SYLLABUS FOR MATH 274Y
SPECTRAL THEORY AND QUANTUM SPIN SYSTEMS

SPRING 2020

Class: Wednesday and Friday, 10:30 – 11:45 AM, in Science Center 310

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Course website: https://canvas.harvard.edu/courses/67764

Course Description: Quantum spin systems lie at the interface of theoretical physics and mathematical physics. They allow us to study true many-body phenomena, like phase transitions, within a solid mathematical framework. In this topics class, we will study the basic notions, main results, and open problems, of quantum spin systems. Two highlights will be the the proof of long-range order in the Heisenberg antiferromagnet by Dyson, Lieb, and Simon and the derivation of a spectral gap in the AKLT chain by Affleck, Kennedy, Lieb, and Tasaki. We will mainly use mathematical methods from spectral theory, complex analysis, and matrix analysis, as well as some representation theory of SU(2).

The main goal of the class is to present students with examples of rigorous, mathematical argumentation with consequences for models of interest to modern theoretical physics. The class will focus on the mathematics and discuss the physical motivations and implications only as needed.

References.
The course will follow an unpublished manuscript written by Hal Tasaki which will be available to students registered for the course.

For students interested in some preliminary reading, a 10-page summary of the subject of quantum spin systems can be found under the following link.

`https://arxiv.org/pdf/math-ph/0409006.pdf`

This summary overlaps with the material covered in this course, but is by no means in one-to-one correspondence. For instance, the course will not develop the C^*-algebraic formulation of quantum spin systems in order to avoid technical issues.

Prerequisites.
Graduate students and advanced undergraduate students that have a firm background in real analysis, complex analysis, linear algebra, and basic operator theory should be well-prepared for Ma 274Y. In particular, a good working knowledge of the spectral theorem and the tensor product of vector spaces is required. Some knowledge of representation theory of matrix groups is helpful, but not essential. Background in the theory of unbounded operators is not required because the class will focus on large, finite systems to avoid technical issues.

Knowledge of quantum physics (specifically angular momentum and spin) and statistical mechanics will not be assumed directly. *However, having a basic understanding of both of these areas is strongly recommended because it aids in understanding the physical rationale behind various concepts and arguments that appear in the class.*

Homework.
There are no homework sets. For the benefit of students interested in practicing the material through problems, exercises that suggest themselves from the flow of the lectures will regularly be stated in class.

Final projects.
There will be final projects for undergraduate students (and graduate students who have not yet passed the qualifying exam) who wish to receive a grade for the course. A list of possible project topics will be distributed in the middle of the term, and the projects will be due at the end of term.

The final project will involve a short written summary and an oral presentation to the class and may be completed in small groups of two to three students. There is great flexibility in how the projects are treated: On the one hand, a pure literature review is completely acceptable. On the other hand, many of the projects can lead to publishable research output.

Detailed outline of the course.
The course plan is to cover the following material. The basics (1) will be covered within about two weeks and then the rest of the semester will be divided roughly evenly among the two main topics (2) and (3).

(1) *The Basics.* We set the stage and introduce the key notions and facts for describing the spin of quantum particles. In essence, we present the physicist's way of studying representations of $SU(2)$ which we will see

has the advantage of being very concrete. Afterwards, we introduce the paradigmatic quantum spin systems: the quantum Ising model and the quantum Heisenberg model.

(2) *Long-range order in the Heisenberg antiferromagnet.* We motivate the notion of long-range order by considering some classical spin models. Then we prove a strong version of the Kohn-Hohenberg theorem: long-range order in Heisenberg-type models cannot occur in 1 and 2 dimensions. The first highlight of this course is the presentation of the proof that the 3-dimensional Heisenberg antiferromagnet exhbitis long-range orde by Dyson, Lieb, and Simon (arguably one of the greatest achievements of mathematical physics).

(3) *Spectral gap and topological order in AKLT models.*
In 1983, Haldane presented a completely novel perspective on antiferromagnetic quantum spin chains and concluded using heuristic methods that models of integer spin exhibit a spectral gap. (Haldane's 2016 Nobel prize was largely awarded for this discovery.) In an effort to turn Haldane's physical arguments into rigorous mathematics, Affleck-Kennedy-Lieb-Tasaki (AKLT) introduced a novel class of antiferromagnetic models (now called AKLT models) which have the great technical advantage of being *frustation-free*. They rigorously derived a spectral gap in one dimension, thereby verifying Haldane's conjecture in this context. But the influence of the paradigmatic AKLT models does not stop there: Their ground states (so-called matrix product states) possess a special mathematical structure and constitute the first examples of symmetry-protected topological order (SPT)—a topic of great current interest in physics and mathematics.
We will explore frustration-free models (with a focus on the paradigmatic AKLT models) in some detail. This will include a thorough discussion of spectral gaps (including the seminal result of AKLT in one dimension and recent results in higher dimensions) as well as aspects of symmetry-protected topological order. Since this thematic block has many direct connections to ongoing research, multiple of the final projects will explore topics in this direction.