Contents

1 Hahn-Banach Extension Theorems

Lecture 1

We start with setting up some notation.

1. Let X be a normed space. The dual space of X is denoted by X^* and is the space of all bounded linear functionals on X. Observe that X^* is always a Banach space in the operator norm
$$\|f\| = \sup\{|f(x)| : x \in B_X\}, \quad f \in X^*.$$
Recall that $B_X = \{x \in X : \|x\| \leqslant 1\}$ (the unit ball in X), and $S_X = \{x \in X : \|x\| = 1\}$ (the unit sphere in X).

2. Let X, Y be normed spaces. We write $X \sim Y$ if X, Y are *isomorphic*, i.e. there exists a linear bijection $T : X \to Y$ s.t. T, T^{-1} are continuous in the norm topologies.

3. Let X, Y be normed spaces. We write $X \cong Y$ if X, Y are *isometrically isomorphic*, i.e. there exists a surjective linear map $T : X \to Y$ s.t. $\|Tx\| = \|x\|$ for all $x \in X$.

4. For $x \in X$, we write $\langle x, f\rangle = f(x)$. Note that $\langle x, f\rangle = |f(x)| \leqslant \|f\| \cdot \|x\|$.

Examples 1.1. *1. For $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $\ell_p^* \cong \ell_q$ (isometrically isomorphic)*

2. If H is a Hilbert space, then $H^* \cong H$ (conjugate linear in the complex case).

Definition 1.2. *Let X be a real vector space. A functional $p : X \to \mathbb{R}$ is:*

(i) positive homogeneous if $p(tx) = tp(x)$ for all $x \in X$ and $t > 0$

(ii) sub-additive if $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$.

Theorem 1.3. *Let X be a real vector space and $p : X \to \mathbb{R}$ be positive homogeneous and sub-additive. Let $Y \leqslant X$ and $g : Y \to \mathbb{R}$ be a linear functional s.t. $g(y) \leqslant p(y)$ for all $y \in Y$. Then, there exists linear $f : X \to \mathbb{R}$ s.t. $f|_Y = g$ and $f(x) \leqslant p(x)$ for all $x \in X$.*

Recall now Zorn's lemma, which is needed to prove Theorem 1.3 in complete generality. Let $(P, \leqslant)$ be a poset.

- If $A \subset \mathcal{P}$, $x \in P$, then x is an *upper bound* for A if for all $x \in A$, $a \leqslant x$.
- x is a *maximal element* if for all $y \in P$, $y \geqslant x$ implies $y = x$
- A collection of subsets $\mathcal{C}$ of P is called a *chain* if for any two subsets $C, D \in \mathcal{C}$, either $C \subseteq D$ or vice versa.

Lemma 1.4 (Zorn). *If $P \neq \varnothing$ and every non-empty chain has an upper bound, then P has a maximal element.*

Proof of Theorem 1.3. Let P be the set of pairs (Z, h) where Z is a subspace of X with $Y \subseteq Z$, $h : Z \to \mathbb{R}$ linear, $h|_Y = g$ and for all $z \in Z$, $h(z) \leqslant p(z)$. Observe that P is partially ordered by

$$(Z_1, h_1) \leqslant (Z_2, h_2) \iff Z_1 \subseteq Z_2, \quad h_2|_{Z_1} = h_1.$$

Also, we have $P \neq \varnothing$ since (Y, g) is in P. If $\{(Z_i, h_i)\}_{i\in I}$ is a chain in P with $I \neq \varnothing$, then setting $Z = \bigcup_{i\in I} Z_i$ and $h : Z \to \mathbb{R}$ by requiring that $h|_{Z_i} = h_i$, for $i \in I$, we have that (Z, h) is in P and it is an upper bound for the chain. So by Zorn, P has a maximal element (Z, h).

It suffices to show that $Z = X$. Suppose not, and fix $x \in X/Z$. Let $W = \operatorname{span}(Z \cup \{x\})$ and $f : W \to \mathbb{R}$, $f(z + \lambda x) = h(z) + \lambda\alpha$, for $z \in Z$, $\lambda \in \mathbb{R}$ for some $\alpha \in \mathbb{R}$. We seek $\alpha \in \mathbb{R}$ s.t. for all $w \in W$, $f(w) \leqslant p(w)$. Then, $(W, f) \in P$ and (W, f) is strictly bigger than (Z, h), which would establish a contradiction.

<u>Need</u>: $h(z) + \lambda\alpha \leqslant p(z + \lambda\alpha)$ for all $z, \lambda \in \mathbb{R}$. Since p is positive homogeneous, this is equivalent to:

$$\left\{ \begin{array}{l} h(z) + \alpha \leqslant p(z + x) \\ h(z) - \alpha \leqslant p(z - x) \end{array} \right\} \text{ for all } z \text{ in } Z.$$

That is, $h(y) - p(y - x) \leqslant \alpha \leqslant p(z + x) - h(z)$ for all $y, z \in Z$. This holds since, for $y, z \in Z$:

$$h(y) + h(z) = h(y + z) \leqslant p(y + z) = p(y - x + z + x) \leqslant p(y - x) + p(z + x).$$

□

Definition 1.5. *Let X be a real or complex vector space. A semi-norm on X is a functional $p : X \to \mathbb{R}$ s.t.:*

- *for all $x \in X$, $p(x) \geqslant 0$*
- *for all $x \in X$ and all $\lambda \in \mathbb{R}$, $p(\lambda x) = |\lambda| \cdot p(x)$*
- *for all $x, y \in X$, $p(x + y) \leqslant p(x) + p(y)$.*

Note: $Norm \implies seminorm \implies (sub-additive \,\&\, positive\ homogeneous)$

Theorem 1.6 (Hahn Banach)**.** *Let X be a real of complex vector space and p be a semi-norm on X. Let Y be a subspace of X and $g : Y \to \mathbb{C}$ linear s.t. for all $y \in Y$ $|g(y)| \leqslant p(y)$. Then there exists linear functional f on X s.t. $f|_Y = g$ and for all $x \in X$ $|f(x)| \leqslant p(x)$.*

Lecture

Proof. Real case: for all $y \in Y$ $g(y) \leqslant |g(y)| \leqslant p(y)$. By Theorem 1.3 there exists linear functional $f : X \to \mathbb{R}$ s.t. $f|_Y = g$ and for all $x \in X$ $f(x) = p(x)$. For $x \in X$, we have also $-f(x) = f(-x) \leqslant p(-x) = p(x)$, so $|f(x)| \leqslant p(x)$.

Complex case: $\operatorname{Re}(g) : Y \to \mathbb{R}$, $(\operatorname{Re})(y) = \operatorname{Re}(g(y))$, is real linear. For all $y \in Y$ $|\operatorname{Re}(g)(y)| \leqslant |g(y)| \leqslant p(y)$. By the real case, there exists a real linear map $h : X \to \mathbb{R}$ s.t. $h|_Y = \operatorname{Re}(g)$ and for all $x \in X$ $|h(x)| \leqslant p(x)$.

Claim: there exists unique complex linear map $f : X \to \mathbb{C}$ s.t. $h = \operatorname{Re}(f)$.

Proof of claim: Uniqueness If we have such an f, then for any $x \in X$, $f(x) = h(x) + \operatorname{Im}(f) = h(x) + \operatorname{Im}(-if(ix)) = h(x) - ih(ix)$. Existence *define* $f(x) = h(x) - ih(ix)$, for $x \in X$. Then f is real linear and $f(x) = if(x)$ for all $x \in X$. Hence, f is complex linear and $\operatorname{Re}(f) = h$, by definition.

We have $f : X \to \mathbb{C}$ linear s.t. $\operatorname{Re}(f) = h$. Then $\operatorname{Re}(f)|_Y = h|_Y = \operatorname{Re}(g)$, so by uniqueness $f|_Y = g$. Given $x \in X$, write $|f(x)| = \lambda f(x)$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$; now, $|f(x)| = \lambda f(x) = f(\lambda x) = \operatorname{Re}(f)(\lambda x)$ [a] $= h(\lambda x) \leqslant p(\lambda x) = |\lambda| p(x) = p(x)$. □

[a] $|f(x)| \in \mathbb{R}$.

Remark. For a complex vector space Y, let $Y_{\mathbb{R}}$ be Y viewed as a real vector space. The proof above shows that for a normed space, X, the map $f \mapsto \operatorname{Re}(f) : (X^*) \to (X^*_{\mathbb{R}})$ is an isometric isomorphism.

Corollary 1.7. *Let X be a real or complex vector space, p a semi-norm on X and $x_0 \in X$. Then there exists linear functional f on X s.t. $f(x_0) = p(x_0)$ and for all $x \in X$ $|f(x)| \leqslant p(x)$.*

Proof. Let $Y = \operatorname{span}\{x_0\}$, define $g : Y \to (\mathbb{R}, \mathbb{C})$ $g(\lambda x_0) = \lambda p(x_0)$. Then g is linear and $g(x_0) = p(x_0), |g(\lambda x_0)| = |\lambda| \cdot |p(x_0)| = p(\lambda x_0)$. So for all $y \in Y$ $|g(y)| \leqslant p(y)$. By Theorem 1.6, there exists linear function f on X s.t. $f|_Y = g$ and for all $x \in X$ $|f(x)| \leqslant p(x)$. Note that $f(x_0) = g(x_0) = p(x_0)$. □

Theorem 1.8 (Hahn-Banach)**.** *Let X be a real or complex normed space.*

- *It Y is a subspace of X and $g \in Y^*$ then there exists $f \in X^*$ s.t. $f|_Y = g$ and $\|f\| = \|g\|$*
- *Given $x_0 \in X/\{0\}$, there exists $f \in S_{X^*}$ [a] s.t. $f(x_0) = \|x_0\|$.*

[a] unit sphere.

Proof. (i) let $p(x) = \|g\| \cdot \|x\|$, for $x \in X$. Then p is a semi-norm on X and for all $y \in Y$, $|g(y)| \leqslant \|g\| \cdot \|y\|$. By Theorem 1.6 there exists linear functional $f : X \to (\mathbb{R}, \mathbb{C})$ s.t. $f|_Y = g$ and for all $x \in X$ $|f(x)| \leqslant p(x) = \|g\| \cdot \|x\|$, which implies $\|f\| \leqslant \|g\|$; since $f|_Y = g$, we also have $\|f\| \geqslant \|g\|$, so we have the desired equality $\|f\| = \|g\|$.

(ii) Apply Corollary 1.7 with $p(x) = \|x\|$, to get a linear functional f on X s.t. for all $x \in X$ $|f(x)| \leqslant \|x\|$ and $f(x_0) = \|x_0\|$. It follows that $\|f\| = 1$.

□

Remark. 1. part (i) is a sort of linear version of Tietze's extension theorem: given K compact, Hausdorff, $L \subseteq K$ closed, $g : K \to (\mathbb{R}, \mathbb{C})$ continuous, there exists continuous $f : K \to (\mathbb{R}, \mathbb{C})$

s.t. $f|_L = g$ and $\|f\|_\infty = \|g\|_\infty$.

2. part (i) shows that for all $x \neq y \in X$ there exists $f \in X^*$ s.t. $f(x) \neq f(y)$ (use $x_0 = x - y$). X^* separates points of X. (This is a sort of linear version of Uryshon's lemma: $C(K)$ separates points of K, K compact, Hausdorff).

3. The f in part (ii) is called a norming functional for x_0. It shows that $\|x_0\| = \max\{|\langle x_0, g\rangle| : g \in B_{X^*}\}$. Another name for f: support functional at x_0. Assume X is real, $\|x\| = 1$. Then, $B_X \subseteq \{x \in X : f(x) \leqslant 1\}$.

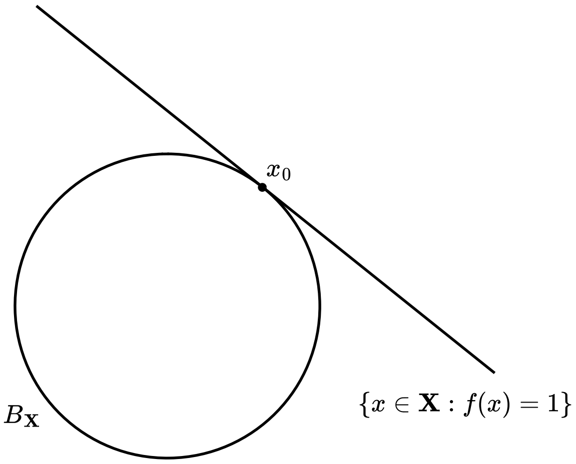

Figure 1: Illustration of support a functional, see the remark above. The pre-image of 1 under f is tangent to B_X at x_0.

Bidual Let X be a normed space. Then $X^{**} = (X^*)^*$ is called the bidual or second dual of X. For $x \in X$, we define $\widehat{x} : X^* \to$ scalar, by $\widehat{x}(f) = f(x)$, for all $f \in X^*$ (evaluation at x). Then $\widehat{x}$ is linear, and $|\widehat{(f)}| = |f(x)| \leqslant \|f\| \cdot \|x\|$, so $\widehat{x} \in X^{**}$ and $\|\widehat{x}\| \leqslant \|x\|$. The map $x \mapsto \widehat{x} : X \to X^{**}$ is called the canonical embedding of X into X^{**}.

Theorem 1.9. *The canonical embedding of X into X^{**} is an isometric isomorphism into X^{**}.*

Proof. Linearity: $(\widehat{\lambda x + \mu y})(f) = f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) = \lambda\widehat{x}(f) + \mu\widehat{y}(f)$ for all $x, y \in X$, λ, μ scalars and $f \in X^*$

Isometry: if $x \in X\backslash\{0\}$, then there exists norming functional f of x and so $\|\widehat{x}\| \geqslant |\widehat{x}(f)| = |f(x)| = \|x\|$. $\square$

Remark. 1. In bracket notation: $\langle f, \widehat{x}\rangle = \langle x, f\rangle$ (for $x \in X$ and $f \in X^*$).

2. Let $\widehat{X} = \{\widehat{x} : x \in X\}$-the image of $X \in X^*$. Then, Theorem 1.9 says that $X \cong \widehat{X} \subseteq X^{**}$. We often identify $\widehat{X}$ with X and think of X isometrically as a subspace of X^{**}. Note that X is complete $\iff$ $\widehat{X}$ is closed in X^{**}.

3. More generally, $\overline{\widehat{X}}$ is a Banach space (closed in X^{**}) containing an isometric copy of X as a dense subspace. We thus proved that normed spaces have completions.

Definition 1.10 (Reflexivity)**.** *A normed space X is called reflexive if the canonical embedding $X \hookrightarrow X^{**}$ is surjective.*

Examples 1.11. *Examples: (Reflexivity)*

1. $\ell_p, 1 < p < \infty$
   Hilbert spaces
   finite-dimensional normed spaces
   $L_p(\mu), 1 < p < \infty$ *(later!)*
2. $c_0, \ell_1, \ell_\infty, L_1[0,1]$ *are not reflexive.*

Remark. If X is reflexive, then $X \cong X^{**}$. Note however that there exist Banach spaces X s.t. $X \cong X^{**}$ but X is not reflexive.

1.1 Dual Operators

Lecture 3 Let X, Y be normed spaces. Recall

$$\mathcal{B}(X,Y) = \{T : X \to Y : T \text{ is linear and bounded}\}.$$

This is a normed space in the operator norm:

$$\|T\|_{X\to Y} = \sup_{\|x\|_X \leq 1} \|Tx\|_Y .$$

If Y is complete, then so is $(\mathcal{B}(X,Y), \|\cdot\|_{X\to Y})$. For $T \in \mathcal{B}(X,Y)$, the dual operator of T, is the map $T^* : X^* \to Y^*, T^*g = g \circ T$ for $g \in Y^*$[1]. In the bracket notation;

$$\langle x, T^*g\rangle = \langle Tx, g\rangle, \quad \text{for } x \in X,\ g \in Y^*.$$

T^* is linear:

$$\begin{aligned}
\langle x, T^*(\lambda g + \mu h)\rangle &= \langle Tx, \lambda g + \mu h\rangle \\
&= \lambda\langle Tx, g\rangle + \mu\langle Tx, h\rangle \\
&= \lambda\langle x, T^*g\rangle + \mu\langle x, T^*h\rangle \\
&= (\lambda T^*g + \mu T^*h)(x) \\
&= \langle x, \lambda T^*g + \mu T^*h\rangle.
\end{aligned}$$

T^* is bounded:

$$\begin{aligned}
\|T^*\| &= \sup_{\|g\|_{Y^*} \leq 1} \|T^*g\| \\
&= \sup_{\|g\|_{Y^*} \leq 1} \sup_{\|x\|_X \leq 1} \|g \circ T(x)\| \\
&= \sup_{\|x\|_X \leq 1} \sup_{\|g\|_{Y^*} \leq 1} \|g \circ T(x)\| \\
&= \sup_{\|x\|_X \leq 1} \|Tx\| = \|T\| .
\end{aligned}$$

Remark. If X, Y are Hilbert spaces, and identify X, Y with X^* and Y^* respectively, then $T^* : Y \to X$ is the adjoint of T.

Examples 1.12. *Example:* $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$, $R : \ell_p \to \ell_q$, *the right shift* $R(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ *then* $R^* : \ell_q \to \ell_p$ *is the left shift.*

Properties:

1. $(\mathrm{Id}_X)^* = \mathrm{Id}_X^*$.
2. $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$ for $S, T \in \mathcal{B}(X,Y)$, and λ, μ scalars.
3. $(ST)^* = T^*S^*$ for $T \in \mathcal{B}(X,Y)$ and $S \in \mathcal{B}(\boldsymbol{Y}, \boldsymbol{Z})$
   $(ST)^*(h \in Z^*) = h \circ S \circ T = T^*h \circ S = T^*S^*(h)$

[1] well-defined.

4. $T \mapsto T^* : \mathcal{B}(X,Y) \to \mathcal{B}(Y^*,X^*)$ is an into isometric isomorphism.

5. Let $x \in X$ then $\langle g, T^*\widehat{x}\rangle = \langle T^*g, \widehat{x}\rangle = \langle x, T^*g\rangle = \langle Tx, g\rangle = \langle g, \widehat{Tx}\rangle$ for all $g \in Y^* \implies T^{**}\widehat{x} \equiv \widehat{Tx}$. In other words, the following diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, hook, "\iota_X"] & Y \arrow[d, hook, "\iota_Y"] \\ X^{**} \arrow[r, "T^{**}"] & Y^{**} \end{tikzcd}$$

commutes (vertical arrows are canonical embeddings).

Remark. From the (above) properties, if $X \sim Y$ then $X^* \sim Y^*$.

1.2 Quotient Spaces

Let X be a normed space and Y be a closed subspace. Then the quotient space X/Y becomes a normed space in the quotient norm:

$$\|x+Y\|_{X/Y} = d(x,Y) = \inf_{y\in Y} \|x+y\|.$$

The quotient map : $q : X \to X/Y, q(x) = x + Y$ is linear and bounded with $\|q(x)\|_{X/Y} \leqslant \|x\|_X$ for all $x \in X$, so $\|q\| \leqslant 1$. It maps the open unit ball $B_X = \{x \in X : \|x\| < 1\}$ onto $B^\circ_{X/Y}$. Indeed, for $x \in B^\circ_X$, then $\|q(x)\| \leqslant \|x\| < 1$. Conversely, if $z \in B_{X/Y}$ and $z = q(x)$, then $\|z\| < 1 \implies \inf_{y\in Y} \|x+Y\| < 1 \implies$ there exists $y \in Y$ s.t. $\|x+y\| < 1 \implies x+y \in B^\circ_X$ and $q(x+y) = q(x) = z$. It follows that q is an open map and $\|q\| = 1$ (provided $Y \neq X$).

If Z is another normed space, $T \in \mathcal{B}(X,Z)$ and $Y \subseteq \ker(T)$, then there exists a unique map $\tilde{T} : X/Y \to Z$ such that

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "q"] & Z \\ X/Y \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

commutes. Hence, $T = \tilde{T} \circ q$; moreover, $\tilde{T}$ is linear and $\tilde{T}(B^\circ_{X/Y}) = \tilde{T}(q(B^\circ_X)) = T(B^\circ_X)$ and so it follows that $\left\|\tilde{T}\right\| = \|T\|$.

Theorem 1.13. *Let X be a normed space. If X^* is separable, then so is X.*

Remark. The converse is false in general. For instance, $X = \ell_1, X^* = \ell_\infty$.

Proof. Since X^* is separable, then so is S_{X^*}. Let $\{f_n : n \in \mathbb{N}\}$ be a dense subset of S_{X^*}. For all n there exists $x_n \in B_X$ s.t. $f_n(x_n) > 1/2$. Let $Y = \overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}}$.

Claim: suffices to show $Y = X$.

Suppose not: Then, by Theorem 1.8 we can pick $g \in (X/Y)^*$ with $\|g\| = 1$, that is a norming functional. Let $f = g \circ q$ ($q : X \to X/Y$ is the quotient map). Then $\|f\| = \|g\| = 1 \implies f \in S_{X^*}$. By density, we have that there exists $n \in \mathbb{N}$ s.t. $\|f - f_n\| < \frac{1}{10}$ (something small). So

$$|(f - f_n)(x_n)| \leqslant \|f - f_n\| \cdot \|x_n\| < \frac{1}{10},$$

but

$$|(f - f_n)(x_n)| = |f_n(x_n)| > \frac{1}{2}, \quad n \to \infty,$$

a contradiction. □

Theorem 1.14. *Let X be a separable normed space. Then X embeds isometrically into ℓ_∞.*

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be dense in X and for all $n \in \mathbb{N}$ let $f_n \in S_{X^*}$ with $f_n(x_n) = \|x_n\|$ (wlog $X \neq \{0\}$). Define $T : X \to \ell_\infty, Tx = (f_n(x_n))$. It is clear that T is linear.

Well-defined: $|f_n(x)| \leqslant \|f_n\| \cdot \|x\| \leqslant \|x\|$, for all $n \in \mathbb{N}$ which implies $\|Tx\|_\infty \leqslant \|x\| < \infty$, hence $Tx \in \ell_\infty$.

T isometric: already $\|Tx\|_\infty \leqslant \|x\|$ for all x. Also, $\|Tx_n\|_\infty = \|x_n\|$ for all n. By density and continuity, $\|Tx\|_\infty = \|x\|$ for all $x \in X$. □

Lecture 4 **Remark.**

1. The result says ℓ_infty is isometrically universal for the class of separable Banach spaces, $\mathcal{SB}$.
2. Dual result: every separable Banach space is a quotient of ℓ_1 (see the Example sheets).

Theorem 1.15 (Vector-valued Liouville)**.** *Let X be a complex Banach space and $f : \mathbb{C} \to X$ be holomorphic and bounded, then f is constant.*

Proof. We have that there exists $M \geqslant 0$, s.t. for all $z \in \mathbb{C}$, $\|f(z)\| \leqslant M$. Also, for $w \in \mathbb{C}$, $\lim_{z \to w} \frac{f(z) - f(w)}{z - w}$ exists in X and we denote this by $f'(z)$. Fix $\phi \in X^*$ and consider $\phi \circ f : \mathbb{C} \to \mathbb{C}$. This is holomorphic and bounded.

Bounded: $|\phi(f(z))| \leqslant \|\phi\| \cdot \|f(z)\| \leqslant \|\phi\| \cdot \|z\|$ for all $z \in \mathbb{C}$.

Holomorphic:

$$\frac{\phi(f(z)) - \phi(f(w))}{z - w} =^{a} \phi\left(\frac{f(z) - f(w)}{z - w}\right) \to \phi(f'(z)), \quad \text{as } w \to^{b} z.$$

Now, by scalar Liouville, $\phi \circ f$ is constant. Hence, $\phi \circ f(z) = \phi(f(0))$ for all $z \in \mathbb{C}$. Fix$z \in \mathbb{C}$, $\phi(f(z)) = \phi(f(0))$, for all $\phi \in X^*$. Since X^* separates points of X, $f(z) = f(0)$ for all $z \in \mathbb{C}$. □

[a] linearity.
[b] ϕ is continuous.

1.3 Locally Convex Spaces

Definition 1.16 (Locally convex space (LCS))**.** *A locally convex space is a pair $(X, \mathcal{P})$, where X is a real/complex vector space and $\mathcal{P}$ is a family of semi-norms on X that separate points of X in the sense that for all $x \in X \neq \{0\}$ there exists semi-norm $P_X \in \mathcal{P}$ s.t. $P_X \neq 0$. The family $\mathcal{P}$ defines a topology on X:*

$$\mathcal{U} \subseteq X \text{ is open} \iff \forall x \in \mathcal{U}\ \exists n \in \mathbb{N} \exists p_1, \ldots, p_n \in \mathcal{P}$$
$$\exists \epsilon > 0 \text{ s.t. } \{y \in X : p_k(y - x) < \epsilon, 1 \leqslant k \leqslant n\} \subseteq \mathcal{U}.$$

Remark.

1. Vector addition and scalar multiplication are continuous.
2. This topology is Hausdorff.
3. $x_n \to x \in X \iff$ for all $p \in \mathcal{P}$, $p(x - x_n) \to 0$.

4. Let Y be a subspace of X. Let $\mathcal{P}_Y = \{p|_Y : p \in \mathcal{P}\}$. Then the pair $(Y, \mathcal{P}_Y)$ is a LCS and its topology is the subspace topology induced by $(X, \mathcal{P})$.
5. Let $\mathcal{P}, \mathcal{Q}$ be families of semi-norms on X both separating points of X. We say $\mathcal{P}, \mathcal{Q}$ are equivalent, write $\mathcal{P} \sim \mathcal{Q}$ if they define the same topology on X. Then $(X, \mathcal{P})$ is metrisable iff there exists countable family $\mathcal{Q} \sim \mathcal{P}$.

Definition 1.17 (Fréchet space)**.** *A Fréchet space is a complete metrisable locally convex space.*

Examples 1.18. *1. A normed space $(X, \|\cdot\|)$ is a LCS (here $\mathcal{P} = \{\|\cdot\|\}$).*

2. Let $\mathcal{U} \subseteq \mathbb{C}$ be non-empty open. Let $\mathcal{O}(\mathcal{U}) = \{f : \mathcal{U} \to \mathbb{C} : f \text{ holomorphic}\}$. For $K \subseteq \mathcal{U}$ define $\mathcal{P}_K(f) = \sup_{z \in K} |f(z)|$. Let $\mathcal{P} = \{\mathcal{P}_K : K \subseteq \mathcal{U}, K \text{ compact}\}$. Then $(\mathcal{O}(\mathcal{U}), \mathcal{P})$ is a LCS. Note further that there exists K_n, $n \in \mathbb{N}$, a sequence of compact subsets of $\mathcal{U}$ s.t. $\mathcal{U} = \bigcup_{n \in \mathbb{N}} K_n$ and for all $n \in \mathbb{N}$ $K_n \subset (K_{n+1})^\circ$ (a compact exhaustion of $\mathcal{U}$). Montel's Theorem from complex analysis gives that $(\mathcal{O}(\mathcal{U}), \mathcal{P})$ is not normable: there is no norm on $\mathcal{O}(\mathcal{U})$ that gives the same topology, that is the topology of local uniform convergence. To see this, suppose for a contradiction that there exists norm s.t. $\|\cdot\| \sim \mathcal{P}$, then for all $f \in B_{\mathcal{O}(\mathcal{U})}$, for all $p \in \mathcal{P}$, $p(f) \leqslant C_p \cdot \|f\| := C_p < \infty$ (since $\tau_{\mathcal{P}} = \tau_{\mathcal{O}(\mathcal{U})}$) which implies that that unit ball is compact (by the above and Montel's Theorem), hence sequentially compact due to the metrisability of the norm topology on $\mathcal{O}(\mathcal{U})$. So we conclude that $\mathcal{O}(\mathcal{U})$ is finite-dimensional, a contradition.

3. Fix $d \in \mathbb{N}$ and a non-empty open set $\Omega \subseteq \mathbb{R}^d$. Let $\mathcal{C}^\infty = \{f : \Omega \to \mathbb{R}^d : f \text{ is infinitely differentiable}\}$. Given a multi-index, namely, a $d-$tuple $\alpha \in \mathbb{N}^d$, it defines a differential operator:

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

For a compact set $K \subset \Omega$, $\alpha \in \mathbb{N}^d$, define $p_{k,\alpha}(f) = \sup\{|D^\alpha f(z)| : z \in K\}$. Let $\mathcal{P} = \{p_{k,\alpha} : K \subset \Omega, K \text{ compact}, \alpha \in \mathbb{N}^d\}$. Then $(\mathcal{C}^\infty(\Omega), \mathcal{P})$ is a LCS. It's a Fréchet space and non-normable.

Lemma 1.19. *Let $(X, \mathcal{P}), (Y, \mathcal{Q})$ be LCS and $T : X \to Y$ be a linear map. Then the following are equivalent (TFAE):*

(i) T is continuous.

(ii) T is continuous at 0.

(iii) For all $q \in \mathcal{Q}$ there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$, $c > 0$ s.t.

$$q(Tx) \leqslant C \cdot \max_{1 \leqslant k \leqslant n} p_k(x) \text{ for all } x \in X.$$

Proof. (i) $\iff$ (ii): translation is continuous since vector addition is continuous.

(ii) $\iff$ (iii): given $q \in \mathcal{Q}$, let $\mathcal{V} = \{y \in Y : q(y) \leqslant 1\}$. Then $\mathcal{V}$ is a neighbourhood of zero in Y, so there exists a nbhd of zero in X s.t. $T(X) \subseteq \mathcal{V}$. Then there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$, $\epsilon > 0$ s.t. wlog $\mathcal{U} = \{x \in X : p_k(x) \leqslant \epsilon, 1 \leqslant k \leqslant n\}$. Let $p(x) = \max_{1 \leqslant k \leqslant n} p_k(x)$, for $x \in X$. If $p(x) = 1$ then $p(\epsilon x) = \epsilon \implies \epsilon x$ is in $\mathcal{U}$. So $q(Tx) \leqslant 1 \implies q(Tx) \leqslant \frac{1}{\epsilon} p(x)$ by homogeneity for any x s.t. $p(x) > 0$. If $p(x) = 0 \implies p(\lambda x) = 0$ for all λ scalars giving $q(T(\lambda x)) \leqslant 1$ for all λ scalars. Hence, $q(Tx) \leqslant \frac{1}{\epsilon} p(x)$, concluding the proof of this equivalence.

(iii) $\iff$ (ii): Let $\mathcal{V}$ be a nbhd of zero in Y. Then, there exists $n \in \mathbb{N}$, $q_1, \ldots, q_n \in \mathcal{Q}$ and $\epsilon > 0$

s.t. wlog $\mathcal{V} = \{y \in Y : q_j(y) \leqslant \epsilon \text{ for } 1 \leqslant j \leqslant n\}$. For each $1 \leqslant j \leqslant n$, there exists $m_j \in \mathbb{N}$, $p_{j1}, \ldots, p_{jm_j} \in \mathcal{P}$, $C_j > 0$ s.t. $q_j(Tx) \leqslant C_j \cdot \max_{1 \leqslant i \leqslant m_j} p_{ji}(x)$ for all $x \in X$. Finally, let
$\mathcal{U} = \{x \in X : p_{ji}(x) < \frac{\epsilon}{C_j}, 1 \leqslant i \leqslant m_j, 1 \leqslant j \leqslant n\}$ so then $T(\mathcal{U}) \subseteq \mathcal{V}$. □

Definition 1.20. *Let $(X, \mathcal{P})$ be a LCS. The dual space of X is the space X^* of all linear functionals which are continuous wrt the topology $(X, \mathcal{P})$.*

Lecture 5 **Lemma 1.21.** *Let f be a linear functional on a LCS $(X, \mathcal{P})$. Then f is in $X^* \iff \ker f$ is closed.*

Proof. $\Longleftarrow$: $\ker f = f^{-1}(\{0\})$ is closed if f is continuous.

$\Longrightarrow$: If $\ker f = X$, then $f \equiv 0$ is continuous.
Assume $\ker f \neq X$ and fix $x_0 \in X \backslash \ker f$. Since $X \backslash \ker f$ is open, there exists $n \in \mathbb{N}, p_1, \ldots, p_n \in \mathcal{P}$ and $\epsilon > 0$ s.t. $\{x \in X : p_k(x - x_0) < \epsilon, 1 \leqslant k \leqslant n\} \subseteq X \backslash \ker f$. Let $\mathcal{U} = \{x \in X : p_k(x) < \epsilon, 1 \leqslant k \leqslant n\}$. Then $\mathcal{U}$ is a nbhd of zero in X, and $(x_0 + \mathcal{U}) \cap \ker f = \varnothing$.

