Part III Stochastic Calculus
Based on lectures by J. Miller

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Contents

Lecture 1

1 Motivation

This course is about developing a theory of calculus which is applicable to continuous time stochastic processes, e.g. Brownian motion. Why do we need a special theory?

Brownian motion is **not differentiable**.

Ordinary calculus	**Stochastic calculus**
Integral	Itô (stochastic) integral
Derivative	Itô (stochastic) derivative
ODEs	SDEs

Example: Suppose that we have a gambler who repeatedly tosses a fair coin, betting \$1 on getting a heads for each toss. Let

$$\xi_k = \begin{cases} +1, & \text{heads on } k\text{th toss} \\ -1, & \text{otherwise}\,. \end{cases}$$

That is, the (ξ_k) are i.i.d. Bernoulli(± 1). Let

$$X_n = \sum_{k=1}^{n} \xi_k$$

be the net winnings of the gambler. Note that (X_n) is a simple random walk and $X_0 = 0$, hence is a martingale (MG) w.r.t. $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$. Suppose that at the nth toss, bet h_k on heads. Then

$$(H \cdot X)_n = \sum_{k=1}^{n} h_k (X_k - X_{k-1})\,.$$

We interpret $(H \cdot X)_n$ as the gains process from a self-financing strategy H which gives the net winnings after n tosses. Assume that (H_n) is a deterministic sequence.

<u>Claim:</u> $(H \cdot X)_n$ is an $\mathcal{F}_n$-martingale.

(a) H_k is integrable ✓

(b) H_k is adapted ✓

(c) $\mathbb{E}[(H \cdot X)_{n+1} - (H \cdot X)_n \mid \mathcal{F}_n] = H_{n+1} \cdot \mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] = 0$.

More generally, the same is true if we take H_{n+1} to be $\mathcal{F}_n$-measurable *(and integrable)*. This is called a **previsible process**. As before, $(H \cdot X)$ gives the net winnings of the gambler. This is called a **martingale transform**.

<u>Goal for first part of the course:</u> Extend this reasoning to define the *stochastic integral*

$$(H \cdot X)_t = \int_0^t H_s \, dX_s \qquad (\spadesuit)$$

where H is previsible and X is a continuous martingale (e.g., Brownian motion). Crucially, one cannot use the Lebesgue–Stieltjes integral to define ($\spadesuit$), since this requires X to have finite variation, and the only continuous martingales with finite variation are *constant*, as we will show later in the course. Thus, our strategy to define the Itô Integral will be to set

$$(H \cdot X)_t \coloneqq \lim_{\varepsilon \to 0} \sum_{k=1}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon}(X_{k\varepsilon} - X_{(k-1)\varepsilon})$$

We need to be careful about the type of limit since X in general will be rough (not differentiable), like Brownian motion. To get convergence, we need to take advantage of cancellations. For example, if X is a Brownian motion and H is a deterministic and continuous process. We have

$$\mathbb{E}\left[\left(\sum_{k\varepsilon \le t} H_{k\varepsilon}(X_{(k+1)\varepsilon} - X_{k\varepsilon})\right)^2\right] = \mathbb{E}\left[\sum_{k=0}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon}^2 (X_{(k+1)\varepsilon} - X_{k\varepsilon})^2 + \sum_{k \ne \ell} H_{k\varepsilon} H_{\ell\varepsilon}(X_{(k+1)\varepsilon} - X_{k\varepsilon})(X_{(\ell+1)\varepsilon} - X_{\ell\varepsilon})\right]$$

$$= \mathbb{E}\left[\sum_{k=0}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon}^2 (X_{(k+1)\varepsilon} - X_{k\varepsilon})^2\right] = \sum_{k=0}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon}^2 \cdot \varepsilon \to \int_0^t H_s^2 \, ds \quad \text{as } \varepsilon \to 0.$$

Cancellations come from martingale orthogonality and are what make it possible to define the Itô integral.

Next, learn about properties of the integral:

- Stochastic analogue of the chain rule,
- Stochastic analogue of integration by parts.

Formulas look like those in regular calculus but with an extra term to reflect that X is rough (quadratic variation).

$$Y_t = \int_0^t H_s \, dX_s \iff \mathrm{d}Y_t = H_t \, \mathrm{d}X_t \,.$$

Itô's formula will tell us how to write $df(Y_t)$ in terms of $\mathrm{d}Y_t$ for $f \in \mathcal{C}^2$. It has many applications, for example the Dubins–Schwarz theorem which states that *any continuous martingale is a time-change of Brownian motion.*

Next look at stochastic differential equations (SDEs), i.e.,

$$\mathrm{d}X_t = b(t, X_t) \, dt + \sigma(t, X_t) \, \mathrm{d}B_t,$$

where b, σ are "nice" and B is a Brownian motion. For $\sigma \equiv 0$, just an ODE. For $\sigma \not\equiv 0$, corresponds to adding noise which depends on time and the state of the system.

Last part of the course: diffusion processes and how they are related to SDEs, and how they can be used to solve PDEs involving 2^{nd} order elliptic equations (e.g., Δ).

Next time we will start with some preliminaries (càdlàg processes, function of finite variation, integral against a function/process of finite variation).

2 Preliminaries

Lecture 2

2.1 Càdlàg processes, functions of finite variation

Recall that $\alpha : [0,\infty) \to \mathbb{R}$ is càdlàg if α is right-continuous and has left-hand limits:

$$\lim_{s\to t^+} \alpha(s) = \alpha(t), \quad \lim_{s\to t^-} \alpha(s) \text{ exists}.$$

Let $\alpha(x-), x \in [0,\infty)$ be right-hand limit, and set

$$\Delta\alpha(x) \coloneqq \alpha(x) - \alpha(x^-)\,, x \in [0,\infty)]\,.$$

Suppose that α is non-decreasing, càdlàg and $a(0) = 0$. Then there exists a unique Borel measure $\mathrm{d}\alpha$ on $([0,t],\mathcal{B})$ with

$$\mathrm{d}\alpha((s,t]) \coloneqq \alpha(t) - \alpha(s)\,, \text{ for all } 0 \le s < t\,.$$

For f measurable and integrable, then the Lebesgue–Stieltjes integral of f w.r.t. α is defined by

$$\int_{(0,t]} f(s)\,\mathrm{d}\alpha(s) \quad \forall t \ge 0.$$

Then, by dominated convergence $t \mapsto \int_{[0,t]} f(s)\,\mathrm{d}\alpha(s)$ is a right-continuous function. Moreover, if f is continuous, then $t \mapsto \int_0^t f(s)\,\mathrm{d}\alpha(s)$ is continuous so we can write

$$\int_0^t f(s)\,\mathrm{d}\alpha(s) \coloneqq \int_{(0,t]} f(s)\,\mathrm{d}\alpha(s).$$

We want to integrate more general functions. Suppose that α^+, a^- are functions satisfying the same conditions as before, and set $a = a^+ - a^-$. Define $(f\cdot a)(t) = (f\cdot a^+)(t) - (f\cdot a^-)(t)$ for all f measurable so that both terms on the RHS are finite. This class of functions (i.e., differences of càdlàg non-decreasing functions) coincides with the càdlàg functions with *finite variation*.

Definition 2.1. *Let $\alpha : [0,\infty) \to \mathbb{R}$ be càdlàg. For each $n \in \mathbb{N}, t \ge 0$, let*

$$v^n(t) \coloneqq \sum_{k=0}^{\lceil 2^n t\rceil - 1} \left| \alpha\left(\frac{(k+1)t}{2^n}\right) - \alpha\left(\frac{kt}{2^n}\right)\right|\,. \tag{$\circledast$}$$

Then the limit $v(t)_t \coloneqq \lim_{n\to\infty} v^n(t)$ exists and is called the total variation of α on $[0,t]$.

If $v(t)_t < \infty$ then we say that α has finite variation on $[0,t]$. If $v(t)_t < \infty$ for all $t \ge 0$, we say that α is a càdlàg function of finite variation.

To see that $\lim v^n(t)$ exists, fix $t > 0$ and let

$$t_n^+ = 2^{-n}\lceil 2^n t\rceil \quad \text{so that } t_n^+ \ge t \ge t_n^- \quad \forall n$$

$$t_n^- = 2^{-n}\left(\lfloor 2^n t\rfloor - 1\right)$$

and

$$v^n(t) = \sum_{k=0}^{2^n t_n^- - 1} \left|a((k+1)\cdot 2^{-n}) - a(k\cdot 2^{-n})\right| + \left|a(t_n^+) - a(t_n^-)\right|.$$

The triangle inequality gives that the first term is non-decreasing in n. The càdlàg property implies that the second term converges to the jump $|\Delta a(t)|$, hence $v^n(t)$ converges as $n \to \infty$.

Lemma 2.1. *Let a be a càdlàg function of finite variation. Then v is also càdlàg of finite variation with $\Delta v(t) = |\Delta a(t)|$ for all $t \geq 0$ and v is non-decreasing. In particular, if a is continuous, then so is v.*

Proof. [ ] . □

Proposition 2.1. *A càdlàg function can be decomposed as a difference of two non-decreasing right-continuous functions if and only if it has finite variation.*

Proof. Assume that $a = a^+ - a^-$ are càdlàg, non-decreasing. NTS: a has finite variation. Note,

$$|a(t) - a(s)| \leq \Big(a^+(t) - a^+(s)\Big) + (a^-(t) - a^-(s)) \qquad \forall\, 0 \leq s < t.$$

Plug this into $\circledast$ and use that the sum telescopes for monotone functions to get that

$$v^n(t) \leq \Big(a^+(t_n^+) - a^+(0)\Big) + \Big(a^-(t_n^+) - a^-(0)\Big).$$

Since a^+, a^- are right-continuous,

$$\text{RHS} \overset{n\to\infty}{\longrightarrow} \Big(a^+(t) - a^+(0)\Big) + (a^-(t) - a^-(0))$$

which gives that a has finite variation, as desired.

Now the reverse direction. Assume that $v(t) < \infty$ for all $t > 0$. Set $a^+ = \frac{1}{2}(v + a)$, $a^- = \frac{1}{2}(v - a)$. Then $a = a^+ - a^-$ and a^+, a^- are càdlàg since v, a are càdlàg. NTS: $a^\pm$ are non-decreasing.

Fix $0 \leq s < t$, define $t_n^{+/-}$ as before and $s_n^{+/-}$ analogously. Then:

$$\begin{aligned} a^+(t) - a^+(s) &= \lim_{n\to\infty} \frac{1}{2}\Big(v^n(t) - v^n(s) + a(t_n^+) - a(s_n^+)\Big) \\ &= \lim_{n\to\infty} \frac{1}{2}\Bigg[\sum_{k=2^n s_n^+}^{2^n t_n^- - 1} \left(|a((k+1)\cdot 2^{-n}) - a(k\cdot 2^{-n})| + a((k+1)\cdot 2^{-n}) - a(k\cdot 2^{-n})\right) \\ &\quad + |a(t_n^+) - a(t_n^-)| + (a(t_n^+) - a(t_n^-))\Bigg] \geq 0. \end{aligned}$$

Same argument works for a^-. □

2.2 Random integrands

We now discuss integration against random functions of finite variation.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ filtered probability space. Recall that $X_t(\omega), t \in [0,\infty) \to \mathbb{R}$ is *adapted* to $(\mathcal{F}_t)_{t\geq 0}$ if $X_t = X(\cdot, t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$. X is *càdlàg* if $X(\omega, \cdot)$ is càdlàg for all $\omega \in \Omega$.

Definition 2.2. *Given a càdlàg, adapted process $A\colon \Omega \times [0,\infty) \to \mathbb{R}$, its total variation process $V\colon \Omega \times [0,\infty) \to \mathbb{R}$ is pathwise by setting $V(\omega, t)$ to be the total variation of $A(\omega, \cdot)$.*

Lemma 2.2. *If A is càdlàg, adapted, and of finite variation then V is càdlàg, adapted, and non-decreasing.*

Proof. Only NTS V is adapted. For $t \geq 0$, $t_n^- = 2^{-n}(\lceil 2^n t \rceil - 1)$

$$\tilde{V}_t^n = \sum_{k=0}^{2^n t_n^- - 1} \left| A_{(k+1)\cdot 2^{-n}} - A_{k\cdot 2^{-n}} \right| ,$$

$\tilde{V}_t^n$ adapted for all n since $t_n^- \leq t$.

$$V_t = \lim_{n\to\infty} \left(\tilde{V}_t^n + |\Delta A(t)| \right)$$

which shows that V_t is $\mathcal{F}_t$-measurable. $\square$

Lecture 3 We now seek a class of functions so that the integral is adapted.

Recall from the introduction that a discrete-time process $(H_n)_n$ is called previsible w.r.t. $(\mathcal{F}_n)$ if H_{n+1} is measurable w.r.t. $\mathcal{F}_n$ for all n.

Definition 2.3. *The previsible σ-algebra $\mathcal{P}$ on $\Omega \times (0, \infty)$ is the σ-algebra which is generated by sets of the form $E \times (s, t]$ where $E \in \mathcal{F}_s$, $s < t$. A process $H\colon \Omega \times (0, \infty) \to \mathbb{R}$ is previsible if it is measurable with respect to $\mathcal{P}$.*

Examples:

1. $H(\omega, t) = Z(\omega) \cdot \mathbf{1}_{(t_1,t_2]}(t)$, $t_1 < t_2$, Z is $\mathcal{F}_{t_1}$-measurable.
2. $H(\omega, t) = \sum_{k=0}^{n-1} Z_k(\omega) \cdot \mathbf{1}_{(t_k,t_{k+1}]}(t)$, for $0 = t_0 < \cdots < t_n$ and Z_k is $\mathcal{F}_{t_k}$-measurable.

A *simple process*, will be important for the construction of the Itô integral.

Remark. *Simple processes are left-continuous and adapted. It turns out that $\mathcal{P}$ is the smallest σ-algebra on $\Omega \times (0, \infty)$ so that all left-continuous processes are measurable, . In general, càdlàg processes are not previsible, but their left-continuous modification is.*

Proposition 2.2. *Let X be a càdlàg, adapted process and let $H_t = X_{t-}$, $t \geq 0$. Then H is previsible.*

Proof. Since X is càdlàg and adapted, it is clear that H is left-continuous and adapted. For each n, set

$$H_t^n = \sum_{k=0}^{\infty} H_{k\cdot 2^{-n}} \cdot \mathbf{1}_{(k\cdot 2^{-n},(k+1)\cdot 2^{-n}]}(t)$$

Then H^n is previsible for all n and since H is a left-continuous process,

$$\lim_{n\to\infty} H_t^n = H_t \quad \forall t \Rightarrow H \text{ is previsible as a limit of previsible processes.} \quad \square$$

$\square$

Remark. *The proposition above implies that continuous, adapted processes are previsible.*

Proposition 2.3. *If H is previsible, then H_t is measurable w.r.t. $\mathcal{F}_{t-} = \sigma(\mathcal{F}_s : s < t)$, $\forall t \geq 0$.*

Proof. . $\square$

Remark. *The Poisson process* (N_t) *is* not *previsible since* N_t *is not* $\mathcal{F}_{t-}$*-measurable, where* $(\mathcal{F}_t)$ *is the natural filtration.*

Now we are going to see that integrating a previsible process H against a càdlàg process with a.s. finite variation A yields a well-defined and adapted càdlàg process of finite variation.

Theorem 2.1. *Let* $A : \Omega \times (0, \infty) \to \mathbb{R}$ *be a càdlàg process which is adapted and has finite variation* V*. Let* H *be a previsible process with*

$$\int_{0<s\leq t} |H(\omega, s)| \, dV(s) < \infty \quad \forall t > 0, \, \omega \in \Omega. \tag{2.1}$$

Then the process $H \cdot A : \Omega \times (0, \infty) \to \mathbb{R}$ *given by*

$$(H \cdot A)(\omega, t) = \int_{(0,t]} H(\omega, s) \, dA(\omega, s), \tag{2.2}$$

with

$$(H \cdot A)(\omega, 0) = 0,$$

is càdlàg, adapted and has finite variation.

Proof. The integral in 2.2 is well-defined due to 2.1. Indeed, let $H^+ = \max(H, 0)$, $H^- = \max(-H, 0)$, and

$$A^{\pm} = \tfrac{1}{2}(V \pm A).$$

Then $H = H^+ - H^-$ and $A = A^+ - A^-$ and

$$H \cdot A = (H^+ - H^-) \cdot (A^+ - A^-) = H^+ \cdot A^+ + H^- \cdot A^- - H^+ \cdot A^- - H^- \cdot A^+.$$

All terms on RHS are finite by 2.1. Need to show:

1. $H \cdot 1$ is càdlàg,
2. adapted,
3. finite variation.

Step 1. Note that $\mathbf{1}_{(0,s]} \to \mathbf{1}_{(0,t]}$ as $s \downarrow t$ and $\mathbf{1}_{(0,s]} \to \mathbf{1}_{(0,t]}$ as $s \nearrow t$. By definition,

$$(H \cdot A)_t = \int H_s \cdot \mathbf{1}_{(s \in (0,t])} \, dA_s.$$

Hence,

$$(H \cdot A)_t = \int H_s \cdot \lim_{r \downarrow t} \mathbf{1}_{(s \in (0,r])} \, dA_s$$

$$\overset{\text{(DCT)}}{=} \lim_{r \downarrow t} \int H_s \cdot \mathbf{1}_{(s \in (0,r])} \, dA_s = \lim_{r \downarrow t} (H \cdot A)_r$$

giving right-continuity. An analogous argument gives that $H \cdot A$ has left-limits, hence is càdlàg. Also,

$$\Delta(H \cdot A)_t = \int H_s \cdot \mathbf{1}_{(s=t)} \, dA_s = H_t \cdot \Delta A_t$$

Step 2. "Monotone class" style argument. Suppose

$$H = \mathbf{1}_{B\times(s,u]}, \quad B \in \mathcal{F}_s, \quad s < u.$$

Then

$$(H \cdot A)_t = \mathbf{1}_B \cdot (A_{t\wedge u} - A_{t\wedge s}), \text{ which is } \mathcal{F}_t\text{-measurable.}$$

Let $\mathcal{A} = \{Z \in \mathcal{P} : \mathbf{1}_Z \cdot A \text{ is adapted}\}$. Want to show: $\mathcal{A} = \mathcal{P}$. Let

$$\Pi = \{B \times (s,u] : B \in \mathcal{F}_s,\ s < u\}.$$

We have shown $\Pi \subseteq \mathcal{A}$, and know that Π is a π-system generating $\mathcal{P}$. Not difficult to see that $\mathcal{A}$ is also a d-system, and by Dynkin's lemma we deduce

$$\mathcal{P} = \sigma(\Pi) \subseteq \mathcal{A} \subseteq \mathcal{P} \Rightarrow \mathcal{A} = \mathcal{P}.$$

Now suppose that $H \geq 0$ so previsible. Set

$$H^n = \left(2^{-n} \lfloor 2^n H \rfloor\right) \wedge n$$

$$= \sum_{k=0}^{n2^n-1} 2^{-n} \cdot k \cdot \mathbf{1}\left(H \in \left[\tfrac{\Sigma - nk}{2^n}, \tfrac{\Sigma - n(k+1)}{2^n}\right)\right) + n \cdot \underbrace{\mathbf{1}(H \geq n)}_{\in\mathcal{P}}.$$

This implies that H^n is a finite linear combination of functions of the form $\mathbf{1}_C$, where $C \in \mathcal{P}$ which in turn implies that $(H^n \cdot A)_t$ is $\mathcal{F}_t$-measurable for all t. By the monotone convergence theorem, $(H^n \cdot A)_t \to (H \cdot A)_t$ as $n \to \infty$. For general H, write $H = H^+ - H^-$, where $H^\pm = \max(\pm H, 0)$, and use that

$$(H \cdot A)_t = (H^+ \cdot A)_t - (H^- \cdot A)_t \qquad \text{(both } \mathcal{F}_t\text{-measurable).}$$

Step 3. To show that $H \cdot A$ has finite variation, observe that

$$H \cdot A = (H^+ - H^-) \cdot (A^+ - A^-) = (H^+ \cdot A^+ + H^- \cdot A^-) - (H^- \cdot A^+ + H^+ \cdot A^-)$$

is a difference of non-decreasing functions. □

Next, we will introduce and generalise our theory of stochastic integration to integrating against Martingales.

Lecture 4

3 Local Martingales.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space.

Definition 3.1. *We say that $(\mathcal{F}_t)_{t\geq 0}$ satisfies the **usual conditions** if:*

1. *$\mathcal{F}_t$ contains all $\mathbb{P}$-null sets.*
2. *$(\mathcal{F}_t)_{t\geq 0}$ is right-continuous: $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$.*

Throughout, assume that $(\mathcal{F}_t)$ satisfies the usual conditions. Recall that an integrable adapted process X is an $(\mathcal{F}_t)$ martingale if

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s \quad \text{a.s.}$$

supermartingale if

$$\mathbb{E}[X_t \mid \mathcal{F}_s] \leq X_s \quad \text{a.s.}$$

submartingale if

$$\mathbb{E}[X_t \mid \mathcal{F}_s] \geq X_s \quad \text{a.s.}$$

for all $0 \leq s < t$.

A random variable T is called a *stopping time* if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. If X is càdlàg and adapted to $(\mathcal{F}_t)$ and we set

$$\mathcal{F}_T = \{E \in \mathcal{F} : E \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$$

then X_T is an $\mathcal{F}_T$-measurable random variable.

If X is a martingale then $X_t^T = X_{t\wedge T}$ is also a martingale.

Theorem 3.1 (Optional Stopping Theorem (OST))**.** *Let X be an adapted, càdlàg and integrable process. Then the following are equivalent:*

1. *X is a martingale.*
2. *$X^T := (X_{t\wedge T})_{t\geq 0}$ is a martingale for every stopping time T.*
3. *For all bounded stopping times $S \leq T$, we have*

$$\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S \quad a.s.$$

4. *For all bounded stopping times T, we have that*

$$\mathbb{E}[X_T] = \mathbb{E}[X_0].$$

Definition 3.2. *A càdlàg adapted process X_t is called a local martingale if there exists a sequence $(T_n)_{n\geq 0}$ of stopping times with $T_n \nearrow \infty$ a.s. (non-decreasing), and for every n, such that the stopped process X^{T_n} is a (true) martingale for all $n \geq 1$. In this case, we say that (T_n)* ***reduces*** *X.*

Note that a MG is a local martingale as any deterministic sequence $T_n \nearrow \infty$ will reduce it.

Example. Let B be a standard Brownian motion in $\mathbb{R}^3$. Let $M_t = \frac{1}{|B_t|}$. ( )

(i) $(M_t)_{t\geq 1}$ is L^2-bounded: $\sup_{t\geq 1} \mathbb{E}[M_t^2] < \infty$.

