HEISENBERG CALCULUS IN QUANTUM TOPOLOGY
MATH256X

MWF 12:00-1:00 pm SC 411

INSTRUCTOR

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COURSE DESCRIPTION

This course aims to provide a hands-on introduction to the Heisenberg calculus of Beals-Greiner and Taylor. This pseudodifferential calculus provides a natural method for studying non-elliptic problems where the classical abelian symbol calculus is not applicable. This theory is of independent interest and also makes connections with diverse topics such as harmonic analysis, noncommutative geometry, number theory, partial differential equations, representation theory, sub-Riemannian geometry, quantum field theory, and shows up as a key tool in the Julg-Kasparov proof of the $\mathrm{SU}(n,1)$ Baum-Connes conjecture. Some key topics to be covered are: Heisenberg groups, model operators, principal symbols, kernels, pseudodifferential operators, hypoellipticity, parametrices, the Rockland condition, contact/CR manifolds, Hörmander's sum of squares, Kohn Laplacian, horizontal sub-Laplacian, and the Rumin (contact) Laplacian. A secondary goal for the course is to consider applications of the Heisenberg calculus in quantum Chern-Simons theory and to motivate some problems of current research interest. Some further possible topics are: asymptotic expansion conjecture, eta/zeta functions, Rumin-Seshadri analytic torsion, and perturbative quantum invariants.

GOALS OF THE COURSE

The main goal of this course is for students to produce a collaborative original/review research article in the area of Quantum Topology that is based on a course project included in the list below. After taking this course, you should be able to read, interpret and recognize articles of importance related to a particular problem in Quantum Topology, and utilize tools from the Heisenberg pseudodifferential calculus in your general research.

GRADING

Your grade will be based on the following:

- (25%) First midterm evaluation of your research article. At this stage your article should provide a review of the main problem and how it fits into the broader research landscape.

- (25%) Second midterm evaluation of your research article. At this stage your article should be extended to include an in depth study of a specific example, computation, theorem, etc... that builds on your first midterm review.

- (50%) Final evaluation of your research article. A final version of your article should be completed. The main component of this evaluation will be a class presentation based on your work.

TEXTBOOKS AND COURSE MATERIAL

We will not have a standard text for this course and material will be taken from several sources:

- Beals, R., Greiner, P.C. "Calculus on Heisenberg Manifolds," Annals of Mathematics Studies, vol. 119. Princeton University Press, Princeton, NJ, 1988.
- Folland, G., Stein, E. "Estimates for the $\overline{\partial}_b$ complex and analysis on the Heisenberg group," Comm. Pure Appl. Math. 27 (1974) 429-522.
- Ponge, R. "Heisenberg calculus and spectral theory of hypoelliptic operators on Heisenberg manifolds," Mem. Amer. Math. Soc. 194 (2008), no. 906.
- Rockland, C. "Hypoellipticity on the Heisenberg group-representation-theoretic criteria," Trans. Amer. Math. Soc. 240 (1978) 1-52.
- Taylor, M.E. "Noncommutative microlocal analysis. I," Mem. Amer. Math. Soc. 52 (1984), no. 313.

Note that this does not comprise a complete list of sources and other possible references may be included throughout the course.

COURSE PROJECTS

NOTE: Numerically indicate your top 5 choices in the left hand column from greatest = 1 to least = 5.

______ **Canonical Quantization of Contact Chern-Simons Theory**
Review canonical quantization on moduli space of flat connections on a surface and develop the analogue of §3,

- Witten, E. "Quantum field theory and the Jones polynomial," Commun. Math. Phys. 121 (1989), No. 3, 351-399.

with respect to the contact Chern-Simons functional introduced in §3.1,

- Beasley, C. and Witten, E. "Non-abelian localization for Chern-Simons theory," J. Differential Geom. 70 (2005), 183-323.

Constructive Quantum Field Theory and Contact Chern-Simons Theory
Develop an analogue of the work,

- Albeverio, S., Schäfer, J. "Abelian Chern-Simons theory and linking numbers via oscillatory integrals," J. Math. Phys. 36 No. 5 (1994) 2135-2169.
- Sahlmann, H., Thiemann, T. "Abelian Chern-Simons theory, Stokes' theorem, and generalized connections," Journal of Geometry and Physics, Vol. 62, No. 2, (2010) 204-212.

for the abelian Chern-Simons partition function with action given in §'s $3, 4$,

- Jeffrey, L., McLellan, B. "Eta-invariants and anomalies in U(1) Chern-Simons theory," Chern-Simons Gauge Theory, 20 Years After, Conf. Proc., (2010).

