] e^{i\lambda\Phi(\theta)}\, d\theta$$

where

$$L^t = -\frac{1}{i\lambda}\frac{d}{d\theta}\left[\frac{1}{\Phi'}\cdot\right] \quad \text{“formal adjoint of } L\text{”.}$$

Can do this as many times as we please to get

$$\left|\int e^{i\lambda\Phi}\varphi\, d\theta\right| = \left|\int (L^t)^N \varphi(\theta) e^{i\lambda\Phi}\, d\theta\right| \lesssim_N \langle\lambda\rangle^{-N}, \quad N = 0, 1, 2, \ldots$$

Expect to get dominant contribution from pts at which $\Phi' = 0$ (stationary pts).

Lemma 6.2 (Stationary phse lemma)**.** *Let $\Phi \in C^\infty(\mathbb{R})$ such that $\Phi' \neq 0$ on $\mathbb{R} \setminus \{0\}$ and $\Phi(0) = \Phi'(0) = 0$, $\Phi''(0) \neq 0$. Then for $\varphi \in \mathcal{D}(\mathbb{R})$:*

$$\left|\int e^{i\lambda\Phi(\theta)}\varphi(\theta)\, d\theta\right| \lesssim \frac{1}{|\lambda|^{1/2}}, \quad |\lambda| \to \infty.$$

Proof. Fix $\rho \in \mathcal{D}(\mathbb{R})$ such that $\rho = 1$ on $|\theta| < 1$ and $\rho = 0$ on $|\theta| > 2$. Write

$$\int e^{i\lambda\Phi}\varphi(\theta)\, d\theta = \int e^{i\lambda\Phi}\rho(\theta)\varphi(\theta/\delta)\, d\theta \qquad (\text{call this } I_1) + \int e^{i\lambda\Phi}(1-\rho(\theta/\delta))\varphi(\theta)\, d\theta \qquad (\delta > 0, \text{ call this } I_2)\,.$$

Since $\rho(\theta/\delta) = 0$ on $|\theta| > 2\delta$, we get simple estimate $|I_1| \lesssim \delta$. Note $(1 - \rho(\delta\theta)) = 0$ on $|\theta| \leq \delta$. So we're integrating over $|\theta| \geq \delta$, so

$$L = \frac{1}{i\lambda\Phi'(\theta)}\frac{d}{d\theta}$$

is well-defined and $Le^{i\lambda\Phi} = e^{i\lambda\Phi}$. So

$$I_2(\lambda) = \int e^{i\lambda\Phi}(L^t)^2\left[(1-\rho(\delta\theta))\rho(\theta)\right] d\theta$$

where

$$L^t = -\frac{1}{i\lambda}\frac{d}{d\theta}\left[\frac{1}{\Phi'}\cdot\right]$$

Note if

$$\mathcal{P} = \frac{d}{d\theta}\left[a\cdot\right], \quad \mathcal{P}^2 = \frac{d}{d\theta}\left[a\frac{d}{d\theta}(a\cdot)\right]$$

$$= a^2\frac{d^2}{d\theta^2} + 3a\frac{da}{d\theta}\frac{d}{d\theta} + (aa')'$$

$$(L^t)^2 = -\frac{1}{\lambda^2}\left[\frac{1}{(\Phi')^2}\frac{d^2}{d\theta^2} - 3\frac{\Phi''}{(\Phi')^3}\frac{d}{d\theta} - \left(\frac{\Phi''}{(\Phi')^3}\right)'\right]$$

$$\left(\text{since}\frac{\Phi'''}{(\Phi')^3}+3\frac{(\Phi'')^2}{(\Phi')^4}\right).$$