(ii) $\mathbb{E}[M_t] \to 0$ as $t \to \infty$.

(iii) M is a supermartingale.

M cannot be a martingale, otherwise its expectation would vanish by (ii), but this cannot be true since $M_t > 0$ a.s.

For each $n \geq 1$, set:

$$\begin{aligned} T_n &= \inf\left\{t \geq 1 : |B_t| < \tfrac{1}{n}\right\} \\ &= \inf\left\{t \geq 1 : M_t > n\right\}. \end{aligned}$$

We want to show

1) $(M_{t\wedge T_n})_{t\geq 1}$ is a martingale for all n.

2) $T_n \to \infty$ as $n \to \infty$ a.s.

Note that

$$n \leq M_1 \Rightarrow T_n = 1, \qquad n > M_1 \Rightarrow T_n > 1.$$

Since $|B_t|$ cannot hit $1/n$ before hitting $|B_1|$, have that T_n is non-decreasing. Now, recall from **Advanced Probability:** $f \in C_0^\infty(\mathbb{R})$

$$f(B_t) - \frac{1}{2}\int_0^t \Delta f(B_s)\, ds \text{ is a martingale.}$$

Note that $f(x) = \frac{1}{|x|}$ is a harmonic function in $\mathbb{R}^3 \setminus \{0\}$. Let $(f_n)_{n\geq 1}$ be a sequence of $C_c^\infty(\mathbb{R}^3)$ with $f_n(x) = f(x)$ on $\{|x| \geq \frac{1}{n}\}$. If

$$0 < |B_t| < \frac{1}{n}, \text{ then } T_n = 1 \text{ and so } M_{t\wedge T_n} = M_t \text{ is a martingale.}$$

Since $B_1 \neq 0$ a.s., we have that $|B_1| > \frac{1}{n}$ for all n sufficiently large enough, in which case

$$f(B_{t\wedge T_n}) = f^n(B_{t\wedge T_n}) \quad \forall t \geq 1.$$

Thus:

$$\begin{aligned} M_{t\wedge T_n} &= f(B_{t\wedge T_n}) - f(B_t) + f(B_1) \\ &= \left[f(B_{t\wedge T_n}) - f(B_t) - \frac{1}{2}\int_1^{t\wedge T_n} \Delta f(B_s)\ \mathrm{d}s\right] + f(B_1) \\ &= \left[f^n(B_{t\wedge T_n}) - f^n(B_t) - \frac{1}{2}\int_1^{t\wedge T_n} \Delta f^n(B_s)\ \mathrm{d}s\right] + f^n(B_1) \end{aligned}$$

and so we conslude $M_{T_n} = (M_{t\wedge T_n})_{t\geq 1}$ is a martingale.

We also need to show that $T_n \nearrow \infty$ as $n \to \infty$. Now, as $T_n \leq T_{n+1}$, it remains to show that $\lim_{n\to\infty} T_n = \infty$ a.s.. For each R, let

$$S_R = \inf\{t \geq 1 : |B_t| \geq R\} = \inf\{t \geq 1 : M_t < 1/R\}.$$

Then $S_R \to \infty$ as $R \to \infty$.

$$\mathbb{P}\left(\lim_{n\to\infty} T_n < \infty\right) \leq \mathbb{P}\left(\exists R : T_n < S_R\, \forall n\right) = \lim_{R\to\infty}\lim_{n\to\infty} \mathbb{P}(T_n < S_R).$$

The OST implies

$$\mathbb{E}[M_{T_n \wedge S_R}] = \mathbb{E}[M_1] = N \in (0, \infty).$$

and so the LHS becomes

$$n\mathbb{P}(T_n < S_R) + \frac{1}{R}\mathbb{P}(S_R \leq T_n) = \frac{N}{R} \Rightarrow \mathbb{P}(T_n < S_R) = \frac{N - \frac{1}{R}}{n - \frac{1}{R}} \to 0 \text{ as } n \to \infty.$$

Therefore, $(M_t)_{t \geq 0}$, non-negative local martingale but not a martingale, a supermartingale and in L^2-bounded.

Observe from the preceding discussion that by only requiring non-negativity, the first two properties actually give that M is a super martingale.

Proposition 3.1. *If X is a local martingale, $X_t \geq 0$ for all $t \geq 0$, then X is a supermartingale.*

Proof. Suppose that (T_n) is a reducing sequence. Then for any $s \leq t$, we know that

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}\left[\lim_{n\to\infty} X_{t \wedge T_n} \mid \mathcal{F}_s\right] \overset{\textbf{(Fatou)}}{\leq} \liminf_{n\to\infty} \mathbb{E}[X_{t\wedge T_n} \mid \mathcal{F}_s] = \liminf_{n\to\infty} X_{s \wedge T_n} = X_s \quad \text{a.s.}$$

□

Often work with local martingales instead of martingales, so as to not have to worry about integrability.

Lecture 5 We will now answer the following

1. When is a local MG a MG?
2. Continuous local MGs with finite variation in time.

Definition 3.3. *A collection $\mathcal{G}$ of random variables in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ is called uniformly integrable (UI) if*

$$\sup_{X \in \mathcal{G}} \mathbb{E}[|X|\mathbf{1}_{|X|>M}] \to 0 \quad \text{as } M \to \infty.$$

Examples of UI families:

1. Uniformly bounded random variables: If $\mathcal{G} \subseteq L^1$ is bounded in L^2, then $\mathcal{G}$ is UI.
2. L^p bounded for $p > 1$: $\sup_{X \in \mathcal{G}} \mathbb{E}[|X|^p] < \infty$.
3. there exists Y integrable so that $|X| \leq Y$ for all $X \in \mathcal{G}$.

Lemma 3.1. *Suppose that $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then*

$$\mathcal{X} := \{\mathbb{E}[X \mid \mathcal{G}] : \mathcal{G} \text{ is a sub-}\sigma\text{-algebra of } \mathcal{F}\}$$

is also a uniformly integrable family.

Proof. . □

Proposition 3.2. *The following are equivalent:*

i) *X is a martingale.*

ii) *X is a local martingale and for all $t > 0$, the family*

$$\mathcal{X}_t = \{X_T : T \text{ is a stopping time with } T \leq t\}$$

is uniformly integrable.

Proof. *i)* $\Rightarrow$ *ii)*: Suppose X is a martingale. By OST, if T is a stopping time with $T \leq t$, then

$$\mathbb{E}[X_t \mid \mathcal{F}_T] = X_T \Rightarrow X_t \text{ is UI.}$$

ii) $\Rightarrow$ *i)*: Suppose that X is a local martingale and X_t is UI for all $t \geq 0$. To show that X is a martingale, by OST it suffices to show that for all bounded stopping times T, we have

$$\mathbb{E}[X_T] = \mathbb{E}[X_0].$$

Let (T_n) be a reducing sequence for X and let $T \leq t$ be a stopping time. Then

$$\mathbb{E}[X_0] = \mathbb{E}[X_0^{T_n}] \overset{\textbf{OST}}{=} \mathbb{E}[X_T^{T_n}] \overset{\textbf{(def'n of } X^{T_n}\textbf{)}}{=} \mathbb{E}[X_{T \wedge T_n}].$$

Since $\{X_{T \wedge T_n} : n \geq 0\}$ is UI and $X_{T \wedge T_n} \to X_T$ a.s.,

$$\text{Advanced Probability } \Rightarrow X_{T \wedge T_n} \to X_T \text{ in } L^1 \text{ as } n \to \infty.$$

Therefore, $\mathbb{E}[X_{T \wedge T_n}] \to \mathbb{E}[X_T]$ as $n \to \infty$. Hence $\mathbb{E}[X_0] = \mathbb{E}[X_T]$. OST finally implies X is a martingale. ☐

Corollary 3.1. *A bounded local martingale is a martingale. More generally, if X is a local martingale and there exists Y integrable such that $|X_t| \leq Y$ for all $t \geq 0$, then X is a martingale.*

Theorem 3.2. *Let X be a continuous local martingale with $X_0 = 0$. If X has finite variation, then $X \equiv 0$ a.s.*

Proof. Let V be the total variation process for X. Then $V_0 = 0$, and V is continuous, adapted and non-decreasing. Let

$$T_n := \inf\{t \geq 0 : V_t = n\}$$

for all $n \in \mathbb{N}$. Then $T_n \nearrow \infty$ as $n \to \infty$, since X has finite variation. Moreover,

$$|X_t^{T_n}| = |X_{t \wedge T_n}| \leq V_{t \wedge T_n} \leq n.$$

Therefore X^{T_n} is a bounded local martingale and hence is a proper MG.

To prove that $X \equiv 0$, note: $X^{T_n} \equiv 0$ for all $T_n \nearrow \infty$ as $n \to \infty$. Fix $n \in \mathbb{N}$, let $Y := X^{T_n}$. Y is a continuous bounded martingale with $Y_0 = 0$. To prove that $Y \equiv 0$, it suffices to show that $\mathbb{E}[Y_t^2] = 0$ for all $t \geq 0$. This implies that $Y_t = 0$ for all $t \geq 0$, $t \in \mathbb{Q}$ a.s., so $Y \equiv 0$ by continuity. Fix $t \geq 0$, $N \in \mathbb{N}$, let

$$t_k := \frac{k}{N} t \quad \text{for } k \leq N.$$

Compute

$$\mathbb{E}[Y_t^2] = \mathbb{E}\left[\sum_{k=0}^{N-1} (Y_{t_{k+1}} - Y_{t_k})^2\right] \overset{\text{(MG orthogonality)}}{=} \mathbb{E}\left[\sum_{k=0}^{N-1} (Y_{t_{k+1}} - Y_{t_k})^2\right]$$

$$\leq \mathbb{E}\Bigg[\overbrace{\max_{0\leq k\leq N-1}|Y_{t_{k+1}}-Y_{t_k}|}^{\leq V_{t\wedge T_n}\leq n}\underbrace{\sum_{k=0}^{N-1}|Y_{t_{k+1}}-Y_{t_k}|}_{\leq V_{t\wedge T_n}}\Bigg]\leq n^2.$$

Since Y is continuous,

$$\lim_{N\to\infty}\left(\max_{0\leq k\leq N-1}|Y_{t_{k+1}}-Y_{t_k}|\right)=0 \quad \text{a.s.}$$

Bounded convergence finally gives $\mathbb{E}[Y_t^2]=0$. □

Remark. *(i) The proof requires continuity, in particular* not *true without continuity.*

(ii) Theorem implies Brownian motion has infinite variation, so cannot use Lebesgue–Stieltjes integral to define the integral against a BM.

For continuous local martingales, there is always an explicit way of choosing the reducing sequence.

Proposition 3.3. *Let X be a continuous local martingale with $X_0=0$. Then*

$$T_n := \inf\{t\geq 0 : |X_t|=n\}$$

reduces X.

Proof. **Step 1:** T_n is a stopping time.

Let $t\geq 0$, then:

$$\{T_n\leq t\}=\{\sup_{0\leq s\leq t}\{|X_s|\geq n\}=\bigcup_{k=1]}^{\infty}\bigcup_{s\in\mathbb{Q},s\leq t}\overbrace{\{|X_q|\geq n-1/k\}}^{\in\mathcal{F}_t}.$$

Step 2: $T_n\nearrow\infty$ as $n\to\infty$.

Since

$$\sup_{0\leq s\leq t}|X_s(\omega)|<\infty\Rightarrow \text{there exists } n(\omega,t)\in\mathbb{N} \text{ such that } n(\omega,t)\geq\sup_{0\leq s\leq t}|X_s(\omega)|.$$

$$\Rightarrow n\geq n(\omega,t)\Rightarrow T_n(\omega)>t\Rightarrow T_n(\omega)\to\infty \text{ as } n\to\infty.$$

Step 3: (T_n) reduces X.

Let (T_n^*) be a reducing sequence (exists since X is a local martingale). Then $X^{T_n^*}$ is a martingale for all n. Need to show: X^{T_n} is a martingale. The Optional stopping theorem implies $X^{T_n\wedge T_m^*}$ is a martingale for all n,

$$X^{T_n} \text{ is a local martingale with reducing sequence } (T_m^*).$$

Since X^{T_n} is in addition bounded, it is a martingale; concluding the proof. □

We now move on to construct the stochastic integral proper.

Lecture 6

4 The Stochastic Integral

Goal: Be able to integrate against a continuous local MG. How does one construct an integral (Riemann / Lebesgue)?

An integral is a linear map

$$\mathcal{I} : X \to Y \quad \text{where } X, Y \text{ are normed vector spaces.}$$

Steps:

1. Define it on a dense set $\mathcal{D} \subseteq X$

2. Show that it is a continuous linear map:

$$\exists C > 0 \text{ such that } \|\mathcal{I}(f)\|_Y \leq C\|f\|_X \quad \forall f \in \mathcal{D}.$$

$$\Rightarrow \mathcal{I} \text{ extends by continuity to } X.$$

Need to

$$\underbrace{\text{1 specify } \mathcal{D}, X, Y}_{\text{simple processes, quadratic variation}} \quad , \quad \underbrace{\text{prove 2}}_{\text{Itô isometry}}.$$

Theorem 4.1. *Let X be a càdlàg, L^2-bounded MG (i.e., $\sup_t \mathbb{E}[X_t^2] < \infty$). Then there exists $X_\infty \in L^2$ such that:*

$$X_t \to X_\infty \quad \text{a.s. and in } L^2, \quad \text{and} \quad X_t = \mathbb{E}[X_\infty \mid \mathcal{F}_t] \quad (X_\infty \text{ is called the final value of } X).$$

Proposition 4.1 (Doob's L^2 inequality)**.** *Let X be a càdlàg, L^2-bounded MG. Then:*

$$\mathbb{E}\left[\sup_t |X_t|^2\right] \leq 4\,\mathbb{E}\left[X_\infty^2\right].$$

Define:

- $\mathcal{M}^2 = \{L^2\text{-bounded càdlàg MGs}\}$.
- $\mathcal{M}^2_{\mathcal{C}} = \{L^2\text{-bounded, continuous MGs}\}$.
- $\mathcal{M}^2_{c,loc} = \{L^2\text{-bounded, continuous local MGs}\}$.

Definition 4.1. *A process $H : [0,\infty) \times \Omega \to \mathbb{R}$ is called a simple process if there exist $0 = t_0 < t_1 < \cdots < t_n$, and bounded, $\mathcal{F}_{t_i}$-measurable random variables Z_i, such that:*

$$H_t = \sum_{i=0}^{n-1} Z_i \mathbf{1}_{(t_i, t_{i+1}]}(t).$$

Let $\mathcal{S}$ be the set of simple processes. We will proceed to

- define $\left(\int_0^t H_s \, dM_s\right)$ for $H \in \mathcal{S}$, $M \in \mathcal{M}^2$.
- Extend the integral to more general integrands ($M \in \mathcal{M}^2_{\mathcal{C}}$).

Proposition 4.2. *If $H \in \mathcal{S}$, $M \in \mathcal{M}^2$, then $H \cdot M \in \mathcal{M}^2$. Moreover,*

$$\mathbb{E}[(H \cdot M)_\infty^2] = \sum_{k=0}^{n-1} \mathbb{E}\left[Z_k^2(M_{t_{k+1}} - M_{t_k})^2\right] \leq 4\,\|H\|_\infty^2\, \mathbb{E}\left[(M_\infty - M_0)^2\right].$$

Proof. **Step 1:**$H \cdot M$ is a martingale.

Suppose that $t_k \leq s < t < t_{k+1}$. Then we have that

$$(H \cdot M)_t - (H \cdot M)_s = Z_k(M_t - M_s),$$

so that

$$\mathbb{E}[(H \cdot M)_t - (H \cdot M)_s|\mathcal{F}_s] = Z_k \mathbb{E}[M_t - M_s|\mathcal{F}_s] = 0$$

since $Z_k \in \mathcal{F}_s$ and $M \in \mathcal{M}^2$.

Suppose that $0 \leq t_i \leq s \leq t_j \leq t \leq t_k$. Then

$$\mathbb{E}[(H \cdot M)_t - (H \cdot M)_s|\mathcal{F}_s]$$

$$= \mathbb{E}\left[\sum_{i=0}^{k-1} Z_i(M_{t_{i+1}} - M_{t_i}) + Z_k(M_t - M_{t_k}) - \left(\sum_{i=0}^{j-1} Z_i(M_{t_{i+1}} - M_{t_i}) + Z_j(M_s - M_{t_j})\right)\middle|\mathcal{F}_s\right].$$

$$= \sum_{i=j+1}^{k-1} \mathbb{E}[Z_i(M_{t_{i+1}} - M_{t_i})|\mathcal{F}_s] + \mathbb{E}[Z_j(M_{t_j} - M_s)|\mathcal{F}_s] + \mathbb{E}[Z_k(M_t - M_{t_k})|\mathcal{F}_s].$$

Since

$$\begin{aligned}
\mathbb{E}[Z_i(M_{t_{i+1}} - M_{t_i})|\mathcal{F}_s] &= Z_i \mathbb{E}[M_{t_{i+1}} - M_{t_i}|\mathcal{F}_s] = 0, \quad j+1 \leq i \leq k-1, \\
\mathbb{E}[Z_j(M_{t_j} - M_s)|\mathcal{F}_s] &= Z_j \mathbb{E}[M_{t_j} - M_s|\mathcal{F}_s] = 0, \\
\mathbb{E}[Z_k(M_t - M_{t_k})|\mathcal{F}_s] &= \mathbb{E}[Z_k \mathbb{E}[M_t - M_{t_k}|\mathcal{F}_{t_k}]|\mathcal{F}_s] = 0\,.
\end{aligned}$$

Step 2: $H \cdot M$ is L^2-bounded.

If $j < k$, then we have that

$$\mathbb{E}\left[Z_j(M_{t_{j+1}} - M_{t_j})Z_k(M_{t_{k+1}} - M_{t_k})\right] = \mathbb{E}\left[\mathbb{E}[Z_j(M_{t_{j+1}} - M_{t_j})|\mathcal{F}_{t_k}]Z_k(M_{t_{k+1}} - M_{t_k})\right] = 0.$$

So,

$$\mathbb{E}[(H \cdot M)_t^2] = \mathbb{E}\left[\left(\sum_{k=0}^{n-1} Z_k(M_{t_{k+1}} - M_{t_k})\right)^2\right] \overset{\text{MG orthogonality}}{=} \mathbb{E}\left[\sum_{k=0}^{n-1} Z_k^2(M_{t_{k+1}} - M_{t_k})^2\right]$$

$$\leq \|H\|_\infty^2 \sum_{k=0}^{n-1} \mathbb{E}\left[(M_{t_{k+1}} - M_{t_k})^2\right] \overset{\text{Doob's } L^2 \text{ inequality}}{\leq} 4\,\|H\|_\infty^2\, \mathbb{E}[(M_\infty - M_0)^2].$$

This bound is uniform in t, so $H \cdot M$ is L^2 bounded, so $H \cdot M \in \mathcal{M}^2$.

Step 3:

$$\mathbb{E}[(H \cdot M)_\infty^2] \leq \liminf_{t \to \infty} \mathbb{E}[(H \cdot M)_t^2] \leq \sup_{t \geq 0} \mathbb{E}[(H \cdot M)_t^2] \leq 4\,\|H\|_\infty^2\, \mathbb{E}[(M_\infty - M_0)^2].$$

□

4.1 Space of integrators

For X càdlàg and adapted, define the norm:

$$|||X||| = \|X^*\|_{L^2}, \quad X^* = \sup_{t\geq 0} |X_t|.$$

$$\mathcal{C}^2 = \{X \text{ càdlàg, adapted processes } X \text{ with } |||X||| < \infty\}.$$

Define the norm on $\mathcal{M}^2$ is given by

$$|||X||| = \|X_\infty\|_{L^2}.$$

Clearly $\|\cdot\|$ is a seminorm. To see that it is a norm, suppose that

$$|||X||| = \|X_\infty\|_{L^2} = 0 \Rightarrow X_\infty = 0 \text{ a.s.} \Rightarrow X_t = \mathbb{E}[X_\infty|\mathcal{F}_t] = 0 \text{ a.s. for all } t \geq 0.$$

Càdlàg property implies $X \equiv 0$ a.s.

Setup:

$$\mathcal{M} = \{\text{càdlàg martingales}\}$$

$$\mathcal{M}_c = \{\text{continuous martingales}\}$$

$$\mathcal{M}_{\text{c, loc}} = \{\text{cont. loc. martingales}\}$$

Lecture 7

Proposition 4.3.

a) *$(\mathcal{C}^2, |||X\cdot|||$ is complete.*

b) *$\mathcal{M}^2 = \mathcal{M} \cap \mathcal{C}^2$*

c) *$(\mathcal{M}^2, \|\cdot\|)$ is a Hilbert space.*

d) *$\mathcal{M}_c^2 := \mathcal{M}_c \cap \mathcal{M}^2$ is a closed subspace.*
The map

$$\mathcal{M}^2 \to L^2(\mathcal{F}_\infty), \qquad X \mapsto X_\infty$$

is an isometry, where

$$\mathcal{F}_\infty = \sigma(\mathcal{F}_t : t > 0).$$

Remark. *We can identify an element of L^2 with its final value, so $(\mathcal{M}^2, \|\cdot\|)$ inherits the Hilbert space structure of $(L^2(\mathcal{F}_\infty), \|\cdot\|_{L^2})$. Since $(\mathcal{M}_c^2, \|\cdot\|)$ is a closed linear subspace of $(\mathcal{M}^2, \|\cdot\|)$, it is also a Hilbert space. This is the space of processes against which we will integrate.*

Proof. **(a)** Suppose that (X^n) is Cauchy with respect to $\|\cdot\|$. Then there exists a subsequence $(X^{n_k})_{k=1}$ of (X^n) such that

$$\sum_k \|X^{n_k} - X^{n_{k+1}}\| < \infty.$$

Thus,

$$\left\| \sum_k \sup_t |X_t^{n_k} - X_t^{n_{k+1}}| \right\|_{L^2} \leq \sum_k \|X^{n_k} - X^{n_{k+1}}\| < \infty$$

$$\Rightarrow \sum_{k\geq 0} \sup_{t\geq 0} |X_t^{n_k} - X_t^{n_{k+1}}| < \infty \text{ a.s.}$$

$\Rightarrow (X^{n_k})_{t\geq 0}$ is uniformly Cauchy on $[0,\infty)$ a.s., hence converges to a càdlàg limit X.

