# Math 118r: Dynamical Systems, Spring 2024

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**Prerequisites:** Strong command of multivariate calculus and linear algebra at the level of Math 21a and 21b or higher. Some prior exposure to brief mathematical proofs and/or "epsilon-delta formalism" (the theoretical underpinnings of calculus) is desirable, but not mandatory.

**Course Description:** Many natural phenomena involve complex interactions between time-dependent dynamic variables. Mathematical models of such phenomena are often presented as differential equations[1] that govern how dependent variables evolve over time. In Math 118r, we will develop a rigorous theoretical framework for studying continuous-time and discrete-time dynamical systems, following in the footsteps of Poincaré, Lyapunov, and other (more modern) contributors. We will apply the theory to problems of genuine scientific interest, drawn from a variety of disciplines (biology, chemistry, physics, engineering, medicine, economics). Topics covered include: Linear constant-coefficient systems of ordinary differential equations (ODEs); nonlinear ODEs: existence and uniqueness, local theory; nonlinear ODEs: global existence theory, nullclines, trapping regions, dependence upon parameters and initial conditions; flows on manifolds (mainly cylinder and torus); behavior near equilibrium, local stability, Lyapunov functions, stable and unstable manifolds; existence and stability of limit cycles; bifurcation from equilibrium, Lyapunov-Schmidt reduction, steady-state and Hopf bifurcation theorems; global bifurcations; chaotic dynamics through examples (Lorenz, discrete logistic map, etc...), formal definition of chaos, topological conjugacy and equivalent dynamical systems, Sharkovsky theorems.

**Course Goals and Learning Objectives:** By the end of Math 118r, students will

- understand and apply key theorems to predict (qualitative) dynamical behavior associated with systems of differential equations and their discrete-time counterparts;
- contrast generic versus non-generic behaviors, linear versus nonlinear dynamical systems, and local versus global results;
- classify and/or analyze certain types of special solutions (equilibria, periodic solutions, heteroclinic/homoclinic solutions, chaotic solutions)
- deepen the ability to synthesize notions from linear algebra, multivariate calculus, and rigorous analysis;
- explore dynamical behavior of previously-unseen model equations from a variety of scientific disciplines, applying relevant qualitative theory as needed;
- interpret bifurcation phenomena both mathematically and practically;
- develop an ability to create and/or evaluate proofs involving applied real analysis.

**Textbooks:** Three good references for our course are

- D. G. Schaeffer and J. W. Cain, *Ordinary Differential Equations: Basics and Beyond*, Springer, New York, 2016

[1] Or their discrete-time counterparts...

- S. Sternberg, *Dynamical Systems*, Dover, New York, 2014.
- B. Hasselblatt and A. Katok, *A First Course in Dynamics*, Cambridge University Press, 2003.

Note: Harvard's library subscriptions allow you to download/view the electronic versions of these book for free! Be sure to download the Schaeffer and Cain book from Springer's website, as that's the reference we'll use the most.

**Tutorials and Office Hours:** Outside of the regular lectures, you will have wonderful opportunities to solidify the concepts discussed in class. I insist that every student attend office hours every so often, if for no other reason than to help me tailor the class meetings to your individuals needs. Because the backgrounds of the students in Math 118r vary considerably (some have deep experience with applications but not theory, and vice-versa), it is likely that I will record a series of several tutorial videos outside of our regular class meetings. One will be a review of important notions from linear algebra (eigenstuff, diagonalization, Jordan normal form). One will be an overview of differential calculus of functions from $\mathbb{R}^n \to \mathbb{R}^m$. One will be an overview of $\epsilon - \delta$ formalism and key definitions/theorems from elementary analysis.

**Problem Sets:** There is no question that the best way to learn math is by doing math, and homework exercises are an essential part of any math course. There are a few things you should do in order to get the most out of your homework. First, make sure to look over each new problem set *on the same day that it's posted*. Even if you don't plan to start the problem set until later, try to at least get a sense of what each problem is asking you to do. That way, if you need to ask clarifying questions during office hours, you'll have time to do so before the weekend. Second, try out a few of the problems on your own. If you get stuck or confused, jot down some questions to bring to office hours or to other students. Third, after you have tried problems on your own, feel free to consult with study groups of your classmates. It's perfectly fine to discuss your work with each other, as long as you write up your solutions independently and acknowledge any collaborators. Fourth, after you have written up a solution to a problem, allocate one or two minutes to reflect upon the problem, and "say back to yourself" what strategy/ideas you used. This will help solidify your understanding of important ideas, and will save you considerable time when you prepare for exams. Finally, after submitting a problem set, be sure to look over any hints/solutions that I post on Canvas (see next paragraph).

Typically, there will be one problem set due each week, submitted electronically via Canvas file uploads by 12:00pm on the due date. Time permitting, I will then post hints/solutions to the homework on the course website. Check the solutions so that you can learn from your work. In order for me to post solutions as soon as possible (and in light of the fact that falling behind in a math class is one of the most uncomfortable things you can do to yourself), **homework must be turned in on time**. Your lowest problem set grade will be dropped, so please do not request extensions on homework assignments. Claiming to have submitted the wrong version of a problem set and/or technical issues with Canvas file submission is never grounds to submit work after a deadline. If you ever have issues with Canvas file uploads, you can email me your work as a file attachment prior to the deadline.