Note that

$$\Phi'(\theta)-\Phi'(0)=\int_0^\theta \Phi''(t)dt=\theta\int_0^1 \Phi''(\theta t)dt$$

i.e.,

$$\frac{\Phi'(\theta)}{\theta}=\int_0^1 \Phi''(\theta t)dt$$

LHS $\neq 0$ at $\theta \neq 0$ and $\Rightarrow \Phi''(c\theta)\neq 0$ as $\theta \to 0$. i.e. $|\Phi'(\theta)| \gtrsim |\theta|$ on $\operatorname{supp}\varphi$. So,

$$(L^t)^2\left[\rho(\theta)(1-\rho(\theta\delta))\right]=O\left(\frac{1}{\lambda^2\theta^2\delta^2}\right)+O\left(\frac{1}{\lambda\theta\delta^3}\right)+O\left(\frac{1}{\lambda^2\delta^4}\right).$$

Integrating over $|\theta|>\delta$ gives

$$|I_2(\lambda)|=O\left(\frac{1}{\lambda^2\delta^2}\right)+O\left(\frac{1}{\lambda\delta^3}\right)+O\left(\frac{1}{\lambda^2\delta^3}\right).$$

Matching with $I_2(\lambda)=O(\delta)$, want

$$\delta^2=\frac{1}{\lambda\delta^3}\Rightarrow\delta=\frac{1}{|\lambda|^{1/2}}.$$

□

Remark. *This estimate is sharp, for example, consider*

$$\int e^{i\lambda\theta^2}\rho(\theta)\,d\theta,\quad \theta=\varphi/\sqrt{\lambda}$$

$$=\frac{1}{\sqrt{\lambda}}\int e^{i\phi^2}\varphi\left(\frac{\theta}{\sqrt{\lambda}}\right)d\theta\sim\frac{const}{\sqrt{\lambda}}+\textit{lower order terms as }\lambda\to\infty.$$

Using this result, we expect

$$u(x)=\int e^{i\lambda\Phi(x,\theta)}a(x,\theta)\,d\theta$$

to be "badly behaved" at $x_0\in X$, for which $\nabla_\theta\Phi(x_0,\theta)=0$ for some $\theta\in\mathbb{R}^n$. We will show that

$$\operatorname{sing\,supp}(u)\subset\{x\in X:\nabla_\theta\Phi(x,\theta)=0\text{ for some }\theta\in\mathbb{R}^n\setminus\{0\}\}.$$

Lecture 14

Definition 6.1. *Let $X\subset\mathbb{R}^k$ open. A smooth function $a: X\times\mathbb{R}^n\to\mathbb{C}$ is called a symbol of order $N\in\mathbb{R}$ if for each compact $K\subset X$,*

$$|\partial_x^\alpha\partial_\theta^\beta a(x,\theta)|\lesssim_{K,\alpha,\beta}\langle\theta\rangle^{N-|\beta|}\quad\textit{for all }(x,\theta)\in K\times\mathbb{R}^n.$$

Call the space of all such symbols $\mathrm{Sym}(X,\mathbb{R}^n;N)$.

For example, if $\{\varphi_\alpha\}\in C^\infty(X)$, then

$$a(x,\theta)=\sum_{|\alpha|\leq N}\varphi_\alpha(x)\cdot\theta^\alpha\in\mathrm{Sym}(X,\mathbb{R}^n;N).$$

Only care about behaviour of symbols for large $|\theta|$, since for any compact $L\subset\mathbb{R}^n$, if $a\in C^\infty(X\times\mathbb{R}^n)$, then

$$(x,\theta)\mapsto\frac{D_x^\alpha D\theta^\beta(a(x,\theta))}{\langle\theta\rangle^{N-|\beta|}}\quad\text{compact in }X$$

will always be bounded on $K\times L$ (K compact in X).

Lemma 6.3.
- *If $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$, then $\partial_x^\alpha \partial_\theta^\beta a \in \mathrm{Sym}(X, \mathbb{R}^k; N - |\beta|)$.*
- *If $a_i \in \mathrm{Sym}(X, \mathbb{R}^k; N_i),\ i = 1, 2$ then $a_1 a_2 \in \mathrm{Sym}(X, \mathbb{R}^k; N_1 + N_2)$.*

