MATH 268Z: Kashiwara Crystals and Quantum Integrable Systems

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Fall 2022

Meeting time:

Tuesday 10:30 AM - 11:45 AM; Thursday 10:30 AM - 11:45 AM

Overview

Kashiwara crystals are a combinatorial model of the tensor category of finite-dimensional representations of a semisimple complex Lie algebra $\mathfrak{g}$, where the weight basis vectors in the representation are represented by points (marked by the weights of the representation), and the action of the Chevalley generators by arrows (marked by simple roots). This can be also regarded as the $q \to 0$ limit of finite-dimensional $U_q(\mathfrak{g})$-modules. Crystals form a *coboundary* monoidal category in which the tensor product is not symmetric, but the tensor products of two crystals in different orders are still connected by some functorial isomorphism (similar to braiding but different). The role of the braid group here is played by the cactus group J_n – the S_n-equivariant fundamental group of the Deligne-Mumford compactification $\overline{M_{0,n+1}}(\mathbb{R})$ moduli spaces of real stable rational curves with $n+1$ marked points. The famous combinatorial algorithms such as the Robinson-Schensted-Knuth correspondence and the Schützenberger involution become very natural in terms of this monoidal category. The main goal of this course is to explain how combinatorial structures like crystals arise from quantum integrable systems (more specifically, from the Gaudin magnet chain). This gives a new definition of the category of Kashiwara crystals for a given semisimple Lie algebra $\mathfrak{g}$ in terms of $\mathfrak{g}^\vee$-opers (with respect to the Langlands dual of the Lie algebra $\mathfrak{g}^\vee$) on a rational curve with regular singularities at marked points. In particular, this explains the appearance of the Deligne-Mumford space in the theory of Kashiwara crystals.

Prerequisites

Good background in Representation Theory of Lie groups and Lie algebras, basic Algebraic Geometry and basic Topology.

Exams and grading

There will be a take-home exam at the end of the semester for those who need a grade for this course.

Course outline and references

Roughly, the course consists of the following four parts:

- Preliminaries on Representation Theory: representations of the symmetric group, Schur-Weyl duality, representations of semisimple Lie algebras, Weyl character formula, related combinatorial formulas. The material is standard, I will mainly use the books P. Etingof et al, *Introduction to representation theory* and A. Kirillov Jr., *Introduction to Lie Groups and Lie Algebras*.

- Quantum groups and crystal bases. Kashiwara crystals and their presentations. Crystals as a coboundary monoidal category. I will mainly follow the book D. Bump, A. Schilling, *Crystal Bases: Representations And Combinatorics*, and the survey of A. Savage, *Braided and coboundary monoidal categories*.

- Gaudin magnet chain and similar quantum integrable systems. Algebraic Bethe ansatz and its representation-theoretic meaning. The main references are B. Feigin, E. Frenkel, N. Reshetikhin, *Gaudin model, Bethe ansatz and critical level*, Comm. Math. Phys. 166(1): 27-62 (1994) and B. Feigin, E. Frenkel, L. Rybnikov, *Opers with irregular singularity and spectra of the shift of argument subalgebra*, Duke Math. J. 155(2): 337-363.

- Crystal structure on solutions of Bethe ansatz. Combinatorial version of the Drinfeld-Kohno theorem: monodromy of solutions of Bethe ansatz equations is given by crystal commutors. The main references are A. Henriques and J. Kamnitzer, *Crystals and coboundary categories*, Duke Math. J. 132(2): 191-216 and I. Halacheva, J. Kamnitzer, L. Rybnikov, A. Weekes, *Crystals and monodromy of Bethe vectors*, Duke Math. J. 169(12): 2337-2419.