**Grading and Exams:** There will be two midterm exams and a final exam. Your course grade will be determined as follows:

| **Component** | **Date** | **Percentage** |
|---|---|---|
| Homework | see course website | 35 % |
| Midterm I | February 28th | 15% or 20%* |
| Midterm II | April 10th | 15% or 20%* |
| Final Exam | May 2nd | 30% |

*Your best midterm score will count for 20%, and your worst midterm score will count for 15%.

When I calculate your final grade at the end of the course, I will do so using a 0-100 point scale using the scores that you have obtained during the course, and using the percentages given above. Your course grade will then be obtained using this table:

| Cutoff | Corresponding letter |
|---|---|
| 92.5 | A |
| 89.5 | A- |
| 86.5 | B+ |
| 82.5 | B |
| 79.5 | B- |
| 76.5 | C+ |
| 72.5 | C |
| 69.5 | C- |
| 59.5 | D |
| below 59.5 | E |

**Academic Integrity:** Discussion and the exchange of ideas are essential to doing academic work. For assignments in this course, you are encouraged to consult with your classmates as you work on problem sets. However, after discussions with peers (or course instructional staff such as tutors, TF/TAs, course assistants), make sure that you can work through the problem yourself and ensure that any answers you submit for evaluation are the result of your own efforts. In addition, you must cite any books, articles, websites, lectures, etc that have helped you with your work using appropriate citation practices. Similarly, you must list the names of students with whom you have collaborated on problem sets. You should never attempt to persuade a CA to accept late work, and any grade-related questions should be communicated to your course head, not directly to a CA.

During exams, all forms of communication (written, verbal, electronic, etc.) related to our course material with anyone other than your course head is forbidden.

Generative artificial intelligence (GenAI) tools such as ChatGPT can sometimes help you explore a concept, but beware—these tools can give answers that sound confident regardless of whether they are correct. You should never ask such tools to solve problems on any of our graded assessments.

Redistribution of any of our course materials (class notes, readings, homework assignments, exams, and so on) is strictly forbidden unless written permission from your course head is granted.

**Disabilities Requiring Accommodation:** If you need accommodation or assistance for a documented disability, please contact the course head as soon as you can so that we can make the necessary arrangements. Please note that for graded assessments, **accommodations for extended time do not apply** in this course. (For take-home exams, the deadlines already include extended time, and for homework, extensions are not granted for the reasons outlined in the "Problem Sets" section above.) Refer to

https://aeo.fas.harvard.edu

for more information.

# Syllabus[2] for Math 118r, Spring 2024

| Date | Due | Math Topics |
|---|---|---|
| Mon 01/22 | | Course Orientation, basic notions, first-order ODEs |
| Wed 01/24 | (PS01 due 01/26) | Constant-coefficient systems, matrix norms |
| Mon 01/29 | | Matrix exponential, asymptotic behavior, phase portraits |
| Wed 01/31 | PS02 | Nonlinear systems of ODEs, notions from basic real analysis |
| Mon 02/05 | | Cauchy sequences, completeness, Contraction Mapping Principle |
| Wed 02/07 | PS03 | Picard-Lindelöf existence and uniqueness; Gronwall's inequality |
| Mon 02/12 | | Uniqueness via Gronwall; maximal interval of existence |
| Wed 02/14 | PS04 | Sufficient conditions for global existence; trapping regions |
| Wed 02/21 | PS05 | Nullclines; dependence on parameters and initial conditions |
| Mon 02/26 | | Stability of equilibria, theorem and associated terminology |
| Wed 02/28 | **Exam 1** | Lyapunov functions and Lasalle's invariance principle |
| Mon 03/04 | | Stable and unstable manifolds |
| Wed 03/06 | PS06 | Periodic solutions, special behavior in two dimensions |
| Mon 03/18 | | Limit sets, Poincaré-Bendixson Theorem, Dulac's theorem |
| Wed 03/20 | PS07 | Stability of limit cycles, Poincaré maps |
| Mon 03/25 | | Bifurcation from equilibrium; implicit function theorem |
| Wed 03/27 | PS08 | Pitchfork, transcritical, saddle-node bifurcation; Lyapunov-Schmidt |
| Mon 04/01 | | Hopf bifurcation theorem |
| Wed 04/03 | PS09 | Global bifurcations (homoclinic, invariant torus, etc.) |
| Mon 04/08 | | Discrete dynamical systems and period-doubling bifurcation |
| Wed 04/10 | **Exam 2** | Chaos in the Lorenz system and discrete logistic map |
| Mon 04/15 | | Chaotic dynamical systems: formal definition, examples |
| Wed 04/17 | PS10 | Topological conjugacy and equivalent dynamical systems |
| Mon 04/22 | | More examples of chaotic dynamical systems |
| Wed 04/24 | PS11 | Sharkovsky Theorems |
| 04/25–05/01 | | Reading period |
| **Thu 05/02** | **Final Exam** | Final exam |

[2] The dates listed here are approximate, so don't worry if there are times when we're a bit ahead/behind what the syllabus indicates. Due dates for certain problem sets may be adjusted in order to keep your workload as uniform as possible!