Note that $\mathcal{U}$ is convex and, in the real case, symmetric ($x \in \mathcal{U}$ implies $-x \in \mathcal{U}$). In the complex case, balanced ($x \in \mathcal{U}$, $|\lambda| \leqslant 1$ implies $\lambda x \in \mathcal{U}$), and hence so is $f(\mathcal{U})$ as f is linear. If $f(\mathcal{U})$ is not bounded, then $f(\mathcal{U})$ is the whole scalar field, and hence so is $f(x_0 + \mathcal{U}) = f(x_0) + f(\mathcal{U})$, a contradiction as zero is not in $f(x_0 + \mathcal{U})$. So there exists $M > 0$ s.t. $|f(x)| < M$ for all $x \in \mathcal{U}$. So given $\delta > 0$, $\frac{\delta}{M}\mathcal{U}$ is a nbhd of zero in $\mathcal{X}$ and $f\left(\frac{\delta}{M}\mathcal{U}\right) \subseteq \{\lambda \text{ scalar}, \lambda < \delta\}$. Thus, f is continuous at zero, hence everywhere. Thus f is in X^*. □

Theorem 1.22. *Let $(X, \mathcal{P})$ be a LCS.*

(i) Given a subspace Y of X and $g \in Y^*$, there exists $f \in X^*$ s.t. $f|_Y = g$.

(ii) Given a closed subspace Y of X and $x_0 \in X \backslash Y$, there exists $f \in X^*$ s.t. $f|_Y = 0$ and $f(x_0) \neq 0$.

Remark. So X^* separates the points of X.

Proof. (i) by lemma 1.19, there exists $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$ and $C > 0$ s.t. for all $y \in Y$ $|g(y)| \leqslant C \cdot \max_{1 \leqslant k \leqslant n} p_k(y)$. Let $p(x) = C \max_{1 \leqslant k \leqslant n} p_k(x)$, for $x \in X$. Then, p is a semi-norm on X and for all $y \in Y$ $|g(y)| \leqslant p(y)$. By Theorem 1.6, there exists a linear functional f on X s.t. $f|_Y = g$ and for all $x \in X$, $|f(x)| \leqslant p(x)$. Now, finally observe that by lemma 1.19, f is in X^*.

(ii) Let $Z = \text{span}(Y \cup \{x_0\})$ and define a linear functional g on Z by $g(y + \lambda x_0) = \lambda$, for $y \in Y$ and λ scalar. Then $g|_Y = 0, g(x_0) = 1 \neq 0$ and $\ker g = Y$ is closed, so $g \in Z^*$ by lemma 1.21. By part (i), there exists $f \in X^*$ s.t. $f|_Z = g$ and this works. □

2 Dual Spaces of $L_p(\mu)$ and $\mathcal{C}(K)$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For $\leqslant< p < \infty$,

$$L_p(\mu) = \left\{f : \Omega \to \text{scalar} : f \text{ is measurable and } \int_\Omega |f|^p d\mu < \infty\right\}$$

This is a normed space in the L_P norm $\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}$.

$p = \infty$: A measurable function $f : \Omega \to$ scalar is essentially bounded if there is $N \in \mathcal{F}$, $\mu(N) = 0$, and $f\big|_{\Omega \backslash N}$ is bounded.

$L_\infty(\mu) = \{f : \Omega \to \text{scalar} : f \text{ measurable and essentially bounded}\}$. This is again a normed space in the $L^\infty-$ norm:

$$\|f\|_\infty = \text{essup}\,|f| = \inf\left\{\sup_{\Omega\setminus N}|f| : N \in \mathcal{F}, \mu(N) = 0\right\}.$$

The inf is attained: there exists $N \in \mathcal{F}$, $\mu(N) = 0$, $\|f\|_\infty = \sup_{\Omega\setminus N}|f|$.

In all the cases, we identify functions f, g if $f = g$ a.e.

Theorem 2.1. *$L_p(\mu)$ is complete for $1 \leqslant p \leqslant \infty$.*

Proof. Can be found in any standard reference in measure theory, see the literature provided. □

2.1 Complex Measures

Let Ω be a set, $\mathcal{F}$ a $\sigma-$field on Ω. A complex measure on $\mathcal{F}$ is a countably additive function $\nu : \mathcal{F} \to \mathbb{C}$. For $A \in \mathcal{F}$, the total variation measure $|\nu|$ of ν is defined as follows:

$$|\nu(A)| = \sup\left\{\sum_{k=1}^{n} : A = \bigcup_{k=1}^{n} A_k \text{ is a measurable partition of } A\right\}^2.$$

Then, $|\nu| : \mathcal{F} \to [0, \infty]$ is a positive measure. Later we see that $|\nu|$ is a finite measure. The total variation of ν is $\|\nu\|_1 = |\nu|(\Omega)$.

Continuity: if ν is a complex measure on $\mathcal{F}$ and $(A_n) \subseteq \mathcal{F}$, then:

(i) if $A_n \subseteq A_{n+1}$, then $\nu(\bigcup_n A_n) = \lim_{n\to\infty} \nu(A_n)$

(ii) if $A_{n+1} \subseteq A_n$, then $\nu(\cap_n) = \lim_{n\to\infty} \nu(A_n)$.

Signed measure: Ω a set, $\mathcal{F}$ a $\sigma-$algebra on Ω.
A signed measure on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{R}$.

Theorem 2.2 (Hahn decomposition)**.** *Let Ω be a set, $\mathcal{F}$ a $\sigma-$algebra on Ω, ν a signed measure on $\mathcal{F}$. Then there exists a measurable partition $P \cup N$ of Ω s.t. for all $A \in \mathcal{F}$, $A \subseteq P$ implies $\nu(A) \geqslant 0$ and for all $A \in \mathcal{F}$, $A \subseteq N$ implies $\nu(A) \leqslant 0$.*

Remark. 1. The decomposition $\Omega = P \cup N$ is called the Hahn decomposition of ν (or of Ω).

2. Lets us define $\nu^+(A) = \nu(A \cap P), \nu^-(A) = -\nu(A \cap N)$, for $A \in \mathcal{F}$. Then ν^+, ν^- are finite positive measures such that $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$. These determine ν^+, ν^- uniquely and $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν.

3. If ν is a complex measure on $\mathcal{F}$ then $\text{Re}(\nu), \text{Im}(\nu)$ are signed measures with Jordan decompositions $\nu_1 - \nu_2 + i(\nu_3 - \nu_4)$-the Jordan decomposition of ν. Then $\nu_k \leqslant |\nu|, 1 \leqslant k \leqslant 4$ and $|\nu| \leqslant \nu_1 + \nu_2 + \nu_3 + \nu_4$. So $|\nu|$ is a finite measure.

4. If ν is a signed measure on $\mathcal{F}$ with Jordan decomposition $\nu^+ - \nu^-$, then $\nu^+(A) = \sup\{\nu(B) : B \in \mathcal{F}, B \subseteq A\}$, for $A \in \mathcal{F}$.

Proof of Theorem 2.2. The strategy is to define $\nu^+(A) = \sup\{\nu(B) : B \in \mathcal{F}, B \subseteq A\}$ for $A \in \mathcal{F}$. Then $\nu^+ \geqslant 0$ and ν^+ is finitely additive.

Key step: $\nu^+(\Omega) \geqslant 0$.

By contradiction, assume not; construct sequences $(A_n), (B_n)$ with $A_0 = \Omega$, $\nu^+(A_n) = \infty$,

[2] $A_k \in \mathcal{F}, A_j \cap A_k = \varnothing\ \forall j \neq k$.

$B_n \subset A_n$ and $\nu(B_n) > n$. Now by the finite additivity of ν^+, pick $A_{n+1} = B_n$ or $A_n \backslash B_n$, to ensure the initial condition ($\nu^+(A_{n+1}) = \infty$) is satisfied.

Claim: this will contradict $\sigma-$additivity.

To see this, note that (A_n) is by construction a decreasing sequence wrt inclusion. By $\sigma-$additivity of ν, $\nu(\cap_n A_n) = \lim_{n\to\infty} \nu(A_n)$. Thus, it cannot be the case that $A_{n+1} = B_n$ infinitely often, since $\nu(\cap_n A_n) < \infty$ (being a signed measure). Thus, there exists $N \in \mathbb{N}$ s.t. for all $n \geqslant N$, $A_{n+1} = A_n \backslash B_n$. Now, $\nu(A_k) = \nu(A_k \backslash B_k) + \nu(B_k) > \nu(A_{k+1}) + k > \nu(A_{k+1})$ for $k \geqslant N$ and so $\nu(A_k) < \nu(A_{k-1}) - k < \nu(A_N) - k, k \to -\infty$, a contradiction.

Claim: there exists $P \in \mathcal{F}$ s.t. $\nu^+(\Omega) = \nu(P)$.

By approximation, take (A_n) s.t. $\nu(A_n) > \nu^+(\Omega) - 2^{-n}$. We will see that the choice $P = \bigcup_n \bigcap_{m\geqslant n} A_m$ works. Let $N = \Omega\backslash P$. By $\sigma-$additivity of ν, have that $\nu(P) = \lim_{n\to\infty} \nu\left(\bigcap_{m\geqslant n} A_m\right)$. Now, for $j \geqslant n$, consider $\bigcap_{n\leqslant m\leqslant j} A_m$, we first see that

$$\begin{aligned} \nu\left(\bigcap_{n\leqslant m\leqslant n+1} A_m\right) &= -\nu(A_n \cup A_{n+1}) + \nu(A_n) + \nu(A_{n+1}) \\ &> -\nu^+(\Omega) + 2\nu^+(\Omega) - 2^{-n} - 2^{-n-1} > \nu^+(\Omega) - 2^{-n-1}. \end{aligned}$$

By inducting, we see that:

$$\nu\left(\bigcap_{n\leqslant m\leqslant n+p} A_m\right) > \nu^+(\Omega) - \sum_{m=0}^{p} 2^{-n-m}.$$

and so

$$\nu\left(\bigcap_{n\leqslant m} A_m\right) = \lim_{p\to\infty} \nu\left(\bigcap_{n\leqslant m\leqslant n+p} A_m\right) > \nu^+(\Omega) - \sum_{m=0}^{\infty} 2^{-n-m} = \nu^+(\Omega) - 2^{-n}.$$

which allows us to conclude that $\nu(P) = \nu^+(\Omega)$ upon taking limits.

Now, with $N = \Omega\backslash P$, define the set functions $\overline{\nu}_\pm : \mathcal{F} \to \mathbb{R}$ by $\overline{\nu}_+(E) = \nu(E \cap P)$ and $\overline{\nu}_-(E) = \nu(E \cap N)$ for $E \in \mathcal{F}$.

Observe first that $\overline{\nu}_- \leqslant 0$. Indeed, suppose there exists $E \in \mathcal{F}$ such that $\nu(E \cap N) > 0$. Then, we see that $\nu^+(\Omega) = \nu^+(P) < \nu(E \cap N) + \nu(P) = \nu((E \cap N) \cup P) \leqslant \nu^+(\Omega)$, a contradiction. Thus, $\overline{\nu}_-$ is a negative measure.

Claim: $\nu(N) = \inf\{\nu(E) : E \in \mathcal{F}\}$.

Suppose otherwise, then there exists $E \in \mathcal{F}$ s.t. $\nu(E) < \nu(N)$, which implies $\nu(\Omega\backslash E) = \nu(\Omega) - \nu(E) > -\nu(N) + \nu(\Omega) = \nu(P)$ and so $\nu(\Omega\backslash E) > \nu(P)$, a contradiction.

Now, we can prove $\overline{\nu}_+ \geqslant 0$. Indeed, suppose there exists $E \in \mathcal{F}$ such that $\nu(E \cap P) < 0$. Then, we see that $\nu(N) \leqslant \nu((E \cap P) \cup N) = \nu(N) + \nu(E \cap P) < \nu(N)$, a contradiction. Thus, $\overline{\nu}_+$ is a positive measure.

Claim: $\overline{\nu}_-(E) = \inf\{\nu(A) : A \subseteq E, A \in \mathcal{F}\}$.

Suppose otherwise, then there exists $E \in \mathcal{F}$ s.t. $\nu(A) < \nu(E \cap N)$, which implies $\nu(A \cap N) < \nu(A \cap P) + \nu(A \cap N) = \nu(A) < \nu(E \cap N)$ and so $\nu(A \cap N) < \nu(E \cap N)$ and so $\nu((E\backslash A) \cap N) = \nu(E \cap N\backslash A \cap N) > 0$ a contradiction.

Finally, we observe that

Claim: $\overline{\nu}_+(E) = \sup\{\nu(A) : A \subseteq E, A \in \mathcal{F}\}$.

Suppose otherwise, then there exists $E \in \mathcal{F}$ s.t. $\nu(A) > \nu(E \cap P)$, which implies $\nu(E \cap P) < \nu(A) = \nu(A \cap P) + \nu(A \cap N) < \nu(A \cap P)$ and so $\nu(E \cap P) < \nu(A \cap P)$ and so $\nu((E \backslash A) \cap P) = \nu(E \cap P \backslash A \cap)) < 0$ a contradiction, and we finally obtain the desired decomposition. □

Lecture 6 **Definition 2.3** (Absolute Continuity). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $\nu : \mathcal{F} \to \mathbb{C}$ be a complex measure. ν is absolutely continuous wrt μ, written $\nu << \mu$ if for all $A \in \mathcal{F}$, $\mu(A) = 0 \implies \nu(A) = 0$.*

Remark.

1. $\nu << \mu \implies |\nu| << \mu$. So if ν has Jordan decomposition $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ and $\nu << \mu$, then $\nu_k << \mu, 1 \leqslant k \leqslant 4$.
2. If $\nu << \mu$, then for all $\epsilon > 0$, there exists $\delta > 0$ s.t. for all $A \in \mathcal{F}$ $\mu(A) < \delta \implies |\nu(A)| < \epsilon$.

Examples 2.4. *Example: For f in $L_1(\mu)$ define $\nu(A) = \int_A f d\mu$, $A \in \mathcal{F}$. By the Theorem of Dominated Convergence (DCT), ν is a complex measure and $\mu(A) = 0 \implies \nu(A) = 0$, i.e. $\nu << \mu$.*

Definition 2.5. *A set in $\mathcal{F}$ is said to be a $\sigma-$finite set (wrt μ) if there is a sequence $(A_n)_{n\in\mathbb{N}} \in \mathcal{F}$ s.t. $A = \bigcup_{n\in\mathbb{N}} A_n$ and for all $n \in \mathbb{N}$, $\mu(A_n) < \infty$. We say μ is $\sigma-$finite if Ω is a $\sigma-$finite set.*

Theorem 2.6 (Radon-Nikodym). *Let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma-$finite emasure space and $\nu : \mathcal{F} \to \mathbb{C}$ be a complex measure s.t. $\nu << \mu$. Then there exists a unique $f \in L_1(\mu)$ s.t. $\nu(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$. Moreover, f takes values in $\mathbb{C}/\mathbb{R}/\mathbb{R}^+$ according to whether ν is a complex, signed or positive measure respectively.*

Proof. Uniqueness: standard.

Existence: wlog ν is a finite positive measure (Jordan decomposition) and wlog μ is a finite measure ($\sigma-$finiteness).

Let $\mathcal{H} = \{h : \Omega \to \mathbb{R}^+ : h \text{ integrable and } \int_A h d\mu \leqslant \nu(A)\ \forall A \in \mathcal{F}\}$. $\mathcal{H} \neq \varnothing$ $(0 \in \mathcal{H})$ and $h_1, h_2 \in \mathcal{H}$ implies $h_1 \vee h_2 = \max\{h_1, h_2\}$ is in $\mathcal{H}$. Also, if (h_n) are in $\mathcal{H}$ s.t. $h_n \uparrow h$, then h is in $\mathcal{H}$. Let $\alpha = \sup_{h\in\mathcal{H}} \int_\Omega f d\mu, 0 \leqslant \alpha \leqslant \nu(\Omega)$.

Claim: there exists $f \in \mathcal{H}$ s.t. $\alpha = \int_\Omega f d\mu$.

We construct such an $f \in \mathcal{H}$. Take $f_n \in \mathcal{H}$ s.t. $\int_A g_n d\mu \leqslant \nu(A)$, for all $A \in \mathcal{F}$ and $\int_\Omega f_n d\mu \to \alpha$. The same holds if we replace f_n by $f_1 \vee \cdots \vee f_n$, and so wlog we can assume the sequence is non-decreasing. Now, by induction, there exists sets $E_1, \ldots, E_n \in \mathcal{F}$ pairwise disjoint s.t. $\bigcup_{k=1}^n E_k = \Omega$ and $g_n = \sum_{k=1}^n f_j \mathbf{1}_{E_j}$ and $\int_A g_n d\mu = \sum_{j=1}^n \int_{E_j\cap A} f_n d\mu \leqslant \sum_{j=1}^n \nu(E_j \cap A) = \nu(A)$. Since g_n is non-decreasing, take the pointwise supremum to obtain $f_0 := \sup_n g_n$, which is in $\mathcal{H}$ by the above, and is seen to work by inspection.

Now consider the signed measures $\nu_1, \nu_2, (\lambda_n)_{n\in\mathbb{N}} : \mathcal{F} \to \mathbb{R}$, s.t. $\nu_1 = \nu - \nu_2, \nu_2(A) = \int_A f_0 d\mu$ and $\lambda_n(A) = \nu_1(A) - \frac{1}{n}\mu(A)$ for all $A \in \mathcal{F}$. Then there exist (Hahn decomposition) $(P_n), (N_n)$ in $\mathcal{F}$ s.t. $\Omega = P_n \cup N_n, P_n = \Omega \backslash N_n$, s.t. $\lambda_n(E) \geqslant 0$ for all $E \in \mathcal{F}$ s.t. $E \subseteq P_n$. Now, for such

E, we have $\lambda_n(E) = \nu_1(E) - \frac{1}{n}\mu(E) \geqslant 0$ and so $\nu(E) = \nu_1(E) + \nu_2(E) \geqslant \int_E f_0 d\mu + \frac{1}{n}\int_E d\mu$. Let $\tilde{f}_n = f_0 + \frac{1}{n}\mathbf{1}(P_n)$. Observe that for all $E \in \mathcal{F}$, $\int_E \tilde{f}_n d\mu = \int_E f_0 d\mu + \frac{1}{n}\int_{E\cap P_n} d\mu \leqslant \nu(E)$ by the above and the fact that μ is a positive measure. We have by the above that $\tilde{f}_n$ is in $\mathcal{H}$ for all $n \in \mathbb{N}$ and $\alpha \leqslant \int_\Omega \tilde{f}_n d\mu \leqslant \alpha$ and so $\mu(P_n) = 0$ for all $n \in \mathbb{N}$. Thus, by $\sigma-$additivity, $\mu\left(\bigcup_n P_n\right) = 0$.

Let $N = \Omega \backslash \bigcup_n P_n$, then for all $E \in \mathcal{F}$ s.t. $E \subseteq N$, $\lambda_n(E) = \nu_1(E) - \frac{1}{n}\mu(E) \leqslant 0$ for all $n \in \mathbb{N}$ and so $\nu_1(E) \leqslant 0$, i.e. $\nu(E) \leqslant \nu_2(E)$. The reverse inequality is obtained by observing that f_0 is in $\mathcal{H}$ and so we see that $\nu(E) = \nu_2(E)$ for such E. Finally, since $\nu << \mu$, for all $E \in \mathcal{F}$, $\nu(E) = \nu(E\cap N) = \nu_2(E\cap N) = \nu_2(E) = \int_E f_0 d\mu$, which concludes the proof. □

Remark.

1. Without assuming $\nu << \mu$, the proof shows that there exists a decomposition (Lebesgue decomposition) $\nu = \nu_1 + \nu_2$, where $\nu_2(A) = \int_A f d\mu$, and $\nu_2 \perp \mu$ (orthogonal), i.e. there exists a measurable partition $\Omega = P \cup N$, s.t. $\mu(P) = 0$ ($\mu(A) = 0$, for all $A \subseteq P$), $|\nu_2(P)| = 0$ ($\nu_2(A) = 0$, for all $A \subseteq N$).
2. The unique f in Theorem 2.6 is the Radon-Nikodym derivative of ν wrt μ, denoted $\frac{d\nu}{d\mu}$. The result says that $\nu(A) = \int_\Omega \mathbf{1}_A d\nu = \int_A f d\mu = \int_\Omega \mathbf{1}_A \frac{d\nu}{d\mu} d\nu$. Hence a measurable function g is $\nu-$integrable iff $g\frac{d\nu}{d\mu}$ is $\mu-$integrable and then $\int_\Omega g d\nu = \int_\Omega g\frac{d\nu}{d\mu} d\nu$.

2.2 The dual space of L_p

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $1 \leqslant p < \infty$ and $1 < q \leqslant \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L_q = L_q(\mu)$, define $\phi_g : L_p \to$ scalars by $\phi_g(f) = \int_\Omega f g d\mu$, for $f \in L_p$. By Hölder, the product fg is in $L_1(\mu)$ and $|\phi_g(f)| \leqslant \|f\|_p \cdot \|g\|_q$. So ϕ_g is well-defined and clearly linear, also bounded with $\|\phi_g\| \leqslant \|g\|_q$ and so ϕ_g is an element of L_p^*. So we have the map

$$\begin{aligned} \phi : L_q &\to L_p^* \\ g &\mapsto \phi_g. \end{aligned}$$

This map is linear and bounded with $\|\phi\| \leqslant 1$.

Theorem 2.7. *Let $(\Omega, \mathcal{F}, \mu), p, q, \phi$ be as above.*

(i) If $1 < p < \infty$, then ϕ is an isometric isomorphism. So $L_p^* \cong L_q$.

(ii) If $p = 1$ and μ is a $\sigma-$finite, then $L_1^* \cong L_\infty$.

Proof. Proof of (i): ϕ is isometric. Fix $g \in L_q$. We know $\|\phi_g\| \leqslant \|g\|_q$. Let λ be a measurable function s.t. $|\lambda| = 1$ and $\lambda g = |g|$. Let $f = \lambda|g|^{q-1}$. Then, $\|f\|_p^p = \int_\Omega |f|^p d\mu = \int_\Omega |g|^{p(q-1)} d\mu$ $= \int_\Omega |g|^q d\mu = \|g\|_q^q$. Hence, $\|g\|_q^{\frac{q}{p}} \cdot \|\phi_g\| \geqslant |\phi_g(f)| = \int_\Omega |g|^q d\mu = \|g\|_q^q$, so $\|\phi_g\| \geqslant \|g\|_q^{q-\frac{q}{p}} = \|g\|_q$.

ϕ is onto: Fix $\psi \in L_p^*$. We seek $g \in L_q$ s.t. $\psi = \phi_g$ (Idea: $\psi(\mathbf{1}_A) = \int_A g d\mu$).

Case 1: μ is finite.

Then for $A \in \mathcal{F}$ and $\mathbf{1}_A \in L_p$ so can define $\nu(a) = \psi(\mathbf{1}_A)$. It is an easy check using the DCT that $\nu : \mathcal{F} \to \mathbb{C}$ is indeed a complex measure and $\nu << \mu$. If $A \in \mathcal{F}$, $\mu(A) = 0$, then $\mathbf{1}_A = 0$ almost everywhere (a.e.) in $L_p(\mu)$, so $\nu(A) = \psi(\mathbf{1}_A) = 0$. Then $\nu << \mu$. By Theorem 2.6, there exists $g \in L_1(\mu)$ s.t. $\nu(A) = \int_A g d\mu$ for all $A \in \mathcal{F}$. So $\psi(\mathbf{1}_A) = \int_\Omega \mathbf{1}_A g d\mu$, for $A \in \mathcal{F}$. Hence, $\psi(f) = \int_\Omega f g d\mu$ for all simple functions f. Now given $f \in L_\infty(\mu)$, there exists simple $f_n \to f \in L_\infty(\mu)$ (hence in $L_p(\mu)$ since μ is finite). So $\psi(f_n) \to \psi(f)$ and $f_n g \to fg \in L_1(\mu)$, using Hölder for $p = 1, \infty$. So $\psi(f) = \int_\Omega f g d\mu$ for all $f \in L_\infty(\mu)$. For $n \in \mathbb{N}$, let $A_n = \{|g| \leqslant n\}$ and $f_n = \lambda \cdot \mathbf{1}_{A_n}|g|^{q-1}$, where $|\lambda| = 1, \lambda g = |g|$.

Now, $\int_\Omega f_n g d\mu = \int_{A_n} |g|^q d\mu = \psi(f_n)$ (as f_n is in L_∞). $\psi(f_n) \leqslant \|\psi\| \cdot \|f_n\|_p = \|\psi\| \left(\int_{A_n} |g|^q\right)^{\frac{1}{p}}$.

By monotone convergence, we deduce that $\left(\int_{A_n}|g|^q\right)^{\frac{1}{q}} \leqslant \|\psi\|$ and hence that g is in L_q. Given $f \in L_p$, there exists $f_n \to f$ simple in L_p. So $\psi(f_n) \to \psi(f)$ and $f_n g \to fg \in L_1$ (Hölder for the pair (p,q)). Hence, $\psi(f) = \int_\Omega fg d\mu$, concluding the case where μ is finite.

Before we treat the more general case, observe that for $A \in \mathcal{F}$, let $\mathcal{F}_A = \{B \in \mathcal{F} : B \subseteq A\}$ and $\mu_A = \mu|_{\mathcal{F}_A}$, $(A, \mathcal{F}_A, \mu_A)$ is a measure space. Then $L_p(\mu_A) \subseteq L_p(\mu)$ (where we identify $f \in L_p(\mu_A)$ with $f \cdot \mathbf{1}_A \in L_p(\mu)$; this is an isometric embedding). Let $\psi_* = \psi\big|_{L^p(\mu_A)}$.

Lecture 7

Claim: If A, B are in $\mathcal{F}$ s.t. $A \cap B$ is empty, then $\left\|\psi_{A\,\cup\,B}\right\| = (\|\psi_A\|^q + \|\psi_A\|^q)^{\frac{1}{q}}$.
Observe that [a]

$$\begin{aligned}(\|\psi_A\|^q + \|\psi_A\|^q)^{\frac{1}{q}} &= \sup\{a\,\|\psi_A\| + b\,\|\psi_B\| : a, b, \geqslant 0, a^p + b^p \leqslant 1\}\\ &= \sup\{a|\psi_A(f)| + b|\psi_B(g)| : a, b, \geqslant 0, a^p + b^p \leqslant 1, f \in B_{L_p(\mu_A)}, g \in B_{L_p(\mu_B)}\}\\ &= \sup\{|a\psi_A(f) + b\psi_B(g)| : a, b, \geqslant 0, a^p + b^p \leqslant 1, f \in B_{L_p(\mu_A)}, g \in B_{L_p(\mu_B)}\}.\end{aligned}$$

Now, $a\psi_A(f) + b\psi_B(g) = \psi_{A\,\cup\,B}(af+bg)$ (embed $f, g \in L_p(\mu)$ be extending f, g to zero outside A, B respectively). Now, continuing the above we obtain

$$= \sup\{|\psi_{A\,\cup\,B}(h)| : h \in B_{L_p(\mu_{A\,\cup\,B})}\} = \left\|\psi_{A\,\cup\,B}\right\|$$

as required, concluding the proof of the finite case.

Case 2: μ is $\sigma-$finite.

There exists a measurable partition $\Omega = \bigcup_{n\in\mathbb{N}} A_n$, of Ω, s.t. $\mu(A_n) < \infty$ for all n. By Case 1, for all $n \in \mathbb{N}$, there exists $g_n \in L_q(\mu_A)$ s.t. $\psi_{A_n} = \psi_{g_n}$, i.e. $\psi(f) = \int_{A_n} f g_n d\mu$, for all $f \in L_p(\mu_{A_n})$. By Claim 2, $\sum_{k=1}^{n} \|g_k\|_q^q = \sum_{k=1}^{n} \|\psi_{A_n}\|^q = \left\|\psi_{\bigcup_{k=1}^n A_k}\right\|^q \leqslant \|\psi\|^q$. If we define g on Ω by setting $g = g_n$ on A_n, then g is in L_q. Thus, $\psi(f) = \psi_g(f)$ for all $f \in L_p(\mu_n)$, for all n. Hence, $\psi(f) = \phi_g(f)$ on $\operatorname{span}\{\bigcup_{n\in\mathbb{N}} L_p(\mu_n)\} = L_p(\mu)$.

Case 3: general μ.

First assume that for $f \in L_p(\mu)$, $\{f \neq 0\}$ is $\sigma-$ finite. Indeed, $\{f \neq 0\} = \bigcup_{n\in\mathbb{N}}\{|f| > \dfrac{1}{n}\}$ and $\mu(\{|f| > \frac{1}{n}\}) \leqslant n^p \cdot \|f\|_p^p < \infty$ by Markov's inequality.
Chose $(f_n) \in B_{L_p}$ s.t. $\psi(f_n) \to \|\psi\|$. Then $A = \bigcup_{n\in\mathbb{N}}\{f_n \neq 0\}$ is $\sigma-$finite and $\|\psi_A\| = \|\psi\|$. By the claim previously established, $\|\psi\| = \left(\|\psi_A\|^q + \left\|\psi_{\Omega\setminus A}\right\|^q\right)^{\frac{1}{q}}$. By case 2, there exists a $g \in L_q(\mu_A) \subseteq L_q(\mu)$ s.t. $\phi_A = \phi_g$. So for all $f \in L_p(\mu)$, $\psi(f) = \psi_A(f|_A) + \psi_{\Omega\setminus A}(f\big|_{\Omega\setminus A}) = \int_A f|_A g d\mu = \int_\Omega fg d\mu$ (extend g in the usual sense.).

Proof of (ii) (μ is $\sigma-$ finite).

ϕ is isometric: Let $g \in L_\infty$. We know already that $\|\phi_g\| \leqslant \|g\|_\infty$ (Hölder). Fix $s < \|g\|_\infty$. Then $\mu(\{|g| > s\}) > 0$. Since μ is $\sigma-$finite, there exists $A \subseteq \{|g| > s\}$ s.t. $0 < \mu(A) < \infty$. Choose a measurable function λ s.t. $|\lambda| = 1$ and $\lambda g = |g|$. Then λg is in $L_1(\mu)$, $\|\lambda g\|_1 = \mu(A)$. Now, $\mu(A) \cdot \|\phi_g\| \geqslant |\phi_g(\lambda \mathbf{1}_A)| = \int_A |g| \geqslant s\mu(A)$. We deduce $\|\phi_g\| > s$ and so $\|\phi_g\| \geqslant s$ and hence $\|\phi_g\| \geqslant \|g\|_\infty$.

ϕ is onto: Fix $\psi \in L_1^*$. Seek $g \in L_\infty$ s.t. $\psi = \phi_g$.

Case 1: μ is finite. Define $\nu(A) = \psi(\mathbf{1}_A)$ for all $A \in \mathcal{F}$ and proceed in the same way as for $p > 1$.

Case 2: μ is $\sigma-$finite. This time we prove

Claim: If A, B are in $\mathcal{F}$ s.t. $A \cap B$ is empty, then $\left\|\psi_{A \cup B}\right\| = \max\{\|\psi_A\|, \|\psi_B\|\}$.
Observe like before that [b]

$$\begin{aligned}
\max\{\|\psi_A\|, \|\psi_B\|\} &= \sup\{a\|\psi_A\| + b\|\psi_B\| : a, b, \geqslant 0, a^p + b^p \leqslant 1\} \\
&= \sup\{a|\psi_A(f)| + b|\psi_B(g)| : a, b, \geqslant 0, a + b \leqslant 1, f \in B_{L_1(\mu_A)}, g \in B_{L_1(\mu_B)}\} \\
&= \sup\{|a\psi_A(f) + b\psi_B(g)| : a, b, \geqslant 0, a + b \leqslant 1, f \in B_{L_1(\mu_A)}, g \in B_{L_1(\mu_B)}\} \\
&= \sup\{|\psi_{A \cup B}(h)| : h \in B_{L_1(\mu_{A\cup B})}\} \\
&= \|\psi_{A\cup B}\|
\end{aligned}$$

as required.

