SYLLABUS FOR MATH 266Y
MATRIX INEQUALITIES AND QUANTUM ENTROPY

SPRING 2019

Class: Tuesday and Thursday, 3:00 – 4:15 PM, in Science Center 310

Instructor: Marius Lemm
Email: `mlemm[at]math.harvard.edu`
Office hours: Tuesday 4:30 – 5:30 PM, in Science Center 237

Course website: https://canvas.harvard.edu/courses/50697/

Goals of the course.
Inequalities are at the core of many analytical arguments in mathematics. Matrix inequalities, specifically, are useful because they can provide analytical control on the non-commutativity inherent to matrices. Accordingly, the theory of matrix inequalities is intricate and beautiful, and when a matrix inequality is applicable, it can have profound and unexpected consequences.

The course has two main goals: (1) To give an introduction to the most basic matrix inequalities. A good working knowledge of these basic inequalities (which are mostly inequalities involving the traces of matrices) can be of benefit to research in mathematics, physics and computer science. (2) To present some modern applications of matrix inequalities, with a particular focus on quantum information science and random matrix theory, and thereby link up to current research in these fields.

Literature.
About half of the course material will come from the formidable lecture notes "Trace Inequalities and Quantum Entropy: An Introductory Course" by Eric Carlen. These notes are available online for free under the link:

http://www.mathphys.org/AZschool/material/AZ09-carlen.pdf

In addition, the course will explore applications of matrix inequalities on the basis of recent research articles from quantum information science and random matrix theory. The relevant articles are available online and references will be provided as we come across them.

Prerequisites.
Students that have a firm background in real analysis and linear algebra, and basic knowledge of complex analysis, should be well-prepared for Ma 266Y. In particular, knowledge of the spectral theorem will be assumed.

Background in the following fields is *not* required: Operator theory (the class will focus on finite-dimensional matrices to avoid technicalities) and physics (all the relevant notions related to physics will be defined in class).

Detailed outline of the course.
Time permitting, the course will cover the following material.

- *The Basics.* We set the stage and introduce basic vocabulary, including the key notions from quantum mechanics: quantum states, expectation values of observables, and quantum entropy. We also provide a toolbox of the most basic and versatile matrix inequalities: Peierls inequality, Klein's inequality, and the Gibbs variational principle.
- *Operator monotone functions and Löwner's theorem.* We give a thorough discussion of operator monotone and operator convex functions. In particular, we discuss the Löwner-Heinz theorem which answers the question which of the power functions $A \mapsto A^p$ are operator convex or monotone. A highlight of this part will be a proof of Löwner's theorem which completely characterizes operator monotone functions. (There is a book by Barry Simon coming out which contains 11 different proofs of Löwner's theorem! We will be content with one proof.) A brief discussion on joint convexity will lead us to some open problems (Audenaert-Datta conjecture)[1].
- *Trace inequalities for matrix exponentials.* We build up, starting from the famous Golden-Thompson inequality, to Lieb's triple matrix inequality, to the recent n-matrix inequalities by Sutter-Berta-Tomamichel. Time permitting, we will discuss various different proof methods (Lieb's method, the pinching method and complex interpolation). These particular trace inequalities have applications to large deviations for random matrices (as observed by Ahlswede-Winter and by Tropp), and we will point out some open problems in this vein.
- *Tensor products and partial trace.* Finally, we focus on quantum entropy, in particular on strong subadditivity (SSA) and its refinements. Such refinements have been a hot topic in quantum information science in recent years, e.g., because they provide access to topological properties of quantum states. We will present several papers establishing refinements of SSA and point out some open problems.

[1]We will skip the discussion of joint convexity because the Audenaert-Datta conjecture was recently solved by Zhang https://arxiv.org/pdf/1811.01205.pdf

The parts of the course are strongly interrelated, and there are many cases where one trace inequality implies a seemingly very different one in a few short steps. We will emphasize these relations and the universality of the developed methods along the way.

Homework.
There are no homework sets. For the benefit of students interested in practicing the material through problems, exercises that suggest themselves from the flow of the lectures will regularly be pointed out in class.

Projects.
There will be projects for students wishing to receive a grade for the course. A list of possible project topics will be available in the middle of the term, and the projects will be due at the end of term. There is great flexibility in how the projects are treated: On the one hand, a pure literature review is completely acceptable. On the other hand, many of the projects can lead to publishable research output.