QUANTITATIVE BROWNIAN REGULARITY OF THE KPZ FIXED POINT WITH ARBITRARY INITIAL DATA

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Abstract. We show that the spatial increments of the KPZ fixed point starting from arbitrary initial data, exhibit strong quantitative comparison against rate two Brownian motion on compacts. The above estimates are uniform for uniformly bounded continuous, compactly supported initial data and countably many narrow wedges with supports contained in a fixed compact set.

Contents

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1. Introduction

1.1. **Motivation.** In 1986, Kardar, Parisi and Zhang [KPZ86] predicted that many planar random growth processes possess universal scaling behaviour. In particular, models in the KPZ universality class have an analogue of the height function which is conjectured to converge at large time and small length scales under the KPZ $1:2:3$ scaling (i.e. $h(t,x) \mapsto \epsilon h(\epsilon^{-3}t, \epsilon^{-2}x)$, as $\epsilon \searrow 0$) to a universal object $h_t(\cdot)$ called the KPZ fixed point. Matetski-Quastel-Remenik [MQR16] constructed the KPZ fixed point as a Markov process in t, and they showed that it is a limit of the height function evolution of the totally asymmetric simple exclusion process (TASEP) with arbitrary initial condition. Later in [NQR20], Nica-Quastel-Remenik constructed the KPZ fixed point as a scaling limit of Brownian last passage percolation (LPP). For an introduction to the KPZ universality class, see [Qua11], [FS10], [Rom15], [Cor16], [WFS17], [Gan21] and [Zyg22].

The directed landscape $\mathcal{L}$ was constructed from Brownian last passage percolation (BLPP) in [DJOBV22] as a four-parameter scaling limit of the Brownian last passage value from different spatial locations and curves in the Brownian environment. It is conjectured to be the full scaling limit of all KPZ models. Reinforcing this claim, in [DZ25], the authors provide a general framework for proving convergence to the directed landscape and apply it to a range of models, proving convergence. It is a random continuous function from

$$\mathbb{R}^4_\uparrow = \{(p;q) = (x,s;y,t) \in \mathbb{R}^4 : s < t\}$$

to $\mathbb{R}$. They showed that the KPZ fixed point also admits a variational formula in terms of the directed landscape, where for initial data $h_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ the KPZ fixed point can be expressed as

$$h_t(y) = \sup_{x\in\mathbb{R}} (h_0(x) + \mathcal{L}(x,0;y,t))\,,$$

for all $y \in \mathbb{R}$ almost surely. This, and the metric composition law inherited from Brownian LPP, means the directed landscape can be interpreted as a stochastic semi-group. For the narrow wedge initial condition, $h_0(0) = 0$ and $h_0(x) = -\infty$ elsewhere, $h_1(\cdot) = \mathcal{A}_1(\cdot)$ is the parabolic Airy$_2$ process, that is the top line of the Airy line ensemble. For $h_0 \equiv 0$, the flat initial condition, $h_1(\cdot)$ is called the Airy$_1$ process.

The directed landscape at unit time $\mathcal{L}(\cdot,0;\cdot,1)$, is also called the *Airy sheet*, and denoted by $\mathcal{S}(\cdot\,,\,\cdot)$. In [DJOBV22], the authors obtain a coupling between the Airy sheet and the differences in last passage values with respect to the Airy line ensemble.

In [SV21] the authors show that the spatial increments of the KPZ fixed point at any fixed time for general initial data are absolutely continuous with respect to Brownian motion on compacts. One would like to know for which $p \in (1,\infty)$, the Radon-Nikodym derivative of spatial increments of the KPZ fixed point is in L^p. This would be a desirable property to have since it would quantitatively strengthen the relationship between low-probability events of Brownian motion and that of the KPZ fixed point [CHH19]. More generally, in the setting of two finite measures $\mu \ll \nu$ (μ absolutely continuous with respect to ν), one wants if possible to quantify the relationship between the $\delta > 0$ and $\epsilon > 0$ so that the implication $\nu(A) < \delta$ guarantees $\mu(A) < \epsilon$ for all measurable A[1]. This can be achieved if, for instance, one imposes that the Radon-Nikodym derivative $\mathrm{d}\mu/\,\mathrm{d}\nu \in L^p(\nu)$, for some $p > 1$. Then, for A measurable,

$$\mu(A) = \int_A \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\,\mathrm{d}\nu \le \Big(\int_A \Big(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)^p d\nu\Big)^{\frac{1}{p}} (\nu(A))^{\frac{p-1}{p}} = \Big\|\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big\|_{L^p(\nu)} (\nu(A))^{1-\frac{1}{p}}\,, \tag{1.1}$$

by applying Hölder's inequality. One can also easily verify that the above inequality is also sufficient to deduce that the Radon-Nikodym derivative exists and $\mathrm{d}\mu/\,\mathrm{d}\nu \in L^p$. One can relax

[1] Recall the definition of absolute continuity of measures μ with respect to ν, namely, that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all A measurable, if $\nu(A) < \delta$, then $\mu(A) < \epsilon$.

this type of inequality and impose the following comparison of two measures for all A measurable (in an appropriate measure space) for some Borel function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\lim_{t \searrow 0} f(t) = 0$,

$$\mu(A) = O(f(\nu(A)))\,. \tag{1.2}$$

When ν is replaced with various restrictions of the Wiener measure on compacts, we will call this type of estimate a form of *quantitative Brownian regularity* with *rate function* f.

The variational characterisation of the KPZ fixed point and the coupling of the Airy sheet with the Airy line ensemble and the so-called Gibbs property enjoyed by the Airy line ensemble, together imply that a question on Brownian absolute continuity of the KPZ fixed point can ultimately be transferred to that of an 'inhomogeneous' Brownian LPP, see Definition 3.6. This was done in [SV21, Theorem 4.3], where it was shown that away from zero, inhomogeneous Brownian LPP is absolutely continuous with respect to Brownian motion on compacts. A quantitative Brownian regularity of the KPZ fixed point would thus require, as a first step, a strong control on the Radon-Nikodym derivative of the inhomogeneous BLPP with respect to Brownian motion. This is established in our companion paper [TS25, Theorem 7.1] (stated here as Theorem 3.7), where it is shown that the Radon-Nikodym derivative of the law of the spatial increments (with endpoints away from zero) of the inhomogeneous BLPP against the Wiener measure μ on compacts is in $L^{\infty-}(\mu)$, and in particular, that for any fixed $p > 1$, one has that the L^p norm is at most of the order $O_p(\mathrm{e}^{d_p m^2 \log m})$ for some p-dependent constant $d_p > 0$.

Before proceeding further, we need to discuss a bit about the initial condition of the KPZ fixed point. First, for $t > 0$, recall the definition of *t-finitary* initial data.

Definition 1.1. *(t-finitary) Let $h_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a locally bounded measurable function such that*

$$\lim_{|x| \to \infty} \frac{h_0(x) - x^2/t}{|x|} = -\infty\,.$$

This condition on the initial data (for any $t > 0$ fixed) is both necessary and sufficient to guarantee that the KPZ fixed point (at time $t > 0$) is globally finite, see [SV21, Proposition 6.1].

Next, we need an appropriate definition of 'support' compatible with the 'max-plus' nature of the directed landscape.

Definition 1.2. *(max-plus support) Let $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a Borel function. We define the* ***max-plus*** *support of f to be the set*

$$\mathrm{supp}_{-\infty}(f) := \{x \in \mathbb{R} : f(x) \neq -\infty\}\,.$$

Using the definition of the Airy sheet, the **KPZ fixed point** at unit time $h_1(\cdot)$ starting from initial data $h_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ with max-plus support $\mathrm{supp}_{-\infty}(h_0)$ and 1-finitary, has the following variational representation

$$h_1(y) = \sup_{x \in \mathrm{supp}_{-\infty}(h_0)} (h_0(x) + \mathcal{L}(x, 0; y, 1))\,.$$

Now, the coupling of the Airy sheet with the Airy line ensemble, see 4.2, allows us to express the KPZ fixed point (at unit time) on a compact interval $[1, y_0]$, for $y_0 > 1$ as

$$h(y) \;=\; \max_{x \in \mathrm{supp}_{-\infty}(h_0)} (h_0(x) + \mathcal{S}(x, y))\,, \qquad \text{for } y \in [1, y_0]\,. \tag{1.3}$$

The right hand side of the equation (1.3) can be expressed as a maximisation problem of a random depth of last passage percolation values over the Airy line ensemble with random boundary data.

Now, what remains is to obtain quantitative control of the random depth and the boundary data. Both require a refinement of the picture of the geodesic geometry in the Airy line ensemble. The former will amount to obtaining concentration bounds for the transversal fluctuations of semi-infinite geodesics, which is Theorem 4.5. The latter will provide control on semi-infinite geodesic coalescence in the Airy line ensemble, see Definition 4.3.

Now we state the main result of the paper informally. See Theorem 6.6 for the proper statement of the result.

Theorem 1.3 (Quantitative Brownian regularity)**.** *The spatial increments of the KPZ fixed point started from arbitrary (finitary in the above sense) initial data on a fixed interval, exhibit a form of quantitative Brownian regularity with rate function (as defined in* (1.2)*) of the form*

$$f(\nu(A)) = \exp\left(-d\log^r \log\left(1/\nu(A)\right)\right) ,$$

for all A Borel measurable sets on paths and some positive constants $d > 0, r \in (0,1)$, where ν denotes an appropriate restriction of the rate two Wiener measure.

We also show that when the ‘max-plus’ support of the initial data is contained in a fixed reference compact set and is countable and the data itself is uniformly pointwise bounded, the aforementioned quantitative Brownian comparison holds uniformly. Note that this set of initial data includes compactly supported continuous functions and countably many narrow wedges. This is the content of Theorem 6.5.

We believe that the bounds in Theorem 1.3 can be improved to the point where the exponents $r > 1$ and $d > 0$ can be tuned appropriately to reduce to the type of bound as in (1.1), which would give the desired $L^{\infty-}(\nu)$ control.

1.2. **Organisation of the paper.** First, in Section 2 we establish notation that will be used throughout. In Section 3 we provide necessary background material including estimates of Radon-Nikodym derivatives of the laws of Brownian bridges against that of Brownian motion and some path-wise properties of the Airy line ensemble and its last passage values. Section 4 is devoted to studying geodesic geometry in the Airy line ensemble. In particular, we obtain *exponentially stretched* tail bounds on intercepts of semi-infinite geodesics, Theorem 4.5 and also *uniform* coalescence time tail bounds for semi-infinite geodesics with ‘sufficiently concentrated’ asymptotic directions (or ‘speeds’), see Definition 4.8 and Theorem 4.14. Then, in Section 5, we use the variational formula for the KPZ fixed point and the coupling in Definition 4.2 and rely on a series of favourable events and technical inputs from [Wu25], thereby reducing the problem to estimating the Radon-Nikodym derivatives of inhomogeneous Brownian LPP with non-decreasing initial data. For this we use [TS25, Theorem 7.1] as a crucial input to obtain an analytically tractable quantitative comparison of finite depth truncations of the KPZ fixed point against the rate two Wiener measure in Theorem 5.5. In Section 6 we combine the above estimates to obtain the quantitative Brownian regularity of the KPZ fixed point started from compactly initial data with countable support, namely Theorem 6.5. It is here where the tail bounds on the transversal fluctuations and control on coalescence depths of semi-infinite geodesics play a critical role. Then, by a localisation argument, we extend this quantitative comparison to all finitary initial data, see Theorem 6.6. Finally, in Section 7 we briefly outline possible avenues of strengthening the comparison of the KPZ fixed point with respect to Brownian motion, specifically in refining our understanding of the geodesic geometry in the Airy line ensemble and its last passage values. Below is a flowchart depicting the main ingredients in the proof of Theorem 6.6.

1.3. **Related works.** The Brownian nature of models in the KPZ universality class, including the Airy Line ensemble and the KPZ fixed point, has been a subject of intense research in recent times. Aside from integrable inputs, see for instance [BDJ99, MQR16, Liu19] and [JR19, Joh17, Joh18], probabilistic and geometric methods have featured prominently ever since Corwin and Hammond proved in [CH14] that the parabolic Airy line ensemble admits a Brownian

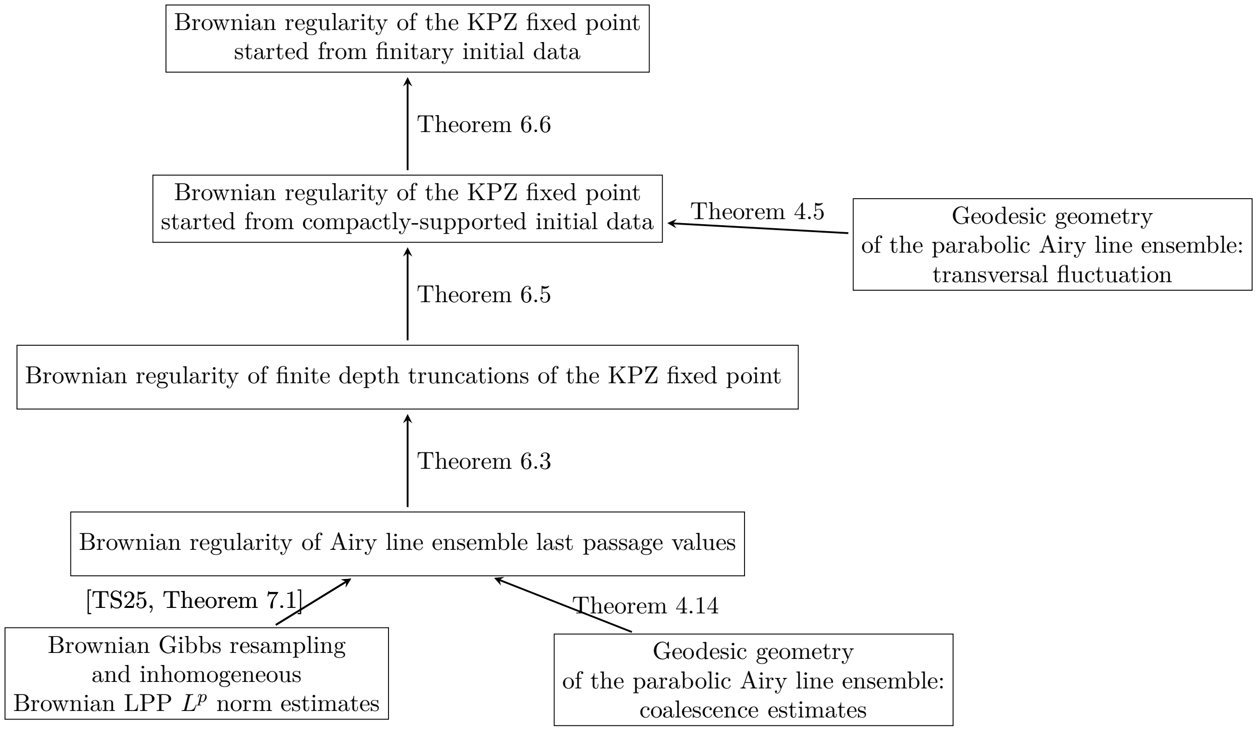

FIGURE 1. Flowchart of main steps in the proof of Theorem 6.6.

Gibbs resampling property. For a more detailed account of recent developments, one can consult the work of Calvert, Hammond and Hegde [CHH19] and the references therein.

One version of local Brownianness is to show that the local limits of the Airy_2 process (the narrow wedge solution to the KPZ fixed point at unit time) converge in law to a Brownian motion, [Häg08], [CP15], [QR13]. In fact, [QR13] also establishes Hölder 1/2- continuity of the Airy_2 and Airy_1 processes (solution to KPZ fixed point at unit time started from flat, i.e. $h_0 \equiv 0$ initial data). The Hölder 1/2- continuity and the locally Brownian nature (in terms of convergence of the finite dimensional distributions) were established in [MQR16]. Such Hölder continuity results and local limits for certain initial conditions have also been established in [Pim18] and [Pim20] (see also [Joh17], [Joh18]). A stronger notion of the locally Brownian nature is absolute continuity with respect to Brownian motion on compact intervals, which we call Brownian on compacts. That the Airy_2 process is Brownian on compacts was first proved in [CH14] using the Brownian Gibbs property.

Very recently, the locally Brownian nature of the Airy line ensemble (and so for Airy_2 process) has been considerably strengthened in [Dau24], where Dauvergne gave an explicit form for the density of the finite depth truncations of increments of the Airy line ensemble against Brownian motion on compacts and established ways of estimating inverse acceptance probabilities following ideas from the 'tangent method'. This geometric approach, first introduced in [Agg22] was also used in [GH24] to provide sharp estimates for the one-point density of the KPZ fixed point started from fairly general initial data at the origin. Also, in [Wu25], Wu introduced ideas from the theory of optimal transportation to the study of spatial regularity of the Airy line ensemble and has provided sub-Gaussian tail bounds (with universal coefficients) on the modulus of continuity of any given level of the Airy line ensemble on a compact interval.

However, for general initial conditions, the picture is less clear with more questions open. A result providing a more quantitative notion of Brownian regularity, called *patchwork quilt of*

Brownian fabrics, was established in Hammond [Ham19] and [CHH19]. Roughly the result states that the KPZ fixed point $h(\cdot)$ on a unit interval is the result of 'stitching' a random number of profiles, where each profile is absolutely continuous with respect to a Brownian motion with Radon-Nikodym derivative in L^p for all $p < 3$. The authors conjectured (Conjectured 1.3 in [Ham19]) that one can dispense with these random patches and establish L^p estimates for all $p > 1$ for the Radon-Nikodym derivative, a problem which remains open. A first step in this direction was the proof of absolute continuity on a single non-random patch for general initial conditions, which has been established in [SV21, Theorem 1.2], using methods different from those in [Ham19].

Our main result in Theorem 6.5 of this paper strengthens quantitatively the absolute continuity result of [SV21] for finitarty initial data (see Definitions 1.2 and 1.1) within a *single* patch. Our proof of this result crucially depends upon refining certain aspects of the construction of the directed landscape in [DJOBV22], the variational characterisation of the KPZ fixed point from [SV21], the Brownian Gibbs property of the parabolic Airy line ensemble established in [CH14], the strong comparison against Brownian motion on compacts of inhomogeneous Brownian LPP (the Radon-Nikodym derivative of the law of the spatial increments against the Wiener measure μ on compacts being in $L^{\infty-}(\mu)$) established in [TS25] (stated here as Theorem 3.7), as well as technical inputs from [DV21] and [Wu25] used in estimating Brownian inverse acceptance probabilities with random boundary points and global modulus of continuity estimates for the stationary version of the Airy line ensemble respectively.

Geodesic coalescence in the Airy line ensemble also features prominently in our work. More generally, one can consider such coalescence in other geodesic environments; it is a well-studied and significant problem in probability. They have been studied in various models of the KPZ universality class: the prelimiting models of the last passage percolation [Pim16], [BSS18], [SS20], [Zha20], [BF22], in polymer models [RASS23], [GHZ23] and in the related first passage percolation models [New95], [DEP24] and in first passage percolations in higher dimensions [Ale23]. Geodesic networks are widely investigated in other random surface models, like the Brownian map [MQ23], [AKM17], [Gal10] and the Liouville quantum gravity [Gwy21], that arise as scaling limits of random planar maps.

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2. Notation

We introduce some notation and conventions we will be using throughout.

When in some estimates a constant appears that will depend on some parameters $a, b, c, \cdots$, it will be denoted by $C_{a,b,c,\cdots}$, unless otherwise specified. Constants without subscripts are deemed to be universal. Additionally, for ease of notation, such constants are allowed to change from line to line. Moreover, for ease of notation such constants may be dropped and instead replaced with the symbols $\lesssim_{a,b,c,\cdots}$ ($\equiv O_{a,b,c,\cdots}(\cdot)$) and $\gtrsim_{a,b,c,\cdots}$ for some parameters $a, b, c, \cdots$ which stand for $\leq C_{a,b,c,\cdots}$ and $\geq C'_{a,b,c,\cdots}$ for some positive constants $C_{a,b,c,\cdots}, C'_{a,b,c,\cdots}$ respectively.

We take the set of natural numbers $\mathbb{N}$ to be $\{1, 2, \ldots\}$. For $k \in \mathbb{N}$, we use an underbar to denote a k-vector, that is, $\underline{x} \in \mathbb{R}^k$. We denote the integer interval $\{i, i+1, \ldots, j\}$ by $[\![i, j]\!]$. A k-vector $\underline{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ is called a k-decreasing list if $x_1 > x_2 > \ldots > x_k$. For a set $I \subseteq \mathbb{R}$, let $I^k_> \subseteq I^k$ be the set of k-decreasing lists of elements of I, and $I^k_\geq$ be the analogous set of k-non-increasing lists.

The symbols $\cdot \wedge \cdot, \cdot \vee \cdot$ denote $\min\{\cdot, \cdot\}$ and $\max\{\cdot, \cdot\}$ respectively. For any $a \in \mathbb{R}$, a_+ denotes $a \vee 0$.

We define the affinely shifted bridge version, that is zero at both endpoints, of a real-valued function f on an interval $[a,b]$, $f^{[a,b]}:[a,b]\to\mathbb{R}$ by

$$f^{[a,b]}(x) := f(x) - \frac{x-a}{b-a}\cdot f(b) - \frac{b-x}{b-a}\cdot f(a) \tag{2.1}$$

for $x\in[a,b]$.

We now turn to some notational conventions for the path spaces that will be used throughout. For general domains of paths J, we denote the space of continuous paths, in the usual topologies, by $C_{*,*}(J,\mathbb{R})$. More specifically, if the domain is an interval $[a,b]\subseteq\mathbb{R}$, we denote the space of continuous functions with domain $[a,b]$ which vanish at a by $C_{0,*}([a,b],\mathbb{R})$. For random functions taking values in these spaces, we will always endow them with their respective Borel σ-algebras generated by the topologies of uniform convergence (which makes them into Polish spaces). Similarly, for $k\geq 1, a<b$, define $C^k_{*,*}([a,b],\mathbb{R}) := \bigtimes_{i=1}^k C_{*,*}([a,b],\mathbb{R})$ and equip it with the product of the uniform topologies. Furthermore, for $a<b$, $k\in\mathbb{N}$ and $\underline{x},\underline{y}\in\mathbb{R}^k_>$, let $C^k_{\underline{x},\underline{y}}([a,b],\mathbb{R})$ denote the space $\left\{g\in C^k_{*,*}([a,b],\mathbb{R}) : \forall i\in[\![1,k]\!],\ g_i(a)=x_i \text{ and } g_i(b)=y_i\right\}$.

We say that a Brownian motion or a Brownian bridge has *rate v* if its quadratic variation in an interval $[s,t]$ is equal to $v(t-s)$. We say that a Dyson's Brownian motion or a Brownian k-melon has rate v if the component Brownian motions have rate v. From now on, all Brownian motions are rate two unless stated otherwise.

For $0\leq a<b$, in analogy to the above, let $\mathfrak{B}^{[a,b]}_{*,*}(\cdot)$ denote the law of a rate two Brownian motion on $[0,\infty)$ starting from the origin restricted to the interval $[a,b]$ (the two star symbols indicate that the Brownian motion starts from the origin at time zero, which might be outside of the interval $[a,b]$). When $k\geq 1$ independent copies are considered, we will be using the usual product measure notation $(\mathfrak{B}^{[a,b]}_{*,*})^{\otimes k}$. Moreover, for $\underline{x},\underline{y}\in\mathbb{R}^k$ let $\mathfrak{B}^{[a,b]}_{\underline{x},\underline{y}}(\cdot)$ denote the law of k independent rate two Brownian bridges on $[a,b]$ with endpoints $(a,\underline{x})$ and $(b,\underline{y})$, hence it is a measure on $C^k_{\underline{x},\underline{y}}([a,b],\mathbb{R})$ equipped with the usual Borel σ-algebra on the product topology of local uniform convergence.

For $k\in\mathbb{N}$, $a<b$, $\underline{x},\underline{y}\in\mathbb{R}^k_>$ and $f:[a,b]\to\mathbb{R}$ a measurable function such that $x_k>f(a)$ and $y_k>f(b)$, the *non-crossing* event on any fixed union of finite sub-intervals $J\subseteq[a,b]$ is denoted by

$$\mathrm{NoInt}(J\,,f) := \left\{g\in C^k_{*,*}(J,\mathbb{R}) : \forall r\in J, g_i(r)>g_j(r) \text{ for all } 1\leq i<j\leq k \text{ and } g_k(r)>f(r)\right\}. \tag{2.2}$$

In what is to follow, the probability $\mathfrak{B}^{[a,b]}_{\underline{x},\underline{y}}(\mathrm{NoInt}(J\,,f))$ is called an *acceptance probability.* Roughly speaking, it is the probability of the event that a collection of k independent Brownian bridges on $[a,b]$ with endpoints $\underline{x},\underline{y}$ do not intersect, and also stay above the 'lower barrier' f on J. We note this event has a positive probability owing to standard facts of Brownian bridges, see Section 2.2.2 in [CH14].

3. Preliminaries

In this section, we will recall some basic definitions that appear in the KPZ universality class, namely, last passage percolation, the Pitman transform and melons; and collect some basic results that will be useful later on including some elementary estimates involving Brownian bridge, Radon-Nikodym derivatives (against Brownian motion) estimates and pathwise properties of the Airy line ensemble. We start with the central probabilistic object of study, namely random line ensembles.

3.1. **Line ensembles.** The following definition makes precise the notion of a *random line ensemble*, a probabilistic object of central importance in the KPZ universality class. It is a random variable taking values in an indexed (at most countably infinite) family of continuous paths defined on a common subset of $\mathbb{R}$.

Definition 3.1 (Random ensemble). *Let Σ be a (possibly infinite) interval of $\mathbb{Z}$, and let Λ be an interval of $\mathbb{R}$. Consider the set $X := C^{\Sigma}$ of continuous functions $f : \Sigma \times \Lambda \to \mathbb{R}$. We endow it with the topology of uniform convergence on compact subsets of $\Sigma \times \Lambda$. Let $\mathscr{C}$ denote the sigma-field generated by Borel sets in X.*

A Σ-indexed line ensemble L is a random variable defined on a probability space $(\Omega, \mathfrak{B}, \mathbb{P})$, taking values in X such that L is a $(\mathfrak{B}, \mathscr{C})$-measurable function. Furthermore, we write $L_i := (L(\omega))(i, \cdot)$ for the line indexed by $i \in \Sigma$.

3.2. **Last passage percolation.** We begin with the collection of some preliminary facts regarding last passage percolation (sometimes abbreviated as LPP in the paper) over ensembles of functions following [DJOBV22].

Formally, let $I \subset \mathbb{Z}$ be a possibly finite index set and define the space C^I of sequences of continuous functions with real domains, that is, the space

$$f : \mathbb{R} \times I \to \mathbb{R} \quad (x, i) \mapsto f_i(x)\,.$$

Definition 3.2 (Path). *Let $x \le y \in \mathbb{R}$, and $m \le \ell \in \mathbb{Z}$ respectively. A **path**, from (x, ℓ) to (y, m) is a non-increasing function $\pi : [x, y] \to \mathbb{N}$ which is cadlag on (x, y) and takes the values $\pi(x) = \ell$ and $\pi(y) = m$.*

Remark. *The convention that the paths be non-increasing is so that they match the natural indexing of the Airy line ensemble, see Section 3.6.*

In what is to follow, since we will primarily be considering the Airy line ensemble (see Section 3.6 for a definition), we will take the indexing set to be $I = \mathbb{N}$. We now define an important quantity associated to each such path, namely, its *length* as the sum of increments of f along π. This also leads one to naturally define a derived quantity, namely the *last passage value*.

Definition 3.3 (Length). *Let $x \le y \in \mathbb{R}$ and $m < k \in \mathbb{Z}$. For each $m \le i < k$, let t_{k-i} denote the jump of the path π, on an ensemble $(f_i)_{i \in I}$, from f_{i+1} to f_i. Then the length of π is defined as*

$$\ell(\pi) = f_m(y) - f_m(t_{k-m}) + \sum_{i=1}^{\ell - m - 1} (f_{k-i}(t_{i+1}) - f_{k-i}(t_i)) + f_k(t_1) - f_k(x)\,.$$

Definition 3.4 (Last passage value). *With $x \le y, m < k$ as before and $f \in C^I$, define the **last passage value** of f from (x, k) to (y, m) as*

$$f[(x, k) \to (y, m)] := \sup_{\pi} \ell(\pi)\,,$$

where the supremum is over precisely the paths π from (x, k) to (y, m).

Remark. *Any path π from (x, k) to (y, m) such that its length is equal to its last passage value is called a **geodesic**. To establish the existence of geodesics one can proceed by first noticing that the length of a path $\ell(\pi)$, can be viewed as a function on the subset $\mathcal{Z}$ of non-increasing cadlag functions with fixed endpoints in $\mathbf{D}$, the space of cadlag functions $\mathbf{D} := \mathbf{D}([x, y], \mathbb{N})$. When endowed with respect to the Skorokhod topology, which is metrisable, the above function is continuous. Since $\mathcal{Z}$ is closed with respect to the above topology of "jump times", a compactness argument using Arzela-Ascoli, see* [Bil13, ch. 3]*, implies that the supremum over admissible paths is indeed attained.*

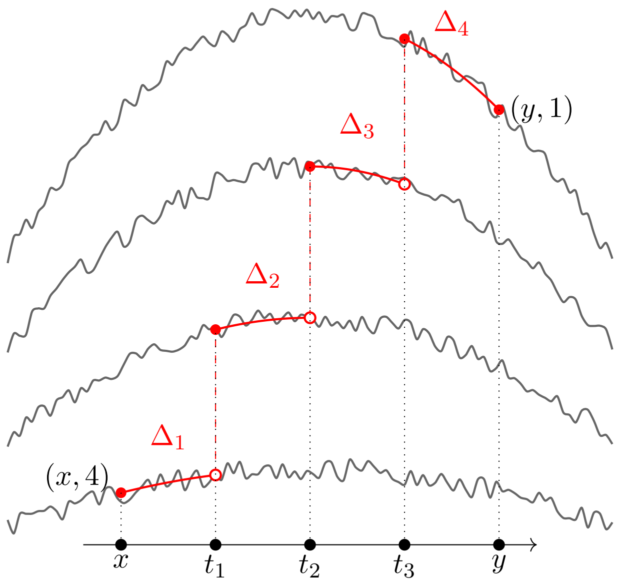

FIGURE 2. Visualisation of a possible path (red) "embedded" on the Airy line ensemble, here $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4)$ from top to bottom, and $m = 1, k = 4$ (see Section 3.6). Here $\Delta_1 = \mathcal{A}_4(t_1) - \mathcal{A}_4(x)$, $\Delta_2 = \mathcal{A}_3(t_2) - \mathcal{A}_3(t_1)$, $\Delta_3 = \mathcal{A}_2(t_3) - \mathcal{A}_2(t_2)$, $\Delta_4 = \mathcal{A}_1(y) - \mathcal{A}_1(t_3)$ and $\ell = \sum_{i=1}^{4} \Delta_i$.

Last passage percolation enjoys the following **metric composition law**, Lemma 3.2 in DOV [DJOBV22].

