MAT 289Y: Topics in Geometric PDE

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1 Course Overview

In this course, we will develop the tools and techniques for the study of nonlinear elliptic PDEs and their applications to geometric problems. We will begin with the analysis of the Laplace equation on $\mathbb{R}^n$ and discuss basic estimates via mean-value properties, maximum principles, and integral methods. Next, we will see how each of these methods can be applied to the study of linear elliptic equations and also discuss the classical estimates of Schauder. Then we will move on to the theory of DeGiorgi-Nash-Moser and its role in the study of semi-linear equations and the solution of Hilbert's 19th problem. Lastly, I would like to discuss the modern theory of fully nonlinear equations. Throughout the course, we will also discuss various examples of how the theory can be applied to the study of specific problems arising from geometry and physics, such as the regularity of minimal surfaces, the Minkowski problem in differential geometry, and the Calabi-Yau problem in Kähler geometry.

2 Syllabus

Possible topics include but are not limited to the following

1. Laplacian in $\mathbb{R}^n$, mean value property, fundamental solutions
2. Dirichlet problem, Energy method for producing weak solution, regularity of weak solutions

3. Maximum principles, Harnack inequalities, Perron's method for solving Laplace equation
4. General 2nd order elliptic equations, Schauder theory
5. Laplace operator on manifolds
6. Semi-linear equations, regularity of Minimal surfaces
7. Hilbert's 19th problem and Nash-DeGiorgi-Moser theory
8. Fully nonlinear equations, Monge-Ampere equations
9. Concave equation and the Evans-Krylov theory
10. Minkowski problem for prescribing Gauss curvature
11. Existence of optimal transport maps
12. Kähler geometry and the complex Monge-Ampere equation
13. Kähler-Einstein metrics and the Calabi-Yau Theorem

3 Prerequisite

Students should know multivariable calculus. Some knowledge of L^p and Hölder spaces will be useful, but the relevant results will be reviewed in class.

4 Homework and Final Presentation

There will be regular homework problems assigned for this class and a possibility to do a final project or presentation at the end of the semester. Students who wish to receive a grade should submit solutions to the homework problems, and do a final project.

5 Textbooks

We will use a variety of sources including a mix of books, online lecture notes, and original research papers. I will provide lecture notes for the class. Some useful references for this part of the course:

- Gilbarg-Trudinger, Elliptic Partial Differential Equation of Second Order. Springer, 2001.
- Fernandez-Real/Ros-Oton, Regularity theory of elliptic PDE: `https://arxiv.org/abs/2301.01564`
- Qing Han, Fanghua Lin, Elliptic Partial Differential Equation.
- Connor Mooney's notes: `https://www.math.uci.edu/~mooneycr/EllipticPDE_BasicTheory.pdf`
- S.Y. Cheng, S.T. Yau, *On the regularity of the solution of the n-dimensional Minkowski problem.* Comm. Pure Appl. Math., 1976.
- S.T. Yau, *On The Ricci Curvature of a Compact Kähler Manifold and the complex Monge-Ampere equation.* Comm. Pure Appl. Math., 1978.