NTS: $X^n \to X$ with respect to $|||\cdot|||$.

$$|||X - X^n|||^2 = \mathbb{E}\left[\sup_{t\geq 0} |X_t - X_t^n|^2\right] = \mathbb{E}\left[\lim_{k\to\infty} \sup_{t\geq 0} |X_t^n - X_t^{n_k}|^2\right]$$

$$\overset{\textbf{Fatou}}{\leq} \liminf_{k\to\infty} \mathbb{E}\left[\sup_{t\geq 0} |X_t^n - X_t^{n_k}|^2\right] \leq \left(\liminf_{k\to\infty} |||X^n - X^{n_k}|||\right)^2 \to 0 \quad \text{a.s.}$$

Since X^m is Cauchy.

(b) Suppose that $X \in \mathcal{C}^2 \cap \mathcal{M}$. Then

$$|||X||| < +\infty \Rightarrow \sup_{t\geq 0} \|X_t\|_{L^2} \overset{\textbf{Jensen}}{\leq} \left\|\sup_{t\geq 0} |X_t|\right\|_{L^2} < \infty \Rightarrow X \in \mathcal{M}^2$$

Suppose that $X \in \mathcal{M}^2$. By Doob's L^2-inequality,

$$|||X||| \leq 2\|X_\infty\|_{L^2} \Rightarrow 2\|X\| < \infty \Rightarrow X \in \mathcal{C}^2 \cap \mathcal{M}$$

and so

$$\mathcal{M}^2 = \mathcal{M} \cap \mathcal{C}^2$$

(c) Note that $\langle X, Y\rangle := \mathbb{E}[X_\infty Y_\infty]$ defines an inner product on L^2. For $X \in \mathcal{M}^2$,

$$|||X||| \leq \|X_\infty\|_{L^2} \leq 2|||X||| \quad \text{(Doob's } L^2\text{-inequality)}$$

which shows that

$$\|\cdot\|, |||\cdot||| \text{ are equivalent norms on } \mathcal{M}^2$$

To show that $(\mathcal{M}^2, |||\cdot|||)$ is complete, it suffices to show that $(\mathcal{M}^2, |||\cdot|||)$ is complete. To see this, let X^n be a sequence in $\mathcal{M}^2$ such that

$$|||X^n - X||| \to 0 \text{ as } n\to\infty \text{ where } X \in \mathcal{C}^2$$

(Suffices to show $\mathcal{M}$ is closed.) We know that X is càdlàg, adapted, L^2-bounded since $X \in \mathcal{C}^2$.

NTS: $X \in \mathcal{M}^2$.

Fix $s \leq t$, we have that

$$\begin{aligned}
\|\mathbb{E}[X_t \mid \mathcal{F}_s] - X_s\|_{L^2} &\overset{X^n \text{ is MG}}{=} \|\mathbb{E}[X_t - X_t^n \mid \mathcal{F}_s] + X_s^n - X_s\|_{L^2} \\
&\leq \|\mathbb{E}[X_t - X_t^n \mid \mathcal{F}_s]\|_{L^2} + \|X_s^n - X_s\|_{L^2} \\
&\overset{\text{Jensen}}{\leq} \|X_t^n - X_t\|_{L^2} + \|X_s^n - X_s\|_{L^2} \leq 2\cdot |||X^n - X||| \xrightarrow{n\to\infty} 0
\end{aligned}$$

which implies

$$X \in \mathcal{M}^2 \Rightarrow \mathcal{M}^2 \text{ is closed in } \mathcal{C}^2.$$

(d) True by definition. $\square$

4.2 Space of integrals

Let (X^n) be a sequence of processes. We say that

$$X^n \xrightarrow{\text{ucp}} X \quad \text{uniformly on compact sets in probability (ucp)}$$

if for every $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{s\le t} |X_s^n - X_s| > \varepsilon\right) \to 0 \quad \text{a.s. as } n \to \infty.$$

Theorem 4.2. *Suppose that $M \in \mathcal{M}_{loc}$. There exists a unique (up to indistinguishability), continuous, adapted, non-decreasing process $[M_t]$ such that:*

$$[M]_0 = 0, \quad M^2 - [M] \in \mathcal{M}_{loc}.$$

Moreover, if we set:

$$[M]_t^n = \sum_{k=0}^{\lceil 2^n t\rceil - 1} \left(M_{(k+1)2^{-n}} - M_{k2^{-n}}\right)^2,$$

then

$$[M]^n \xrightarrow{ucp} [M] \quad \text{as } n \to \infty.$$

The process $[M]$ is called the quadratic variation of M.

Example. *Let B be a standard Brownian motion. Then $(B_t^2 - t)_{t\ge 0}$ is a martingale, which implies that $[B]_t = t$. We will prove later that Brownian motion is characterized by this property, i.e., if $M \in \mathcal{M}_{c,loc}$, and $[M]_t = t$ for all $t \ge 0$, then M is a Brownian motion. (Lévy characterization of Brownian motion.)*

Proof. Replace M_t with $M_t - M_0$, so without loss of generality $M_0 = 0$.

Step 1: *Uniqueness.* Suppose that A, A' are two non-decreasing, continuous, adapted processes satisfying the conditions in the theorem. Then

$$A_t - A_t' = (M_t^2 - A_t) - (M_t^2 - A_t').$$

LHS: continuous, bounded variation. RHS: process in $\mathcal{M}_{\text{c,loc}} \Rightarrow A - A'$ constant. Since $A_0 = A_0' = 0 \Rightarrow A = A'$. □

Before we proceed with the proof of existence, we start with a lemma.

Lecture 9

Lemma 4.1. *Suppose that $M \in \mathcal{M}_{c,loc}$ is bounded. Then for any $N \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_N < \infty$, we have that:*

$$\mathbb{E}\left[\left(\sum_{k=0}^{N-1} \underbrace{(M_{t_{k+1}} - M_{t_k})}_{:=\Delta_k}\right)^2\right] \le 48 \cdot \|\!|M|\!\|_{L^\infty}^4 .$$

Proof. First write

$$\mathbb{E}\left[\left(\sum_{k=0}^{N-1}\Delta_k\right)^2\right] \overset{\circledast}{=} \sum_{k=0}^{N-1}\mathbb{E}\left[(\Delta_k)^4\right] + 2\sum_{k=0}^{N-1}\mathbb{E}\left[\Delta_k^2\sum_{j=k+1}^{N-1}\Delta_j^2\right].$$

For each fixed k, we have that:

$$\mathbb{E}\left[\Delta_k^2\sum_{j=k+1}^{N-1}\Delta_j^2\right] = \mathbb{E}\left[\Delta_k^2\mathbb{E}\left[\sum_{j=k+1}^{N-1}\Delta_j^2\,\middle|\,\mathcal{F}_{t_{k+1}}\right]\right]$$

$$\overset{\text{MG orthogonality}}{=} \mathbb{E}\left[\Delta_k^2\mathbb{E}\left[\sum_{j=k+1}^{N-1}\Delta_j^2\,\middle|\,\mathcal{F}_{t_{k+1}}\right]\right]$$

$$= \mathbb{E}\left[\Delta_k^2\mathbb{E}\left[(M_{t_N}-M_{t_{k+1}})^2\,\middle|\,\mathcal{F}_{t_{k+1}}\right]\right] = \mathbb{E}\left[\Delta_k^2\cdot(M_{t_N}-M_{t_{k+1}})^2\right].$$

Hence,

$$\circledast \le \mathbb{E}\left[\left(\max_{0\le j\le N-1}|M_{t_{j+1}}-M_{t_j}|^2\right) + 2\cdot\max_{0\le j\le N-1}|M_{t_N}-M_{t_j}|^2\cdot\left(\sum_{k=0}^{N-1}\Delta_k^2\right)\right]$$

and using the inequality $(a+b)^2 \le 2(a^2+b^2)$, we obtain:

$$\circledast \le 12\cdot\|\!|M|\!\|_{L^\infty}^2\cdot\mathbb{E}\left[\sum_{k=0}^{N-1}\Delta_k^2\right] = 12\cdot\|\!|M|\!\|_{L^\infty}^2\cdot\mathbb{E}\left[\left(\sum_{k=0}^{N-1}\Delta_k\right)^2\right]$$

$$= 12\cdot\|\!|M|\!\|_{L^\infty}^2\cdot\mathbb{E}\left[(M_{t_N}-M_{t_0})^2\right] \le 48\cdot\|\!|M|\!\|_{L^\infty}^4.$$

□

Proof of Theorem 4.2 (Cont'd). Uniqueness

WLOG $M_0 = 0$ (by replacing M_t with $M_t - M_0$ if necessary).

Step 2: $M \in \mathcal{M}_c$ bounded ($M \in \mathcal{M}_c^2$). Fix $T > 0$ and set:

$$H_t^n = \sum_{k=0}^{\lceil 2^n T\rceil - 1} M_{k2^{-n}}\mathbf{1}_{(k2^{-n},(k+1)2^{-n}]}(t).$$

Then $H^n \in \mathcal{S}$ for all n, and set

$$X_t^n = (H^n\cdot M)_t = \sum_{k=0}^{\lceil 2^n T\rceil - 1} M_{k2^{-n}}(M_{(k+1)2^{-n}} - M_{k2^{-n}}).$$

Then $X^n \in \mathcal{M}_c$, bounded implies $X^n \in \mathcal{M}_c^2$. We will show that (X^n) is Cauchy in $(\mathcal{M}_c^2, \|\cdot\|)$, hence has a limit in $\mathcal{M}_c^2$. Fix $n > m \ge 1$ and write

$$H := H^n - H^m \quad \text{so that} \quad X^n - X^m = (H^n - H^m)\cdot M = H\cdot M.$$

Then,

$$\|X^n - X^m\|^2 = \mathbb{E}[(H\cdot M)_\infty^2]$$

$$
\begin{aligned}
&= \mathbb{E}[(H \cdot M)_T^2] \\
&= \mathbb{E}\left[\left(\sum_{k=0}^{\lceil 2^n T\rceil -1} H_{k2^{-n}}(M_{(k+1)2^{-n}} - M_{k2^{-n}})\right)^2\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\lceil 2^n T\rceil -1} H_{k2^{-n}}^2(M_{(k+1)2^{-n}} - M_{k2^{-n}})^2\right] \quad \text{(MG orthogonality)} \\
&\leq \mathbb{E}\left[\sup_{t\in[0,T]} |H_t|^2 \cdot \sum_{k=0}^{\lceil 2^n T\rceil -1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2\right] \\
&\leq \left(\mathbb{E}\left[\sup_{t\in[0,T]} |H_t|^4\right]\right)^{1/2} \cdot \left(\mathbb{E}\left[\left(\sum_{k=0}^{\lceil 2^n T\rceil -1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2\right)^2\right]\right)^{1/2}.
\end{aligned}
$$

First term: (A) $\sup_{t\in[0,T]} |H_t|^4 = \sup_{t\in[0,T]} |H_t^n - H_t^m|^4 \leq 16 \cdot \|M\|_{L^\infty}^4$.

(B) $\sup_{t\in[0,T]} |H_t^n - H_t^m| \to 0$ as $n, m \to \infty$.

Since M is continuous, by the Bounded Convergence Theorem, first term $\to 0$ as $n, m \to \infty$.

Second term:$\leq \left(48 \cdot \|M\|_{L^\infty}^4\right)^{1/2} < \infty \Rightarrow \|X^n - X^m\| \to 0 \quad$ as $n, m \to \infty$. Since $(\mathcal{M}_c^2, \|\cdot\|)$ is complete, there exists $Y \in \mathcal{M}_c^2$ such that

$$X_n \to Y \quad \text{as } n \to \infty \text{ in } \mathcal{M}_c^2.$$

For any n and $1 \leq k \leq \lceil 2^n T\rceil$, we have that

$$
\begin{aligned}
M_{k2^{-n}}^2 - 2X_{k2^{-n}}^n &= \sum_{j=0}^{k-1} (M_{(j+1)2^{-n}} - M_{j2^{-n}})^2 \\
&= [M^n]_{k2^{-n}}.
\end{aligned}
$$

Hence, for all n, $M^2 - 2X^n$ is non-decreasing when restricted to times of the form $\{k2^{-n} : 1 \leq k \leq \lceil 2^n T\rceil\}$. To prove the same is also true for $M^2 - 2Y$, it suffices to show that $X^n \to Y$ a.s. uniformly, at least along a subsequence. This follows from the equivalence of norms $\|\cdot\|$, $|||\cdot|||$.

Set $[M]_t := M_t^2 - 2Y_t$. Then $[M]$ is continuous, adapted, non-decreasing and

$$M^2 - [M] = 2Y \in \mathcal{M}_c.$$

Can extend to all times by applying the above $T = k$, $\forall k \in \mathbb{N}$. *Uniqueness*implies the process obtained with $T = k$, $T = k+1$ restricted to $[0, k]$ is the same.

Step 3: $[M^n] \to [M]$ ucp as $n \to \infty$.

Observe that

$$X^n \to Y \quad \text{in } (\mathcal{M}_c^2, \|\cdot\|) \Rightarrow \sup_{0\leq t\leq T} |X_t^n - Y_t| \to 0 \text{ as } n \to \infty \text{ in } L^2$$

since $\|\cdot\|$, $|||\cdot|||$ are equivalent which implies $\sup_{0\leq t\leq T} |X_t^n - Y_t| \to 0$ in probability.

Now, $[M]^n_t = M^2_{2^{-n}\lceil 2^n t\rceil} - 2X^n_{2^{-n}\lceil 2^n t\rceil}$. So,

$$\sup_{0\le t\le T} |[M]^n_t - [M]_t| \le \sup_{0\le t\le T} \left| M^2_{2^{-n}\lceil 2^n t\rceil} - M^2_t \right| \tag{4.1}$$

$$+ 2\cdot \sup_{0\le t\le T} \left| X_{2^{-n}\lceil 2^n t\rceil^n} - Y_{2^{-n}\lceil 2^n t\rceil} \right| + 2\cdot \sup_{0\le t\le T} \left| Y_{2^{-n}\lceil 2^n t\rceil} - Y_t \right|. \tag{4.2}$$

Each term on RHS converges to zero in probability and so we obtain the ucp convergence.

Lecture 9

Step 4: Let $M_n \in \mathcal{M}_{c,loc}$. *"Localization argument".*

For each $n \in \mathbb{N}$, let $\tau_n = \inf\{t \ge 0 : |M_t| \ge n\}$. Then (τ_n) reduces M and $M_n := M^{\tau_n}$ is a bounded MG for all n. Therefore, there exists a unique continuous, adapted and non-decreasing process $[M^{T_n}]$ such that

$$[M^{T_n}]_0 = 0 \quad \text{and} \quad (M^{T_n})^2 - [M^{T_n}] \in \mathcal{M}_{c,loc}\,.$$

Let $A^n := [M^{T_n}]$. By uniqueness, $(A^{n+1}_{t\wedge T_n}, A^n_t)$ are indistinguishable. Let A be the process such that

$$A_{t\wedge T_n} = A^n_t\,, \text{ for all } n \ge 1\,.$$

Then $M^2_{t\wedge T_n} - A_{t\wedge T_n} \in \mathcal{M}$ for all $n \in \mathbb{N}$ and so $M^2 - A \in \mathcal{M}_{c,loc}$ with reducing sequence (T_n) giving $[M] = A$.

We know that $[M^{T_k}]^n \to [M^{T_k}]$ in ucp as $n \to \infty$ for all k. In other words, for all

$$\varepsilon, T > 0: \quad \mathbb{P}\left[\sup_{0\le t\le T} |[M^{T_k}]^n_t - [M^{T_k}]_t| > \varepsilon\right] \to 0 \quad \text{as } n \to \infty.$$

On $\{T_k \le T\}$, $[M^n]_t = [M^{T_k}]^n_t$ and $[M]_t = [M^{T_k}]_t$. Thus,

$$\mathbb{P}\left[\sup_{0\le t\le T} |[M]^n_t - [M]_t| > \varepsilon\right] \le \mathbb{P}[T_k \le T] + \mathbb{P}\left[\sup_{0\le t\le T} |[M^{T_k}]^n_t - [M^{T_k}]_t| > \varepsilon\right] \to 0$$
$$\text{as } n \to \infty, \text{ then } k \to \infty.$$

LHS $\to 0$ as $n \to \infty$. $\square$

Theorem 4.3. *Let $M \in \mathcal{M}^2_c$. Then $M^2 - [M]$ is a UI martingale.*

Proof. Let $T_n := \inf\{t \ge 0 : [M]_t \ge n\}$ for $n \in \mathbb{N}$. Then $T_n \nearrow \infty$ as $n \to \infty$, T_n is a stopping time, $[M]_{t\wedge T_n} \le n$ and (noting $M^{T_n} \in \mathcal{M}_{c,loc}$, for all $n \ge 1$)

$$|M^2_{t\wedge T_n} - [M]_{t\wedge T_n}| \le n + \sup_{u\ge 0} M^2_u.$$

By Doob's inequality the RHS is integrable and so

$$M^2_{t\wedge T_n} - [M]_{t\wedge T_n} \in \mathcal{M}_c\,.$$

The Optional Stopping Theorem (OST) also gives

$$\mathbb{E}\left[M^2_{t\wedge T_n} - [M]_{t\wedge T_n}\right] = 0 \Rightarrow \mathbb{E}\left[[M]_{t\wedge T_n}\right] = \mathbb{E}\left[M^2_{t\wedge T_n}\right].$$

Send $t \to \infty$; the Monotone Convergence Theorem (MCT) implies

$$\text{LHS} \overset{t\to\infty}{\longrightarrow} \mathbb{E}\left[[M]_{T_n}\right],$$

and the Dominated Convergence Theorem (MCT) also implies

$$\text{RHS} \overset{t\to\infty}{\longrightarrow} \mathbb{E}\left[M_{T_n}^2\right].$$

and so

$$\mathbb{E}\left[[M]_{T_n}\right] = \mathbb{E}\left[M_{T_n}^2\right].$$

Finally, send $n \to \infty$. MCT implies the LHS converges to $\mathbb{E}\left[[M]_\infty\right]$, and the RHS converges to

$$\mathbb{E}\left[M_\infty^2\right] \Rightarrow \mathbb{E}\left[[M]_\infty\right] = \mathbb{E}\left[M_\infty^2\right] < \infty \Rightarrow \mathbb{E}\left[[M]_\infty\right] \text{ is integrable.}$$

Moreover,

$$|M_t^2 - [M]_t| \leq \sup_{u\geq 0} M_u^2 + [M]_\infty.$$

So we conclude the RHS is integrable $\Rightarrow M^2 - [M] \in \mathcal{M}_c$ and UI as it is dominated by an integrable r.v. $\square$

4.3 The Space $L^2(M)$, $M \in \mathcal{M}_c^2$

Recall that $\mathcal{P}$ = previsible σ-algebra:

$$\mathcal{P} = \sigma(\{E \times (s,t] : E \in \mathcal{F}_s,\ s < t\}).$$

For $A \in \mathcal{P}$, define

$$\mu(A) = \mathbb{E}\left[\int_0^\infty \mathbf{1}_A(\omega, s)\, d[M]_s\right].$$

Then μ is a measure on $(\Omega \times [0,\infty), \mathcal{P})$. Moreover, it is uniquely determined by

$$\mu(E \times (s,t]) = \mathbb{E}\left[\mathbf{1}_E\left([M]_t - [M]_s\right)\right] \quad \text{for } s < t,\ E \in \mathcal{F}_s,$$

since $\mathcal{P}$ is generated by sets of this form and they form a π-system. If $H \geq 0$ is previsible, then:

$$\int_{\Omega\times[0,\infty)} H\, d\mu = \mathbb{E}\left[\int_0^\infty H_s\, d[M]_s\right].$$

Definition 4.2. *Let $L^2(\mu) := L^2(\Omega \times [0,\infty), \mathcal{P}, \mu)$.*

Write $\|H\|_{L^2(\mu)} = \|H\|_\mu := \left(\mathbb{E}\left[\int_0^\infty H_s^2\, d[M]_s\right]\right)^{1/2}$. Then $L^2(\mu)$ = previsible processes with $\|H\|_\mu < \infty$, a Hilbert space. This is the space of integrands.

Remark. *$(L^2(\mu), \|\cdot\|_\mu)$ depends on M, since μ depends on M, but the simple processes are always*

$$\mathcal{S} \subseteq L^2(M) \quad \forall M \in \mathcal{M}_c^2.$$

(here $\mathcal{S}$ denotes simple processes)

4.4 Itô integrals

Recall that for

$$H = \sum_{k=0}^{n-1} Z_k \mathbf{1}_{(t_k, t_{k+1}]} \in \mathcal{S}, \quad M \in \mathcal{M}_c^2,$$

we set

$$(H \cdot M)_t := \sum_{k=0}^{n-1} Z_k (M_{t_{k+1} \wedge t} - M_{t_k \wedge t}) \in \mathcal{M}_c^2.$$

This map defines a map

$$L^2(M) \supseteq \mathcal{S} \longrightarrow \mathcal{M}_c^2.$$

We will prove that it defines an isometry between

$$(L^2(\mu), \|\cdot\|_\mu) \quad \text{and} \quad (\mathcal{M}_c^2, \|\cdot\|),$$

when restricted to $\mathcal{S} \subset L^2(M)$. (Itô isometry). Indeed, compute

$$\begin{aligned} \|H \cdot M\|^2 &= \|(H \cdot M)_\infty\|_{L^2}^2 \quad \text{(see calculation from before)} \\ &= \sum_{k=0}^{n-1} \mathbb{E}\left[Z_k^2 (M_{t_{k+1}} - M_{t_k})^2\right]. \end{aligned}$$

Since $M^2 - [M]$ is a martingale, we have that

$$\begin{aligned} \mathbb{E}\left[Z_k^2 (M_{t_{k+1}} - M_{t_k})^2\right] &= \mathbb{E}\left[Z_k^2 \mathbb{E}\left[(M_{t_{k+1}} - M_{t_k})^2 \mid \mathcal{F}_{t_k}\right]\right] \\ &= \mathbb{E}\left[Z_k^2 \mathbb{E}\left[M_{t_{k+1}}^2 - M_{t_k}^2 \mid \mathcal{F}_{t_k}\right]\right] \\ &= \mathbb{E}\left[Z_k^2 \mathbb{E}\left[[M]_{t_{k+1}} - [M]_{t_k} \mid \mathcal{F}_{t_k}\right]\right] \\ &= \mathbb{E}\left[Z_k^2 ([M]_{t_{k+1}} - [M]_{t_k})\right]. \end{aligned}$$

Hence,

$$\begin{aligned} \|H \cdot M\|^2 &= \mathbb{E}\left[\sum_{k=0}^{n-1} Z_k^2 ([M]_{t_{k+1}} - [M]_{t_k})\right] \\ &= \mathbb{E}\left[\int_0^\infty H_s^2 \, d[M]_s\right] = \|H\|_M^2. \end{aligned}$$

Lecture 10

Theorem 4.4 (Itô Isometry)**.** *There exists a unique isometry $I : L^2(M) \to \mathcal{M}_c^2$ such that*

$$I(H) = H \cdot M \quad \textit{for all simple } H (\in \mathcal{S}).$$

Definition: For $M \in \mathcal{L}^2$, $H \in L^2(M)$, let

$$H \cdot M := I(H) \quad \text{where } I \text{ is from the theorem.}$$

To prove the theorem, we first prove that the simple processes are dense in $L^2(M)$.