Reshetikhin-Turaev Invariants and Contact Topology
Provide a review of the Reshetikhin-Turaev invariants,

- Reshetikhin, N., Turaev, V.G. "Invariants of 3-manifolds via link polynomials and quantum groups," Invent. Math. 103 (1991), No. 3, 547-597.

and investigate a contact geometric version involving ingredients from,

- Ding, F., Geiges, H. "A Legendrian surgery presentation of contact 3-manifolds," Math. Proc. Cambridge Philos. Soc., 136 No. 3 (2004) 583-598
- Ding, F., Geiges, H. "Handle moves in contact surgery diagrams," J. Topol. 2 (2009) 105-122.

Witten Laplacian in Contact Geometry
Introduce supersymmetry and the Witten Laplacian,

- Witten, E. "Supersymmetry and Morse theory," J. Diff. Geom. 17 (1982), 661-692.

and investigate a possible contact geometric analogue by considering a Witten-type deformation of the Rumin operator,

- Rumin, M., Seshadri, N. "Analytic torsions on contact manifolds," Annales de l'Institut Fourier, 61 (2011).

Combinatorial Contact R-Torsion

Introduce the Müller-Cheeger theorem,

– Müller, W. "Analytic torsion and R-torsion for unimodular representations," J. Amer. Math. Soc. 6 (1993), 721-743.

and investigate a combinatorial analogue for the contact analytic torsion,

– Rumin, M., Seshadri, N. "Analytic torsions on contact manifolds," Annales de l'Institut Fourier, 61 (2011).

ρ-Invariants and Heisenberg Calculus

Introduce the ρ-invariant of Atiyah-Patodi-Singer,

– Atiyah, M. F., Patodi, V.K., Singer, I.M. "Spectral asymmetry and Riemannian geometry, I," Proc. Cambridge Philos. Soc. 77 (1975), 43-69.

and investigate a contact analogue for the Rumin complex,

– Rumin, M., Seshadri, N. "Analytic torsions on contact manifolds," Annales de l'Institut Fourier, 61 (2011).

Analytic Contact Torsion on Manifolds with Boundary

Introduce the analytic torsion on manifolds with boundary,

– Müller, W. "Analytic torsion and R-torsion for unimodular representations," J. Amer. Math. Soc. 6 (1993), 721-743.

and investigate an analogue for the contact analytic torsion,

– Rumin, M., Seshadri, N. "Analytic torsions on contact manifolds," Annales de l'Institut Fourier, 61 (2011).

Theta Functions and Legendrian Knots

Introduce theta functions and abelian Chern-Simons theory,

– Gelca, R. "Theta functions and knots," World Scientific Publishing Co. Pte. Ltd. (2014)

and investigate a Legendrian analogue for abelian contact Chern-Simons theory,

– Jeffrey, L., McLellan, B. "Eta-invariants and anomalies in U(1) Chern-Simons theory," Chern-Simons Gauge Theory, 20 Years After, Conf. Proc., (2010).

Non-Abelian Localization and $\mathrm{SL}(2,\mathbb{C})$-Chern-Simons Theory
Introduce non-abelian localization for Chern-Simons theory with compact gauge group,

- Beasley, C. and Witten, E. "Non-abelian localization for Chern-Simons theory," J. Differential Geom. 70 (2005), 183-323.

and investigate an analogous construction for $\mathrm{SL}(2,\mathbb{C})$-Chern-Simons Theory,

- Witten, E. "Quantization of chern-simons gauge theory with complex gauge group," Commun. Math. Phys. 137 (1991) 29-66.

Instanton Floer Homology and the Contact Chern-Simons Functional
Introduce Instanton Floer Homology,

- Floer, A. "An instanton-invariant for 3-manifolds," Comm. Math. Phys. 118 (1988), No. 2, 215-240.

and investigate an analogous construction using the contact Chern-Simons functional,

- Beasley, C. and Witten, E. "Non-abelian localization for Chern-Simons theory," J. Differential Geom. 70 (2005), 183-323.