Proof. Obviously $D_x^\alpha D_\theta^\beta a(x, \theta)$ is smooth on $X \times \mathbb{R}^k$. For $K \subset X$ compact,

$$|D_x^{\alpha'} D_\theta^{\beta'} [D_x^\alpha D_\theta^\beta a]| = |D_x^{\alpha+\alpha'} D_\theta^{\beta+\beta'} a| \lesssim_{K,\alpha',\beta'} \langle\theta\rangle^{N-|\beta|-|\beta'|} \Rightarrow D_x^\alpha D_\theta^\beta a \in \mathrm{Sym}(X, \mathbb{R}^k; N - |\beta|).$$

Again for $K \subset X$ compact,

$$\begin{aligned} |D_x^\alpha D_\theta^\beta (a_1 a_2)| &= \left| \sum_{\alpha' \le \alpha} \sum_{\beta' \le \beta} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} (D_x^{\alpha'} D_\theta^{\beta'} a_1)(D_x^{\alpha-\alpha'} D_\theta^{\beta-\beta'} a_2) \right| \\ &\lesssim_{K,\alpha,\beta} \sum_{\alpha' \le \alpha} \sum_{\beta' \le \beta} \langle\theta\rangle^{N_1 - |\beta'|} \langle\theta\rangle^{N_2 - |\beta-\beta'|} \lesssim_{K,\alpha,\beta} \langle\theta\rangle^{N_1+N_2-|\beta|}, \end{aligned}$$

whence $a_1 a_2 \in \mathrm{Sym}(X, \mathbb{R}^k; N_1 + N_2)$. □

Lemma 6.4. *If $a \in C^\infty(X \times \mathbb{R}^n)$ and a is positively homogeneous of degree N (in θ) for $|\theta|$ sufficiently large, then*

$$a \in \mathrm{Sym}(X, \mathbb{R}^n; N).$$

Proof. For $|\theta|$ sufficiently large, $a(x, t\theta) = t^N a(x, \theta)$ for $t > 0$. So for $|\theta|$ large

$$t^N D_x^\alpha D_\theta^\beta [a(x, \theta)] = D_x^\alpha D_\theta^\beta [a(x, t\theta)] = t^{|\beta|} (D_x^\alpha D_\theta^\beta a)(x, t\theta).$$

I.e. $D_x^\alpha D_\theta^\beta a$ is positively homogeneous of degree $N - |\beta|$, for $|\theta|$ large. For $K \subset X$ compact,

$$\begin{aligned} |D_x^\alpha D_\theta^\beta a(x, \theta)| &= |D_x^\alpha D_\theta^\beta a(x, |\theta|\omega)|, \quad \omega = \theta/|\theta| \in \mathbb{S}^{k-1} \\ &= |\theta|^{N-|\beta|} |D_x^\alpha D_\theta^\beta a(x, \omega)| \lesssim_{K,\alpha,\beta} \langle\theta\rangle^{N-|\beta|} \quad (t = 1/|\theta|,\ \theta \text{ large}) \end{aligned}$$

□

Definition 6.2. $\Phi : X \times \mathbb{R}^k \to \mathbb{R}$ *is called a* <u>*phase function*</u> *if:*

i) *Φ is continuous on $X \times \mathbb{R}^k$ and positively homogeneous of degree 1 in θ, i.e. $\Phi(x, t\theta) = t\Phi(x, \theta),\ t > 0$,*

ii) *Φ is smooth on $X \times (\mathbb{R}^k \setminus \{0\})$,*

iii) *$d\Phi = \nabla_\theta \Phi \cdot d\theta + \nabla_x \Phi \cdot dx \neq 0$ on $X \times (\mathbb{R}^k \setminus \{0\})$.*

Want to make sense of

$$D_\varphi^\alpha(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\theta} \theta^\alpha \, d\theta$$

i.e. $\Phi(x, \theta) = x \cdot \theta$, $a(x, \theta) = (2\pi)^{-n} \theta^\alpha \in \mathrm{Sym}(\mathbb{R}^n, \mathbb{R}^n; |\alpha|)$ and more generally, if $x \in X$

$$\int e^{i \overbrace{\Phi(x,\theta)}^{\text{phase function}}} \underbrace{a(x, \theta)}_{\in \mathrm{Sym}(X, \mathbb{R}^k; N)} \, d\theta\,.$$