Lemma 3.5 (Metric composition law)**.** *Let $x \le y \in \mathbb{R}$, $m < \ell \in \mathbb{Z}$ and $f \in C^I$. If $k \in \{m, \dots, \ell\}$, then we have*

$$f[(x,\ell) \to (y,m)] = \sup_{z \in [x,y]} \left(f[(x,\ell) \to (z,k)] + f[(z,k) \to (y,m)] \right),$$

and if $k \in \{m+1, \dots, \ell\}$, then

$$f[(x,\ell) \to (y,m)] = \sup_{z \in [x,y]} \left(f[(x,\ell) \to (z,k)] + f[(z,k-1) \to (y,m)] \right).$$

Furthermore for any $z \in [x,y]$,

$$f[(x,\ell) \to (y,m)] = \sup_{k \in \{m,\dots,\ell\}} \left(f[(x,\ell) \to (z,k)] + f[(z,k) \to (y,m)] \right) \tag{3.1}$$

We are now in a position to state the main result of [TS25] which gives pathwise estimates for the Radon-Nikodym derivatives of Brownian LPP started from inhomogeneous 'initial data', that will be crucial in obtaining quantitative Brownian regularity of the KPZ fixed point.

First we define inhomogeneous Brownian LPP started from non-increasing initial data.

Definition 3.6. *(Inhomogeneous Brownian LPP) Fix $m \ge 1$, $B_1, \cdots, B_m$ be independent Brownian motions starting from the origin, $\underline{g} = (g_\ell)_{\ell=1}^m \in \mathbb{R}^m_{\ge}$ and $B = (B_1, \dots, B_m)$. Then, the process*

$$\max_{1 \le \ell \le m} \left(g_\ell + B[(0,\ell) \to (y,1)] \right), \qquad y \in [0,\infty)$$

is called the ***inhomogeneous Brownian LPP*** *started from initial data $\underline{g}$.*

Now we can proceed to the statement of the main result of [TS25].

Theorem 3.7. *([TS25, Theorem 7.1]) Fix $m \geq 1$, $(g_\ell)_{\ell=1}^m \in \mathbb{R}^m_\geq$ and let $H(\cdot)$ be the inhomogeneous Brownian LPP started from initial data $\underline{g}$. Then, for all $0 < \ell < r < \infty$, we have that the Radon-Nikodym derivative of the law of $H(\cdot)$ against a rate two Brownian motion starting from the origin μ on $[\ell, r]$ is in $L^{\infty-}(\mu|_{[\ell,r]})$. In particular, with $\xi_{\ell,r,m,\underline{b}}$ denoting the law of H as defined above on $[\ell, r]$*

$$\left\| \frac{\mathrm{d}\xi_{\ell,r,m,\underline{b}}}{\mathrm{d}\mu|_{[\ell,r]}} \right\|_{L^p(\mu|_{[\ell,r]})} = O_p(\mathrm{e}^{d_p m^2 \log m}), \quad \text{forall} \quad p > 1.$$

for some universal in $m \in \mathbb{N}$ (though possibly p-dependent) constant $d_p > 0$ for all $p > 1$.

In particular, we obtain the estimates

$$\begin{aligned}\left\| \tfrac{\mathrm{d}\xi_{\ell,r,m,\underline{b}}}{\mathrm{d}\mu} \right\|_{L^p(\mu)} &= \prod_{i=1}^m \exp\big(-(b_i - b_m)^2/(4\ell)\big) \cdot \left(\frac{(b_1 - b_m)}{2\ell} \vee 1 \right)^{m^2} \\ &\quad \cdot O_{p,\ell,r}\left(\mathrm{e}^{dm^2 \log m + c_\ell \left(\sum_{i=1}^m (b_i - b_m) \right)^2} \right),\end{aligned}$$

for some constants $c_{\ell,r}, d > 0$ independent of $m \in \mathbb{N}$ and all $p > 1$.

3.3. **Pitman transform.** Recall that with $f = (f_1, f_2)$ where $f_i : [0, \infty) \mapsto \mathbb{R}$ for $i = 1, 2$, for $f \in C^2_{*,*}([0, \infty))$, we define $Wf = (\mathrm{W}f_1, \mathrm{W}f_2) \in C^2_{*,*}([0, \infty))$, the **Pitman transform** of f as follows. For $x < y \in [0, \infty)$, define the maximal gap size

$$G(f_1, f_2)(x, y) := \max \left(\max_{s \in [x,y]} (f_2(s) - f_1(s)), 0 \right).$$

Then define

$$\begin{aligned} \mathrm{W}f_1(t) &= f_1(t) + G(f_1, f_2)(0, t), \\ \mathrm{W}f_2(t) &= f_2(t) - G(f_1, f_2)(0, t), \end{aligned} \tag{3.2}$$

for all $t \in [0, \infty)$.

One can express the top line of the Pitman transform in terms of last passage values.

Lemma 3.8. *Let $f \in C^2_+$ and let $Wf = (Wf_1, Wf_2)$ be as above. Then for all $t \in [0, \infty)$,*

$$Wf_1(t) = \max_{i=1,2} \{ f_i(0) + f[(0, i) \to (t, 1)] \}.$$

Proof. By definition,

$$\begin{aligned} Wf_1(t) &= f_1(t) + G(f_1, f_2)(0, t) \\ &= f_1(t) + \max\{ \max_{s \in [x,y]} (f_2(s) - f_1(s)), 0 \} \\ &= \max\{ \max_{s \in [x,y]} (f_2(s) + f_1(t) - f_1(s)), f_1(t) \}. \end{aligned}$$

From 3.1, we get $f_1(t) = f_1(0) + f[(0, 1) \to (t, 1)]$ and

$$\max_{s \in [0,t]} (f_2(s) + f_1(t) - f_1(s))\} = f_2(0) + f[(0, 2) \to (t, 1)].$$

Combining the above gives the result. □

Particularly in the case where $f_1(0) = f_2(0) = 0$, we obtain that

$$Wf_1(t) = f[(0, 2) \to (t, 1)].$$

Wf is commonly referred to as the **2-melon** (which will be generalised in the following section to the so-called n-melons) of f, since paths in Wf avoid each other and thus resemble the stripes of a watermelon.

3.4. **Dyson Brownian motion.** Fix any $\epsilon, t > 0$ and let B^n be the collection of n independent Brownian motions with initial conditions $B_i^n(0) = 0$ conditioned not to intersect on $[\epsilon, t]$ (note the non-intersection event has positive probability). Then, as $\epsilon \searrow 0$, $t \nearrow \infty$, Kolmogorov's extension theorem gives that the B^n converges in law to a limiting process, namely, n-level Dyson Brownian motion.

An alternative construction is to first take $x \in \mathbb{R}^n_>$ and with $\mathbb{P}_x$ denoting the law of n independent Brownian motions B started at x and $\hat{\mathbb{P}}_x$ the law of the Doob's h-transform of B started at x, where $h(x_1, x_2, \cdots, x_n) = \prod_{1 \le i < j \le n}(x_i - x_j)_+$. Then the weak limit of $\hat{\mathbb{P}}_x$ as $\mathbb{R}^n_> \ni x \to 0$ can be realised as a random ensemble with law on paths $\hat{\mathbb{P}}_{0+}$ which agrees with the n-level Dyson Brownian motion starting from the origin. The advantage of this construction is that it is more amenable to Radon-Nikodym derivative estimates.

It is worth mentioning that the Dyson Brownian motion was initially described as the eigenvalues of $n \times n$ time-dependent Hermitian matrices with entries independent complex-valued Brownian motion, [Dys62].

3.5. **Melons.** An application of the above that is of interest is that of two independent standard Brownian motions (starting from zero) $B = (B_1, B_2)$. Let $\hat{B} = (\hat{B}_1, \hat{B}_2)$ be two independent Brownian motions conditioned not to collide, in the sense of Doob (a 2-Dyson Brownian motion). Then, the law of the melon WB as defined above in (3.2) is the same as that of $\hat{B}$. In [OY02], a generalisation was proved for n Brownian motions, using a continuous analogue of the Robinson–Schensted–Knuth (RSK) correspondence, where each level in the *n-melon* $WB^n = (WB_1^n, WB_2^n, \cdots, WB_n^n)$ is obtained from a family of n Brownian motions by a sequence of deterministic operations that are analogous to the sorting algorithm 'bubble sort' where the top curve WB_1^n coincides with the top level of an n-Dyson Brownian motion. The term melon comes from the ordering of paths: for some continuous $n-$tuple f, $(Wf)_1^n \ge (Wf)_2^n \ge \cdots \ge (Wf)_n^n$ and their initial value which is 0, which means they look like stripes on a watermelon. When clear from context, we will abuse notation and drop the superscript, writing instead Wf.

In particular, [DJOBV22, Proposition 4.1] gives an important property of melon paths in that they preserve last passage values (with no restriction on their starting point). In particular,

$$WB[(0,n) \to (t,1)] = B[(0,n) \to (t,1)]\,, \forall t \ge 0\,.$$

Using the fact that $WB^n(0) = \underline{0}$ and the ordering of melon paths, one gets that the left-hand-side of the above equation is just $WB_1(t)$. Thus the top line of melon paths is completely characterised in terms of Brownian last passage percolation. For a more complete definition of melons involving the remaining lines, see [DJOBV22, sec. 2] and [OY02].

3.6. **Airy line ensemble and the Brownian Gibbs property.** After appropriate rescaling, WB^n converges in law as $n \to \infty$ to a non-intersecting random ensemble $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ in $C^{\mathbb{N}}$ (see Theorem 2.1 in [DJOBV22]), such that $\mathcal{A}_1 > \mathcal{A}_2 > \cdots$. The random ensemble $\mathcal{A}$ is called the **(parabolic) Airy line ensemble**. It was introduced by Prähofer and Spohn [PS02] in the version $(\mathcal{A}_i^{\text{stat}})_{i\in\mathbb{N}} := (\mathcal{A}_i(\cdot) + (\cdot)^2)_{i\in\mathbb{N}}$, which is stationary in time, see also [CH14] and [CS14]. We will thus call it the **stationary Airy line ensemble**. The top line $\mathcal{A}_1$ is known as the parabolic Airy$_2$ process that appears as the limiting spatial fluctuation of random growth models starting from a single point.

Theorem 3.9. *Let WB^n be a Brownian n-melon. Define the rescaled melon $A^n = (A_1^n, \dots, A_n^n)$ by*

$$A_i^n(y) = n^{1/6}\left((WB^n)_i(1 + 2yn^{-1/3}) - 2\sqrt{n} - 2yn^{1/6}\right).$$

Then A^n converges to a random sequence of functions $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots) \in C^{\mathbb{N}}$ in law with respect to product of uniform-on-compact topology on $C^{\mathbb{N}}$. For every $y \in \mathbb{R}$ and $i < j$, we have $\mathcal{A}_i(y) > \mathcal{A}_j(y)$. The function $\mathcal{A}$ is called the ***(parabolic) Airy line ensemble.***

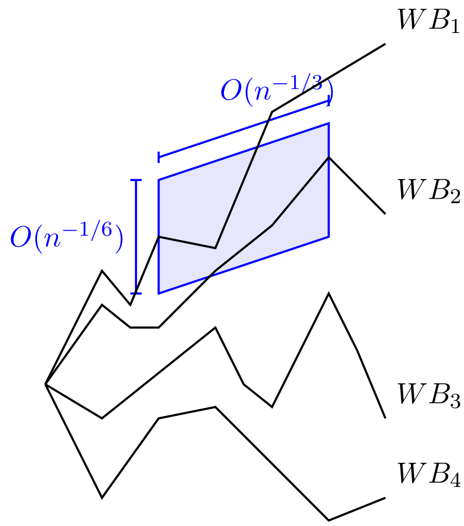

FIGURE 3. Brownian melon scaling limit. Above is a realisation of the WB^4 melon. 'Zooming in' on the parallelogram at small scales and taking the limit as $n \to \infty$ yields the convergence in law to the (parabolic) Airy line ensemble.

We now recall the Brownian Gibbs resampling property enjoyed by the Airy line ensemble (see Figure 4), first established in [CH14]. Informally, it states that for $a < b$, $k \in \mathbb{N}$, the law of the Airy line ensemble restricted to $\{1, 2, \cdots, k\} \times (a, b)$, $\mathcal{A}|_{\{1,2,\cdots,k\}\times(a,b)}$, conditionally on all the data generated by the Airy line ensemble outside of this region, $\mathcal{F}_k := \sigma(\{A_i(x) : (i, x) \notin [\![1, k]\!] \times (a, b)\})$, is given by non-intersecting Brownian bridges with entry data $\underline{x} = (\mathcal{A}_i(a))_{1\le i\le k}$, $\underline{y} = (\mathcal{A}_i(b))_{1\le i\le k}$ and also conditioned to stay above $f = \mathcal{A}_{k+1}$ on (a, b).

More precisely, the Brownian Gibbs property allows us to specify the regular conditional distribution

$$\mathrm{Law}\Big(\mathcal{A}|_{\{1,2,\cdots,k\}\times(a,b)}\,\mathrm{conditioned\,on}\,\mathcal{F}_k\Big) = \mathfrak{B}^{f,[a,b]}_{\underline{x},\underline{y}}\,,$$

where

$$\mathfrak{B}^{f,[a,b]}_{\underline{x},\underline{y}} := \frac{\mathfrak{B}^{[a,b]}_{\underline{x},\underline{y}}(\cdot \cap \mathrm{NoInt}([a, b], f)}{\mathfrak{B}^{[a,b]}_{\underline{x},\underline{y}}(\mathrm{NoInt}([a, b], f))}\,.$$

Notice that for fixed data $\underline{x}, \underline{y}, f$, the measure $\mathfrak{B}^{f,[a,b]}_{\underline{x},\underline{y}}$ is absolutely continuous with respect to $\mathfrak{B}^{[a,b]}_{\underline{x},\underline{y}}$, that is the law of k independent Brownian bridges on $[a, b]$ starting at (a, x_i) and ending at (b, y_i) respectively, for $1 \le i \le k$.

We now include the following global modulus of continuity result from [Wu25] obtained using techniques from optimal transport, using the fact that the Dyson Brownian motion can be viewed as a log-concave perturbation of Brownian motion, and is inherited by a large class of random ensembles, including the stationary Airy line ensemble (see Theorem 3.9 and the remark thereafter). It essentially shows that lines in the stationary Airy line ensemble have the same modulus of continuity as that of Brownian motion.

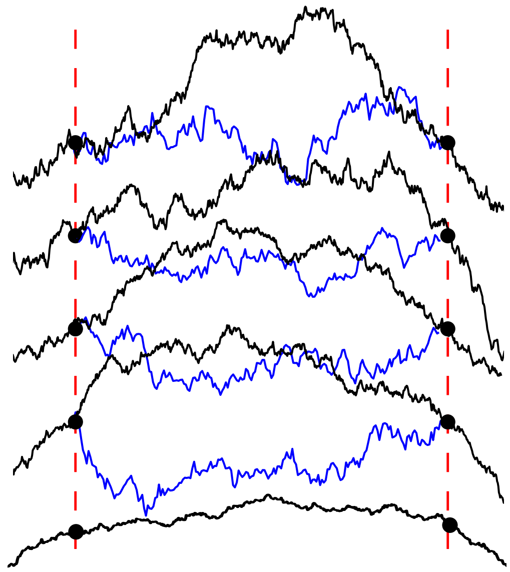

FIGURE 4. Figure illustrating the Brownian Gibbs property on the first four lines of the parabolic Airy Line ensemble $\mathcal{A} = \{\mathcal{A}_1 > \mathcal{A}_2 > \dots\}$ (in **black**) between two points (indicated by the red vertical dashed lines). The blue curves represent resampled versions of first four lines in the ensemble between the endpoints, conditioning on everything else and avoiding the fifth line.

Proposition 3.10. *(*[Wu25, Corollary 1.4]*) There exist universal constants $C_1, C_2 > 0$ such that for any $a < b$, $j \in \mathbb{N}$, and $K \geq 0$, it holds that*

$$\mathbb{P}\left(\sup_{t,s\in[a,b],t\neq s} \frac{|\mathcal{A}_j(t) - \mathcal{A}_j(s) + t^2 - s^2|}{\sqrt{|t-s|\log(2(b-a)/|t-s|)}} > K\right) \leq C_1 \mathrm{e}^{-C_2 K^2}. \tag{3.3}$$

These improved bounds on the modulus of continuity of the stationary line ensemble allow us to state the following refinement of [Dau24, Lemma 2.3] that will be needed in the later sections. It gives sub-Gaussian tails for the fluctuations of the Airy lines across indices, while also improving the dependence on the depth of the Airy line ensemble.

Corollary 3.11. *Fix $t > 0$, then for every $m \in \mathbb{N}$, let*

$$M = \max_{r,r'\in[-t,t]} |\mathcal{A}_{m+1}(r) - \mathcal{A}_{m+1}(r')| + \max_{i\in[\![1,m]\!]} |\mathcal{A}_i(t) - \mathcal{A}_i(-t)|\,.$$

We have that there exist some positive constants $C_1, C_2, d > 0$ independent of t, m such that for all $a > 0$,

$$\mathbb{P}(M > a) \leq C_1 m \mathrm{e}^{dt^3} \mathrm{e}^{-C_2 a^2/t}\,.$$

Proof. To prove the bound on M, we apply Proposition 3.10 to the process $\mathcal{A}_{m+1}(r), r \in [0,t]$ using the estimates for $r, r+\epsilon \in [0,t]$ from Proposition 3.10 (with $a = 0, b = t$ in that proposition), to obtain for all $s > 0$

$$\begin{aligned}
&\mathbb{P}\left(\max_{r,r+\epsilon\in[0,t]} |\mathcal{A}_{m+1}(r) - \mathcal{A}_{m+1}(r+\epsilon)| > s\sqrt{t}\right) \\
&\leq \mathbb{P}\left(\max_{r,r+\epsilon\in[0,t]} |\mathcal{A}_{m+1}(r) + r^2 - \mathcal{A}_{m+1}(r+\epsilon) - (r+\epsilon)^2| + \epsilon^2 + 2\epsilon r > s\sqrt{t}\right)
\end{aligned}$$

$$\leq \mathbb{P}\left(\max_{r,r+\epsilon\in[0,t]} |\mathcal{A}_{m+1}(r)+r^2-\mathcal{A}_{m+1}(r+\epsilon)-(r+\epsilon)^2|+2\epsilon t > s\sqrt{t}\right) \quad (\text{since } \epsilon^2+2\epsilon r \leq 2\epsilon t)\,.$$

Moreover, as $\epsilon \in [0,t]$,

$$\sqrt{\epsilon \log 2t/\epsilon} \leq \sup_{x\in(0,1]} \sqrt{x\log 2/x}t^{1/2} \leq Ct^{1/2}$$

for some constant $C>0$, multiplying and dividing both sides of the term containing the Airy process by $\sqrt{\epsilon \log 2t/\epsilon}$ shows that the above probability is

$$\begin{aligned}
&\leq \mathbb{P}\left(\max_{r,r+\epsilon\in[0,t]} \frac{|\mathcal{A}_{m+1}(r)+r^2-\mathcal{A}_{m+1}(r+\epsilon)-(r+\epsilon)^2|}{\sqrt{\epsilon\log(2t/\epsilon)}} > c(s-2t^{3/2})_+\right)\\
&\overset{\text{Prop.3.10}}{\leq} C_1\exp\left(-C_2(s-2t^{3/2})_+^2\right)\\
&\leq C_1 \mathrm{e}^{dt^3}\exp(-C_2s^2)\,,
\end{aligned}$$

for some $C_1,C_2,c,d>0$ universal constants. We thus obtain by a union bound that there exist some $C_1,C_2,d>0$ independent of t,m such that for all $a>0$,

$$\mathbb{P}\left(\max_{r,r'\in[-t,t]} |\mathcal{A}_{m+1}(r)-\mathcal{A}_{m+1}(r')|>a\right) \leq C_1\mathrm{e}^{dt^3}\mathrm{e}^{-C_2a^2/t}\,.$$

Applying (a simpler version of) the same argument for each of the Airy lines $\mathcal{A}_i, i\in[\![1,m]\!]$, and using union bounds, we get the result. $\square$

We also obtain the following proposition which is a slight variation of Proposition 3.10, giving sub-Gaussian concentration for the modulus of continuity of the parabolic Airy line ensemble over a fixed interval at any given depth.

Proposition 3.12. *Fix $0<s<t$, then for every $m\in\mathbb{N}$, the following tail bounds hold for all $a>0$.*

$$\begin{aligned}
&\mathbb{P}\left(\max_{r,r+\epsilon\in[s,t]} |\mathcal{A}_m(r)-\mathcal{A}_m(r+\epsilon)|>a\right)\\
&\leq C_1\mathrm{e}^{dt^2(t-s)}\exp(-C_2a^2/(t-s))\,,
\end{aligned}$$

for some $C_1,C_2,c,d>0$ universal constants.

Proof. To prove the tail bound for the process $\mathcal{A}_m(r), r\in[s,t]$, first recall the definition of the stationary Airy line ensemble $\mathcal{A}^{\text{stat}}$ from the Remark after Theorem 3.9. Now, use Proposition 3.10 (applied for $r,r+\epsilon\in[s,t]$) to obtain for all $a>0$

$$\begin{aligned}
&\mathbb{P}\left(\max_{r,r+\epsilon\in[s,t]} |\mathcal{A}_m(r)-\mathcal{A}_m(r+\epsilon)|>a\sqrt{t-s}\right)\\
&\leq \mathbb{P}\left(\max_{r,r+\epsilon\in[s,t]} |\mathcal{A}_m(r)+r^2-\mathcal{A}_m(r+\epsilon)-(r+\epsilon)^2|+\epsilon^2+2\epsilon r>a\sqrt{t-s}\right)\\
&\leq \mathbb{P}\left(\max_{r,r+\epsilon\in[s,t]} |\mathcal{A}_m(r)+r^2-\mathcal{A}_m(r+\epsilon)-(r+\epsilon)^2|+2t(t-s)>a\sqrt{t-s}\right) \quad (\text{since } \epsilon^2+2\epsilon r\leq 2t(t-s))\,.
\end{aligned}$$

Moreover, as $\epsilon\in[0,t-s]$,

$$\sqrt{\epsilon\log 2(t-s)/\epsilon} \leq \sup_{x\in(0,1]}\sqrt{x\log 2/x}(t-s)^{1/2} \leq C(t-s)^{1/2}$$

for some constant $C > 0$, multiplying and dividing both sides of the term containing the Airy process by $\sqrt{\epsilon \log 2(t-s)/\epsilon}$ shows that the above probability is

$$\begin{aligned}
&\leq \mathbb{P}\left(\max_{r, r+\epsilon \in [0,t]} \frac{|\mathcal{A}_m(r) + r^2 - \mathcal{A}_m(r+\epsilon) - (r+\epsilon)^2|}{\sqrt{\epsilon \log(2t/\epsilon)}} > c(a - 2t(t-s)^{1/2})_+\right) \\
&\overset{\text{Prop.3.10}}{\leq} C_1 \exp\left(-C_2(a - 2t(t-s)^{1/2})_+^2\right) \\
&\leq C_1 \mathrm{e}^{dt^2(t-s)} \exp(-C_2 a^2)\,,
\end{aligned}$$

for some $C_1, C_2, c, d > 0$ universal constants. $\square$

3.7. **Brownian bridge properties and lemmas.** Here we put together a few standard facts and basic lemmas on Brownian bridges, that will be needed in the later sections.

We first record a key monotonicity lemma for Brownian bridges.

Lemma 3.13. *(Monotonic coupling) Let $[s,t], J$ be closed intervals in $\mathbb{R}$ with $J \subseteq [s,t]$, let $\underline{x}^1 \leq \underline{x}^2, \underline{y}^1 \leq \underline{y}^2 \in \mathbb{R}^k_>$ where $\leq$ is the coordinate-wise partial order, and let g_1, g_2 be two bounded Borel measurable functions from $[s,t] \to \mathbb{R} \cup \{-\infty\}$ such that $g_1(x) \leq g_2(x)$ for all $x \in [s,t]$. For $i = 1,2$, let B^i be a k-tuple of Brownian bridges from $(s, \underline{x}^i)$ to $(t, \underline{y}^i)$, conditioned on the event* $\mathrm{NoInt}(J, g_i)$ *(recall the definition from (2.2)). Then there exists a coupling such that $B^1_j(r) \leq B^2_j(r)$ for all $r \in [s,t], j \in [\![1,k]\!]$.*

For a sketch of a proof, see the proof of Lemmas 2.6 and 2.7 in [CH14]. For a more complete argument, see the proof of Lemma 2.15 in [DM21]. The key idea behind their proof is to first establish a similar result in the discrete setting of random walk bridges which is easier, and then pass to a suitable limit where the random walks converge to Brownian bridges.

The following basic lemma computes the Radon-Nikodym derivative of a Brownian bridge with respect to a Brownian motion.

Lemma 3.14. *Fix $0 < x < y$, $m \in \mathbb{N}$ and let $W(\cdot)$ be a rate two Brownian bridge on $[0,y]$ with endpoints $\underline{0}, \underline{a} \in \mathbb{R}^m$, with law $\mathfrak{B}^{[0,y]}_{\underline{0},\underline{a}}(\cdot)$ on $C_{\underline{0},\underline{a}}([0,y])$. Then the law $\mathfrak{B}^{[0,y]}_{\underline{0},\underline{a}}(\cdot)$ restricted to $[0,x]$ is absolutely continuous with respect to that of a rate two Brownian motion with law $\mathfrak{B}^{[0,x]}_{\underline{0},*}(\cdot)$ with Radon-Nikodym derivative for $\mathfrak{B}^{[0,x]}_{\underline{0},*}$-almost all ω in $C_{\underline{0},*}([0,x])$,*

$$\frac{\mathrm{d}\mathfrak{B}^{[0,y]}_{\underline{0},\underline{a}}|_{[0,x]}}{\mathrm{d}\mathfrak{B}^{[0,x]}_{\underline{0},*}}(\omega) = (y/(y-x))^{\frac{m}{2}} \cdot \exp\left(-\frac{y\,\|\omega(x) - x/y\underline{a}\|^2}{4x(y-x)}\right) \cdot \exp\left(\frac{\|\omega(x)\|^2}{4x}\right).$$

Moreover, we have that $\mathrm{d}\mathfrak{B}^{[0,y]}_{\underline{0},\underline{a}}|_{[0,x]}/\,\mathrm{d}\mathfrak{B}^{[0,x]}_{\underline{0},*}$ is in $L^\infty(\mathfrak{B}^{[0,x]}_{\underline{0},*})$ with norm estimates

$$\left\|\frac{\mathrm{d}\mathfrak{B}^{[0,y]}_{\underline{0},\underline{a}}|_{[0,x]}}{\mathrm{d}\mathfrak{B}^{[0,x]}_{\underline{0},*}}\right\|_{L^p\left(\mathfrak{B}^{[0,x]}_{\underline{0},*}\right)} = \frac{(y/(y-x))^{\frac{m}{2}}}{(px/(y-x)+1)^{\frac{m}{2}}} \cdot \exp\left(\frac{x\|\underline{a}\|^2}{4(y-x)}\left(\frac{p}{(p-1)x+y} - \frac{1}{y}\right)\right)$$

for all $p > 1$ and letting $p \to \infty$,

$$\left\|\frac{\mathrm{d}\mathfrak{B}^{[0,y]}_{\underline{0},\underline{a}}|_{[0,x]}}{\mathrm{d}\mathfrak{B}^{[0,x]}_{\underline{0},*}}\right\|_{L^\infty\left(\mathfrak{B}^{[0,x]}_{\underline{0},*}\right)} \leq (y/(y-x))^{\frac{m}{2}} \cdot \exp\left(\frac{\|\underline{a}\|^2}{4y}\right).$$

Proof. Recalling the notation $f^{[a,b]}$ for an affine shift of a function f on an interval $[a,b]$ vanishing at its endpoints, see (2.1) in Section 2, we can couple a Brownian motion B and a Brownian bridge W with endpoints $\underline{0}, \underline{a} \in \mathbb{R}^m$ on $[0,y]$ by performing an affine shift and setting

$$W(\cdot) = B^{[0,y]}(\cdot) + \frac{(\cdot)}{y}\underline{a}$$

which we can re-express as

$$W(\cdot) = B^{[0,x]}(\cdot) + \frac{(\cdot)}{x}N$$

on $[0,x]$ for some $m-$dimensional Gaussian vector N with independent entries having mean $x\underline{a}/y$ and variance $2(y-x)x/y$, that is independent of the affine shift $B^{[0,x]}(\cdot)$ on $[0,x]$ (this can be seen by simply checking that the covariances vanish). Observe that if one were to replace N with B_x, one would recover the original Brownian motion; now, a straight-forward computation shows that

$$\frac{\mathrm{d}\mathfrak{B}^{[0,y]}_{\underline{0},\underline{a}}|_{[0,x]}}{\mathrm{d}\mathfrak{B}^{[0,x]}_{\underline{0},*}} = \frac{\mathrm{d}N}{\mathrm{d}B_x}$$

whence we derive the desired almost sure equality for the Radon-Nikodym derivative and conclude the proof of the first part. Now we fix any $p > 1$ and compute

$$\begin{aligned}\left\|\frac{\mathrm{d}\mathfrak{B}^{[0,y]}_{\underline{0},\underline{a}}|_{[0,x]}}{\mathrm{d}\mathfrak{B}^{[0,x]}_{\underline{0},*}}\right\|^p_{L^p\left(\mathfrak{B}^{[0,x]}_{\underline{0},*}\right)} &= \frac{(y/(y-x))^{\frac{pm}{2}}}{(4\pi x)^{\frac{m}{2}}}\int_{\mathbb{R}^m}\exp\left(-\frac{py\left\|\underline{z}-x/y\underline{a}\right\|^2}{4x(y-x)}\right)\cdot\exp\left(\frac{(p-1)\left\|\underline{z}\right\|^2}{4x}\right)\mathrm{d}\underline{z}\\
&= \frac{(y/(y-x))^{\frac{pm}{2}}}{(4\pi x)^{\frac{m}{2}}}\int_{\mathbb{R}^m}\exp\left(-\left(\frac{py}{4x(y-x)}-\frac{(p-1)}{4x}\right)\left\|\underline{z}\right\|^2\right)\\
&\qquad\cdot\exp\left(\tfrac{p}{2(y-x)}\underline{z}\cdot\underline{a}-\tfrac{px}{4y(y-x)}\left\|a\right\|^2\right)\mathrm{d}\underline{z}\\
&= \frac{(y/(y-x))^{\frac{pm}{2}}}{(4\pi x)^{\frac{m}{2}}}\int_{\mathbb{R}^m}\exp\left(-\left(\frac{p}{4(y-x)}+\frac{1}{4x}\right)\left\|\underline{z}\right\|^2\right)\\
&\qquad\cdot\exp\left(\tfrac{p}{2(y-x)}\underline{z}\cdot\underline{a}-\tfrac{px}{4y(y-x)}\left\|a\right\|^2\right)\mathrm{d}\underline{z}\\
&= \frac{(y/(y-x))^{\frac{pm}{2}}}{(px/(y-x)+1)^{\frac{m}{2}}}\cdot\exp\left(\tfrac{px\left\|\underline{a}\right\|^2}{4(y-x)}\left(\tfrac{p}{(p-1)x+y}-\tfrac{1}{y}\right)\right).\end{aligned}$$

We now have a uniform bound which allows us to pass to $p \to \infty$ and conclude. $\square$

We now slightly generalise the above, comparing the Brownian bridge in an interval in the interior of its domain to Brownian motion.