Lemma 4.2. *Let ν be any finite measure on $\mathcal{P}$. Then $\mathcal{S}$ is dense in $L^2(\mathcal{P},\nu)$. In particular, if $M \in \mathcal{M}_{c,loc}$ and we take $\nu = \mu$, we have that $\mathcal{S}$ is dense in $L^2(M)$.*

Proof. Since $H \in \mathcal{S} \Rightarrow \|H \cdot M\|_{L^\infty} < \infty$, it follows that $\mathcal{S} \subseteq L^2(\mathcal{P},\nu)$. Let $\overline{\mathcal{S}}$ be the closure of $\mathcal{S}$ in $L^2(\mathcal{P},\nu)$. We wish to show: $\overline{\mathcal{S}} = L^2(\mathcal{P},\nu)$. Let $\mathcal{A} := \{A \in \mathcal{P} : \mathbf{1}_A \in \overline{\mathcal{S}}\}$.

We wish to show: $\mathcal{A} = \mathcal{P}$. It is obvious that $\mathcal{A} \subseteq \mathcal{P}$. To see why the other direction holds, note that:

A contains the π-system $\{E \times (s,t] : E \in \mathcal{F}_s,\ s < t\}$, which generates $\mathcal{P}$,

B $\mathcal{A}$ is a λ-system.

By Dynkin's lemma, it follows that $\mathcal{P} \subseteq \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{P}$. Thus, the lemma follows since linear combinations of such indicators are dense in $L^2(\mathcal{P},\nu)$. □

Proof of Itô Isometry. Take $H \in L^2(M)$. The above lemma implies there exists $(H^n) \subset \mathcal{S}$ such that

$$\|H^n - H\|_{L^2(M)} \to 0 \quad \text{as } n \to \infty.$$

This implies (H^n) is a Cauchy sequence with respect to $\|\cdot\|_{L^2(M)}$.

Need to show: $I(H^n)$ is Cauchy with respect to $\|\cdot\|$.

$$\begin{aligned}\|I(H^n) - I(H^m)\| &= \|H^n \cdot M - H^m \cdot M\| \quad \text{(linearity)}\\ &= \|(H^n - H^m) \cdot M\| = \|H^n - H^m\|_M \quad \text{(isometry)}\\ &\to 0 \quad \text{as } n,m \to \infty.\end{aligned}$$

Therefore, $(I(H^n))$ converges with respect to $\|\cdot\|$ to an element in $\mathcal{M}_c^2$. Since $(\mathcal{M}_c^2, \|\cdot\|)$ is complete, set $I(H)$ to be this element.

NTS: I is well-defined.

Suppose that $(K^n) \subset \mathcal{S}$ converges to H with respect to $\|\cdot\|_{L^2(M)}$. Then

$$\begin{aligned}\|I(H^n) - I(K^n)\| &= \|H^n \cdot M - K^n \cdot M\|\\ &= \|H^n - K^n\|_M \le \|H^n - H\|_M + \|K^n - H\|_M \to 0\end{aligned}$$

as $n \to \infty$, so that the limits of $I(H^n), I(K^n)$ are indistinguishable.

NTS: I is an isometry $L^2(M) \to \mathcal{M}_c^2$

$(H^n) \subset \mathcal{S}, H^n \to H \in L^2(M), \|I(H)\| = \lim \|H^n \cdot M\| = \lim \|H^n\|_M = \|H\|_M$. □

From now on, we write

$$I(H)_t = (H \cdot M)_t = \int_0^t H_s \, \mathrm{d}M_s$$

This process $H \cdot M$ is the Itô (stochastic) integral of H with respect to M.

Extensions: Our goal now is to extend the definition of $H \cdot M$ to the setting that H is locally bounded and $M \in \mathcal{M}_{c,\text{loc}}$. Need to understand how the integral behaves under stopping.

Proposition 4.4. *Let $H \in \mathcal{S}, M \in \mathcal{M}$. Then for any stopping time T, we have that*

$$(H \cdot M^T) = (H \cdot M)^T.$$

Proof. We have that:

$$\begin{aligned}(H \cdot M^T)_t &= \sum_{k=0}^{n-1} Z_k (M^T_{t \wedge t_{k+1}} - M^T_{t \wedge t_k}) \\ &= \sum_{k=0}^{n-1} Z_k \left(M_{t \wedge (t_{k+1} \wedge T)} - M_{t \wedge (t_k \wedge T)} \right) \\ &= (H \cdot M)_{t \wedge T} = (H \cdot M)^T_t.\end{aligned}$$

□

Proposition 4.5. *Let $M \in \mathcal{M}_c^2$, $H \in L^2(M)$, and T a stopping time. Then*

$$(H \cdot M)^T = (H \cdot \mathbf{1}_{(0,T]}) \cdot M = (H \cdot M^T).$$

Proof. First note that if $H \in L^2(M)$, then $H \cdot \mathbf{1}_{(0,T]} \in L^2(M)$ and $H \in L^2(M^T)$, so the integrals make sense.

<u>**Step 1:**</u> Let $H \in \mathcal{S}, M \in \mathcal{M}_c^2$, and T takes on finitely many values. Then $H \cdot \mathbf{1}_{(0,T]} \in \mathcal{S}$ and

$$(H \cdot M)^T = (H \cdot \mathbf{1}_{(0,T]}) \cdot M = H \cdot M^T.$$

<u>**Step 2:**</u> Let $H \in \mathcal{S}, M \in \mathcal{M}_c^2$, and T a general stopping time. Previous proposition implies $(H \cdot M)^T = (H \cdot \mathbf{1}_{(0,T]}) \cdot M$. **Need to show:** $(H \cdot M)^T = (H \cdot \mathbf{1}_{(0,T]}) \cdot M$. Will prove via an approximation argument.

For $m, n \in \mathbb{N}$, let $T_{n,m} = (2^{-n} \lceil 2^n T \rceil) \wedge m$. Then $T_{n,m}$ takes finitely many values and $T_{n,m} \searrow T \wedge m$ as $n \to \infty$. Thus,

$$\left\| H \cdot \mathbf{1}_{(0,T_{n,m}]} - H \cdot \mathbf{1}_{(0,T \wedge m]} \right\|^2_{L^2(M)} = \mathbb{E}\left[\int_0^\infty H_t^2 \cdot \mathbf{1}_{(T_{n,m}, T \wedge m]} \, d[M]_t \right] \to 0,$$

as $n \to \infty$ by the Dominated Convergence Theorem, with dominating function H_t^2. Therefore, $(H \cdot \mathbf{1}_{(0,T_{n,m}]}) \cdot M \to (H \cdot \mathbf{1}_{(0,T \wedge m]}) \cdot M$ in $\mathcal{M}_c^2$ as $n \to \infty$.

<u>**Step 3:**</u>

$$\text{LHS} = (H \cdot M)^{T_{n,m}}, \quad (H \cdot M)^{T_{n,m}} \to (H \cdot M)^{T \wedge m}$$

pointwise almost surely by continuity of $H \cdot M$. Thus,

$$(H \cdot \mathbf{1}_{(0,T \wedge m]}) \cdot M \to (H \cdot M)^{T \wedge m}.$$

Repeat the same argument, send $n \to \infty$

$$\Rightarrow H \cdot \mathbf{1}_{(0,T]} \cdot M = (H \cdot M)^T.$$

Step 3: Let $H \in L^2(M), M \in \mathcal{M}_c^2, T$ a general stopping time.

Let (H^n) be a sequence in $\mathcal{S}$ with $H^n \to H$ in $L^2(M)$. Then,

$$\left\| (H^n \cdot M)^T - (H \cdot M)^T \right\|_{\mathcal{M}_c^2} = \| (H^n \cdot M)_T - (H \cdot M)_T \|_{L^2}$$

$$
\begin{aligned}
&\leq \left\| \sup_{t\leq T} (H^n \cdot M)_t - (H\cdot M)_t \right\|_{L^2} \\
&\leq 2 \cdot \|(H^n \cdot M)_\infty - (H\cdot M)_\infty\|_{L^2} \quad \text{(Doob's } L^2 \text{ inequality)} \\
&= 2 \cdot \|(H^n - H)\cdot M\| = 2\cdot \|H^n - H\|_M \to 0 \text{ as } n\to\infty
\end{aligned}
$$

(by Itô isometry) and so

$$(H^n \cdot M)^T \to (H\cdot M)^T \text{ in } \mathcal{M}_c^2.$$

On the other hand,

$$
\begin{aligned}
\left\| H^n \cdot \mathbf{1}_{(0,T]} - H\cdot \mathbf{1}_{(0,T]} \right\|_M^2 &= \mathbb{E}\left[\int_0^\infty (H_t^n - H_t)^2 \cdot \mathbf{1}_{(0,T]}\, d[M]_t \right] \\
&\leq \mathbb{E}\left[\int_0^\infty (H_t^n - H_t)^2 \, d[M]_t \right] = \|H^n - H\|_M^2 \to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Hence,

$$H^n \cdot \mathbf{1}_{(0,T]} \cdot M \to H\cdot \mathbf{1}_{(0,T]}\cdot M \text{ in } \mathcal{M}_c^2 \text{ by the Itô isometry.}$$

Since $H^n \cdot \mathbf{1}_{(0,T]} \cdot M = (H^n\cdot M)^T$ for all n, we have that

$$(H\cdot M)^T = H\cdot \mathbf{1}_{(0,T]} \cdot M.$$

$\square$

NTS: $(H\cdot M)^T = (H\circ M^T)$. Assume there exists (H^n) in $\mathcal{S}$ such that $H^n \to H$ in $L^2(\mu)$.

$$
\begin{aligned}
\|H^n - H\|_{\mu^T}^2 &= \mathbb{E}\left[\int_0^\infty (H_s^n - H_s)^2\, d[M^T]_s \right] \\
&= \mathbb{E}\left[\int_0^\infty (H_s^n - H_s)^2 \cdot \mathbf{1}_{(0,T]}\, d[M]_s \right] \\
&\leq \|H^n - H\|_\mu^2 \to 0 \text{ as } n\to\infty.
\end{aligned}
$$

$$\Rightarrow H^n \circ M^T \to H\circ M^T \text{ in } \mathcal{M}_c^2 \text{ by Itô isometry.}$$

Since $(H^n\cdot M)^T = H^n \circ M^T$ for all n, we get that

$$(H\cdot M)^T = (H\circ M^T). \qquad \square$$

Definition 4.3. *We say that a previsible process H is* <u>*locally bounded*</u> *if there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of stopping times where $S_n \nearrow \infty$ as $n\to\infty$ and $H\cdot \mathbf{1}_{(0,S_n]}$ is bounded for all n.*

Remark. *Every continuous adapted process is previsible and locally bounded.*

Definition 4.4. *Let H be a locally bounded, previsible process with $H\cdot \mathbf{1}_{[0,S_n]}$ bounded for all n, where (S_n) is a sequence of stopping times with $S_n \nearrow \infty$ as $n\to\infty$. Let $M\in\mathcal{M}_{c,loc}$ with $M_0 = 0$ and let*

$$S_n' := \inf\{t\geq 0 : |M_t| \geq n\}$$

so that $M^{S_n'} \in \mathcal{M}_c^2$ for all n. Let $T_n := S_n \wedge S_n'$, and set

$$(H\cdot M)_t := (H\mathbf{1}_{(0,T_n]}\cdot M^{T_n})_t, \quad \forall t\in[0,T_n].$$

Using the previous proposition, this definition is well-defined, and is consistent with the Itô integral with $M\in\mathcal{M}_c^2$, $H\in L^2(M)$.

Proposition 4.6. *Let $M \in \mathcal{M}_{c,loc}$, H locally bounded and previsible, then $H \cdot M \in \mathcal{M}_{c,loc}$ where the sequence (T_n) is a reducing sequence. Moreover, for any stopping time T, we have that*

$$(H \cdot M)^T = H\mathbf{1}_{(0,T]} \cdot M = H \cdot M^T.$$

Proof. That $H \cdot M \in \mathcal{M}_{c,\text{loc}}$ with reducing sequence (T_n) follows from the definition of $H \cdot M$. For any stopping time T,

$$(H \cdot M)^T = \lim_{n\to\infty} (H\mathbf{1}_{(0,T_n]} \cdot M^{T_n})^T \quad \text{(pointwise limit)}.$$

By the previous proposition,

$$(H \cdot M)^T = \lim_{n\to\infty} (H\mathbf{1}_{(0,T]} \cdot \mathbf{1}_{(0,T_n]} \cdot M^T = H \cdot \mathbf{1}_{(0,T]} \circ M.$$

The same argument shows that $(H \cdot M)^T = H \cdot M^T$. □

Lecture 12

Today we will show

$$[H \cdot M] = H^2 \cdot [M], \quad H \cdot (K \cdot M) = (HK) \cdot M,$$

for semimartingales.

Proposition 4.7. *Let $M \in \mathcal{M}_{c,loc}$ and H locally bounded and previsible. Then*

$$\underbrace{[H \cdot M]}_{\textit{Itô}} = \underbrace{H^2 \cdot [M]}_{\textit{Lebesgue-Stieltjes}}.$$

Proof. Suppose that T is a bounded stopping time. Then H, M are uniformly bounded. Then

$$\begin{aligned}
\mathbb{E}\left[(H \cdot M)_T^2\right] &= \mathbb{E}\left[\left((H \cdot \mathbf{1}_{(0,T]}) \cdot M\right)_\infty^2\right] \\
&= \mathbb{E}\left[\left(H^2 \cdot \mathbf{1}_{(0,T]} \cdot [M]\right)_\infty\right] && \text{(Itô isometry)} \\
&= \mathbb{E}\left[\left(H^2 \cdot [M]\right)_T\right].
\end{aligned}$$

OST: $(H \cdot M)^2 - H^2 \cdot [M] \in \mathcal{M}_c$. Uniqueness of quadratic variation implies

$$[H \cdot M] = H^2 \cdot [M].$$

Now assume that H is locally bounded, previsible, and $M \in \mathcal{M}_{c,\text{loc}}$. Let (T_n) be a sequence of stopping times so that $H \cdot \mathbf{1}_{(0,T_n]}, M^{T_n}$ are bounded, and $T_n \to \infty$ as $n \to \infty$. Then

$$\begin{aligned}
[H \cdot M] &= \lim_{n\to\infty} [H \cdot M]^{T_n} \\
&= \lim_{n\to\infty} [(H \cdot M)^{T_n}] && \text{(uniqueness of quadratic variation)} \\
&= \lim_{n\to\infty} [(H\mathbf{1}_{(0,T_n]}) \cdot M] \\
&= \lim_{n\to\infty} H^2\mathbf{1}_{(0,T_n]} \cdot [M^{T_n}] \\
&= H^2 \cdot [M] \quad \text{(applying MCT)}. \quad \square
\end{aligned}$$

□

Since $H \cdot M \in \mathcal{M}_{c,\text{loc}}$ for $M \in \mathcal{M}_{c,\text{loc}}$, H locally bounded, previsible, we can integrate against it.

Proposition 4.8. *Let $M \in \mathcal{M}_{c,loc}, H, K$ locally bounded, previsible. Then:*

$$H \cdot (K \cdot M) = (HK) \cdot M.$$

Proof. Elementary to check that this holds for H, K simple processes, . Note that by linearity in each argument, it suffices to check for H, K consisting of single time intervals and noticing that for $0 \leq s'' < s' < t'$, $0 < s < t$,

$$\mathbf{1}_{(s'' \wedge t', t' \wedge t]} - \mathbf{1}_{(s \wedge t', t' \wedge s'')} = \mathbf{1}_{(s'' \wedge t', t')} \cdot \mathbf{1}_{(s', t]}$$

Now suppose that H, K, M are uniformly bounded. **NTS:** $H \in L^2(K \cdot M)$, $HK \in L^2(M)$.

$$\begin{aligned}
\|H\|^2_{L^2(K \cdot M)} &= \mathbb{E}\left[(H^2 \cdot [K \cdot M])_\infty\right] \\
&= \mathbb{E}\left[\Big(H^2 \cdot (K^2 \cdot [M])\Big)_\infty\right] \\
&= \mathbb{E}\left[\Big((HK)^2 \cdot [M]\Big)_\infty\right] && \text{(Lebesgue–Stieltjes)} \\
&= \|HK\|^2_{L^2(M)} \\
&\leq \min\left\{\|H\|^2_\infty \|K\|^2_{L^2(M)}, \|K\|^2_\infty \|H\|^2_{L^2(M)}\right\} < \infty.
\end{aligned}$$

Let $(H^n), (K^n)$ be sequences in $\mathcal{S}$ which converge to H, K in $L^2(M)$ and where $(H^n), (K^n)$ uniformly bounded. Then

$$H^n \cdot (K^n \cdot M) = (H^n K^n) \cdot M.$$

Then

$$\begin{aligned}
\|H^n \cdot (K^n \cdot M) - H \cdot (K \cdot M)\| &\leq \|(H^n - H) \cdot (K^n \cdot M)\| + \|H \cdot ((K^n - K) \cdot M)\| \\
&= \|H^n - H\|_{L^2(K^n \cdot M)} + \|H\|_{L^2((K^n - K) \cdot M)} && \text{(Itô isomet} \\
&= \|(H^n - H) \cdot K^n\|_{L^2(M)} + \|H \cdot (K^n - K)\|_{L^2(M)} \\
&\leq \|K^n\|_\infty \|H^n - H\|_{L^2(M)} + \|H\|_\infty \|K^n - K\|_{L^2(M)} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

A similar argument shows $(H^n K^n) \cdot M \to (HK) \cdot M$ as $n \to \infty$ in $\mathcal{M}_c$ yielding

$$H \cdot (K \cdot M) = (HK) \cdot M \quad \text{(bounded case)}.$$

Now suppose that H, K are locally bounded, previsible and $M \in \mathcal{M}_{c,\text{loc}}$. Let (T_n) be a sequence of stopping times so that

$$H\mathbf{1}_{[0,T_n]},\ K\mathbf{1}_{[0,T_n]},\ M^{T_n} \text{ are bounded and } T_n \nearrow \infty \text{ as } n \to \infty.$$

Then

$$HK\mathbf{1}_{[0,T_n]} \cdot M^{T_n} = \left(H\mathbf{1}_{[0,T_n]}\right) \cdot \left(K\mathbf{1}_{[0,T_n]} \cdot M^{T_n}\right).$$

Also,

$$K\mathbf{1}_{[0,T_n]} \cdot M^{T_n} = (K \cdot M)^{T_n}.$$

Hence,

$$H\mathbf{1}_{[0,T_n]} \cdot (K\mathbf{1}_{[0,T_n]} \cdot M)^{T_n} = H\mathbf{1}_{[0,T_n]} \cdot (K \cdot M)^{T_n} = (H \cdot (K \cdot M))^{T_n} \to H \cdot (K \cdot M) \quad \text{as } n \to \infty.$$

Also,

$$(HK\mathbf{1}_{[0,T_n]}) \cdot M^{T_n} = (HK \cdot M)^{T_n} \to (HK \cdot M) \quad \text{as } n \to \infty$$

which finally gives

$$H \cdot (K \cdot M) = (HK) \cdot M.$$

Remark. *We have repeatedly used a "localisation" argument to reduce everything to the setting of a bounded integrand and martingale. This is a standard procedure; will omit in later arguments.*

5 Semimartingales

Definition 5.1. *A continuous, adapted process X is a* semimartingale *if it can be decomposed as*

$$X = X_0 + M + A$$

where $M \in \mathcal{M}_{c,loc}$, A is of finite variation, and $M_0 = A_0 = 0$.

"Doob–Meyer decomposition": For a continuous semi-martingale $X = X_0 + M + A$, define the quadratic variation by $[X]_t := [M]_t$. Justified since once can compute ( )

$$\sum_{k=0}^{\lceil 2^n t \rceil - 1} \left(X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}} \right)^2 \xrightarrow[n\to\infty]{ucp} [M]_t\,.$$

Definition 5.2. *For H locally bounded and previsible, and $X = X_0 + M + A$ a continuous semimartingale, define (Here, the first term is the Itô integral, the second is Lebesgue–Stieltjes.)*

$$H \cdot X := H \cdot M + \int H_s \, dA_s\,.$$

Then $H \cdot X$ is also a semimartingale.

Proposition 5.1. *Let X be a continuous semimartingale and H locally bounded, left-continuous and adapted. Then:*

$$\sum_{k=0}^{\lceil 2^n t \rceil - 1} H_{k\cdot 2^{-n}} (X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}}) \xrightarrow[n\to\infty]{ucp} (H \cdot X)_t$$

Proof. . Hint: use a localisation argument first. Show that the Itô integral of H can be approximated by discretely approximating H by simple processes. □

Lecture 13

Summary of the Stochastic Integral

Step 1: $H \in \mathcal{S}$, $H_t = \sum_{k=0}^{n-1} Z_k \cdot \mathbf{1}_{(t_k, t_{k+1}]}(t)$,
Z_k bounded, $\mathcal{F}_{t_k}$-measurable, $M \in \mathcal{M}_c^2$ set:

$$(H \cdot M)_t = \sum_{k=0}^{n-1} Z_k (M_{t\wedge t_{k+1}} - M_{t\wedge t_k}).$$

Then $H \cdot M \in \mathcal{M}_c^2$.

Step 2: Equip $\mathcal{M}_c^2$ with a Hilbert space structure with norm $\|M\| = \|M_\infty\|_{L^2}$, $M \in \mathcal{M}_c^2$.

Step 3: Establish the existence of $[M] \in \mathcal{M}_{\text{c, loc}}$, where $[M]$ is the unique adapted, non-decreasing continuous process with $[M]_0 = 0$ so that $M^2 - [M] \in \mathcal{M}_{\text{c, loc}}$.

Step 4: For $M \in \mathcal{M}_c^2$, use $[M]$ to define a Hilbert space $(L^2(M), \|\cdot\|_M$ where

$$\|H\|_M = \left(\mathbb{E}\left[\int_0^\infty H_s^2 \, d[M]_s\right]\right)^{1/2}$$

Step 5: Extend the integral to $H \in L^2(M)$, $M \in \mathcal{M}_c^2$ using the Itô isometry:

$$\|H \cdot M\| = \|H\|_{\mathcal{H}_M}$$

$H \cdot M \in \mathcal{M}_c^2$ for all $H \in L^2(M)$, $M \in \mathcal{M}_c^2$.