Could define a linear form $I_\Phi(a) \colon \mathcal{D}(X) \to \mathbb{C}$ by

$$\langle I_\Phi(a), \varphi \rangle = \iint e^{i\Phi(x,\theta)} a(x, \theta) \varphi(x) \, dx \, d\theta$$

But too cumbersome because of lack of absolute integrability of the double integral. Instead, fix $\chi \in D(\mathbb{R}^n)$ s.t. $\chi = 1$ on $|\theta| \le 1$ and set

$$I_\Phi^\varepsilon(a)(x) := \int e^{i\Phi(x,\theta)} a(x, \theta) \chi(\varepsilon\theta) \, d\theta$$

Then define $I_\Phi^\varepsilon(a) = \lim_{\varepsilon \searrow 0} I_\Phi^\varepsilon(a)$ in $D'(X)$.

Lemma 6.5. *If L_Φ has the form*

$$L = \sum_{j=1}^{n} a_j(x,\theta)\frac{\partial}{\partial\theta_j} + \sum_{j=1}^{n} b_j(x,\theta)\frac{\partial}{\partial x_j} + c(x,\theta)$$

with $a_j \in \mathrm{Sym}(X,\mathbb{R}^k;0)$, $b_j, c \in \mathrm{Sym}(X,\mathbb{R}^k;-1)$ then L^t has same form.

Proof.

$$L^t = -\sum_j \frac{\partial}{\partial\theta_j}(a_j\cdot) - \sum_j \frac{\partial}{\partial x_j}(b_j\cdot) + c$$

$$= \sum \tilde{a}_j \frac{\partial}{\partial\theta_j} + \tilde{b}_j \frac{\partial}{\partial x_j} + \tilde{c} \quad \text{where } \tilde{a}_j \in \mathrm{Sym}(X,\mathbb{R}^k;0), \quad \tilde{b}_j \in \mathrm{Sym}(X,\mathbb{R}^k;-1)$$

$$-\sum \frac{\partial a_j}{\partial\theta_j} - \sum \frac{\partial b_j}{\partial x_j} + c \in \mathrm{Sym}(X,\mathbb{R}^k;-1) \quad \text{(use symbol lemma).}$$

□

If we could find such an L for which $Le^{i\Phi} = e^{i\Phi}$ then

$$\langle I^\varepsilon_\Phi(a), \varphi, =\rangle \iint (L^N e^{i\Phi}) a(x,\theta)\chi(\varepsilon\theta)\varphi(x)\,dx\,d\theta$$

$$= \iint e^{i\Phi}(L^t)^N \left[a(x,\theta)\chi(\varepsilon\theta)\varphi(x)\right]\,dx\,d\theta.$$

From form of L, (L^t) should lower order of $[a(x,\theta)\chi(\varepsilon\theta)\varphi(x)]$ by 1 each time.

Lecture 15

Lemma 6.6. *There exists a differential operator L of the form*

$$L = (\cdots)$$

such that

$$L^t e^{i\Phi} = e^{i\Phi},$$

where Φ is any (fixed) phase function.

Proof. Clearly,

$$\frac{\partial}{\partial\theta_j}e^{i\Phi} = i\frac{\partial\Phi}{\partial\theta_j}e^{i\Phi}, \quad \frac{\partial}{\partial x_j}e^{i\Phi} = i\frac{\partial\Phi}{\partial x_j}e^{i\Phi}$$

so

$$\left(\sum_{j=1}^{n} -i|\theta|^2\frac{\partial\Phi}{\partial\theta_j}\frac{\partial}{\partial\theta_j} + \sum_{j=1}^{n} i\frac{\partial\Phi}{\partial x_j}\frac{\partial}{\partial x_j}\right)e^{i\Phi} = \left(|\theta|^2|\nabla_\theta\Phi|^2 + |\nabla_x\Phi|^2\right)e^{i\Phi}.$$

Note, since $\Phi(x,t\theta) = t\Phi(x,\theta),\ t > 0$

$$\Rightarrow \frac{\partial}{\partial x_j}\Phi(x,t\theta) = \frac{\partial}{\partial x_j}\left(t\Phi(x,\theta)\right) = \frac{\partial\Phi}{\partial x_j}(x,t\theta)$$