Lemma 3.15. *Fix $x < y < z < w$, $m \in \mathbb{N}$ and let $W(\cdot)$ be a rate two Brownian bridge on $[x,w]$ with endpoints $\underline{a}, \underline{b} \in \mathbb{R}^m$ with law $\mathfrak{B}^{[x,w]}_{\underline{a},\underline{b}}(\cdot)$ on $C_{\underline{a},\underline{b}}([x,w])$. Then the law of $W(\cdot) - W(y)$ restricted to $[y,z]$ is absolutely continuous with respect to that of a rate two Brownian motion with law $\mathfrak{B}^{[y,z]}_{\underline{0},*}(\cdot)$ with Radon-Nikodym derivative for $\mathfrak{B}^{[y,z]}_{\underline{0},*}$-almost all ω in $C_{\underline{0},*}([y,z])$,*

$$\begin{aligned}\frac{\mathrm{d}\mathfrak{B}^{[x,w]}_{\underline{a},\underline{b}}|_{[y,z]}}{\mathrm{d}\mathfrak{B}^{[0,z-y]}_{\underline{0},*}}(\omega) &= ((w-x)/(w-x-z+y))^{\frac{m}{2}}\cdot\exp\left(-\tfrac{(w-x)\left\|\omega(z-y)-(z-y)/(w-x)\underline{a}\right\|^2}{4(z-y)(w-x-z+y)}\right)\\
&\qquad\cdot\exp\left(\tfrac{\left\|\omega(z-y)\right\|^2}{4(z-y)}\right).\end{aligned}$$

Moreover, we have that $\mathrm{d}\mathfrak{B}^{[x,w]}_{\underline{a},\underline{b}}|_{[y,z]}/\,\mathrm{d}\mathfrak{B}^{[y,z]}_{\underline{0},*}$ is in $L^\infty(\mathfrak{B}^{[y,z]}_{\underline{0},*})$ with norm estimates

$$\left\|\frac{\mathrm{d}\mathfrak{B}^{[x,w]}_{\underline{a},\underline{b}}|_{[y,z]}}{\mathrm{d}\mathfrak{B}^{[y,z]}_{\underline{0},*}}\right\|_{L^p\left(\mathfrak{B}^{[y,z]}_{\underline{0},*}\right)} = \frac{((w-x)/(w-x-z+y))^{\frac{m}{2}}}{(px/(w-x-z+y)+1)^{\frac{m}{2}}}\cdot\exp\left(\tfrac{(z-y)\left\|\underline{a}\right\|^2}{4(w-x-z+y)}\left(\tfrac{p}{(p-1)(z-y)+w-x}-\tfrac{1}{w-x}\right)\right)$$

for all $p > 1$ and letting $p \to \infty$,

$$\left\| \frac{\mathrm{d}\mathfrak{B}^{[x,w]}_{\underline{a},\underline{b}}|_{[y,z]}}{\mathrm{d}\mathfrak{B}^{[y,z]}_{\underline{0},*}} \right\|_{L^\infty\left(\mathfrak{B}^{[y,z]}_{\underline{0},*}\right)} \leq ((w-x)/(w-x-z+y))^{\frac{m}{2}} \cdot \exp\left(\frac{\|\underline{a}-\underline{b}\|^2}{4(w-x)}\right).$$

Proof. By translation, it suffices to prove the lemma for $x = 0$. Observe we can realise a Brownian bridge W with endpoints $\underline{a}, \underline{b} \in \mathbb{R}^m$ on $[0, w]$ using a Brownian motion B by performing an affine shift and setting

$$W(\cdot) = B^{[0,w]}(\cdot) + \frac{(\cdot)}{w}\underline{b} + \frac{w-\cdot}{w}\underline{a}.$$

Thus, we observe that on $[0, z-y]$, $W(\cdot + y) - W(y)$ has the law of m independent Brownian bridges starting from $\underline{0}$ and

$$W(z) - W(y) = B(z) - B(y) - \frac{z-y}{w}B_y + \frac{z-y}{w}\underline{a} - \frac{z-y}{w}\underline{b},$$

which has the distribution of a Gaussian vector having independent entries with mean $\frac{z-y}{w}\underline{a} - \frac{z-y}{w}\underline{b}$ and variance $2(z-y)(1-z/w+y/w)$. Hence, we obtain the decomposition

$$W(\cdot + z) - W(y) = (W(\cdot + z) - W(y))^{[0,z-y]} + \frac{(\cdot)}{z-y}(W(z) - W(y))$$

on $[0, z-y]$, where the terms on the right hand side are independent (zero mean and one can check the covariance vanishes). Observe that if one were to replace $(W(z) - W(y))$ with an independent m-dimensional Gaussian vector with coordinatewise independent entries with mean zero and variance $2(z-y)$, one would recover the original Brownian motion; now, a straight-forward computation shows that

$$\frac{\mathrm{d}\mathfrak{B}^{[0,w]}_{\underline{a},\underline{b}}|_{[y,z]}}{\mathrm{d}\mathfrak{B}^{[0,z-y]}_{\underline{0},*}} = \frac{\mathrm{d}(W(z) - W(y))}{\mathrm{d}N}, \tag{3.4}$$

whence we derive the desired almost sure equality for the Radon-Nikodym derivative and conclude the proof of the first part. For the remaining parts, one proceeds as in the previous lemma. □

We finally end with a standard result regarding the maximum of a rate two Brownian bridge vanishing at its endpoints.

Lemma 3.16. *Let $T > 0$ and consider a rate two Brownian bridge $(W_t)_{t\in[0,T]}$ vanishing at both endpoints, then there is a universal constant $c > 0$ such that for all $a > 0$,*

$$\mathbb{P}\left(\max_{0\leq t\leq T} |W_t| \leq a\right) \geq c\exp\left(-\frac{\pi^2 T}{2a^2}\right).$$

Proof. Observe that for a rate two Brownian motion $(B_t)_{t\geq 0}$, one has that $(B_t)_{t\geq 0} \overset{d}{=} (B'_{\sqrt{2}t})_{t\geq 0}$ where $(B'_t)_{t\geq 0}$ is a standard Brownian motion. By Brownian scaling and the above, we thus obtain the distributional identities

$$(W_t)_{t\in[0,T]} \overset{d}{=} (B_t - tB_1)_{t\in[0,T]} \overset{d}{=} (B'_{\sqrt{2}t} - tB'_{\sqrt{2}})_{t\in[0,T]} \overset{d}{=} (2B'_t - 2tB'_1)_{t\in[0,T]} \overset{d}{=} (2W'_t)_{t\in[0,T]},$$

where $(W'_t)_{t\in[0,T]}$ is a standard Brownian bridge vanishing at both endpoint. Hence, by another application of Brownian scaling, we have the distributional equality

$$\max_{0\leq t\leq T} |W_t| \overset{d}{=} 2\sqrt{T} \max_{0\leq t\leq 1} |\tilde{W}_t|$$

where $\tilde{W}$ is a standard (rate one) Brownian bridge vanishing at 0 and 1 and so it suffices to prove the lower bound for a rate one Brownian bridge and $T = 1$. Recall that we can realise the Brownian bridge as

$$\tilde{W}_t = B'_t - tB'_1, \quad t \in [0,1]$$

where $(B'_t)_{t\in[0,1]}$ is a standard Brownian motion. Hence, we can estimate from below

$$\mathbb{P}(\max_{0\le t\le 1} |\tilde{W}_t| \le a) \ge \mathbb{P}(\max_{0\le t\le 1} |B'_t| \le a/2) \tag{†}$$

Now, from [Fel91, p.342, eq. 1.1.8] one can express

$$\begin{aligned}\mathbb{P}\left(\sup_{0<t<1} |B'_t| \le a\right) &= \frac{4}{\pi}\sum_{n\ge 0}\frac{1}{2n+1}\exp\left(-\frac{(2n+1)^2\pi^2}{8a^2}\right)\sin\frac{(2n+1)\pi}{2} \\ &= \frac{4}{\pi}\sum_{n\ge 0}(-1)^n\frac{1}{2n+1}\exp\left(-\frac{(2n+1)^2\pi^2}{8a^2}\right), \text{ for all } a > 0.\end{aligned}$$

Observe that for sufficiently small $a > 0$, the dominant contribution comes from the first term, whence we can estimate from below

$$\mathbb{P}\left(\sup_{0<s<1} |B'_s| \le a\right) \ge c\exp\left(-\frac{\pi^2}{8a^2}\right)$$

for some $c > 0$, which we can trivially extend to all $a > 0$ (after possibly making $c > 0$ smaller). Finally, observe that

$$\mathbb{P}\left(\max_{0\le t\le T} |W_t| \le a\right) = \mathbb{P}\left(\max_{0\le t\le 1} |\tilde{W}_t| \le \frac{a}{2\sqrt{T}}\right) \ge c\exp\left(-\frac{\pi^2 T}{2a^2}\right),$$

concluding the proof. □

3.8. **Airy line ensemble.** Using the refined modulus of continuity estimates for the Airy line ensemble in Proposition 3.10, one can obtain control over the fluctuations of the Airy last passage values about the typical Brownian counterpart after some normalisation. This follows from sub-additivity properties of last passage percolation and the Brownian bridge representation for the Airy line ensemble as delineated in [DV21]. In the following theorem, to ease notation, we will write for $a < b$ and $k \in \mathbb{N}$, the last passage percolation values of the Airy line ensemble to the first line by

$$\langle (a,k) \to b\rangle := \mathcal{A}[(a,k) \to (b,1)]\,. \tag{3.5}$$

Now, there are two regimes regarding the fluctuations of the value of the Airy line ensemble LPP around its Brownian counterpart's mean on compact intervals,

$$\frac{|\langle (0,k) \to x\rangle - 2\sqrt{2kx}|}{k^{1/2}} > \epsilon \qquad \text{for } \epsilon > 0, k \ge 1, x > 0\,,$$

in which we will be interested: namely when $\epsilon < k^{1/126}$ and when $\epsilon > O_x(1) \vee k^{1/126}$. We will be exploiting the bridge representation to study the former and concentration of measure plus sub-Gaussian tails of the moduli of continuity of lines in the Airy line ensemble for the latter. Also note that the parameters in the tail exponents were not optimised and so it may most likely be possible to improve them. This is the content of the following theorem.

Theorem 3.17. *Fix $x > 0$, and recall that $\langle (0,k) \to x\rangle$ is the last passage value across the Airy line ensemble $\mathcal{A}$ from line k at time 0 to line 1 at time x. Then for all $\epsilon < k^{1/126}$,*

$$\mathbb{P}\left(\frac{|\langle (0,k) \to x\rangle - 2\sqrt{2kx}|}{k^{1/2}} > \epsilon\right) \le ck^2 \mathrm{e}^{dx^3}\left(\exp(-d\epsilon^{1/2}k^{1/28}) + \exp\left(-d\epsilon k^{1/126}/(x \wedge x^{3/4})\right)\right),$$

for some universal $c, d > 0$. Alternatively, in the regime where $\epsilon > 4\sqrt{2x}$, then

$$\mathbb{P}\left(\frac{|\langle(0,k)\to x\rangle - 2\sqrt{2kx}|}{k^{1/2}} > \epsilon\right) \le C\mathrm{e}^{dx^3} k \exp\left(-d\frac{\epsilon^2}{kx}\right) \qquad k \ge 1\,,$$

for some universal $C, d > 0$.

Proof of Theorem 3.17. We will essentially adapt the arguments from the proof of [DJOBV22, Theorem 6.7] making use of the improved modulus of continuity estimates for the Airy line ensemble from [Wu25], which simplify parts of the proof, paying close attention to tail probabilities.

First, consider the regime where $\epsilon > 4\sqrt{2x}$, one estimates using Proposition 3.12 and a union bound,

$$\begin{aligned}&\mathbb{P}\left(\frac{|\langle(0,k)\to x\rangle - 2\sqrt{2kx}|}{k^{1/2}} > \epsilon\right) \le \mathbb{P}\left(|\langle(0,k)\to x\rangle| > \frac{\epsilon}{2}k^{1/2}\right)\\ &\le \sum_{i=1}^{k}\mathbb{P}\left(\sup_{r,r+\theta\in[0,x]}|\mathcal{A}_i(r)-\mathcal{A}_i(r+\theta)| > \frac{\epsilon}{2k^{1/2}}\right) \le C_1\mathrm{e}^{dx^3}k\exp\left(-d\frac{\epsilon^2}{kx}\right) \qquad k\ge 1\,,\end{aligned}$$

for some positive constant $d > 0$.

In the remainder, set $x = 1$ for notational simplicity as the value of x plays no important role. Further note that in all estimates the dependence of coefficients of tail bounds on x is continuous so one can harmlessly take suprema of such bounds for x in compacts at the expense of some weaker constants, keeping the functional form of the tails the same.

Let $\mathfrak{B}^k = \mathfrak{B}^k(x, \lceil k^{2/3+\gamma}\rceil, k^{-1/3-\gamma/4})$ be the bridge representation induced the division of time $\{s_r = rx/\lceil k^{2/3+\gamma}\rceil : r \in \{1, \dots, \lceil k^{2/3+\gamma}\rceil\}\}$ and the graph

$$G_{2k} = G_{2k}(x, \lceil k^{2/3+\gamma}\rceil, k^{-1/3-\gamma/4}).$$

Here $\gamma \in (0, 1/3)$ is a parameter that we will optimize over later in the proof. By [DV21]Theorem 7.2, we can couple all the representations $\mathfrak{B}^k$ with the Airy line ensemble $\mathcal{A}$ so that for some universal constant d and all $k \ge 1$

$$\mathbb{P}\left(\mathfrak{B}^k|_{\{1,\dots,k\}\times[0,x]} \ne \mathcal{A}|_{\{1,\dots,k\}\times[0,x]}\right) \le x\lceil k^{2/3+\gamma}\rceil \mathrm{e}^{-d\gamma k^{\gamma/12}}\,. \tag{3.6}$$

Hence it suffices to analyse the last passage time $L(\mathfrak{B}^k)$ from $(0, k)$ to $(x, 1)$.

Step 1: Splitting up the paths. By representing each of the Brownian bridges used to create $\mathfrak{B}^k = (\mathfrak{B}_{k,1}, \dots, \mathfrak{B}_{k,k})$ as a Brownian motion minus a random linear term, we can write

$$\mathfrak{B}_{k,i} = H_{k,i} + R_{k,i} + X_{k,i}$$

Here the k-tuple $H_k = (H_{k,1}, \dots, H_{k,k})$ consists of k independent Brownian motions of variance 2 on $[0, x]$. The functions $R_{k,i}$ are piecewise linear with pieces defined on the time intervals $[s_{r-1}, s_r]$ for $r \in \{0, \dots, \lceil k^{2/3+\gamma}\rceil\}$, and the error term $X_{k,i}$ is equal to zero except for on intervals $[s_{r-1}, s_r]$ where the vertex (i, r) is in a component of size greater than one in the graph G_{2k}. On such intervals, $X_{k,i}$ is the difference between a Brownian bridge from 0 to 0 and a Brownian bridge conditioned to avoid $U_{i,r} - 1$ other Brownian bridges with certain start and endpoints. Here $U_{i,r}$ is the size of the component of (i, r) in G_{2k} and the two Brownian bridges used in the definition of $X_{k,i}$ are independent.

By [DJOBV22]Lemma 6.9 applied twice, we have that the last passage values, here denoted by $L(\cdot)$,

$$L(H_k) + F(R_k) + F(X_k) \le L(\mathfrak{B}^k) \le L(H_k) + L(R_k) + L(X_k). \tag{3.7}$$

By Theorem 2.5 in [DV21], the main term

$$L(H_k) = 2\sqrt{2kx} + Y_k k^{-1/6}, \tag{3.8}$$

where $\{Y_k\}_{k\in\mathbb{N}}$ is a sequence of random variables satisfying a tail bound

$$\mathbb{P}(|Y_k| > m) \leq c\mathrm{e}^{-dm^{3/2}/x^{3/4}}$$

for $c, d > 0$ universal constants. To translate Theorem 2.5 in [DJOBV22] to a bound on last passage values, we have used the preservation of last passage values under the melon operation.

Step 2: Bounding the piecewise linear term. First, we have the bound

$$|L(R_k)|, |F(R_k)| \leq M_k,$$

where M_k is the maximum absolute slope of any of the piecewise linear segments in R_k. The slopes in R_k come from increments in the Airy line ensemble minus the increments of the Brownian motions H_k on the grid points. Recalling that $S_k(\ell) = \{1, \dots, k\} \times \{1, \dots, \ell\}$, $\ell_k = \lceil k^{2/3+\gamma} \rceil$ and $s_i = ix/\ell_k\,, i \in \{0, \cdots, \ell_k\}$, we have the following upper bound for M_k:

$$\lceil k^{2/3+\gamma} \rceil \left[\max_{(i,r)\in S_k(\lceil k^{2/3+\gamma}\rceil)} |H_{k,i}(s_r) - H_{k,i}(s_{r-1})| + \max_{(i,r)\in S_k(\lceil k^{2/3+\gamma}\rceil)} |\mathcal{A}_i(s_r) - \mathcal{A}_i(s_{r-1})| \right].$$

By a standard Gaussian bound on the first term and Proposition 3.12 for the second term, for some $d \in \mathbb{N}$ we have that for all $\delta \in (0, 1/2 - 1/3 - \gamma/2)$

$$\mathbb{P}\left(M_k \geq \epsilon k^{1/3+\gamma/2+\delta}\right) \leq c\mathrm{e}^{dx^3} k \lceil k^{2/3+\gamma} \rceil \exp\left(-d\epsilon^2 k^{2/3+\gamma+2\delta}/x\right)\,, \qquad k \geq 1\,, \tag{3.9}$$

for some possibly δ-dependent $c, d > 0$.

Step 3: Bounding the large component error. To bound $L(X_k)$ and $F(X_k)$, we divide $\{1, \dots, k\}$ into $n = \lceil k^{2/3+\gamma} \rceil$ intervals

$$I_{k,i} = \left\{ \lfloor \frac{(i-1)k}{n} \rfloor + 1, \dots, \lfloor \frac{ik}{n} \rfloor \right\}, \qquad i \in \{1, \dots, n\}.$$

This, and the division of time into the intervals $[s_{r-1}, s_r]$ for $r \in \{1, \dots, n\}$ breaks the line ensemble X_k into n^2 boxes. Each last passage path can meet at most $2n-1$ of these boxes. So we have that

$$L(X_k) \leq (2n-1)Z_k, \tag{3.10}$$

where Z_k is the maximal last passage value among all values that start and end in the same box (including the boundary). Specifically,

$$Z_k = \max_{(i,r)\in[1,n]^2} \max\left\{X_k[(\ell_1, t_1) \to (\ell_2, t_2)] : \ell_1, \ell_2 \in I_{k,i},\ t_1, t_2 \in [s_{r-1}, s_r]\right\}.$$

We have that $Z_k \leq N_k D_k$, where

$$N_k = \max_{(i,r)\in[1,n]^2} \mathrm{card}\Big\{\ell \in I_{k,i} : X_{k,\ell}|_{[s_{r-1},s_r]} \neq 0\Big\} \quad \text{and}$$

$$D_k = \max\Big\{|X_{k,\ell}(t) - X_{k,\ell}(m)| : \ell \in [1,k], t, m \in [s_{r-1}, s_r] \text{ for some } r \in \{1, \dots, n\}\Big\}$$

and card denotes the cardinality of a (finite) set, i.e. the number of elements it contains.

That is, N_k is the maximum number of non-zero line segments in any box, and D_k is the maximum increment over any line segment in a box. Since $X_{k,\ell} = \mathfrak{B}_{k,\ell} - H_{k,\ell} - R_{k,\ell}$, we can bound D_k in terms of the deviations of the other paths. To bound the deviation of $R_{k,\ell}$, we use the bound on M_k above. The deviation of $H_{k,\ell}$ can be bounded with standard bounds on

Gaussian random variables. On the event where $\mathfrak{B}^k|_{\{1,\dots,k\}\times[0,x]} = \mathcal{A}|_{\{1,\dots,k\}\times[0,x]}$, we can bound the deviation of $\mathfrak{B}_{k,\ell}$ using Proposition 3.10. Thus, we have for all $\delta > 0$

$$\mathbb{P}\left(D_k > \epsilon k^{-1/3-\gamma/2+\delta},\ \mathfrak{B}^k = \mathcal{A}|_{\{1,\dots,k\}\times[0,1]}\right) \le ck\lceil k^{2/3+\gamma}\rceil \exp\left(-d\epsilon^2 k^{2/3+\gamma+2\delta}/x\right), \qquad k \ge 1 \tag{3.11}$$

for some $d > 0$. Combining equations (3.11) and (3.6) gives for all $\delta > 0$

$$\mathbb{P}\left(D_k > \epsilon k^{-1/3-\gamma/2+\delta}\right) \le ck\lceil k^{2/3+\gamma}\rceil \left(\exp\left(-d\epsilon^2 k^{2/3+\gamma+2\delta}/x\right) + \mathrm{e}^{-d\gamma k^{\gamma/12}}\right), \qquad k \ge 1\,. \tag{3.12}$$

The quantity N_k is equal to the maximum number of edges in the graph G_k in a region of the form $I_{k,i} \times \{r\}$ for some $r \in \{1, \dots, n\}$. This can be bounded by using [DJOBV22, Proposition 6.7] and a union bound, which yields for all $\delta > 0, \epsilon < k^{1/6-\gamma/2-\delta}$

$$\mathbb{P}\left(N_k > \epsilon k^{1/3-\gamma} k^{-3\gamma/4} k^{\delta}\right) \le cx\lceil k^{2/3+\gamma}\rceil \exp(-d\epsilon k^{\delta})\,,$$

for some constant $d > 0$. Combining this with the bound in (3.10) and (3.12) implies that for all $\delta > 0, \epsilon < k^{1/6-\gamma/2-\delta}$

$$\begin{aligned}\mathbb{P}\left(L(X_k) > \epsilon k^{2/3-5\gamma/4} k^{\delta}\right) &\le ck\lceil k^{2/3+\gamma}\rceil \\ &\quad \times \left(\exp\left(-d\epsilon k^{2/3+\gamma+\delta/2}/x\right) + \exp(-d\epsilon^{1/2} k^{\delta/2}) + \mathrm{e}^{-d\gamma k^{\gamma/12}}\right), \quad k \ge 1\,.\end{aligned} \tag{3.13}$$

We can symmetrically bound $F(X_k)$.

Step 4: Putting it all together. By combining the inequalities (3.7), (3.8), (3.9) and (3.13), we get that for all $\delta > 0$ and $\epsilon < k^{1/6-\gamma/2-\delta}$

$$\begin{aligned}&\mathbb{P}\left(|L(\mathfrak{B}^k) - 2\sqrt{2kx}| > \epsilon k^{2/3-5\gamma/4+\delta} + \epsilon k^{1/3+\gamma/2+\delta}\right) \\ &\le ck\lceil k^{2/3+\gamma}\rceil \mathrm{e}^{dx^3} \left(\exp\left(-d\epsilon k^{2/3+\gamma+\delta/2}/(x \wedge x^{3/4})\right) + \exp(-d\epsilon^{1/2} k^{\delta/2}) + \mathrm{e}^{-d\gamma k^{\gamma/12}}\right)\end{aligned}$$

for positive constants c, d. Taking $\gamma = 4/21, \delta = 1/14 - 1/126$ completes the proof of the first regime of 'small' ϵ. ☐

4. Geometry of semi-infinite geodesics in the Airy line ensemble: deviation and coalescence

In this section, we study geodesic geometry in the Airy line ensemble. In Theorem 4.5, we obtain *exponentially stretched* tail bounds on intercepts of semi-infinite geodesics. Moreover, in Theorem 4.14, we also obtain uniform coalescence time tail bounds for semi-infinite geodesics with asymptotic directions, or 'speeds' in some 'meagre' set, see Definition 4.8. We start with providing the concentration result for semi-infinite geodesic intercepts in the Airy line ensemble.

4.1. **Tail bounds on geodesic intercepts.** Recall Theorem 3.9 which gives the Airy line ensemble as a scaling limit of rescaled Brownian melons. With this in mind, we will start in the prelimiting environment and obtain some more refined structural properties of the prelimiting jump times of geodesics on such melons. By the weak convergence already established, they easily translate to the limiting objects.

Now, using the notation established in [DJOBV22], for $n \in \mathbb{N}$, let

$$\underline{x} = 2xn^{-1/3}\,, \qquad \text{and} \qquad \hat{y} = 1 + 2yn^{-1/3}\,.$$

Furthermore, let $\gamma_n := \pi\{\underline{x} \to \hat{y}\}_n$ be the rightmost last passage path between $\underline{x}$ and $\hat{y}$ in the melon WB^n. For $n \in \mathbb{N}$ and $k \in \{1, 2, \dots, n\}$, let $Z^n_k(x, y)$ be the supremum of w so that (w, k)

lies along γ_n. Then, by [DJOBV22], Lemma 4.1, it follows that for each $k \in \mathbb{N}$, the sequence $\{Z_k^n(x,y)\}_n$ is tight. Let $Z_k(x,y)$ denote the subsequential limits of $\{Z_k^n(x,y)\}_n$ for any x, y.

Lemma 4.1. *Let K be a compact countable subset of $(0,\infty) \times \mathbb{R}$. Then for any $\epsilon > 0$*

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(x,y)\in K}\left|Z_k(x,y) + \sqrt{\frac{k}{2x}}\right| \geq \epsilon\sqrt{k}\right) \\
&\leq C(\epsilon^2 \vee 1/\epsilon)\left(\sup_{x\in K}\mathbb{P}\left(\frac{\left|\mathcal{A}[(0,k)\to(x,1)] - 2\sqrt{2kx}\right|}{k^{1/2}} > \epsilon\right)\right. \\
&\quad \left. + \exp\left(-d\frac{\epsilon^{3/4}k^{3/4}}{\left(\sup_{(x,y)\in K}(|x|+|y|+1)\right)^2}\right)\right), \quad k \geq 1\,.
\end{aligned}$$

for some universal $d > 0$ and $C > 0$.

Proof. First, fix $x, y \in K$. Now, rescale by $n^{1/6}$ and centre so that the triangle inequality

$$\{\bar{x} \to (\hat{z},k)\} + \{(\hat{z},k) \to \hat{y}\} \leq \{\bar{x} \to \hat{y}\}$$

reads

$$F_k^n(x,z) + G_k^n(z,y) \leq H_n(x,y) \tag{4.1}$$

with

$$\begin{aligned}
H_n(x,y) &= n^{1/6}\{\bar{x}\to\hat{y}\} - 2n^{2/3} - 2(y-x)n^{1/3}, \\
F_k^n(x,z) &= n^{1/6}(\{\bar{x}\to(\hat{z},k)\} - W_k^n(\hat{z})) + 2xn^{1/3}, \\
G_k^n(z,y) &= n^{1/6}(W_k^n(\hat{z}) + \{(\hat{z},k)\to\hat{y}\}) - 2yn^{1/3} - 2n^{2/3}.
\end{aligned}$$

The basic proof strategy for bounding $Z_k^n(x,y)$ is as follows. On the one hand,

$$F_k^n(x, Z_k^n(x,y)) + G_k^n(Z_k^n(x,y), y) = H_n(x,y)$$

We will show that for every $\epsilon \in (0,1)$ we have

$$\sup_{z:\,|z+\sqrt{k/(2x)}|>\epsilon\sqrt{k}} F_k^n(x,z) + G_k^n(z,y) \leq -\epsilon^2\sqrt{kx}/2 + \mathfrak{o}(\sqrt{k})\,, \tag{4.2}$$

where the error term $\mathfrak{o}(\sqrt{k})$ is asymptotically 'small' with respect to the scale of $\sqrt{k}$. By [DJOBV22, Lemma 3.3], $F_k^n(x,\cdot)$ is monotonically increasing and $G_k^n(\cdot,y)$ is monotonically decreasing. We can use this monotonicity to bound the left hand side of (4.2) by a supremum over a finite set. Let $A = (12\epsilon^2)^{-1}\mathbb{Z} \cap [1/4, 2]$, and for $z \in [-n^{1/3}+x, y]$, define

$$\begin{aligned}
&\lfloor z \rfloor_{n,k} = \max\{w \in -\sqrt{k/x}A \cup \{x - n^{1/3}\} : w < z\} \qquad \text{and} \\
&\lceil z \rceil_{n,k} = \min\{w \in -\sqrt{k/x}A \cup \{y\} : w > z\}.
\end{aligned}$$

We also set $\lfloor x - n^{1/3} \rfloor_{n,k} = x - n^{1/3}$ and $\lceil y \rceil_{n,k} = y$. The monotonicity of $F_k^n(x,\cdot)$ and $G_k^n(\cdot,y)$ implies that the left hand side of (4.2) is bounded above by

$$\sup_{z:\,|z+\sqrt{k/(2x)}|>\epsilon\sqrt{k}} F_k^n(x, \lceil z \rceil_{n,k}) + G_k^n(\lfloor z \rfloor_{n,k}, y)\,. \tag{4.3}$$

Notice that the number of terms is uniformly bounded in n and k, so it is enough to control the terms individually. There are three cases to consider, namely,

$$\begin{cases} F_k^n(x, z_{k,a}) + G_k^n(x - n^{1/3}, y) \le -\epsilon^2 \sqrt{kx}/2 + \mathfrak{o}(\sqrt{k}) \\ F_k^n(x, z_{k,a}) + G_k^n(z_{k,a}, y) \le -\epsilon^2 \sqrt{kx}/2 + \mathfrak{o}(\sqrt{k}) \\ F_k^n(x, y) + G_k^n(z_{k,a}, y) \le -\epsilon^2 \sqrt{kx}/2 + \mathfrak{o}(\sqrt{k}) \,, \end{cases} \tag{4.4}$$

for every fixed $a \in A$, with $z_{k,a} = -a\sqrt{k/x}$.

To prove (4.4), we establish pointwise bounds on F_k^n and G_k^n. [DJOBV22, Proposition 6.1] gives that for a fixed $a > 0$ we have

$$F_k^n(x, z_{k,a}) \le 2\sqrt{kx}(\sqrt{2} - a) + R_{n,k}^{1,a} \,.$$

[DJOBV22, Proposition 6.1] also yields the bound

$$F_k^n(x, y) = 2\sqrt{2kx} + R_{n,k}^2 \,. \tag{4.5}$$

Observe that Theorem 3.17, [DJOBV22, Proposition 6.1] and weak convergence give the following uniform bounds with respect to y for any fixed $\epsilon > 0$

$$\limsup_{n\to\infty} \left(\mathbb{P}(R_{n,k}^{1,a} > \epsilon\sqrt{k}) + \mathbb{P}(R_{n,k}^2 > \epsilon\sqrt{k}) \right) \le 2\mathbb{P}\left(\frac{|\mathcal{A}[(0,k) \to (x,1)] - 2\sqrt{2kx}|}{k^{1/2}} > \epsilon \right)$$

The triangle inequality (4.1) with $x' = x/(2a^2)$ gives

$$G_k^n(z_{k,a}, y) \le H_n(x', y) - F_k^n(x', z_{k,a}). \tag{4.6}$$

Now, $H_n(x', y)$ is equal to a rescaled and shifted Brownian last passage value by Proposition [DJOBV22, Proposition 4.1]. Therefore, by Theorem [DJOBV22, Theorem 2.5] which gives bounds on single Brownian last passage values, it is tight in n and hence $H_n(x', y) = \mathfrak{o}(\sqrt{k})$. In particular, making this more quantitative gives for any $\epsilon > 0$

$$\limsup_{n\to\infty} \mathbb{P}\left(\frac{|H_n(x,y)|}{\sqrt{k}} \ge \epsilon \right) \lesssim \exp\left(-d \frac{\epsilon^{3/8} k^{3/4}}{(|x-y| \vee 1)^{3/2}} \right) \,,$$

for some universal $d > 0$.