To conclude, proceed in an entirely analogous way using a measurable partition of $\Omega = \bigcup_{n\in\mathbb{N}} A_n$, s.t. μ_{A_n} is a finite measure for all $n \in \mathbb{N}$. By Case 1, for all $n \in \mathbb{N}$, there exists $g_n \in L_\infty(\mu_{A_n})$ s.t. $\psi_{A_n} = \psi_{g_n}$, i.e. $\psi(f) = \int_{A_n} f g_n d\mu$, for all $f \in L_1(\mu_{A_n})$. Now, by the previous claim,

$$\left\|\sum_{k=1}^n g_k \mathbf{1}_{A_k}\right\|_\infty = \left\|\psi_{\bigcup_{k=1}^n A_k}\right\| = \max_{1\leqslant k\leqslant n} \|\psi_{A_k}\| \leqslant \|\psi\|.$$

If we define g on Ω by setting $g = g_n$ on A_n, then g is in L_∞ with $\|g\|_\infty \leqslant \|\psi\|$. Thus, $\psi(f) = \psi_g(f)$ for all $f \in L_1(\mu_{A_n})$, for all n. Hence, $\psi(f) = \phi_g(f)$ on $\text{span}\{\bigcup_{n\in\mathbb{N}} L_1(\mu_n)\} = L_1(\mu)$.

□

[a] using the fact that $(\ell_q^2)^* \equiv \ell_p^2$.
[b] using the fact that $(\ell_1^2)^* \equiv \ell_\infty^2$.

Corollary 2.8. *For $1 < p < \infty$, for a measure space $(\Omega, \mathcal{F}, \mu)$ $L_p(\mu)$ is reflexive.*

Proof. Let ψ be in L_p^{**}. then $g \mapsto \psi(\phi_g) : L_q \to$ scalars is in L_q^* ($\frac{1}{p} + \frac{1}{q} = 1$). By Theorem 2.7($i$), there exists $f \in L_p$ s.t. $\langle \phi_g, \psi\rangle = \int_\Omega fg d\mu = \langle f, \phi_g\rangle = \langle \phi_g, \widehat{f}\rangle$ for all $g \in L_q$. Then $\psi = \widehat{f}$, since $L_p^* = \{\phi_g : g \in L_q\}$. □

2.3 $\mathcal{C}(K)$ spaces

Throughout, K is a compact, Hausdorff topological space. Define

$$\mathcal{C}(K) = \{f : K \to \mathbb{C} : f \text{ continuous}\},$$

a complex Banach space in the sup-norm: $\|f\|_\infty = \sup_K |f|$.

$$\mathcal{C}^{\mathbb{R}}(K) = \{f : K \to \mathbb{R} : f \text{ continuous},\}$$

is a real Banach space with norm $\|f\|_\infty = \sup_K |f|$.

$$\mathcal{C}^+(K) = \{f \in \mathcal{C}(K) : f \geqslant 0\}.$$

Moreover,

$$\mathcal{M}(K) = \mathcal{C}(K)^*,$$

is a complex Banach space in the operator norm.

$$\mathcal{M}^{\mathbb{R}}(K) = \{\phi \in \mathcal{M}(K) : \phi(f) \in \mathbb{R}, \forall f \in \mathcal{C}^R(K)\},$$

is a closed, real-linear subspace of $\mathcal{M}(K)$.

$$\mathcal{M}^+(K) = \{\phi : \mathcal{C}(K) \to \mathbb{C} : \phi \text{ is linear} \phi(f) \geqslant 0, \forall f \in \mathcal{C}^+(K)\}.$$

Elements of $\mathcal{M}^+(K)$ are called positive linear functionals.

Aim: identify $\mathcal{M}(K), \mathcal{M}^{\mathbb{R}}(K)$.

Lemma 2.9. *(i) For all $\phi \in \mathcal{M}(K)$, there exist unique $\phi_1, \phi_2 \in \mathcal{M}^R(K)$, $\phi = \phi_1 + i\phi_2$*

(ii) $\phi \mapsto \phi\big|_{\mathcal{C}^{\mathbb{R}}(K)} : \mathcal{M}^{\mathbb{R}}(K) \to (\mathcal{C}^{\mathbb{R}}(K))^*$ is an isometric isomorphism.

(iii) $\mathcal{M}^+(K) \subset \mathcal{M}(K)$ and $\mathcal{M}^+(K) = \{\phi \in \mathcal{M}(K) : \|\phi\| = \phi(\mathbf{1}_K)\}$.

(iv) For all $\phi \in \mathcal{M}^{\mathbb{R}}(K)$, there exist unique $\phi^+, \phi^- \in \mathcal{M}^+(K)$ s.t. $\phi = \phi^+ - \phi^-$ and $\|\phi\| = \|\phi^+\| + \|\phi^-\|$.

Proof. (i) Let ϕ be in $\mathcal{M}(K)$. Define $\overline{\phi} : \mathcal{C}(K) \to \mathbb{C}$, by $\overline{\phi}(f) = \phi(\overline{f})$. Then, $\overline{\phi}$ is in $\mathcal{M}(K)$ and ϕ is in $\mathcal{M}^{\mathbb{R}}(K) \iff \phi = \overline{\phi}^a$.

Uniqueness: assume $\phi = \phi_1 + i\phi_2$ where $\phi_1, \phi_2 \in \mathcal{M}^{\mathbb{R}}(K)$. Then $\overline{\phi} = \phi_1 - i\phi_2$ so $\phi_1 = \frac{\phi+\overline{\phi}}{2}, \phi_2 = \frac{\phi-\overline{\phi}}{2}$.

Existence: check that the above works.

(ii) Let ϕ be in $\mathcal{M}^{\mathbb{R}}(K)$. The fact that $\left\|\phi\big|_{\mathcal{C}^{\mathbb{R}}(K)}\right\| \leqslant \|\phi\|_{\mathcal{C}(K)}$ is clear. Let f be in $B_{\mathcal{C}(K)}$. Choose $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and $\lambda\phi(f) = |\phi(f)|$. So $|\phi(f)| = \phi(\lambda f) = \phi(\mathrm{Re}(\lambda f)) + i\phi(\mathrm{Im}(\lambda f)) = \phi(\mathrm{Re}(\lambda f)) \leqslant \left\|\phi\big|_{\mathcal{C}^{\mathbb{R}}(K)}\right\| \cdot \|\mathrm{Re}(\lambda f)\|_\infty \leqslant \left\|\phi\big|_{\mathcal{C}^{\mathbb{R}}(K)}\right\|$.
Hence, $\left\|\phi\big|_{\mathcal{C}^{\mathbb{R}}(K)}\right\| \geqslant \|\phi\|$. Finally, given $\psi \in (\mathcal{C}^{\mathbb{R}}(K))^*$, define $\phi(f) = \psi(\mathrm{Re}(f)) + i\psi(\mathrm{Im}(f))$, for $f \in \mathcal{C}(k)$. Then ϕ is in $\mathcal{M}(K)$ and $\phi\big|_{\mathcal{C}^{\mathbb{R}}(K)} = \psi$.

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(iii) $\mathcal{M}^+(K) \subset \mathcal{M}(K)$: let ϕ be in $\mathcal{M}^+(K)$.
For $f \in \mathcal{C}^{\mathbb{R}}(K)$, $\|f\|_\infty \leqslant 1$ we have $\mathbf{1}_K \pm f \geqslant 0$, so $\phi(\mathbf{1}_K \pm f) \geqslant 0$. So $\phi(f)$ is in $\mathbb{R}$ and $|\phi(f)| \leqslant \phi(\mathbf{1}_K)$. So $\phi\big|_{\mathbf{C}^{\mathbb{R}}(K)}$ is in $(\mathcal{C}^{\mathbb{R}}(K))^*$ and $\left\|\phi\big|_{\mathbf{C}^{\mathbb{R}}(K)}\right\| = \phi(\mathbf{1}_K)$. By *(ii)*, ϕ is in $\mathcal{M}(K)$, $\|\phi\| = \phi(\mathbf{1}_K)$.

$\mathcal{M}^+(K) = \{\phi \in \mathcal{M}(K) : \|\phi\| = \phi(\mathbf{1}_K)\}$ ("$\supseteq$"): let ϕ be in $\mathcal{M}(K)$ with $\|\phi\| = \phi(\mathbf{1}_K)$. Wlog, $\|\phi\| = \phi(\mathbf{1}_K) = 1$. Fix $f \in B_{\mathcal{C}^{\mathbb{R}}(K)}$, let $\phi(f) = \alpha + i\beta$, with $\alpha, \beta \in \mathbb{R}$.

Need: $\beta = 0$. For $t \in \mathbb{R}$, $|\phi(f + it\mathbf{1}_K)|^2 = \alpha^2 + (\beta + t)^2 = \alpha^2 + \beta^2 + 2\beta t \leqslant \|f + it\mathbf{1}_K\|_\infty^2 \leqslant 1 + t^2$, so $\beta = 0$. Given $f \in \mathcal{C}^+(K)$, with $0 \leqslant f \leqslant 1$ on K it follows that $|2f - \mathbf{1}_K| \leqslant 1$, so $\|2f - \mathbf{1}_K\|_\infty \leqslant 1$. So $|\phi(2f - \mathbf{1}_K)| \leqslant 1$, i.e. $-1 \leqslant 2\phi(f) \leqslant 1$, which implies $\phi(f) \geqslant 0$.

(iv) Let ϕ be in $\mathcal{M}^{\mathbb{R}}(K)$. Assume $\phi = \psi_1 - \psi_2$, where $\psi_1, \psi_2 \in \mathcal{M}^+(K)$. For $f, g \in \mathcal{C}^+(K)$ with $0 \leqslant g \leqslant f$, $\psi_1(f) \geqslant \psi_1(g) = \phi(g) + \psi_2(g) \geqslant \phi(g)$. So $\psi_1(f) \geqslant \sup\{\phi(g) : 0 \leqslant g \leqslant f\}$. Define for $f$in $\mathcal{C}(K)$

$$\phi^+(K) = \sup\{\phi(g) : 0 \leqslant g \leqslant f\}$$

Note that $\phi^+(f) \geqslant 0$, $\phi^+(f) \leqslant \|\phi\| \cdot \|f\|_\infty$, $\phi^+(f) \geqslant \phi(f)$. Furthermore, it is easy to check that $\phi^+(t_1 f_1 + t_2 f_2) = t_1\phi^+(f_1) + t_2\phi^+(f_2)$ for all $f_1, f_2 \in \mathcal{C}^+(K)$, $t_1, t_2 \in \mathbb{R}^+$. Next, for $f \in \mathcal{C}^{\mathbb{R}}(K)$, write $f = f_1 - f_2$, both in $\mathcal{C}^+(K)^b$ and define $\phi^+(f) = \phi^+(f_1) - \phi^+(f_2)$. This is well-defined and $\mathbb{R}-$linear (check). Finally, for f in $\mathcal{C}(K)$, let $\phi^+(f) = \phi^+(\mathrm{Re}\, f) + i\phi^+(\mathrm{Im}\, f)$. Then ϕ^+ is $\mathbb{C}-$linear and since $\phi^+(f) \geqslant 0$ for all $f \in \mathcal{C}^+(K)$, we have ϕ^+ is in $\mathcal{M}^+(K)$. Define $\phi^- = \phi^+ - \phi$. For $f \in \mathcal{C}^+(K)$, $\phi^+(f) \geqslant \phi(f)$ implies that ϕ^- is in $\mathcal{M}^+(K)$ and $\phi = \phi^+ - \phi^-$. $\|\phi\| \leqslant \|\phi^+\| + \|\phi^-\| = \phi^+(\mathbf{1}_K) + \phi^-(\mathbf{1}_K) = 2\phi^+(\mathbf{1}_K) - \phi(\mathbf{1}_K)$. Given $f \in \mathcal{C}^+(K)$ with $0 \leqslant f \leqslant 1$, $-1 \leqslant 2f - 1 \leqslant 1$, so $2\phi(f) - \phi(\mathbf{1}_K) = \phi(2f - \mathbf{1}_K) \leqslant \|\phi\|$. Taking the supremum over f, we deduce that $2\phi^+(\mathbf{1}_K) - \phi(\mathbf{1}_K) = \phi(2f - \mathbf{1}_K)$. So $\|\phi\| = \|\phi^+\| + \|\phi^-\|$.

Uniqueness: Assume $\phi = \psi_1 - \psi_2$, where ψ_1, ψ_2 are in $\mathcal{M}^+(K)$ and $\|\phi\| = \|\psi_1\| + \|\psi_2\|$. From initial observation, $\psi_1 \geqslant \phi^+$ on $\mathcal{C}^+(K)$ and so $\psi_2 = \psi_1 - \phi \geqslant \phi^+ - \phi = \phi^-$ on $\mathcal{C}^+(K)$. Hence, $\psi_1 - \psi^+ = \psi_2 - \phi^-$ is in $\mathcal{M}^+(K)$. By *(iii)*, $\|\psi_1 - \psi^+\| + \|\psi_2 - \psi^-\| = \psi_1(\mathbf{1}_K) - \phi^+(\mathbf{1}_K) + \psi_2(\mathbf{1}_K) - \phi^-(\mathbf{1}_K) = \|\psi_1\| + \|\psi_2\| - \|\phi^+\| - \|\phi^-\| = \|\phi\| - \|\phi\| = 0$. Thus, $\psi_1 = \phi^+, \psi_2 = \phi^-$.

□

[a]check!
[b]e.g. $f_1 = f \vee 0, f_2 = (-f) \vee 0$.

2.4 Topological Preliminaries

We begin with some definitions and key topological results that will be useful in obtaining the characterisation of the dual spaces $(\mathcal{C}(K))^*$.

1. K being compact, Hausdorff is normal: given disjoint closed sets E, F there exists disjoint open sets $\mathcal{U}, \mathcal{V} \in K$ s.t. $E \subset \mathcal{U}$, $F \subset \mathcal{V}$. Equivalently, given $E \subset \mathcal{U} \subseteq K$, E closed, $\mathcal{U}$ open, there exists $\mathcal{V}$ open s.t. $E \subset \mathcal{V} \subseteq \mathcal{U}$ (use normality in $E, K\backslash\mathcal{U}$).

2. Urysohn Lemma: given disjoint closed sets $E, F \in K$, there exists a continuous function $f : K \to [0,1]$ s.t. $f|_E = 0$ and $f|_F = 1$.

3. Notation: $f \prec \mathcal{U}$ means $\mathcal{U} \subseteq K$ open $f : K \to [0,1]$ is continuous and the support of f $\text{supp}(f) = \overline{\{x \in K : f(x) \neq 0\}} \subseteq \mathcal{U}$. $E \prec \mathcal{U}$ means E is a closed subset of K, $f : K \to [0,1]$ continuous and $f|_E = 1$.
Urysohn says: $E \subseteq \mathcal{U} \subseteq K$, E closed, $\mathcal{U}$ open, then there exists a continuous function f s.t. $E \prec f \prec \mathcal{U}$ ($E \subseteq \mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq \mathcal{U}$, $\mathcal{V}$ open and apply Urysohn to $E, F = K\backslash\mathcal{V}$).

Lemma 2.10. *Let $E, \mathcal{U}_1 \ldots \mathcal{U}_n$ be subsets of K $(n \in \mathbb{N})$, E closed, $\mathcal{U}_j$ open for $1 \leqslant j \leqslant n$ s.t. $E \subseteq \cup_{j=1}^n \mathcal{U}_j$. Then*

(i) there exist open sets $\mathcal{V}_j, 1 \leqslant j \leqslant n$, s.t. $\overline{\mathcal{V}}_j \subseteq \mathcal{U}_j$ for all j and $E \subseteq \cup_{j=1}^n \mathcal{V}_j$.

(ii) there exist $f_j \prec \mathcal{U}_j, 1 \leqslant j \leqslant n$, s.t. $0 \leqslant \sum_{j=1}^n f_j \leqslant 1$ on K and $\sum_{j=1}^n f_j = 1$ on E.

Proof. (i) We proceed by induction on n.

$n = 1$: is just a restatement of normality of K.

$n > 1$: $E\backslash\mathcal{U}\backslash \subseteq \bigcup_{j<n} \mathcal{U}_j$, so by induction there exist open sets $\mathcal{V}_j, j < n$, s.t. $\overline{\mathcal{V}}_j \subseteq \mathcal{U}_j$ and $E\backslash\mathcal{U}_n \subseteq \cup_{j<n}\mathcal{V}_j$. So $E\backslash \cup_{j<n} \mathcal{V}_j \subseteq \mathcal{U}_n$ and so by normality, there exists open $\mathcal{V}_n$ s.t. $E\backslash \cup_{j<n} \mathcal{V}_j \subseteq \mathcal{V}_n \subseteq \overline{\mathcal{V}}_n \subseteq \mathcal{U}_n$.

(ii) Let $\mathcal{V}_j$ be as in part (i). By Urysohn, there exists h_j s.t. $\overline{\mathcal{V}}_j \prec h_j \prec \mathcal{U}_j$ for $1 \leqslant j \leqslant n$, and there exists h_0 s.t. $K\backslash \bigcup_{j=1}^n \mathcal{V}_j \prec h_0 \prec K\backslash E$.

Let $h = h_0 + \sum_{j=1}^n h_j$. Then $h \geqslant 1$ on K. Let $f_j = \frac{h_j}{h}$ for all j. Then $0 \leqslant \sum_{j=1}^n f_j \leqslant 1$ on K and $\sum_{j=1}^n f_j = 1$ on E where $f_j \prec \mathcal{U}_j$ for all j.

□

2.5 Borel Measures

Let X be a Hausdorff space. Let $\mathcal{G}$ be the family of open sets in X. The Borel $\sigma-$algebra of X is $\mathcal{B} = \sigma(\mathcal{G})$, the $\sigma-$algebra generated by $\mathcal{G}$. members of $\mathcal{B}$ are called Borel sets. A Borel measure on X is a (positive) measure μ on $\mathcal{B}$. we say μ is regular if

(i) $\mu(E) < \infty$ for all $E \subseteq X$, E compact.

(ii) $\mu(A) = \inf\{\mu(\mathcal{U}) : A \subseteq \mathcal{U} \in \mathcal{G}\}$ for all $A \in \mathcal{B}$.

(iii) $\mu(\mathcal{U}) = \sup\{\mu(E) : E \subseteq \mathcal{U}, E \text{ compact}\}$.

A complex Borel measure ν is regular if $|\nu|$ is regular. If X is compact, Hausdorff, then a Borel measure μ on X is regular

$$\begin{aligned}&\iff \mu(X) < \infty \text{ and } \mu(A) = \inf\{\mu(\mathcal{U}) : A \subseteq \mathcal{U} \in \mathcal{G}\} \text{ for all } A \in \mathcal{B}.\\ &\iff \mu(X) < \infty \text{ and } \mu(A) = \sup\{\mu(E) : E \subseteq A, E \text{ closed}\} \text{ for all } A \in \mathcal{B}.\end{aligned}$$

2.6 Integration with respect to complex measures

Let Ω be a set, $\mathcal{F}$ a $\sigma-$algebra on Ω and ν a complex measure on $\mathcal{F}$. Then ν has Jordan decomposition $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$. Say a measurable function $f : \Omega \to \mathbb{C}$ is $\nu-$integrable if f is $|\nu|-$integrable (i.e. $\int_\Omega |f|d|\nu| < \infty$) iff f is ν_k-integrable for all k. So we define

$$\int_\Omega f d\nu = \int_\Omega f d\nu_1 - \int_\Omega f d\nu_2 + i\int_\Omega f d\nu_3 - i\int_\Omega f d\nu_4.$$

Lecture 9 Properties:

1. $\int_\Omega \mathbf{1}_A d\nu = \nu(A)$, for all $A \in \mathcal{F}$.
2. Linearity: if $f, g : \Omega \to \mathbb{C}$ are $\nu-$integrable, $a, b \in \mathbb{C}$, then $af + bg$ is $\nu-$integrable and $\int_\Omega (af + bg)d\nu = a\int_\Omega f d\nu + b\int_\Omega g d\nu$.
3. Dominated Convergence (DC): let $(f_n)_{n\in\mathbb{N}}$, f, g, be emasurable functions s.t. $f_n \to f$ a.e. (wrt $|\nu|$) and g is in $L_1(|\nu|)$ and for all n $|f_n| \leqslant g$ then f is $\nu-$integrable and $\int_\Omega f_n d\nu \to \int_\Omega f d\nu$ (True for ν_k for all k, so true for ν).
4. $\left|\int_\Omega f d\nu\right| \leqslant \int_\Omega |f|d|\nu|$ for all $f \in L_1(\nu)$ (True for simple functions by 1&2 and for general f, use DCT).

Let ν be a complex Borel measure on K (compact, hausdorff). Then for f continuous, then

$$\int_K |f|d|\nu| \leqslant \|f\| \cdot |\nu|(K).$$

So, f is $\nu-$integrable. Define $\phi : \mathcal{C}(K) \to \mathbb{C}$ by $\phi(f) = \int_K f d\nu$. Then ϕ is in $\mathcal{M}(K)$ and $\|\phi\| \leqslant |\nu|(K) = \|\nu\|_1$ (TV norm). If ν is a signed measure, then ϕ is a member of $\mathcal{M}^{\mathbb{R}}(K)$. If ν is a positive measure, then ϕ is in $\mathcal{M}^+(K)$.

Theorem 2.11 (Riesz Representation Theorem)**.** *For every $\phi \in \mathcal{M}^+(K)$, there exists a unique regular Borel measure μ on K that represents ϕ, i.e. $\phi(f) = \int_K f d\mu$ for all continuous f. Moreover,*

$$\|\phi\| = \mu(K) = \|\mu\|_1 \ \ \textit{TV norm of } \mu.$$

Proof. Uniqueness: Assume μ_1, μ_2 both represent ϕ. Let $E \subseteq \mathcal{U} \subseteq K$, where E is closed and $\mathcal{U}$ is open, then by Urysohn, there exists f continuous s.t. $E \prec f \prec \mathcal{U}$. Now, $\mu_1(E) \leqslant \int_K f d\mu_1 = \phi(f) = \int_K f d\mu_2 \leqslant \mu_2(\mathcal{U})$. Take infemum over $\mathcal{U}$ open and use regularity to deduce that $\mu_1(E) \leqslant \mu_2(E)$, and by symmetry $\mu_1(E) = \mu_2(E)$ agree on closed sets, and we conclude that $\mu_1 = \mu_2$ for all $A \in \mathcal{F}$ by regularity from below.

Existence: Define for $\mathcal{U} \in \mathcal{G}$ (i.e. $\mathcal{U}$ open), $\mu^*(\mathcal{U}) = \sup\{\phi(f) : f \prec \mathcal{U}\}$. Note that $\mu^*(\mathcal{U}) \geqslant 0$, and for $\mathcal{V} \supseteq \mathcal{U}$, $\mathcal{U}, \mathcal{V} \in \mathcal{G}$, then $\mu^*(\mathcal{V}) \geqslant \mu^*(\mathcal{U})$ and hence $\mu^*(\mathcal{U}) \leqslant \mu^*(\mathcal{K})$ but $\mu^*(K) = \phi(\mathbf{1}_K)$ ($f \prec K$ implies $f \leqslant \mathbf{1}_K$ and ϕ is in $\mathcal{M}^+(K)$). It follows that for $\mathcal{U} \in \mathcal{G}$, $\mu^*(\mathcal{U}) = \inf\{\mu^*(\mathcal{U}) : A \subseteq \mathcal{U} \in \mathcal{G}\}$. Extend the definition of μ^*: for $A \subseteq K$ let $\mu^*(A) = \inf\{\mu^*(\mathcal{U}) : A \subseteq \mathcal{U} \in \mathcal{G}\}$.

Claim:μ^* is an outer measure.

We easily have that $\mu^*(\varnothing) = 0$ and for all $A \subseteq B \subseteq K$ $\mu^*(A) \leqslant \mu^*(B)$. It remains to show that if for all n in $\mathbb{N}$

$$(A_n) \subseteq K, \text{ then } \mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leqslant \sum_n \mu^*(A_n).$$

To see this, first fix $\mathcal{U}_n \in \mathcal{G}$ for $n \in \mathbb{N}$ and let $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$. Fix $f \prec \mathcal{U}$ and let $E = \operatorname{supp} f$. Then $E \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, by compactness, $E \subseteq \bigcup_{k=1}^{n} \mathcal{U}_k$ for some $n \in \mathbb{N}$. By lemma 2.10, there exist $h_j \prec \mathbf{U}_j$, $1 \leqslant j \leqslant n$, $\sum_{j=1}^{n} h_j \leqslant 1$ on K and is equal to 1 on E. So $f = \sum_{j=1}^{n} fh_j$ and hence $\phi(f) = \sum_{j=1}^{n} \phi(fh_j) \leqslant \sum_{j=1}^{n} \mu^*(\mathcal{U}_j) \leqslant \sum_{j=1}^{\infty} \mu^*(\mathcal{U}_j)$ as $fh_j \prec \mu^*(\mathcal{U}_j)$ for all j.

Taking the supremum of f, we deduce $\mu^*(\mathcal{U}) \leqslant \sum_{j=1}^{\infty} \mu^*(\mathcal{U}_j)$. It follows easily that $\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leqslant \sum_n \mu^*(A_n)$ for arbitrary sets (just approximate using an $\frac{\epsilon}{2^n}$ argument).
We now let $\mathcal{M}$ be the set of μ^*-measurable subsets of K, then $\mathcal{M}$ is a $\sigma-$algebra and $\mu^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$.
Next we show that $\mathcal{B} \subseteq \mathcal{U}$. Enough to show that $\mathcal{G} \subseteq \mathcal{M}$. Let $\mathcal{U}$ be in $\mathcal{G}$. We need to show: $\mu^*(A) \geqslant \mu^*(A \cap \mathcal{U}) + \mu^*(A \backslash \mathcal{U})$ for all $A \subseteq K$. First let $A = \mathcal{V} \in \mathcal{G}$. Fix $f \prec \mathcal{V} \cap \mathcal{U}$, fix $g \prec \mathcal{V} \backslash \operatorname{supp} f$. Then $f + g \prec \mathcal{V}$, and thus $\mu^*(\mathcal{V}) \geqslant \phi(f+g) = \phi(f) + \phi(g)$. Taking the supremum over g, we get $\mu^*(\mathcal{V}) \geqslant \phi(f) + \mu^*(\mathcal{V} \backslash \operatorname{supp} f) \geqslant \phi(f) + \mu^*(\mathcal{V} \cap \mathcal{U})$. Now let $A \subset K$ be arbitrary. Fix $\mathcal{V} \in \mathcal{G}$ s.t. $A \subseteq \mathcal{V}$, then $\mu^*(\mathcal{V}) \geqslant \mu^*(\mathcal{V} \cap \mathcal{U}) + \mu^*(\mathcal{V} \cap \mathcal{U}) \geqslant \mu^*(A \cap \mathcal{U}) + \mu^*(A \backslash \mathcal{U})$. Taking the infinum over all such $\mathcal{V}$, we have that $\mu^*(A) \geqslant \mu^*(A \cap \mathcal{U}) + \mu^*(A \backslash \mathcal{U})$.

Now, $\mu := \mu^*|_{\mathcal{B}}$ is a Borel measure on K. We have that $\mu(K) = \phi(\mathbf{1}_K) = \|\phi\| < \infty$ and by definition, μ is regular. It remains to show that

$$\phi(f) = \int_K f d\mu$$

for all continuous f. It is enough to check that for all $f \in \mathcal{C}^{\mathbb{R}}(K)$ and then to show that $\phi(f) \leqslant \int_K f d\mu$ (by applying the it to $-f$).

Fix $a < b \in \mathbb{R}$ s.t. $f(K) \subseteq [a,b]$. Wlog, $a > 0$, since $\phi(\mathbf{1}_K) = \int_K \mathbf{1}_K d\mu$. Let $\epsilon > 0$; choose $0 \leqslant y_0 < a \leqslant y_1 < \cdots < y_n = b$ s.t. $y_j < y_{j+1} + \epsilon$ for all $1 \leqslant j \leqslant n$. Let $A_j = f^{-1}((y_{j-1}, y_j])$. Then, $K = \bigcup_{j=1}^{n} A_j$ and this is a measurable partition. Choose closed sets E_j and open sets $\mathcal{U}_j$ s.t. $E_j \subseteq A_j \subseteq \mathcal{U}_j$ and $\mu(\mathcal{U}_j \backslash E_j) < \frac{\epsilon}{n}$ (by regularity) and $f(\mathcal{U}_j) \subseteq (y_{j-1}, y_j + \epsilon)$. By lemma 2.10 there exist $h_j \prec \mathcal{U}_j$, $1 \leqslant j \leqslant n$, $\sum_{j=1}^{n} h_j \leqslant 1$ on K. Now [a]

$$\begin{aligned}
\phi(f) = \sum_{j=1}^{n} \phi(fh_j) \leqslant \sum_{j=1}^{n} (y_j + \epsilon)\phi(h_j) \\
&\leqslant \sum_{j=1}^{n} (y_j + \epsilon)\mu(\mathcal{U}_j) \leqslant \sum_{j=1}^{n} (y_{j-1} + 2\epsilon)\left(\mu(\mathcal{U}_j) + \frac{\epsilon}{n}\right) \\
&\leqslant \sum_{j=1}^{n} y_{j-1}\mu(\mathcal{U}_j) + \epsilon(b+\epsilon) + 2\epsilon\mu(K) + 2\epsilon^2 \\
&= \int_K \sum_{j=1}^{n} y_{j-1}\mathbf{1}_{E_j} d\mu + \mathcal{O}(\epsilon) \\
&\leqslant \int_K f d\mu + \mathcal{O}(\epsilon).
\end{aligned}$$

Hence, $\phi(f) \leqslant \int_K f d\mu$, since $\epsilon > 0$ was arbitrary.

□

[a]using that $f \leqslant y_j \leqslant \epsilon$ and $h_j \prec \mathcal{U}_j$ and $\phi \in \mathcal{M}^+(K)$.

Corollary 2.12. *For every $\phi \in \mathcal{M}(K)$, there exists a unique regular complex Borel measure ν on K that represents ϕ, namely, $\phi(f) = \int_K f d\nu$ for all continuous f. Moreover, $\|\phi\| = \|\nu\|_1$ and if ϕ is in $\mathcal{M}^{\mathbb{R}}(K)$, then ν is a signed measure.*

Proof. Existence: Apply lemma 2.9 and theorem 2.11 to obtain a regular complex Borel measure ν that represents ϕ.

Need: $\|\nu\|_1 = \|\phi\|$.