So $\frac{\partial\Phi}{\partial x_j} + \nabla_\theta$ homogeneous of deg 1.

$$t\frac{\partial}{\partial\theta_j}\Phi(x,\theta) = \frac{\partial\Phi}{\partial\theta_j}(x,t\theta) \Rightarrow \frac{\partial\Phi}{\partial\theta_j} \text{ is } +\text{vely homogeneous of deg 0.}$$

Now define

$$P = \sum_{j=1}^{n} -\frac{i}{|\theta|^2|\nabla_\theta\Phi|^2 + |\nabla_x\Phi|^2}\frac{\partial\Phi}{\partial\theta_j}\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_j} + \sum_{j=1}^{n} -\frac{i\partial\Phi}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j},$$

from which we easily verify $Pe^{i\Phi} = e^{i\Phi}$. We see that the coefficients above are positively homogeneous of deg 0, -1 respectively. Note that denominators can vanish at $\theta = 0$. Fix $\rho \in D(\mathbb{R}^k)$, $\rho = 1$ on $|\theta| < 1$ and $\rho = 0$ on $|\theta| > 2$. Define

$$L^t = (1-\rho)P + \rho$$

Then

$$L^t e^{i\Phi} = (1-\rho)e^{i\Phi} + \rho e^{i\Phi} = e^{i\Phi}$$

and by Lemmas 6.3 and 6.5

$$L = \sum_{j=1}^{k} a_j \frac{D}{D\theta_j} + \sum_{j=1}^{n} b_j \frac{D}{Dx_j} + c \quad (:= (L^+)^+)$$

$$a_j \in \mathrm{Sym}(X, \mathbb{R}^k; 0), \quad b_j \in \mathrm{Sym}(X, \mathbb{R}^k; -1)$$

Note that

$$L : \mathrm{Sym}(X, \mathbb{R}^k; N) \to \mathrm{Sym}(X, \mathbb{R}^k; N-1)$$

Also, more generally,

$$L^M \left[a(x,\theta)\varphi(x)\right] = \sum_{|\alpha| \le M} a_\alpha(x,\theta) D^\alpha \varphi \quad \text{where } a_\alpha \in \mathrm{Sym}(X, \mathbb{R}^k; N-M)$$

which can be seen by inducting on M.

□

Theorem 6.1. *If Φ is a phase function and $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then*

$$I_\Phi(a) = \lim_{\varepsilon \searrow 0} I_\Phi^\varepsilon(a) \in D'(X)$$

and

$$\mathrm{ord}(I_\Phi(a)) \le \mathbb{N} + k + 1.$$

Proof. For each $\varepsilon > 0$,

$$I_\Phi^\varepsilon(a) = \int e^{i\Phi(x,\theta)} a(x,\theta)\chi(\varepsilon\theta)\, d\theta \qquad \big(\chi \in D(\mathbb{R}^k),\ \chi = 1 \text{ on } |\theta| < 1, \chi = 0 \text{ on } |\theta| > 2\big)$$

So for $\varphi \in \mathcal{D}(X)$

$$\langle I_\Phi^\varepsilon(a), \varphi, =\rangle \iint e^{i\Phi(x,\theta)} a(x,\theta)\chi(\varepsilon\theta)\varphi(x)\, dx\, d\theta$$

$$= \iint \left[(L^t)^M e^{i\Phi}\right] a(x,\theta)\chi(\varepsilon\theta)\varphi(x)\, dx\, d\theta \qquad (\ L^t \text{ as in Lemma 6.6})$$

$$= \iint e^{i\Phi} L^M \left[a(x,\theta)\chi(\varepsilon\theta)\varphi(x)\right]\, dx\, d\theta$$