Moreover, [DJOBV22, Proposition 6.1] gives that

$$F_k^n(x', z_{k,a}) = 2\sqrt{2kx'} + 2z_{k,a}x' + \mathfrak{o}(\sqrt{k}) = \frac{\sqrt{kx}}{a} + \mathfrak{o}(\sqrt{k})$$

and so

$$G_k^n(z_{k,a}, y) \le -\frac{\sqrt{kx}}{a} + R_{n,k}^{3,a}.$$

where the following uniform bounds wrt y for any fixed $\epsilon > 0$ are satisfied

$$\limsup_{n\to\infty} \mathbb{P}(R_{n,k}^{3,a} > \epsilon\sqrt{k}) \le \mathbb{P}\left(\frac{|\mathcal{A}[(0,k) \to (x,1)] - 2\sqrt{2kx}|}{k^{1/2}} > \epsilon \right) \,.$$

We also have the bound

$$G_k^n(x - n^{1/3}, y) \le H_n(0, y) - F_k^n(0, x - n^{1/3}) = H_n(0, y) = \mathfrak{o}(\sqrt{k}).$$

The first equality here follows from the fact that $F_k^n(0, \cdot) = 0$, and the second equality again follows from [DJOBV22, Theorem 2.5].

Having now established the bound in (4.4), one obtains by a union bound and the convergence in distribution of $Z_k^n(x,y) \xrightarrow{d} Z_k(x,y) \,, n \to \infty$,

$$
\begin{aligned}
&\mathbb{P}\left(\left|Z_k(x,y)+\sqrt{\frac{k}{2x}}\right|\geq \epsilon\sqrt{k}\right)\leq \liminf_{n\to\infty}\mathbb{P}\left(\left|Z_k^n(x,y)+\sqrt{\frac{k}{2x}}\right|\geq \epsilon\sqrt{k}\right)\\
&\leq \liminf_{n\to\infty}\mathbb{P}\left(H^n(x,y)\leq -\epsilon^2\sqrt{kx}/2+\max_{a\in A}\sum_{i=1}^3|R^{i,a}_{n,k}|\right)\\
&\leq \liminf_{n\to\infty}\sum_{a\in A}\mathbb{P}\left(|H^n(x,y)|+|R^{1,a}_{n,k}|+|R^{2}_{n,k}|+|R^{3,a}_{n,k}|\geq \epsilon^2\sqrt{xk}/2\right)\\
&\leq C(\epsilon^2\vee 1)\left(\mathbb{P}\left(\frac{|\mathcal{A}[(0,k)\to(x,1)]-2\sqrt{2kx}|}{k^{1/2}}>\epsilon\right)+\exp\left(-d\frac{(\epsilon^{3/4})k^{3/4}}{(\sup_{(x,y)\in K}(|x|+|y|+1)^2}\right)\right)\,,
\end{aligned}
$$

for some universal $C>0$. Now, to obtain the uniform bound, observe that by the monotonicity of $Z_k(\cdot,\cdot)$ in each of its of its arguments and the continuity of $1/\sqrt{2\cdot}$, it suffices to fix any ϵ cover of K with at most $\lceil 1/\epsilon\rceil$ elements and use the pointwise estimates for fixed $x,y\in K$ at the expense of the $\lceil 1/\epsilon\rceil$ term that comes from a union bound. $\square$

We now introduce the coupling between the Airy sheet and the Airy line ensemble last passage values quoted from [DJOBV22, Definition 1.2], that will be used throughout the paper.

Definition 4.2. *(Airy sheet coupling) The Airy sheet $\mathcal{S}(\cdot,\cdot)=\mathcal{L}(\cdot,0;\cdot,1)$ can be coupled with the (parabolic) Airy line ensemble $\mathcal{A}$ so that $\mathcal{S}(0,\cdot)=\mathcal{A}_1(\cdot)$ and almost surely for all $(x,y,z)\in K\subseteq\mathbb{R}^+\times\mathbb{R}^2$, there exists a random integer $K_{x,y,z}$ such that for all $k\geq K_{x,y,z}$*

$$\mathcal{A}[x_k\to(z,1)]-\mathcal{A}[x_k\to(y,1)]=\mathcal{S}(x,z)-\mathcal{S}(x,y)\,,$$

where $x_k=(-\sqrt{k/2x},k)$.

We shall use this coupling of the Airy sheet throughout the paper. For $x\leq y\in\mathbb{R}$ and $\ell\in\mathbb{N}$, we shall denote the rightmost geodesic between (x,ℓ) and $(y,1)$ in the Airy line ensemble $\mathcal{A}$ by $\pi[(x,\ell)\to y]$. Next we define the infinite geodesics in the Airy line ensemble.

Definition 4.3. *For any $x\in\mathbb{R}^+$ and $y\in\mathbb{R}$ with $x_k=(-\sqrt{k/2x},k)$, we define the geodesic $\pi[x\to y]$ as the almost sure pointwise limit of $\pi[x_k\to y]$ as $k\to\infty$, whenever the limit exists. We define the length of the geodesic $\pi[x\to y]$ as $\mathcal{S}(x,y)$. We call the variable x the 'speed' of the geodesic $\pi[x\to y]$.*

Remark. *The fact that these limits exist almost surely for all x,y in a countable dense set of $\mathbb{R}^+\times\mathbb{R}^2$ is the content of* [SV21, Lemma 3.4].

In the absolute continuity paper of [SV21], the authors obtain, using a coupling with the Airy sheet, the following characterisation of the Airy sheet in terms of the intercept of semi-infinite geodesics with the vertical axis $\{x=0\}$ in the Airy line ensemble. For an illustration, see Figure 5. For a proof of the following lemma, see [SV21, Lemma 3.10].

Lemma 4.4. *Let $x_0>1$ and $y_0>1$ and $K\subseteq\mathbb{R}$ be a countable dense set. Let*

$$L_0=\pi[x_0'\to y_0'](0)\,,$$

for some $x_0',y_0'\in K$ with $x_0'\geq x_0$ and $y_0'\geq y_0$. Then almost surely for all $x\in[1,x_0]\cap K$ and all $y\in[1,y_0]$,

$$\mathcal{S}(x,y)=\max_{1\leq\ell\leq L_0}\left(\mathcal{A}[x\to(0,\ell)]+\mathcal{A}[(0,\ell)\to(y,1)]\right).$$

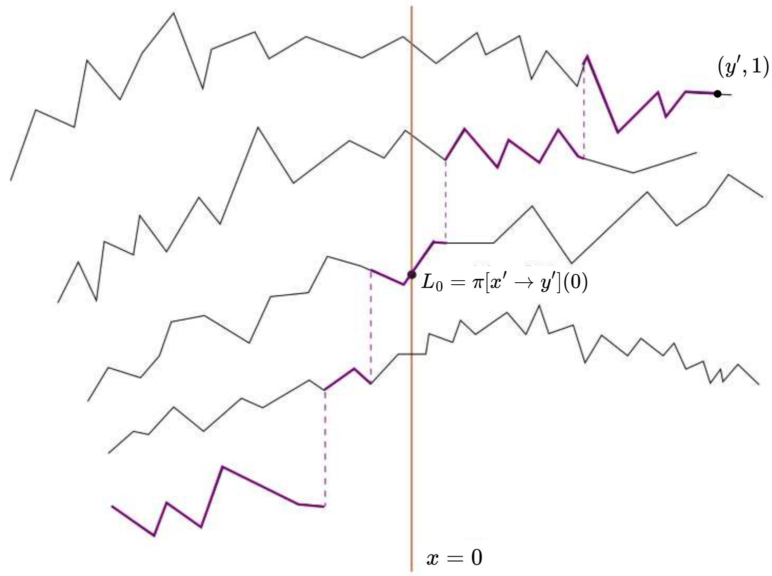

FIGURE 5. Above is displayed the point $(0, L_0)$ at which the last passage path $\pi[x' \to y']$ on the Airy line ensemble $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \cdots)$ (purple) meets with the axis $\{x = 0\}$, where $y' > 1$. Here $L_0 = 3$ and the first four lines of $\mathcal{A}$ are shown. The last passage path $\pi[x' \to y']$ is defined in Definition 3.3 in [SV21] and the definition 4.3.

Thus, obtaining good control over L_0 should translate to control over the Airy sheet and hence the KPZ fixed point, owing to its variational characterisation as a stochastic semi-group using the Airy sheet as an evolution (random) kernel. The structure of jump times of (semi-infinite) geodesics on the Airy line ensemble and Lemma 4.1 give the following Theorem which is the main result of this subsection.

Theorem 4.5. *For any $x_0' > 1, y_0' > 1$, there exists a $d > 0$ such that the semi-infinite geodesic intercept $L_0 = \pi[x_0' \to y_0']$ as given in the statement of Lemma 4.4 satisfies the tail bounds*

$$\sup_{k \in \mathbb{N}} \exp(dk^{1/126}) \cdot \mathbb{P}(L_0 \geq k) < \infty\,.$$

for some universal x_0', y_0'-dependent constant $d > 0$. Keeping track of the x_0', y_0' dependence (at the expense of the exponent) gives that there exist universal $\theta, \eta > 0$ such that for all $k \geq 1$

$$\mathbb{P}(L_0 \geq k) \leq \mathrm{e}^{c(x_0+y_0)^\theta} \mathrm{e}^{-dk^\eta}\,,$$

for some universal $c, d > 0$.

Proof. First observe that for any $k \in \mathbb{N}$, by the Skorokhod coupling in [DJOBV22, p.43], the almost sure pointwise limits $Z_k(x_0', y_0')$ of the jump times $Z_k^n(x_0', y_0')$ correspond to the jump times of the semi-infinite geodesic $\pi[x_0' \to y_0']$. Thus, by Lemma 4.1 for $k \geq 1$ and $\epsilon = 1/\sqrt{2x_0'}$,

$$\mathbb{P}(L_0 \geq k) = \mathbb{P}(Z_k(x_0', y_0') \geq 0) \leq \liminf_{n \to \infty} \mathbb{P}(Z_k^n(x_0', y_0') \geq 0)$$
$$\leq \mathbb{P}\left(\left|Z_k(x_0', y_0') + \sqrt{\frac{k}{2x_0'}}\right| \geq \epsilon\sqrt{k}\right)$$
$$\leq c\sqrt{x_0' \vee 1/x_0'}\left(\mathbb{P}\left(\frac{|\mathcal{A}[(0,k) \to (x,1)] - 2\sqrt{2kx}|}{k^{1/2}} > \epsilon\right) + \exp\left(-d\frac{\epsilon^{3/4}k^{3/4}}{(|x_0'| + |y_0'| + 1)^2}\right)\right)$$

$$\stackrel{\text{Thrm}3.17}{\leq} c\exp(d(x_0')^3)k^2\left(\exp(-d\epsilon^{1/2}k^{1/28})+\exp\left(-d\epsilon k^{1/126}/(x_0')^{3/8}\right)+\exp\left(-d\frac{\epsilon^{3/4}k^{3/4}}{(|x_0'|+|y_0'|+1)^2}\right)\right),$$

for some $c, d > 0$ and all $1/\sqrt{2x_0'} < k^{1/126}$, whence the result follows. □

4.2. **Uniform tail bounds on coalescence depths with respect to 'meagre' data.** The following propositions aim to obtain uniform control over the likelihood of geodesic non-coalescence on bounded intervals, which will translate into a uniform control of coalescence depths for semi-infinite geodesics, provided the semi-infinite geodesic 'speeds' are concentrated, or 'meagre' in a sense to be made precise below.

We start with some preliminary results. In particular, having refined the fluctuations of infinite-geodesic jump times around their 'typical' parabolic values, we are in a position to prove that the last jump time of semi-infinite geodesics is unlikely to be very small. First we need to obtain a last jump time anti-concentration result for Brownian last passage percolation, which is the content of the following lemma.

Lemma 4.6. *Fix $m \in \mathbb{N}$, $B_1, B_2, \cdots, B_m$ independent rate two Brownian motions starting from the origin, $\epsilon \in (0, 1/2)$ and consider the events*

$$A_{\epsilon,m} := \{\text{Last jump time of } B[(0,m) \to (1,1)] \geq -\epsilon\}.$$

We then have the bound

$$\mathbb{P}(A_{\epsilon,m}) \leq c\mathrm{e}^{dm^2\log m}\epsilon^{1/4},$$

for universal $c, d > 0$.

Proof. Observe that by the metric composition law for LPP, the last jump time Z_1^m is the a.s. unique maximiser of

$$A(z) = W(B_{2:m})_1^{m-1}(z) + B_1(1) - B_1(z), \qquad z \in [0,1],$$

where $W(B_{2:m})^{m-1}$ ($\equiv WB^{m-1}$ for short) is a Brownian $m-1$-melon and B' an independent Brownian motion. Thus, we can estimate

$$\mathbb{P}(A_{\epsilon,m}) \leq \mathbb{P}\left(\mathrm{argmax}_{z\in[1/2,1]}(WB^{m-1}(z) + B_1(1) - B_1(z)) \geq 1-\epsilon\right).$$

Now, by [TS25, Proposition 4.1] and Theorem 3.7, the law of $WB^{m-1}(\cdot)$ restricted to $[1/2, 1]$ is absolutely continuous with respect to that of a standard Brownian motion starting from 0 restricted to $[1/2, 1]$. Furthermore, one has the norm estimates of the Radon-Nikodym derivative of WB^{m-1} with respect to the Wiener measure restricted to $[1/2, 1]$ is bounded above by

$$\left\|\frac{\mathrm{dLaw}\, WB^{m-1}}{\mathrm{d}\mu|_{[\ell,r]}}\right\|_{L^2(\mu|_{[1/2,1]})} = O(\mathrm{e}^{dm^2\log m}).$$

Thus, using Cauchy-Schwarz, we have

$$\mathbb{P}(A_{\epsilon,m}) \leq \mathbb{P}\left(\mathrm{argmax}_{z\in[1/2,1]}(\tilde{B}(z) - B(z)) \geq 1-\epsilon\right)^{1/2} \cdot \left\|\frac{\mathrm{dLaw}\, WB^{m-1}}{\mathrm{d}\mu|_{[\ell,r]}}\right\|_{L^2(\mu|_{[1/2,1]})}$$
$$\leq O(\mathrm{e}^{dm^2\log m})\mathbb{P}\left(\mathrm{argmax}_{z\in[1/2,1]}(\tilde{B}(z) - B(z)) \geq 1-\epsilon\right)^{1/2},$$

where B and $\tilde{B}$ are independent rate two Brownian motions starting from the origin. Now, by Lévy's arcsine law and the fact that $(\tilde{B}(\cdot) - B(\cdot))/\sqrt{2} \stackrel{d}{=} B \stackrel{d}{=} \tilde{B}$, we estimate

$$\mathbb{P}(A_{\epsilon,m}) \leq c\mathrm{e}^{dm^2\log m}\epsilon^{1/4},$$

for universal $c, d > 0$. □

Remark. *Note that the exponent* $\epsilon^{1/4}$ *is artificial and using Hölder inequality, it could have been chosen to lie in* $(0, 1/2)$*, at the expense of the* m*-dependent constant. Such an estimate does not impact our estimates qualitatively.*

We are now in a position, in a manner analogous to the previous lemma, to control the probabilities of events where the first jump time of a semi-infinite geodesic starting from the top line of the Airy line ensemble at the origin is very 'small', which is the content of the following lemma.

Lemma 4.7. *Fix* $i \geq 1$, $x \in K \subset \mathbb{R}^+$ *countable and dense. Then there exists a possibly* K*-dependent* $d > 0$ *such that for all* $\epsilon \in (0,1)$

$$\mathbb{P}(Z_1(x,0) \geq -\epsilon) \leq C_x \exp\left(-d_x \log^{1/882}(1/\epsilon)\right) .$$

Proof. Now we estimate for all $i \geq 1$, $x, \epsilon > 0$

$$\begin{aligned}\mathbb{P}(Z_1(x,0) \geq -\epsilon) &\leq \mathbb{P}(Z_1(x,0) \geq -\epsilon, Z_i(x,0) \leq -1) \\ &\quad + \mathbb{P}\left(\left|\frac{Z_i(x,0)}{\sqrt{i}} + \sqrt{\frac{1}{2x}}\right| > \sqrt{\frac{1}{2x}} - \frac{1}{\sqrt{i}}\right) .\end{aligned}$$

Now, using the Brownian Gibbs property on the larger interval $[0, 2]$ and arguing as in Theorem 5.5, one obtains

$$\mathbb{P}(Z_1(x,0) \geq -\epsilon) \leq \sum_{1 \leq j \leq i} \mathbb{P}(\text{Last jump time of Airy LPP}[(-1,j) \to (0,1)] \geq -\epsilon) \tag{4.7}$$

$$+ \mathbb{P}\left(\left|\frac{Z_i(x,0)}{\sqrt{i}} + \sqrt{\frac{1}{2x}}\right| > \sqrt{\frac{1}{2x}} - \frac{1}{\sqrt{i}}\right) . \tag{4.8}$$

Thus, by Hölder, with

$$A_{\epsilon,i} := \bigcup_{1 \leq j \leq i} \{\text{Last jump time of LPP}[(-1,j) \to (0,1)] \geq -\epsilon\} ,$$

the first unconditional probability in (4.7) can be estimated as follows

$$\mathbb{P}(A_{\epsilon,i}) \leq \mathbb{E}\left[\frac{1}{\mathfrak{B}^{[0,1]}_{(\mathcal{A}_j(0))_{j=1}^i, (\mathcal{A}_i(t))_{j=1}^i}(\mathrm{NoInt}(i,[0,2],\mathcal{A}_{i+1}))}\right]^{1/2} \cdot \mathbb{E}\left[\mu^{\mathcal{A}(0),\mathcal{A}(2)}(A_{\epsilon,i})\right]^{1/2} ,$$

where $\mu^{\mathcal{A}(0),\mathcal{A}(2)}(\cdot)$ denotes the law of an ensemble of i independent Brownian bridges with starting and ending points $(0, \mathcal{A})$ and $(2, \mathcal{A}(2))$ respectively which from Lemma 3.15, can be compared to Brownian motions on $[0, 1]$ with Radon-Nikodym bounded by

$$\left\|\frac{\mathrm{d}\mathfrak{B}^{[0,2]}_{\underline{0},\underline{\mathcal{A}}}|_{[0,1]}}{\mathrm{d}\mathfrak{B}^{[0,1]}_{\underline{0},*}}\right\|_{L^\infty\left(\mathfrak{B}^{[0,1]}_{\underline{0},*}\right)} = 2^{\frac{i}{2}} \cdot \exp\left(\frac{\left\|\mathcal{A}^i(2) - \mathcal{A}^i(0)\right\|^2}{8}\right) .$$

Thus, a localisation argument and Hölder give for all $a > 0$

$$\begin{aligned}&\mathbb{E}\left[\mu^{\mathcal{A}(0),\mathcal{A}(2)}(A)^2\right]^{1/2} \\ &= \mathbb{E}\left[\mu^{\mathcal{A}(0),\mathcal{A}(2)}(A)\mathbf{1}\left(\max_{1 \leq j \leq i} |\mathcal{A}(2) - \mathcal{A}(0)| < a\right)\right]^{1/2} \\ &\quad + \mathbb{P}\left(\max_{1 \leq j \leq i} |\mathcal{A}(2) - \mathcal{A}(0)| \geq a\right)^{1/2} .\end{aligned}$$

Now, combining the two estimates above, we obtain using Proposition 3.11

$$\begin{aligned}
&\mathbb{E}\left[\mu^{\mathcal{A}(0),\mathcal{A}(2)}(A)\right]^{1/2}\\
&\le \mathbb{E}\left[(2)^{\frac{i}{2}}\cdot\exp\left(\frac{\left\|\mathcal{A}^i(2)-\mathcal{A}^i(0)\right\|^2}{8}\right)\cdot\mathbf{1}\left(\max_{1\le j\le i}|\mathcal{A}(1)-\mathcal{A}(0)|<a\right)\right]^{1/2}\cdot\mu(A_{\epsilon,i})^{1/2}\\
&\quad+\mathbb{P}\left(\max_{1\le j\le i}|\mathcal{A}(2)-\mathcal{A}(0)|\ge a\right)^{1/2}\\
&\le 2^{\frac{i}{2}}\cdot\exp\left(\frac{ia^2}{8}\right)\cdot\mu(A_{\epsilon,i})^{1/2}+\mathbb{P}\left(\max_{1\le j\le i}|\mathcal{A}_j(2)-\mathcal{A}_j(0)|\ge a\right)^{1/2}\\
&\le c\exp\left(dia^2\right)\cdot\epsilon^{1/8}+ci^{1/2}\left(\exp\left(-Ca^2\right)\right).
\end{aligned}$$

for some universal $C,c,d>0$.

Using Lemma 4.6, we obtain by a union bound

$$\mathbb{P}(A_{\epsilon,i})\le c\exp\left(di^7+dia^2\right)\cdot\epsilon^{1/8}+c\exp\left(di^7\right)\left(\exp\left(-Ca^2\right)\right),$$

for some constants $C,c,d>0$. Moreover, Lemma 4.1 and Theorem 3.17 give for $i\ge 1$

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{Z_i(x,0)}{\sqrt{i}}+\sqrt{\frac{1}{2x}}\right|>\sqrt{\frac{1}{2x}}-\frac{1}{\sqrt{i}}\right)\\
&\le C_x\left(\mathbb{P}\left(\frac{|\langle(0,i)\to x\rangle-2\sqrt{2kx}|}{k^{1/2}}>1/\sqrt{2x}\right)+\exp\left(-d_x i^{3/4}\right)\right)\\
&\le C_x i^2\exp\left(-d_x i^{1/126}\right)\\
&\le C_x\exp\left(-d_x i^{1/126}\right),\qquad k\ge 1
\end{aligned}$$

after changing d_x, for some x-dependent $d_x>0$ and $C_x>0$.

Combining the above, we thus obtain for all $i\ge 1,\epsilon\in(0,1/2),a>0$ using Lemma 4.1

$$\begin{aligned}
&\mathbb{P}(Z_1(x,0)\ge-\epsilon)\le c\exp\left(di^7+dia^2\right)\cdot\epsilon^{1/8}\\
&+c\exp\left(di^7\right)\left(\exp\left(-Ca^2\right)\right)+C_x\exp\left(-d_x i^{1/126}\right),
\end{aligned}$$

for some universal $C_x,d_x>0$. Now, for $\theta\in(0,1)$ with

$$\begin{cases}
i=\left\lceil c_x\log^{126}(C_x/\theta)\right\rceil\\
a=d_x\left\lceil\log^{1/2}(c/\theta)+i^{7/2}\right\rceil\\
\epsilon=c_x\mathrm{e}^{-di^7-dia^2},
\end{cases}$$

for some positive c,d, we obtain

$$\mathbb{P}(Z_1(x,0)\ge-\epsilon)\le\theta.$$

Thus, we have for all $\theta\in(0,1)$

$$\sup\{\epsilon>0:\mathbb{P}(Z_1(x,0)\ge-\epsilon)\le\theta\}\ge\exp\left(-d_x\log^{882}(C_x/\theta)\right)$$

for some $C_x,d_x>0$. This gives the tails for all $\epsilon\in(0,1/2)$

$$\mathbb{P}(Z_1(x,0)\ge-\epsilon)\le C_x\exp\left(-d_x\log^{1/882}(1/\epsilon)\right),$$

for some $C_x,d_x>0$, which concludes the proof.

□

We start by making the following definition of a 'meagre' subset of $\mathbb{R}^n, n \geq 1$ that is sufficiently rich at 'all scales'. The following definition and discussion up to but not including Definition 4.11 is not strictly relevant for the proof of the main result, though we think it is useful as it provides uniform coalescence bounds on semi-infinite geodesics with sufficiently concentrated asymptotic directions.

Definition 4.8. *(meagreness criterion) Fix $n \in \mathbb{N}, M > 0, r > 0$ and let A be a bounded subset of $\mathbb{R}^n$, then A is called (M, r)-**meagre** if*

$$\limsup_{\epsilon \to 0} \frac{N(\epsilon)}{\exp\left(\log^r 1/\epsilon\right)} < M\,,$$

where $N(\epsilon)$ denotes the infimum of all cardinalities of ϵ-covers of A. For $r > 0$, a set is called $(\infty-, r)$-meagre if it is (M, r)-meagre for all $M > 0$.

Examples of $(\infty-, 1/\sigma)$-meagre sets for $\sigma > 1$ include finite sets and finite unions of rapidly convergent sequences, for instance $\{1/\mathrm{e}^{n^\sigma} : n \in \mathbb{N}\}$. A class of less trivial examples of **compact, perfect and nowhere dense** (hence uncountable) $(\infty-, 1/\sigma)$-meagre sets for $\sigma > 1$ include generalised Cantor sets where at each stage from each interval, a 'middle third' interval is removed. Thus, at stage $n \geq 1$, the set is contained in 2^n many intervals of some finite length ϵ_n (monotone in $n \geq 1$) small enough that $\limsup_{n\to\infty} 2^n/\exp(\log^{1/\sigma}(1/\epsilon_n)) < \infty$. One can then control

$$\limsup_{\epsilon \to 0} \frac{N(\epsilon)}{\exp\left(\log^{1/\sigma}(1/\epsilon)\right)} \leq 2 \limsup_{n \to \infty} \frac{2^n}{\exp\left(\log^{1/\sigma}(1/\epsilon_n)\right)} < \infty\,.$$

Moreover, the set of $(\infty-, 1/\sigma)$-meagre sets for $\sigma > 1$ is stable under finite unions and arbitrary intersections and under composition by Lipschitz maps. Moreover, all finite sets are (M, r) meagre for all $M, r > 0$.

Definition 4.9. *(Set Projection) Let X, Y be sets and consider $K \subseteq X \times Y$. Define the projection 'onto the first coordinate' $\mathrm{Pr}(K)_1$ by*

$$\mathrm{Pr}(K)_1 := \{x \in X : \exists y \in Y\,, (x, y) \in K\}\,.$$

In the following lemma, we refine Lemma 7.2 in [DJOBV22] to provide uniform control on no-coalescence of geodesics with arbitrarily close left endpoints in the prelimiting melon environments in 3.9. Note the uniformity concerns the midpoints of the geodesic endpoints which are in a meagre set.

Lemma 4.10. [DJOBV22] *Let $K \subseteq \mathbb{R}^+ \times \mathbb{Q}^2_>$ be compact and countable such that the projection of K onto its first coordinate $\mathrm{Pr}(K)_1$, has is (M, r)-**meagre** for some $M > 1$ and $r < 1/882$. Then*

$$\lim_{\epsilon \to 0^+} \limsup_{n\to\infty} \mathbb{P}\Bigg(\bigcup_{(x,y_1,y_2)\in K} \pi\{\overline{x} - \overline{\epsilon}, \hat{y}_1\} \quad \text{and} \quad \pi\{\overline{x} + \overline{\epsilon}, \hat{y}_2\} \textit{ are disjoint}\Bigg) = 0. \tag{4.9}$$

Hence, there exists a random dyadic valued random variable ϵ such that almost surely, for all triples $(x, y, z) \in K$,

$$\pi\{\overline{x - \epsilon} \to \hat{y}\}_n \qquad \textit{and} \qquad \pi\{\overline{x + \epsilon} \to \hat{z}\}_n$$

are not disjoint for all large enough n.

Proof. Note we can take some compact $K' \subseteq \mathbb{Q}$ that contains all the y_1, y_2 such that $(y_1, y_2) \in \mathrm{Pr}(K)_{2,3}$ with $\mathrm{diam}(K') \leq \mathrm{diam}(K)$. By the monotonicity of last passage paths, the inclusion

$$\bigcup_{(x,y_1,y_2)\in K} \{\pi\{\overline{x} - \overline{\epsilon}, \hat{y}_1\} \quad \text{and} \quad \pi\{\overline{x} + \overline{\epsilon}, \hat{y}_2\} \text{ are disjoint}\}$$

$$\subseteq \bigcup_x \{\pi\{\overline{x}-\bar{\epsilon}, \widehat{\inf_{x\in K}|x|'}\} \quad \text{and} \quad \pi\{\overline{x}+\bar{\epsilon}, \widehat{\sup_{x\in K}|x|'}\} \text{ are disjoint}\}\,.$$

Thus, it suffices to show that

$$\lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\mathbb{P}\Big(\bigcup_x \{\pi\{\overline{x}-\bar{\epsilon}, \widehat{\inf_{x\in K}|x|'}\} \quad \text{and} \quad \pi\{\overline{x}+\bar{\epsilon}, \widehat{\sup_{x\in K}|x|'}\} \text{ are disjoint}\}\Big) = 0\,.$$

In fact, we will prove a stronger statement, for $x \in \Pr(K)_1$, with the leftmost last passage path $\pi^-\{\overline{x}-\bar{\epsilon}, \widehat{\inf_{x\in K}|x|'}\}$ replacing one of the rightmost paths $\pi\{\overline{x}-\bar{\epsilon}, \widehat{\inf_{x\in K}|x|'}\}$. Disjointness of $\pi\{\overline{x}-\bar{\epsilon}, \widehat{\inf_{x\in K}|x|'}\}$ and $\pi\{\overline{x}+\bar{\epsilon}, \widehat{\sup_{x\in K}|x|'}\}$ implies disjointness of $\pi^-\{\overline{x}-\bar{\epsilon}, \widehat{\inf_{x\in K}|x|'}\}$ and $\pi\{\overline{x}+\bar{\epsilon}, \widehat{\sup_{x\in K}|x|'}\}$ by monotonicity.