Lecture 10

This will give uniqueness, if ν_1, ν_2 represent ϕ, then $\nu_1 - \nu_2$ represents $\phi - \phi = 0$, then $\|\nu_1 - \nu_2\|_1 = 0$, hence $\nu_1 = \nu_2$. $\|\phi\| \leqslant \|\nu\|_1$, was already done before Theorem 2.11. Take a measurable partition $K = \bigcup_{j=1}^n A_j$. Fix $\epsilon > 0$ and closed sets E_j, open sets $\mathcal{U}_j$ s.t. $E_j \subseteq A_j \subseteq \mathcal{U}_j$, $|\nu|(\mathcal{U}_j \backslash E_j) < \frac{\epsilon}{n}$ ($|\nu|$ is regular). Can also assume that $\mathcal{U}_j \subseteq K \backslash \bigcup_{i \neq j} E_i$, for all $1 \leqslant j \leqslant n$. Fix $\lambda_j \in \mathbb{C}$ s.t. $|\lambda_j| = 1$, $\lambda_j \nu(E_j) = |\nu(E_j)|$, $1 \leqslant j \leqslant n$. By lemma 2.10, there exist $h_j \prec \mathcal{U}_j$, $1 \leqslant j \leqslant n$, $\sum_{j=1}^n h_j \leqslant 1$ on K. then for all j $E_j \prec h_j$. Hence,

$$\begin{aligned}\left|\int_K \left(\sum_{j=1}^n \lambda_j \mathbf{1}_{E_j} - \sum_{j=1}^n \lambda_j h_j\right) d\nu\right| &\leqslant \sum_{j=1}^n \int_K |\mathbf{1}_{E_j} - \sum_{j=1}^n h_j| d|\nu| \\ &\leqslant \sum_{j=1}^n |\nu|(\mathcal{U}_j \backslash E_j) < \epsilon.\end{aligned}$$

Now,

$$\begin{aligned}\sum_{j=1}^n |\nu(A_j)| &\leqslant \sum_{j=1}^n |\nu(E_j)| + \epsilon = \sum_{j=1}^n \lambda_j \nu(E_j) + \epsilon \\ &= \int_K \sum_{j=1}^n \lambda_j \mathbf{1}_{E_j} d\nu + \epsilon \leqslant \left|\int_K \left(\sum_{j=1}^n \lambda_j h_j\right) d\nu\right| + 2\epsilon \\ &= \left|\phi\left(\sum_{j=1}^n \lambda_j h_j\right)\right| + 2\epsilon \\ &= \|\phi\| \cdot \left\|\textstyle\sum_{j=1}^n \lambda_j h_j\right\|_\infty + 2\epsilon \leqslant \|\phi\| + 2\epsilon\end{aligned}$$

using the fact the the expression in the second to last line is a convex combination of function with sup norm equal to one. Hence, it follows that $\|\nu\|_1 \leqslant \|\phi\|$. ☐

Corollary 2.13. *The space of regular complex Borel measures is a complex Banach space in the $\|\nu\|_1$ (total variation norm) and is isometrically isomorphic $\mathcal{M}(K)$.*
The space of regular real Borel measures is a real Banach space in the $\|\nu\|_1$ (total variation norm) and is isometrically isomorphic $\mathcal{M}^{\mathbb{R}}(K)$.

3 Weak Topologies

Let X be a set and $\mathcal{F}$ be a family of function s.t. each $f \in \mathcal{F}$ is a function $f : X \to Y_f$, where Y_f is a topological space.

The weak topology $\sigma(X, \mathcal{F})$ on X generated by $\mathcal{F}$ is the smallest topology on X s.t. each $f \in \mathcal{F}$ is continuous (is easily see to exist).

Remark. 1. $\mathcal{S} = \{f^{-1}(\mathcal{U}) : f \in \mathcal{F}, \mathcal{U} \subseteq Y_f open\}$ is a sub-base of $\sigma(X, \mathcal{F})$. So $\mathcal{V} \subseteq X$ is open, i.e. it is in $\sigma(X,\mathcal{F})$ iff for all $x \in \mathcal{V}$, there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in \mathcal{F}$ and open sets $\mathcal{U}_j \subseteq Y_{f_j}$ (open nbhds of $f_j(x)$) for $1 \leqslant j \leqslant n$ s.t. x is in $\bigcap_{j=1}^{n} f^{-1}(\mathcal{U}_j) \subseteq \mathcal{V}$.

2. If $\mathcal{S}_f$ is a sub-base in Y_f, then $\{f^{-1(\mathcal{U})} : f \in \mathcal{F}, \mathcal{U} \in \mathcal{S}_f\}$, is a sub-base for $\sigma(X,\mathcal{F})$.
3. If Y_f is Hausdorff for all $f \in \mathcal{F}$ and $\mathcal{F}$ separates points in X (i.e., for all $x \neq y$, there exists $f \in \mathcal{F}$ s.t. $f(x) \neq f(y)$). Then $\sigma(X,\mathcal{F})$ is Hausdorff (easy to check).
4. $Y \subseteq X$, let $\mathcal{F}_Y = \{f|_Y : f \in \mathcal{F}\}$. Then $\sigma(Y,\mathcal{F}_Y) = \sigma(X,\mathcal{F})|_Y$ (check!).
5. Universal property: let Z be a topological space and $g : Z \to X$ be a function. Then g is continuous iff $f \circ g : Z \to Y_f$ is continuous for all $f \in \mathcal{F}$.

Examples 3.1. *1. Let X be a topological space, let $Y \subseteq X$ and $\iota : Y \to X$ be the inclusion map. Then, $\sigma(Y,\{\iota\})$ is the subspace topology of Y.*

2. let Γ be a set, X_γ a topological space for all $\gamma \in \Gamma$ and $X = \prod_{\gamma\in\Gamma} X_\gamma = \{X :$ X is a function on Γ s.t. $\forall\gamma \in \Gamma, x(\gamma) \in X_\gamma\}$. For $x \in X$, $\gamma \in \Gamma$ we often write x_γ for $x(\gamma)$. We think of x as the "$\Gamma-$tuple", $(x_\gamma)_{\gamma\in\Gamma}$. For each γ we have $\pi_\gamma : X \to X_\gamma, x \mapsto x_\gamma$ $((x_\delta)_{\delta\in\Gamma})$ the evaluation at γ, or projection onto X_γ. The weak topology $\sigma(X, \{\pi_\gamma : \gamma \in \Gamma\})$ is called the product topology on X. $\mathcal{V}$ is open iff for all $x = (x_\gamma)_{\gamma\in\Gamma} \in \mathcal{V}$, there exist $n \in \mathbb{N}$, $\gamma_1, \dots, \gamma_n \in \Gamma$ and open neighbourhoods $\mathcal{U}_j$ of $x_{\gamma_j} in X_{\gamma_j}$ s.t.

$$\{y = (y_\gamma)_{\gamma\in\Gamma} \in X : y_{\gamma_j} \in \mathcal{U}_j, 1 \leqslant j \leqslant n\} \subseteq \mathcal{V}$$

.

Proposition 3.2. *Let X be a set. For each $n \in \mathbb{N}$, let (Y_n, d_W) be a metric space and $f_n : X \to Y_n$ be a function s.t. $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ separates points of X. Then $\sigma(X,\mathcal{F})$ is metrisable.*

Proof. Define

$$d(x,y) = \sum_{n=1}^{\infty} \min(|f_n(x) - f_n(y)|, 1) \cdot 2^{-n}, \text{ for } x, y \text{ in } X.$$

This is a metric on X (easy to check) ($\mathcal{F}$ separating points implies that for $x \neq y$, $d(x,y) > 0$). Fiven $\epsilon \in (0,1)$ and $d(x,y) < \frac{\epsilon}{2^n}$, then $|f_n(x) - f_n(y)| < \epsilon$. So each f_n is continuous wrt the topology τ induced by d. So $\sigma = \sigma(X,\mathcal{F}) \subseteq \tau$. Fix $x \in X$, then $y \mapsto \min(|f_n(x)-f_n(y)|, 1)\cdot 2^{-n}$ is $\sigma-$continuous. By the Weierstrass M-test, $\sum_{n=1}^{\infty} \min(|f_n(x) - f_n(y)|, 1) \cdot 2^{-n}$ is univormly convergent, hence $\sigma-$continuous. So, $\{y \in X : d(y,x) < \epsilon\}$ is $\sigma-$open. Hence, $\tau \subseteq \sigma$ and $\tau = \sigma$. □

Theorem 3.3 (Tychonov)**.** *The product of compact topological spaces is compact in the product topology.*

Proof. We have $X = \prod_{\gamma\in\Gamma} X_\gamma$ as in examples 3.1. Assume each X_γ is compact. Let $\mathcal{F}$ be a family of closed subsets of X with the finite intersection property (FIP). We need to show that $\bigcap_{F\in\mathcal{F}} F \neq \varnothing$ (equivalent to compactness).
By Zorn, there exists a maximal family $\mathcal{A}$ of subsets of X s.t. $\mathcal{F} \subseteq \mathcal{A}$ and $\mathcal{A}$ has the FIP ($\mathcal{M} = \{\mathcal{A} \subseteq \mathcal{P}(X) : \mathcal{A} \supseteq \mathcal{F}$ & $\mathcal{A}$ has the FIP$\}$, and every chain has a maximal element. Check!).
We will show that $\bigcap_{A\in\mathcal{A}} \overline{A} \neq \varnothing$.

<u>Note</u>:

1. $A_1, \ldots, A_n \in \mathcal{A}$ implies that $A = \bigcap_{i=1}^{n} A_i$ is in $\mathcal{A}$.
   Indeed, for all $B_1, \ldots, B_m \in \mathcal{A}$, s.t. $A \cap B_1 \cap \cdots \cap B_m \neq \varnothing$ so $\mathcal{A} \cup \{A\}$ has the FIP. Hence, A is in $\mathcal{A}$.

2. $B \subseteq X$, $B \cap A \neq \varnothing$ for all $A \in \mathcal{A}$ implies B is in $\mathcal{A}$. Indeed, for $A_1, \ldots, A_n \in \mathcal{A}$ s.t. $\bigcup_{i=1}^{n} A_i \neq \varnothing$ and $B \cap \bigcup_{i=1}^{n} A_i \neq \varnothing$, then $A \cup \{B\}$ has the FIP and using maximality, we conclude that B is in $\mathcal{A}$.

Let $\gamma \in \Gamma$. Then $\{\pi_\gamma(A) : A \in \mathcal{A}\}$ has the FIP. Since X_γ is compact, $\bigcap_{A \in \mathcal{A}} \overline{\pi_\gamma(A)} \neq \varnothing$. Fix $x_\gamma \in \bigcap_{A \in \mathcal{A}} \overline{\pi_\gamma(A)} \neq$. Let $x = (x_\gamma)_{\gamma \in \Gamma}$ and $\mathcal{U}$ be an open neighbourhood of $\underline{x}$. We show that $\mathbf{U} \cap A \neq \varnothing$ for all $A \in \mathcal{A}$. Then $x \in A$, for all $A \in \mathcal{A}$. Wlog, $\mathcal{U} = \bigcup_{j=1}^{n} \pi_{\gamma_j}^{-1}(\mathcal{U}_j)$ for $n \in \mathbb{N}, \gamma_1, \ldots, \gamma_n \in \mathcal{F}, \mathcal{U}_j$ is an open neighbourhood of $x_{\gamma_j} \in X_{\gamma_j}$. So $\mathcal{U}_j \cap \bigcup_{j=1}^{n} \pi_{\gamma_j}^{-1}(A_j) \neq \varnothing$ for all $A \in \mathcal{A}$, so $\pi_{\gamma_j}^{-1}(\mathcal{U}_j) \in \mathcal{A}$ by note 2 above. By 1 above, $\mathcal{U} \in \mathcal{A}$ and hence, $\mathcal{U} \cap A \neq \varnothing$ for all $A \in \mathcal{A}$. We have thus demonstrated that for all $A \in \mathcal{A}$, $x \in \overline{A}$, which concludes the proof. $\square$

Lecture 11

3.1 Weak topologies on vector spaces

Let E be a real or complex vector space. Let F be a subspace of the space of all linear functionals on E that separates points, i.e. for all $x \in E, x \neq 0$, then there exists $f \in F, f(x) \neq 0$. Consider the weak topology $\sigma(E, F)$. So $\mathcal{U} \subseteq E$ is open iff for all $x \in \mathcal{U}$, there exists $n \in \mathbb{N}, f_1, \ldots, f_n \in F, \epsilon > 0$ s.t. $\{y \in E : |f_j(y - x)| < \epsilon, 1 \leqslant j \leqslant n\} \subseteq \mathcal{U}$. For $f \in F, x \in E, p_f(x) = |f(x)|$. Let $\mathcal{P} = \{p_f : f \in F\}$. Then $(E, \mathcal{P})$ is a locally convex space (LCS) whose topology is $\sigma(E, F)$. So $\sigma(E, F)$ is Hausdorff and vector addition and scalar multiplication are continuous.

Lemma 3.4. *Let E be as above, ler $f, g_1, \ldots, g_n$ be linear functionals on E s.t. $\bigcup_{j=1}^{n} \ker g_j \subseteq \ker f$. Then $f \in \operatorname{span}\{g_1, \ldots, g_n\}$.*

Proof. Let $\mathbb{K}$ be the scalar field. Define $T : E \to \mathbb{K}^n$ by $Tx = (g_j(x))_{j=1}^{n}$. Then $\ker(T) = \bigcup_{j=1}^{n} \ker g_j \subseteq \ker f$ and hence we have a factorisation

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "T"] & \mathbb{K} \\ \mathbb{K}^n \arrow[ur, "h"'] & \end{tikzcd}$$

with h linear, $f = h \circ T$. Then there exists $(a_j(x))_{j=1}^{n} \in \mathbb{K}^n$ s.t. $h(y) = \sum_{j=1}^{n} a_j y_j$ for all $y \in \mathbb{K}^n$. So for all $x \in E, f(x) = h(Tx) = \sum_{j=1}^{n} a_j g_j(x)$. So $f = \sum_{j=1}^{n} a_j g_j$ as required. $\square$

Proposition 3.5. *Let E, F be as above, let f be a linear function on E. Then f is $\sigma(E,F)-$continuous iff $f \in F$. So, $(E, \sigma(E,F))^* = F$.*

Proof. $\underline{\Longleftarrow}$: holds by definition.

$\underline{\Longrightarrow}$: there exists an open neighbourhood $\mathcal{U}$ of 0 in E s.t. for all $x \in \mathcal{U}$, $|f(x)| < 1$. Wlog, (shrink $\mathcal{U}$ if necessary) $\mathcal{U} = \{x \in E : |g_j(x)| < \epsilon, 1 \leqslant j \leqslant n\}$ for some $n \in \mathbb{N}, g_1, \dots, g_n \in F, \epsilon > 0$. If $x \in \bigcup_{j=1}^{n} \ker g_j$, then $ambx \in \mathcal{U}$ for all scalars λ and hence $|f(x)| = |\lambda| \cdot |f(x)| < 1$ for all λ. So $f(x) = 0$. By lemma 3.4, $f \in \text{span}\{g_1, \dots, g_n\}$. □

Examples 3.6.
1. *Let X be a normed space. The weak topology on X is the topology $\sigma(X, X^*)$ on X. (X^* annihilates points of X by Hahn-Banach). We sometimes write, (X, w) for $(X, \sigma(X, X^*))$. Open sets in $\sigma(X, X^*)$ are called weak open, or $w-$open. $\mathcal{U} \subseteq X$ is $w-$open $\iff$ for all $x \in \mathcal{U}$, there exists $n \in \mathbb{N}, f_1, \dots, f_n \in X^*, \epsilon > 0$ s.t. $\{y \in X : |f_j(y-x)| < \epsilon, 1 \leqslant j \leqslant n\}$.*
2. *Let X be a normed space. The weak star topology or w^*-topology on X^* is the topology $\sigma(X^*, X)$ on X^*. Here, we are identifying X with its image in X^{**} under the canonical embedding. Open sets in $\sigma(X^*, X)$ are called w^*-open and $\mathcal{U} \subseteq X^*$ is weak-* open iff for all $f \in \mathcal{U}$, there exist $n \in \mathbb{N}, x_1, \dots, x_n \in X, \epsilon > 0$ s.t. $\{y \in X^* : |g(x_j) - f(x_j)| < \epsilon, 1 \leqslant j \leqslant n\} \subseteq \mathcal{U}$.*

Properties:

1. (W, w) and (X^*, w^*) (this is $(X^*, \sigma(X^*, X))$) are LCS and hence Hausdorff with continuous vector space operations.
2. $\sigma(X, X^*) \subseteq \|\cdot\|-$topology with equality iff $\dim X < \infty$.
3. $\sigma(X, X^*) \subseteq \sigma(X^*, X) \subseteq \|\cdot\|$, where equality in the first inclusion is achieved iff X is reflexive, and for the latter iff $\dim X^* = \dim X < \infty$.
4. Let Y be a subspace of X. Then, $\sigma(X, X^*)|_Y = \sigma(Y, \{f \in X^*\}) = \sigma(Y, Y^*)$ by Hahn-Banach. Similarly, $\sigma(X^{**}, X^*)|_X = \sigma(X, X^*)$. So in other words, the canonical embedding $X \to X^{**}$ is also a weak-to-weak-* homeomorphism between X and $\widehat{X}$.

Proposition 3.7. *Let X be a normed space.*

(i) A linear functional f on X is continuous in the weak topology iff $f \in X^*$. So $(X, w)^* = X^*$.

(ii) A linear functional f on X^* is w^*- continuous iff $f \in X$, i.e. $f = \widehat{x}$ for some $x \in X$. So $(X^*, w^*)^* = X$. It follows that $\sigma(X^*, X) = \sigma(X^*, X^{**})$ iff X is reflexive.

Definition 3.8 (Weak Boundedness)**.** *Let X be a normed space, then a subset $A \subseteq X$ is weakly bounded if $\{f(x) : x \in A\}$ is bounded for all $f \in X^*$ (iff for all w-neighbourhood of 0 in X, there exists $\lambda > 0$ s.t. $A \subseteq \lambda\mathcal{U}$).*
A subset $B \subseteq X^*$ is weak- bounded if $\{f(x) : x \in B\}$ is bounded for all $x \in X$ (iff iff for all w^*-neighbourhood of 0 in X^*, there exists $\lambda > 0$ s.t. $B \subseteq \lambda\mathcal{U}$).*

3.2 Principle of Uniform Boundedness (PUB)

Let X be a Banach space, Y be a normed space and $\mathcal{T} \subseteq \mathcal{B}(, \mathcal{Y})$. If $\mathcal{T}$ is pointwise bounded $\left(\sup_{T \in \mathcal{T}} \|Tx\| \text{ for all } x \in X\right)$, then T is uniformly bounded $\left(\sup_{T \in \mathcal{T}} \|T\| < \infty\right)$.

Proposition 3.9. *(i) $A \subseteq X$ is weakly bounded implies that A is $\|\cdot\|$ −bounded.*

(ii) $B \subseteq X^*$ is weak- bounded and X is complete implies that B is $\|\cdot\|$ −bounded.*

Proof. (ii) $B \subseteq X^* = \mathcal{B}(X, \text{scalars})$, B weak-* bounded says B is pointwise bounded. So done by PUB.
(i) $\widehat{A} = \{\widehat{x} : x \in A\} \subseteq X^{**} = \mathcal{B}(X^*, \text{scalars})$. A weakly bounded iff $\widehat{A}$ is pointwise bounded and so can conclude again by PUB. □

Notation: We write $x_n \xrightarrow{w} x$ if $(x_n)_{n\in\mathbb{N}}$ converges to x in the weak topology (in some normed space). Note that $x_n \xrightarrow{w} x$ in X iff $\langle x_n, f\rangle \to \langle x, f\rangle$ for all $f \in X^*$. We write $f_n \xrightarrow{w^*} f$ in X^* if $(f_n)_{n\in\mathbb{N}}$ converges to f in the weak-* topology (in some dual space) iff $\langle x, f_n\rangle \to \langle x, f\rangle$ for all $x \in X$.

Consequences of PUB: Let X be a Banach space, Y a normed space, (T_n) a sequence in $\mathcal{B}(X, Y)$. If $T : X \to Y$ is a function s.t. $T_n \to T$ pointwise on X (i.e. $T_n x \to Tx$ for all $x \in X$), then $T \in \mathcal{B}(X, X)$, $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$ and $\|T\| \leqslant \liminf_n \|T_n\|$.

Proposition 3.10. *Let X be a normed space.*

(i) If $x_n \xrightarrow{w} x$ in X, then $\sup_n \|x_n\| < \infty$ and $\|x\| \leqslant \liminf_n \|x_n\|$.

(ii) If $f_n \xrightarrow{w^*} f$ in X^* and X is complete, then $\sup_n \|f_n\| < \infty$ and $\|f\| \leqslant \liminf_n \|f_n\|$.

Proof. (ii) We have that $f_n \to f$ pointwise in $X^* = \mathcal{B}(X, \text{scalars})$. Result follows by PUB.
(i) Since $x_n \xrightarrow{w} x$, $\widehat{x}_n \to \widehat{x}$ pointwise in $X^{**} = \mathcal{B}(X^*, \text{scalars})$ and we conclude by PUB again. □

Lecture 12 For the above, the converse is not true. We can find a sequence that converges weakly but not in the norm topology. For instance,

Examples 3.11. *Example: In $\ell_p, 1 < p < \infty$, $e_n = (0, \ldots, 0, \overset{\overset{n^{th}\text{-entry}}{\downarrow}}{1}, 0, \ldots, 0)$, $e_n \xrightarrow{w} 0$, but clearly $e_n \overset{\|\cdot\|}{\nrightarrow} 0$.*

3.3 Hahn-Banach Separation Theorems

Let $(X, \mathcal{P})$ be a LCS. Let $\mathcal{C}$ be a convex subspace of X, s.t. $0 \in \operatorname{int}\mathcal{C}$. Then define $\mu_{\mathcal{C}} : X \to \mathbb{R}, \mu_{\mathcal{C}} = \inf\{t > 0 : x \in t\mathcal{C}$.

Well-defined: $\frac{1}{n}x \to 0$ as $n \to \infty$, so there exists $n \in \mathbb{N}$ s.t. $\frac{1}{n}x \in \mathcal{C}$. $\mu_{\mathcal{C}}$ is the Minkowski functional (gauge functional) of $\mathcal{C}$.

Examples 3.12. *Example: If X is a normed space and $\mathcal{C} = B_X$, then $\mu_{\mathcal{C}} = \|\cdot\|$.*

Lemma 3.13. *$\mu_{\mathcal{C}}$ is positive homogeneous and sub-additive. Moreover, $\{x : \mu_{\mathcal{C}} < 1\} \subset \mathcal{C} \subset \{x : \mu_{\mathcal{C}} \leqslant 1\}$. The first inclusion is an equality if $\mathcal{C}$ open.*

Proof. Positive homogeneous: for $x \in X, s, t, > 0$ we have $sx \in st\mathcal{C} \iff x \in t\mathcal{C}$. Hence, $\mu_{\mathcal{C}}(sx) = s\mu_{\mathcal{C}}(x)$. A;so holds for $s = 0$, since $\mu_{\mathcal{C}}(0) = 0$.

Subadditivity: First an observation: $\mu_{\mathcal{C}} < t$ implies $x \in t\mathcal{C}$. Indeed, there exists $t' < t$ s.t. $x \in t'\mathcal{C}$. Then, $\frac{x}{t} = (1 - \frac{t'}{t}) \cdot 0 + \frac{t'}{t} \cdot \frac{x}{t'} \in \mathcal{C}$ by the convexity of $\mathcal{C}$.

Now, let $x, y \in X$. Fix $s > \mu_{\mathcal{C}}(x)$, $t > \mu_{\mathcal{C}}(y)$. Then $x \in s\mathcal{C}$, $y \in t\mathcal{C}$. So, $x + y = \left(\frac{s}{s+t} \cdot \frac{x}{s} + \frac{t}{s+t} \cdot \frac{x}{t}\right)(s+t) \in (s+t)\mathcal{C}$ by convexity. So $\mu_{\mathcal{C}}(x+y) \leqslant s+t$, and hence $\mu_{\mathcal{C}}(x+y) \leqslant \mu_{\mathcal{C}}(x) + \mu_{\mathcal{C}}(y)$.

Next, if $\mu_{\mathcal{C}}(x) < 1$, then $x \in \mathcal{C}$ by above. If $\mathcal{C}$ is open and $x \in \mathcal{C}$, then there exists $n \in \mathbb{N}$ s.t. $(1+\frac{1}{n})x \in \mathcal{C}$, since $(1+\frac{1}{n})x \xrightarrow{n\to\infty} x$ and $\mathcal{C}$ open. Hence, $\mu_{\mathcal{C}}(x) \leqslant \frac{1}{1+\frac{1}{n}} < 1$.

Finally, $x \in \mathcal{C}$ implies that $\mu_{\mathcal{C}}(x) \leqslant 1$. Then, by homogeneity, $\mu_{\mathcal{C}}((1-\frac{1}{n})x) < 1$ for all n, so $(1-\frac{1}{n})x \in \mathcal{C}$ for all n, since $(1-\frac{1}{n})x \to x$, the in case $\mathcal{C}$ is closed $x \in \mathcal{C}$.
Positive homogeneous: for $x \in \mathbb{R}^n, s, t, > 0$ we have $sx \in st\mathcal{D} \iff x \in t\mathcal{D}$. Hence, $\mu_{\mathcal{D}}(sx) = s\mu_{\mathcal{D}}(x)$. A;so holds for $s = 0$, since $\mu_{\mathcal{D}}(0) = 0$.

Subadditivity: First an observation: $\mu_{\mathcal{D}} < t$ implies $x \in t\mathcal{D}$. Indeed, there exists $t' < t$ s.t. $x \in t'\mathcal{D}$. Then, $\frac{x}{t} = (1-\frac{t'}{t}) \cdot 0 + \frac{t'}{t} \cdot \frac{x}{t'} \in \mathcal{D}$ by the convexity of $\mathcal{D}$.

Now, let $x, y \in \mathbb{R}^n$. Fix $s > \mu_{\mathcal{D}}(x)$, $t > \mu_{\mathcal{D}}(y)$. Then $x \in s\mathcal{D}$, $y \in t\mathcal{D}$. So, $x + y = \left(\frac{s}{s+t} \cdot \frac{x}{s} + \frac{t}{s+t} \cdot \frac{x}{t}\right)(s+t) \in (s+t)\mathcal{D}$ by convexity. So $\mu_{\mathcal{D}}(x+y) \leqslant s+t$, and hence $\mu_{\mathcal{D}}(x+y) \leqslant \mu_{\mathcal{D}}(x) + \mu_{\mathcal{D}}(y)$.

Next, if $\mu_{\mathcal{D}}(x) < 1$, then $x \in \mathcal{D}$ by above. If $\mathcal{D}$ is open and $x \in \mathcal{D}$, then there exists $n \in \mathbb{N}$ s.t. $(1+\frac{1}{n})x \in \mathcal{D}$, since $(1+\frac{1}{n})x \xrightarrow{n\to\infty} x$ and $\mathcal{D}$ open. Hence, $\mu_{\mathcal{D}}(x) \leqslant \frac{1}{1+\frac{1}{n}} < 1$.

Finally, $x \in \mathcal{D}$ implies that $\mu_{\mathcal{D}}(x) \leqslant 1$. Then, by homogeneity, $\mu_{\mathcal{D}}((1-\frac{1}{n})x) < 1$ for all n, so $(1-\frac{1}{n})x \in \mathcal{D}$ for all n, since $(1-\frac{1}{n})x \to x$, the in case $\mathcal{D}$ is closed $x \in \mathcal{D}$. □

Remark. If $\mathcal{C}$ is symmetric (in real case) or balanced (in complex case)m then $\mu_{\mathcal{C}}$ is a semi-norm. If, in addition $\mathcal{C}$ is bounded, then $\mu_{\mathcal{C}}$ is a norm.

Theorem 3.14. *Hahn-Banach Separation Theorem Let $(\mathbb{R}^n, \mathbb{P})$ be a LCS and $\mathcal{C}$ be an open convex subset of X with $0 \in \operatorname{int}\mathcal{C}$. let $x_0 \in X\backslash\mathcal{C}$. Then there exists $f \in X^*$ s.t. $f(x_0) > f(x)$ for all $x \in \mathcal{C}$.(In complex case: $\operatorname{Re}(f(x_0))) > \operatorname{Re}(f(x))$ for all $x \in \mathcal{C}$).*

Remark. From now on we work with real scalars and the complex case will follow, since

$$f \mapsto \operatorname{Re} f : X^* \to X^*_{\mathbb{R}}$$

is a real linear injection.

Proof. Consider $\mu_{\mathcal{C}}$. By lemma 3.13, $\mathcal{C} = \{s : \mu_{\mathcal{C}}(x) < 1\}$ and so $\mu_{\mathcal{C}}(x_0) \geqslant 1$. Let $Y = \operatorname{span}\{x_0\}$ and $g : Y \to \mathbb{R}$, $g(\lambda x_0) = 1 \leqslant \mu_{\mathcal{C}}(x_0)$. Hence, $g \leqslant \mu_{\mathcal{C}}$ on Y.
By Theorem 1.3, there exists linear $f : X \to \mathbb{R}$ s.t. $f|_Y = g$ and $f \leqslant \mu_{\mathcal{C}}$ on X. For all $x \in \mathcal{C}$, $f(x) \leqslant \mu_{\mathcal{C}}(x) < 1 = f(x_0)$. We also gave $f(x) < 1$ on $\mathcal{C}$ and so $|f(x)| < 1$ on $\mathcal{C} \cap (-\mathcal{C})$. Since $\mathcal{C} \cap (-\mathcal{C})$ is an open neighbourhood of 0, we have that $f \in X^*$. □

Theorem 3.15. *Let $(X, \mathcal{P})$ be a LCS. Let $A, B \neq \varnothing$, disjoint convex subsets of X.*

(i) If A is open, there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ s.t. $f(x) < \alpha \leqslant f(y)$ for all $x \in A, y \in B$.

(ii) If A is compact, and B is closed, then there exists $f \in X^*$ s.t. $\sup_A f < \inf_B f$.

Proof. (i) Fix $\alpha \in A, b \in B$. Let $\mathcal{C} = A - B + b - \alpha$ and $x_0 = b - \alpha$. Then $\mathcal{C}$ is open, convex, $0 \in \mathcal{C}$ and $x_0 \notin \mathcal{C}$ $(A \cap B = \varnothing)$. By Theorem 3.14, there exists $f \in X^*$ s.t. $f(z) < f(x_0)$ for all $z \in \mathcal{C}$. So for all $x \in A, y \in B$ $f(x - y + x_0) < f(x_0)$, i.e $f(x) < f(y)$. In particular, $f \neq 0$. Let $\alpha = \inf f$. Then $\alpha \leqslant f(y)$ for all $y \in B$. Since $f \neq 0$, there exists $u \in X$ s.t. $f(u) > 0$.Now, given $x \in A$, $x + \frac{1}{n}u \to x$ and since A is open, there exists $n \in \mathbb{N}$ s.t. $x + \frac{1}{n}u \in A$. Then $f(x) < f(x + \frac{1}{n}u) \leqslant \alpha$.

(ii) Claim: there exists open, convex neighbourhood of 0 in X,$\mathcal{U}$ s.t. $(A+\mathcal{U}) \cap B = \varnothing$

Indeed, for $x \in A$, there exists open neighbourhood $\mathcal{U}_x$ of 0 s.t. $(x+\mathcal{U}_x) \cap B = \varnothing$ (B is closed). Since $0+0=0$ and "+" is continuous, there exists open neighbourhood $\mathcal{V}_x$ of 0 s.t. $\mathcal{V}_x + \mathcal{V}_x \subseteq \mathcal{U}_x$. Wlog, $\underline{\mathcal{V}_x}$ is convex and symmetric. By compactness, there exist $x_1, \ldots, x_n \in A$ s.t. $A \subseteq \bigcup_{i=1}^{n}(x_i + \mathcal{V}_{x_i})$. Let $\mathcal{U} = \bigcap_{i=1}^{n} \mathcal{V}_{x_i}$. Given $x \in A$, there exists i s.t. $x \in x_i + \mathcal{V}_{x_i}$. So, $x + \mathcal{U} \subseteq x \in x_i + \mathcal{V}_{x_i} + \mathcal{U} \subseteq x \in x_i + \mathcal{V}_{x_i} + \mathcal{V}_{x_i} \subseteq x_i + \mathcal{U}_{x_i}$ is disjoint from B. So, $A+\mathcal{U}$ is disjoint from B.

Now, apply part (i) with $A+\mathcal{U}, B$ to show that there exists $f \in X^*$ s.t. $f(x+u) < f(y)$ for all $x \in A, y \in B, u \in \mathcal{U}$. In particular, $f \neq 0$ so there exists $z \in X$ s.t. $f(z) > 0$. Also, $\frac{1}{n}z \xrightarrow{n\to\infty} 0$, so there exists $n \in \mathbb{N}$ s.t. $\frac{1}{n}z \in \mathcal{U}$. So $f(x) + \frac{1}{n}f(z) < f(y)$ for all $x \in A, y \in B$. It follows that $\sup_A f < \inf_B f$.

□

Theorem 3.16 (Mazur)**.** *Let $\mathcal{C}$ be a convex subset of a normed space X. Then $\overline{\mathcal{C}}^{\|\cdot\|} = \overline{\mathcal{C}}^{w}$. In particular, $\mathcal{C}$ is $\|\cdot\|$-closed iff $\mathcal{C}$ is weakly closed.*

Proof. Wlog, $\mathcal{C} \neq \varnothing$.