Note that, since $\chi \in \mathcal{D}(\mathbb{R}^k)$,

$$\left|\left(\frac{\partial}{\partial\theta}\right)^\alpha \chi(\varepsilon\theta)\right| = \varepsilon^{|\alpha|} \left|(\partial^\alpha \chi)(\varepsilon\theta)\right| \lesssim_\alpha \varepsilon^{|\alpha|} \langle \varepsilon\theta \rangle^{-|\alpha|}$$

$$= C_\alpha \cdot \frac{\varepsilon^{|\alpha|}}{(1 + \varepsilon^2|\theta|^2)^{|\alpha|/2}} = C_\alpha \cdot \frac{1}{\left(\frac{1}{\varepsilon^2} + |\theta|^2\right)^{|\alpha|/2}}$$

So, for $0 < \varepsilon \le 1$:

$$\left|\left(\frac{\partial}{\partial\theta}\right)^\alpha \chi(\varepsilon\theta)\right| \lesssim_\alpha \langle\theta\rangle^{-|\alpha|} \quad \text{i.e.,} \quad \chi(\varepsilon\theta) \in \mathrm{Sym}(X, \mathbb{R}^k; 0) \quad \text{uniformly in } \varepsilon.$$

So

$$a(x,\theta)\chi(\varepsilon\theta) \in \mathrm{Sym}(X,\mathbb{R}^k;N) \quad \Rightarrow \quad L^M\left[a(x,\theta)\chi(\varepsilon\theta)\varphi(x)\right] = \sum_{|\alpha|\le M} a_\alpha(x,\theta,\varepsilon)D^\alpha\varphi$$

$$a_\alpha \in \mathrm{Sym}(X,\mathbb{R}^k;N-M).$$

And also $a_\alpha(x,0) := a_\alpha(x,\theta;0) \in \mathrm{Sym}(X,\mathbb{R}^k;N-M)$. Choose M sufficiently large, i.e.,

$$N - M \le -(k+1) \quad \text{for all } x,\theta,$$

i.e., enough to take $M = N+k+1$. So, by the Dominated Convergence Theorem (DCT),

$$\langle I_{\tilde\Phi}(\alpha),\varphi\rangle = \lim_{\varepsilon\to 0}\langle I^\varepsilon_\Phi(\alpha),\varphi\rangle = \sum_{|\alpha|\le N+k+1}\iint e^{i\Phi(x,\theta)}a_\alpha(x,\theta)\,\partial^\alpha\varphi\,dx\,d\theta.$$

If $\mathrm{supp}(\varphi)\subset K$, then

$$|\langle I_\Phi(\alpha),\varphi\rangle| \le \sum_{|\alpha|\le N+k+1}\iint |a_\alpha(x,\theta)|\cdot|\partial^\alpha\varphi|\,dx\,d\theta \lesssim_K \sum_{|\alpha|\le N+k+1}\sup|\partial^\alpha\varphi| \quad \text{(as } a \text{ is integrable).}$$

So $I_{\Phi(\alpha)}\in\mathcal{D}'(X)$ and $\mathrm{ord}(I_{\tilde\Phi(\alpha)}) \le N+k+1$. □

Given $I_\Phi(\alpha)\in\mathcal{D}'(X)$,

$$\int e^{i\tilde\Phi(x,\theta)}a(x,\theta)\,d\theta$$

We can show that $\partial/\partial x_j\, I_{\tilde\Phi}(\alpha)$ coincides with the oscillatory integral

$$\int e^{i\Phi(x,\theta)}\left[i\,\partial_j\tilde\Phi(x,\theta) + \frac{\partial}{\partial x_j}a(x,\theta)\right]d\theta$$

★ Since $[\cdots]$ might fail to be smooth at $\theta=0$, write

$$\int e^{i\Phi(x,\theta)}\rho(\theta)a(x,\theta)\,d\theta + \int e^{i\Phi(x,\theta)}(1-\rho(\theta))a(x,\theta)\,d\theta$$

where $\rho\in\mathcal{D}(\mathbb{R}^k)$, $\rho=1$ on $|\theta|<1$ and $\rho=0$ on $|\theta|>2$. Because of this technicality, we often assume that the support of $a(x,\theta)$ lies in $|\theta|>1$.