Step 6: Extended to H locally bounded & previsible, $M \in \mathcal{M}_{\text{c, loc}}$ by setting

$$(H \cdot M)_t = (H \mathbf{1}_{[0,\tau_n]} \cdot M^{\tau_n})_t \quad \forall t \le \tau_n$$

Step 7: Extend to H locally bounded, previsible and $X = X_0 + M + A$ a continuous semimartingale by setting

$$H \cdot X = \underbrace{H \cdot M}_{\text{Itô}} + \underbrace{H \cdot A}_{\text{Lebesgue–Stieltjes}}$$

then $H \cdot X$ is a continuous semimartingale.

Stochastic Calculus

Definition 5.3. *For $M, N \in \mathcal{M}_{c,loc}$, define the covariation of M, N by setting:*

$$[M, N] := \frac{1}{4}\left([M+N] - [M-N]\right).$$

(Polarization identity). Note that: $[M, M] = [M]$.

Theorem 5.1. *Let $M, N \in \mathcal{M}_{c,loc}$. Then:*

(a) *$[M, N]$ is the unique process (up to indistinguishability), continuous, adapted, finite-variation process with $[M, N]_0 = 0$, so that $MN - [M, N] \in \mathcal{M}_{c,loc}$.*

(b) *For $n \in \mathbb{N}$, set*

$$[M, N]_t^n := \sum_{k=0}^{\lceil 2^n t \rceil - 1} \left(M_{(k+1)2^{-n}} - M_{k2^{-n}}\right)\left(N_{(k+1)2^{-n}} - N_{k2^{-n}}\right).$$

Then $[M, N]_t^n \to [M, N]_t$ as $n \to \infty$, almost surely and locally uniformly in t.

(c) *If $M, N \in \mathcal{M}_c^2$, then $MN - [M, N]$ is a UI martingale.*

(d) *For H locally bounded, previsible,*

$$[H \cdot M, N] + [M, H \cdot N] = 2H \cdot [M, N].$$

Proof. (a) $MN = \frac{1}{4}\left((M+N)^2 - (M-N)^2\right)$. So

$$\circledast\, MN - [M,N] = \frac{1}{4}\Big((M+N)^2 - [M+N] - (M-N)^2 + [M-N]\Big), \quad \in \mathcal{M}_{c,\mathrm{loc}}.$$

Therefore, $MN - [M,N] \in \mathcal{M}_{c,\mathrm{loc}}$. By definition, $[M,N]$ is continuous, adapted and finite-variation (difference of non-decreasing functions). Same argument used to prove the uniqueness of covariation.

(b) Note that

$$\begin{array}{ccccc} [M,N]_t^n & = \frac{1}{4} & ([M+N]_t^n & - & [M-N]_t^n) \\ \downarrow_{\mathrm{ucp}} & & \downarrow_{\mathrm{ucp}} & & \downarrow_{\mathrm{ucp}} \\ [M,N] & & [M+N] & & [M-N] \end{array}$$

So $[M,N]_t^n \to [M,N]_t$ ucp.

(c) $MN - [M,N]$ is a UI martingale for $M,N \in \mathcal{M}_c^2$, follows from the identity $\circledast$ and the corresponding property for quadratic variation.

(d)

$$[H\cdot(M+N)] = H^2 \cdot [M+N],$$

so

$$[H\cdot M, H\cdot N] = H\cdot[M,N].$$

Moreover,

$$(H+1)^2\cdot[M,N] = [(H+1)\cdot M, (H+1)\cdot N]$$

by bilinearity ( )

$$\begin{aligned} &= [H\cdot M + M, H\cdot N + N] \\ &= [H\cdot M, H\cdot N] + [H\cdot M, N] + [M, H\cdot N] + [M,N], \end{aligned}$$

and

$$\begin{aligned} (H+1)^2\cdot[M,N] &= (H^2+2H+1)\cdot[M,N] \\ &= H^2\cdot[M,N] + 2H\cdot[M,N] + [M,N]. \end{aligned}$$

giving

$$2H\cdot[M,N] = [M, H\cdot N] + [H\cdot M, N].$$

□

Proposition 5.2 (Kunita–Watanabe identity)**.** *Let $M,N \in \mathcal{M}_{c,\mathrm{loc}}$, H locally bounded, previsible. Then*

$$[H\cdot M, N] = H\cdot[M,N].$$

Proof. **<u>NTS:</u>** $[H\cdot M, N] = [N, H\cdot M]$, as then we can apply part (d) of the previous theorem.

Now, use that

$$\begin{aligned} (H\cdot M)N - [H\cdot M, N] &\in \mathcal{M}_{c,\mathrm{loc}}, \\ M(H\cdot N) - [M, H\cdot N] &\in \mathcal{M}_{c,\mathrm{loc}}. \end{aligned}$$

We will show that

$$(H\cdot M)N - M(H\cdot N) \in \mathcal{M}_{c,\mathrm{loc}}.$$

This suffices, since then $[H \cdot M, N] - [M, H \cdot N] \in \mathcal{M}_{c,\text{loc}}$ with finite variation and starts from 0, so

$$[H \cdot M, N] = [M, H \cdot N].$$

<u>Localisation:</u> WLOG $M, N \in \mathcal{M}_c^2$, H bounded.

By optional stopping, it suffices to show that for bounded stopping time T,

$$\mathbb{E}[(H \cdot M)_T N_T] = \mathbb{E}[M_T (H \cdot N)_T].$$

LHS $= \mathbb{E}[(H \cdot M)_\infty^T N_\infty^T]$, RHS $= \mathbb{E}[M_\infty^T (H \cdot N)_\infty^T]$. So it suffices to show that

$$\mathbb{E}[(H \cdot M)_\infty N_\infty] = \mathbb{E}[M_\infty (H \cdot N)_\infty]$$

for all $M, N \in \mathcal{M}_c^2$, bounded H. Suppose now that $H = Z\mathbf{1}_{(s,t]}$, Z $\mathcal{F}_s$-measurable, bounded. We then compute

$$\begin{aligned}
\mathbb{E}[(H \cdot M)_\infty N_\infty] &= \mathbb{E}\left[Z(M_t - M_s) N_\infty\right] \\
&= \mathbb{E}[Z M_t \mathbb{E}[N_\infty \mid \mathcal{F}_t] - Z M_s \mathbb{E}[N_\infty \mid \mathcal{F}_s]] \\
&= \mathbb{E}\left[Z(M_t N_t - M_s N_s\right] \\
&= \mathbb{E}[M_\infty (H \cdot N)_\infty],
\end{aligned}$$

Same argument the same argument gives

$$\mathbb{E}[M_\infty (H \cdot N)_\infty] = \mathbb{E}[M_\infty (H \cdot N)_\infty]$$

for $H = \sum Z\mathbf{1}_{(s,t]}$. Linearity gives $\circledast$ for $H \in \mathcal{S}$.

Lecture 14

Suppose now that H is a bounded predictable process. Then there exists a sequence $(H^n) \subset \mathcal{S}$ so that $H^n \to H$ in $L^2(M), L^2(N)$ (in the lemma where we showed that $\mathcal{S}$ are dense in $L^2(\mathbb{P}, \nu)$, ν finite, to be given by $\nu(E) = \mathbb{E}\left[\int_0^\infty \mathbf{1}_E (\mathrm{d}[M]_s + \mathrm{d}[N]_s)\right]$). Hence,

$$H^n \cdot M \to H \cdot M, \quad H^n \cdot N \to H \cdot N \text{ in } \|\cdot\|\text{-norm}$$

and so

$$H^n \cdot M)_\infty \to (H \cdot M)_\infty \text{ and in } L^2$$

and

$$(H^n \cdot N)_\infty \to (H \cdot N)_\infty \quad \text{as } n \to \infty$$

Thus,

$$\begin{aligned}
\|\mathbb{E}[(H^n \cdot M)_\infty N_\infty] - \mathbb{E}[(H \cdot M)_\infty N_\infty]\|_{L^1} &\overset{\text{C-S}}{\leq} \|(H^n \cdot M)_\infty - (H \cdot M)_\infty\|_{L^2} \|N_\infty\|_{L^2} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus,

$$\mathbb{E}[(H^n \cdot M)_\infty N_\infty] \overset{n\to\infty}{\longrightarrow} \mathbb{E}[(H \cdot M)_\infty N_\infty]$$

Same works with M, N swapped which finally gives $\circledast$. $\square$

Definition 5.4. *For continuous semi-martingales X, Y, define $[X, Y]$ to be the covariation of their martingale parts.*

- *This is justified as*

$$[X,Y]_t^n = \sum_{k=0}^{\lceil 2^n t\rceil - 1} (X_{(k+1)2^{-n}} - X_{k2^{-n}})(Y_{(k+1)2^{-n}} - Y_{k2^{-n}})$$

$$\xrightarrow{ucp} [X,Y]_t \text{ as } n \to \infty$$

- *Kunita–Watanabe also holds for semi-martingales.*

Proposition 5.3. *Let X, Y be independent semi-martingales. Then their covariation $[X,Y] = 0$.*

Proof. [ ] . □

5.1 Itô's formula

Theorem 5.2 (Integration by parts)**.** *Let X, Y be continuous semi-martingales. Then*

$$X_tY_t - X_0Y_0 = \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + [X,Y]_t. \qquad \circledast$$

Proof. Note that the integrals are well-defined since any continuous adapted process is locally bounded and predictable.

Note that for $s \le t$, we have

$$\begin{aligned} X_tY_t - X_sY_s &= X_s(Y_t - Y_s) + Y_s(X_t - X_s) \\ &\quad + (X_t - X_s)(Y_t - Y_s). \end{aligned}$$

Since the LHS and RHS of identity $\circledast$ are continuous, it suffices to prove the result for t of the form

$$t = m \cdot 2^{-j}, \quad m, j \in \mathbb{N}, \quad (n \ge j),$$

$$\begin{aligned} X_tY_t - X_0Y_0 = \sum_{k=0}^{m\cdot 2^{n-j}-1} & (X_{k\cdot 2^{-n}}(Y_{(k+1)\cdot 2^{-n}} - Y_{k\cdot 2^{-n}}) + Y_{k\cdot 2^{-n}}(X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}}) \\ & + (X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}})(Y_{(k+1)\cdot 2^{-n}} - Y_{k\cdot 2^{-n}})). \\ & \xrightarrow{\text{ucp}} (X \cdot Y)_t + (Y \cdot X)_t + [X,Y]_t \text{ as } j \to \infty. \end{aligned}$$

□

Note that the $[X,Y]$ term does not appear if either X, Y are independent or if X or Y does not have a martingale part.

Theorem 5.3 (Itô's Formula)**.** *Let $(X^1, \dots, X^d)$ where each X^i, for $1 \le i \le d$, is a continuous semi-martingale. Let $f : \mathbb{R}^d \to \mathbb{R}$ be C^2. Then,*

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s)\,dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\,d[X^i, X^j]_s.$$

Remark. 1. *Integration by parts is a special case of Itô's formula with $f(x,y) = x \cdot y$.*

2. For $d=1$, Itô's formula reads:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d[X]_s.$$

It is possible to derive this using Taylor expansions, since:

$$f(X_t) = f(X_0) + \sum_{k=0}^{\lfloor 2^n t\rfloor - 1} \Big(f(X_{(k+1)2^{-n}}) - f(X_{k2^{-n}}) \Big)$$

$$= f(X_0) + \sum_{k=0}^{\lfloor 2^n t\rfloor - 1} f'(X_{k2^{-n}})(X_{(k+1)2^{-n}} - X_{k2^{-n}}) + \frac{1}{2}\sum_{k=0}^{\lfloor 2^n t\rfloor - 1} f''(X_{k2^{-n}})(X_{(k+1)2^{-n}} - X_{k2^{-n}})^2 + \textit{error}.$$

$$\longrightarrow f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d[X]_s \quad \textit{(ucp as } n \to \infty\textit{)}.$$

We will prove it a different way, since the extra error term is inconvenient to deal with.

Examples.

1. Let $X = B$, a standard Brownian motion, and $f(x) = x^2$. Then:

$$f(X_t) = f(B_t) = f(B_0) + \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,d[B]_s$$

$$= 0 + \int_0^t 2B_s\,dB_s + \frac{1}{2}\int_0^t 2\,ds = 2\int_0^t B_s\,dB_s + t$$

which gives

$$B_t^2 - t = 2\int_0^t B_s\,dB_s \in \mathcal{M}_{c,\mathrm{loc}}.$$

2. Let $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be $C^{1,2}$, and define

$$X_t = (t, B_t^1, \ldots, B_t^d)$$

where $B_t^1, \ldots, B_t^d$ are independent Brownian motions. By Itô's formula:

$$f(t, B_t) - f(0, B_0) = \int_0^t \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta \right) f(s, B_s)\,ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, B_s)\,dB_s^i \in \mathcal{M}_{c,\mathrm{loc}}.$$

Here, Δ is the Laplacian in the spatial coordinates.

If f does not depend on t and is harmonic in spatial variables, then $f(B_t) \in \mathcal{M}_{c,\mathrm{loc}}$. If f is bounded, then $f(B_t)$ is a martingale.

Lecture 15

Proof (Itô's Formula). We are doing the proof for $d = 1$; the case $d > 1$ is just notationally more cumbersome but the same argument essentially applies, . Let

$$X = X_0 + M + A$$

and let V be the total variation of A. Let

$$T_r = \inf\{t \geq 0 : |X_t| + V_t + [M]_t > r\}$$

for each $r > 0$. Then (T_r) is a sequence of stopping times with $T_r \nearrow \infty$ as $r \to \infty$.

It suffices to prove the formula on $[0, T_r]$ for each $r > 0$. Let $\mathcal{A}$ be the subset of $C_c^2(\mathbb{R})$ such that the formula holds. We show $\mathcal{A} = C_c^2(\mathbb{R})$.

We will prove this by showing

(a) $\mathcal{A}$ contains $f(x) \equiv 1$, $f(x) \equiv x$.

(b) $\mathcal{A}$ is a vector space.

(c) $\mathcal{A}$ is an algebra, i.e., $f, g \in \mathcal{A} \Rightarrow fg \in \mathcal{A}$.

(d) If $(f_n) \subset \mathcal{A}$ with

$$f_n \to f \text{ in } C^2(\overline{B_r}) \text{ for each } r > 0$$

(where $B_r = \{x \in \mathbb{R} : |x| < r\}$), then $f \in \mathcal{A}$.

Here, $f_n \to f$ in $C^2(\overline{B_r})$ means that with

$$\Delta_{n,r} := \sup_{x \in \overline{B_r}} |f_n - f| + \sup_{x \in \overline{B_r}} |f_n' - f'| + \sup_{x \in \overline{B_r}} |f_n'' - f''|,$$

we have $\Delta_{n,r} \to 0$ as $n \to \infty$ for each $r > 0$.

(a), (b), (c) imply that polynomials are in $\mathcal{A}$. The Weierstrass approximation theorem gives that polynomials are dense in $C^2(\overline{B_r})$ $\forall r > 0$, so (d) implies that $\mathcal{A} = C_c^2(\mathbb{R})$. That (a), (b) hold is easy to see, .

<u>Proof of (c)</u>: Suppose $f, g \in \mathcal{A}$. Let $F_t = f(X_t)$, $G_t = g(X_t)$. Itô's formula holds for f, g give that F, G are continuous semi-martingales. Integration by parts also gives

$$F_t G_t - F_0 G_0 = \int_0^t F_s \, dG_s + \int_0^t G_s \, dF_s + [F, G]_t.$$

Since Itô's formula holds for f, g, we have:

$$\int_0^t F_s \, dG_s = \int_0^t F_s d\left(\int_0^s g'(X_u) \, dX_u + \frac{1}{2}\int_0^s g''(X_u) \, d[X]_u\right). \tag{1}$$

$$\overset{\text{K-W}}{=} \int_0^t f(X_s) g'(X_s) \, dX_s + \frac{1}{2}\int_0^t f(X_s) g''(X_s) \, d[X]_s \tag{2}$$

Also,

$$\int_0^t G_s \, dF_s = \int_0^t f'(X_s) g(X_s) \, dX_s + \frac{1}{2}\int_0^t f''(X_s) g(X_s) \, d[X]_s \tag{3}$$

$$\begin{aligned}[F, G]_t = [f(X), g(X)]_t &= [f'(X) \cdot X,\ g'(X) \cdot X] \quad \text{(by def. of cov. and Itô formula)} \\ &= \int_0^t f'(X_s) g'(X_s) \, d[X]_s \quad \text{(Kunita-Watanabe)}\end{aligned} \tag{4}$$

Plug (2)–(4) into (1) gives Itô's formula for fg, i.e., $fg \in \mathcal{A}$.

Proof of (d): Suppose that (f_n) is a sequence in $\mathcal{A}$ and $f_n \to f$ in $C^2(\overline{B_r})$ for all $r > 0$.
WTS: Itô's formula for f, i.e., $f \in \mathcal{A}$. Since Itô's formula holds for f_n:

$$f_n(X_t) = f_n(X_0) + \int_0^t f_n'(X_s)\, dA_s + \frac{1}{2}\int_0^t f_n''(X_s)\, d[M]_s + \int_0^t f_n'(X_s)\, dM_s.$$

Finite variation part:

$$\int_0^{t\wedge T_r} \left(f_n'(X_s) - f'(X_s)\right)\, dV_s + \frac{1}{2}\int_0^{t\wedge T_r} \left(f_n''(X_s) - f''(X_s)\right)\, d[M]_s$$

$$\leq \Delta_{n,r} \cdot \left(V_{t\wedge T_r} + \frac{1}{2}[M]_{t\wedge T_r}\right) \leq 2r \cdot \Delta_{n,r} \to 0 \quad \text{as } n \to \infty$$

which implies that

$$\xrightarrow{n\to\infty} \int_0^{t\wedge T_r} f_n'(X_s)\, dA_s + \frac{1}{2}\int_0^{t\wedge T_r} f_n''(X_s)\, d[M]_s \to \int_0^{t\wedge T_r} f'(X_s)\, dA_s + \frac{1}{2}\int_0^{t\wedge T_r} f''(X_s)\, d[M]_s \quad \text{uniformly in } t.$$

MG part: $M^r \in \mathcal{M}_c^2$ since $[M]_T \leq r$.

$$\left\|\left(f_n'(X)\cdot M\right)^{T_r} - \left(f'(X)\cdot M\right)^{T_r}\right\|_2^2 = \mathbb{E}\left[\int_0^{T_r} \left(f_n'(X_s) - f'(X_s)\right)^2 d[M]_s\right]$$

$$\leq \Delta_{n,r}^2 \cdot \mathbb{E}\left[[M]_{T_r}\right] \leq r\Delta_{n,r}^2 \to 0 \quad \text{as } n \to \infty$$

which implies that

$$\left(f_n'(X)\cdot M\right)^{T_r} \to \left(f'(X)\cdot M\right)^{T_r} \text{ in } \mathcal{M}_c \text{ as } n \to \infty$$

finally giving

$$f(X_{t\wedge T_r}) = f(X_0) + \int_0^{t\wedge T_r} f'(X_s) dA_s + \frac{1}{2}\int_0^{t\wedge T_r} f''(X_s) d[M]_s + \int_0^{t\wedge T_r} f'(X_s) dM_s.$$

□

5.2 Stratonovich Integral

Let X, Y be continuous semi-martingales. The Stratonovich integral of X against Y is defined as:

$$\int_0^t X_s \partial Y_s := \underbrace{\int_0^t X_s\, dY_s}_{\text{(Itô)}} + \frac{1}{2}[X,Y]_t.$$

This is essentially a 'midpoint approximation' since one can show

$$\sum_{k=0}^{\lfloor 2^n t\rfloor - 1} \left(\frac{X_{k2^{-n}} + X_{(k+1)2^{-n}}}{2}\right)\left(Y_{(k+1)2^{-n}} - Y_{k2^{-n}}\right) \overset{ucp}{\to} \int_0^t X_s \partial Y_s.$$

Proposition 5.4. *Let $X^1, \dots, X^d$ be continuous semi-martingales and let $f : \mathbb{R}^d \to \mathbb{R}$ be C^3. Then*

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s) \partial X_s^i$$

In particular, integration by parts is given by:

$$X_tY_t - X_0Y_0 = \int_0^t X_s \partial Y_s + \int_0^t Y_s \partial X_s.$$

This shows that the Stratonovich integral satisfies the usual rules of calculus. But the Stratonovich integral against $\mathcal{M}_c \cap \mathcal{M}_{loc}$ is <u>not</u> in $\mathcal{M}_{\text{c, loc}}$.

For example,

$$\int_0^t B_s \partial B_s = \int_0^t B_s \, dB_s + \frac{1}{2}t = \frac{1}{2}B_t^2 \notin \mathcal{M}_{\text{c, loc}}$$

for B a standard Brownian motion.

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Proposition 5.5. *Let $X^1, \dots, X^d$ be continuous semi-martingales, $X = (X^1, \dots, X^d)$, and let $f : \mathbb{R}^d \to \mathbb{R}$ be C^3. Then*

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s) \partial X_s^i$$

In particular,

$$X_tY_t - X_0Y_0 = \int_0^t X_s \partial Y_s + \int_0^t Y_s \partial X_s$$

Proof: $d = 1$; $d > 1$ is similar, . Itô's formula gives,

$$(1) \quad f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2}\int_0^t f''(X_s) d[X]_s$$

$$(2) \quad f'(X_t) = f'(X_0) + \int_0^t f''(X_s) dX_s + \frac{1}{2}\int_0^t f^{(3)}(X_s) d[X]_s$$

$$[f'(X), X]_t \overset{(2)}{=} [f''(X) \cdot X, X]_t = f''(X) \cdot [X]_t \quad \text{(Kunita–Watanabe)}$$

giving

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2}[f'(X), X]_t = f(X_0) + \int_0^t f'(X_s) \partial X_s$$

□

Before we proceed with some applications of the theory developed so far, we will make the following notational conventions.