"$\overline{\mathcal{C}}^{\|\cdot\|} \subseteq \overline{\mathcal{C}}^{w}$": is true since the weak topology is weaker than the $\|\cdot\|$-topology.

"$\overline{\mathcal{C}}^{\|\cdot\|} \supseteq \overline{\mathcal{C}}^{w}$": If $x \notin \overline{\mathcal{C}}^{\|\cdot\|}$, then apply Theorem 3.14 (ii) to $A = \{x\}$, $B = \overline{\mathcal{C}}^{\|\cdot\|}$ to obtain $f \in X^*$ s.t. $f(x) < \inf_B f := \alpha$. Then, $\{y : f(y) < \alpha\}$ is a weakly open neighbourhood of X, disjoint from B (and hence from $\mathcal{C}$). So $x \notin \mathcal{C}$. □

Corollary 3.17 (Mazur)**.** *If $x_n \xrightarrow{w} 0$ in a normed space X, then for all $\epsilon > 0$, there exists $x \in \operatorname{conv}\{x_n : n \in \mathbb{N}\}$ s.t. $\|x\| \leqslant \epsilon$.*

Proof. $0 \in \overline{\operatorname{conv}\{x_n : n \in \mathbb{N}\}}^{w} = \overline{\operatorname{conv}\{x_n : n \in \mathbb{N}\}}^{\|\cdot\|}$ by Mazur. □

Remark. It follows from this that there exist $p_1 < q_1 < p_2 < q_2 \ldots$ and convex combinations $z_n = \sum_{i=p_n}^{q_n} t_i x_i$ s.t. $z_n \to 0$ in $\|\cdot\|$.

Lecture 13 **Theorem 3.18** (Banach-Alaoglu)**.** *For any normed space X, (B_{X^*}, w^*) is compact.*

Proof. For $x \in X$, let $K_x = \{\lambda : \lambda \text{ scalar }, |\lambda| \leqslant \|x\|\}$. Let $K = \prod_{x\in X} K_x$ in he product topology.

Let $\pi_x : K \to K_x$ be the projection $(\lambda_y)_{y\in X} \mapsto \lambda_x$.

Note $K = \{\lambda : X \to \text{scalars} : |\lambda(x)| \leqslant \|x\|\}$, so $B_{X^*} \subseteq K$.

The subspace topology on B_{X^*} is $\sigma(K, \{\pi_x : x \in X\})\big|_{B_{X^*}} = \sigma(B_{X^*}, \{\pi_x\big|_{B_{X^*}} : x \in X\}) = \sigma(B_{X^*}, \{\widehat{x}\big|_{B_{X^*}} : x \in X\}) = \sigma(X^*, X)\big|_{B_{X^*}}$, the weak-* topology. By Theorem 3.3, K is compact. So all we need to show is that B_{X^*} is closed in K. Now,

$$\begin{aligned} B_{X^*} &= \{\lambda \in K : \lambda_{ax+by} = a\lambda_x + b\lambda_y \forall x, y \in X, \forall a, b \in \text{scalars}\} \\ &= \bigcap_{x,y,a,b} \{\lambda \in K : \pi_{ax+by}(\lambda) = a\pi_x(\lambda) + b\pi_y(\lambda)\} \\ &= \bigcap_{x,y,a,b} \{\lambda \in K : \pi_{ax+by}(\lambda) - a\pi_x(\lambda) - b\pi_y(\lambda)^{-1}(\{0\})\} \end{aligned}$$

closed in K as each π_x is continuous. $\square$

Proposition 3.19. *Let X be a normed space and K be a compact, Hausdorff space.*

(i) X is separable (in the $\|\cdot\|-$top) iff (B_{X^*}, w^*) is metrisable.

(ii) $\mathcal{C}(K)$ is separable iff K is metrisable.

Proof. (i)$\underline{" \implies "}$: Fix a dense sequence (x_n) in X. Let $\mathcal{F} = \{\widehat{x_n}|_{B_{X^*}} : n \in \mathbb{N}\}$. Then $\mathcal{F}$ separates the points of X, so $\sigma(B_{X^*}, \mathcal{F})$ is Hausdorff and is contained in the weak-* topology. So

$$\mathrm{Id} : (B_{X^*}, w^*) \to (B_{X^*}, \sigma(B_{X^*}, \mathcal{F}))$$

is a continuous bijection from a compact space to a Hausdorff space, and hence a homeomorphism. So $\sigma(B_{X^*}, \mathcal{F})$ is the weak-* topology on B_{X^*}. This is metrisable by proposition 3.2.

(i)$\underline{" \implies "}$: By above, $(B_{\mathcal{C}(K)^*}, w^*)$ is metrisable. For $k \in K$, define $\delta_k : \mathcal{C}(K) \to$ scalars by $\delta_k(f) = f(k)$ for all $f \in \mathcal{C}(K)$. Then $\delta_k \in B_{\mathcal{C}(K)^*}$. Hence

$$\begin{aligned} \delta :&\to (B_{\mathcal{C}(K)^*}, w^*) \\ k &\mapsto \delta_k \end{aligned}$$

$\underline{\delta \text{ is continuous}}$: let $f \in \mathcal{C}(K)$. Is $\widehat{f} \circ \delta$ continuous? For $k \in K$, $(\widehat{f} \circ \delta)(k) = \delta_k(f) = f(k)$. Then, $\widehat{f} \circ \delta = f$. This $\underline{\text{is}}$ continuous on K. By the universal property of the weak topology, δ is continuous.

$\underline{\delta \text{ is injective}}$: $\mathcal{C}(K)$ separates points of K by Urysohn.

Now, $\delta : K \to (\delta(K), w^*)$ is a continuous bijection from compact to Hausdorff, and hence a homeomorphism. Hence K is metrisable.

(ii)$\underline{" \impliedby "}$: K compact metrisable, so K is separable. Fix a dense sequence (x_n) in K. Let $(f_n) = d(x, x_n)$ (d is a metric inducing the topology of K). Let A be the sub-algebra of $\mathcal{C}(K)$ generated by $f_n, n \in \mathbb{N}$ and $\mathbf{1}_K$. The A is separable, A separates points of K, $\mathbf{1}_K \in A$ and in complex case, closed under complex conjugate. By Stone Weierstrass, $\overline{A} = \mathcal{C}(K)$, so $\mathcal{C}(K)$ is separable.

(i)$\underline{" \impliedby "}$: let $K = (B_{X^*}, w^*)$. This is compact, by Theorem 3.18. Since K is metrisable, $\mathcal{C}(K)$ is separable. We prove that $X \hookrightarrow \mathcal{C}(K)$ isometrically. Then done. Let $T : X \to \mathcal{C}(K)$ be $Tx = \widehat{x}|_{B_{X^*}}$. then T is linear and $\|Tx\|_\infty = \|\widehat{x}\| = \|x\|$. $\square$

Remark. 1. If X is separable, then (B_{X^*}, w^*) is compact, metrisable and hence weak-* sequentially compact(+separable).

2. X is separable implies that X^* is weak-* separable ($X^* = \bigcup_{n\in\mathbb{N}} nB_{X^*}$).
   By mazur, X is separable iff X is weakly separable (weak closure of span of some (x_n) weakly dense in X is $\|\cdot\|-$closure by Mazur, since it is convex).
   So X weakly separable implies X^* is weak-* separable. The converse is not true in general (e.g. ℓ_∞).

3. The proof shows $(B_{\mathcal{C}(K)^*}, w^*)$ contains a homeomorphic copy of K.

4. Proof also shows that for every normed space X there exists compact, hausdorff K s.t. $X \hookrightarrow \mathcal{C}(K)$ isometically ($K = (B_{X^*}, w^*)$).

Proposition 3.20. *Let X be a normed space. Then X^* is separable iff (B_X, w) is metrisable.*

Proof. "$\Longrightarrow$": By proposition 3.19 (i), $(B_{X^{**}}, w^*)$ is metrisable. Hence, $(B_X, w) = (B_{X^{**}}, w^*)|_{B_X}$ is metrisable.

"$\Longleftarrow$": let d metrise (B_X, w). Then for all $n \in \mathbb{N}$, there exists finite $F_n \subseteq X^*$ and $\epsilon_n > 0$ s.t. $\mathcal{U}_n = \{x \in B_X : |f(x)| < \epsilon_n \forall f \in F_n\} \subseteq \{x : d(x,0) < \frac{1}{n}\}$. Let $Z = \operatorname{span} \bigcup_{n\in\mathbb{N}} F_n$.

Claim: $\overline{Z} = X^*$, then done.

Indeed, let $g \in X^*$ and fix $\epsilon > 0$. Then $\{x \in B_X : |g(x)| < \epsilon\}$ is a weak neighbourhood of 0 in B_X and hence contains $\mathcal{U}_n$ for some $n \in \mathbb{N}$. Let $Y = \bigcap_{f\in F_n} \ker f$, then for $x \in B_Y, x \in \mathcal{U}_n$, so, $g(x) < \epsilon$. So $\|g|_{Y^*}\| \leqslant \epsilon$. Now $Y = \bigcap_{f\in F_n} \ker f \subseteq \ker(g-h)$, so by lemma 3.4 $g-h \in \operatorname{span} F_n \subseteq Z$ implies $d(g,z) \leqslant \epsilon$ which gives $g \in \overline{Z}$. □

Theorem 3.21 (Goldstine)**.** *For any normed space X, $\overline{B_X}^{w^*} = B_{X^{**}}$ ($\overline{B_X}^{w^*}$ is the closure in (X^{**}, w^*) of B_X).*

Proof. $B_{X^{**}}$ is weak-* closed (follows from Theorem 3.18) and $B_X \subseteq B_{X^{**}}$ so $\overline{B_X}^{w^*} \subseteq B_{X^{**}}$. Now let $\phi \in X^{**} \backslash \overline{B_X}^{w^*}$. Apply Theorem 3.15 (ii) to (X^{**}, w^*), $A = \{\phi\}, B = \overline{B_X}^{w^*}$ (show weak-* closure of convex set is closed). Now, there exists $f \in X^*$ s.t. $\phi(f) > \sup_B \widehat{f}$ (real case), $[\operatorname{Re}(\phi(f))] > \sup_B \operatorname{Re}(\widehat{f})$, $\|\phi\| \cdot \|f\| > \sup_{B_X} f$. So $\|\phi\| > 1$. □

Examples 3.22. *Example: Note that $\overline{X}^{w^*} = X^{**}$. So X separable implies X^* is weak-* separable. For instance, $\ell_\infty^* = \ell_1^{**}$ is weak-* separable, but ℓ_∞ is NOT separable.*

Indeed, we have that the map

$$\psi : \ell_\infty \to \ell_1^*$$
$$x \mapsto \left(f_x : \ell_1 \to \ \textit{scalars} : y \mapsto \sum_{n\in\mathbb{N}} x_n y_n \right)$$

is an isometric isomorphism (in the norm topologies). It suffices to show that

$$(\ell_\infty^*, \sigma(\ell_\infty^*, \ell_\infty)) \xrightarrow{\phi} (\ell_1^{**}, \sigma(\ell_1^{**}, \ell_1^*))$$

is a homeomorphism. Observe that $\phi = (\psi^{-1})^*, \phi^{-1} = (\psi)^*$, both dual maps. ψ being an isometric isomorphism in the norm topology implies that the same holds for ϕ. By the previous observation, it suffices to show that for all $y \in \ell_1^*$, $\widehat{y} \circ \phi : (\ell_\infty^*, \sigma(\ell_\infty^*, \ell_\infty)) \to \textit{scalars}$ is continuous. Indeed, observe that for $f \in \ell_\infty^*$, $\widehat{y} \circ \phi(f) = \phi(f)(y) = (\psi^{-1})^*(f)(y) = f(\psi^{-1}(y)) = \widehat{\psi^{-1}(y)}(f)$, and so $\widehat{y} \circ \phi = \widehat{\psi^{-1}(y)}$, which is weak- continuous by the universal property of the weak topology, hence we are done.*

Lecture 14

Theorem 3.23. *Let X be a Banach space. Then TFAE:*

(i) X is reflexive.

(ii) (B_X, w) is compact.

(iii) X^* is reflexive.

Proof. $\underline{(i) \implies (ii)}$: using the canonical embedding (a $w - w^*$ homeomorphism), $(B_X, w) = (B_{X^{**}}, w^*)$ B_X is compact by Banach-Alaoglu (Theorem 3.18).

$\underline{(ii) \implies (i)}$: $(B_X, w) = (B_{X^{**}}, w^*)$, so B_X is compact in the weak-* topology of X^{**}. So B_X is weak-* closed in X^{**}. By Goldstine, $B_{X^{**}} \supseteq \overline{B_X}^{w^*} = B_X$.

$\underline{(i) \implies (iii)}$: $(B_{X^*}, w) = (B_{X^*}, w^*)$ by reflexivity and is compact by Theorem 3.18. By $\underline{(ii) \implies (i)}$, X^* is reflexive.

$\underline{(iii) \implies (i)}$: By what we have just proved, X^{**} is reflexive. By the implication $(i) \implies (ii)$, $\underline{(B_{X^{**}}, w)}$ is compact. Since, X is complete, X is closed in X^{**}, and hence weakly closed in X^{**} (by Mazur). Hence, $B_X = X \cap B_{X^{**}}$ is a weakly. closed subset of $B_{X^{**}}$ and thus weakly compact[a]. By $(ii) \implies (i)$, X is reflexive. □

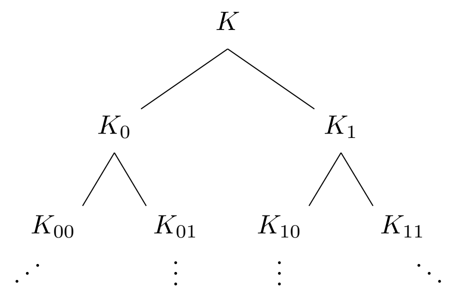
[a] $B_{X^{**}}$ is weak-* compact by Banach-Alaoglu and the map $\iota : (B_X, w) \to (\hat{B}_X, w^*)$ is a homeomorphism.

Remark. If X is separable and reflexive, then (B_X, w) is compact, metrisable. Hence, B_X is weakly sequentially compact.

Lemma 3.24. *Let (K, d) be a non-empty compact metric space. Then there exists a continuous surjection $\phi : \{0,1\}^{\mathbb{N}} \to K$, where $\{0,1\}^{\mathbb{N}}$ is given the product topology.*

Proof. Since compact and metric imply totally bounded, that is if $A \subseteq K$ is non-empty, closed and $\epsilon > 0$, then there exist non-empty closed sets $B_1, \ldots, B_n$ s.t. $A = \bigcup_{j=1}^{n} B_j$ and $\operatorname{diam}(B_j) < \epsilon$ for all j.

Applying this[a], there exists a non-empty closed subset K_ϵ of K for all $\epsilon \in \Sigma = \bigcup_{n=1}^{\infty} \{0,1\}^n$ s.t. $K_\varnothing = K$, $K_\epsilon = K_{\epsilon,0} \cup K_{\epsilon,1}$ and $\max_{\epsilon\{0,1\}^n} \operatorname{diam} K_\epsilon \to 0$ as $n \to \infty$. Imagine some picture like the one below:



Define $\phi : \{0,1\}^{\mathbb{N}} \to K$, $\phi((\epsilon_i)_{i=1}^{\infty})$ = the unique point in $\bigcap_{n=1}^{\infty} K_{\epsilon_1,\ldots,\epsilon_n}$ (is well-defined by compactness and nestedness of K_ϵ's).

$\underline{\phi \text{ is onto}}$: given $x \in K$, inductively construct $\epsilon_1, \ldots, \epsilon_n$ s.t. for all n $x \in K_{\epsilon_1,\ldots,\epsilon_n}$.

$\underline{\phi \text{ is continuous}}$: for $\epsilon = (\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$, let $n \in \mathbb{N}$, then for all $\delta = (\delta_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ if $\delta_i = \epsilon_i$ for all $1 \leqslant i \leqslant$, then $d(\phi(d), \phi(\epsilon)) \leqslant \operatorname{diam} K_{\epsilon_1,\ldots,\epsilon_n} \to 0$ as $n \to \infty$. □

[a] at each branching point $\epsilon \in \Sigma$, can cover K_ϵ by balls of diameter $\operatorname{diam} K_\epsilon/2$, 'shedding balls' until only the intersection with one remains, hence halving the diameter in a finite depth and proceed like so recursively.

Remark. $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the middle third Cantor set Δ via the map

$$(\epsilon_i)_{i=1}^{\infty} \mapsto \sum_{i=1}^{\infty} (2\epsilon_i) \cdot 3^{-i}.$$

Theorem 3.25. *Every separable Banach space X embeds isometrically into $\mathcal{C}[0,1]$. So $\mathcal{C}[0,1]$ is isometrically universal for the class of separable Banach spaces ($\mathcal{SB}$).*

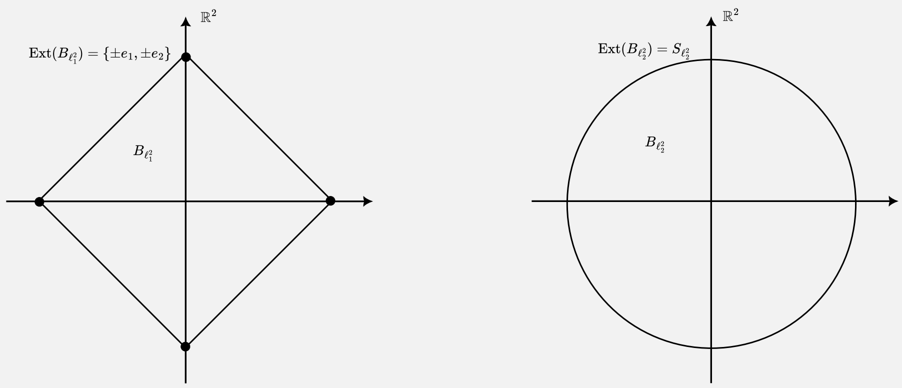
Proof. From the proof of proposition 3.19 that $X \hookrightarrow \mathcal{C}(K)$ isometrically where $K = (B_{X^*}, w^*)$. Since X is separable, K is metrisable. By lemma 3.24, there exists a continuous surjection $\phi : \Delta \to K$. Hence, $\mathcal{C}(K) \hookrightarrow \mathcal{C}(\Delta)$ isometrically via $f \mapsto f \circ \phi$. Also have $\mathcal{C}(\Delta) \hookrightarrow \mathcal{C}([0,1])$ isometrically via $f \mapsto \tilde{f}_1$.

Write $[0,1] \backslash \Delta$ as a disjoint union $\bigcup_{n=1}^{\infty} (a_n, b_n)$. Then $\tilde{f}|_\Delta = f$ for all n, $\tilde{f}$ is linear on $[a_n, b_n]$ with $\tilde{f}(a_n) = f(a_n), \tilde{f}(b_n) = f(b_n)$. □

4 Convexity

Let X be a real or complex vector space and $K \subseteq X$ be a convex set. A point $x \in K$ is an extreme point of K if whenever $x = (1-t)y + tz$ for $t \in (0,1)$, $y, z \in K$, we have $y = z = x$. Let $\operatorname{Ext} K$ be the set of extreme points of K.

Examples 4.1.



(a) $B_{\ell_1^2}$ (b) $B_{\ell_2^2}$

Figure 2: Above are displayed balls and their extreme points in ℓ_1^2, ℓ_1^2 respectively.

Furthermore, for the sequence space c_0, have that $\operatorname{Ext}(B_{c_0}) = \varnothing$.

Indeed, given $x = (x_n) \in B_{c_0}$. Fix $N \in \mathbb{N}$ s.t. $|x_N| < \frac{1}{2}$. Let $y_n = z_n = x_n$ for all $n \neq N \in \mathbb{N}$ and $y_N = x_N + \frac{1}{2}, z_N = x_N - \frac{1}{2}$. Then $y = (y_n)_{n\in\mathbb{N}}, z = (z_n)_{n\in\mathbb{N}} \in B_{c_0}$ and $x = \frac{1}{2}y + \frac{1}{2}z, y \neq x, z \neq x$.

Theorem 4.2 (Krein-Milman)**.** *Let $(X, \mathcal{P})$ be a LCS. Let K be a compact, convex subset of X. Then $K = \overline{\operatorname{conv}}(\operatorname{Ext} K)$. In particular,* $\operatorname{Ext} K \neq \varnothing$ *provided $K \neq \varnothing$.*

Corollary 4.3. *If X is a normed space, then $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{Ext} K)$ and $\operatorname{Ext} B_{X^*} \neq \varnothing$. Note c_0 is not a dual spce isometrically, i.e. there exists no normed space X s.t. $c_0 \cong X^*$.*

Definition 4.4. *Let K be a compact convex set in a LCS $(X, \mathcal{P})$. A face of K is a non-empty, compact convex set $E \subseteq K$ s.t. if $y, z \in K$, $t \in (0,1)$, $(1-t)y + tz \in E$, then $y, z \in E$.*

Examples 4.5. *1. K is a face of K. For $x \in K$, $x \in \operatorname{Ext} K \iff \{x\}$ is a face of K.*

2. let $f \in X^*$, $\alpha = \sup_K f$, $E = \{x \in K : f(x) = \alpha\}$ is a face.
($E \neq \varnothing$, convex, compact and if $y, z \in K$, $t \in (0,1)$ and $(1-t)y + tz \in E$, then $\alpha = f((1-t)y+tz) = (1-t)f(y) + tf(z) \geqslant \alpha$ giving equality, hence $f(y) = f(z) = \alpha$, hence $y, z \in E$).

[In the complex case, use $\operatorname{Re} f$. From now on, we only use real scalars.]

3. Let E be a face of K. If F is a face of E, then F is a face of K. So if $x \in \operatorname{Ext} E$, then $x \in \operatorname{Ext} K$.

Proof. Proof of Theorem 4.2 Let E be a face of K. We show $\operatorname{Ext} E \neq \varnothing$.

By Zorn, lemma 1.4, there exists a minimal (wrt inclusion) face F of E. If $|F| > 1$, then pick $x \neq y \in F$ and $f \in X^*$ s.t. $f(x) > f(y)$ (by Hahn-Banach). Then $\mathbb{G} = \{z \in F : f(z) = \sup_F f\}$ is a face of $F, y \notin \mathcal{G}$ so $\mathcal{G} \subsetneq F$, a contradiction. So F is a singleton which means $\operatorname{Ext} E \neq \varnothing$.

Now, let $L = \overline{\operatorname{conv}} \operatorname{Ext} K$. then $L \neq \varnothing$, convex, compact, $L \subseteq K$. Assume $x_0 \in K \backslash L$. By Theorem 3.14, there exists $f \in X^*$ s.t. $f(x_0) > \sup_L f$. Let $\alpha = \sup_K f$, then $E = \{x \in K : f(x) = \alpha\}$ is a face of K. So there's an extreme point z of K with $X \in E$. Since $\alpha \geqslant f(x_0)$, $E \cap L \neq \varnothing$, a contradiction. So $z \notin L$. □

Lecture 15 **Lemma 4.6.** *let $(X, \mathcal{P})$ be a LCS, let $K \subseteq X$ be compact and $x_0 \in K$. Then for a neighbourhood $\mathcal{V}$ of x_0 in X, there exist $f_1, \ldots, f_n \in X^*$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ s.t. $x_0 \in \{x \in X : f_i(x) < \alpha_i, 1 \leqslant i \leqslant n\} \cap K \subseteq \mathcal{V}$.*

Proof. let τ be the topology of X defined by $\mathcal{P}$ let $\sigma = \sigma(X, X^*)$. Then $\operatorname{Id} : (K, \tau) \to (K, \sigma)$ is a continuous bijection ($\sigma \subseteq \tau$) from compact to Hausdorff (as X^* separates points of X by Hahn-Banach), so it is a homeomorphism, i.e. $\sigma = \tau$ on K. □

Lemma 4.7. *let $(X, \mathcal{P})$ be a LCS, let $K \subseteq X$ be compact and convex. $x_0 \in \operatorname{Ext} K$. Then for a neighbourhood $\mathcal{V}$ of x_0 in X, there exists $f \in X^*$, $\alpha \in \mathbb{R}$ s.t. $x_0 \in \{x \in X : f(x) < \alpha\} \cap K \subseteq \mathcal{V}$.*

Proof. Let $n, f_1, \ldots, f_n \in X^*$, $\alpha_1, \ldots, \alpha_n$ be as in lemma 4.6 and $K_1 = \{x \in K : f_i(x) \geqslant \alpha_i\}$. This is compact and convex. Observe $\bigcup_{i=1}^n K_i \supseteq K \backslash \mathcal{V}$ and $x_0 \notin \bigcup_{i=1}^n K_i$. Also,

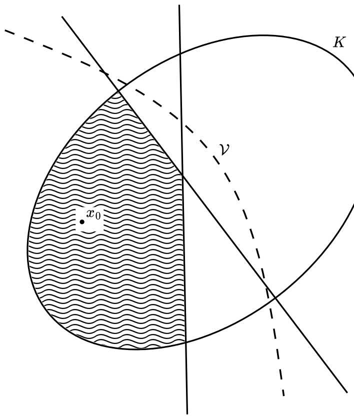
$$\operatorname{conv} \bigcup_{i=1}^n K_i = \left\{ \sum_{i=1}^n t_i x_i : x_i \in K_i, t_i \geqslant 0, \sum_{i=1}^n t_i = 1 \right\}.$$

Since x_0 is an extreme point of K, $x_0 \notin \operatorname{conv} \bigcup_{i=1}^n K_i$ (the case $n = 2$ is true by definition, and use induction to arrive at the general case).

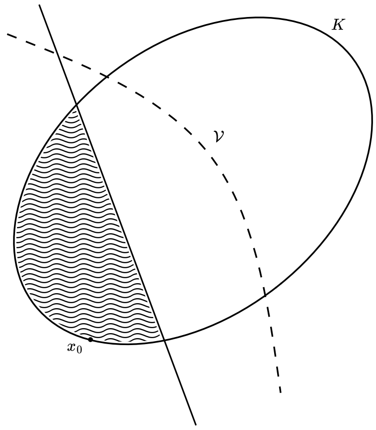
Furthermore,

$$K_1 \times \ldots K_n \times \left\{ (t_i) \in \mathbb{R}^n : t_i \geqslant 0 \forall i, \sum_{i=1}^n t_i = 1 \right\}$$

is compact and $(x_1, \ldots, x_n, (t_i)_{i=1}^n) \mapsto \sum_{i=1}^n t_i x_i$ is continuous (algebraic operaitons "$+, \times$" are continuous in LCS), so the image $B = \operatorname{conv} \bigcup_{i=1}^{n} K_i$ is compact. By Theorem 3.14, there exists $f \in X^*$ s.t. $f(x_0) < \inf_B f$. Choose $\alpha \in \mathbb{R}$ with $f(x_0) < \alpha < \inf_B f$. Then $x_0 \in \{x \in X : f(x) < \alpha\} \cap K$, which is disjoint from B and hence from $\bigcup_{i=1}^{n} K_i$ and so is contained in $\mathcal{V}$. $\square$



(a) Illustration of lemma 4.6.



(b) Illustration of lemma 4.7.

Figure 3

Theorem 4.8. *Let $(X, \mathcal{P})$ be a locally convex space, $K \subseteq X$ compact, convex and $S \subseteq K$. If $K = \overline{\operatorname{conv}} S$, then $\overline{S} \supseteq \operatorname{Ext} K$.*

Remark. The closure is necessary. For instance, let S be a dense subset of $S_{\ell_2^2}$. Then $\overline{\operatorname{conv}} S_{\ell_2^2} = B_{\ell_2^2}$ and $\operatorname{Ext} B_{\ell_2^2} = S_{\ell_2^2}$. Also, $\operatorname{Ext} K$ need not be closed. E.g. in $\mathbb{R}^3$,

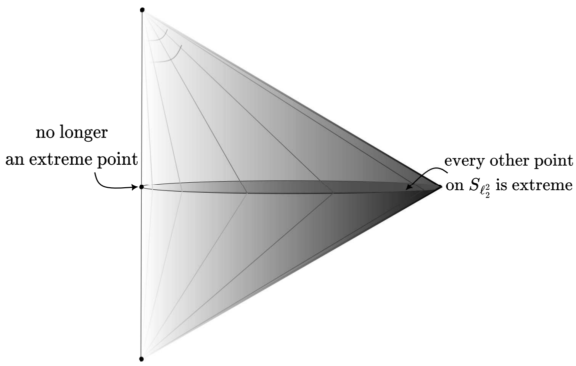


Figure 4: Illustration of extreme points of a double cone in $\mathbb{R}^3$ (which include top and bottom vertices).

Proof. Proof of Theorem 4.8 Assume $x_0 \in \operatorname{Ext} K \backslash \overline{S}$. Apply lemma 4.7 with $\mathcal{V} = X \backslash \overline{S}$. So, $f \in X^*$, $\alpha \in \mathbb{R}$ s.t. $x_0 \in \{x \in X : f(x) < \alpha\} \cap K \subseteq \mathcal{V}$. Then, $L = \{x \in K : f(x) \geqslant \alpha\}$ is compact, convex with $L \supseteq S$. Hence, $L \supseteq \overline{\operatorname{conv}} S = K$, a contradiction since $x_0 \notin L$. Thus, $x_0 \in S$. $\square$

Remark. One can show that $\operatorname{Ext} B_{\mathcal{C}(K)^*} = \{\lambda \delta_k : |\lambda| = 1, k \in K\}$ $(\delta_k(f) = f(k))$, where K is compact, Hausdorff. Can use Theorem 4.8 for "$\subseteq$".

Theorem 4.9 (Banach-Stone)**.** *Let K, L be compact, Hausdorff spaces, then $\mathcal{C}(K) \cong \mathcal{C}(L)$ $\iff$ L and K are homeomorphic.*

Proof. "$\Longleftarrow$": If $\phi : K \to L$ is a homeomorphism then

$$\begin{aligned} \phi^* : \mathcal{C}(L) &\cong \mathcal{C}(K) \\ f &\mapsto f \circ \phi \end{aligned}$$

is an isometric isomorphism.

"$\Longrightarrow$": let $T : \mathcal{C}(L) \cong \mathcal{C}(K)$ be an isometric isomorphism. Then so is its dual $T^* : \mathcal{C}(K)^* \cong \mathcal{C}(L)^*$. So $T^*(B_{\mathcal{C}(K)^*}) = B_{\mathcal{C}(L)^*}$ and $T^*(\operatorname{Ext} B_{\mathcal{C}(K)^*}) = \operatorname{Ext} B_{\mathcal{C}(L)^*}$. Thus, for each $k \in K$, $T^*(\delta_k) = \lambda(k) \cdot \delta_{\phi(k)}$ for some scalar $\lambda(k), |\lambda(k)| = 1$ and some $\phi(k) \in L$. So we have functions

$$\begin{aligned} \lambda : K &\to \text{ scalars} \\ \phi : K &\to L \end{aligned}$$

Now, for all $k \in K$, $\lambda(k) = T^*(\delta_k)(\mathbf{1}_L) = T(\mathbf{1}_L)(k)$, which means $\lambda = T(\mathbf{1}_L) \in \mathcal{C}(K)$, so λ is continuous. Recall, $\delta : K \to (\mathcal{C}(L)^*, w^*)$ is continuous (indeed, it is a homeomorphism between K and $\delta(K)$). Also, $T^* : \mathcal{C}(K)^* \to \mathcal{C}(L)^*$ is $w*-w^*$ continuous. hence, $h \mapsto \overline{\lambda(k)} \cdot T^*(\delta_k) = \delta_{\phi(k)} : K \to (\mathcal{C}(L)^*, w^*)$ is continuous. Since $\phi : K \xrightarrow{T^*} (\delta(L), w^*) \xrightarrow{\delta^{-1}} L$ is a composition of continuous maps, hence continuous.

ϕ is into: Assume $\phi(k_1) = \phi(k_2)$. So $\overline{\lambda(k)} \cdot T^*(\delta_{k_1}) = \overline{\lambda(k)} \cdot T^*(\delta_{k_1})$. Evaluate at $T^{-1}(\mathbf{1}_K)$ to get $\lambda(k_1) = \lambda(k_2)$ and so $\delta_{k_1} = \delta_{k_2}$ (as T^* is injective) which finally gives $k_1 = k_2$.