Lecture 16

Consider

$$I_\Phi(a) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\theta}\,d\theta, \quad (x,\theta)\in\mathbb{R}^n\times\mathbb{R}^n \qquad (X\to\mathbb{R}^n,\mathbb{R}^k\to\mathbb{R}^n).$$

For $\varphi\in\mathcal{D}(\mathbb{R}^n)$

$$\langle I_\Phi(a),\varphi\rangle = \lim_{\varepsilon\to 0}\iint e^{ix\cdot\theta}\chi(\varepsilon\theta)\varphi(x)\,dx\,d\theta$$

$$(x,\theta)\mapsto(x,\varepsilon\theta) \Rightarrow \lim_{\varepsilon\to 0}\frac{1}{(2\pi)^n}\iint e^{ix\cdot\theta}\chi(\theta)\varphi(\varepsilon x)\,dx\,d\theta = \lim_{\varepsilon\to 0}\int\frac{1}{(2\pi)^n}\widehat{\chi}(-x)\varphi(\varepsilon x)\,dx$$

$$= \varphi(0)\cdot\int\frac{1}{(2\pi)^n}\widehat{\chi}(-x)\,dx = \varphi(0)\cdot\chi(0) = \varphi(0)$$

i.e.

$$\delta_0(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\theta}\,d\theta$$

This gives

$$D^\alpha\delta_0(x) = \frac{1}{(2\pi)^n}\int\theta^\alpha e^{ix\cdot\theta}\,d\theta$$

Natural to ask when $I_\Phi(a)\in D'(X)$ can be identified with a smooth function.

Definition 6.3. *Let $Y \subset X$ be open. Say $u \in D'(X)$ is* <u>*smooth on Y*</u> *if there exists $f \in C^\infty(Y)$ such that $\langle u, \varphi \rangle = \int f\varphi \, dx$ for all $\varphi \in D(Y)$. Define the* <u>*singular support*</u> *of $u \in D'(X)$ by:*

$$\operatorname{sing\,supp}(u) = X \setminus \bigcup_{\substack{Y \subset X \\ open}} \{Y : u \text{ is smooth on } Y\}$$

i.e. the complement of the largest open set on which u is smooth.

<u>**Example.**</u> $\operatorname{sing\,supp}(\delta_0) = \{0\}$, [ ] .

When looking at $\operatorname{sing\,supp}(I_\Phi(a))$ the following lemma allows us to assume $a(x, \theta) = 0$ on $|\theta| < 1$ wlog.

Lemma 6.7. *If Φ is a phase function, a symbol, then the function*

$$x \mapsto \int e^{i\Phi(x,\theta)} \rho(\theta) a(x, \theta) \, d\theta$$

is smooth for any $\rho \in D(\mathbb{R}^k)$.

Fix $\rho \in D(\mathbb{R}^k)$, $\rho = 1$ on $|\theta| < 1$ and $\rho = 0$ on $|\theta| > 2$, can write

$$I_\Phi(a) = \underbrace{I_\Phi(\rho a)}_{\text{smooth function}} + I_\Phi((1-\rho)a) \qquad I_\Phi(\underbrace{(1-\rho)a}_{\tilde{a}}).$$

Note $a \in \operatorname{Sym}(X, \mathbb{R}^k; N) \Rightarrow \tilde{a} \in \operatorname{Sym}(X, \mathbb{R}^k; N)$ and

$$\operatorname{sing\,supp} I_\Phi(a) = \operatorname{sing\,supp} I_\Phi(\tilde{a})$$

Clearly $a(x, \theta) = 0$ on $|\theta| < 1$. Consider again

$$\int e^{i\Phi(x,\theta)} a(x, \theta) \, d\theta$$

Expect this to be "bad" at $x \in X$ for which $\nabla_\theta \Phi(x, \theta) = 0$ for some $\theta \in \mathbb{R}^\kappa$.