By Lemma 4.5 in [DJOBV22], disjointness of the paths $\pi^-\{\overline{x}-\bar{\epsilon}, \widehat{\inf_{x\in K}|x|'}\}$ and $\pi\{\overline{x}+\bar{\epsilon}, \widehat{\sup_{x\in K}|x|'}\}$ is equivalent to disjointness of the original Brownian last passage paths $\pi^-[\overline{x}-\bar{\epsilon}, \widehat{\inf_{x\in K}|x|'}]$ and $\pi[\overline{x}+\bar{\epsilon}, \widehat{\sup_{x\in K}|x|'}]$. Here $\pi^-[\overline{x}-\bar{\epsilon}, \widehat{\inf_{x\in K}|x|'}]$ is the leftmost last passage path in B^n from $\overline{x}-\bar{\epsilon}$ to $\widehat{\inf_{x\in K}|x|'}$. Hence the probability in (4.9) is bounded above by

$$\mathbb{P}\left(\bigcup_{x\in\Pr(K)_1} \pi^-[\overline{x}-\bar{\epsilon}, \widehat{\inf_{x\in K}|x|'}] \quad \text{and} \quad \pi[\overline{x}+\bar{\epsilon}, \widehat{\sup_{x\in K}|x|'}] \text{ are disjoint}\right). \tag{4.10}$$

By time-reversal symmetry of the increments of Brownian motion under the map $t\mapsto 1-t$ the probability in (4.10) equals

$$\mathbb{P}\Big(\bigcup_x \{\pi^-[-\overline{\sup_{x\in K}|x|'}, 1-\overline{x}-\bar{\epsilon}] \quad \text{and} \quad \pi[-\overline{\inf_{x\in K}|x|'}, 1-\overline{x}+\bar{\epsilon}] \text{ are disjoint}\}\Big). \tag{4.11}$$

Now, let $N(\epsilon)$ be a family $N(\epsilon)$ of neighbourhoods in $\mathbb{R}$ that cover $\Pr(K)_1$ where each neighbourhood has diameter bounded above by ϵ. Then, we can estimate by a union bound

$$\mathbb{P}\Big(\bigcup_{x\in\Pr(K)_1} \{\pi^-[-\overline{\sup_{x\in K}|x|'}, 1-\overline{x}-\bar{\epsilon}] \quad \text{and} \quad \pi[-\overline{\inf_{x\in K}|x|'}, 1-\overline{x}+\bar{\epsilon}] \text{ are disjoint}\}\Big) \tag{4.12}$$

$$\leq \sum_{U\in N(\epsilon)}\mathbb{P}\Big(\bigcup_{x\in U} \{\pi^-[-\overline{\sup_{x\in K}|x|'}, 1-\overline{x}-\bar{\epsilon}] \quad \text{and} \quad \pi[-\overline{\inf_{x\in K}|x|'}, 1-\overline{x}+\bar{\epsilon}] \text{ are disjoint}\}\Big). \tag{4.13}$$

Now, fixing such $U \in N(\epsilon)$, observe that by translation invariance and Brownian scaling, the probability (4.11) remains unchanged if the points $-\overline{\sup_{x\in K}|x|'}, 1-\overline{x}-\bar{\epsilon}, -\overline{\inf_{x\in K}|x|'}, 1-\sup_{x\in U}\overline{x}+\bar{\epsilon}$ are replaced by their images under any linear function $L(t) = at+b$ for some $a>0, b\in\mathbb{R}$. In particular, for each n we may choose the linear function $L = L_{n,\epsilon}$ sending $-\overline{\sup_{x\in K}|x|'} \mapsto 2(\overline{\sup_{x\in K}|x|'} - \overline{\inf_{x\in K}|x|'})$ and $1-\sup_{x\in U}\bar{x}+\bar{\epsilon}\mapsto 1$. For $t\in[-1,2]$, we have

$$L_{n,\epsilon}(t) = (1-2\overline{\sup_{x\in K}|x|'} + \overline{\inf_{x\in K}|x|'} + \sup_{x\in U}\bar{x} - \bar{\epsilon})t + 2\overline{\sup_{x\in K}|x|'} - \overline{\inf_{x\in K}|x|'} + O(n^{-2/3}).$$

Therefore for all large enough n, we have for all $x \in \Pr(K)_1$, the projection of K onto its first co-odrinate,

$$L_{n,\epsilon}(-\overline{\inf_{x\in K}|x|'}) \geq 2(\overline{\sup_{x\in K}|x|'} - \overline{\inf_{x\in K}|x|'}) + O(n^{-2/3}) \geq 0, \tag{4.14}$$

$$L_{n,\epsilon}(1-\bar{x}-\bar{\epsilon}) \geq 1 - 2\bar{\epsilon} + O(n^{-2/3}) \geq 1 - 3\bar{\epsilon}, \quad x \in U\,. \tag{4.15}$$

and

$$L_{n,\epsilon}(1-\bar{x}+\bar{\epsilon}) \leq 1 + \bar{\epsilon} + O(n^{-2/3}) \leq 1 + 2\bar{\epsilon}\,, \qquad x \in U\,.$$

After translating back to melon paths we get that the probability in (4.11) is equal to

$$\mathbb{P}\Big(\bigcup_{x\in U} \pi^-\{L_{n,\epsilon}(-\overline{\sup_{x\in K}|x|'}), L_{n,\epsilon}(1-\bar{x}-\bar{\epsilon})\} \text{ and } \pi\{L_{n,\epsilon}(-\overline{\inf_{x\in K}|x|'}), L_{n,\epsilon}(1-\bar{x}+\bar{\epsilon})\} \text{ are disjoint}\Big).$$

By monotonicity of last passage paths, [DJOBV22, Lemma 3.6], and (4.14), this is bounded above by

$$\mathbb{P}(\pi^-\{0, 1-3\bar{\epsilon}\} \text{ and } \pi\{\overline{2\mathrm{diam}(K)}, \hat{\epsilon}\} \text{ are disjoint}) \tag{4.16}$$

for n large enough. Now, the path $\pi^-\{0, 1-3\bar{\epsilon}\}$ starts at zero and therefore simply follows the top line in the melon, so the paths $\pi^-\{0, 1-3\bar{\epsilon}\}$ and $\pi\{\overline{2\mathrm{diam}(K)}, \hat{\epsilon}\}$ are disjoint if and only if $\pi\{\overline{2\mathrm{diam}(K)}, \hat{\epsilon}\}$ jumps up to line 1 after time $1-3\bar{\epsilon}$. This jump time is $\hat{Z}_1^n(2\mathrm{diam}(K), \epsilon)$, so (4.16) is equal to

$$\mathbb{P}(Z_1^n(2\mathrm{diam}(K), \epsilon) \geq -3\epsilon).$$

Thus, a union bound and the above gives

$$\begin{aligned}\liminf_{n\to\infty}(4.12) &\leq \limsup_{n\to\infty}|N(\epsilon)|\mathbb{P}(Z_1^n(2\mathrm{diam}(K), \epsilon) \geq -3\epsilon) \\ &\leq C_{\mathrm{diam}(K)} M \exp(-\log^{1/882-r}(1/\epsilon)) \xrightarrow{\epsilon\to 0} 0\end{aligned}$$

for some $C_{\mathrm{diam}(K)} > 0$, concluding the proof. □

Now, Lemma 4.10 allows us to strengthen the coupling in (points $2., 3., 4.$ of the itemized list in page 37 of [DJOBV22]) to the following. Fix $K \subseteq \mathbb{R}^+ \times \mathbb{R}^2$ compact and countable. By Skorokhod representation theorem and Lemma 4.10, there exists a coupling of the process WB^n and $\mathcal{A}$ and a subsequence, such that along that subsequence, almost surely

1. The melon WB^n in the scaling in Theorem 3.9 converges to the Airy line ensemble $\mathcal{A}$ uniformly on compact sets in $\mathbb{Z} \times \mathbb{R}$.
2. We have for all $(x, y) \in \Pr(K)_{1,2} \subseteq \mathbb{R}^+ \times \mathbb{R}$,
$$Z_k^n(x, y) \to Z_k(x, y) \qquad \text{for all } k \in \mathbb{N}.$$
Moreover, as $k \to \infty$,
$$Z_k(x, y)/\sqrt{k} \to -1/\sqrt{2x}\,.$$
3. There exist random $\epsilon \in (0, 1) \cap \Pr(K)_1$ such that for every triple $(x, y, z) \in K \subseteq$ with $\Pr(K)_1$ being (M, r)-meagre for some $M > 1, r < 1/882$ with $y \leq z$
$$\pi\{\overline{x-\epsilon} \to \hat{y}\}_n \qquad \text{and} \qquad \pi\{\overline{x+\epsilon} \to \hat{z}\}_n$$
are not disjoint for all large enough n.

Combining the above, we obtain the uniform coalescence of semi-infinite geodesics on the Airy line ensemble starting from any fixed level with the ones starting from the top. Crucially, we take their speeds to be contained in a meagre set. Note individual coalescence depths (and times) are always finite due to [SV21]. But first we make an important definition, making precise what we mean by 'geodesic coalescence depth'.

Definition 4.11. *(Geodesic coalescence depth) Fix $x \in (0,\infty)$ and $\ell \in \mathbb{N}$. Then, with the π-notation as in Definition 4.3, define the geodesic coalescence depth $K_{x,\ell} \in \mathbb{N}$ as*

$$K_{x,\ell} := \inf\left\{k \geq 1 : \pi[x \to (0,1))](s) = \pi[x \to (0,\ell))](s), \forall s \leq -\sqrt{k/(2x)}\right\}.$$

We now prove that the coalescence depths of semi-infinite geodesics with speeds that are concentrated in an appropriately meagre set are uniformly bounded.

Proposition 4.12. *Let $K \subseteq \mathbb{R}_+ \times \mathbb{R}^2_>$ be compact and countable with $\mathrm{Pr}(K)_1 \subset [1,\infty)$ (M,r)-meagre for some $M > 1, r < 1/882$ and let $\ell \in \mathbb{N}$. Then, with $K_{x,\ell}$ as in Definition 4.11, one has for fixed $\ell \in \mathbb{N}$*

$$\sup_{x\in \mathrm{Pr}(K)_1} K_{x,\ell} < +\infty\,, \qquad \text{a.s.}$$

Proof. Then we define $\pi[x,y] : (-\infty,y] \to \mathbb{Z}$ as the non-increasing cadlag function given by

$$\pi[x,y](t) = \min\{k \in \mathbb{N} : Z_{k+1}(x,y) \leq t\}$$

for all $t \in (-\infty,y]$. Thus, $Z_k(x,y)$ is the supremum of w so that (w,k) lies along $\pi[x,y]$. The path $\pi[x,y]$ is an almost sure pointwise limit of γ_n over the subsequence. Moreover, Property 1 above guarantees that $\pi[x,y]$ is a rightmost last passage path when restricted to any compact interval.

Now fix any $(x,y,z) \in K, y < z$ as in the statement of Proposition 4.12. Let $\epsilon > 0$ be as in Property 3 above, that is, $\pi\{\overline{x-\epsilon} \to \hat{y}\}_n$ and $\pi\{\overline{x+\epsilon} \to \hat{y}\}_n$ are not disjoint for all large enough n. Observe that from Lemma 4.1, one has has the jump times of semi-infinite geodesics for $(x,y) \in \mathrm{Pr}(K)_{1,2}$ satisfy

$$\sup_{(x,y)\in K}\left|\frac{Z_\ell(x,y)}{\sqrt{\ell}} + \sqrt{\frac{1}{2x}}\right| \overset{\ell\to\infty}{\longrightarrow} 0 \qquad \text{a.s.}.$$

This means that there exists a random $N \in \mathbb{N}$ such that (after possibly augmenting K, keeping it compact and countable)

$$-\sqrt{k/2x} \in \left(Z_k(x-\epsilon,y), Z_k(x+\epsilon,z)\right),$$

for all $(x,y,z) \in K, k \geq N$.

Claim: $\sup_{x\in \mathrm{Pr}(K)_1} K_{x,\ell} \leq N$.

First recall from [SV21, Lemma 3.4] that for any $x \in \mathrm{Pr}(K)_1, y < z, (y,z) \in \mathrm{Pr}(K)_{2,3}$, almost surely there exists a random $T \leq y \in \mathbb{R}$ (depending on x,y,z) such that

$$\pi[x \to y](T) = \pi[x \to z](T) = \pi[x_k \to y](T) = \pi[x_k \to z](T)\,, \tag{4.17}$$

for all $k \geq N$. That is, the paths $\pi[x_k \to y]$, $\pi[x_k \to z]$, $\pi[x \to y]$ and $\pi[x \to z]$ intersect for all large k. Moreover, for all $t \geq T$ and $k \geq N$,

$$\pi[x \to y](t) = \pi[x_k \to y](t) \qquad \text{and} \qquad \pi[x \to z](t) = \pi[x_k \to z](t)\,.$$

Finally let the common value in (4.17) be denoted by $d(T)$.

Indeed, observe that $\pi[x,y]$ and $\pi[x_k \to y]$ restricted to $[T,y]$ are both rightmost geodesics between $(T,d(T))$ and $(y,1)$. Hence for all $t \geq T$ and $k \geq K$,

$$\pi[x,y](t) = \pi[x_k \to y](t) \qquad \text{and} \qquad \pi[x,z](t) = \pi[x_k \to z](t)\,. \tag{4.18}$$

Next we claim that for $(x,y) \in \mathrm{Pr}(K)_{1,2}$, almost surely for all $r \in \mathbb{Z}; r < y$, there exists a random $K \in \mathbb{N}$ (depending on x,y,r) such that for all $t \in [r,y]$ and all $k \geq N$,

$$\pi[x,y](t) = \pi[x_k \to y](t)\,.$$

Indeed, by (4.18) with $x \in \mathbb{Q}^+$ and $r < y$, we have that there exists a random $T \leq r$ and $K \in \mathbb{N}$ such that for all $t \in [T, y]$ and all $k \geq K$,

$$\pi[x, y](t) = \pi[x_k \to y](t)\,.$$

Since $T \leq r$ and $[r, y] \subseteq [T, y]$, the claim follows.

Fix any $x \in \mathrm{Pr}(K)_1$, $\ell \geq 1$. Using [SV21, Lemma 3.7], we get a random $Y_\ell \in \mathrm{Pr}(K)_1$ such that $Z_\ell(x, Y) > 0$ almost surely. More precisely, we can take

$$Y_\ell = \min\{n \in \mathbb{N} : Z_\ell(\inf \mathrm{Pr}(K)_1, n) > 0\} < +\infty \qquad \text{a.s.}\,. \tag{4.19}$$

Moreover, this can be done uniformly over $x \in \mathrm{Pr}(K)_1$ by the monotonicity of the jump times $Z_\ell(\cdot, \cdot)$. By the above argument, we have that there exist $(T, d(T))$ such that almost surely for all $k \geq N$, the paths $\pi[x \to 0], \pi[x \to Y], \pi[x_k \to 0]$ and $\pi[x_k \to Y]$ intersect at $(T, d(T))$. Since $T \leq 0$, and $Z_\ell(x, Y) > 0$,

$$d(T) > \ell\,.$$

Since

$$Z_\ell(x, 0) \leq 0 < Z_\ell(x, Y)\,,$$

by ordering of geodesics, for all $k \geq N$, $\pi[x_k \to (0, \ell)]$ also passes through $(T, d(T))$. Thus for all $k \geq N$,

$$\mathcal{A}[x_k \to (0, \ell)] - \mathcal{A}[x_k \to (0, 1)] = \mathcal{A}[(T, d(T)) \to (0, \ell)] - \mathcal{A}[(T, d(T)) \to (0, 1)]\,.$$

Hence we establish the uniform upper bound on the coalescence depths $K_{x,\ell}$ for $x \in \mathrm{Pr}(K)_1$. $\square$

Proposition 4.12 gives a roadmap for obtaining tails of the coalescence depths of semi-infinite geodesics by localising on a series of favourable events that depend on jump times thereof, leading to Theorem 4.14. But before we proceed with the proof of Theorem 4.14, we start with some preliminary results that control these favourable events.

Now, we can obtain the following tail bounds on a random threshold $\epsilon > 0$ in Lemma 4.10 which ensures that any two geodesics with endpoints closer than ϵ meet, uniformly over points in a meagre set. This is the content of the next proposition.

Proposition 4.13. *Let $K \subseteq \mathbb{R}^+ \times \mathbb{R}^2$ compact such that the projection of K onto its first coordinate $\mathrm{Pr}(K)_1$, has is (M, r)-**meagre** for some $M > 1$ and $r < 1/882$. With the random ϵ as in Lemma 4.10 one obtains the following for $\epsilon_0 > 0$ sufficiently small (possibly depending on K)*

$$\mathbb{P}(\epsilon < \epsilon_0) \leq C_K M \exp\left(-d_K \log^{1/882 - r}(1/\epsilon_0)\right)$$

for some $C_K, d_K > 0$.

Combining the above, we are now in a position to prove the second theorem of this section, giving uniform coalescence of semi-infinite geodesics with speeds located on some 'meagre' set.

Theorem 4.14. *Let $K \subseteq [1, \infty)$ be countable, bounded and (M, r)-**meagre** for some $M > 1$ and $r < 1/882$. Then, for any $\theta \in (0, 1)$,*

$$\begin{aligned}
&\inf\left\{m \geq 1 : \mathbb{P}(\sup_{x \in K} K_{x,\ell} \geq m) \leq \theta\right\} \\
&\leq C_K \ell^{256} \left(\exp\left(d_K M \log^{1/(1/882 - r)}(1/\theta)\right)\right)\,,
\end{aligned}$$

for some $C_K, d_K > 0$.

Proof. Fix any $m \in \mathbb{N}$, going through the Skorokhod coupling and Y_ℓ, ϵ as in the proof of Proposition 4.12, one observes for any $n \geq 1, \epsilon_0 > 0$, the inclusion of the event

$$\{Y_\ell \leq n\} \cap \{\epsilon \geq \epsilon_0\} \cap \left\{ \sup_{(x,y) \in K \times [0,n] \cap \mathbb{Q}} \left| \frac{Z_m(x,y)}{\sqrt{m}} + \sqrt{\frac{1}{2x}} \right| \leq c_K \epsilon_0 \right\} \subseteq \{\sup_{x \in K} K_{x,\ell} \leq m\},$$

with $c_K = 1/(2\sqrt{\sup_{x \in K} |x|})$. Thus, we can estimate using Proposition 4.13 and (4.19) the tails of the maximum coalescence depth by a union bound for any $n \geq 1, \epsilon_0$

$$\begin{aligned}
\mathbb{P}(\sup_{x \in K} K_{x,\ell} \geq m) &\leq \mathbb{P}\left(\sup_{(x,y) \in K \times [0,n] \cap \mathbb{Q}} \left| \frac{Z_m(x,y)}{\sqrt{m}} + \sqrt{\frac{1}{2x}} \right| > c_K \epsilon_0 \right) \\
&\quad + \mathbb{P}(Y_\ell > n) + \mathbb{P}(\epsilon < \epsilon_0) \\
&\leq \mathbb{P}\left(\sup_{(x,y) \in K \times [0,n] \cap \mathbb{Q}} \left| \frac{Z_m(x,y)}{\sqrt{m}} + \sqrt{\frac{1}{2x}} \right| > c_K \epsilon_0 \right) \\
&\quad + \mathbb{P}(Z_1(2\mathrm{diam}(K), 0) \geq -4\epsilon_0) + \mathbb{P}(Y_\ell > n) \\
&\leq \mathbb{P}\left(\sup_{(x,y) \in K \times [0,n] \cap \mathbb{Q}} \left| \frac{Z_m(x,y)}{\sqrt{m}} + \sqrt{\frac{1}{2x}} \right| > c_K \epsilon_0 \right) \\
&\quad + \mathbb{P}\left(\sup_{x \in K} \left| \frac{Z_\ell(x,0)}{\sqrt{\ell}} + \sqrt{\frac{1}{2x}} \right| > \frac{n}{\sqrt{\ell}} - \sqrt{\frac{1}{2x}} \right) \\
&\quad + 2\mathbb{P}(Z_1(\inf_{x \in K} |x|, 0) \geq -4\epsilon_0),
\end{aligned}$$

for some constant $c > 0$. Now, using Lemmas 4.1, 4.7 and Theorem 3.17, we estimate for $\epsilon \in (0, 1/c_K \wedge 1)$ and $m \geq 1$

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(x,y) \in K \times [0,n] \cap \mathbb{Q}} \left| \frac{Z_m(x,y)}{\sqrt{m}} + \sqrt{\frac{1}{2x}} \right| > c_K \epsilon_0 \right) \\
&\leq C_K/\epsilon_0 \left(\sup_{x \in K} \mathbb{P}\left(\frac{|\langle (0,m) \to x \rangle - 2\sqrt{2mx}|}{m^{1/2}} > C_K \epsilon_0 \right) \right. \\
&\left. + \exp\left(-d_K \epsilon_0^{3/4} \frac{m^{3/4}}{(\sup_{x \in K} |x| + |n| + 1)^2} \right) \right) \\
&\leq C_K m^2 \exp\left(-d_K \frac{\epsilon_0 m^{1/126}}{(\sup_{x \in K} |x| + |n| + 1)^2} \right)
\end{aligned}$$

for some $C_K, d_K > 0$. Similarly we estimate for $n \geq 2\sqrt{\ell}/\sqrt{2 \inf_{x \in K} |x|}$

$$\begin{aligned}
&\mathbb{P}\left(\sup_{x \in K} \left| \frac{Z_\ell(x,0)}{\sqrt{\ell}} + \sqrt{\frac{1}{2x}} \right| > \frac{n}{\sqrt{\ell}} - \sqrt{\frac{1}{2x}} \right) \\
&\leq C_K n^2/\ell \left(\sup_{x \in K} \mathbb{P}\left(\frac{|\langle (0,\ell) \to x \rangle - 2\sqrt{2\ell x}|}{\ell^{1/2}} > \frac{n}{2\sqrt{\ell}} \right) \right. \\
&\left. + \exp\left(-d_K \frac{n}{\ell^{1/2}}^{3/4} \frac{m^{3/4}}{(\sup_{x \in K} |x| + 1)^2} \right) \right)
\end{aligned}$$

$$\leq C_K n^2 \left(\exp\left(-d_K \frac{n^{3/4}\ell^{3/8}}{(\sup_{x\in K}|x|+1)^2}\right) + \exp\left(-d\frac{n}{\ell^2}\right)\right)$$
$$\leq C_K n^2 \left(\exp\left(-d_K\frac{n}{\ell^2}\right)\right)$$

for some $C_K, d_K > 0$. Additionally, using Lemma 4.1, we estimate for $\epsilon_0 \in (0, \epsilon_K)$ with some $\epsilon_K > 0$

$$\mathbb{P}(Z_1(\inf_{x\in K}|x|, 0) \geq -4\epsilon_0) \leq C_K \exp\left(-d_K \log^{1/882}(1/\epsilon_0)\right)$$

for some $C_K, d_K > 0$.

Combining the above, we estimate for all $m \geq 1$, $\epsilon_0 \in (0, \epsilon_K), n \geq 2\sqrt{\ell}/\sqrt{2\inf_{x\in K}|x|}$ for some $\epsilon_K > 0$

$$\begin{aligned}
\mathbb{P}(\sup_{x\in K} K_{x,\ell} \geq m) &\leq C_K \left(m^2 \exp\left(-d_K \frac{\epsilon_0 m^{1/126}}{(\sup_{x\in K}|x|+|n|+1)^2}\right)\right.\\
&\left.+n^2\exp\left(-d_K\frac{n}{\ell^2}\right) + M\exp\left(-d_K \log^{1/882-r}(1/\epsilon_0)\right)\right)\\
&\leq C_K \left(\frac{n^{512}}{\epsilon_0^{256}}\exp\left(-d_K \frac{\epsilon_0 m^{1/126}}{(\sup_{x\in K}|x|+|n|+1)^2}\right)\right.\\
&\left.+\ell^4\exp\left(-d_K\frac{n}{\ell^2}\right) + M\exp\left(-d_K \log^{1/882-r}(1/\epsilon_0)\right)\right).
\end{aligned}$$

for some $C_K, d_K > 0$.

Now, fix $\theta \in (0,1)$. With

$$\begin{cases}
n = \left\lceil C_K\left(\log(\ell/\theta) + \sqrt{\ell}\right)\right\rceil \\
\epsilon_0 = C_K' \exp\left(-d_K \log^{1/(1/882-r)}(M/\theta)\right) \\
m = \left\lceil C_K \frac{n^{512}}{\epsilon_0^{256}} \log^{126}(\frac{\epsilon_0}{\theta n})\right\rceil
\end{cases}$$

for some positive $C_K, C_K', d_K > 0$, we deduce

$$\mathbb{P}(\sup_{x\in K} K_{x,\ell} \geq m) \leq \theta\,.$$

We thus deduce that for any $\theta \in (0,1)$,

$$\begin{aligned}
\inf\left\{m \geq 1 : \mathbb{P}(\sup_{x\in K} K_{x,\ell} \geq m) \leq \theta\right\} &\leq m(\epsilon_0, n, \ell, \theta)\\
&\leq C_K \ell^{256}\left(\exp\left(d_K M \log^{1/(1/882-r)}(1/\theta)\right)\right),
\end{aligned}$$

for some $C_K, d_K > 0$, concluding the proof. □

5. Regularity of finite-depth truncations of the KPZ fixed point

In this section, we obtain a quantitative comparison of the spatial increments of 'finite depth truncations' of the KPZ fixed point in terms of the Wiener measure and last passage values of semi-infinite geodesics in the Airy line ensemble. We crucially use the variational formula for the KPZ fixed point and the coupling in Definition 4.2. This is achieved through the Brownian Gibbs property of the Airy line ensemble, which further reduces the problem to estimating the

Radon-Nikodym derivatives of inhomogeneous Brownian LPP with non-decreasing initial data. This is done in Theorem 3.7. Technical input from [Dau24] allows us to estimate inverse acceptance probabilities that appear in the estimates. Combining the above leads to Theorem 5.5.

By $1:2:3$ scaling, we lose no generality in considering the KPZ fixed point at unit time, $h_1(\cdot)$ with initial data $h_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, which can be written more explicitly as

$$h(y) = \sup_{x \in \operatorname{supp}_{-\infty}(h_0)} \left(h_0(x) + \mathcal{S}(x, y) \right),$$

where $\mathcal{S}(\cdot, \cdot)$ denotes the Airy sheet, see Definition 4.2. Recall also Definition 1.2 for the max-plus support. We also do not lose generality if we translate the support of the initial data to lie in $[1, x_0]$, for some $x_0 > 1$. We additionally set the interval of comparison to $[1, y_0]$ with $y_0 > 1$ as in Lemma 4.4, at no loss of generality. Henceforth, we will make the following assumptions on the initial data:

- $\operatorname{supp}_{-\infty}(h_0)$ is bounded and countable
- $\operatorname{supp}_{-\infty}(h_0) \subseteq [1, \infty)$.

Now, from Lemma 4.4 with $K = \operatorname{supp}_{-\infty}(h_0)$ and $x_0, y_0 > 1$ as above, there is a random constant L_0 with tails as in Theorem 4.5, such that almost surely for all $y \in [1, y_0]$

$$h(y) = \sup_{x \in \operatorname{supp}_{-\infty}(h_0)} \left(h_0(x) + \max_{\ell \leq L_0} \mathcal{A}[x \to (0, \ell)] + \mathcal{A}[(0, \ell) \to (y, 1)] \right)$$

where $\mathcal{A}$ is an Airy line ensemble that is coupled to the Airy sheet $\mathcal{S}$ as in Definition 4.2 and

$$\mathcal{A}[x \to (0, \ell)] := \begin{cases} \mathcal{S}(x, 0) & \ell = 1 \\ \lim\limits_{k \to \infty} \mathcal{A}[x_k \to (0, \ell)] - \mathcal{A}[x_k \to (0, 1)] + \mathcal{S}(x, 0) & \ell > 1\,, \end{cases} \tag{5.1}$$

for $x \in \mathbb{Q}^+ \cup \operatorname{supp}_{-\infty}(h_0)$ with $x_k = (-\sqrt{k/2x}, k), k \in \mathbb{N}$.

The fact that this limit exists and is well defined is the crux of Theorem 3.7 in [SV21] and uses geometric properties of geodesics in the Airy line ensemble. In some sense, the downward parabolic curvature of the Airy line ensemble $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \cdots)$, which follows from the observation that for $i \in \mathbb{N}$, the process $\mathcal{A}_i(\cdot) + (\cdot)^2$ is stationary (see [SV21]), forces rightmost last passage paths (rightmost geodesics) to almost surely eventually intersect in the far left end of the plane (Lemma 3.4 in [SV21]). For $\ell = 1$, $\mathcal{A}[x \to (0, \ell)] = \mathcal{S}(x, 0)$, which has Hölder $1/2-$ continuous sample paths and is the parabolic Airy$_2$ process. For $\ell > 1$, the pathwise properties of $\mathcal{A}[x \to (0, \ell)]$ for $x > 0$ become a lot less clear. However, the uniform modulus of continuity estimates for the Airy line ensemble 3.10 and geodesic coalescence time tail bounds, see Theorem 4.14, allow us to obtain that the process $x \mapsto \mathcal{A}[x \to (0, \ell)]$ is *continuous in probability*, see Section A in the appendix.

Now, exchanging the sup and max gives

$$h(y) = \max_{\ell \leq L_0} (G_\ell + \mathcal{A}[(0, \ell) \to (y, 1)])\,,$$

where

$$G_\ell := \sup_{x \in \operatorname{supp}_{-\infty}(h_0)} (h_0(x) + \mathcal{A}[x \to (0, \ell)])\,.$$

For any $\ell \in \mathbb{N}$, G_ℓ enjoys the following two properties, namely, that almost surely $G_\ell < \infty$ and that it is measurable with respect to the sigma algebra $\mathcal{F}_- := \sigma(\{\mathcal{A}_i(x) : x \leq 0, i = 1, 2, \cdots\})$. This is the content of Lemmas 3.8 and 3.9 in [SV21] respectively. The latter property essentially follows from the Definition 5.1.

One can observe by inspecting the proof of Lemma 3.10 in [SV21] that for any $m \in \mathbb{N}$, on the event $\{L_0 \leq m\}$ one has the almost sure equality

$$\mathcal{S}(x,y) = \max_{1\le \ell\le m} \left(\mathcal{A}[x\to(0,\ell)] + \mathcal{A}[(0,\ell)\to(y,1)]\right)$$

for all $x \in \mathrm{supp}_{-\infty}(h_0)$ and all $y \in [1, y_0]$. Thus, on the event $\{L_0 \le m\}$ the KPZ fixed point has the expression (at unit time)

$$\begin{aligned} h(y) &= \max_{x\in\mathrm{supp}_{-\infty}(h_0)} (h_0(x) + \mathcal{S}(x,y)) \\ &= \max_{\ell\le m}(G_\ell + \mathcal{A}[(0,\ell)\to(y,1)]) \qquad \text{for } y\in[1,y_0] \end{aligned}$$

where $G_\ell := \max_{x\in\mathrm{supp}_{-\infty}(h_0)} (h_0(x) + \mathcal{A}[x\to(0,\ell)]) < \infty$ almost surely.

Thus, being able to control the tails of L_0, we can apply a localisation argument and derive a priori L^p, $p>1$ estimates for the law of the 'truncated' profile

$$H_m(\cdot) = \max_{\ell\le m}(G_\ell + \mathcal{A}[(0,\ell)\to(\cdot,1)])\,, \qquad m\ge 1\,.$$

against that of rate two Brownian motion on the interval $[1, y_0]$. Below is a sketch of the estimates to follow. First note that the absolute continuity relation $\mathrm{dLaw}_{H_m} \ll \mathrm{d}\mathfrak{B}_{*,*}^{[1,y_0]}$ has already been established in [SV21, Proposition 5.1].