ϕ is onto: Given $l \in L$, since T^* is onto, there exists a scalar μ, $|\mu| = 1, k \in K$ s.t. $T^*(\mu\delta_k) = \delta_l$. So $\mu\lambda(k)\delta_{\phi(k)} = \delta_l$. Evaluate at $\mathbf{1}_L$ to get $\mu\lambda(k) = 1$ and so $\phi(k) = l$. $\square$

5 Banach Algebras

A real or complex algebra is a real or resp. complex vector space A with multiplication $A \times A :\to A$, $(a, b) \mapsto a \cdot b$ s.t.

(i) $a(bc) = (ab)c$

(ii) $a(b + c) = ab + ac$, $(a + b) \cdot c = ac + bc$

(iii) $\lambda(ab) = (\lambda a)b = a(\lambda b)$

for all $a, b, c \in A, b$ scalar.

A is unital if there exists $\mathbf{1} \in A$ s.t. $1 \neq 0$ and for all $x \in A$ $\mathbf{1}a = a\mathbf{1} = a$. This element is unique, called the unit of A.

An algebra norm on A is a norm on A s.t. for all $a, b \in A$, $\|ab\| \leqslant \|a\| \cdot \|b\|$. A normed algebra is an algebra with an algebra norm. note that multiplication is continuous (as well as addition and scalar multiplication). A Banach algebra (BA) is a complete normed algebra.

A unital normed algebra is a normed algebra, A with an element $\mathbf{1} \in A$ s.t. for all $x \in A$, $\mathbf{1}a = a\mathbf{1} = a$ and s.t. $\|\mathbf{1}\| = 1$ ($\|\mathbf{1}\| \leqslant \|\mathbf{1}\| \cdot \|\mathbf{1}\|$ and $1 \leqslant \|\mathbf{1}\|$). If A is a normed algebra which is also a unital algebra (but not assuming $\|\mathbf{1}\| = 1$), then $|||a||| = \sup\{\|ab\| : \|b\| \leqslant 1\}$ defines an equivalent norm on A that makes A a unital normed algebra.

A unital Banach algebra is a complete unital normed algebra. A linear map $\theta : A \to B$ between algebras is a homomorphism if for all $a, b \in A$ $\theta(ab) = \theta(a) \cdot \theta(b)$. If in addition A and B are unital with units $\mathbf{1}_A$ and $\mathbf{1}_B$ and $\theta(\mathbf{1}_A) = \mathbf{1}_B$, then θ is a unital homomorphism. In the category of normed algebras, an isomorphism will mean a continuous homomorphism with continuous inverse. BUT, homomorphisms are not assumed continuous.

Lecture 16 Note: from now on, the scalar field is $\mathbb{C}$.

Examples 5.1. *1. $\mathcal{C}(K)$, K compact Hausdorff, is a commutative, unital BA with pointwise multiplication in the uniform norm.*

2. Let K be compact, Hausdorff, A uniform algebra on K is a closed sub-algebra of $\mathcal{C}(K)$ that separates points of K and contains the constant functions.

3. The disk algebra $A(\Delta) = \{f \in \mathcal{C}(\Delta) : f$ holomorphic on the interior of $\Delta\}$, $\Delta = \{z \in \mathbb{C} : |z| \leqslant 1\}$.

More generally, let $K \subseteq \mathbb{C}, K \neq \varnothing$ compact. We have the following uniform algebras on K : $\mathcal{P}(K) \subseteq \mathcal{R}(K) \subseteq \mathcal{O}(K) \subseteq A(K) \subseteq \mathcal{C}(K)$, where $\mathcal{P}(K), \mathcal{R}(K), \mathcal{O}(K)$ are the closures in $\mathcal{C}(K)$ of respectively, polynomials, rational functions with no pole in K, functions holomorphic on some open neighbourhood of K. $A(K) = \{f \in \mathcal{C}(K) :$ f holomorphic on $\operatorname{int}(K)\}$. Later, $\mathcal{R}(K) = \mathcal{O}(K)$ say, $\mathcal{R}(K) = \mathcal{R}(K)$ if and only of $\mathbb{C}\backslash K$ is connected. In general $A(K) \neq \mathcal{O}(K)$, $A(K) = \mathcal{C}(K) \iff \operatorname{int}(K) = \varnothing$.

4. $L_1(\mathbb{R})$ with the L_1-norm and convolution $f * g(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$ is a commutative Banach algebra without a unit (Riemann-Lebesgue lemma).

5. If X is a Banach space, then $\mathcal{B}(X)$ with composition an operator norm is a unital Banach algebra. It is not commutative if $\dim X > 1$.

special case: if X is a Hilbert space, then $\mathcal{B}(X)$ is a C^*-algebra (see later).

5.1 Elementary constructions

1. If A is a unital algebra with unit $\mathbf{1}$, then a unital sub-algebra is a sub-algebra B of A s.t. $\mathbf{1} \in B$. If A is a normed algebra, then the closure of a sub-algebra of A is a sub-algebra of A.

2. Unitisation: The unitisation of an algebra A is the vector space direct sum $A_+ = A \bigoplus \mathbb{C}$ with multiplication $(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda, \mu)$. Then A_+ is a unital algebra with unit $\mathbf{1} = (0, 1)$.

   The ideal $\{(a, 0) : a \in A\}$ is isomorphic to A and will always be identified with A/ We can write $A = \{a + \lambda \mathbf{1} : a \in A, \lambda \in \mathbb{C}\}$. If A is a normed algebra, then A_+ becomes a unital normed algebra with $\|a + \lambda\mathbf{1}\| = \|a\| + |\lambda|$. Then A is a closed ideal of A_+. If A is a Banach algebra, then A_+ is a unital Banach algebra.

3. The closure of an ideal of a normed algebra is an ideal. If $\mathcal{J}$ is a closed ideal of the normed algebra of A, then $A\backslash\mathcal{J}$ is a normed algebra in the quotient norm. If A is a unital normed algebra and $\mathcal{J}$ is a proper closed ideal of $A(\mathcal{J} \neq A)$, then $A\backslash\mathcal{J}$ is a unital normed algebra with $\mathbf{1} + \mathcal{J}$ ($\|\mathbf{1} + \mathcal{J}\| \leqslant \|\mathbf{1}\| = 1$ and $\|\mathbf{1} + \mathcal{J}\| \geqslant 1$ from an earlier observation).

4. let $\tilde{A}$ be the Banach space completion of a normed algebra. Then $\tilde{A}$ is a Banach algebra with the following multiplication: given $a, b \in \tilde{A}$, choose sequences $(a_n), (b_n)$ in A s.t. $a_n \to a, b_n \to b$ and define $a \cdot b = \lim_{n\to\infty} a_n \cdot b_n$.

5. Let A be a unital Banach algebra. Let $X = A$ thought of as a Banach space. For $a \in A$, define $L_a : X \to X$, $L_a(x) = a \cdot x$. Then $L_a \in \mathcal{B}(X)$ and $\|L_a\| = \|a\|$. The map $L : A \to \mathcal{B}(X)$, $a \mapsto L_a$, is an isometric unital HM (homomorphism).

Lemma 5.2. *Let A be a unital Banach algebra and $a \in A$. Ift $\|\mathbf{1} - a\| < 1$, then a is invertible (there exists $b \in A$ s.t. $ab = ba = \mathbf{1}$) and $\left\|a^{-1}\right\| \leqslant \frac{1}{1-\|\mathbf{1}-a\|}$.*

Proof. For all $n \in \mathbb{N}$, $\|(\mathbf{1} - a)^n\| \leqslant \|\mathbf{1} - a\|^n$, so $\sum_{n=0}^{\infty} \|(\mathbf{1} - a)^n\| < \infty$. Hence, $\sum_{n=0}^{\infty} (1 - a)^n$ converges

$((1-a)^0 = 1)$.

Let $b = \sum_{n=0}^{\infty} (\mathbf{1}-a)^n$. Then $(\mathbf{1}-a)b = b(\mathbf{1}-a) = \sum_{n=1}^{\infty} (\mathbf{1}-a)^n = b-1$, and so $ab = ba = \mathbf{1}$. So, $b = a^{-1}$ and $\|a^{-1}\| = \|b\| \leqslant$ Let $b = \sum_{n=0}^{\infty} (\mathbf{1}-a)^n$. Then $(\mathbf{1}-a)b = b(\mathbf{1}-a) = \sum_{n=1}^{\infty} (\mathbf{1}-a)^n = b-1$, and so $ab = ba = \mathbf{1}$. So, $b = a^{-1}$ and $\|a^{-1}\| = \|b\| \leqslant \sum_{n=1}^{\infty} \|(\mathbf{1}-a)^n\| \leqslant \sum_{n=1}^{\infty} \|\mathbf{1}-a\|^n = \dfrac{1}{1-\|\mathbf{1}-a\|}$. □

Notation: we let $\mathcal{G}(A)$ denote the group of invertibles of a unital algebra A.

Corollary 5.3. *Let A be a unital Banach algebra.*

(i) $\mathcal{G}(A)$ is open in A.

(ii) $x \mapsto x^{-1}$ is a continuous function on $\mathcal{G}(A)$.

(iii) Assume $(x_n) \subseteq \mathcal{G}(A)$, $x_n \to x \in A\backslash\mathcal{G}(A)$. Then $\|x_n^{-1}\| \to \infty$ as $n \to \infty$.

(iv) If $x \in \partial\mathcal{G}(A) = \overline{\mathcal{G}(A)}\backslash\mathcal{G}(A)$, then there exists (z_n) in A s.t. $\|z_n\| = 1$ for all n and $z_n \cdot x \to 0$ and $x \cdot z_n$ as $n \to \infty$. It follows that x has no left or right inverse in A, not even in any unital algebra B containing A as a (not necessarily unital) sub-algebra.

Proof. (i) Let $x \in \mathcal{G}(A)$. If $y \in A$ and $\|y-x\| \leqslant \frac{1}{\|x^{-1}\|}$, then $\|\mathbf{1} - x^{-1}y\| = \|x^{-1}(x-y)\| \leqslant \|x^{-1}\| \cdot \|x-y\| < 1$. Hence, by lemma 5.2, $x^{-1}y \in \mathcal{G}(A)$, which implies that $y = x \cdot x^{-1}y \in \mathcal{G}(A)$.

(ii) Let us fix $x \in \mathcal{G}(A)$. For $y \in \mathcal{G}(A)$ $y^{-1} - x^{-1} = y^{-1}(x-y)x^{-1}$ so $\|y^{-1} - x^{-1}\| \leqslant \|y^{-1}\| \cdot \|x^{-1}\| \cdot \|x-y\|$. If $\|x-y\| < \frac{1}{2\|x^{-1}\|}$, then $|\,\|y^{-1}\| - \|x^{-1}\|\,| \leqslant 2 \cdot \|x^{-1}\|^2 \cdot \|x-y\| \to 0$ as $y \to x$.

(iii) From proof of (i), if $\|x - x_n\| < \frac{1}{x_n^{-1}}$, then $x \in \mathcal{G}(A)$, a contradiction. So $\|x - x_n\| \geqslant \frac{1}{x_n}$. Since, $\|x - x_n\| \to 0$, the result follows.

(iv) Given $x \in \partial\mathcal{G}(A)$, there exists a sequence $(x_n) \subseteq \mathcal{G}(A)$, $x_n \to x$. By part (iii) $\|x_n\| \to \infty$, let $z_n = \frac{x_n^{-1}}{\|x_n^{-1}\|}$, for all $n \in \mathbb{N}$. Then $z_n x = z_n x_n + z_n(x - x_n) = \frac{1}{\|x_n^{-1}\|} + z_n(x-x_n) \to 0$, by the above and since $\|z_n(x-x_n)\| \leqslant \|z_n\| \cdot \|x - x_n\| \to 0$. Similarly, $xz_n \to 0$.

Assume that B is a unital BA and A is a sub-algebra of B. If $y \in A$ and $yx = \mathbf{1}_B$, then $yxz_n = z_n$. So $\|z_n\| = 1 = \|yxz_n\| \leqslant \|y\| \cdot \|xz_n\|$, $n \to \infty$, a contradiction. Similarly, there is no $y \in B$ s.t. $xy = \mathbf{1}_B$. □

Lecture 17 **Definition 5.4.** *Let A be an algebra (always complex) and let $x \in A$. The spectrum $\sigma_A(x)$ of x in A is defined as follows: if A is unital, then $\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin \mathcal{G}(A)\}$ and if A is non-unital then $\sigma_A(x) := \sigma_{A_+}(x)$.*

Examples 5.5. *1. $A = M_n(\mathbb{C})$, $x \in A$, $\sigma_A(x)$ is the set of eigenvalues (evals) of x.*

2. $A = \mathcal{C}(K)$, K compact Hausdorff, $f \in A$, $\sigma_A(f) = f(K)$.

3. X a Banach space, $A = \mathcal{B}(X)$, $T \in A$, then $\sigma_A(T) = \{\lambda \in \mathbb{C} : \lambda \operatorname{Id} - T$ not an isomorphism$\}$.

Theorem 5.6. *Let A be a Banach algebra, $x \in A$. Then $\sigma_A(x)$ is a non-empty, compact subset of $\{\lambda \in \mathbb{C} : |\lambda| \leqslant \|x\|\}$.*

Proof. Wlog, A is a unital Banach algebra. If $|\lambda| > \|x\|$, then $\|x\| < 1$, so by lemma 5.2, $\mathbf{1} - \frac{x}{\lambda} \in \mathcal{G}(A)$ and so $\lambda\mathbf{1} - x = \lambda(\mathbf{1} - \frac{x}{\lambda}) \in \mathcal{G}(A)$. Hence, $\sigma_A(x) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leqslant \|x\|\}$. Also, $\sigma_A(x)$ is the inverse image of the closed set $A \backslash \mathcal{G}(A)$ (corollary 5.3(i)) under the continuous function $\lambda \mapsto \mathbb{C} \to A : \lambda\mathbf{1} - x$ and hence $\sigma_A(x)$ is closed. It follows that $\sigma_A(x)$ is compact.

$\sigma_A(x)$ is non-empty: consider $f : \mathbb{C}\backslash\sigma_A(x) \to A$, $f(\lambda) = (\lambda\mathbf{1} - x)$. By corollary 5.3($ii$) f is continuous and for $\lambda \neq \mu$:

$$\begin{aligned} f(\lambda) - f(\mu) = f(\lambda)((\mu\mathbf{1} - x) - (\lambda\mathbf{1} - x))f(\mu) \\ &= f(\lambda)(\mu - \lambda)f(\mu) \\ &= (\mu - \lambda)f(\lambda)f(\mu). \end{aligned}$$

So $\frac{f(\lambda)-f(\mu)}{\lambda-\mu} = -f(\lambda)f(\mu) \to -f(\mu)^2$ as $\lambda \to \mu$ because f is continuous. Thus, f is holomorphic. If $|\lambda| > \|x\|$ then $\lambda\mathbf{1} - x \in \mathcal{G}(A)$ and $\left\|(\lambda\mathbf{1} - x)^{-1}\right\| = \frac{1}{|\lambda|}\left\|\left(\mathbf{1} - \frac{x}{\lambda}\right)^{-1}\right\| \leqslant \frac{1}{|\lambda|}\frac{1}{1-\left\|\frac{x}{\lambda}\right\|} = \frac{1}{|\lambda|-\|x\|} \to 0$ as $|\lambda| \to \infty$. If $\sigma_A(x)$ where empty, then f is a bounded entire function, so by vector-valued Liouville, f is constant, and since $f(\lambda) \to 0$ as $|\lambda| \to \infty$, $f \equiv 0$, a contradiction. ☐

Corollary 5.7 (Gelfand-Mazur)**.** *A complex unital normed division ($\mathcal{G}(A) = A\backslash\{0\}$) algebra is isometrically isomorphic to $\mathbb{C}$.*

Proof. Let us define the map $\theta : \mathbb{C} \to A$, $\theta(\lambda). = \lambda \cdot \mathbf{1}$. then θ is an isomtric homomorphism. To show that it is onto, fix any $x \in A$. Let B be the completion of A. Then B is a unital Banach algebra. Then by Theorem 5.6, $\sigma_B(x)$ is non-empty which implies that there exists $\lambda \in \mathbb{C}$ s.t. $\lambda\mathbf{1} - x$ is NOT invertible in B, hence $\lambda\mathbf{1} - x$ is not in $\mathcal{G}(A)$ which means that $\lambda\mathbf{1} - x = 0$ and so $\theta(\lambda) = x$. ☐

Definition 5.8 (Spectral radius)**.** *Let A be a Banach algebra and $x \in A$. The spectral radius $r_A(x)$ of x in A is $r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$. From Theorem 5.6, $r_A(x)$ is well-defined and $r_A(x) \leqslant \|x\|$.*

Note: let x, y be comuting elements of a unital algebra A. Then $x \cdot y \in \mathcal{G}(A) \iff x \in \mathcal{G}(A)$ and $y \in \mathcal{G}(A)$ (use the fact that $z(xy) = (xy)z = 1$ gives $yzx = yzx \cdot yxz = yxz = 1$).

Lemma 5.9 (Spectral Mapping Theorem for polynomials)**.** *Let A be a unital Banach algebra and $x \in A$. Then for a complex polynomial $p = \sum_{k=0}^{n} a_k z^k$ we have*

$$\sigma_A(p(x)) = \{p(\lambda) : \lambda \in \sigma_A(x)\} = p(\sigma_A(x))$$

where $p(x) = \sum_{k=0}^{n} a_k z^k$ and $x^0 = \mathbf{1}_A$.

Proof. Wlog $n \neq 1$ and $a_n \neq 0$ ($\sigma_A(\lambda \mathbf{1}) = \{\lambda\}$). Fix $\mu \in \mathbb{C}$. Write $\mu - p(z) = c \cdot \prod_{k=1}^{n}(\lambda_k - z)$ for some $c, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $c \neq 0$. note that $\{\lambda : p(\lambda) = \mu\} = \{\lambda_1, \ldots, \lambda_n\}$. Now $\mu \notin \sigma_A(p(x))$ $\iff \mu\mathbf{1} - p(x) = \prod_{k=1}^{n}(\lambda_k \mathbf{1} - x)$ is invertible $\iff \lambda_k - x\lambda_k \mathbf{1} - x$ is invertible (use previous note on commutativity and invertibility) $\iff$ there exists no $\lambda \in \sigma_A(x)$ s.t. $p(\lambda) = \mu$. The result now follows. □

Theorem 5.10 (Beurling-Gelfand Spectral Radius Formula (SRF))**.** *Let A be a unital Banach algebra, $x \in A$. Then*

$$r_A(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} .$$

Proof. Wlog A is unital. By lemma 5.9, if $\lambda \in \sigma_A(x)$ and $n \in \mathbb{N}$, then $\lambda^n \in \sigma(x^n)$. By Theorem 5.6, $|\lambda^n| \leqslant \|x^n\|$ and $|\lambda| \leqslant \|x^n\|^{\frac{1}{n}}$. It thus follows that $r_A(x) \leqslant \inf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}}$.

Consider $f : \mathbb{C}\backslash\sigma_A(x) \to A, f(\lambda) = (\lambda\mathbf{1} - x)^{-1}$, by the proof of Theorem 5.6, f is holomorphic. Note that $\mathbb{C}\backslash\sigma_A(x) \supseteq \{|\lambda| > r_A(x)\} \supseteq \{\lambda : |\lambda| > \|x\|\}$. If $|\lambda| > \|x\|$, then $f(\lambda) = \frac{1}{\lambda}\left(\mathbf{1} - \frac{x}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{x^n}{\lambda^n}$ (by the proof of lemma 5.2).

Fix $\phi \in A^*$ (Banach space dual). Then $\phi \circ f$ is holomorphic on $\mathbb{C}\backslash\sigma_A(x)$ and if $|\lambda| > \|x\|$, then $\phi \circ f(\lambda) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{\phi(x^n)}{\lambda^n}$. This Laurent expansion must also be valid on $\{|\lambda| > r_A(x)\}$. So for $|\lambda| > r_A(x)$ and for $\phi \in A^*$, $\phi\left(\frac{x^n}{\lambda^n}\right) \to 0$ as $n \to \infty$. So for $|\lambda| > r_A(x)$, $\frac{x^n}{\lambda^n} \xrightarrow{w} 0$. By proposition 3.10, there exists $M > 0$ s.t. for all $n \in \mathbb{N}$, $\left\|\frac{x^n}{\lambda^n}\right\| \leqslant M^{\frac{1}{n}}$ and so $\limsup \|x^n\|^{\frac{1}{n}} \leqslant |\lambda|$. We have thus proved that $r_A(x) \leqslant \inf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leqslant \liminf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leqslant \limsup_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leqslant r_A(x)$. □

Theorem 5.11. *Let A be a unital Banach algebra and B be a closed, unital sub-algebra of A. Let $x \in B$. Then, $\sigma_B(x) \supseteq \sigma_A(x)$ and $\partial\sigma_B(x) \subseteq \sigma_A(x)$. It follows that $\sigma_B(x)$ is the union of $\sigma_A(x)$ and <u>some</u> of the bounded components of $\mathbb{C}\backslash\sigma_A(x)$.*

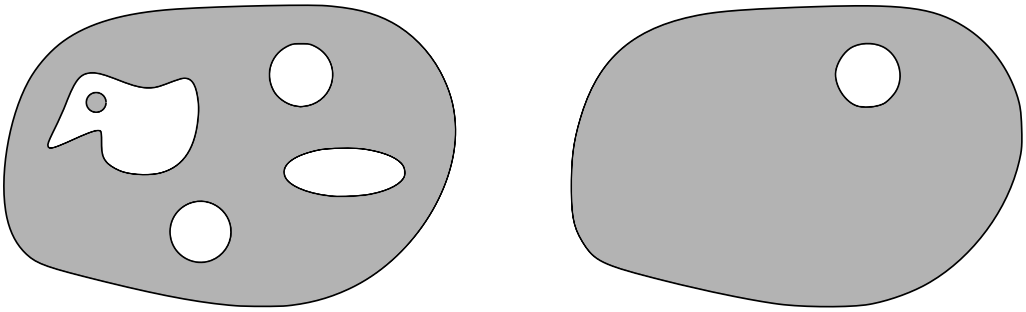
Before we proceed with the proof of the above, we prove a topological lemma.

Lemma 5.12. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are open sets in some topological space X s.t. $\mathcal{V} \subseteq \mathcal{W}$ and $\mathcal{W}$ contains non boundary points of $\mathcal{V}$. Then $\mathcal{V}$ is a union of components of $\mathcal{W}$.*

Proof. Let Ω be a component of W that intersects $\mathcal{V}$. Let $\mathcal{U}$ be the complement of $\overline{\mathcal{V}}$. Since $\mathcal{W}$ contains no boundary point of $\mathcal{V}$, Ω is the union of two disjoint open sets $\Omega \cap \mathcal{V}$ and $\Omega \cap \mathcal{U}$. Since Ω is connected, $\Omega \cap \mathcal{U}$ is empty and so it follows that $\Omega \subseteq \mathcal{V}$. □

Proof of Theorem 5.11. $\sigma_B(x) \supseteq \sigma_A(x)$ holds since an element invertible in B is also invertible in A. Let $\lambda \in \partial\sigma_B(x)$. then, there exist $(\lambda_n) \subseteq \mathbb{C}\backslash\sigma_B(x)$ s.t. $\lambda_n \to \lambda$. So $\lambda_n\mathbf{1} - x \in \mathcal{G}(B)$ and $\lambda_n\mathbf{1} - x \to \lambda\mathbf{1} - x \in B\backslash\mathcal{G}(B)$, which means $\lambda\mathbf{1} - x \in \partial\mathcal{G}(B)$. By corollary 5.3($iv$), $\lambda\mathbf{1} - x$ is not invertible in A, that is $\lambda \in \sigma_A(x)$.

To conclude, let Ω_A, Ω_B be the complements in $\mathbb{C}$ of $\sigma_A(x), \sigma_B(x)$ respectively. The preceding discussion implies that $\partial\Omega_B \subseteq \sigma_A(x)$ and so can use the topological lemma with $\mathcal{V} = \Omega_B, \mathcal{W} = \Omega_A$. Thus, Ω_B is the union of components of Ω_A. This means that $\sigma_B(x)$ is the union of $\sigma_A(x)$ and some bounded components of $\Omega_A = \sigma_A(x)\backslash\sigma_A(x)$. □



(a) $\sigma_A(x)$ (b) $\sigma_B(x)$

Figure 5: Illustration of Thheorem 5.11 for a sub-algebra $B \subseteq A$, $x \in B$.

Lecture 18

Proposition 5.13. *Let A be a unital Banach algebra and C a maximal commutative sub-algebra of A (wrt inclusion). Then C is a unital closed sub-algebra of A. Moreover, for all $x \in C$, $\sigma_C(x) = \sigma_A(x)$.*

Proof. $\overline{C}$ is a commutative sub-algebra of A. $\overline{C} \supseteq C$ and by maximality $\overline{C} = C$ is closed. $C + \mathbb{C} \cdot \mathbf{1}$ is a commutative sub-algebra of A contains C, so by maximality $C = C + \mathbb{C} \cdot \mathbf{1}$, i.e. $\mathbf{1} \in C$. Fix $x \in C$. We know that $\sigma_C(x) \supseteq \sigma_A(x)$. Assume $\lambda \in \mathbb{C} \backslash \sigma_A(x)/$ Let $y = (\lambda \mathbf{1} - x)^{-1}$ (in A). Have for all $z \in C$, $z(\lambda \mathbf{1} - x) = (\lambda \mathbf{1} - x)z$ as C is commutative and hence $yz = zy$. It follows that the sub-algebra generated by $C \cup \{y\}$ is commutative, so by maximality it is in C and so $y \in C$ and $\lambda \notin \sigma_C(x)$. Hence, $\sigma_C(x) \subseteq \sigma_A(x)$. □

Definition 5.14. *A non-zero homomorphism $\phi : A \to \mathbb{C}$ on an algebra A is called a character on A. Let Φ_A be the set of all characters on A. If A is unital, then $\phi(\mathbf{1}_A) = 1$ for all characters ϕ.*

Lemma 5.15. *Let A be a Banach algebra and $\phi \in \Phi_A$. Then ϕ is continuous and $\|\phi\| \leqslant 1$. Moreover, if A is a unital Banach algebra, then $\|\phi\| = 1$.*

Proof. Wlog, A is a unital Banach algebra: can define $\phi_+ : A_+ \to \mathbb{C}$ by $\phi_+(a + \lambda \mathbf{1}) = \phi(a) + \lambda$. Then $\phi_+ \in \Phi_{A_+}$ and $\phi_+|_A = \phi$. Now assume that A is a unital Banach algebra and $\phi \in \Phi_A$. Let $x \in A$ and assume $\phi(x) > \|x\|$. By Theorem 5.6, $\phi(x) \notin \sigma_A(x)$. So $\phi(x)\mathbf{1} - x \in \mathcal{G}(A)$. So $1 = \phi(x) = \phi((\phi(x)\mathbf{1} - x) \cdot (\phi(x)\mathbf{1} - x)^{-1}) = (\phi(x)\mathbf{1} - x) = 0$, a contradiction. So $|\phi(x)| \leqslant \|X\|$, giving $\|\phi\| \leqslant 1$. In fact $\|\phi\| = 1$ since $\phi(\mathbf{1}) = 1$. □

Lemma 5.16. *Let A be a unital Banach algebra and $\mathcal{J}$ be a proper ideal of A. Then $\overline{\mathcal{J}}$ is also a proper ideal. In particular, maximal ideals are closed.*

Proof. Since $\mathcal{J}$ is proper, $\mathcal{J} \cap \mathcal{G}(A)$ is empty. By corollary 5.3, $\mathcal{G}(A)$ is open giving that $\overline{\mathcal{J}} \cap \mathcal{G}(A)$ is empty, hence $\overline{\mathcal{J}}$ is proper. We have shown that if $\mathcal{M}$ is a maximal ideal of A, then $\mathcal{M}$ is proper and hence so is $\overline{\mathcal{M}}$. By maximality, $\mathcal{M} = \overline{\mathcal{J}}$ is closed. □

Notation: For an algebra A, we let $\mathcal{M}_A$ be the set of all maximal ideals of A.

Theorem 5.17. *Let A be a commutative unital Banach algebra. Then the map*

$$\begin{aligned} \Phi_A &\to \mathcal{M}_A \\ \phi &\mapsto \ker \phi \end{aligned}$$

is a bijection.

Proof. Well-defined: let $\phi \in \Phi_A$. Since ϕ is a homomorphism, $\ker\phi$ is an ideal of A. Since ϕ is a non-zero linear functional, $\ker\phi$ is a $1-$codimensional sub-space. So $\ker\phi$ is a maximal ideal.

Injective: assume $\phi, \psi \in \Phi_A$ and $\ker\phi = \ker\psi$. For $x \in A$, $\phi(x)\mathbf{1} - x \in \ker\phi = \ker\psi$, which implies $\psi(\phi(x)\mathbf{1} - x) = 0$ giving $\phi(x) \cdot \psi(\mathbf{1}) = \psi(x) = \phi(x)$.

Surjective: let $\mathcal{M} \in \mathcal{M}_A$. By lemma 5.16, $\mathcal{M}$ is closed, so $A\backslash\mathcal{M}$ is a unital Banach algebra in the quotient norm. From algebra, $A\backslash\mathcal{M}$ is a field, so a division algebra. By corollary 5.7 (Galfand-Mazur), $A\backslash\mathcal{M} \cong \mathbb{C}$. So the quotient map $q : A \to A\backslash\mathcal{M}$ "is" a character and $\ker q = \mathcal{M}$. ☐

Corollary 5.18. *Let A be a commutative unital Banach algebra and $x \in A$. Then*

(i) $x \in \mathcal{G}(A) \iff$ for all $\phi \in \Phi_A$, $\phi(x) \neq 0$.

(ii) $\sigma_A(x) = \{\phi(x) : \phi \in \Phi_A\}$.

(iii) $r_A(x) = \sup\{|\phi(x)| : \phi \in \Phi_A\}$

Proof. (i) If $x \in \mathcal{G}(A)$, then for all characters ϕ, $1 = \phi(\mathbf{1}) = \phi(x \cdot x^{-1}) = \phi(x) \cdot (\phi(x))^{-1}$ implying that $\phi(x) \neq 0$.
Assume that $x \notin \mathcal{G}(A)$, then $\mathcal{J} = xA = \{xa : a \in A\}$ is a proper ideal of A, and so is contained in a maximal ideal which is $\ker\phi$ for some character ϕ by Theorem 5.17. So $\phi(x) = 0$ since $x \in \mathcal{J} \subseteq \ker\phi$.

(ii) $\lambda \in \sigma_A(x) \iff (\lambda\mathbf{1} - x) \notin \mathcal{G}(A) \iff$ (by (i)) there exists $\phi \in \Phi_A$ s.t. $\phi(\lambda\mathbf{1} - x) = 0$, i.e. $\lambda = \phi(x)$.

(iii) Is immediate from (ii).