<u>Shorthand:</u>

$$Z_t = Z_0 + \int_0^t H_s dX_s \quad \Leftrightarrow \quad \mathrm{d}Z_t = H_t \, \mathrm{d}X_t$$

$$Z_t = Z_0 + \int_0^t H_s \partial X_s \quad \Leftrightarrow \quad \partial Z_t = H_t \partial X_t$$

$$Z_t = [X, Y]_t = \int_0^t d[X, Y]_s \quad \Leftrightarrow \quad \partial Z_t = \mathrm{d}X_t \, \mathrm{d}Y_t$$

<u>Computational rules</u>

$$H_t d(K_t \, \mathrm{d}X_t) = (H_t K_t) \, \mathrm{d}X_t \quad \text{[Iterated integral]}$$

$$
\begin{aligned}
H_t d(X_t \, \mathrm{d}Y_t) &= d(H_t X_t) \, \mathrm{d}Y_t \quad \text{[Kunita–Watanabe]} \\
d(X_t Y_t) &= X_t \, \mathrm{d}Y_t + Y_t \, \mathrm{d}X_t + d[X, Y]_t \quad \text{[Integration by parts]} \\
df(X_t) &= \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(X_t) \, \mathrm{d}X_t^i + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_t) \, \mathrm{d}X_t^i \, \mathrm{d}X_t^j \quad \text{[Itô's formula]}
\end{aligned}
$$

6 Applications

Theorem 6.1 (Lévy Characterisation)**.** *Let $X^1, \dots, X^d \in \mathcal{M}_{c,\mathrm{loc}}$, and set $X = (X^1, \dots, X^d)$. Suppose $X_0 = 0$, and*

$$[X^i, X^j]_t = \delta_{ij} t \quad \forall i, j, \ t \geq 0.$$

Then X is a standard Brownian motion.

Proof. We need to show: for all $0 \leq s \leq t < \infty$, $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the law of $\mathcal{N}(0, (t-s)\mathrm{Id})$, where Id is the $d \times d$ identity matrix. quivalently, for all $\theta \in \mathbb{R}^d$,

$$\mathbb{E}\left[\exp(i\langle \theta, X_t - X_s \rangle) \mid \mathcal{F}_s\right] = \exp\left(-\frac{1}{2}|\theta|^2(t-s)\right)$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $|\theta|^2 = \langle \theta, \theta \rangle$. To see this, let $A \in \mathcal{F}_s$, $\mathbb{P}(A) \neq 0$ and define the probability measure

$$\mathbb{P}_A(\cdot) := \mathbb{P}(A)^{-1}\mathbb{P}(\cdot \cap A).$$

Then, by the tower property,

$$\mathbb{E}_{\mathbb{P}_A}\left[\exp(i\langle \theta, X_t - X_s \rangle)\right] = \mathbb{E}\left[\exp(i\langle \theta, X_t - X_s \rangle)\right]$$

which implies that the law of $X_t - X_s$ under $\mathbb{P}_A$ is the same under $\mathbb{P}$. hence, for all bounded and measurable $f : \mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E}\left[\mathbf{1}_A \cdot f(X_t - X_s)\right] = \mathbb{P}(A) \cdot \mathbb{E}\left[f(X_t - X_s)\right]$$

which implies $X_t - X_s \perp\!\!\!\perp \mathcal{F}_s$.

For $\theta \in \mathbb{R}^d$, set $Y_t = \langle \theta, X_t \rangle = \sum_{j=1}^d \theta_j X_t^j$. Then $Y \in \mathcal{M}_{c,\mathrm{loc}}$ since $\mathcal{M}_{c,\mathrm{loc}}$ is a vector space. Moreover,

$$[Y]_t = [Y, Y]_t = \left[\sum_{j=1}^{d} \theta_j X^j, \sum_{k=1}^{d} \theta_k X^k\right]_t = \sum_{j,k=1}^{d} \theta_j \theta_k [X^j, X^k]_t = |\theta|^2 t.$$

Let

$$Z_t = \exp\left(iY_t + \frac{1}{2}[Y]_t\right) = \exp\left(i\langle \theta, X_t \rangle + \frac{1}{2}|\theta|^2 t\right).$$

By Itô's formula applied to $W_t = iY_t + \frac{1}{2}[Y]_t$, with $f(w) = e^w \in C^2$, we have:

$$\mathrm{d}Z_t = Z_t\left(i \; \mathrm{d}Y_t + \frac{1}{2}d[Y]_t\right) - \frac{1}{2}Z_t \, d[Y]_t = iZ_t \; \mathrm{d}Y_t.$$

which implies $Z \in \mathcal{M}_{c,\mathrm{loc}}$ since $Y \in \mathcal{M}_{c,\mathrm{loc}}$. Since Z is bounded on $[s, t]$ for $t < \infty$, $Z \in \mathcal{M}$. Thus, $\mathbb{E}[Z_t \mid \mathcal{F}_s] = Z_s$ and so

$$\mathbb{E}\left[\exp(i\langle \theta, X_t - X_s \rangle) \mid \mathcal{F}_s\right] = \exp\left(-\frac{1}{2}|\theta|^2(t-s)\right).$$

$\square$

Theorem 6.2 (Dubins–Schwarz)**.** *Let $M \in \mathcal{M}_{c,\text{loc}}$ with $M_0 = 0$, $[M]_\infty = \infty$. Set*

$$\tau_s := \inf\{t \geq 0 : [M]_t > s\}, \quad B_s := M_{\tau_s}, \quad \mathcal{G}_s := \mathcal{F}_{\tau_s}.$$

Then (τ_s) is an $(\mathcal{F}_t)$-stopping time and $[M]_{\tau_s} = s$ for all $s \geq 0$. Moreover, B is a $(\mathcal{G}_s)$-Brownian motion with $M_t = B_{[M]_t}$.

This means that every continuous local martingale starting from 0 is a time-change of a standard Brownian motion.

Proof. Since $[M]$ is continuous and adapted, τ_s is a stopping time for each $s \geq 0$. Since $[M]_\infty = \infty$, τ_s is a finite stopping time $\forall s \geq 0$. Moreover, $(\mathcal{G}_s)$ is a filtration since if S, T are stopping times with $s \leq t$, then $\tau_s \leq \tau_t \Rightarrow \mathcal{F}_{\tau_s} \subseteq \mathcal{F}_{\tau_t} \Rightarrow \mathcal{G}_s \subseteq \mathcal{G}_t$.

Step 1: B is adapted to $(\mathcal{G}_s)$. NTS: M_{τ_s} is $\mathcal{F}_{\tau_s}$-measurable $\forall s \geq 0$.

Recall that, ( ) if X is càdlàg, adapted, and T a stopping time, then $X_T \mathbf{1}_{\{T<\infty\}}$ is $\mathcal{F}_T$-measurable.

Now, apply for $X = M$ and $T = \tau_s$, and use that $\mathbb{P}(\tau_s < \infty) = 1$.

Step 2: B is continuous.

Since $s \mapsto \tau_s$ is non-decreasing and càdlàg, it follows that B is càdlàg (since $B_s = M_{\tau_s}$). To prove that B is continuous, it suffices to show

$$B_{s^-} = B_s \quad \forall s \geq 0 \quad \iff \quad M_{\tau_s^-} = M_{\tau_s} \quad \forall s \geq 0.$$

where $\tau_s^- := \inf\{t \geq 0 : [M]_t = s\}$. If $\tau_s = \tau_s^-$, there is nothing to prove. If $\tau_s > \tau_s^-$, then $[M]_t$ is constant on $[\tau_s^-, \tau_s]$.

NTS: If $[M]_t$ is constant on any interval, then M_t is constant as well. For each rational $q \in \mathbb{Q}$, define

$$S_q := \inf\{t > q : [M]_t > [M]_q\}.$$

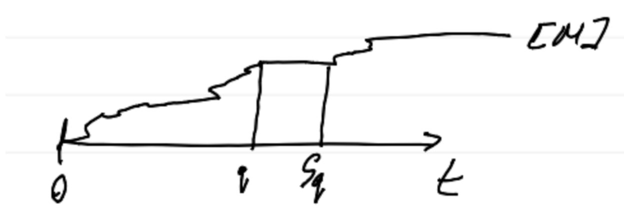

Figure 1: Illustration of times S_q.

Lecture 17 We continue working on step 2, which is the continuity of B. Need to prove that if $[M]$ is constant on a given interval, then M is constant on the same interval. By localisation, WLOG, $M \in \mathcal{M}_c^2$. Suppose that $q \in \mathbb{Q}, q > 0$. It suffices to show that M is a.s. constant on each $[q, S_q]$.

We know that $M^2 - [M]$ is a local martingale since $M \in \mathcal{M}_c$. By OST, we have that:

$$\mathbb{E}\left[M_{S_q}^2 - [M]_{S_q} \mid \mathcal{F}_{S_q}\right] = M_q^2 - [M]_q. \quad \circledast$$

Since $M \in \mathcal{M}_c^2$, we also have that

$$\text{(MG orthog.)} \quad \mathbb{E}\left[(M_{S_q} - M_q)^2 \mid \mathcal{F}_{S_q}\right] = \mathbb{E}[M_{S_q}^2 - M_q^2 \mid \mathcal{F}_{S_q}]$$

$$(*) \quad = \mathbb{E}\left[M_{S_q}^2 - [M]_{S_q} \mid \mathcal{F}_{S_q}\right] = 0 \quad \text{since } [M]_{S_q} = [M]_q.$$

Therefore $M_{S_q} - M_q = 0$ a.s. which implies M is a.s. constant on $[q, S_q]$ since for all $t \geq q$,

$$M_{t \wedge S_q} = \mathbb{E}[M_{S_q} \mid \mathcal{F}_t] = \mathbb{E}[M_q \mid \mathcal{F}_q] = M_q \text{ , a.s.}$$

Step 3: B is a $(\mathcal{G}_s)$-BM.

Fix $s > 0$. Then we know that $[M^{\tau_s}]_\infty = [M]_{\tau_s} = s$. Therefore $M^{\tau_s} \in \mathcal{M}_c^2$, since $\mathbb{E}[[M^{\tau_s}]_\infty] < \infty$. Therefore $(M^2 - [M])^{\tau_s}$ is a UI MG. By OST, for $0 \leq t \leq s < \infty$, we have that:

(i) $\mathbb{E}[B_s \mid \mathcal{G}_t] = \mathbb{E}[M_{\tau_s} \mid \mathcal{F}_{\tau_t}] = M_{\tau_t} = B_t$.

(ii) $\mathbb{E}[B_s^2 - s \mid \mathcal{G}_t] = \mathbb{E}[(M^2 - [M])_{\tau_s} \mid \mathcal{F}_{\tau_t}] = M_{\tau_t}^2 - [M]_{\tau_t} = B_t^2 - t$

Thus, (i) implies $B \in \mathcal{M}_c$, and (ii) implies $[B]_s = s$ and so

$$B \text{ is a } (\mathcal{G}_s)\text{-BM by the Lévy characterisation.}$$

□

Dubins–Schwarz requires $[M]_\infty = \infty$. One can also provide an extension thereof for the case that $[M]_\infty < \infty$:

Theorem 6.3. *$M \in \mathcal{M}_{loc}, M_0 = 0$. Let β be a BM which is independent of M. Set:*

$$B_s = \begin{cases} M_{\tau_s} & \text{if } s \leq [M]_\infty \\ M_\infty + \left(\beta_s - \beta_{[M]_\infty}\right) & \text{if } s > [M]_\infty \end{cases}$$

Then B is a standard BM and $M_t = B_{[M]_t}$ for all $t \geq 0$.

Examples.

(i) Let B be a standard BM, h deterministic, measurable in $L^2([0,\infty))$. Let

$$M_t = \int_0^t h(s)\, dB_s.$$

Then $M_0 = 0$, $M \in \mathcal{M}_{\text{loc}}$, and

$$[M]_t = \int_0^t h(s)^2\, ds.$$

Moreover,

$$M_\infty \stackrel{d}{=} B_{\int_0^\infty h(s)^2 ds} \quad \text{(Dubins–Schwarz)} \sim \mathcal{N}(0, \|h\|_{L^2}^2).$$

(ii) Let $M \in \mathcal{M}_{\text{loc}}$. Then,

$$\{[M]_\infty < \infty\} = \left\{\lim_{t\to\infty} M_t \text{ exists}\right\},$$

$$\{[M]_\infty = \infty\} = \left\{\liminf_{t\to\infty} M_t = -\infty, \limsup_{t\to\infty} M_t = \infty\right\}.$$

6.1 Exponential MGs

Let $M \in \mathcal{M}_{\text{loc}}, M_0 = 0$. Set
$$Z_t = \exp\left(M_t - \tfrac{1}{2}[M]_t\right).$$

By Itô's formula,

$$\mathrm{d}Z_t = Z_t \left(\mathrm{d}M_t - \tfrac{1}{2}d[M]_t\right) + \tfrac{1}{2}d[M]_t = Z_t \,\mathrm{d}M_t$$

giving $Z \in \mathcal{M}_{\text{loc}}, \quad Z_0 = 1$.

Definition 6.1 (Exponential MG)**.** *In the setting above, the process* $\mathcal{E}(M)_t = Z_t = \exp\left(M_t - \frac{1}{2}[M]_t\right)$ *is the stochastic exponential or exponential martingale associated with* M

Note that $\mathcal{E}(M) \in \mathcal{M}_{\text{loc}},\ d\mathcal{E}(M)_t = \mathcal{E}(M)_t \,\mathrm{d}M_t$.

Proposition 6.1. *Let* $M \in \mathcal{M}_{loc}, M_0 = 0$. *If* $[M]_\infty$ *is bounded, then* $\mathcal{E}(M)$ *is a UI martingale.*

Proposition 6.2. *Let* $M \in \mathcal{M}_{loc}, M_0 \geq 0$. *For all* $\varepsilon, \delta > 0$, *we have that*

$$\mathbb{P}\left(\sup_{t\geq 0} M_t \geq \varepsilon,\ [M]_\infty < \delta\right) \leq e^{-\frac{\varepsilon^2}{2\delta}}.$$

Proof. Fix $\varepsilon > 0$ and let $T = \inf\{t \geq 0 : M_t \geq \varepsilon\}$. Fix $\theta > 0$ and set $Z_t = \mathcal{E}(\theta M^T)_t$, i.e.

$$Z_t = \exp\left(\theta M_t^T - \frac{\theta^2}{2}[M^T]_t\right) \in \mathcal{M}_{\text{loc}}.$$

Note that $|Z_t| \leq e^{\theta\varepsilon}$ for all $t \geq 0$. So Z is a bounded MG, hence $\mathbb{E}[Z_\infty] = Z_0 = 1$. For $\delta \geq 0$, we have that

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\geq 0} M_t \geq \varepsilon, [M]_\infty \leq \delta\right) &= \mathbb{P}\left(\sup_{t\geq 0} \theta M_t^T \geq \theta\varepsilon, [M^T]_\infty \leq \delta\right) \\
&\leq \mathbb{P}\left(\sup_{t\geq 0} Z_t \geq Ce^{\theta\varepsilon - \frac{\theta^2}{2}\delta}\right) \quad \text{(Doob's inequality)} \\
&\leq C\exp\left(-\theta\varepsilon + \frac{\theta^2}{2}\delta\right).
\end{aligned}$$

Optimising over θ gives the claimed bound. □

Proof of (previous) proposition. We will show that $\mathcal{E}(M)$ is bounded by an integrable random variable. Note that
$$\sup_{t\geq 0} \mathcal{E}(M)_t \leq \exp\left(\sup_{t\geq 0} M_t\right) \quad \text{(since } [M]_t \geq 0\text{)}.$$

NTS: RHS is integrable. Let $C > 0$ so that $[M]_\infty \leq C$. Then:

$$\mathbb{P}\left(\sup_{t\geq 0} M_t \geq \varepsilon\right) = \mathbb{P}\left(\sup_{t\geq 0} M_t \geq \varepsilon,\ [M]_\infty \leq C\right) \leq \exp\left(-\frac{\varepsilon^2}{2C}\right)$$

which implies

$$\mathbb{E}\left[\exp\left(\sup_{t\geq 0} M_t\right)\right] = \int_0^\infty \mathbb{P}\left(\exp\left(\sup_{t\geq 0} M_t\right) \geq \lambda\right) \mathrm{d}\lambda = \int_0^\infty \mathbb{P}\left(\sup_{t\geq 0} M_t \geq \log\lambda\right) \mathrm{d}\lambda$$

$$\leq 1 + \int_1^\infty \exp\left(-\frac{(\log\lambda)^2}{2C}\right) \mathrm{d}\lambda < \infty$$

finally giving that $\mathcal{E}(M)$ is UI. □

Lecture 18 Suppose that Q, P are probability measures on $(\Omega, \mathcal{F})$. Say that Q is absolutely continuous w.r.t. P, denoted by $Q \ll P$, if for any $A \in \mathcal{F}$ with

$$P(A) = 0 \Rightarrow Q(A) = 0.$$

Recall from measure theory that this implies the existence of a random variable $Z \geq 0$ such that

$$Q(A) = \mathbb{E}[Z \cdot \mathbf{1}_A] \quad \text{for all } A \in \mathcal{F}.$$

Z is called the *Radon-Nikodym derivative* of Q w.r.t. P and is denoted by $Z = \frac{dQ}{dP}$.

Example. Suppose that $X \sim \mathcal{N}(0,1)$, $\mu \in \mathbb{R}$. Let

$$Z = \exp\left(\mu X - \frac{\mu^2}{2}\right).$$

Then $A \mapsto \mathbb{E}[\mathbf{1}_A Z]$ defines a probability measure Q, and under Q, $X \sim \mathcal{N}(\mu, 1)$.

The Girsanov Theorem generalizes this idea to the setting of semi-martingales, except instead of changing the mean, we will change the semi-martingale decomposition.

Theorem 6.4 (Girsanov)**.** *Let $M \in \mathcal{M}_{c,loc}$, $M_0 = 0$, and assume that $Z = \mathcal{E}(M)$ is uniformly integrable. Then we can construct a new probability measure $\tilde{\mathbb{P}} \ll \mathbb{P}$ on $(\mathcal{F}_t)$ by setting*

$$\tilde{\mathbb{P}}(A) := \mathbb{E}[Z_\infty \mathbf{1}_A] \quad \forall A \in \mathcal{F}.$$

If $X \in \mathcal{M}_{c,loc}(\mathbb{P})$, then $X - [X, M] \in \mathcal{M}_{c,loc}(\tilde{\mathbb{P}})$.

'A change of measure induces a change of drift'.

Girsanov. Since Z is UI, hence that Z_∞ exists and $Z_\infty \geq 0$ with $\mathbb{E}[Z_\infty] = 1$ and so $\tilde{\mathbb{P}}$ defines a probability measure with $\tilde{\mathbb{P}} \ll \mathbb{P}$. Suppose that $X \in \mathcal{M}_{c,\mathrm{loc}}(\mathbb{P})$ and set

$$T_n := \inf\{t \geq 0 : |X_t - [X, M]_t| \geq n\}.$$

Since $X - [X, M]$ is continuous (starts from zero), we have that

$$\mathbb{P}(T_n \nearrow \infty) = 1 \Rightarrow \tilde{\mathbb{P}}(T_n \nearrow \infty) = 1 \quad (\text{since } \tilde{\mathbb{P}} \ll \mathbb{P}).$$

To prove that $Y := X - [X, M] \in \mathcal{M}_{c,\mathrm{loc}}(\tilde{\mathbb{P}})$, it suffices to show that $Y^{T_n} := X^{T_n} - [X, M]^{T_n} \in \mathcal{M}_c(\tilde{\mathbb{P}})$. In what follows, write X, Y in place of X^{T_n}, Y^{T_n}.

Using Itô's product rule (IBP):

$$\mathrm{d}(Z_t Y_t) = Y_t \,\mathrm{d}Z_t + Z_t \,\mathrm{d}Y_t + \mathrm{d}Y_t \,\mathrm{d}Z_t.$$

Now,

$$
\begin{aligned}
\mathrm{d}Z_t &= Z_t \,\mathrm{d}M_t, \\
\mathrm{d}Y_t &= \mathrm{d}X_t - \mathrm{d}[X, M]_t, \\
\mathrm{d}Y_t \,\mathrm{d}Z_t &= Z_t \,\mathrm{d}[M, Y]_t = Z_t \,\mathrm{d}[X, M]_t.
\end{aligned}
$$

Thus,

$$
d(Z_t Y_t) = Y_t Z_t \,\mathrm{d}M_t + Z_t(\mathrm{d}X_t - \mathrm{d}[X, M]_t) + Z_t \,\mathrm{d}[X, M]_t = Z_t \,\mathrm{d}X_t + Y_t Z_t \,\mathrm{d}M_t
$$

giving that $ZY \in \mathcal{M}_{c,\mathrm{loc}}(\mathbb{P})$.

Moreover, $ZY : T \leq t$ is a stopping time, and is UI for each $t > 0$, [✍] . Since Y is bounded, we also have that

$$
ZY \cdot \mathbf{1}_{\{T \leq t\}} \text{ is a stopping time and UI} \Rightarrow ZY \in \mathcal{M}_c(\mathbb{P}).
$$

For $s \leq t$, we have that

$$
\mathbb{E}\left[Y_t - Y_s \mid \mathcal{F}_s\right] = \frac{1}{Z_s}\mathbb{E}\left[Z_t Y_t - Z_s Y_s \mid \mathcal{F}_s\right] = 0 \quad \text{(tower property)}.
$$

Since $ZY \in \mathcal{M}_c(\mathbb{P})$ we finally obtain $Y \in \mathcal{M}_c(\tilde{\mathbb{P}})$. □

Remark. *The quadratic variation does not change when performing a change of measures,* [✍] .

Corollary 6.1. *Let B be a standard Brownian motion under $\mathbb{P}$, $M \in \mathcal{M}_{c,loc}$, $M_0 = 0$. Suppose that*

$$
Z = \mathcal{E}(M) \text{ is UI}, \quad \text{and } \mathbb{Q}(A) = \mathbb{E}[1_A Z_\infty] \text{ for all } A \in \mathcal{F}.
$$

Then $\widetilde{B} := B - [B, M]$ is a $\mathbb{Q}$-Brownian motion.

Proof. Since $\widetilde{B} \in \mathcal{M}_{c,\mathrm{loc}}(\mathbb{Q})$ by the Girsanov theorem, and $[\widetilde{B}]_t = [B - [B, M]]_t = t$, it follows from the Lévy characterisation that $\widetilde{B}$ is a $\mathbb{Q}$-Brownian motion. □ □

Example. *Suppose that B is a $\mathbb{P}$-Brownian motion, $\mu \in \mathbb{R}$, $T > 0$, and let $M_t = \mu B_t$, so that*

$$
Z_t = \mathcal{E}(M)_t = \exp\left(\mu B_t - \mu^2 t/2\right).
$$

Then

$$
\mathbb{Q}(A) = \mathbb{E}\left[Z_T \cdot \mathbf{1}_A\right] = \mathbb{E}\left[\exp\left(\mu B_T - \mu^2 T/2\right) \mathbf{1}_A\right] \quad \forall A \in \mathcal{F}.
$$

You render under $\mathbb{P}$ that $B_t = \widetilde{B}_t + \mu t$ for $t \in [0, T]$, and $\widetilde{B}$ is a $\mathbb{Q}$-Brownian motion.