For $a<b\in\mathbb{R}, k\in\mathbb{N}$ set $\mathcal{F}_k^{[a,b]} := \sigma(\{A_i(x) : (i,x)\notin [\![1,m]\!]\times(a,b)\})$. Now we turn our attention towards bounding

$$\left\| \frac{\mathrm{dLaw}_{H_m}}{\mathrm{d}\mathfrak{B}_{*,*}^{[1,y_0]}} \right\|_{L^p(\mathfrak{B}_{*,*}^{[1,y_0]})}$$

for $m\ge 1$. We start with a quick estimate that will motivate the rest of this section. Fix $A\subseteq C_{*,*}([1,y_0])$ Borel measurable and compute

$$\mathbb{P}(H_m(\cdot)\in A) = \mathbb{P}\left(\max_{\ell\le m}(G_\ell + \mathcal{A}[(0,\ell)\to(\cdot,1)])\in A\right).$$

Now, conditioning on the sigma algebra $\mathcal{F}_k^{[0,y_0+1]}$ we get

$$\mathbb{P}(H_m(\cdot)\in A) \le \mathbb{E}\left[\mathbb{P}\left(\max_{\ell\le m}(G_\ell + (\mathcal{A}-\mathcal{A}(0))[(0,\ell)\to(y,1)])\in A\middle|\mathcal{F}\right)\cdot \mathbf{1}_{\mathbf{Fav}}\right] + \mathbb{P}(\mathbf{Fav}^c)$$

where we used the fact that Brownian LPP ignores constant shifts to the environment, for some favourable event **Fav**, whose probability we wish to be able to control (perhaps as a function of some parameter which we will be free to choose). The next natural thing one could do is to apply the **Brownian Gibbs property** which the parabolic Airy line ensemble satisfies, however this leads to the technical challenge of estimating Brownian inverse acceptance probabilities with Airy line ensemble endpoints, see Subsection 3.6.

The crucial technical input that allows us to overcome this challenge comes from [Dau24], in order to estimate the inverse acceptance probability that comes from using the Brownian Gibbs property. In particular, we will need the following slight modification of [Dau24, Lemmas 3.2, 3.3] The goal of the next few lemmas is to estimate $\mathfrak{B}_{\underline{x},\underline{y}}^{[a,b]}(\mathrm{NoInt}([a,b],f))$ as given above in terms of a few simple $\mathcal{F}_m^{[-T_m,U_m]}$-measurable random variables for some $-a<T_m,U_m<b$.

Lemma 5.1. [Dau24, Lemma 3.3] *Fix $t>1$, $a<s<t<b$ and let $\underline{x},\underline{y}\in\mathbb{R}^m_>$. Let $g\in C_{*,*}([a,b])$ be such that $g(a)<x_m, g(b)<y_m$.*

Let B be a m-tuple on independent Brownian bridges from $(a,\underline{x})$ to $(b,\underline{y})$, conditioned on the event

$$\mathrm{NoInt}([a,s]\cup[t,b],g) \quad \text{or} \quad \mathrm{NoInt}([a,b],g)$$

Fix $\epsilon \in (0,1)$ and define $\iota = (1/m, 1/(m+1), \cdots, 1/(2m)), \mathbf{1} = (1,1,\ldots,1) \in \mathbb{R}^m$, and for $\alpha, \beta \geq 0$ define $f^{\alpha,\beta} \in C^m_{*,*}([a,b])$ by letting

$$f^{\alpha,\beta}(a) = 0, \quad f^{\alpha,\beta}(s) = f^{\alpha,\beta}(t) = \alpha\iota + \beta\mathbf{1}, \quad f^{\alpha,\beta}(b) = 0,$$

and so that $f^{\alpha,\beta}$ is linear on each of the pieces $[a,s],[s,t],[t,b]$.

Then for $f \in \mathrm{NoInt}([a,s]\cup[t,b],g)$ (or $\mathrm{NoInt}([a,b],g)$) we have the pointwise lower bound on the density of the law μ'_B of $B - f^{\alpha,\beta}$ against the law μ_B of B, on the set $\mathrm{NoInt}([a,1]\cup[t,b],g)$ (or $\mathrm{NoInt}([a,b],g)$) where μ'_B is absolutely continuous with respect to μ_B

$$\frac{\mathrm{d}\mu'_B}{\mathrm{d}\mu_B}(f) \geq \exp\left(-\zeta^2\frac{m(\alpha/m+\beta)^2}{4} - \zeta\frac{(\alpha/m+\beta)\sum_{i=1}^m[(f_i(s)-x_i)^+ + (f_i(t)-y_i)^+]}{4}\right) \tag{5.2}$$

$$\geq \exp\left(-\zeta^2\frac{m(\alpha/m+\beta)^2}{4} - \zeta\frac{(\alpha/m+\beta)\sum_{i=1}^m[(f_i(s)-x_i)^2_+ + (f_i(t)-y_i)^2_+]}{4}\right), \tag{5.3}$$

where $\zeta = \frac{1}{\min(s-a,b-t)}$.

Proof. Let ν be the law of m independent Brownian bridges from $(a,\underline{x})$ to $(b,\underline{y})$. Observe first that $f^{\alpha,\beta}$ being piecewise linear, it is in the Sobolev space $W^{1,2}([a,b])$ and so by Girsanov's Theorem, for a rate two Brownian motion W starting from $x \in \mathbb{R}$ on $[a,b]$, the Radon-Nikodym derivative of the process $W - f^{\alpha,\beta}$ against W is given by

$$\exp\left(-\frac{1}{2}\int_{[a,b]}\dot{f}^{\alpha,\beta}(s)\,\mathrm{d}W_s - \frac{1}{4}\int_{[a,b]}(\dot{f}^{\alpha,\beta})^2(s)\,\mathrm{d}s\right)$$

$$= \exp\left(-\frac{1}{2}(W_s - x)\frac{f^{\alpha,\beta}(s)}{s-a} + \frac{1}{2}(W_b - W_t)\cdot\frac{f^{\alpha,\beta}(t)}{b-t} - \frac{1}{4}\left(\left(\frac{f^{\alpha,\beta}(s)}{s-a}\right)^2 + \left(\frac{f^{\alpha,\beta}(t)}{b-(t)}\right)^2\right)\right).$$

Now conditioning on $W_b = y \in \mathbb{R}$ and using the uniqueness of regular conditional distributions and the regularity of the conditional measures for Brownian bridges and the above Radon-Nikodym transform thereof, we can conclude by independence that μ_B, μ'_B are absolutely continuous with respect to ν with densities

$$\frac{d\mu_B}{d\nu}(f) = \frac{1}{Z}\mathbf{1}(f \in \mathrm{NoInt}([a,s]\cup[t,b],g)),$$

$$\frac{d\mu'_B}{d\nu}(f) = \frac{1}{Z}\mathbf{1}(f + f^{\alpha,\beta} \in \mathrm{NoInt}([a,s]\cup[t,b],g))$$
$$\cdot\exp\left(-c\frac{2(f(s)-\underline{x})\cdot\frac{f^{\alpha,\beta}(s)}{s-a} + \|\frac{f^{\alpha,\beta}(s)}{s-a}\|^2}{4} - c\frac{2(f(t)-\underline{y})\cdot\frac{f^{\alpha,\beta}(t)}{b-t)} + \|\frac{f^{\alpha,\beta}(t)}{b-t}\|^2}{4}\right). \tag{5.4}$$

where $Z = \mathbb{P}_{a,b}(\underline{x},\underline{y},g,[a,s]\cup[t,b])$ is a normalizing factor. Now, if f is in the set $\mathrm{NoInt}([a,s]\cup[t,b],g)$, then so is $f + f^{\alpha,\beta}$. Hence the right-hand side of (5.2) is bounded below by the exponential factor (5.4). We can bound (5.4) below by using $0 \leq f^{\alpha,\beta} \leq (\alpha/m+\beta)\mathbf{1}$, which yields the desired bound. □

We now record [Dau24, Lemma 3.1], stated slightly more generally, which allows one to estimate the conditional inverse acceptance probability by the inverse of a conditional probability over a resampled ensemble, by 'stepping out' of the original interval and applying the Brownian Gibbs on that larger interval. For $a < b \in \mathbb{R}, k \in \mathbb{N}$ set $\mathcal{F}_k^{[a,b]} := \sigma(\{A_i(x) : (i,x) \notin [\![1,m]\!] \times (a,b)\})$.

Lemma 5.2. [Dau24, Lemma 3.1] *For every $a < y_0 + 1 < y_0 + 2 < b$ $k \in \mathbb{N}$, we have for any non-negative measurable functional F on $\mathcal{C}([\![1,k]\!] \times [a, y_0+2] \cup [y_0+1, b])$ and any $\mathcal{F}_m^{[a,b]}$-measurable U*

$$\begin{aligned}&\mathbb{E}_{\mathcal{F}_k^{[a,b]}}\left(\frac{F(U, \mathcal{A}|_{[\![1,k]\!]\times[a,y_0+1]\cup[y_0+2,b]})}{\mathbb{P}_{y_0+1,y_0+2}(\mathcal{A}^k(y_0+1), \mathcal{A}^k(y_0+2), \mathcal{A}_{k+1})}\right) \\ &= \left(\frac{\mathbb{E}_{\mathcal{F}_k^{[a,b]}}[F(U, \mathfrak{B}|_{[\![1,k]\!]\times[a,y_0+1]\cup[y_0+2,b]})]}{\mathbb{E}_{\mathcal{F}_k^{[a,b]}}[\mathbb{P}_{y_0+1,y_0+2}(\mathfrak{B}^k(y_0+1), \mathfrak{B}^k(y_0+2), \mathcal{A}_{k+1})]}\right),\end{aligned} \tag{5.5}$$

where $\mathfrak{B}$ is a line ensemble constructed by letting $\mathfrak{B}_i(r) = \mathcal{A}_i(r)$ for $(i,r) \notin [\![1,k]\!] \times [a,b]$, and let $\mathfrak{B}^k|_{[a,b]}$ have the conditional law given $\mathcal{A}$ outside of $[\![1,k]\!] \times [a,b]$ of k independent Brownian bridges from $(a, \mathcal{A}^k(a))$ to $(b, \mathcal{A}^k(b))$ conditioned on the event $\mathrm{NoInt}(\mathcal{A}_{k+1}, [a, y_0+1] \cup [y_0+2, b])$.

The following lemma is a refinement of [Dau24, lemma 3.2], where non-intersection probabilities are estimated from below by more analytically tractable quantities. They are in turn are controlled by the modulus of continuity estimates for the Airy line ensemble in Corollary 3.11.

Lemma 5.3. [Dau24, Lemma 3.2] *Fix $a < s < t < b$, $\epsilon > 0$ and define $\mathcal{F}_m^{[a,b]}$-measurable random variables*

$$\begin{aligned}D &= D(m,t) = 1 + \max_{r,r' \in [s,t]} |\mathcal{A}_{m+1}(r) - \mathcal{A}_{m+1}(r')|, \\ M &= M(m, y_0) = 1 + \max_{r,r' \in [a,b]} |\mathcal{A}_{m+1}(r) - \mathcal{A}_{m+1}(r')| + \max_{i \in [\![1,m]\!]} |\mathcal{A}_i(b) - \mathcal{A}_i(a)|.\end{aligned}$$

Then with $\mathfrak{B}$ as in Lemma 5.4, we have

$$\begin{aligned}&\mathbb{E}_{\mathcal{F}_m^{[a,b]}}[\mathbb{P}_{s,t}(\mathfrak{B}^m(s), \mathfrak{B}^m(t), \mathcal{A}_{m+1})] \\ &\geq C \exp\left(-m^{1+\epsilon}(\zeta^2 D^2 + \zeta M D) - m^{2+\epsilon}\zeta D - cm\log(b-a)\right) \\ &\cdot \exp\left(-\frac{dm^{3-\epsilon}(t-s)}{\epsilon^2}\right)\end{aligned}$$

for some constant $c, d, C > 0$ independent of $m \in \mathbb{N}, \epsilon > 0$, and $\zeta = \frac{3}{\min(s-a, t-b)}$.

Proof. All statements in the proof are conditional on $\mathcal{F}_m^{[a,b]}$. Define the $\mathcal{F}_m^{[a,b]}$-measurable vector

$$\underline{z} = (D + m^{\epsilon/2}, D + (m-1)^{\epsilon/2}, \ldots, D+1)$$

and the $\mathcal{F}_m^{[a,b]}$-measurable set

$$O = \{(\underline{x}, \underline{y}) \in \mathbb{R}^m_> \times \mathbb{R}^m_> : x_m > \mathcal{A}_{m+1}(s), y_m > \mathcal{A}_{m+1}(t)\}. \tag{5.6}$$

By the definition of D, for $(\underline{x}, \underline{y}) \in O$ we have noting that $m^{\epsilon/2} - (m-1)^{\epsilon/2} \geq \epsilon/(2m^{1-\epsilon/2})$, $m \geq 2$ inclusion and independence

$$\mathbb{P}_{s,t}(\mathrm{NoInt}(\underline{x} + \underline{z}, \underline{y} + \underline{z}, \mathcal{A}_{m+1})) \geq \mathbb{P}\Big(\sup_{s \leq r \leq t} |B(r)| \leq \epsilon/(4m^{1-\epsilon/2})\Big)^m, \tag{5.7}$$

where B is a rate two Brownian bridge from $(s,0)$ to $(t,0)$. By Lemma 3.16, we have

$$\mathbb{P}(\max_{s \leq r \leq t} |B_t| \leq \epsilon/(4m^{1-\epsilon/2})) \geq c\exp\left(-\frac{dm^{2-\epsilon}(t-s)}{\epsilon^2}\right), \quad \text{for all } \epsilon > 0, m \geq 1$$

and so the right hand side in (5.7) is bounded below by

$$c\exp\left(-\frac{dm^{3-\epsilon}(t-s)}{\epsilon^2}\right)$$

for some positive constant $c > 0$ independent of $m \in \mathbb{N}$, which may change from line to line. Therefore letting $\mu_{\mathfrak{B}}$ denote the conditional law of $(\mathfrak{B}^m(s), \mathfrak{B}^m(t))$ given $\mathcal{F}_m^{[a,b]}$, to complete the proof it suffices to find a set $A \subseteq O$ such that $\mu_{\mathfrak{B}}(A + (\mathbf{z}, \mathbf{z}))$ is large. Fix $\Delta > 0$ and let A_Δ be the $\mathcal{F}_m^{[a,b]}$-measurable subset of $(\underline{x}, \underline{y}) \in O$ where

$$x_i \leq \mathcal{A}_i(a) + \Delta, \qquad y_i \leq \mathcal{A}_i(b) + \Delta$$

for all $i \in [\![1, m]\!]$. Then by Lemma 5.1 with $\alpha = 1, \beta = D$, we have

$$\begin{aligned} &\mu_{\mathfrak{B}}(A_\Delta + (\mathbf{z}, \mathbf{z})) \\ &\geq \mu_{\mathfrak{B}}(A_\Delta) \inf_{(\underline{x},\underline{y}) \in A_\Delta} \exp\Big(-\frac{\zeta^2 m^{1+\epsilon}(1+D)^2}{4} \\ &\qquad - \zeta \frac{m^{\epsilon/2}(1+D) \sum_{i=1}^m ((x_i - \mathcal{A}_i(a))^+ + (y_i - \mathcal{A}_i(b))^+)}{4} \Big) &(5.8) \\ &\geq \mu_{\mathfrak{B}}(A_\Delta) \exp\Big(-\zeta^2 m^{1+\epsilon} D^2 - \zeta m^{1+\epsilon/2} D \Delta \Big) &(5.9) \end{aligned}$$

where $\zeta = \frac{3}{\min(s-a, b-t)}$. In the final line we have used that $1 + D \leq 2D$. It remains to find Δ where $\mu_{\mathfrak{B}}(A_\Delta)$ is large.

Define vectors $\underline{w}^{a,b}$ for $i \in [\![1, m]\!]$ at a, b respectively, where

$$\underline{w}_i^{a,b} = M + i + \mathcal{A}_i(\{a, b\}).$$

By a monotonic coupling for Brownian bridges, see 3.13, on the interval $[a, b]$, the m-tuple $(\mathfrak{B}_1, \ldots, \mathfrak{B}_m)$ is stochastically dominated by m independent Brownian bridges $B = (B_1, \ldots, B_m)$ from $(a, \underline{w}^a)$ to $(b, \underline{w}^b)$ conditioned on the event

$$\mathrm{NoInt}([a, s] \cup [t, b], \mathcal{A}_{m+1}).$$

Now, let $L \in C^m([a, b])$ be the function whose ith coordinate L_i is the linear function satisfying $L_i(a, b) = w_i^{a,b}$. By [Dau24, Lemma 2.5], we have $f \in \mathrm{NoInt}([a, s] \cup [t, b], \mathcal{A}_{m+1})$ for any sequence of bridges f from $(a, \underline{w}^a)$ to $(b, \underline{w}^b)$ when $\|f - L\|_{\infty,[a,b]} \leq 1/100$ with probability bounded below by $c\mathrm{e}^{-dm\log(b-a)}$ for positive constants $c, d > 0$. This allows us to estimate

$$\begin{aligned} &\mathbb{P}(B_i(r) \leq M + i + 2 + \mathcal{A}_i(-b) \vee \mathcal{A}_i(b) \quad \forall i \in [\![1, m]\!], r = s, t) \\ &\qquad \geq \mathbb{P}(\|B - L\|_{\infty,[a,b]} < 1/100) \\ &\qquad \geq c\mathrm{e}^{-dm\log(b-a)}. \end{aligned}$$

Observing that

$$M + i + 2 + \mathcal{A}_i(a) \vee \mathcal{A}_i(b) - \mathcal{A}_i(\{a, b\}) \leq 5M + 3m$$

for all i, we can conclude that

$$\mu_{\mathfrak{B}}(A_{5M+3m}) \geq \mathbb{P}((B(s), B(t)) \in A_{5M+3m}) \geq c\mathrm{e}^{-dm\log(b-a)}.$$

Combining this with the bound on (5.7) and (5.9) and simplifying yields the result. $\square$

Before proving the quantitative Brownian regularity of finite depth truncatios of the KPZ fixed point against Brownian motion, we need one final preliminary result estimating the expected value of the inverse acceptance probability that appears in the conditioning when applying the Brownian Gibbs property to the Airy line ensemble, which is the content of the following lemma.

Lemma 5.4. *Fix $m \in \mathbb{N}$, $t > 0$ $\epsilon > 0$, then there exists some universal $\eta > 0$ such that the following estimate holds.*

$$\mathbb{E}\left[\frac{1}{\mathfrak{B}^{[0,t]}_{(\mathcal{A}_i(0))_{i=1}^m, (\mathcal{A}_i(t))_{i=1}^m}(\mathrm{NoInt}([0, t], \mathcal{A}_{m+1}))} \right] = O_t(\mathrm{e}^{d_{t,\epsilon} m^{6+\epsilon}}) = O(\mathrm{e}^{d_\epsilon t^\eta} \mathrm{e}^{dm^\theta}).$$

for some constants $\theta, d_\epsilon > 0$ independent of m.

Proof. We first begin by 'stepping outside' of the interval $[0,t]$ and condition on $\mathcal{F}^{[-T_m,U_m]} \subseteq \mathcal{F}_m^{[0,t]}$, for $T_m, U_m > 0$ sufficiently large, to be chosen later. To control the inverse acceptance probability (5.10) conditional on $\mathcal{F}^{[-T_m,U_m]}$, we use Lemma 5.3 and the lower bound provided by Lemma 5.3 to obtain for all $\epsilon \in (0,1)$

$$\mathbb{E}\left[\frac{1}{\mathfrak{B}^{[0,t]}_{(\mathcal{A}_i(0))_{i=1}^m,(\mathcal{A}_i(t))_{i=1}^m}(\mathrm{NoInt}([0,t],\mathcal{A}_{m+1}))}\right]$$
$$\leq \mathbb{E}\left[\exp\left((m^{1+\epsilon}(\zeta^2 D^2 + \zeta M D + m^{2+\epsilon}\zeta D)) + cm\log(U_m+T_m)\right)\cdot \exp\left(\frac{dm^{3-\epsilon}t}{\epsilon^2}\right)\right]$$
$$= \exp\left(cm\log(U_m+T_m)\right)\cdot\exp\left(\frac{dm^{3-\epsilon}t}{\epsilon^2}\right)\cdot\mathbb{E}\left[\exp\left(m^{1+\epsilon}(\zeta^2D^2) + \zeta MD + m^{2+\epsilon}\zeta D\right)\right]$$

where $c > 0$ is some $\epsilon-$dependent constant and

- the $\mathcal{F}_m^{[-T_m,U_m]}$-measurable random variables

$$D = D(m,y_0) = 1 + \max_{r,r'\in[0,t]}|\mathcal{A}_{m+1}(r) - \mathcal{A}_{m+1}(r')|,$$
$$M = M(m,T_m,U_m) = 1 + \max_{r,r'\in[-T_m,U_m]}|\mathcal{A}_{m+1}(r) - \mathcal{A}_{m+1}(r')| + \max_{i\in[\![1,m]\!]}|\mathcal{A}_i(U_m) - \mathcal{A}_i(-T_m)|$$

- $\zeta = \frac{3}{\min(T_m, U_m - t)}$.

We will henceforth take $T_m = U_m = O(m^\alpha) + t$ for some $\alpha > 0$ so that $\zeta = m^{-\alpha}$. In particular, taking $\alpha = 2 + 2\epsilon + \eta$, for some $\eta > 0$ to be chosen later, we estimate using the elementary inequality for $a, b \geq 0$, $2ab \leq a^2 + b^2$

$$\begin{aligned}&\mathbb{E}\left[\exp\left(m^{1+\epsilon}(\zeta^2D^2 + \zeta MD) + m^{2+\epsilon}\zeta D\right)\right]\\ &\leq \tfrac{1}{2}\mathbb{E}\left[\exp\left(2m^{-1}D^2\right)\right] + \tfrac{1}{4}\mathbb{E}\left[\exp\left(4m^{1+\epsilon}\zeta MD\right)\right] + \tfrac{1}{4}\mathbb{E}\left[\exp\left(4m^{2+\epsilon}\zeta D\right)\right]\\ &\leq \mathbb{E}\left[\exp\left(2m^{-1}D^2\right)\right] + \tfrac{1}{2}\mathbb{E}\left[\exp\left(cm^{2+2\epsilon}\zeta^2 M^2\right)\right]\\ &\leq O(\mathrm{e}^{dt^\theta}\mathrm{e}^{dm^2\log m}) + \mathbb{E}\left[\exp\left(cm^{-\eta}\zeta M^2\right)\right]\end{aligned}$$

for some constant $c, d, \theta > 0$.

By Corollary 3.11, we have that there exist some positive $C_1, C_2, d > 0$ independent of t, m such that for all $a > 0$,

$$\mathbb{P}(M > a) \leq C_1 \mathrm{e}^{dm^{3\alpha}}\mathrm{e}^{-C_2 a^2/m^\alpha}.$$

In particular, keeping track of t-dependence, we obtain

$$\mathbb{P}(M > a) \leq C_1 \mathrm{e}^{dm^\eta + dt^\eta}\mathrm{e}^{-C_2a^2},$$

for some absolute constant $\eta > 0$. Thus,

$$\begin{aligned}&\leq O_t(\mathrm{e}^{dm^2\log m}) + \mathbb{E}\left[\exp\left(cm^{-\eta}\zeta M^2\right)\right]\\ &\leq O_t(\mathrm{e}^{dm^2\log m}) + 2c\int_0^\infty a\exp\left(cm^{-2-2\epsilon-2\eta}a^2\right)\mathbb{P}(M>a)\,\mathrm{d}a\\ &\leq O_t(\mathrm{e}^{dm^2\log m}) + O(\mathrm{e}^{dm^{3\alpha}})\int_0^\infty a\exp\left(cm^{-2-2\epsilon-2\eta}a^2 - C_2m^{-2-2\epsilon-\eta}a^2\right)\mathrm{d}a\\ &= O_t(\mathrm{e}^{dm^{6+6\epsilon+6\eta}}),\end{aligned}$$

for positive constants $c > 0$, concluding the proof. One can obtain analogous expressions, keeping track of the t-dependence to finally conclude the proof. □

We are now in a position to obtain the quantitative control of the spatial increments of finite depth truncations of the KPZ fixed point started from Brownian motion in terms of the Wiener measure and Airy line ensemble.

Theorem 5.5. *Fix $m \in \mathbb{N}$, $T_m > 0, U_m > y_0 + 2$ $\epsilon \in (0,1)$ and define the random continuous function*

$$H_m(y) = \max_{\ell \leq m}(G_\ell + \mathcal{A}[(0,\ell) \to (y,1)]), \qquad \text{for } y \in [1, y_0].$$

Then with μ the rate two Wiener measure μ on $[0, y_0 - 1]$, $H_m(\cdot + 1) - H_m(1)$, satisfies for all $p, r > 1$, $A \subseteq C([0, y_0 - 1])$ Borel and $a > 0$ the estimates

$$\begin{aligned}
\mathbb{P}\left(H_m(\cdot+1) - H_m(1) \in A\right) &\leq O_{y_0}(\exp(m^7)) \cdot \exp\left(\tfrac{y_0 m^2 a^2}{4}\left(\tfrac{r/(r-1)}{ry_0/(r-1)+1} - \tfrac{1}{y_0+1}\right)\right) \\
&\cdot \sup_{\max_{1\leq \ell \leq m}|G_\ell - G_1| \leq a} \left\| Q^{m,G} \right\|_{L^{2r/(r-1)}(\mu)} \cdot \mu(A)^{\frac{1}{r}(1-\frac{1}{p})} \\
&+ O_{y_0}(\exp(m^7)) \cdot \mathbb{P}\left(\max_{1\leq \ell \leq m}|G_\ell - G_1| + \max_{1 \leq i \leq m}|\mathcal{A}(y_0+1) - \mathcal{A}(0)| \geq a\right)^{1/p} \\
&= O(\exp(m^\theta + y_0^\theta)) \cdot \exp\left(\tfrac{y_0 m^2 a^2}{4}\left(\tfrac{r/(r-1)}{ry_0/(r-1)+1} - \tfrac{1}{y_0+1}\right)\right) \\
&\cdot \sup_{\max_{1\leq \ell \leq m}|G_\ell - G_1| \leq a} \left\| Q^{m,G} \right\|_{L^{2r/(r-1)}(\mu)} \cdot \mu(A)^{\frac{1}{r}(1-\frac{1}{p})} \\
&+ O(\exp(m^\theta + y_0^\theta)) \cdot \mathbb{P}\left(\max_{1\leq \ell \leq m}|G_\ell - G_1| + \max_{1 \leq i \leq m}|\mathcal{A}(y_0+1) - \mathcal{A}(0)| \geq a\right)^{1/p}.
\end{aligned}$$

for some $\theta > 0$, where

- *$Q^{m,G}$ is the Radon-Nikodym derivative of*
$$Y^{m,G} := \max_{1\leq \ell \leq m}(G_\ell - G_1 + B[(0,\ell) \to (\cdot+1, 1)]) - \max_{1 \leq \ell \leq m}(G_\ell - G_1 + B[(0,\ell) \to (1,1)])$$
against rate two Brownian motion on $[0, y_0 - 1]$
- *G denotes the boundary data $G = (G_\ell)_{\ell=1}^m$, $G_\ell = \max_{x \in \mathrm{supp}_{-\infty}(h_0)}(h_0(x) + \mathcal{A}[x \to (0,\ell)])$*
- *$\zeta = \frac{3}{\min(1+T_m, U_m - t)}$.*

Proof. First, fix $t > y_0$ and condition on the sigma algebra $\mathcal{F}_m^{[0,t]}$.

By the Brownian Gibbs property enjoyed by the Airy line ensemble, we get that conditioning on the sigma algebra $\mathcal{F}_m^{[0,t]}$, the law of $\mathcal{A}$ on $[\![1,k]\!] \times [0,t]$ has the law of m independent Brownian bridges with starting points $(\mathcal{A}_i(0))_{i=1}^m$ and ending at $(\mathcal{A}_i(t))_{i=1}^m$ conditioned to not intersect each other and the bottom line $\mathcal{A}_{m+1}$, an event in $C_{*,*}^m([0,t])$. This conditional law has Radon-Nikodym Derivative against m independent Brownian bridges with starting points $(\mathcal{A}_i(0))_{i=1}^m$ and ending at $(\mathcal{A}_i(t))_{i=1}^m$

$$\frac{\mathbf{1}_{\mathrm{NoInt}([0,t],\mathcal{A}_{m+1})}(\omega)}{\mathfrak{B}^{[0,t]}_{(\mathcal{A}_i(0))_{i=1}^m, (\mathcal{A}_i(t))_{i=1}^m}(\mathrm{NoInt}([0,t], \mathcal{A}_{m+1}))} \tag{5.10}$$

for paths ω in $C_{*,*}^m([0,t])$.

Now, by the by metric composition for LPP, and the $\mathcal{F}_m^{[0,t]}$-measurability of $G_\ell, 1 \leq \ell \leq m$, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\leq \ell \leq m}(G_\ell + \mathcal{A}[(0,\ell) \to (\cdot, 1)]) \in A | \mathcal{F}_m^{[0,t]}\right) \\
&= \mathfrak{B}^{[0,t]}_{(\mathcal{A}_i(0))_{i=1}^m, (\mathcal{A}_i(t))_{i=1}^m}\left(\left\{\omega \in C^m_{(\mathcal{A}_i(0))_{i=1}^m, (\mathcal{A}_i(t))_{i=1}^m}([0,t]) : \max_{1 \leq \ell \leq m}(G_\ell + \omega[(0,\ell) \to (\cdot,1)]) \in A\right\}\right)
\end{aligned}$$

where

$$G_\ell := \max_{x \in \mathrm{supp}_{-\infty}(h_0)}(h_0(x) + \mathcal{A}[x \to (0,\ell)]).$$

Now, by Lemma 3.14, we have that the law of the first m lines of $\mathcal{A}(\cdot)-\mathcal{A}(0)$ on $[0,y_0]$ conditional on $\mathcal{F}_m^{[0,t]}$ is absolutely continuous with respect to the law of m independent rate two Brownian motions on $[0,t]$ with bounded Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathfrak{B}^{[0,t]}_{\underline{0},\underline{\mathcal{A}}}|_{[0,y_0]}}{\mathrm{d}\mathfrak{B}^{[0,y_0]}_{\underline{0},*}}$$

against rate two Brownian motion on paths in $C_{0,*}([0,t-1])^m$ with norms

$$\left\|\frac{\mathrm{d}\mathfrak{B}^{[0,t]}_{\underline{0},\underline{\mathcal{A}}}|_{[0,y_0]}}{\mathrm{d}\mathfrak{B}^{[0,y_0]}_{\underline{0},*}}\right\|_{L^p\left(\mathfrak{B}^{[0,y_0]}_{\underline{0},*}\right)} = \frac{(t/(t-y_0))^{\frac{m}{2}}}{(px/(t-y_0)+1)^{\frac{m}{2}}}\cdot\exp\left(\frac{y_0\|\mathcal{A}^m(t)-\mathcal{A}^m(0)\|^2}{4(t-y_0)}\left(\frac{p}{(p-1)y_0+t}-\frac{1}{t}\right)\right)$$

for all $p>1$ and

$$\left\|\frac{\mathrm{d}\mathfrak{B}^{[0,t]}_{\underline{0},\underline{\mathcal{A}}}|_{[0,y_0]}}{\mathrm{d}\mathfrak{B}^{[0,y_0]}_{\underline{0},*}}\right\|_{L^\infty\left(\mathfrak{B}^{[0,y_0]}_{\underline{0},*}\right)} = (t/(t-y_0))^{\frac{m}{2}}\cdot\exp\left(\frac{\|\mathcal{A}^m(t)-\mathcal{A}^m(0)\|^2}{4t}\right).$$

Combining all of the above, we deduce for any $A\subseteq C_{0,*}([0,y_0-1])$ Borel measurable that

$$\begin{aligned}\mathbb{P}\Big(\max_{1\le\ell\le m}\left(G_\ell+\mathcal{A}[(0,\ell)\to(\cdot+1,1)]\right)-\max_{1\le\ell\le m}\left(G_\ell+\mathcal{A}[(0,\ell)\to(1,1)]\right)\in A\big|\mathcal{F}_m^{[0,t]}\Big)\\ \le\frac{\mu^{\mathcal{A}(0),\mathcal{A}(t),G}(A)^{1-1/p}}{\mathfrak{B}^{[0,t]}_{(\mathcal{A}_i(0))_{i=1}^m,(\mathcal{A}_i(t))_{i=1}^m}(\mathrm{NoInt}([0,t],\mathcal{A}_{m+1}))^{(p-1)/p}}\end{aligned}$$

for all $p>1$, where $\mu^{\mathcal{A}(0),\mathcal{A}(t),G}(\cdot)$ denotes the law of

$$\mu^{\mathcal{A}(0),\mathcal{A}(t),G}(\cdot):=\mathbb{P}\left(\max_{1\le\ell\le m}\left(G_\ell+\mathfrak{B}[(0,\ell)\to(\cdot+1,1)]\right)-\max_{1\le\ell\le m}\left(G_\ell+\mathfrak{B}[(0,\ell)\to(1,1)]\right)\in\cdot\right)$$

where $\mathfrak{B}$ is an ensemble of m independent Brownian bridges with starting and ending points $(0,\mathcal{A})$ and $(t,\mathcal{A}(t))$ respectively.