☐

Corollary 5.19. *Let x, y be commuting elements of a Banach algebra A. Then*

$$r_A(x + y) \leqslant r_A(x) + r_A(y)$$
$$r_A(x \cdot y) \leqslant r_A(x) \cdot r_A(y).$$

Proof. Wlog, A is a commutative unital Banach Algebra. ($A \to A_+$ if necesary and then replace A by a maximal commutative sub-algebra containing x, y and use proposition 5.13). Then for all characters ϕ, $|\phi(x+y)| \leqslant |\phi(x)|+|\phi(y)| \leqslant r_A(x)+r_A(y)$ by corollary 5.18. Taking supremum over all characters ϕ gives $r_A(x + y) \leqslant r_A(x) + r_A(y)$. Argue analogously for the remaining inequality. ☐

Examples 5.20. *1.* $A = \mathcal{C}(K)$, K *compact, Hausdorff.* $\Phi_A = \{\delta_k : k \in K\}$ $(\delta_k(f) = f(k))$. *"$\supseteq$" is easy to check.*

"$\subseteq$": let $\mathcal{M} \in \mathcal{M}_A$. *Seek* $k \in K$ *s.t.* $\mathcal{M} = \ker \delta_k$. *Assume there is non such A. Then for all* $k \in K$, *there exist* $f_k \in \mathcal{M}$ *s.t.* $f_k(k) \neq 0$. *By continuity, there exists open neighbourhoods* $\mathcal{U}_k$ *of* k *s.t.* $f_k|_{\mathcal{U}_k} \neq 0$. *By compactness, there exist* $k_1, \ldots, k_n \in K$ *s.t.* $\bigcup^n \mathcal{U}_{k_j} = K$. *Then* $g = \sum_{j=1}^{n} |f_{k_j}|^2 > 0$ *on* K. *So* $\frac{1}{g} \in \mathcal{C}(K)$. *Also,* $g = \sum_{j=1}^{n} f_{k_j} \cdot \overline{f}_{k_j} \in \mathcal{M}$, *a contradiction.*

2. Let $K \subseteq \mathbb{C}$, K *compact and non-empty. Then* $\Phi_{\mathcal{R}(K)} = \{\delta_w : w \in K\}$.

3. $\Phi_{A(\Delta)} = \{\delta_w : w \in \Delta\}$ *where* $A(\Delta)$ *is the disc algebra.*

4. *Wiener algebra:* $W = \{f \in \mathcal{C}(S^1) : \sum_{n\in\mathbb{Z}} |\widehat{f}_n| < \infty\}$, *where* $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, $\widehat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$. *W is a commutative unital Banach algebra with pointwise operations in the norm* $\|f\|_1 = \sum_{n\in\mathbb{Z}} |\widehat{f}_n|$. *[It is isometrically isomorphic to* $\ell_1(\mathbb{Z})$ *which is a Banach algebra in the* ℓ_1-*norm and convolution product. That is for* $a = (a_n), b = (b_n)$, $(a * b)_n = \sum_{j+k=n} a_k b_j, n \in \mathbb{Z}$. *The isomorphism is given by* $f \mapsto (\widehat{f}_n)_n \in \mathbb{Z}$*]. Have* $\Phi_W = \{\delta_w : w \in S^1\}$, *so* $\sigma_W(f) = f(S^1)$. *So if* $f \in \mathcal{C}(S^1)$ *has absolutely convergence Fourier series and is nowhere zero, then* $\frac{1}{f} \in W$ *and so has an absolutely convergence Fourier series and is nowhere zero (Wiener's Theorem).*

Lecture 19 **Definition 5.21.** *Let A be a commutative unital Banach algebra. Then*

$$\begin{aligned}\Phi_A &= \{\phi \in B_{A*} : \phi(ab) = \phi(a)\phi(b) \forall a, b \in A, \phi(\mathbf{1}_A = 1\} \\ &= B_{A*} \cap (\widehat{ab} - \widehat{a} \cdot \widehat{b})^{-1}(\{0\}) \cap \mathbf{1}_A^{-1}(\{1\})\end{aligned}$$

is weak- closed. (Here for* $x \in A$, $\widehat{x} \in A^{**}$ *is its canonical image in* A^{**}*). Hence,* Φ_A *is* w^*-*compact. The* $w^* -$ *topology on* Φ_A *is called the Gelfand topology.* Φ_A *with the Gelfand topology is the spectrum of A OR the character space of A OR the maximal ideal space of A. For* $x \in A$, $\widehat{x}|_{\Phi_A}$ *is continuous on* Φ_A *wrt the Gelfand topology; we denote* $\widehat{x}|_{\Phi_A}$ *by* $\widehat{x}$. *So* $\widehat{x} \in \mathcal{C}(\Phi_A)$*-called the Gelfand transform of x. The map*

$$\begin{aligned}A &\to \mathcal{C}(\Phi_A) \\ x &\mapsto \widehat{x}\end{aligned}$$

is the Gelfand map.

Theorem 5.22 (Gelfand Representation Theorem)**.** *Let A be a commutative unital Banach algebra, then the Gelfand map is a continuous unital homomorphism* $A \to \mathcal{C}(\Phi_A)$. *For* $x \in A$

(i) $\|\widehat{x}\|_\infty = r_A(x) \leqslant \|x\|$.

(ii) $\sigma_{\mathcal{C}(\Phi_A)}(\widehat{x}) = \sigma(x)$.

(iii) $x \in \mathcal{G}(A) \iff \widehat{x} \in \mathcal{G}(\mathcal{C}(\Phi_A))$.

Proof. The Gelfand map is linear since $x \to \widehat{x} : A \to A^{**}$ is linear.

Homomorphism: for $x, y \in A$ $\widehat{xy}(\phi) = \phi(xy) = \phi(x)\phi(y) = \widehat{x} \cdot \widehat{y}$ for all $\phi \in \Phi_A$, so $\widehat{xy} = \widehat{x}\widehat{y}$.

Unital: $\widehat{\mathbf{1}}_A(\phi) = \phi(\mathbf{1}_A) = 1$ for all $\phi \in \Phi_A$, so $\widehat{\mathbf{1}}_A = \widehat{\mathbf{1}}_{\Phi_A}$.

Continuity: follows once we prove (i).

(i) $\|\widehat{x}\|_{\infty} = \sup\{|\widehat{x}(\phi)| : \phi \in \Phi_A\} \overset{Cor 5.18(iii)}{=} r_A(x) \overset{Thm 5.6}{\leqslant} \|x\|$.

(ii) $\sigma_{\mathcal{C}(\Phi_A)}(\widehat{x}) = \{|\overset{=\phi(x)}{\widehat{x}(\phi)}| : \phi \in \Phi_A\} \overset{Cor 5.18(ii)}{=} \sigma_A(x)$.

(iii) Immediate.

□

Note: the Gelfand map need not be injective or surjective. Using Theorem, 5.10 its kernel is

$$\begin{aligned}\{x \in A : \sigma_A(x) = \{0\}\} &= \{x \in A : \overbrace{\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}}^{\text{quasi-nilpotent}=0}\} \\ &= \bigcap_{\phi\in\Phi_A} \ker\phi \\ &= \underbrace{\bigcap_{\mathcal{M}\in\mathcal{M}_A} \mathcal{M}}_{\text{Jacobson radical of } A, \mathcal{J}(A)} .\end{aligned}$$

Say A is semi-simple if $\mathcal{J}(A) = \{0\}$.

6 Holomorphic Functional Calculus (HFC)

Recall For a non-empty open set $\mathcal{U} \subseteq \mathbb{C}$, $\mathcal{O}(\mathcal{U}) = \{f : \mathcal{U} \to \mathbb{C} : f \text{ holomorphic}\}$ is a LCS with the topology of local uniform convergence induce by the family of semi-nnorms: $f \mapsto \|f\|_K = \sup_K |f|$ for non-empty compact $K \subseteq \mathcal{U}$. $\mathcal{O}(\mathcal{U})$ is also an algebra with pointwise multiplication which is cotinuous wrt the topology of $\mathcal{O}(\mathcal{U})$ [a Fréchet algebra].

Notation: Define $e, u \in \mathcal{O}(\mathcal{U})$ by $e(z) = 1$ and $u(z) = z$ for all $z \in \mathbb{C}$. $\mathcal{O}(\mathcal{U})$ is a untial algebra with unit e.

Theorem 6.1 (Holomorphic Function Calculus)**.** *Let A be a commutative unital Banach algebra, $x \in A$, $\mathcal{U} \subseteq \mathbb{C}$ open and $\sigma_A(x) \subseteq \mathcal{U}$. Then there exists a unique unital homomorphism $\Theta_x : \mathcal{O}(\mathcal{U}) \to A$ s.t. $\Theta_x(u) = x$. Moreover, $\phi(\Theta_x(f)) = f(\phi(x))$ for all $\phi \in \Phi_A, f \in \mathcal{O}(\mathcal{U})$ and $\sigma_A(\Theta_x(f)) = \{f(\lambda) : \lambda \in \sigma_A(x)\}$.*

Note: Think of Θ_x as "evaluation as x"-write $f(x)$ for $\Theta_x(f)$. Then $e(x) = 1$, $u(x) = x$. If p is a polynomial, there exist $n \in \mathbb{N}, a_0, \ldots, a_n \in \mathbb{C}$ s.t. for all $z \in \mathbb{C}$, $p(z) = \sum_{k=0}^{n} a_k z^k$, then $p = \sum_{k=0}^{n} a_k u^k$. So $\Theta_x(p) = p(z) = \sum_{k=0}^{n} a_k (\Theta_x(u))^k = \sum_{k=0}^{n} a_k x^k = p(x)$ as defined in lemma 5.9.
Also, $\phi(f(x)) = f(\phi(x))$ for all $f \in \mathcal{O}(\mathcal{U})$, $\phi \in \Phi_A$ and $\sigma_A(f) = \{f(\lambda) : \lambda \in \sigma_A(x)\} = f(\sigma_A(x))$.

Theorem 6.2 (Runge's Approximation Theorem)**.** *Let K be non-empty and compact. Then $\mathcal{O}(K) = \mathcal{R}(K)$, i.e. if f is a function holomorphic on some open neighbourhood of K then for all $\epsilon > 0$, there exists ration function r with no poles in K s.t. $\|f - r\|_K < \epsilon$.*

More precisely, given a set Λ consisting of one point from each bounded component of $\mathbb{C}\backslash K$, r can be chosen s.t. all its poles are in Λ. If $\mathbb{C}\backslash K$ is connected, then Λ is empty so in fact we get $\mathcal{O}(K) = \mathcal{P}(K)$.

6.1 Vector-valued integration

Let $a < b$ in $\mathbb{R}$, X be a Banach space and $f : [a, b] \to X$ continuous. We define "$\int_a^b f(t)dt$". We choose dissections $\mathcal{D}_n := a = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n}^{(n)} = b$ s.t. $|\mathcal{D}_n| = \max_{1 \leqslant j \leqslant k_n} (t_j^{(n)} - t_{j-1}^{(n)}) \to 0$ as $n \to \infty$.

Since f is uniformly continuous, the limit of

$$\sum_{j=0}^{k_n} f(t_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})$$

exists and is independent of $(\mathcal{D}_n)$. We define $\int_a^b f(t)dt$ to be this limit. It follows that for all $\phi \in X^*$

$$\phi\left(\int_a^b f(t)dt\right) = \int_a^b \phi(f(t))dt.$$

Taking ϕ to be a norming functional for $\int_a^b f(t)dt$, we get

$$\left\|\int_a^b f(t)dt\right\| \leqslant \int_a^b \|f(t)\|\, dt, \quad (\|\phi\| \leqslant 1).$$

Let γ be a path in $\mathbb{C}$ (continuously differentiable), $f : [\gamma] \to X$ be continuous[3]. Define

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Given a chain $\Gamma = (\gamma_1, \ldots, \gamma_n)$[4] and continuous $f : [\gamma] \to X$ define

$$\int_\Gamma f(z)\,\mathrm{d}z = \sum_{j=1}^n \int_a^b f(\gamma(t))\gamma'(t)dt$$

and have for all $\phi \in X^*$

$$\phi\left(\int_\Gamma f(z)\,\mathrm{d}z\right) = \int_\Gamma \phi(f(z))\,\mathrm{d}z.$$

and

$$\left\|\int_\Gamma f(z)\,\mathrm{d}z\right\| \leqslant \ell(\Gamma) \cdot \sup_{z \in [\gamma]} \|f(z)\|.$$

Theorem 6.3 (Vector-valued Cauchy's Theorem)**.** *Let $\mathcal{U} \subseteq \mathbb{C}$ be open, Γ a cycle[a] in $\mathcal{U}$, s.t. $n(\Gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-w}\,\mathrm{d}z = 0$ for all $w \notin \mathcal{U}$ and $f : \mathcal{U} \to X$ holomorphic. Then*

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

[a] a cycle is a chain $\Gamma = (\gamma_1, \ldots, \gamma_n), n \in \mathbb{N}$ of paths $\gamma_j : [a_j, b_j] \to \mathbb{C}$ s.t. there exists a permutation $\rho \in S_n$ s.t. $\gamma_j(b_j) = \gamma_{\rho(j)(a_{\rho(j)})}$ for all $j = 1, \ldots, n$.

Proof. For $\phi \in X^*$, apply the scalar version of Cauchy's Theorem to deduce

$$\phi\left(\int_\Gamma f(z)\,\mathrm{d}z\right) = 0, \qquad \text{for all } \phi \in X^*$$

[3] $[\gamma]$ denotes the path itself in $\mathbb{C}$.
[4] any finite collection of paths defined as above.

and then apply Hahn-Banach to conclude. $\square$

Lemma 6.4. *Let K be a non-empty compact s.t. $K \subseteq \mathcal{U}$, $\mathcal{U} \subseteq \mathbb{C}$ open. Then there is a cycle Γ such that*

$$n(\Gamma, w) = \begin{cases} 1, & w \in K \\ 0, & w \notin \mathcal{U}. \end{cases}$$

Proof. Note that K being compact means that $\mathrm{dist}(K, \mathbb{C}\backslash\mathcal{U}) = \delta > 0$. Thus, there exists an $n \in \mathbb{N}^a$, s.t. K is covered by finitely many (by compactness) boxes in the dyadic lattice $2^{-n}\mathbf{Z}^2$ where any adjacent to them boxes are also $\subseteq \mathcal{U}$, see figure 6. More precisely, $\mathcal{A} = \{(x, y) \in \mathbb{Z}^2 : [x \cdot 2^{-n}, x \cdot 2^{-n} + 2^{-n}] \times [y \cdot 2^{-n}, y \cdot 2^{-n} + 2^{-n}] \cap K \neq \varnothing\}$. Have $|\mathcal{A}| < \infty$. Now, define $\mathcal{B} = \mathcal{A} \cup \{(x \pm 1, y \pm 1) \in \mathbb{Z}^2 : (x, y) \in A)\}$. Let Γ be the boundary of the boxes above, that is

$$\Gamma = \partial \bigcup_{(x,y) \in \mathcal{B}} [x \cdot 2^{-n}, x \cdot 2^{-n} + 2^{-n}] \times [y \cdot 2^{-n}, y \cdot 2^{-n} + 2^{-n}]$$

oriented counter-clockwise (black curve in figure 6), and note that $\Gamma \subseteq \mathcal{U}\backslash K$.

Now, for any $w \in K$, w is either in the interior of a box or the interior of the union of boxes adjacent to it. Regardless, one computes the winding number around such a curve $\tilde{\Gamma}$ (red in figure6), which is seen to be the same as the winding number of Γ around w, by homotopy invariance (Cauchy's Theorem). One argues similarly for $w \in \mathbb{C}\backslash\mathcal{U}$ to obtain $n(\Gamma, w) = 0$. $\square$

[a] for instance, take $n \in \mathbb{N}$ s.t. $2\sqrt{2} \cdot 2^{-n} < \frac{\delta}{2}$.

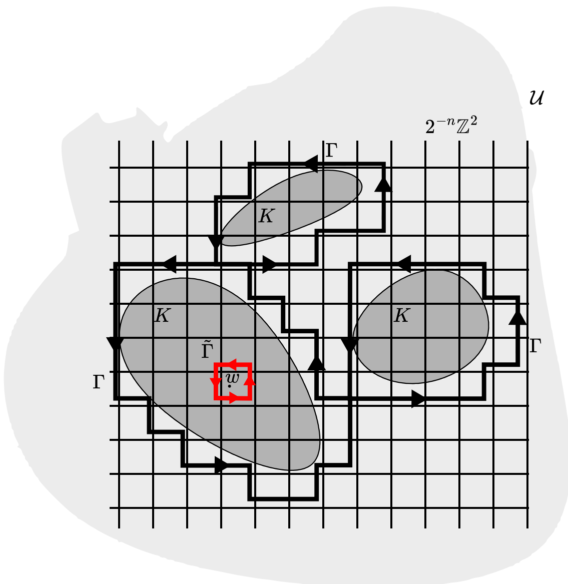


Figure 6: Illustration of proof of Lemma 6.4, where $n \in \mathbb{N}, w \in K$ and $K, \mathcal{U}, \Gamma$ (in black) as in in the lemma.

Lecture 20 **Lemma 6.5.** *Let $A, x, \mathcal{U}$ be as in Theorem 6.1. $K = \sigma_A(x)$ and fix a cycle (guaranteed to exists by Lemma 6.4) Γ in $\mathcal{U}\backslash K$ s.t.*

$$n(\Gamma, w) = \begin{cases} 1, & w \in K \\ 0, & w \notin \mathcal{U}. \end{cases}$$

Define the map

$$\Theta_x : \mathcal{O}(K) \to A$$
$$f \mapsto \Theta_x(f) = \tfrac{1}{2\pi i} \int_\Gamma f(z)(z\mathbf{1} - x)^{-1})\,\mathrm{d}z.$$

Then,

(i) Θ_x is well-defined, linear, continuous.

(ii) For a rational function r with no poles in $\mathcal{U}$, $\Theta_x(r) = r(x)$ in the usual sense.

(iii) $\phi(\Theta_x(f)) = f(\phi(x))$ for all $\phi \in \Phi_A, f \in \mathcal{O}(\mathcal{U})$ and $\sigma_A(\Theta_x(f)) = \{f(\lambda) : \lambda \in \sigma_A(x)\}$.

Remark. So we can think of the HFC as a Banach algebra valued Cauchy integral formula. Lemma 6.5 almost proves the theorem (6.1). It remains to show that Θ_x is a homomorphism and it is unique.

Proof. (i) If $z \in [\Gamma]$ then $z \notin K = \sigma_A(x)$. So $z\mathbf{1} - x \in \mathcal{G}(A)$. By the proof of Theorem 5.6, the map $z \mapsto (z\mathbf{1} - x)^{-1}$ is continuous (indeed, holomorphic). So, Θ_x is well-defined. It's also linear by linearity of integration. We also have the estimate

$$\|\Theta_x(f)\| \leqslant \frac{1}{2\pi}\ell(\Gamma) \cdot \sup_{z \in [\gamma]} |f(z)| \cdot \left\|(z\mathbf{1} - x)^{-1}\right\|.$$

Since the map $z \mapsto \left\|(z\mathbf{1} - x)^{-1}\right\|$ is continuous on the compact set $[\Gamma]$, it is bounded. So there exists $M > 0$ s.t. for all $f \in \mathcal{O}(\mathcal{U})$ $\|\Theta_x(f)\| \leqslant M \cdot \|f\|_{[\Gamma]}$.

By Lemma 1.21, Θ_x is continuous.

(ii) First we show $\Theta_x(e) = 1$.

Fix $R > \|x\|$ and let γ be the anticlockwise boundary of $D(0, R)$. Then γ and Γ are homologous in $\mathbb{C}\backslash K$. So, by Cauchy's Theorem and the proof of Lemma 5.2,

$$\begin{aligned} \Theta_x(e) &= \tfrac{1}{2\pi i} \int_\gamma (z\mathbf{1} - x)^{-1}\,\mathrm{d}z \\ &= \tfrac{1}{2\pi i} \int_\gamma \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{x}{z}\right)^n \mathrm{d}z \\ &\underbrace{= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_\gamma \frac{1}{z} \frac{x^n}{z^{n+1}}\,\mathrm{d}z}_{\text{sum conv. absolutely and uniformly on } \gamma} \\ &= x^0 = 1. \end{aligned}$$

Let $r \in \mathcal{O}(K)$ be a rational function. So $r = \frac{p}{q}$, for polynomials p, q s.t. for all $z \in \mathcal{U}$, $q(z) \neq 0$. By Lemma 5.9, $\sigma_A(q(x)) = \{q(\lambda) : \lambda \in \overset{=K}{\sigma_A(x)}\}$ and so $0 \notin \sigma_A(q(x))$. We define $r(x) = p(x) \cdot q(x)^{-1}$ ("usual sense"). For $z, w \in \mathbb{C}$, $r(z) - r(w) = q(z)^{-1}q(w)^{-1}(q(w)p(z) - q(z)p(w)) = q(z)^{-1}q(w)^{-1}(z - w)s(z, w)$, where s is a polynomial in z, w. Hence, $r(z)\mathbf{1} -$

$r(x) = q(z)^{-1}q(w)^{-1}(z\mathbf{1} - w)s(z, w)$ and

$$\begin{aligned}\Theta_x(r) &= \tfrac{1}{2\pi i}\int_\gamma \underbrace{r(z)}_{r(z)\mathbf{1}-r(x)+r(x)} (z\mathbf{1} - x)^{-1}\,\mathrm{d}z \\ &= \underbrace{\tfrac{1}{2\pi i}\int_\gamma q(z)^{-1}q(w)^{-1}s(z,w)\,\mathrm{d}z}_{=0 \text{ by Cauchy}} + \tfrac{1}{2\pi i}\int_\gamma (z\mathbf{1} - x)^{-1})\,\mathrm{d}z \cdot r(x) \\ &= r(x)\cdot\Theta_x(e) = r(x).\end{aligned}$$

(iii) $\phi\left(\Theta_x(f)\right) = \frac{1}{2\pi i}\int_\gamma f(z)(z\mathbf{1} - x)^{-1}\,\mathrm{d}z = f(\phi(x))$ by Cauchy's integral formula. and so

$$\sigma_A\left(\Theta_x(f)\right) \overset{\text{Cor 5.18}}{=} \{\overbrace{\phi\left(\Theta_x(f)\right)}^{=f(\phi(x))} \colon \phi \in \Phi_A\} = \{f(\lambda) : \lambda \in \sigma_A(x)\}.$$

□

Proof. Proof of Theorem 6.2 Let $A = \mathcal{R}(K)$. Let $x \in A$ be the element $x(z) = z$ for all $z \in K$. $\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda \in \sigma_A(x)\} = K$ (for $\lambda \notin K$, $\frac{1}{\lambda - z}$ is the inverse to $\lambda\mathbf{1} - x$).

Let f be holomorphic on some open set $\mathcal{U} \supseteq K$. Let $\Theta_x : \mathcal{O}(\mathcal{U}) \to A$ be given by Lemma 6.5. $\Theta_x(f)(w) = \delta_w\left(\Theta_x(f)\right) = f(\delta_w(x)) = f(w)$ for all $w \in K$. So $\Theta_x(f) = f|_K \in \mathcal{R}(K)$. So $\mathcal{O}(K) = \mathcal{R}(K)$.

Let us now fix Λ as in the statement of Theorem 6.2. Let B be the closed sub-algebra of A generated by $1, x, (\lambda\mathbf{1} - x)^{-1}, \lambda \in \Lambda$. So $B =$ closure in $\mathcal{C}(K)$ of rational functions with poles in Λ. By Theorem 5.11, $\sigma_B(x)$ is the union of $\sigma_A(x)$ and some of the bounded components of $\mathbb{C}\backslash K$. Since for any such component D there exists $\lambda \in \Lambda \cap D$, so $\lambda \cdot \mathbf{1} - x \in \mathcal{G}(A)$. So $\sigma_B(x) = \sigma_A(x)$. So $\Theta_x(f)$ takes values in B, i.e. $f|_K \in B$. □

Corollary 6.6. *Let Let $\mathcal{U} \subseteq \mathbb{C}$ be non-empty and open. Then the algebra $\mathcal{R}(\mathcal{U})$ of rational functions with no poles in $\mathcal{U}$ is dense in $\mathcal{O}(\mathcal{U})$.*

Proof. Let $f \in \mathcal{O}(\mathcal{U})$ and $\mathcal{V}$ be a neighbourhood of f in $\mathcal{O}(\mathcal{U})$. We need $\mathcal{V} \cap \mathcal{R}(\mathcal{U}) \neq \varnothing$.

Wlog, $\mathcal{V} = \{g \in \mathcal{O}(\mathcal{U}) : \|g - f\|_K < \epsilon\}$ for some non-empty, compact $K \subseteq \mathcal{U}$ and $\epsilon > 0$. Let $\widehat{K}$ be the union of K and those bounded components $\mathcal{D}$ of $\mathbb{C}\backslash K$ that are combined in $\mathcal{U}$.

If D is a bounded component of $\mathbb{C}\backslash\widehat{K}$, then D is a bounded component of $\mathbb{C}\backslash K$ s.t. $D\backslash\mathcal{U} \neq \varnothing$ so we can fix $\lambda_0 \in D\backslash\mathcal{U}$. Let Λ be the set of all these λ_0's. By Theorem 6.2, there exists rational function r s.t. $\|r - f\|_{\widehat{K}} < \epsilon$ and the poles of r are in Λ. Hence, $r \in \mathcal{V} \cap \mathcal{R}(K)$. □

Combining the above results, we can now embark on a proof of Theorem 6.1, which we started this section with.

Proof. Let Θ_x be as in lemma 6.5. Then for all $f, g \in \mathcal{R}(\mathcal{U})$, $\Theta_x(fg) \overset{Lemma 6.5\text{(ii)}}{=} (f \cdot g)(x)$ $= f(x)g(x) = \Theta_x(f)\cdot\Theta_x(g)$ and conclude by density of $\mathcal{R}(\mathcal{U})$ in $\mathcal{O}(\mathcal{U})$ and continuity that Θ_x is a homomorphism.

For uniqueness, assume $\Psi : \mathcal{O}(\mathcal{U}) \to A$ is a continuous unital homomorphism and $\psi(x) = x$. Then for all polynomials p, $\Psi(p) = p(x) = \Theta_x(p)$ and so for all rational $r \in \mathcal{R}(\mathcal{U})$ $\Psi(r) = r(x) = \Theta_x(r)$ and hence $\Psi \equiv \Theta_x$ by density and continuity. □

7 C^*-algebras

A C^*- algebra is a complex algebra A with an involution: a map $A \to A, x \mapsto x^*$ s.t.

(i) $(\lambda x + \mu y)^* = \overline{\lambda}x^* + \overline{\mu}y^*$

(ii) $(xy)^* = y^* x^*$

(iii) $x^{**} = x$

for all $x, y \in A, \lambda, \mu \in \mathbb{C}$. If A is unital, then $\mathbf{1}^* = \mathbf{1}$. A C^*-algebra is a Banach algebra with an involution s.t. the C^*-equation holds:

$$\|x^* x\| = \|x\|^2, \quad \text{for all } x \in A.$$

A complete algebra norm on a *-algebra that satisfies the C^*-equation is a C^*-norm. So a C^*-algebra is a *-algebra with a C^*-norm on it.

Lecture 21 **Remark.**

1. If A is a C^*-algebra, and $x \in A$, then $\|x^*\| = \|x\|$ ($\|x\|^2 = \|x^* x\| \leqslant \|x^*\| \cdot \|x\|$ so $\|x\| \leqslant \|x^*\|$ and hence $\|x^*\| \leqslant \|x^{**}\| = \|x\|$). So the involution is continuous.

   A Banach algebra with an involution s.t. $\|x^*\| = \|x\|$ for all x.

2. If A is a C^*-algebra and if A has a multiplicative identity $\mathbf{1} \neq 0$, then automatically A is a unital C^*-agebra, $\|\mathbf{1}\| = 1$ ($\|\mathbf{1}\|^2 = \|\mathbf{1}^*\mathbf{1}\| = \|\mathbf{1}\|$).

Definition 7.1. *A *-sub-algebra of a *-algebra A is a sub-algebra B of A s.t. for all $x \in B$, $x^* \in B$. A C^*-sub-algebra of a C^*-algebra is a closed *-algebra. So a C^*-sub-algebra of a C^*-algebra is a C^*-algebra. The closure of a *-algebra of a C^*-algebra is a *-sub-algebra, so a C^*-algebra.*

*A *-homomorphism between *-algebras is a homomorphism $\theta : A \to B$ s.t. $\theta(x^*) = \theta(x)^*$ for all $x \in A$. A *-isomorphism is a bijective *-homomorphism.*

Examples 7.2. *1. $\mathcal{C}(K)$, K compact Hausdorff, is a commutative, unital C^*-algebra with involution $f \mapsto f^*$, where $f^*(k) = \overline{f(k)}$ for all $k \in K$, $f \in \mathcal{C}(K)$.*

2. $\mathcal{B}(H)$, H Hilbert space is a unital C^*-algebra with involution $T \mapsto T^*$ where T^* is the adjoint.

3. Any C^*-sub-algebra of $\mathcal{B}(H)$, (H any Hilbert space) is a C^*-algebra.

...And that's all folks!

Remark. the Gelfand-Naimark Theorem says that if A is a C^*-algebra then there exists a Hilbert space H s.t. A is isometrically *-isomorphic to some C^*-sub-algebra of $\mathcal{B}(H)$. We will prove the commutative version.

Definition 7.3. *Let A be a C^*-algebra and $x \in A$. We say x is*

(i) hermitian or self-adjoint if $x^* = x$

(ii) unitary if (A is unital and) $x^* x = x x^* = \mathbf{1}$

(iii) normal if $x^* x = x x^*$

Examples 7.4. *1. $\mathbf{1}$ is both hermitian and unitary. In general, hermitian and unitary are normal.*

2. $f \in \mathcal{C}(K)$ is Hermitian $\iff$ $f(K) \subseteq \mathbb{R}$ and unitary $\iff$ $f(K) \subseteq S^1$. (Recall: $f(K) = \sigma_{\mathcal{C}(K)}(f)$).

Remark. 1. If A is a C^*-algebra and $x \in A$. Then there exist unique hermitian $h, k \in A$ s.t. $x = h + ik$. [If $x = h + ik$ then $x^* = h - ik$, so $h = \frac{x+x^*}{2}, k = \frac{x-x^*}{2i}$ and conversely, this choice for h, k works].

2. If A is a unital C^*-algebra and $x \in A$, then $x \in \mathcal{G}(A) \iff x^* \in \mathcal{G}(A)$ and in this case $(x^*)^{-1} = (x^{-1})*$.

   It follows that $\sigma_A(x^*) = \{\overline{\lambda} : \lambda \in \sigma_A(x)\}$ $(\lambda\mathbf{1} - x \in \mathcal{G}(A) \iff (\lambda\mathbf{1} - x)^* = \overline{\lambda} \cdot \mathbf{1} - x^* \in \mathcal{G}(A))$ so $\sigma_A(x^*) = \overline{\sigma_A(x)}$.

Lemma 7.5. *Let A be a C^*-algebra and $x \in A$. Then $r_A(x) = \|x\|$ provided x is normal.*

Proof. Assume x is hermitian. Then $\|x\|^2 = \left\|x^2\right\|$ and inductively, $\|x\|^{2^n} = \left\|x^{2^n}\right\|$ for all n. By the spectral radius formula (Theorem 5.10), $r_A(x) = \lim_{n\to\infty} \left\|x^{2^n}\right\|^{\frac{1}{2^n}} = \|x\|$.
If x is normal, then $\|x^*x\| = r_A(x^*x)$ because x^*x is hermitian.
Now, $r_A(x^*x) \leqslant r_A(x*)r_A(x) \overset{(Cor 5.19)}{\leqslant} \|x^*\| \cdot \|x\|$. But $\|x\|^2 = \|x^*x\|$. So we have equality throughout and so $\|x\| = r_A(x)$. □

Lemma 7.6. *Let A be a unital C^*-algebra and $x \in A$. Then $\phi(x^*) = \overline{\phi(x)}$ for all $\phi \in \Phi_A$.*

Proof. Wlog we can assume that x is hermitian. [For general x, write $x = h + ik$, h, k hermitian. Then $\phi(x^*) = \phi(h - ik) = \phi(h) - i\phi(k) = \overline{\phi(x)}$ $(\phi(h), \phi(k)$ real)]. Now assume x is hermitian $\phi \in \Phi_A$ and write $\phi(x) = a + ib$, $a, b \in \mathbb{R}$.

Need: For $t \in \mathbb{R}$,

$$\begin{aligned}
|\phi(x + it\mathbf{1})|^2 &= |a + i(b + t)|^2 \\
&= a^2 + (b + t)^2 = a^2 + b^2 + 2bt + t^2 \\
&\leqslant \|x + it\mathbf{1}\|^2 = \|(x + it)^*(x + it)\| \\
&= \|(x - it)^*(x + it)\| = \left\|x^2 + t^2\mathbf{1}\right\| \leqslant \left\|x^2\right\| + t^2.
\end{aligned}$$

Hence, $b = 0$. □

Corollary 7.7. *Let A be a unital C^*-algebra.*

(i) If $x \in A$ is hermitian, then $\sigma_A(x) \subseteq \mathbb{R}$.

(ii) If $x \in A$ is unitary, then $\sigma_A(x) \subseteq S^1$.

(iii) If B is a unital C^*-sub-algebra of A and $x \in B$ is normal then $\sigma_B(x) = \sigma_A(x)$.