7 Stochastic Differential Equations

Let $\mathbb{M}^{d \times m}(\mathbb{R})$ denote the space of $d \times m$ matrices with real entries. Suppose that

$$
\sigma : \mathbb{R}^d \to \mathbb{M}^{d \times m}(\mathbb{R}), \quad b : \mathbb{R}^d \to \mathbb{R}^d
$$

are measurable functions which are bounded on compact sets. Write $\sigma(x) = (\sigma_{ij}(x))$. Consider the SDE:

$$
dX_t = \sigma(X_t)\ \mathrm{d}B_t + b(X_t)\, dt \tag{$\circledast$}
$$

Equivalently,

$$dX_t^i = \sum_{j=1}^m \sigma_{ij}(X_t)\, dB_t^j + b_i(X_t)\, dt.$$

A solution to ⊛ consists of:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions.
- An $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $B = (B^1, \dots, B^m) \in \mathbb{R}^m$.
- An $(\mathcal{F}_t)_{t\geq 0}$-adapted continuous process $X = (X_t^1, \dots, X_t^d) \in \mathbb{R}^d$ such that

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds.$$

When in addition $X_0 = x \in \mathbb{R}^d$, we say that X is *started from* x.

- We say that an SDE has a <u>weak solution</u> if for all $x \in \mathbb{R}^d$, there is a solution starting from x.
- There is <u>uniqueness in law</u> if all solutions starting from each x have the same distribution.
- There is <u>pathwise uniqueness</u> if, when we fix $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and B, then any two solutions X, X' with $X_0 = X_0'$ are indistinguishable:

$$\mathbb{P}(X_t = X_t' \text{ for all } t \geq 0) = 1.$$

- We say that a solution started from x is a <u>strong solution</u> if X is adapted to the filtration generated by B.

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<u>**Example.**</u> It is possible to have the existence of a weak solution and uniqueness in law without having pathwise uniqueness. Suppose that β is a standard Brownian motion in $\mathbb{R}$ with $\beta_0 = x$. Set

$$B_t = \int_0^t \operatorname{sgn}(\beta_s)\, ds, \quad \operatorname{sgn}(x) = 1_{\{(0,\infty)\}}(x) - 1_{\{(-\infty,0]\}}(x).$$

Note that $\operatorname{sgn}(\beta_s)$ is measurable and bounded, hence the integral is well-defined. Then,

$$x + \int_0^t \operatorname{sgn}(\beta_s)\, d\beta_s = x + \int_0^t (\operatorname{sgn}(\beta_s))^2\, d\beta_s = x + \int_0^t d\beta_s = \beta_t.$$

Therefore, β solves the SDE

$$\begin{cases} dX_t = \operatorname{sgn}(X_t)\ \mathrm{d}B_t, \\ X_0 = x. \end{cases}$$

This SDE has a weak solution. By the Lévy characterisation, any solution to this SDE is a Brownian motion (it is in $\mathcal{M}_{c,loc}$ with quadratic varitaion $[\cdot]_t = t$) which gives uniqueness in law. However, we do not have pathwise uniqueness. To see this, take $X = x = 0$.

<u>**Claim:**</u> $\beta_t, -\beta_t$ are solutions.

Indeed, β_t is a solution. For $-\beta_t$, we also obtain

$$-\beta_t = -\int_0^t \operatorname{sgn}(\beta_s)\, ds = \int_0^t \operatorname{sgn}(-\beta_s)\, d(-\beta_s)$$

$$= \int_0^t \operatorname{sgn}(-\beta_s)\, dB_s + 2\int_0^t 1_{\{\beta_s=0\}}\, dB_s.$$

The last term on the RHS is in $\mathcal{M}_{c,\text{loc}}$, starts from 0, and has quadratic variation

$$4\int_0^t 1_{\{\beta_s=0\}}\, ds = 0 \quad \text{a.s.}$$

because $\mathbb{P}(\beta_s = 0) = 0\ \forall s > 0$, and then one can apply Fubini's theorem to obtain that its expectation vanishes. Therefore $\beta_t, -\beta_t$ are both solutions on the same probability space with the same Brownian motion. So we do *not* have pathwise uniqueness.

7.1 Lipschitz Coefficients

Recall that for $U \subset \mathbb{R}^d$ open, $f : U \to \mathbb{R}^d$, we say that f is **Lipschitz** if there exists $K < \infty$ such that

$$|f(x) - f(y)| \le K|x - y| \quad \forall x, y \in U.$$

For $d, m \ge 1$, we equip $\mathcal{M}_{d\times m}(\mathbb{R})$ with the Frobenius norm. If $A \in \mathcal{M}_{d\times m}(\mathbb{R})$, $A = (a_{ij})$, then

$$|A| = \left(\sum_{i=1}^d \sum_{j=1}^m a_{ij}^2\right)^{1/2}.$$

Let $f : U \to \mathcal{M}_{d\times m}(\mathbb{R})$. Say that f is Lipschitz if there exists $K < \infty$ such that

$$|f(x) - f(y)| \le K|x - y| \quad \forall x, y \in U.$$

Theorem 7.1 (Existence and Uniqueness)**.** *Suppose that*

$$\sigma : \mathbb{R}^d \to \mathcal{M}_{d\times m}(\mathbb{R}), \quad b : \mathbb{R}^d \to \mathbb{R}^d$$

are Lipschitz. Then there is pathwise uniqueness for the SDE

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt.$$

Moreover, for each filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ satisfying the usual conditions and each $(\mathcal{F}_t)$-Brownian motion B, $x \in \mathbb{R}^d$, there is a strong solution starting from x.

The proof is analogous to the existence/uniqueness theorem for ODEs. Recall some results from analysis/ODEs.

Theorem 7.2 (Banach Fixed Point Theorem)**.** *Let (X, d) be a complete metric space.*

(a) Suppose that $F : X \to X$ is a contraction, i.e., $\exists r \in (0, 1)$ such that

$$d(F(x), F(y)) \le r\, d(x, y) \quad \forall x, y \in X.$$

Then F has a unique fixed point.

(b) Suppose that $F : X \to X$, and there exists $n \in \mathbb{N}$ so that $F^{(n)}$ is a contraction. Then F has a unique fixed point.

Lemma 7.1 (Gronwall)**.** *Let $T > 0$ and $f : [0,T] \to [0,\infty)$ be a bounded and measurable function. If there exist $a, b > 0$ such that*

$$f(t) \leq a + b \int_0^t f(s)\, ds \quad \forall t \in [0,T],$$

then $f(t) \leq ae^{bt}$ for all $t \in [0,T]$.

Proof. [ ] . ∎

Proof of Existence and Uniqueness We will assume that dim $= 1$ and will let K be such that

$$|\sigma(x) - \sigma(y)| \leq K|x-y|, \quad |b(x) - b(y)| \leq K|x-y| \quad \forall x, y \in \mathbb{R}.$$

Proof of Uniqueness. Suppose that X, X' are two solutions on the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and Brownian motion B. WTS: $\mathbb{P}(X_t = X'_t \ \forall t \geq 0) = 1$.

Fix $M > 0$ and let

$$\tau = \inf\left\{t \geq 0 : |X_t| \vee |X'_t| \geq M\right\}.$$

Then,

$$X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} \sigma(X_s)\, dB_s + \int_0^{t\wedge\tau} b(X_s)\, ds,$$
$$X'_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} \sigma(X'_s)\, dB_s + \int_0^{t\wedge\tau} b(X'_s)\, ds.$$

Fix $T > 0$. If $t \in [0,T]$, we have that

$$\mathbb{E}\left[(X_{t\wedge\tau} - X'_{t\wedge\tau})^2\right] \leq 2 \cdot \mathbb{E}\left[\left(\int_0^{t\wedge\tau} (\sigma(X_s) - \sigma(X'_s))\, dB_s\right)^2\right] + 2 \cdot \mathbb{E}\left[\left(\int_0^{t\wedge\tau} (b(X_s) - b(X'_s))\, ds\right)^2\right]$$

$$\leq 2 \cdot \mathbb{E}\left[\int_0^{t\wedge\tau} (\sigma(X_s) - \sigma(X'_s))^2\, ds\right] + 2T \cdot \mathbb{E}\left[\frac{1}{T}\int_0^{t\wedge\tau} (b(X_s) - b(X'_s))^2\, ds\right] \quad \text{(Itô isometry + Cauchy–Schwarz)}$$

$$\leq 2K^2(1+T) \cdot \mathbb{E}\left[\int_0^{t\wedge\tau} |X_s - X'_s|^2\, ds\right]$$

$$= 2K^2(1+T) \int_0^t \mathbb{E}\left[|X_{s\wedge\tau} - X'_{s\wedge\tau}|^2\right] ds.$$

Let $f(t) := \mathbb{E}[|X_{t\wedge\tau} - X'_{t\wedge\tau}|^2]$. Then:

$$0 \leq f(t) \in 4M^2 \text{ and } f(t) \leq 2K^2(1+T) \int_0^t f(s)\, ds \quad \forall t \in [0,T].$$

By Gronwall's inequality, $f(t) = 0$ for all $t \in [0,T]$, so

$$\mathbb{P}(X_{t\wedge\tau} = X'_{t\wedge\tau} \ \forall t \in [0,T]) = 1.$$

Since M, T were arbitrary, we conclude:

$$\mathbb{P}(X_t = X'_t \ \forall t \geq 0) = 1.$$

That is, we have established **Pathwise uniqueness**. ∎

Proof of existence. Suppose that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space, B is an $(\mathcal{F}_t)$-Brownian motion, and $(\mathcal{F}_t^B)_{t\geq 0}$ is the filtration generated by B (so that $\mathcal{F}_t^B \subseteq \mathcal{F}_t$). We will use the contraction mapping theorem. Need to specify

1) the space,

2) the map.

For each $T > 0$, let $\mathcal{C}_T = \{\text{continuous, adapted processes } X : [0, T] \to \mathbb{R}\}$, with

$$\|X\|_T := \left(\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^2\right]\right)^{1/2}.$$

We proved before that $\mathcal{C}_T$ is complete. Fix $x \in \mathbb{R}$. Using that σ, b are Lipschitz, we have

$$|\sigma(y)| = |\sigma(y) - \sigma(0) + \sigma(0)| \leq |\sigma(y) - \sigma(0)| + |\sigma(0)| \leq K|y| + |\sigma(0)|, \tag{(1)}$$
$$|b(y)| \leq |b(0)| + K|y| \quad \text{for all } y \in \mathbb{R}. \tag{(2)}$$

Fix $T > 0$, and $X \in \mathcal{C}_T$. Let

$$M_t := \int_0^t \sigma(X_s)\, dB_s, \quad 0 \leq t \leq T.$$

Then,

$$[M]_t = \int_0^t \sigma^2(X_s)\, ds.$$

Thus, by (1),

$$\mathbb{E}[[M]_T] \leq 2T\left(|\sigma(0)|^2 + K^2\|X\|_T^2\right) < \infty.$$

which implies that $M \in \mathcal{M}_c^2$, so by Doob's inequality,

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \sigma(X_s)\, dB_s\right|^2\right] \leq 8T\left(|\sigma(0)|^2 + K^2\|X\|_T^2\right).$$

By (2),

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t b(X_s)\, ds\right|^2\right] \leq \dots$$

$$\leq T \cdot \mathbb{E}\left[\int_0^T b(X_s)^2\, ds\right] \quad \text{(Cauchy-Schwarz)}$$

$$\leq 2T \cdot \mathbb{E}\left[|\sigma(0)|^2 + K^2\|X\|_T^2\right] < \infty$$

The map F on $\mathcal{C}_T$ defined by

$$F(X)_t = x + \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds$$

takes values in $\mathcal{C}_T$.

Suppose that $X, Y \in \mathcal{C}_T$. For $0 \le t \le T$, using similar arguments,

$$\|F(X) - F(Y)\|_t^2 \le 4K^2T \cdot (4+T) \int_0^t \|X - Y\|_s^2 \, ds = C_T \int_0^t \|X - Y\|_s^2 \, ds$$

Iterate n times:

$$\begin{aligned}\left\|F^{(n)}(X) - F^{(n)}(Y)\right\|_T^2 &\le C_T^n \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|X - Y\|_T^2 \, dt_n \cdots dt_1 \\ &\le \frac{C_T^n T^n}{n!} \|X - Y\|_T^2 \end{aligned} \tag{3}$$

Take n sufficiently large so that $\frac{C_T^n T^n}{n!} < 1$. Then by the contraction mapping theorem, there exists a unique fixed point $X^{(T)} \in \mathcal{C}_T$ of F. Pathwise uniqueness $\Rightarrow X_t^{(T)} = X_t^{(T')}$ for all $t \le T \wedge T'$ a.s. Define X_t by setting $X_t = X_t^{(N)}$ where $t \le N$, $N \in \mathbb{N}$. Then X is the pathwise unique solution to the SDE starting from x.

NTS: X is a strong solution, i.e. X is adapted to $(\mathcal{F}_t^B)$. We will prove first that for each fixed T, $X^{(T)}$ is the limit of $(\mathcal{F}_t^B)$-processes. Define $y^0 = x$ and $y^n = F(y^{n-1})$ for each $n \in \mathbb{N}$. Then (y^n) is adapted to $(\mathcal{F}_t^B)$ for each n. As $F^{(n)}(X) = X$, for all $n \ge d$, we have from (3) that:

$$\|X - y^n\|_T^2 = \|F^{(n)}(X) - F^{(n)}(x)\|_T^2 \le \frac{C_T^n T^n}{n!} \|X - x\|_T^2 \to 0 \quad \text{as } n \to \infty.$$

Thus $Y^n \to X$ in C_T as $n \to \infty$. So there exists a subsequence (Y^{n_k}) such that $Y^{n_k} \to X$ uniformly in $[0, T]$ a.s. Therefore, (X_t) is the a.s. limit of $(\mathcal{F}_t^B)$-adapted processes and so is $(\mathcal{F}_t^B)$-adapted. Since $T > 0$ was arbitrary, we have that X is $(\mathcal{F}_t^B)$-adapted. □

Remark. *From the abvoe proof, we also obtain that the pathwise unique strong solution lies in C_T for all $T > 0$.*

Proposition 7.1. *Under the hypotheses of the theorem, there is uniqueness in law for the SDE*

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt.$$

Proof. . □

Example.(Ornstein–Uhlenbeck process) Fix $\lambda \in \mathbb{R}$ and consider the SDE

$$dV_t = dB_t - \lambda V_t dt, \quad V_0 = v_0,$$

$$dX_t = V_t dt.$$

For $\lambda > 0$, this models the movement of a grain of pollen in liquid; $X =$ position of the grain, $V =$ velocity. The term $-\lambda V$ damps the system due to viscosity. When $|V|$ is large, the system moves to reduce $|V|$.

The previous theorem implies that there exists a unique strong solution. We can explicitly solve

$$d(e^{\lambda t} V_t) = e^{\lambda t} dV_t + \lambda e^{\lambda t} V_t dt = e^{\lambda t} dB_t.$$

Hence,

$$e^{\lambda t} V_t = v_0 + \int_0^t e^{\lambda s} dB_s,$$

so that

$$V_t = e^{-\lambda t} v_0 + \int_0^t e^{-\lambda(t-s)} dB_s.$$

Therefore,

$$V_t \sim \mathcal{N}\left(e^{-\lambda t} v_0, \frac{1 - e^{-2\lambda t}}{2\lambda}\right).$$

If $\lambda > 0$, then V_t converges in distribution to $\mathcal{N}\left(0, (2\lambda)^{-1}\right)$ as $t \to \infty$. Hence, $\mathcal{N}(0, (2\lambda)^{-1})$ is the stationary distribution of V, i.e. if $V_0 \sim \mathcal{N}(0, (2\lambda)^{-1})$, then

$$V_t \sim \mathcal{N}(0, (2\lambda)^{-1}) \quad \text{for all } t \geq 0.$$

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7.2 Local solutions

Consider the SDE

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt.$$

A locally defined process is a pair (X, τ) consisting of a stopping time τ together with a map

$$X : \{(\omega, t) \in \Omega \times [0, \infty) : t < \tau(\omega)\} \to \mathbb{R}.$$

It is said to be càdlàg if the map $t \mapsto X_t(\omega)$ from $[0, \tau(\omega))$ to $\mathbb{R}$ is càdlàg for all $\omega \in \Omega$. Let $\Omega_t = \{\omega \in \Omega : t < \tau(\omega)\}$. Then (X, τ) is *adapted* if $X_t : \Omega_t \to \mathbb{R}$ is $\mathcal{F}_t$-measurable. We say that (X, τ) is a locally defined martingale if there exist stopping times $\tau_n \nearrow \tau$ such that X^{τ_n} is a martingale for all n. We say that (H, η) is a locally defined, locally bounded, predictable process if there exist stopping times $S_n \nearrow \eta$ such that $H\mathbf{1}_{\{0 \leq t \leq S_n\}}$ is bounded and predictable for all $n \in \mathbb{N}$. We define $(H \cdot X, \tau \wedge \eta)$

$$(H \cdot X)_t^{T_n \wedge S_n} = (H\mathbf{1}_{(0, S_n \wedge T_n]} \cdot X)_t \quad \text{for each } n.$$

Proposition 7.2 (Local Itô's formula)**.** *Let $X^1, \ldots, X^d$ be continuous semimartingales, let $U \subseteq \mathbb{R}^d$ be open, and let $f : U \to \mathbb{R}$ be C^2. Let $X = (X^1, \ldots, X^d)$ and set*

$$\tau = \inf\{t \geq 0 : X_t \notin U\}.$$

Then for all $t < \tau$, we have that

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s)\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\, d[X^i, X^j]_s.$$

Proof. Apply Itô's formula to X^{τ_n}, where

$$\tau_n = \inf\left\{t \geq 0 : \operatorname{dist}(X_t, U^c) \leq \frac{1}{n}\right\},$$

and note that $\tau_n \nearrow \tau$ as $n \to \infty$. $\square$

Example. *Let $X = B$, where B is a standard Brownian motion with $X_0 = B_0 = 1$, $U = (0, \infty)$, and $f(x) = \sqrt{x}$. Then*

$$\sqrt{B_t} = 1 + \frac{1}{2}\int_0^t B_s^{-1/2}\, dB_s - \frac{1}{8}\int_0^t B_s^{-3/2}\, ds \quad \textit{for all } t < \tau,$$

where

$$\tau = \inf\{t \geq 0 : B_t = 0\}.$$

Let $U \subseteq \mathbb{R}^d$ be open, $\sigma : U \to \mathbb{M}^{d\times m}(\mathbb{R})$, $b : U \to \mathbb{R}^d$ be measurable functions which are bounded on compact subsets of U.

A local solution to the SDE

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt$$

consists of:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions.
- An $(\mathcal{F}_t)$-Brownian motion B in $\mathbb{R}^m$.
- A continuous $(\mathcal{F}_t)$-adapted locally defined process (X, τ), with $X \in \mathbb{R}^d$, such that

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds, \quad \text{for all } t < \tau.$$

We say that (X, τ) is a *maximal local solution* if for any other local solution $(\tilde{X}, \eta)$ on the same space such that

$$X_t = \tilde{X}_t \quad \text{for all } t \leq \tau \wedge \eta,$$

we have that $\eta \leq \tau$.

Locally Lipschitz coefficients: Suppose that $U \subseteq \mathbb{R}^d$ is open. Then a function $f : U \to \mathbb{R}^d$ is locally Lipschitz if for each compact set $C \subseteq U$, we have that $f|_C$ is Lipschitz.

Theorem 7.3. *Suppose $U \subseteq \mathbb{R}^d$ is open and $\sigma : U \to \mathbb{M}^{d\times m}(\mathbb{R})$, $b : U \to \mathbb{R}^d$ are locally Lipschitz. Then for all $x \in U$, the SDE*

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt$$

has a pathwise unique maximal local solution (X, τ) starting from x. Moreover, for all compact sets $C \subseteq U$, on the event that $\tau < \infty$, we have that

$$\sup\{t < \tau : X_t \in C\} < \tau.$$

Lemma 7.2. *Let $U \subseteq \mathbb{R}^d$ be open, $C \subseteq U$ be compact. Then:*

1. *There exists a C^∞ function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\varphi|_C \equiv 1$ and $\varphi|_{U^c} \equiv 0$.*
2. *Given a locally Lipschitz function $f : U \to \mathbb{R}$, then there exists a globally Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$ such that $f|_C = g|_C$.*

Proof: (i) .

(ii) Let φ be as in part (i) and set $g = f \cdot \varphi$. □

□

Proof (Theorem). Assume that $d = m = 1$. Fix $C \subsetneq U$ compact. By the lemma, we can find Lipschitz functions $\tilde{\sigma}, \tilde{b}$ on $\mathbb{R}$ such that $\tilde{\sigma}|_C = \sigma|_C$, $\tilde{b}|_C = b|_C$. Then there exists a pathwise unique strong solution $\tilde{X}$ to:

$$\begin{cases} d\tilde{X}_t = \tilde{\sigma}(\tilde{X}_t)dB_t + \tilde{b}(\tilde{X}_t)dt \\ \tilde{X}_0 = x \end{cases}$$

Let $\tau = \inf\{t \geq 0 : \tilde{X}_t \notin C\}$ and let $X = \tilde{X}|_{[0,\tau)}$. Then (X, τ) is a local solution in C, . If $\tau < \infty$, then $X_{\tau^-} = \lim_{t \to \tau^-} X_t$ exists and is in U^C. Suppose that $(X, \tau), (Y, \eta)$ are both local solutions in C. Let

$$f(t) = \mathbb{E}\left[\sup_{0 \leq s \leq t \wedge \tau \wedge \eta} |X_s - Y_s|^2\right]$$

As b, σ are Lipschitz on C, we can use Gronwall's lemma as before to see that $f \equiv 0$, which implies that $X_t = Y_t$ for all $t \leq \eta \wedge \tau$ almost surely.

Let (C_n) be a sequence of compact sets in U with $C_n \subseteq C_{n+1}$ for all n, and $U = \bigcup_n C_n$. Let (X^n, T_n) be the local solution constructed above with $C = C_n$. If $T_n < \infty$, then $X^n_{T_n} \in U \setminus C_n^\circ$. Observe that on

$$\underbrace{\inf\{t \geq 0 : X^{n+1}_t \notin C_n^\circ\}}_{:= \tilde{T}_n} \wedge T_n := S_n$$

we have

$$X^{n+1}_t = X^n_t \quad \text{almost surely for all } t \leq S_n$$

(by a Gronwall-type argument). Suppose for a contradiction that $\tilde{T}_n < T_n$. Then the above implies

$$X^{n+1}_{\tilde{T}_n} = X^n_{\tilde{T}_n} \quad \text{almost surely,} \quad \text{and } t \leq \tilde{T}_n$$

giving

$$X^{n+1}_{\tilde{T}_n} = X^n_{\tilde{T}_n} \notin C_n^\circ \subseteq C_n$$

Hence

$$T_n \leq \tilde{T}_n \leq T_{n+1} \quad \text{which implies that } (T_n) \text{ is increasing.}$$

Since the T_n are non-decreasing, we have $T_n \nearrow \tau$, i.e., $\tau = \sup_n T_n$.