Thus, by Hölder, the unconditional probability can be estimated as follows

$$\begin{aligned}&\mathbb{P}\left(\max_{1\le\ell\le m}\left(G_\ell+\mathcal{A}[(0,\ell)\to(\cdot+1,1)]\right)-\max_{1\le\ell\le m}\left(G_\ell+\mathcal{A}[(0,\ell)\to(1,1)]\right)\in A\right)\\ &\le\mathbb{E}\left[\frac{1}{\mathfrak{B}^{[0,t]}_{(\mathcal{A}_i(0))_{i=1}^m,(\mathcal{A}_i(t))_{i=1}^m}(\mathrm{NoInt}([0,t],\mathcal{A}_{m+1}))}\right]^{(p-1)/p}\cdot\mathbb{E}\left[\mu^{\mathcal{A}(0),\mathcal{A}(t),G}(A)^{p-1}\right]^{1/p}\end{aligned}$$

for all $p>1$.

Now, to estimate the first term, we 'step outside' of the interval $[0,t]$ and condition on $\mathcal{F}^{[-T_m,U_m]}\subseteq\mathcal{F}_m^{[0,t]}$, for T_m,U_m sufficiently large, to be chosen later. To control the inverse acceptance probability (5.10) conditional on $\mathcal{F}^{[-T_m,U_m]}$, we use Lemma 5.3 and the lower bound provided by Lemmas 5.3 and 5.4 to obtain

$$\begin{aligned}\mathbb{P}\Big(\max_{1\le\ell\le m}\left(G_\ell+\mathcal{A}[(0,\ell)\to(\cdot+1,1)]\right)-\max_{1\le\ell\le m}\left(G_\ell+\mathcal{A}[(0,\ell)\to(1,1)]\right)\in A\Big)\\ \le O(\exp(m^\theta+y_0^\theta))\cdot\mathbb{E}\left[\mu^{\mathcal{A}(0),\mathcal{A}(t),G}(A)^{p-1}\right]^{1/p}\end{aligned}\tag{5.11}$$

for all $p>1$ and some universal constant $c>0$.

To estimate the second term in (5.11), let $Q^{m,G}$ be the Radon-Nikodym derivative of

$$Y^{m,G}:=\max_{1\le\ell\le m}\left(G_\ell-G_1+B[(0,\ell)\to(\cdot+1,1)]\right)-\max_{1\le\ell\le m}\left(G_\ell-G_1+B[(0,\ell)\to(1,1)]\right)$$

against rate two Brownian motion on $[0, y_0 - 1]$ (here we treat the initial data G as fixed in $\mathbb{R}^m_>$). Note that by [DM11, Theorem 58], we can take $Q^{m,G}$ to be jointly measurable in $\tilde{G}$ and paths ξ in Wiener space on $[0, y_0 - 1]$. Now, by [SV21, Theorem 4.3] $Y^{m,G}$ can be expressed as the top line of a sequence of upwardly reflected Brownian motions with boundary data $G_\ell - G_1$ $1 \leq \ell \leq m$, hence its Radon-Nikodym derivative against Brownian motion can be estimated from Theorem 3.7, and in particular, $Q^{m,G} \in L^{\infty-}(\mu)$ for all choices of boundary data G, where μ is the restriction of the (rate two) Wiener measure on $[0, y_0 - 1]$.

Combining all of the above, we deduce the following norm estimates for the Radon-Nikodym derivatives $Q^{\underline{x},\underline{y},G}$ of $\mu^{x,y,G}(\cdot)$ for all data $\underline{x}, \underline{y}, G \in \mathbb{R}^m_>$

$$\begin{aligned}\left\|Q^{\underline{x},\underline{y},G}\right\|_{L^p(\mu)} &\leq \frac{(t/(t-y_0))^{\frac{m}{2}}}{(py_0/(t-y_0)+1)^{\frac{m}{2}}} \cdot \exp\left(\frac{y_0 \left\|\underline{x-y}\right\|^2}{4(t-y_0)}\left(\frac{p}{(p-1)y_0+t} - \frac{1}{t}\right)\right) \\ &\quad \cdot \left\|Q^{m,G}\right\|_{L^{2p}(\mu)} \cdot \mu(A)^{1-\frac{1}{p}}\end{aligned}$$

for all $p > 1$.

For the second term in (5.11), we estimate with $t = y_0 + 1$ for all $a > 0$ by Hölder's inequality

$$\begin{aligned}&\mathbb{E}\left[\mu^{\mathcal{A}(0),\mathcal{A}(t),G}(A)^{p-1}\right]^{1/p} \\ &= \mathbb{E}\left[\mu^{\mathcal{A}(0),\mathcal{A}(t),G}(A)^{\frac{p-1}{p}} \mathbf{1}\left(\max_{1\leq \ell\leq m}|G_\ell - G_1| + \max_{1\leq i\leq m}|\mathcal{A}(y_0+1) - \mathcal{A}(0)| < a\right)\right]^{1/p} \\ &\quad + \mathbb{P}\left(\max_{1\leq \ell\leq m}|G_\ell - G_1| + \max_{1\leq i\leq m}|\mathcal{A}(y_0+1) - \mathcal{A}(0)| \geq a\right)^{1/p}\end{aligned}$$

Now, for $r \in (1, \infty)$, combining the two estimates above, we obtain

$$\begin{aligned}&\mathbb{E}\left[\mu^{\mathcal{A}(0),\mathcal{A}(t),G}(A)^{p-1}\right]^{1/p} \\ &\leq \mathbb{E}\left[\exp\left(\frac{py_0\left\|\underline{\mathcal{A}(y_0+1)-\mathcal{A}(0)}\right\|^2}{4}\left(\frac{r/(r-1)}{ry_0/(r-1)+1} - \frac{1}{y_0+1}\right)\right) \cdot \left\|Q^{m,G}\right\|^p_{L^{2r/(r-1)}(\mu)}\right. \\ &\quad \left.\cdot \mathbf{1}\left(\max_{1\leq \ell\leq m}|G_\ell - G_1| + \max_{1\leq i\leq m}|\mathcal{A}(y_0+1) - \mathcal{A}(0)| < a\right)\right]^{1/p} \cdot \mu(A)^{\frac{1}{r}(1-\frac{1}{p})} \\ &\quad + \mathbb{P}\left(\max_{1\leq \ell\leq m}|G_\ell - G_1| + \max_{1\leq i\leq m}|\mathcal{A}(y_0+1) - \mathcal{A}(0)| \geq a\right)^{1/p} \\ &\leq \exp\left(\frac{y_0 m^2 a^a}{4}\left(\frac{r/(r-1)}{ry_0/(r-1)+1} - \frac{1}{y_0+1}\right)\right) \cdot \sup_{\max_{1\leq \ell\leq m}|G_\ell - G_1| \leq a} \left\|Q^{m,G}\right\|_{L^{2r/(r-1)}(\mu)} \cdot \mu(A)^{\frac{1}{r}(1-\frac{1}{p})} \\ &\quad + \mathbb{P}\left(\max_{1\leq \ell\leq m}|G_\ell - G_1| + \max_{1\leq i\leq m}|\mathcal{A}(y_0+1) - \mathcal{A}(0)| \geq a\right)^{1/p},\end{aligned}$$

which when combined with (5.11), concludes the proof of the second part. □

6. Putting it all together: quantitative Brownian regularity

In this section, we establish the quantitative Brownian regularity of the KPZ fixed point started from arbitrary (finitary) initial data in Theorem 6.6.

First, recall the definition of the semi-infinite last passage values from (5.1). To summarise what we have obtained so far, recall that having established the quantitative comparison in Theorem 5.5, we have estimated for $m \geq 1$, the truncated, finite-depth KPZ fixed point

$$H_m(\cdot) = \max_{1\leq \ell\leq m}(G_\ell + \mathcal{A}[(0,\ell) \to (\cdot, 1)]), \qquad y \in [1, y_0] \tag{6.1}$$

in terms of

- the boundary data $G = (G_\ell)_{\ell=1}^m$, $G_\ell = \max_{x \in \mathrm{supp}_{-\infty}(h_0)} (h_0(x) + \mathcal{A}[x \to (0,\ell)])$
- $Q^{m,\tilde{G}^a}$, the Radon-Nikodym derivatives of
$$Y^{m,\tilde{G}^a} := \max_{1\le\ell\le m} (\tilde{G}_\ell \vee a + B[(0,\ell) \to (\cdot+1,1)]) - \max_{1\le\ell\le m} (\tilde{G}_\ell + B[(0,\ell) \to (1,1)])$$
against rate two Brownian motion on $[0, y_0 - 1]$, where $\tilde{G}^a = (-a \wedge (G_\ell - G_1) \vee a)_{\ell=1}^m$, for some $a > 0$
- and the tails of $\max_{1\le\ell\le m} |G_\ell - G_1|$.

Now, Theorem 3.7 allows us to estimate $L^p(\mu)$-norms of $Y^{m,\tilde{G}^a}$ for all $a > 0$, $p > 1$. Thus, the only missing ingredient is to estimate the tails of
$$G_\ell - G_1 := \max_{x \in \mathrm{supp}_{-\infty}(h_0)} (h_0(x) + \mathcal{A}[x \to (0,\ell)]) - \max_{x \in \mathrm{supp}_{-\infty}(h_0)} (h_0(x) + \mathcal{A}[x \to (0,1)])\,.$$

First observe the following inequalities for any $x_0 \in K \subseteq \mathbb{R}_{>0}$ countable, compact and bounded away from zero. We have from [SV21, Lemma 3.8] that almost surely $\mathcal{A}[x \to (0,\ell)] \le \mathcal{S}(x,0)$ for all $\ell \in \mathbb{N}$ and $x \in K$, which allow us to estimate as follows,
$$\begin{aligned} 0 &\le \max_{x\in K}(h_0(x) + \mathcal{A}[x \to (0,1)]) - \max_{x\in K}(h_0(x) + \mathcal{A}[x \to (0,\ell)]) \\ &\le \max_{x\in K} |h_0(x)| + \max_{x\in K} |\mathcal{S}(x,0)| - h_0(x_0) - \mathcal{A}[x_0 \to (0,\ell)]) \\ &\le 2\max_{x\in K} |h_0(x)| + 2\max_{x\in K} |\mathcal{S}(x,0)| + |\mathcal{A}[x_0 \to (0,\ell)]) - \mathcal{S}(x_0,0)| \\ &= 2\max_{x\in K} |h_0(x)| + 2\max_{x\in K} |\mathcal{S}(x,0)| + \lim_{k\to\infty} |\mathcal{A}[(x_0)_k \to (0,\ell)] - \mathcal{A}[(x_0)_k \to (0,1)]| \\ &= 2\max_{x\in K} |h_0(x)| + 2\max_{x\in K} |\mathcal{S}(x,0)| + \lim_{k\to\infty} |\mathcal{A}^{\mathrm{stat}}[(x_0)_k \to (0,\ell)] - \mathcal{A}^{\mathrm{stat}}[(x_0)_k \to (0,1)]|\,. \end{aligned} \tag{6.2}$$
Here we have used the definition of semi-infinite last passage values from (5.1) and the fact that differences of last passage values remain the same when considering the parabolic and stationary line ensembles over intervals with identical endpoints.

In the following lemma, we prove a concentration result for the differences of the last passage values of semi-infinite geodesics over the stationary Airy line ensemble.

Lemma 6.1. *Fix $m \in \mathbb{N}$, $1 \le \ell \le m$, and some countable compact $K \subseteq \mathbb{R} \setminus \{0\}$. Let $\mathcal{A}^{\mathrm{stat}}(\cdot) = \mathcal{A}(\cdot) + (\cdot)^2$ denote the stationary Airy line ensemble and $K_{x_0,\ell}$ the coalescence depth for the infinite geodesics starting from $(0,1)$ and $(0,\ell)$ with respect to any fixed point $x_0 \in K$, see Definition 4.11. Then, with*
$$a = O(m \cdot \inf\{k > 1 : \mathbb{P}(K_{x_0,m} \ge k) \le 1/m\}^{5/4})\,,$$
one has the bound
$$\mathbb{P}(\max_{1\le\ell\le m} |\mathcal{A}^{\mathrm{stat}}[x_0 \to (0,\ell)] - \mathcal{A}^{\mathrm{stat}}[x_0 \to (0,1)]| \ge a) \lesssim_K \frac{1}{m}\,.$$

Proof. We write $G_\ell, 1 \le \ell \le m$ in place of the last passage values over the stationary line ensemble (taking $h_0(\cdot) \equiv \mathbf{1}_{\{x_0\}}(\cdot)$). We now estimate for any $k \ge m$ using the bounds (6.2) and Proposition 3.10
$$\mathbb{P}\Big(\max_{1\le\ell\le m} |G_\ell - G_1| \ge a\Big) \le \mathbb{P}\Big(G_1 - G_m \ge a, K_{x_0,m} \le k\Big) + \mathbb{P}\Big(K_{x_0,m} \ge k\Big)\,.$$
Now, using Definition 3.3 we estimate
$$\max_{1\le\ell\le m} |\mathcal{A}^{\mathrm{stat}}[(-\sqrt{k/(2\inf_{x_0\in K}|x_0|)}, k) \to (0,\ell)]| \le (k/(2\inf_{x_0\in K}|x_0|))^{\frac14} \log^{\frac12} 2 \cdot \sum_{i=1}^k \omega_i(\mathcal{A}^{\mathrm{stat}})\,,$$

where the moduli of continuity

$$\omega_i(\mathcal{A}^{\mathrm{stat}}) := \sup_{t,s\in[-\sqrt{k/(2\inf_{x_0\in K}|x_0|)},0],t\neq s} \frac{|\mathcal{A}_i^{\mathrm{stat}}(t)-\mathcal{A}_i^{\mathrm{stat}}(s)|}{\sqrt{|t-s|\log(2\sqrt{k/(2\sup_{x_0\in K}|x_0|)}/|t-s|)}}, \qquad 1\le i\le k$$

are sub-Gaussian random variables with uniform bounds on their tails (see Proposition 3.10). A union bound now gives

$$\begin{aligned}
&\mathbb{P}\Big(\max_{x_0\in K}|\mathcal{A}^{\mathrm{stat}}[(-\sqrt{k/(2x)},k)\to(0,\ell)]-\mathcal{A}^{\mathrm{stat}}[(-\sqrt{k/(2x)},k)\to(0,1)]|\ge a\Big)\\
&\le \mathbb{P}\Big(\sum_{i=1}^k \omega_i(\mathcal{A}^{\mathrm{stat}})\ge a\frac{(2\inf_{x_0\in K}|x_0|)^{\frac14}}{2k^{\frac14}\log^{\frac12}2}\Big)\\
&\le C_1 k\exp\Big(-C_2\inf_{x_0\in K}|x_0|^{\frac12}\frac{a^2}{k^{\frac52}}\Big),
\end{aligned}$$

for universal $C_1, C_2>0$. Combining the above, we obtain (after possibly enlarging C_1)

$$\mathbb{P}\Big(\max_{1\le\ell\le m}|G_\ell-G_1|\ge a\Big)\le C_1k\exp\Big(-C_2\inf_{x_0\in K}|x_0|^{\frac12}\frac{a^2}{k^{\frac52}}\Big)+\mathbb{P}(K_{x_0,m}\ge k),$$

for all $k>0$ and some universal $C_1,C_2>0$. Now, with

$$k=\inf\{k'>1:\mathbb{P}(K_{x_0,m}\ge k')\le 1/m\}$$

and $a=O(m\cdot k^{5/4})$ gives

$$\begin{aligned}
\mathbb{P}\Big(\max_{1\le\ell\le m}|G_\ell-G_1|\ge a\Big)&\le C_1m\exp\Big(-C_2\inf_{x_0\in K}|x_0|^{\frac12}m^2\Big)+\frac{C_1}{m}\\
&\lesssim_K \frac{1}{m},
\end{aligned}$$

which concludes the proof. ☐

Now with this concentration result for last passage values in the stationary Airy line ensemble, we obtain an extension thereof to differences in the initial data of finite depth truncations of the KPZ fixed point in the following lemma.

Lemma 6.2. *For $\ell\in\mathbb{N}$, $\alpha>0$, with $G_\ell:=\max_{x\in\mathrm{supp}_{-\infty}(h_0)}(h_0(x)+\mathcal{A}[x\to(0,\ell)])$ with h_0 bounded. Suppose further that the max-plus support of h_0 is in some countable bounded $K\subseteq[1,\infty)$. Then, for some sufficiently large universal constant $\theta>0$ and*

$$a_m=O(m\cdot\inf\{k>1:1\le\ell\le m\,,\mathbb{P}(K_{x_0,m}\ge k)\le 1/m\}^{5/4})$$

for any $x_0\in\mathrm{supp}_{-\infty}(h_0)$,

$$\mathbb{P}\Big(\max_{1\le\ell\le m}|G_\ell-G_1|\ge\theta\mathrm{diam}(K)\,\|h_0\|_{K,\infty}\vee a\Big)\lesssim\frac{1}{m}.$$

Proof. Now, for $\ell\in\mathbb{N}$, we estimate using the bounds (6.2)

$$|G_\ell-G_1|\le 2\max_{x\in K}|h_0(x)|+2\max_{x\in K}|\mathcal{S}(x,0)|+\lim_{k\to\infty}|\mathcal{A}^{\mathrm{stat}}[(x_0)_k\to(0,\ell)]-\mathcal{A}^{\mathrm{stat}}[(x_0)_k\to(0,1)]|,$$

for any $x_0\in K$, where $\mathcal{A}^{\mathrm{stat}}(\cdot)=\mathcal{A}(\cdot)+(\cdot)^2$ is the stationary Airy line ensemble. Now, by the modulus of continuity estimate for the stationary version of the directed landscape (Proposition 10.5 of [DJOBV22]), one has almost surely for all $x,x'\in K$,

$$|\mathcal{S}(x,0)-\mathcal{S}(x',0)+(x-x')^2|\le C_K\mathrm{diam}(K)^{1/4},$$

where $C_K > 0$ is random depending on K with $\mathbb{E}a^{C_K^{3/2}} < \infty$ for some $a > 1$. Thus, we further estimate

$$\max_{1\le\ell\le m} |G_\ell - G_1| \le 2\,\|h_0\|_\infty + C_K + \left|\mathcal{A}^{\mathrm{stat}}[x_0 \to (0,m)] - \mathcal{A}^{\mathrm{stat}}[x_0 \to (0,1)]\right|,$$

after possibly enlarging C_K. Finally applying Lemma 6.1 gives the result. □

Combining the above, we are now in a position to provide a quantitative Brownian comparison for the spatial increments of finite depth truncations of the KPZ fixed point. For the essence of the arguments underlying the following results, one can refer to the proof of Proposition 7.1 under the simplifying assumptions made in Section 7.

Theorem 6.3. *Let $m \in \mathbb{N}, y_0 \ge 1, \alpha > 0$ and let $H_m(\cdot)$ with boundary terms G_ℓ be as in (6.1) with continuous and bounded initial data h_0. Suppose furthermore that the support of h_0 is in some countable bounded K. Then, one also has the estimates with*

$$a = a_m = O(m^3 \cdot \inf\{k > 1 : \mathbb{P}(K_{x_0,m} \ge k) \le 1/m\}^{5/4})$$

for any $x_0 \in \operatorname{supp}_{-\infty}(h_0)$, and all $p > 1$, $r > 1$ and A Borel,

$$\begin{aligned}&\mathbb{P}\left(H_m(\cdot+1) - H_m(1) \in A\right)\\ &\le O_{y_0,K,p,r}\left(\exp(c_{y_0,K,h_0,r}a^2)\cdot\mu(A)^{\frac{1}{r}(1-\frac{1}{p})} + \tfrac{1}{m^{1/p}}\right).\end{aligned}$$

Proof. Theorem 5.5 gives that, with μ the rate two Wiener measure on $[0, y_0 - 1]$, $H_m(\cdot+1) - H_m(1)$ satisfies the norm estimates the following holds for all $p, r > 1$, $A \subseteq C([0, y_0 - 1])$ Borel and $a > 0$

$$\begin{aligned}\mathbb{P}\left(H_m(\cdot+1) - H_m(1) \in A\right) &\le O_{y_0}(\exp(m^7))\cdot\exp\left(\tfrac{y_0 m^2 a^2}{4}\left(\tfrac{r/(r-1)}{ry_0/(r-1)+1} - \tfrac{1}{y_0+1}\right)\right)\\ &\quad\cdot \sup_{\max_{1\le\ell\le m}|G_\ell - G_1|\le a}\left\|Q^{m,G}\right\|_{L^{2r/(r-1)}(\mu)}\cdot\mu(A)^{\frac{1}{r}(1-\frac{1}{p})}\\ &\quad+ O_{y_0}(\exp(m^7))\cdot\mathbb{P}\left(\max_{1\le\ell\le m}|G_\ell - G_1| + \max_{1\le i\le m}|\mathcal{A}(y_0+1) - \mathcal{A}(0)| \ge a\right)^{1/p},\end{aligned}$$

for all $p, r > 1$, $\epsilon \in (0,1)$ and some universal constant $c > 0$, where

- G denotes the boundary data $G = (G_\ell)_{\ell=1}^m$, $G_\ell = \max_{x\in\operatorname{supp}_{-\infty}(h_0)}(h_0(x) + \mathcal{A}[x \to (0,\ell)])$.
- $Q^{m,G}$ is the Radon-Nikodym derivative of
$$Y^{m,G} := \max_{1\le\ell\le m}(G_\ell - G_1 + B[(0,\ell) \to (\cdot+1,1)]) - \max_{1\le\ell\le m}(G_\ell - G_1 + B[(0,\ell) \to (1,1)])$$
against a rate two Brownian motion on $[0, y_0 - 1]$.

Now, taking $a = O(m^2 \cdot \inf\{k > 1 : \mathbb{P}(K_{x_0,m} \ge k) \le 1/m\}^{5/4})$ for some fixed $x \in \operatorname{supp}_{-\infty}(h_0)$, a union and Lemma 6.2 give for some universal $\theta > 0$ the estimates for all $p > 1$, $r > 1$ and A Borel

$$\begin{aligned}&\mathbb{P}\left(H_m(\cdot+1) - H_m(1) \in A\right)\\ &\le O_{y_0,K,h_0}\exp\left(c_{y_0,K,h_0}ma^2\right)\\ &\times \sup_{\max\limits_{1\le\ell\le m}|G_\ell - G_1| \le a}\left\|Q^{m,G}\right\|_{L^4(\mu)}\cdot\mu(A)^{\frac{1}{r}(1-\frac{1}{p})} + O_{y_0,K}\left(\frac{1}{m^{\frac{1}{p}}}\right),\end{aligned}$$

for some $c_{y_0,K,h_0} > 0$. Now, using the control on the Radon-Nikodym derivative of upward reflections of Brownian motion from Theorem 3.7 we obtain,

$$\sup_{\max\limits_{1\le\ell\le m}|G_\ell - G_1| \le a}\left\|Q^{m,G}\right\|_{L^{2r/(r-1)}(\mu)} \le O(y_0, r)a^{m^2}\mathrm{e}^{d_r m^2\log m + c_{y_0,r}ma^2}.$$

We thus have the estimates for all $p > 1$, $r > 1$ and A Borel

$$\begin{aligned}&\mathbb{P}\left(H_m(\cdot+1) - H_m(1) \in A\right)\\ &\leq O_{y_0,K,h_0,r} \exp\left(c_{y_0,K,h_0,r} a^2\right) \cdot \mu(A)^{\frac{1}{r}(1-\frac{1}{p})} + O_{y_0,K} \frac{1}{m^{\frac{1}{p}}} ,\end{aligned}$$

which concludes the proof. □

Having established the quantitative Brownian regularity for spatial increments of finite depth truncations of the KPZ fixed point in the previous theorem, we now translate this to a quantitative comparison of spatial increments of the actual KPZ fixed point at unit time. This comparison is expressed in terms of the geodesic intercept values as defined in Theorem 4.5, which is the final step before obtaining the main result. Note that the comparison is uniform over a wide class of initial data.

Corollary 6.4. *Fix $y_0 > 1$ and let $h(\cdot)$ be the KPZ fixed point on $[1, y_0]$ as defined in (1.3). Then there exists a constant $c_{y_0,K,M} > 0$ such that for all initial data in the class*

$$h_0 \in \mathcal{F}_{M,K} := \{h_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} : \operatorname{supp}_{-\infty}(h_0) \subseteq K \text{ and is countable}, \|h_0\|_\infty \leq M\}$$

for some $M > 0$ and $K \subseteq [1, \infty)$ compact, one has with $(a_m)_{m \in \mathbb{N}}$ as in Theorem 6.3, and

$$m^* = \sup\left\{m \in \mathbb{N} : \left(a_m^2 \leq \log\left(\frac{1}{\mu(A)^{\frac{\epsilon\theta}{c_{y_0,K,M}}}}\right)\right)\right\} < \infty ,$$

the estimates for all A Borel

$$\mathbb{P}(h(\cdot+1) - h(1) \in A) \leq O_{y_0,p,s}\left(\mu(A)^{\frac{\epsilon}{s}(1-\frac{1}{p})} + \tfrac{1}{m^{*1/p}} + \mathbb{P}(L_0 \geq m^*)\right) ,$$

for any $p > 1$, $s > 1$ $\epsilon \in (0, 1)$ with $\theta = \frac{1}{s}(1 - 1/p)$.

Proof of Corollary 6.4. From Theorem 6.3, we have the bounds with

$$a_m = O(m^3 \cdot \inf\{k > 1 : 1 \leq \ell \leq m\,, \mathbb{P}(K_{x_0,m} \geq k) \leq 1/m\}^{5/4})$$

for some fixed $x \in \operatorname{supp}_{-\infty}(h_0)$, all $p > 1$, $s > 1$ and A Borel

$$\begin{aligned}&\mathbb{P}\left(H_m(\cdot+1) - H_m(1) \in A\right)\\ &\leq O_{y_0,K,p,s}\left(\exp(c_{y_0,K,M,s} a^2)) \cdot \mu(A)^{\frac{1}{s}(1-\frac{1}{p})} + \tfrac{1}{m^{1/p}}\right) .\end{aligned}$$

One thus estimates for all A Borel measurable

$$\begin{aligned}\mathbb{P}(h(\cdot+1) - h(1) \in A) &= \inf_{m\in\mathbb{N}} \mathbb{P}(h(\cdot+1) - h(1) \in A, L_0 \leq m) + \mathbb{P}(L_0 \geq m+1)\\ &\leq \inf_{m\in\mathbb{N}} \mathbb{P}(H_m(\cdot+1) - H_m(1) \in A, L_0 \leq m) + \mathbb{P}(L_0 \geq m)\\ &\leq \inf_{m\in\mathbb{N}} O_{y_0,K,p,s}\left(\exp(c_{y_0,K,M,s} a^2)) \cdot \mu(A)^{\frac{1}{s}(1-\frac{1}{p})} + \frac{1}{m^{1/p}} + \mathbb{P}(L_0 \geq m)\right) .\end{aligned}$$

Now, with $\theta = \frac{1}{s}(1 - 1/p)$ and any $\epsilon \in (0, 1)$, let

$$m^* = \sup\left\{m \in \mathbb{N} : \left(a_m^2 \leq \log\left(\frac{1}{\mu(A)^{\frac{\epsilon\theta}{c_{y_0,K,M}}}}\right)\right)\right\} < \infty .$$

Now, one further estimates for all A Borel

$$\mathbb{P}(h(\cdot+1) - h(1) \in A) \leq O_{y_0,K,p,s}\left(\mu(A)^{\frac{\epsilon}{s}(1-\frac{1}{p})} + \tfrac{1}{m^{*1/p}} + \mathbb{P}(L_0 \geq m^*)\right) ,$$

concluding the proof. □

Finally, using the tail bounds on L_0 established in Theorems 4.5 and 4.14, we establish using Corollary 6.4 the uniform quantitative Brownian regularity of spatial increments of the KPZ fixed points started from initial data with bounded and countable 'max-plus' support. The uniformity is with respect to a suitable class of initial data. This is the content of the following theorem, which is the main result of this paper.

Theorem 6.5. *Let $h_t(\cdot) := \mathcal{L}(t; h_0), t \geq 0$ be the KPZ fixed point as defined in* (1.3)*. Then, fixing $t > 0$ and any $\ell < r$ both bounded, with $|\ell| + |r| \leq y_0$ for some $y_0 > 0$, one obtains the estimates for all $p > 1$, $s > 1$, A Borel measurable $A \subseteq C_{0,*}([0, r-\ell])$ with $\mu(A) > 0$*

$$\begin{aligned}&\mathbb{P}(h_t(\cdot + \ell) - h_t(\ell) \in A)\\ &\leq O_{K,t,y_0,M,s}\left(\mu(A)^{\frac{\epsilon}{s}(1-1/p)} + \exp\left(-d_{K,t,y_0,s}\log^{1/1000}\log\left(1/\mu(A)^{c_{K,t,y_0,M,s}}\right)\right)\right),\end{aligned}$$

for some $c_{K,t,y_0,h_0}, d'_{K,t,y_0} > 0$ uniformly in initial data in the class

$$h_0 \in \mathcal{F}_{M,K} := \{h_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} : \operatorname{supp}_{-\infty}(h_0) \subseteq K \textit{ and is countable}, \|h_0\|_\infty \leq M\}$$

for any fixed $M > 0$.