Proof. (i) Let B = C^*-algebra generated by $\mathbf{1}, x$ (check *-sub-alg) $p(x) : p$ poly . B is commutative, so $\sigma)B(x) = \{\phi(x) : \phi \in \Phi_B\}$. By Lemma 7.6, $\sigma_A(x) \subseteq \sigma_B(x) \subseteq \mathbb{R}$.

(ii) Let B = C^*-algebra generated by by $\mathbf{1}, x, x^*$ = $\{p(x, x^*) : p$ poly in two variables $\}$. B is commutative, so $\sigma_B(x) = \{\phi(x) : \phi \in \Phi_B\}$. By Lemma 7.6, $1 = \phi(\mathbf{1}) = \phi(x^*x) = \overline{\phi(x)}\phi(x)$, hence $|\phi(x)|^2 = 1$. So $\sigma_A(x) \subseteq \sigma_B(x) \subseteq S^1$.

(iii) For the last part, assume $x \in B$ is hermitian. Then $\sigma_A(x) \subseteq \mathbb{R}$, so $\mathbb{C}\backslash\sigma_A(x)$ is connected. So it follows by Theorem 5.11 that $\sigma_A(x) = \sigma_B(x)$.

Now assume $x \in B$ is normal. Then for $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
\lambda\mathbf{1} - x \in \mathcal{G}(A) &\iff \lambda\mathbf{1} - x \in \mathcal{G}(A) \;\&\; (\lambda\mathbf{1} - x)^* \in \mathcal{G}(A) \\
&\overset{\text{commuting elements}}{\iff} (\lambda\mathbf{1} - x)(\lambda\mathbf{1} - x)^* \in \mathcal{G}(A) \\
&\overset{\text{hermitian}}{\iff} (\lambda\mathbf{1} - x)(\lambda\mathbf{1} - x)^* \in \mathcal{G}(B) \\
&\overset{\text{commuting elements}}{\iff} \lambda\mathbf{1} - x \in \mathcal{G}(B) \;\&\; (\lambda\mathbf{1} - x)^* \in \mathcal{G}(B) \\
&\iff \lambda\mathbf{1} - x \in \mathcal{G}(B).
\end{aligned}$$

□

Remark. $T \in \mathcal{B}(H)$, T hermitian or unitary, then $\sigma(T) = \partial\sigma(T) \subseteq \sigma_{\text{ap}}(T) =$ set of approximate evals. So $\sigma(T) = \sigma_{\text{ap}}(T)$ (also holds for normal operators).

Lecture 22 **Theorem 7.8.** *Let A be a commutative unital C^*-algebra. Then there exists compact, Hausdorff K s.t. A is isometrically isomorphic to $\mathcal{C}(K)$. In particular, the Gelfand map*

$$\begin{aligned} A &\to \mathcal{C}(\Phi_A) \\ x &\mapsto \widehat{x} \end{aligned}$$

*is an isometric *-isomorphism.*

Proof. By Theorem 5.22, the Gelfand map $G : A \to \mathcal{C}(\Phi_A)$ where $\mathcal{G}(x) = \widehat{x}|_{\Phi_A}$, is a unital homomorphism. It remains to check the following three properties:

G is a *-homomorphism: $\widehat{x^*}(\phi) = \phi(x^*) \overset{Lemma 7.6}{=} \overline{\phi(x)} = \overline{\widehat{x}(\phi)} = (\widehat{x})^*(\phi)$ for all $\phi \in Phi_A$.

G is isometric: $\|G(x)\| = \|\widehat{x}\|_\infty \overset{Thm 5.22(i)}{=} r_A(x) \overset{A \text{ commutative Lemma 7.5}}{=}$ for all $x \in A$.

G is surjective: let $\widehat{A}$ be the image of G. So $\widehat{A} = \{\widehat{x} : x \in A\}$. SInce G is an isometric unital *-homomorphism, it follows that $\widehat{A}$ is a closed sub-algebra of $\mathcal{C}(\Phi_A)$ containing the constant functions and closed under conjugation. Also $\widehat{A}$ separates points of Φ_A: if $\phi \neq \psi$ in Φ_A, then there exists $x \in A$ s.t. $\phi(x) \neq \psi(x)$, i.e. $\widehat{x}(\phi) \neq \widehat{x}(\psi)$. By Stone-Weierstrass, $\widehat{A} = \mathcal{C}(\Phi_A)$. □

Applications:

1. Let A be a unital C^*-algebra and let $x \in A$. Say x is positive if x is hermitain and $\sigma_A(x) \subseteq [0, \infty)$. We show there exists a unique positive $y \in A$ s.t. $y^2 = x$, called the square root of x, denoted $x^{\frac{1}{2}}$.

   Existence: $B = C^*-$sub-algebra generated by by $\mathbf{1}, x = \{p(x) : p \text{ poly }\}$. B is a commutative unital C^*-algebra. By Theorem 7.8, the Gelfand map

$$\begin{aligned} B &\to \mathcal{C}(\Phi_B) \\ w &\mapsto \widehat{w} \end{aligned}$$

   is an *-isomorphism. Now, we compute $\sigma_{\mathcal{C}(\Phi_B)}(\widehat{x}) \overset{Cor 5.18(ii)}{=} \sigma_B(x) \overset{Cor 7.7}{=} \sigma_A(x) \subseteq [0, \infty)$.

   The map $\phi \in \Phi_B$, $\phi \mapsto \sqrt{\widehat{x}(\phi)} \in \mathcal{C}(\Phi_B)$, so there exists a $y \in B$ s.t. $\widehat{y}(\phi) = \sqrt{\widehat{x}(\phi)}$ for all $\phi \in \Phi_B$. $\widehat{y^*} = (\widehat{y})^* = \overline{\sqrt{\widehat{x}}} = \sqrt{\widehat{x}} = \widehat{y}$.The Gelfand map is injective, so $y^* = y$, i.e. y is hermitian. Now, $\sigma_A(y) = \sigma_B(y) = \sigma_{\mathcal{C}(\Phi_B)}(\widehat{y}) \subseteq [0, \infty)$, so y is positive. Finally, $\widehat{y^2} = (\widehat{y})^2 = \widehat{x}$, so $y^2 = x$. Note that y is a limit of sequence of polynomials in x.

   Uniqueness: Assume $z \in A$ is positive and $z^2 = x$. Have $zx = xz = z^3$, so $zp(x) = p(x)z$ for all polynomials p, so $yz = zy$. Let $\tilde{B} = C^*-$sub-algebra generated by $\mathbf{1}, y, z$. Then $\tilde{B}$ is a commutative unital C^*-algebra containing $y, z, x = y^2 = z^2$. Theorem 7.8 gives that the

$$\begin{aligned} \tilde{B} &\to \mathcal{C}(\Phi_{\tilde{B}}) \\ w &\mapsto \widehat{w} \end{aligned}$$

   is an isometric *-isomorphism. $\sigma_{\mathcal{C}(\Phi_{\tilde{B}})}(\widehat{y}) = \sigma_{\tilde{B}}(y) = \sigma_A(y) \subseteq [0, \infty)$. Also, $\widehat{z}^2 = \widehat{z^2} = \widehat{x} = \widehat{y^2} = \widehat{y}^2$ and hence $\widehat{y} = \widehat{z}$ and thus $y = z$.

   This applies to a positive operator $T \in \mathcal{B}(H)$, where H is a Hilbert space (T is positive $\iff$ for all $x \in H\langle Tx, x\rangle \geqslant 0$).

2. Polar decomposition: let H be a Hilbert space, and $T \in \mathcal{B}(H)$ invertible. Then there exists unique operators $\mathcal{R}$,$\mathcal{U}$ s.t. $\mathcal{R}$ is positive, $\mathcal{U}$ is unitary and $T = \mathcal{R}\mathcal{U}$.

   Existence: TT^* is positive ($\langle TT^*x, x\rangle = \|T^*x\|^2 \geqslant 0$). Let $\mathcal{R} = (TT^*)^{\frac{1}{2}}$. So $\mathcal{R}^2 = TT^*$ is invertible, and hence so is $\mathcal{R}$ (being the product of $\mathcal{R}, \mathcal{R}$, commuting elements is invertible $\iff$ $\mathcal{R}$). Let $\mathcal{U} = \mathcal{R}^{-1}T$. Then $\mathcal{U}$ is invertible and $\mathcal{U}\mathcal{U}^* = \mathcal{R}^{-1}TT^*(\mathcal{R}^{-1})^*\ \mathcal{R}^{-1}\ \overset{R^2}{TT^*}\ \mathcal{R}^{-1} = \mathrm{Id}$.

   Uniqueness: if $T = \mathcal{R}\mathcal{U}$, $\mathcal{R}$ positive, $\mathcal{U}$ unitary, then $TT^* = \mathcal{R}\mathcal{U}\mathcal{U}^*\mathcal{R} = \mathcal{R}^2$ so $\mathcal{R} = \sqrt{TT^*}$ and $\mathcal{U} = \mathcal{R}^{-1}T$.

8 Borel Functional Calculus and Spectral Theory

Throughout we fix:

H non-zero, complex Hilbert space.

$\mathcal{B}(H)$ a bounded linear operator on H.

K compact, Hausdorff.

$\mathcal{B}$ Borel $\sigma-$field on K.

8.1 Operator-valued measures

Definition 8.1 (A resolution of the identity of H over K)**.** *A resolution of the identity of H over K (roti of H over K) is a map $P : \mathcal{B} \to \mathcal{B}(H)$ s.t.*

(i) $P(\varnothing) = 0$ and $P(K) = \mathrm{Id}$.

(ii) For all $E \in \mathcal{B}$ $P(E)$ is an orthogonal projection.

(iii) For all $E, F \in \mathcal{B}$ $P(E \cap F) = P(E) \circ P(F) = P(F) \circ P(E)$.

(iv) For all $E, F = \varnothing$. Then, $P(E \cup F) = P(E) + P(F)$.

(v) For all $x, y \in H$ the map $P_{x,y} : \mathcal{B} \to \mathbb{C}$ defined by $P_{x,y}(E) = \langle P(E)x, y\rangle$, $E \in \mathcal{B}$, is a regular complex Borel measure.

Examples 8.2. *Example: $H = L_2[0, 1]$, $K = [0, 1]$, $P(E)f = \mathbf{1}_E f$.*

Simple Properties:

(i) For all $E, F \in \mathcal{B}$ $P(E \cap E), P(F)$ commute (directly follows from (ii) above).

(ii) If $E \cap F = \varnothing$, then $P(E)(H) \perp P(F)(H)$. That is for all $x, y \in H$ $\langle P(E)x, P(F)y\rangle = \langle P(F) \cdot P(E)x, y\rangle\langle P\ (\overset{=\varnothing}{E \cap F})\ x, y\rangle = 0$.

(iii) For $x \in H$, $P_{x,x}$ is a positive measure of total mass $P_{x,x}(K) = \|x\|^2$. ($P_{x,x}(E) = \langle P(E)x, x\rangle = \langle P(E)^2x, x\rangle\ \langle P(E)x, P(E)x\rangle = \|P(E)x\|^2 \geqslant 0$, which equals $\|x\|^2$ if $E = K$).

(iv) p is finitely additive and for $x \in H$, $E \mapsto P(E)x : \mathcal{B} \to H$ is countably additive. That is, for $E_n \in \mathcal{B}, n \in \mathbb{N}$, $E_n \cap E_m = \varnothing$ for all $m \neq n$,

$$\begin{aligned}\left\langle \sum_{n\in\mathbb{N}} P(E_n)x, y \right\rangle &= \sum_{n\in\mathbb{N}}\langle P(E_n)x, y\rangle = \sum_{n\in\mathbb{N}} P_{x,y}(E_n)\\ &= P_{x,y}\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \left\langle P\left(\bigcup_{n\in\mathbb{N}} E_n\right)x, y\right\rangle\end{aligned}$$

for all $y \in H$ so

$$\sum_{n\in\mathbb{N}} P(E_n)x = P\left(\bigcup_{n\in\mathbb{N}} E_n\right)x.$$

Note that $\sum_{n\in\mathbb{N}} \|P(E_n)\|^2 \leqslant \|x\|^2$ be Bessel's inequality since $\left(P\left(\bigcup_{n\in\mathbb{N}} E_n\right)x\right)_{n\in\mathbb{N}}$ are pairwise orthogonal.

(v) P need not be countably additive, but if $P(E_n) = 0$ for all $n \in \mathbb{N}$ then $P\left(\bigcup_{n\in\mathbb{N}} E_n\right) = 0$.

(vi) For $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}$, consider the sequence $F_1 = E_1$, $F_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$, for $n > 1$, then

$$P\left(\bigcup_{n\in\mathbb{N}} E_n\right)x = P\left(\bigcup_{n\in\mathbb{N}} F_n\right)x = \sum_{n\in\mathbb{N}} P(F_n)x = 0, \qquad \text{for all } x \in H.$$

Lecture 23 **Definition 8.3** (The algebra $L_\infty(P)$)**.** *Let P be a resolution of H over K. Say a Borel function $f : K \to \mathbb{C}$ is called $P-$essentially bounded if there exists $E \in \mathcal{B}$ s.t. $P(E) = 0$ and f bounded on $K\setminus E$.*
Then define

$$\|f\|_\infty = \inf\{\|f\|_{K\setminus E} : E \in \mathcal{B}, P(E) = 0\},$$

which is attained (check!).
Let $L_\infty(P)$ be the set of all $P-$essentially bounded Borel functions $f : K \to \mathbb{C}$. This is a commutative, unital C^*-algebra with pointwise operations and $\|\cdot\|_\infty$ [As usual, we identify $f, g \in L_\infty(P)$ $P-$a.e., if there exists $E \in \mathcal{B}$ s.t. $P(E) = 0$, $f = g$ on $K\setminus E$].

Lemma 8.4. *Let P be as above. Then there exists an isometric, unital *-homomorphism $\Phi : L_\infty(P) \to \mathcal{B}(H)$ s.t.*

(i) $\langle \Phi(f)x, y\rangle = \int_K f dP_{x,y}$ *for all* $f \in L_\infty(P)$ *for all* $x, y \in H$.

(ii) $\|\Phi(f)x\|^2 = \int_K |f|^2 dP_{x,x}$ *for all* $f \in L_\infty(P)$ *for all* $x, y \in H$.

(iii) For $\mathcal{S} \in \mathcal{B}(H)$, $\mathcal{S}$ commutes with all $\Phi(f)$, $f \in L_\infty(P)$ $\iff$ $\mathcal{S}$ commutes with all $P(E)$, $E \in \mathcal{B}$.

Note: $\Phi(f)$ is uniquely determined by (i). We denote $\Phi(f)$ by $\int_K f dP$. So it says

$$\left\langle \int_K f dPx, y \right\rangle = \int_K f dP_{x,y}.$$

Proof. Sketch Define $\Phi(\mathbf{1}_E) = \int_K \mathbf{1}_E dP = P(E)$.

For simple functions $s = \sum_{j=1}^n a_j \mathbf{1}_{E_j}$, $\Phi(s) = \int_K s dP = \sum_{j=1}^n a_j P(E_j)$.

Φ is an isometric *-isomorphism, unital, on simple functions. Extend by density. □

Definition 8.5. *let $L_\infty(K)$ be the set of all bounded Borel functions $f : K \to \mathbb{C}$. This is a commutative unital C^*-algebra with pointwise operations and the sup-norm $\|f\|_K = \sup_{z\in K} |f(z)|$. If P is a resolution of the identity of H over K, then $L_\infty(K) \subseteq L_\infty(P)$ and the inclusion is a norm decreasing unital *-homomorphism.*

Theorem 8.6 (Spectral Theorem for commutative C^*-algebras)**.** *Let $A \subseteq \mathcal{B}(H)$ be a commutative unital C^*-algebra of $\mathcal{B}(H)$. Let $K = \Phi_A$. Then there exists a unique resolution of the identity of H over K, s.t.*

$$\int_K \widehat{T} dP = T, \qquad \textit{for all } T \in A.$$

Moreover,

(i) $P(\mathcal{U}) \neq \varnothing$ for any $\neq \varnothing$, open $\mathcal{U} \subseteq K$.

(ii) $\mathcal{S} \in \mathcal{B}(H)$ commutes with all $T \in A \iff \mathcal{S}$ commutes with all $P(E)$, $E \in \mathcal{B}$.

Proof. By Theorem 7.8 the Gelfand map

$$\begin{gathered} A \to \mathcal{C}(K) \\ x \mapsto \widehat{x} \end{gathered}$$

is an isometric *-isomorphism and hence so is its inverse

$$\begin{gathered} \mathcal{G}^{-1} : \mathcal{C}(K) \to A \\ \widehat{T} \mapsto T. \end{gathered}$$

We see a roti P over K which represents $\mathcal{G}^{-1} : \mathcal{G}^{-1}(\widehat{T}) = \displaystyle\int_K \widehat{T} dP$.

This is an operator version of the Riesz Representation Theorem, Theorem 2.11.

Uniqueness: $T = \displaystyle\int_K \widehat{T} dP$ for all T means

$$\langle Tx, y \rangle = \int_K \widehat{T} dP_{x,y}, \qquad \text{for all } T \in A, x, y \in H.$$

By uniqueness in the Riesz Representation Theorem (RRT), $P_{x,y}$ is uniquely determined for all $x, y \in H$. Since $P_{x,y}(E) = \langle P(E)x, y \rangle$, $P(E)$ is uniquely determined for all $E \in \mathcal{B}$.

Existence: For $x, y \in H$, $\widehat{T} \mapsto \langle Tx, y \rangle : \mathcal{C}(K) \to \mathbb{C}$ is in $\mathcal{M}(K) = \mathcal{C}(K)^*$ with norm at most $\|x\| \cdot \|y\|$. By RRT, there exists a unique $\mu_{x,y} \in \mathcal{M}(K)$ s.t.

$$\langle Tx, y \rangle = \int_K \widehat{T} d\mu_{x,y}, \qquad \text{for all } T \in A.$$

$\|\mu_{x,y}\|_1 \leqslant \|x\| \cdot \|y\|$. Now, by linearity

$$= \lambda \int_K \widehat{T} d\mu_{x,z} + \int_K \widehat{T} d\mu_{y,z}.$$

By uniqueness in the RRT, $\mu_{\lambda x + y, z} = \lambda \mu_{x,z} + \mu_{y,z}$. If $\widehat{T}$ is real-valued, then T is hermitian, so

$$\int_K \widehat{T} d\mu_{x,y} = \langle Ty, x \rangle = \overline{\langle Tx, y \rangle} = \int_K \widehat{T} d\overline{\mu_{x,y}}.$$

By uniqueness in the RRT, $\mu_{y,x} = \overline{\mu_{x,y}}$.
Fix $f \in L_\infty(K)$. Then $\Theta : H \times H \times \mathbb{C}$, $\Theta(x, y) = \displaystyle\int_K f d\mu_{x,y}$ is a sesquilinear form and $|\Theta(x, y)| \leqslant \|f\|_\infty \cdot \|\mu_{x,y}\|_1 \leqslant \|f\|_\infty \cdot \|x\| \cdot \|y\|$. So there exists $\Psi(f) \in \mathcal{B}(H)$ s.t. $\langle \psi(f)x, y \rangle = \Theta(x, y) = \displaystyle\int_K f d\mu_{x,y}$ and $\|\Psi(f)\| = \|\Theta\| \leqslant \|f\|_K$.

We now have a map $\Psi : L_\infty(K) \to \mathcal{B}(H)$ s.t.
Ψ is linear: clear by the linearity of $\displaystyle\int_K f d\mu_{x,y}$.

Ψ is bounded: $\|\Psi(f)\| \leqslant \|f\|_K$.

Ψ is a *-map:

$$\begin{aligned}\langle \Psi(\overline{f})x, y\rangle &= \int_K \overline{f} d\mu_{x,y} = \overline{\int_K f d\mu_{y,x}} \\ &= \overline{\langle \Psi(f)y, x\rangle} = \langle x, \Psi(f)y\rangle \\ &= \langle \Psi^*(f)x, y\rangle, \quad \text{for all} x, y \in H.\end{aligned}$$

So $\Psi(\overline{(f)}) = \Psi(f)^*$.

$\Psi\big|_{\mathcal{C}(K)} = \mathcal{G}^{-1}$: have $\langle \Psi(\widehat{T})x, y\rangle = \int_K \widehat{T} d\mu_{x,y} = \langle Tx, y\rangle$ for all x, y. So $\Psi(\widehat{T}) = T = \mathcal{G}^{-1}$.

Ψ is multiplicative: for $S, T \in A$.

$$\begin{aligned}\int_K \widehat{S}\cdot\widehat{T} d\mu_{x,y} &= \int_K \widehat{ST} d\mu_{x,y} \\ &= \langle STx, y\rangle \\ &= \int_K \widehat{S} d\mu_{Tx,y}, \quad \text{S} \in A.\end{aligned}$$

By uniqueness in RRT, $\widehat{T} d\mu_{x,y} = d\mu_{Tx,y}$ as measures. Hence,

$$\begin{aligned}\int_K f\widehat{T} d\mu_{x,y} &= \int_K f d\mu_{Tx,y} = \langle \Psi(f)Tx, y\rangle \\ &= \langle Tx, \Psi(f)^* y\rangle = \int_K \widehat{T} d\mu_{x,\Psi(f)^*y}, \quad \text{for all } T \in A, f \in L_\infty(K).\end{aligned}$$

By uniqueness in RRT, $f d\mu_{x,y} = d\mu_{x,\Psi(f)^*y}$. Finally, for $g \in L_\infty(K)$,

$$\begin{aligned}&\int_K gf d\mu_{x,y} \\ &= \int_K g d\mu_{x,\Psi(f)^*y} \\ &= \langle \Psi(gf)x, y\rangle \\ &= \langle \Psi(g)x, \Psi(f)^* y\rangle \\ &= \langle \Psi(f)\Psi(g)x, y\rangle, \quad \text{for all } x, y \in H.\end{aligned}$$

So $\Psi(fg) = \Psi(f)\cdot\Psi(g)$.

Define $P(E) = \Psi(\mathbf{1}_E)$. Easy to check P is a roti of H over K. $P_{x,x}(E) = \langle P(E)x, y\rangle = \int_K \mathbf{1}_E d\mu_{x,y} = \mu_{x,y}(E)$ for all $E \in \mathcal{B}$. So $P_{x,y} = \mu_{x,y}$. Also,

$$\begin{aligned}\left\langle \int_K \widehat{T} dP_{x,y} \right\rangle &= \int_K \widehat{T} dP_{x,y} \\ &= \langle \Psi(\widehat{T})x, y\rangle \\ &= \langle Tx, y\rangle.\end{aligned}$$

So $\int_K \widehat{T} dP = T$. (Without Lemma 8.4, could define $\int_K f dP = \Psi(f)$ for $f \in L_\infty(K)$).

(i) Fix $\mathcal{U} \subseteq K$, $\mathcal{U}$ open. By Urysohn, there exists $f : K \to [0,1]$ continuous, s.t. $\operatorname{supp} f \subseteq \mathcal{U}$, $f \neq 0$.
There exists $T \in A$, $\sqrt{f} = \widehat{T}$. Then $T \neq 0$ so there exists $x \in H$ s.t. $Tx \neq 0$. $0 < \|Tx\|^2 = \langle T^2 x, x\rangle = \int_K \widehat{T^2} dP_{x,x} = \int_K f dP_{x,x} \leqslant P_{x,x}(\mathcal{U}) = \langle P(\mathcal{U}x), x\rangle$. So $P(\mathcal{U}) \neq 0$.

(ii) Let $\mathcal{S} \in \mathcal{B}(H)$. $\langle \mathcal{S}Tx, y\rangle = \langle Tx, \mathcal{S}^* y\rangle = \int_K \widehat{T} dP_{x,\mathcal{S}^*y}$ and $\langle T\mathcal{S}x, y\rangle = \int_K \widehat{T} dP_{\mathcal{S}x,y}$.

So

$$\begin{aligned}\mathcal{S}T = T\mathcal{S} \text{ for all } T \in A &\iff P_{x,\mathcal{S}^*y} = P_{\mathcal{S}x,y} \text{ for all } x, y \in H. \\ &\iff \langle P(E)x, \mathcal{S}^* y\rangle = \langle P(E)\mathcal{S}x, y\rangle \text{ for all } x, y \in H, E \in \mathcal{B}. \\ &\iff \mathcal{S}P(E) = P(E)\mathcal{S} \text{ for all } E \in \mathcal{B}.\end{aligned}$$

□

Lecture 24

Let A be a united Banach algebra and $x \in A$. We define $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $(x^0 = 1)$ (converges absolutely, so converges in A). If $xy = yx$ in A, then $e^{x+y} = e^x \cdot e^y$.

Lemma 8.7 (Fugledo-Putman-Rosenblum). *Let A be a unital C^*-algebra, $x, y, z \in A$ with x, y normal. If $xz = zy$, then $x^*z = zy^*$.*

Proof. None given. □

Theorem 8.8 (Spectral Theorem for normal operators). *Let $T \in \mathcal{B}(H)$ be normal. Then there exists a unique resolution of the identity of H over $\sigma(T) = \sigma_{\mathcal{B}(H)}(T)$, P s.t. $T = \int_{\sigma(T)} \lambda dP$ (i.e. the spectral decomposition of T). Moreover, $\mathcal{S} \in \mathcal{B}(H)$ commutes with T $\iff$ $\mathcal{S}$ commutes with all $\overbrace{P(E)}^{\text{spectral projections}}$, $E \in \mathcal{B}$.*

Proof. Let A be the unital C^*-sub-algebra of $\mathcal{B}(H)$ generated by T. So $A = \overline{\{p(T, T^*) : p \text{ poly in two variables}\}}$. T normal implies that A is a commutative C^*-sub-algebra. $\sigma_A(T) = \sigma(T)$ by Corollary 7.7. By Lemma 7.6, every $\phi \in \Phi_A$ is uniquely determined by $\phi(T)$. $[\phi(T^*) = \overline{\phi(T)}, \phi(p(T, T^*)) = p(\phi(T), \phi(T^*))]$. Thus, the map

$$\begin{aligned} \Phi_A &\to \sigma(T) \\ \phi &\mapsto \phi(T) \end{aligned}$$

is a continuous bijection (Corollary 5.18) and thus a homeomorphism. $\widehat{T}, \widehat{T^*}$ in $\mathcal{C}(\Phi_A)$ correspond to $\lambda \mapsto \lambda$, $\lambda \mapsto \overline{\lambda}$ in $\mathcal{C}(\sigma(T))$ respectively.

Existence of P: follows from Theorem 8.6.

Uniqueness: if $T = \int_{\sigma(T)} \lambda dP$, then $p(T, T^*) = \int_{\sigma(T)} p(\lambda, \overline{\lambda}) dP$ (Lemma 8.4). So $\langle p(T, T^*)x, y\rangle = \int_{\sigma(T)} p(\lambda, \overline{\lambda}) dP_{x,y}$. Since, $\lambda \mapsto p(\lambda, \overline{\lambda})$ are dense in $\mathcal{C}(\sigma(T))$, by uniqueness in RRT, $P_{x,y}$ are uniquely determined and hence so is P.

If $\mathcal{S}T = T\mathcal{S}$, then $\mathcal{S}T^* = T^*\mathcal{S}$ by Lemma 8.7. Finally, $\mathcal{S}T = T\mathcal{S} \iff \mathcal{S}$ commutes with all elements of A, $\iff \mathcal{S}$ commutes with $P(E)$, for all in $E \in \mathcal{B}$ (Theorem 8.6). □

Theorem 8.9 (Borel Functional Calculus). *Let T be a normal operator, let $K = \sigma(T)$ and P be the roti of H over K given by Theorem 8.8. The map*

$$\begin{aligned} L_\infty(K) &\to \mathcal{B}(H) \\ f &\mapsto f(T) := \int_{\sigma(T)} f(\lambda) dP \end{aligned}$$

has the following properties:

*(i) it is a unital *-homomorphism s.t. $z(T) = T$ (where $z(\lambda) = \lambda$ for all $\lambda \in K$).*

(ii) $\|f(T)\| \leqslant \|f\|_K$ for all $f \in L_\infty(K)$ with equality if $f \in \mathcal{C}(K)$.

(iii) For $\mathcal{S} \in \mathcal{B}(H)$, $\mathcal{S}T = T\mathcal{S} \iff \mathcal{S}f(T) = f(T)\mathcal{S}$ for all $f \in L_\infty(K)$.

(iv) $\sigma(f(T)) \subseteq \overline{f(K)}$ for all $f \in L_\infty(K)$.

Proof. Everything follows from Lemma 8.4, Theorems 8.6 and 8.9. (Note that $f(T) = \Psi(f)$ from Theorem8.6). For (iv), $\sigma(f(T)) \subseteq \sigma_{L_\infty(K)}(f) = \overline{f(K)}$. □

Theorem 8.10 (Polar Decomposition)**.** *Let $T \in \mathcal{B}(H)$ be normal. Then, there exists a positive operator $\mathcal{R}$, unitary $\mathcal{U}$ s.t. $T = \mathcal{R}\mathcal{U}$. Also, $T, \mathcal{R}, \mathcal{U}$ pointwise commute.*

Proof. Define r, u on $\sigma(T)$:

$$r(\lambda) = |\lambda|,\ u(\lambda) = \begin{cases} \frac{\lambda}{|\lambda|}, & \lambda \neq 0 \\ 1, & \text{if } \lambda = 0 \in \sigma(T). \end{cases}$$

Then, $r, u \in L_\infty(\sigma(T))$ and $ru = z$ ($z(\lambda) = \lambda$ for all $\lambda \in \sigma(T)$) let $\mathcal{R} = r(T), \mathcal{U}$. Then $T = Z(T) = \mathcal{R}\mathcal{U}$. r is positive, u is unitary in $L_\infty(\sigma(T))$ and hence $\mathcal{R}$ is positive, $\mathcal{U}$ is unitary in $\mathcal{B}(H)$. Since $L_\infty(K)$ is commutative, $\mathcal{R}, \mathcal{U}, T$ must commute. □

Theorem 8.11 (Unitaries as exponentials)**.** *Let $\mathcal{U} \in \mathcal{B}(H)$ be unitary. Then there exists hermitian Q s.t. $\mathcal{U} = e^{iQ}$.*

Proof. By Corollary 7.7, $\sigma(u) \subseteq S^1$. Let $f : S^1 \to \mathbb{R}$ be in $L_\infty(S^1)$ s.t. $e^{if(t)} = t$ for all $t \in S^1$. Let $Q = f(\mathcal{U})$. Then Q is hermitian since f is hermitian in $L_\infty(K)$.

$$\sum_{k=0}^{n} \frac{(if(t))^k}{k!} \to t, \qquad \text{uniformly on } S^1.$$
$$\sum_{k=0}^{n} \frac{(iQ)^k}{k!} \to \mathcal{U},$$

i.e. $\mathcal{U} = e^{iQ}$. □

Theorem 8.12 (Connectedness of $\mathcal{G}(\mathcal{B}(H))$)**.** *Fix $T \in \mathcal{G}(\mathcal{B}(H))$. $T = \mathcal{R}\mathcal{U}$, $\mathcal{R}$ positive, $\mathcal{U}$ unitary (Theorem 8.10) where $\mathcal{R}, \mathcal{U} \in \mathcal{G}(\mathcal{B}(H))$.*

Proof. Since $\mathcal{R}$ is invertible, $\sigma(\mathcal{R}) \subseteq (0, \infty)$. Let $\mathcal{S} = \log(\mathcal{R}) = \displaystyle\int_{\sigma(\mathcal{R})} \log \lambda dP$ (P is a roti of H over K).

$$e^{\mathcal{S}} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\mathcal{S})^k}{k!} = \lim_{n\to\infty} \overbrace{\sum_{k=0}^{n} \frac{(\log\lambda)^k}{k!}}^{\to \text{ uniformly on } \sigma(\mathcal{R})} (\mathcal{R}) = z(\mathcal{R}) = \mathcal{R}.$$

So $T = e^{\mathcal{S} \cdot e^{iQ}}$. The map $[0,1] \to \mathcal{G}(\mathcal{B}(H)) : t \mapsto e^{t\mathcal{S}} \cdot e^{itQ}$ is a continuous path from Id to T. Hence $\mathcal{G}(\mathcal{B}(H))$ is connected. □

End of lecture course.