Theorem 6.2.

$$\operatorname{sing\,supp}(I_\Phi(a)) \subset \{x \in X : \nabla_\theta \Phi(x, \theta) = 0 \text{ for some } \theta \in \mathbb{R}^k \setminus \{0\}\}$$

Proof. Fix $x_0 \in X$ for which $\nabla_\theta \Phi(x_0, \theta) \neq 0$ for all $\theta \in \mathbb{R}^k \setminus \{0\}$. Note $\theta \mapsto |\nabla_\theta \Phi(x_0, \theta)|$ is homogeneous of deg 0, so is completely determined by values it takes on S^{k-1}. By continuity and compactness,

$$|\nabla_\theta \Phi(x_0, \theta)| \gtrsim 1 \text{ on } \mathbb{R}^k \setminus \{0\}$$

By continuity, $\exists$ small open nbhd Y of x_0 such that

$$|\nabla_\theta \Phi(x, \theta)| \gtrsim 1 \text{ on } Y \times (\mathbb{R}^\kappa \setminus \{0\}).$$

Consider

$$\varphi \mapsto \langle I_\Phi(a), \varphi \rangle, \quad \varphi \in D(Y).$$

The differential operator

$$L^\ell = \sum_{j=1}^k \frac{-i \frac{\partial \Phi}{\partial \theta_j}}{|\nabla_\theta \Phi|^2} \frac{\partial}{\partial \theta_j}$$

is well-defined on $Y \times (\mathbb{R}^k \setminus \{0\})$, and

$$L^\ell e^{i\Phi} = e^{i\Phi}.$$

Since we can assume $a(x, \theta) = 0$ on $|\theta| < 1$ wlog, it follows that L^ℓ is well-defined on

$$(Y \times \mathbb{R}^\kappa) \cap \operatorname{supp}[a(x, \theta)].$$

By same argument as proof of Thm 6.1,

$$L^\ell : \mathrm{Sym}(X, \mathbb{R}^k; N) \longrightarrow \mathrm{Sym}(X, \mathbb{R}^k; N-1).$$

So for $\varphi \in D(Y)$,

$$\langle I_\Phi(a), \varphi \rangle = \lim_{\varepsilon \to 0} \iint e^{i\Phi} L^{\ell M}[a(x,\theta)\chi(\varepsilon\theta)]\varphi(x)\, dx\, d\theta$$

Note that L does not hit $\varphi(x)$. Since $a(x,\theta)\chi(\varepsilon\theta) \in \mathrm{Sym}(X, \mathbb{R}^\kappa; N)$ (uniformly in ε), can choose M large enough so we use DCT to take limit

$$\langle I_\Phi(a), \varphi \rangle = \int \left[\int e^{i\Phi} L^{\ell M} a(x,\theta)\, d\theta \right] \varphi(x)\, dx \qquad (*)$$

Can choose M as large as we please, so differentiation under integral permitted and deduce $I_\Phi(a)$ is smooth on Y. Since $x_0 \in Y$, we conclude the proof of the result. $\square$

Recall we showed earlier that

$$\delta_0(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\theta}\, d\theta\,.$$

Now, we also have

$$\operatorname{sing\,supp}(\delta_0) \subset \left\{ x \in \mathbb{R}^n : \nabla_\theta(x\cdot\theta) = 0 \text{ for some } \theta \in \mathbb{R}^k \setminus \{0\} \right\} = \{x = 0\}$$

Now, suppose want to solve

$$\frac{Du}{Dt} + \mathbf{c}\cdot\nabla u = 0, \quad \lim_{t\to 0} u(x,t) = \delta_0(x)$$

I.e.

$$u(\cdot,t) \in D'(\mathbb{R}^n)\ \forall t \quad \text{and} \quad \lim_{t\to 0} u(\cdot,t) = \delta_0(x)$$

Set $x = (x,t)$. Guess, by F.T.,

$$u(x,t) = \frac{1}{(2\pi)^n} \int e^{i\theta\cdot(x - \mathbf{c}t)}\, d\theta$$

Differentiate under integral, find

$$\frac{\partial u}{\partial t} + \mathbf{c}\cdot\nabla u = 0.$$

and

$$\lim_{t\to 0} u(x,t) = \frac{1}{(2\pi)^n} \int e^{i\theta\cdot x}\, d\theta = \delta_0(x) \quad \text{in } D'(\mathbb{R}^n).$$