Proof of Theorem 6.5. By the $1:2:3$ scaling invariance of the directed landscape, we can without loss of generality assume that $t = \ell = 1$. For ease of notation, let $h(\cdot) := h_1(\cdot)$ denote the KPZ fixed point at unit time.

Observe that the estimates in Theorems 4.5 and 4.14 for $m \geq 1$ give

$$\inf\{k > 1 : 1 \leq \ell \leq m\,, \mathbb{P}(\sup_{x \in K} K_{x,\ell} \geq k) \leq 1/m^2\} \leq C_K m^{256}\left(\exp\left(d_{K,\epsilon} M \log^{1/(1/882-\epsilon)}(m)\right)\right),$$

for some positive constant $d_{K,\epsilon} > 0$.

Recalling notation from Corollary 6.4, we now estimate

$$a_{m^*} = O_K\left((m^*)^{323} \cdot \exp\left(d_{K,\epsilon,M}\log^{1/(1/882-\epsilon)} m^*\right)\right)$$

after possibly changing $d_{K,\epsilon}$ to $d_{K,\epsilon,M}$ and obtain

$$m^* \geq \sup\left\{m \in \mathbb{N} : C_K\left(\exp\left(d_{K,\epsilon,M}\log^{1/(1/882-\epsilon)}(m)\right)\right) \leq \log\left(1/\mu(A)^{1/c_{K,y_0,M,s}}\right)\right\}.$$

This simplifies to the lower bound

$$m^* \geq \left\lceil C_{K,\epsilon}\exp\left(d_{K,\epsilon}\frac{\log^{1/882-\epsilon}\log\left(1/\mu(A)^{1/c_{y_0,K,M,s}}\right)}{M^{1/882-\epsilon}}\right)\right\rceil,$$

for some $C_{K,\epsilon}, d_{K,\epsilon,M} > 0$.

Finally, Corollary 6.4 gives the estimates for all A Borel

$$\begin{aligned}\mathbb{P}(h(\cdot + 1) - h(1) \in A) &\leq O_{y_0,p,s}\Bigg(\mu(A)^{\frac{\epsilon}{2}(1-1/p)}\\ &+ C_{K,\epsilon}\exp\left(-d_{K,\epsilon}\frac{\log^{1/882-\epsilon}\log\left(1/\mu(A)^{1/c_{y_0,K,M,s}}\right)}{M^{1/882-\epsilon}}\right)\Bigg)\\ &\leq O_{K,y_0,s,\epsilon}\left(\mu(A)^{\frac{\epsilon}{s}(1-1/p)} + \exp\left(-d_{K,\epsilon}\frac{\log^{1/882-\epsilon}\log\left(1/\mu(A)^{1/c_{y_0,K,M,s}}\right)}{M^{1/882-\epsilon}}\right)\right),\end{aligned}$$

for some $d_{K,\epsilon,M}, d_{K,\epsilon} > 0$, completing the proof. $\square$

We are now in a position to prove the main result of this paper, namely the extension of this quantitative Brownian regularity to all finitary initial data. In brief, this will be achieved through a localisation argument using global shape estimates enjoyed by the directed landscape, allowing us to control the support of the initial data 'seen' by the KPZ fixed point on compacts with high probability.

Theorem 6.6. *Let $h_t(\cdot) := \mathcal{L}(t; h_0), t \geq 0$ be the KPZ fixed point as defined in* (1.3) *where h_0 is t-finitary. Then, for any fixed $\ell < r$ with $|\ell| + |r| \leq y_0$ for some $y_0 > 0$, there exists some universal $\theta > 0$ such that the estimates for all A Borel measurable $A \subseteq C_{0,*}([0, r-\ell])$ with rate two Wiener measure $\mu(A) > 0$*

$$\mathbb{P}(h_t(\cdot + \ell) - h_t(\ell) \in A) \leq c_{t,y_0,h_0} \exp\left(-d_{t,y_0,h_0} \log^\theta \log\left(1/\mu(A)\right)\right) ,$$

hold for some $c_{t,y_0,h_0}, d_{t,y_0,h_0} > 0$.

Proof of Theorem 6.5. By the $1:2:3$ scaling invariance of the directed landscape, we can without loss of generality assume that $t = 2$. Using the metric composition law, we can now express the KPZ fixed point on $[\ell, r]$

$$h_2(y) = \max_{x \in \mathbb{R}} (\mathfrak{h}(x) + \mathcal{S}(x, y)) , \qquad y \in [\ell, r] .$$

where $\mathfrak{h}$ denotes the *random* initial data

$$\mathfrak{h}(y) = \max_{x \in \mathbb{R}} (h(x) + \mathcal{S}'(x, y))$$

where $\mathcal{S}'(\cdot, \cdot)$ is an Airy sheet independent of $\mathcal{S}(\cdot, \cdot)$. Recall that, being 2-finitary, h_0 satisfies

$$\lim_{|x| \to \infty} \frac{h_0(x) - x^2/2}{|x|} = -\infty .$$

Thus, there exists some $x_0 > 0$ deterministic dependent on h_0 such that

$$h_0(x) - x^2/2 \leq -(y_0 + 1)|x| , \qquad \text{for all } |x| \geq x_0 .$$

Henceforth, we will treat x_0 as fixed (only depending on our domain of comparison, which is fixed). Additionally, from [DSV22] the Airy sheets satisfy almost sure pointwise bounds

$$|\mathcal{S}(x, y) + (x - y)^2| \leq \mathfrak{C} + c \log^{2/3}(2 + |x| + |y|) , \qquad \text{for all } x, y \in \mathbb{R}$$

and

$$|\mathcal{S}'(x, y) + (x - y)^2| \leq \mathfrak{C}' + c \log^{2/3}(2 + |x| + |y|) , \qquad \text{for all } x, y \in \mathbb{R}$$

for some universal constant $c > 0$ and some $\mathfrak{C}, \mathfrak{C}'$ independent (identically distributed) both satisfying $\mathbb{E}[a^{\mathfrak{C}^{3/2}} + a^{\mathfrak{C}'^{3/2}}] < \infty$ for some $a > 1$. By rescaling one obtains analogous estimates for $\mathcal{L}(0, \cdot, ; \, , \cdot, t)$ for any $t > 0$ fixed.

Observe that for $|x| \geq x_0$, $y \in [-y_0, y_0]$

$$h_0(x) + \mathcal{L}(x, 0; y, 2) \leq -c|x| - y^2/2 + c \log(1 + |y|)$$

for some $c > 0$. Now, (assuming without loss of generality that h_0 is supported at the origin)

$$h_0(x) + \mathcal{L}(x, 0; y, 2) \leq h_0(0) + \mathcal{L}(0, 0; y, 2)$$

if

$$|x| \geq \mathfrak{C}''$$

for some $\mathfrak{C}'' > 0$ satisfying $\mathbb{E}[a^{\mathfrak{C}''^d}] < \infty$ for some $a > 1, d > 0$. Thus, there is some random $N \in \mathbb{N}$ satisfying $\mathbb{E}[C_{y_0,h_0}^{N^d}] < \infty$ for some $C_{y_0,h_0} > 1$ and $d > 0$ such that almost surely

$$h_2(y) = \max_{x \in [-N,N]} (h_0(x) + \mathcal{L}(x, 0; y, 2)) , \qquad \text{for all } y \in [-y_0, y_0] .$$

Now, we have that $\mathfrak{h}(\cdot)$ satisfies the following almost sure estimates for $y \in \mathbb{R}$

$$\begin{aligned}\mathfrak{h}(y) &\leq \max_{x \in [-N,N]} (h(x) - x^2 + 2xy + \mathfrak{C}' + c \log^{2/3}(2 + |x| + |y|)) - y^2 \\ &\leq c_{h_0} + \mathfrak{C}' + 2|y|N + \max_{x \in [-N,N]} (-|x| + c \log(1 + |x|)) - y^2 + c \log(1 + |y|) \\ &\leq c_{h_0} + \mathfrak{C}' - y^2 + c|y|N\end{aligned}$$

for some constants $c, c_{h_0} > 0$ (changing from line to line). Arguing as before, we have that for $y \in [-y_0, y_0]$

$$\mathfrak{h}(x) + \mathcal{S}(x,y) \leq \mathfrak{h}(0) + \mathcal{S}(0,y)$$

for all

$$|x| \geq N'$$

for some N' satisfying $\mathbb{E}[C_{y_0,h_0}^{N'^d}] < \infty$ for some $C_{y_0,h_0} > 1$ and $d > 0$.

Summarising, we have that

$$\begin{cases} h_2(y) = \max_{x \in [-N',N']} (\mathfrak{h}(x) + \mathcal{S}(x,y)) \,, & \text{for all } y \in [-y_0, y_0] \,, \\ \mathfrak{h}(x) = \max_{z \in [-N,N]} (h_0(z) + \mathcal{S}'(z,x)) \,, & \text{for all } z \in \mathbb{R} \,, \end{cases} \tag{6.3}$$

for some N, N' satisfying $\mathbb{E}[C_{y_0,h_0}^{N'^d} + C_{y_0,h_0}^{N^d}] < \infty$ for some $C_{y_0,h_0} > 1, d > 0$ and independent Airy sheets $\mathcal{S}, \mathcal{S}'$. Notice that one can use *any* $M \geq N, N'$ in their place. Furthermore, note that $\mathfrak{h}$ is continuous and satisfies almost surely,

$$\mathrm{supp}_{-\infty}(\mathfrak{h}) = \mathbb{R} \,. \tag{6.4}$$

This will be useful later in the argument.

Now, fix some $A \subseteq C_{0,*}([\ell, r]; \mathbb{R})$ Borel measurable. We can thus estimate for all $n \geq 1$

$$\mathbb{P}(h_2(\cdot) - h_2(\ell) \in A) \leq \mathbb{P}(h_2(\cdot) - h_2(\ell) \in A, N + N' \leq n) + \mathbb{P}(N + N' \geq n) \,,$$

effectively localising the support of the initial data in the first term. We can now re-express

$$\mathbb{P}(h_2(\cdot) - h_2(\ell) \in A) \leq \mathbb{P}(h_2^n(\cdot + n + 1 - \ell) - h_2^n(n+1) \in A, N + N' \leq n) + \mathbb{P}(N + N' \geq n) \,,$$

where on the event $N + N' \leq n$

$$\begin{aligned} h_2^n(\cdot) &= \max_{x \in [-n,n]} (\mathfrak{h}(x) + \mathcal{S}(x, \cdot + \ell - n - 1)) \\ &= \max_{x \in [1, 2n+1-\ell]} (\mathfrak{h}(x + \ell - n - 1) + \mathcal{S}(x, \cdot)) \end{aligned}$$

by the skew-symmetry of the Airy sheet. Note that $\mathfrak{h}$ is independent from $\mathcal{S}$ so the last distributional equality (skew-symmetry) is valid almost surely. Note also, on the event $\{N + N' \leq n\}$, the initial data $\mathfrak{h}$ satisfies

$$\mathfrak{h}(\cdot) = \max_{x \in [-n,n]} (h_0(x) + \mathcal{S}'(x, \cdot))$$

where the latter is *independent* of $\mathcal{S}$. Now, using Lemma 4.4 and Theorem 4.5, we have the estimates

$$\begin{aligned} &\mathbb{P}(h_2(\cdot) - h_2(\ell) \in A) \leq \mathbb{P}(\max_{1 \leq k \leq m} (G_k^{\mathfrak{h}}) + \mathcal{A}[(0,k) \to (\cdot + n + 1 - \ell, 1)] \\ &- \max_{1 \leq k \leq m} (G_k^{\mathfrak{h}}) + \mathcal{A}[(0,k) \to (n+1, 1)] \in A) + \mathbb{P}(L_0 \geq m) + \mathbb{P}(N + N' \geq n) \,, \end{aligned}$$

where $\mathcal{A}$ is a parabolic Airy line ensemble independent from $\mathfrak{h}$ and $L_0 = \pi[\lceil 2n + 1 - \ell \rceil, \lceil r - \ell + n + 1 \rceil](0)$ (see Definition 4.3) and

$$G_k^{\mathfrak{h}} = \max_{x \in [-n,n]} (\mathfrak{h}(x + \ell - n - 1) + \mathcal{A}[x \to (0,k)]) \,, \qquad \text{for all } k \geq 1 \,.$$

Since $\mathfrak{h}$ has full support, we now estimate (using the almost sure monotonicity of the $G_k^{\mathfrak{h}}, k \geq 1$ and the fact that h_0 is 2-finitary, meaning it has an absolute quadratic upper bound up to some h_0-dependent deterministic constant)

$$\max_{1 \leq k \leq m} |G_k^{\mathfrak{h}} - G_1^{\mathfrak{h}}| \leq |2 \max_{x \in [-n,n]} \mathfrak{h}(x)| + 2 \max_{x \in [-n,n]} |\mathcal{S}(x, 0)| \tag{6.5}$$

$$+ |\mathcal{A}^{\text{stat}}[1 \to (0,m)] - \mathcal{A}^{\text{stat}}[1 \to (0,1)]| \tag{6.6}$$

$$\leq c_{\ell,h_0} + cn^2 + \mathfrak{C} + \mathfrak{C}' + |\mathcal{A}^{\text{stat}}[n \to (0,m)] - \mathcal{A}^{\text{stat}}[n \to (0,1)]| \,, \tag{6.7}$$

for some $c, c_{h_0} > 0$.

Now, proceeding as in the proof of Corollary 6.4 and using the above discussion, we can now estimate for all $n, m \geq 1$ with $a = O(m^3 \cdot \inf\{k > 1 : \mathbb{P}(K_{n,m} \geq k) \leq 1/m\}^{5/4})$ and some universal $\theta, \eta > 0$

$$\mathbb{P}(h_2(\cdot) - h_2(\ell) \in A) \leq O_{y_0,K,p}\left(\exp\left(d_{\ell,r}(n^\theta + a^\theta)\right) \cdot \mu(A)^\eta + \frac{1}{m^\eta} + \mathbb{P}(L_0 \geq m)\right) + \mathbb{P}(N + N' \geq n)$$

where $\mu(\cdot)$ denotes the rate two Wiener measure on $[\ell, r]$.

Moreover, for $x > 0$ fixed, note that keeping track of the dependence on the geodesic 'speed' (see the remarks around Definition 4.3) in Theorem 4.14 gives that there exists some universal $\theta > 0$ such that for all $a > 2, n, k \geq 1$ the geodesic coalescence times (see Definition 4.11) satisfy

$$\mathbb{P}(K_{n,k} \geq a) \leq k^{256} \mathrm{e}^{dn^\theta} \exp\left(-\log^{1/1000} a\right) \,,$$

for some constant $d > 0$. Finally, proceeding as in the proof of Theorem 6.5 using the stretched exponential tails of L_0, N, N', we obtain the desired conclusion. □

7. Future directions

In this section, we discuss possible ways of strengthening the quantitative Brownian comparison of the KPZ fixed point on compacts.

A key to improving Brownian regularity is to strengthen the estimates satisfied by the truncated versions of the KPZ fixed point. This would include improving the inverse acceptance probability estimates as well as the Radon-Nikodym derivative bounds of inhomogeneous BLPP. Next, a refinement of the picture of geodesic geometry in the Airy line ensemble, in particular improving tail bounds on semi-infinite geodesic intercepts and finer control over geodesic coalescence events on Brownian melons would also help improve Brownian regularity.

In particular, for any given $x > 0$, if one could strengthen the comparison in Theorem 3.17 by showing that for every $\epsilon > 0$

$$\mathbb{P}\left(k^{1/6}|\mathcal{A}[(0,k) \to (x,1)] - 2\sqrt{2kx}| > \epsilon\right) \xrightarrow{\epsilon \to 0} 0 \qquad k \geq 1 \,,$$

then one would obtain improved tail bounds for L_0, compared to those in Theorem 4.5, possibly even showing that for all $\epsilon > 0$, L_0 satisfies the tail bounds

$$\sup_{j \in \mathbb{N}} \mathrm{e}^{j^{(3-\epsilon)}} \cdot \mathbb{P}(L_0 \geq j) < \infty \,.$$

Ultimately, the fruits of such an endeavour would be the following proposition.

Proposition 7.1. *Fix $y_0 > 1$ and let $h(\cdot)$ be the KPZ fixed point at unit time as defined in* (1.3)*. Suppose that there exists some $p > 1$, $d \geq 1, r > 0$ such that we have the estimate for all $m \in \mathbb{N}$ and $A \subseteq C_{*,*}([0, y_0 - 1])$ Borel set,*

$$\mathbb{P}(H_m(\cdot + 1) - H_m(1) \in A) \leq c_p(\mathrm{e}^{m^d} \cdot \mu(A)^{1-1/p} + \mathrm{e}^{-m^r})$$

for all $m \in \mathbb{N}$ with $c_p > 0$ independent of $m \in \mathbb{N}$, where μ denotes the law of a rate two Brownian motion on $[0, y_0 - 1]$ and H_m as defined in (6.1). Furthermore, let L_0 satisfy some tail bound

$$\sup_{j \in \mathbb{N}} \mathrm{e}^{j^r} \cdot \mathbb{P}(L_0 \geq j) < \infty \,.$$

Then, for any Borel $A \subseteq C(0, y_0 - 1)$, $\epsilon \in (0,1)$ with $\mu(A) > 0$

$$\mathbb{P}(h(\cdot+1) - h(1) \in A) \leq c'_{p,\epsilon} \exp\left(-\left\lfloor (\epsilon(1-1/p))^{1/d} \log\left(\frac{1}{\mu(A)}\right)^{\frac{1}{d}} \right\rfloor^r\right), r \leq d$$

and for any $t \in (1,p)$

$$\mathbb{P}(h(\cdot+1) - h(1) \in A) \leq c'_t \cdot \mu(A)^{1-1/t}, r > d$$

for some positive $c'_{p,\epsilon}, c'_t > 0$ independent of $m \in \mathbb{N}$. In other words if $r > d$, then the Radon-Nikodym derivative of the increment process of the KPZ fixed point $h(\cdot+1) - h(1)$ is in $L^{p-}(\mu)$ on $[0, y_0 - 1]$.

Proof. Fix $A \subseteq C_{0,*}([0, y_0 - 1])$ Borel measurable. Then we estimate for all $m \in \mathbb{N}$

$$\begin{aligned} \mathbb{P}(h(\cdot+1) - h(1) \in A) &= \mathbb{P}(h(\cdot+1) - h(1) \in A, L_0 \leq m) + \mathbb{P}(L_0 \geq m+1) \\ &\leq \mathbb{P}(H_m(\cdot+1) - H_m(1) \in A, L_0 \leq m) + \mathbb{P}(L_0 \geq m) \\ &\leq c_p(\mathrm{e}^{m^d} \cdot \mu(A)^{1-1/p} + \mathrm{e}^{-m^r}), \end{aligned}$$

for some universal in $m \in \mathbb{N}$ (though possibly p-dependent) constant $c_p > 0$.

Now, if $d \geq r$ fix any $\epsilon \in (0,1)$ and let

$$m^* = \left\lfloor \log\left(\frac{1}{\mu(A)^{\epsilon(1-1/p)}}\right)^{\frac{1}{d}} \right\rfloor.$$

We can without loss of generality assume $m^* \geq 1$ otherwise, we would have $\mu(A) \in (\mathrm{e}^{-1/(\epsilon(1-1/p))}, 1]$ which gives

$$\nu(A) \leq 1 = \mathrm{e}^{1/(\epsilon(1-1/p))} \cdot \mathrm{e}^{-1/(\epsilon(1-1/p))} \leq \mathrm{e}^{1/(\epsilon(1-1/p))} \cdot \mu(A).$$

Hence, we estimate

$$\begin{aligned} \mathbb{P}(h(\cdot+1) - h(1) \in A) &\leq \inf_{m \in \mathbb{N}} c_p(\mathrm{e}^{m^d} \cdot \mu(A)^{1-1/p} + \mathrm{e}^{-m^r}) \\ &\leq c_p(\mathrm{e}^{m^{*d}} \cdot \mu(A)^{1-1/p} + \mathrm{e}^{-m^{*r}}) \\ &\leq c'_p\left(\mu(A)^{(1-\epsilon)(1-1/p)} + \exp\left(-\left\lfloor (\epsilon(1-1/p))^{1/d} \log\left(\frac{1}{\mu(A)}\right)^{\frac{1}{d}} \right\rfloor^r\right)\right) \\ &\leq c'_{p,\epsilon} \exp\left(-\left\lfloor (\epsilon(1-1/p))^{1/d} \log\left(\frac{1}{\mu(A)}\right)^{\frac{1}{d}} \right\rfloor^r\right) \end{aligned}$$

for some positive $c'_{p,\epsilon} > 0$ independent of $m \in \mathbb{N}$, concluding the proof of the first part.

If on the other hand, $d < r$, then for any $t \in (1,p)$ with

$$m^* = \left\lceil \log\left(\frac{1}{\mu(A)^{1-1/t}}\right)^{\frac{1}{r}} \right\rceil \geq 1,$$

we estimate,

$$\begin{aligned} \mathbb{P}(h(\cdot+1) - h(1) \in A) &\leq \inf_{m \in \mathbb{N}} c_p(\mathrm{e}^{m^d} \cdot \mu(A)^{1-1/p} + \mathrm{e}^{-m^r}) \\ &\leq c_p(\mathrm{e}^{m^{*d}} \cdot \mu(A)^{1-1/p} + \mathrm{e}^{-m^{*r}}) \\ &\leq c'_p\left(\exp\left(\left\lceil \log\left(\frac{1}{\mu(A)^{1-1/t}}\right)^{\frac{1}{r}} \right\rceil^d\right) \cdot \mu(A)^{\frac{p-t}{tp}} + 1\right) \cdot \mu(A)^{1-1/t} \end{aligned}$$

$$
\begin{aligned}
&\leq c'_p \left(\exp\left(\left\lceil \log\left(\frac{1}{\mu(A)^{1-1/t}}\right)^{\frac{1}{r}} \right\rceil^d - \frac{p-t}{tp}\log\frac{1}{\mu(A)} \right) + 1 \right) \cdot \mu(A)^{1-1/t} \\
&\leq c'_p \left(\sup_{z\in[1,\infty)} \exp\left(\lceil (1-1/t)^{\frac{1}{r}} (\log z)^{\frac{1}{r}} \rceil^d - \frac{p-t}{tp}\log z \right) + 1 \right) \cdot \mu(A)^{1-1/t} \\
&\leq c'_p \left(\sup_{z\in[0,\infty)} \exp\left(((1-1/t)^{\frac{1}{r}} z^{\frac{1}{r}} + 1)^d - \frac{p-t}{tp} z \right) + 1 \right) \cdot \mu(A)^{1-1/t} \\
&= c'_t \cdot \mu(A)^{1-1/t}
\end{aligned}
$$

for some positive $c'_p, c'_t > 0$ (possibly changing from line to line) independent of $m \in \mathbb{N}$, concluding the proof of the second part. $\square$

In short, applying Proposition 7.1 with $r > d$, one can convert the finite depth bounds to a bound on spatial increments of the KPZ fixed point of the form

$$\mathbb{P}(h(\cdot + 1) - h(1) \in A) \leq c'_t \cdot \mu(A)^{1-1/t}$$

for any $t \in (1, p)$, for some positive $c'_t > 0$ independent of $m \in \mathbb{N}$. In other words, the Radon-Nikodym derivative of the increment process of the KPZ fixed point $h(\cdot + 1) - h(1)$ is in $L^{p-}(\mu)$ on compacts. We believe that one can achieve $r = 3$ from transversal fluctuation of geodesics in discrete environments. Moreover, we also expect to have $d < 3$ from our already established inhomogeneous Brownian LPP estimates, in addition to an improvement in estimating inverse acceptance probabilities. This would give $r > d$, as required for L^{p-}-regularity.

Appendix A. Stochastic continuity of semi-infinite Airy last passage values

We now make some remarks on the regularity of the semi-infinite Airy last passage values appearing in 5.1. First, recall the definition of stochastic continuity.

Definition A.1. *(Stochastic continuity) A stochastic process $(X_t)_{t\in I}$ defined on some interval $I \subseteq \mathbb{R}$ is called* ***continuous in probability****, if for all $t \in I$ and $a > 0$*

$$\lim_{I\ni s\to t} \mathbb{P}(|X_t - X_s| \geq a) = 0\,.$$

We now prove the following theorem which shows that the process defined by (5.1) on the positive reals, is continuous in probability.

Theorem A.2. *Fix $\ell \in \mathbb{N}$ and consider the stochastic process $(\mathcal{A}[x \to (0,\ell)])_{x\in\mathbb{R}_+}$ given by the almost sure pointwise limits*

$$\mathcal{A}[x \to (0,\ell)] := \begin{cases} \mathcal{S}(x,0) & \ell = 1 \\ \lim\limits_{k\to\infty} \mathcal{A}[x_k \to (0,\ell)] - \mathcal{A}[x_k \to (0,1)] + \mathcal{S}(x,0) & \ell > 1\,, \end{cases}$$

where $x_k = (-\sqrt{k/(2x)}, k)$ for $k \geq 1$. Then, for all $a > 0$, $0 < \varepsilon < x < y \in I$ with $|x - y| \leq O(a^{7/5}/\ell^{7/2})$,

$$
\begin{aligned}
&\mathbb{P}\Big(\frac{|\mathcal{A}[x \to (0,\ell)] - \mathcal{A}[y \to (0,\ell)]|}{|x-y|^{5/7}} \geq a\Big) \\
&\leq C_\epsilon \left(\exp\left(-d_\epsilon \log^{1/1000}(a^{2/5}/\ell^{256}) \right) \right).
\end{aligned}
$$

for some $C_\epsilon, d_\epsilon > 0$. This in particular means that the process $(\mathcal{A}[x \to (0,\ell)])_{x\in\mathbb{R}_+}$ is stochastically continuous.

Note that for $\ell = 1$, this is the parabolic Airy$_2$ process, which is Hölder $1/2-$ continuous. We also expect similar continuity statements to hold for values of ℓ greater than one.

Proof. Denoting the common coalescence depth $K_{x,\ell} \vee K_{y,\ell}$, as defined in Definition 4.11 by K_ℓ We can express for any $a > 0$ and any $m \geq \ell$

$$\begin{aligned}
&\mathbb{P}(|\mathcal{A}[x \to (0,\ell)] - \mathcal{A}[y \to (0,\ell)]| \geq a) \\
&\leq \mathbb{P}\Big(|\mathcal{A}[x \to (0,\ell)] - \mathcal{A}[y \to (0,\ell)]| \geq a, K_\ell \leq m\Big) + \mathbb{P}\Big(K_\ell \geq m\Big) \\
&\leq \mathbb{P}\Big(|\mathcal{A}^{\mathrm{stat}}[(-\sqrt{m/(2x)}, m) \to (0,\ell)] - \mathcal{A}^{\mathrm{stat}}[(-\sqrt{m/(2y)}, m) \to (0,\ell)]| \geq a\Big) \\
&+\mathbb{P}\Big(|\mathcal{A}^{\mathrm{stat}}[(-\sqrt{m/(2x)}, m) \to (0,1)] - \mathcal{A}^{\mathrm{stat}}[(-\sqrt{m/(2y)}, m) \to (0,1)]| \geq a\Big) + \mathbb{P}\Big(K_\ell \geq m\Big).
\end{aligned}$$

Observe we can estimate for $a < b$, $m \geq \ell \geq 1$ the last passage values

$$\begin{aligned}
&|\mathcal{A}^{\mathrm{stat}}[(a,m) \to (0,\ell)] - \mathcal{A}^{\mathrm{stat}}[(b,m) \to (0,\ell)]| \leq \sum_{\ell \leq k \leq m} \sup_{z \in [a,b]} \mathcal{A}_k^{\mathrm{stat}}(z) - \inf_{z\in[a,b]} \mathcal{A}_k^{\mathrm{stat}}(z) \\
&\leq \sup\nolimits_{t,s \in [a,b], t \neq s} \sqrt{|t-s| \log(2(b-a)/|t-s|)} \cdot \sum_{k=\ell}^{m} \omega_k(\mathcal{A}^{\mathrm{stat}}) \\
&\leq \sqrt{b-a} \sup\nolimits_{t \in [0,1]} \sqrt{t \log(2/t)} \cdot \sum_{k=\ell}^{m} \omega_k(\mathcal{A}^{\mathrm{stat}}) \\
&\lesssim \sqrt{b-a} \cdot \sum_{k=\ell}^{m} \omega_k(\mathcal{A}^{\mathrm{stat}}).
\end{aligned} \tag{A.1}$$

where the moduli of continuity

$$\omega_k(\mathcal{A}^{\mathrm{stat}}) := \sup_{t,s \in [a,b], t \neq s} \frac{|\mathcal{A}_k^{\mathrm{stat}}(t) - \mathcal{A}_k^{\mathrm{stat}}(s)|}{\sqrt{|t-s| \log(2(b-a)/|t-s|)}}, \qquad \ell \leq k \leq m$$

are sub-Gaussian random variables with uniform bounds on their tails (see Proposition 3.10).

We now estimate with $a = -\sqrt{m/(2x)}$ and $b = -\sqrt{m/(2y)}$, a union bound and Theorem 4.14

$$\begin{aligned}
&\mathbb{P}(|\mathcal{A}[x \to (0,\ell)] - \mathcal{A}[y \to (0,\ell)]| \geq a) \\
&\leq \mathbb{P}\Big(\sum_{k=\ell}^{m} \omega_k(\mathcal{A}^{\mathrm{stat}}) \geq a \cdot (m/2)^{-1/4} \cdot \frac{xy}{|x-y|}\Big) + \mathbb{P}(K_\ell \geq m) \\
&\leq C_1 \cdot m \cdot \exp\Big(- C_2 a^2 \cdot m^{-5/2} \cdot \tfrac{xy}{|x-y|}\Big) + C_\epsilon \left(\exp\left(-d_\epsilon \log^{1/1000}(m/\ell^{256})\right)\right) \\
&\leq C_1 \cdot m \cdot \exp\Big(- C_2 \epsilon a^2 \cdot m^{-5/2} \cdot \tfrac{1}{|x-y|}\Big) + C_\epsilon \left(\exp\left(-d_\epsilon \log^{1/1000}(m/\ell^{256})\right)\right).
\end{aligned}$$

for some $C_1, C_2, C_\epsilon, d_\epsilon > 0$. Now, taking $m \geq O(a^{2/5}/|x-y|^{2/7})$ we estimate for all $a > 0$, $0 < \epsilon < x < y$ such that $|x-y| \leq O(a^{7/5}/\ell^{7/2})$ (so that $m \geq \ell$)

$$\begin{aligned}
&\mathbb{P}(|\mathcal{A}[x \to (0,\ell)] - \mathcal{A}[y \to (0,\ell)]| \geq a) \\
&\leq C_\epsilon \left(\exp\left(-d_\epsilon \log^{1/1000}(a^{2/5}/(\ell^{256}|x-y|^{2/7}))\right)\right).
\end{aligned}$$

for some $C_\epsilon, d_\epsilon > 0$ concluding the proof. □

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