Define the local solution by setting $X_t = X^n_t$ for all $t < T_n$. This is consistent by the above. We now aim to show that (X, τ) is maximal.

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It thus remains to show

1. maximality,

2. $\sup\{t < \tau : X_t \in C\} < \tau$ on the event $\{\tau < \infty\}$.

Suppose that (Y, η) is another solution on the same probability space. For each n, set

$$S_n = \inf\{t \in [0, \infty) : Y_t \notin C_n\} \wedge \eta.$$

By the uniqueness of the solution in each C_n, we have that $X_t = Y_t$ for all $t \leq S_n \wedge T_n$. Therefore, arguing as before, $S_n \leq T_n$. As $n \to \infty$, $S_n \nearrow \eta$, $T_n \nearrow \tau$, so

$$\eta \leq \tau, \quad X_t = Y_t \text{ for all } t \leq \eta.$$

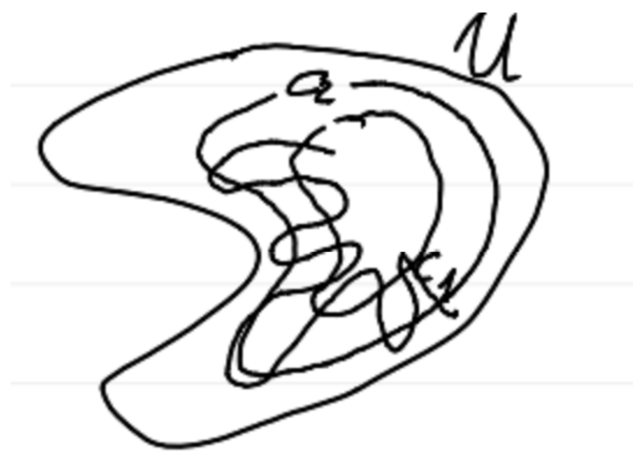
Therefore, (X, τ) is maximal. $\square$

Suppose that C_1, C_2 are compact sets in $\mathcal{U}$ with $C_1 \subseteq C_2^\circ \subseteq C_2 \subseteq \mathcal{U}$. Let $\varphi : \mathcal{U} \to \mathbb{R}$ be a C^∞ function with $\varphi|_{C_1} \equiv 1$, $\varphi|_{(C_2^\circ)^c} \equiv 0$. Let

$$R_0 = \inf\{t \geq 0 : X_t \notin C_2\},$$

$$S_n = \inf\{t \geq R_{n-1} : X_t \notin C_1\} \wedge \tau,$$

$$R_n = \inf\{t \geq S_n : X_t \notin C_2\} \wedge \tau.$$



Let N be the number of crossings that X makes from C_2 to C_1. On the event $\{\tau \leq t,\, N \geq n\}$, we have that:

$$\sum_{k=1}^{n} \left(\varphi(X_{R_k}) - \varphi(X_{S_k})\right) = -n$$

$$= \int_0^t \sum_{k=1}^{n} \mathbf{1}_{(S_k, R_k]}(s) \left(\varphi(X_s)\, dX_s + \frac{1}{2}\varphi''(X_s)\, d[X]_s\right)$$

$$= \int_0^t H_s^n \, dB_s + K_s^n \, ds =: Z_t^n,$$

where H^n, K^n are predictable and bounded uniformly in n. Then:

$$n \cdot \mathbf{1}_{\{\tau \leq t,\, N \geq n\}} \leq (Z_t^n)^2 \Rightarrow \mathbb{P}\left(\tau \leq t,\, N \geq n\right) \leq \frac{1}{n^2}\mathbb{E}\left[(Z_t^n)^2\right]$$

Since H^n, K^n are uniformly bounded and Z_t^n is defined by integrating H^n, K^n over a time-interval which does not depend on n, we have that

$$\mathbb{E}\left[(Z_t^n)^2\right] \leq C \text{ where } C \text{ does not depend on } n \Rightarrow \mathbb{P}\left(\tau \leq t,\, N \geq n\right) \leq \frac{C}{n^2}.$$

Letting $n \to \infty$ gives

$$\mathbb{P}\left(\tau \leq t,\, N = \infty\right) = 0 \Rightarrow \mathbb{P}\left(\tau < \infty,\, N = \infty\right) = 0$$

Therefore, the number of crossings that X makes from C_2 to C_1 is finite on the event $\{\tau < \infty\}$ almost surely.

Example. (Bessel processes) Fix $v \in \mathbb{R}$ and consider the SDE in $U = (0, \infty)$ given by:

$$dX_t = dB_t + \frac{n-1}{2X_t}\, dt, \quad X_0 = x_0 \in U.$$

Then there exists a unique maximal local solution (X, τ) in U and $M_t := \mathbb{P}\left[\exists t \geq 0 : X_t = 0\right] = 0$.
(X, τ) is a **Bessel process of dimension** n.

Suppose that $n \in \mathbb{N}$, β is a Brownian motion in $\mathbb{R}^n$ with $|\beta_0| = x_0 > 0$. Set $X_t := |\beta_t|$ and

$$\tau := \inf\left\{t \geq 0 : \beta_t = 0\right\}.$$

By the local Itô formula, we have that

$$dX_t = (\beta_t, d\beta_t) + \frac{n-1}{2|\beta_t|} dt, \quad t < \tau,$$

where $(\cdot, \cdot)$ is the Euclidean inner product. Then the process

$$W_t := \int_0^t \frac{(\beta_s, d\beta_s)}{|\beta_s|} \quad \text{is a local martingale.}$$

Moreover,

$$d[W]_t = \frac{1}{|\beta_t|^2} \sum_{i,j=1}^n \beta_t^i \beta_t^j \, d[\beta^i, \beta^j]_t = dt.$$

Lévy's characterization implies that W is a standard Brownian motion. Hence,

$$dX_t = dW_t + \frac{n-1}{2X_t} dt, \quad t < \tau.$$

A Bessel process of dimension v describes the true evolution of the norm of an v-dimensional Brownian motion up to when it first hits 0.

7.3 Diffusion Processes

Suppose that $a : \mathbb{R}^d \to \mathbb{M}^{d \times d}(\mathbb{R})$, $b : \mathbb{R}^d \to \mathbb{R}^d$ are bounded, measurable, a is symmetric (i.e., $a(x)$ is symmetric for each x). For $f \in C_b^2(\mathbb{R}^d)$ (i.e., C_b^2 with bounded derivatives), set

$$Lf(x) := \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial f(x)}{\partial x_i}.$$

Let X be a continuous, adapted process in $\mathbb{R}^d$. We say that X is an L-diffusion if for all $f \in C_b^2(\mathbb{R}^d)$ we have that:

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds \quad \text{is a martingale.}$$

(The coefficient a is called the <u>diffusion</u>, and b is the <u>drift</u>.)

<u>**Example.**</u> σ, b constant and $a = \sigma\sigma^\top$. B is standard BM on $\mathbb{R}^d$. Then

$$X_t = \sigma B_t + bt \quad \text{is an } (\sigma, b)\text{-diffusion.}$$

If $\sigma = I_d$, $b = 0$, $X_t = B_t$ is an L-diffusion where $L = \frac{1}{2}\Delta$.

Proposition 7.3. *Suppose that X solves*

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt,$$

let $f \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$ (bounded derivatives, C^1 in the first variable, C^2 in the second variable).

Then,

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t (\partial_s + L)\, f(s, X_s)\, ds \quad \textit{is a martingale,}$$

$a = \sigma\sigma^\top$ and L as above.

If a, b are bounded, then X is an L-diffusion.

Proof. [ ] . □

Lecture 23 **Question:** Which a can be written as $\sigma\sigma^\top$ for such σ? (See proposition from last time.)

Suppose that a, b are Lipschitz, bounded, and there exists $\varepsilon > 0$ so that:

$$(a(x)\xi, \xi) \geq \varepsilon |\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^d.$$

Then a is uniformly positive definite (UPD). Then there exists $\sigma : \mathbb{R}^d \to \mathbb{M}^{d\times d}(\mathbb{R})$ with $\sigma\sigma^\top = a$.

For $d = 1$, take $\sigma = \sqrt{a}$.

For $d \geq 2$, we can write $a(x) = U(x)\Lambda(x)U(x)^\top$ where $\Lambda(x)$ is the diagonal matrix of eigenvalues and $U(x)$ the orthogonal matrix whose columns are eigenvectors of $a(x)$. Take

$$\sigma(x) = U(x)\sqrt{\Lambda(x)}U(x)^\top.$$

That σ is Lipschitz follows from the differentiability of the square root map on the set of UPD matrices.

For such σ, b, the SDE

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt$$

has a unique strong solution which is an (a, b)-diffusion.

Proposition 7.4. *Let X be an L-diffusion and τ a finite stopping time. Set*

$$\tilde{X}_t = X_{\tau+t}, \quad \textit{and} \quad \tilde{\mathcal{F}}_t = \mathcal{F}_{\tau+t}.$$

Then $\tilde{X}$ is an L-diffusion with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$.

Proof. Fix $f \in C_0^2(\mathbb{R}^d)$. Consider the process

$$\tilde{M}_t^f := f(\tilde{X}_t) - f(\tilde{X}_0) - \int_0^t Lf(\tilde{X}_s) ds.$$

$\tilde{M}_t^f$ is adapted to $(\tilde{\mathcal{F}}_t)$ and is integrable. For $A \in \mathcal{F}_s$ and $n \geq 0$ we have that

$$\begin{aligned} \mathbb{E}\left[(\tilde{M}_t^f - \tilde{M}_s^f) \cdot \mathbf{1}_{A\cap\{\tau\leq n\}}\right] &= \mathbb{E}\left[(M_{t+\tau}^f - M_{s+\tau}^f) \cdot \mathbf{1}_{A\cap\{\tau\leq n\}}\right] \\ &= \mathbb{E}\left[(M_{t+\tau}^f - M_{s+\tau}^f) \cdot \mathbf{1}_{A\cap\{\tau\leq n\}\in\mathcal{F}_{\tau+s}}\right] \\ &= 0 \quad \text{(by optional stopping theorem).} \end{aligned}$$

Sending $n \to \infty$ implies

$$\mathbb{E}\left[(\tilde{M}_t^f - \tilde{M}_s^f) \cdot \mathbf{1}_A\right] = 0 \quad \text{(by dominated convergence theorem).}$$

So $\tilde{M}^f$ is a martingale with respect to $(\tilde{\mathcal{F}}_t)$. □

Lemma 7.3. *Let X be an L-diffusion. Then for all $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ the process*

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t (\partial_s + L) f(s, X_s) ds$$

is a martingale.

Proof. Fix $T > 0$ and consider

$$Z_n = \sup_{\substack{0 \le s \le t \le T \\ t-s \le 1/n}} \left| \dot{f}(s, X_t) - \dot{f}(s, X_s) \right| + \sup_{\substack{0 \le s \le t \le T \\ t-s \le 1/n}} \left| Lf(s, X_t) - Lf(t, X_t) \right|.$$

Then Z_n is bounded and $Z_n \to 0$ as $n \to \infty$ by continuity. By the bounded convergence theorem, it follows that

$$\mathbb{E}[Z_n] \to 0 \quad \text{as } n \to \infty.$$

Now,

$$\begin{aligned} M_t^f - M_s^f = &\left(f(t, X_t) - f(s, X_t) - \int_s^t \dot{f}(r, X_t)\, dr \right) \\ &+ \left(f(s, X_t) - f(s, X_s) - \int_s^t Lf(s, X_r)\, dr \right) \\ &+ \left(\int_s^t \dot{f}(r, X_t) - \dot{f}(r, X_r)\, dr \right) \\ &+ \left(\int_s^t Lf(s, X_r) - Lf(r, X_r)\, dr \right). \end{aligned}$$

Choose $s = s_0 < s_1 < \cdots < s_n = t$ such that $s_{k+1} - s_k \le 1/n$ for each k. The first line is equal to 0 by the fundamental theorem of calculus. The second line has expectation equal to 0 given $\mathcal{F}_s$ (since X is an L-diffusion). For the last two lines, we have that

$$\mathbb{E}\left[\left|\mathbb{E}\left[M_t^f - M_s^f \mid \mathcal{F}_s\right]\right|\right] \le (t-s) \cdot \mathbb{E}[Z_n].$$

So,

$$\mathbb{E}\left[\mathbb{E}\left[M_t^f - M_s^f \mid \mathcal{F}_s\right]\right] \le (t-s) \cdot \mathbb{E}[Z_n] \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\mathbb{E}\left[M_t^f \mid \mathcal{F}_s\right] = M_s^f.$$

□

7.4 Dirichlet and Cauchy problem

Assume that a, b are Lipschitz and $a(x)\xi \cdot \xi \geq \varepsilon|\xi|^2$ for some $\varepsilon > 0$, for all $x, \xi \in \mathbb{R}^d$ (i.e., a is uniformly positive definite).

Let $\mathcal{D} \subseteq \mathbb{R}^d$ be a bounded, open domain with smooth boundary. We shall assume the following theorem from PDE.

Theorem 7.4 (Dirichlet Problem)**.** *For all $f \in C(\partial\mathcal{D})$, there exists a unique function $u \in C(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ such that:*

$$\begin{cases} Lu + \varphi = 0 & \text{in } \mathcal{D}, \\ u = f & \text{on } \partial\mathcal{D}. \end{cases}$$

Moreover, there exist continuous functions

$$m : \mathcal{D} \times \partial\mathcal{D} \to [0, \infty), \quad g : \{(x, y) \in \mathcal{D} \times \mathcal{D} : x \neq y\} \to (0, \infty)$$

such that for all f, φ as above, we have

$$u(x) = \int_{\mathcal{D}} g(x, y)\varphi(y)\, dy + \int_{\partial\mathcal{D}} f(y)\, m(x, y)\, \lambda(dy),$$

where g is the Green kernel, and $m(x, y)\, \lambda(dy)$ is the harmonic measure on $\partial\mathcal{D}$ as seen from x.

Theorem 7.5. *Suppose that $u \in C(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ satisfies*

$$\begin{cases} Lu + \varphi = 0 & \text{on } \mathcal{D}, \\ u = f & \text{on } \partial\mathcal{D}, \end{cases}$$

with $f \in C(\partial\mathcal{D}), \varphi \in C(\overline{\mathcal{D}})$. Then for any L-diffusion X starting from $x \in \mathcal{D}$, we have

$$u(x) = \mathbb{E}_x \left[\int_0^\tau \varphi(X_s)\, ds + f(X_\tau) \right],$$

where $\tau = \inf\{t \geq 0 : X_t \notin \mathcal{D}\}$. Moreover, for all Borel sets $A \subseteq \mathcal{D}, B \subseteq \partial\mathcal{D}$, we have

$$\mathbb{E}_x \left[\int_0^\tau \mathbf{1}(X_s \in A)\, ds \right] = \int_A g(x, y)\, dy, \quad \mathbb{P}_x\left[X_\tau \in B\right] = \int_B m(x, y)\, \lambda(dy).$$

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Proof. Fix $n \geq 1$ and let $T_n = \inf\{t \geq 0 : X_t \notin \mathcal{D}_n\}$, where $\mathcal{D}_n = \{x \in \mathcal{D} : \mathrm{dist}(x, \mathcal{D}^c) > 1/n\}$. Consider

$$M_t = u(X_{t \wedge T_n}) - u(X_0) + \int_0^{t \wedge T_n} \varphi(X_s)\, ds.$$

There exists $\tilde{u} \in C_b^2(\mathbb{R}^d)$ with $\tilde{u} = u$ on $\mathcal{D}_n$. Then $M = \tilde{M}^{T_n}$ where:

$$\tilde{M}_t = \tilde{u}(X_t) - \tilde{u}(X_0) - \int_0^t L\tilde{u}(X_s)\, ds.$$

Since X is an L-diffusion, $\tilde{M}$ is a martingale. By the optional stopping theorem, M is a martingale. Hence,

$$u(x) = \mathbb{E}_x \left[u(X_{T_n}) + \int_0^{T_n} \varphi(X_s)\, ds \right]. \tag{$\bigstar$}$$

We want to send $n \to \infty$. First we will show $\mathbb{E}_x[T] < \infty$. Take $\varphi \equiv 1$, $f \equiv 0$, and let $u^{1,0}$ be the solution of the associated Dirichlet problem. Then ($\bigstar$) holds for $u^{1,0}$, so:

$$\mathbb{E}_x\left[T_n \wedge t\right] = u^{1,0}(x) - \mathbb{E}_x\left[u^{1,0}(X_{T_n})\right].$$

Since $u^{1,0}$ is bounded (in $C(\overline{\mathcal{D}})$), $T_n \uparrow T$ as $n \to \infty$, monotone convergence theorem implies $\mathbb{E}_x[T] < \infty$ (as $n \to \infty$, $t \to \infty$).

Now return to the general case in ($\bigstar$). Have that $T_n \wedge t \nearrow T$ as $n, t \to \infty$. Since u is continuous on $\overline{\mathcal{D}}$,

$$u(X_{t\wedge T_n}) \to f(X_T) \quad \text{as } n, t \to \infty.$$

Since u is bounded on $\overline{\mathcal{D}}$ ($\overline{\mathcal{D}}$ compact, u continuous), bounded convergence theorem implies

$$\mathbb{E}_x\left[u(X_{t\wedge T_n})\right] \to \mathbb{E}_x\left[f(X_T)\right] \quad \text{as } t, n \to \infty.$$

Moreover,

$$\mathbb{E}_x\left[\int_0^T |\varphi(X_s)|\, ds\right] \le \|\varphi\|_\infty \cdot \mathbb{E}_x[T] < \infty.$$

By the dominated convergence theorem,

$$\mathbb{E}_x\left[\int_0^{T\wedge t\wedge T_n} \varphi(X_s)\, ds\right] \to \mathbb{E}_x\left[\int_0^T \varphi(X_s)\, ds\right].$$

Thus,

$$u(x) = \mathbb{E}_x\left[f(X_T) + \int_0^T \varphi(X_s)\, ds\right].$$

Final assertions follow by taking limits as $\varphi_n \to \mathbf{1}_A$, $f \equiv 0$ and $f_n \to \mathbf{1}_B$, $\varphi \equiv 0$. $\square$

Theorem 7.6. *For each $f \in C_b^2$, there exists a unique solution $u \in C_b^1(\mathbb{R}_+ \times \mathbb{R}^d)$ such that:*

$$\begin{cases} \partial u/\partial t = Lu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0,x) = f & \text{on } \mathbb{R}^d \end{cases}$$

Moreover, there exists a continuous function $p : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ such that

$$u(t,x) = \int_{\mathbb{R}^d} p(t,x,y) f(y)\, dy \qquad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

where p is the "heat kernel".

Theorem 7.7. *Assume that $f \in C_b^2(\mathbb{R}^d)$. Let u satisfy*

$$\begin{cases} \partial u/\partial t = Lu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0,x) = f & \text{on } \mathbb{R}^d \end{cases}$$

Then for any L-diffusion X starting from x, for all $t \in \mathbb{R}_+$, $0 \le s \le t$, we have that

$$\mathbb{E}_x\left[f(X_t) | \mathcal{F}_s\right] = u(t-s, X_s) \quad \text{almost surely.}$$

In particular,

$$\mathbb{E}_x\left[f(X_t)\right] = u(t,x) = \int_{\mathbb{R}^d} p(t,x,y) f(y)\, dy.$$

Finally, under $\mathbb{P}_x$, the finite-dimensional distributions of X are given by:

$$\mathbb{P}_x\left[X_{t_1} \in dx_1, \ldots, X_{t_n} \in dx_n\right] = p(t_1, x_0, x_1) \cdots p(t_n - t_{n-1}, x_{n-1}, x_n)\, dx_1 \cdots dx_n,$$

for $0 < t_1 < t_2 < \cdots < t_n < \infty$, $x_1, \ldots, x_n \in \mathbb{R}^d$, $x_0 = x$.

Proof. Fix $t \in (0, \infty)$. Consider $g(s, x) = u(t-s, x)$ for $s \leq t$, $x \in \mathbb{R}^d$. Note that

$$\left(\frac{\partial}{\partial s} + L\right) g(s, x) = -\frac{\partial u}{\partial t}(t-s, x) + Lu(t-s, x) = 0.$$

Therefore,

$$M_s^g = g(s, X_s) - g(0, X_0) - \int_0^s \left(\frac{\partial}{\partial r} + L\right) g(r, X_r)\, dr = g(s, X_s) - g(0, X_0)$$

is a martingale for $s \in [0, t)$. By extending g to $\tilde{g} \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ appropriately. Hence, for all $0 \leq s \leq t' < t$, we have

$$\mathbb{E}_x\left[M_{t'}^g | \mathcal{F}_s\right] = M_s^g \text{ almost surely}, \quad \Rightarrow \quad \mathbb{E}_x[M_{t'}^g] = \mathbb{E}_x[M_0^g].$$

Therefore,

$$\mathbb{E}_x[u(t-t', X_{t'})] = u(t, x).$$

Now, as $t' \to t$, by continuity $u(t-t', X_{t'}) \to f(X_t)$ (bounded convergence, $u \in C_b^2$), so

$$\mathbb{E}_x[f(X_t)] = u(t, x).$$

For the second part of the theorem set

$$P_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y)\, dy = u(t, x).$$

<u>Uniqueness of solutions to the Cauchy problem:</u>

$$P_s(P_t f) = P_{s+t} f$$

<u>Claim</u> (by induction):

$$\mathbb{E}_x\left[\prod_{i=1}^n f_i(X_{t_i})\right] = \int_{(\mathbb{R}^d)^n} p(t_1, x_0, x_1) f_1(x_1) \cdots p(t_n - t_{n-1}, x_{n-1}, x_n) f_n(x_n)\, dx_1 \cdots dx_n$$

For induction, we use that:

$$\begin{aligned}
\mathbb{E}_{x_0}\left[\prod_{i=1}^n f_i(X_{t_i})\right] &= \prod_{i=1}^{n-1} f_i(X_{t_i})\, \mathbb{E}\left[f_n(X_{t_n}) \mid \mathcal{F}_{t_{n-1}}\right] \\
&= \prod_{i=1}^{n-1} f_i(X_{t_i})\, P_{t_n - t_{n-1}} f(X_{t_{n-1}})
\end{aligned}$$

Now apply the case $n-1$. □