Math 25a: Theoretical Linear Algebra and Real Analysis I
Fall 2024, MWF 9:00–10:15

Course Head: Dr. John Wesley Cain ("Wes")
Office: Science Center 515
Office Hours: Tuesdays 2:00-3:30, Fridays 2:00-3:30, or by appointment, in Science Center 515
E-mail: jcain2@math.harvard.edu
URL: http://www.math.harvard.edu/~jcain2/
Phone: 617-495-1790

Teaching Fellow (TF): TBA
Office Hours: See our course Canvas site
E-mail: TBA

Course Assistants: TBA
Office Hours: See our course Canvas site
E-mail: TBA

Course Website: https://canvas.harvard.edu/courses/137709

Prerequisites: Strong command of single-variable calculus; interest in mathematical rigor, formalism and proof; and willingness to devote many hours per week to the study of mathematics.

Course Description: Math 25a offers a rigorous introduction to linear algebra, one of the most formative courses in all of undergraduate mathematics. It also introduces some notions from real analysis (the theoretical underpinnings of calculus-based mathematics) and metric space topology; however, those subjects are developed more carefully in Math 25b in the spring semester. Most students who complete Math 25a are seriously considering a Concentration in mathematics or a closely related field. To this end, Math 25a offers a glimpse of what it's like to study higher mathematics, placing more emphasis on theory and conceptual understanding as opposed to computational work. *Students who enroll in Math 25a should expect to devote considerable time and effort to the study of mathematics.*

Course Goals and Learning Objectives: Prior to college, most students experience mathematics as a primarily computational pursuit. Your algebra class probably focused on symbolic manipulation, with less emphasis on algebraic structures such as groups, rings, fields, vector spaces, and so on. In introductory calculus, you were asked to compute linear approximations of differentiable functions $f : \mathbb{R} \to \mathbb{R}$, but how can we generalize linear approximation to functions $f : \mathbb{R}^m \to \mathbb{R}^n$ (i.e., higher-dimensional Euclidean spaces in which we might lack the ability to visualize f graphically)? Professional scientists, mathematicians, engineers, economists, and quantitative social scientists grapple with problems requiring sophisticated mathematics—mathematics for which computation by hand[1] is intractable, and for which "high-dimensionality" impedes our ability to gain geometric intuition.

You are entering a stage of your mathematical training at which the *theoretical* underpinnings of algebra and calculus are essential for your progress, and Math 25 provides an intensive introduction to such theory. Math 25a is devoted primarily to the study of linear algebra and Math 25b is devoted to the study of real analysis[2] and the topology of metric spaces. The main goals of Math 25a are

- To develop your ability to (i) use mathematical language carefully; (ii) recognize what constitutes rigorous proof; (iii) read a variety of mathematical texts with careful attention to detail; and (iv) write logically-sound mathematical proofs.

[1] Often, even computer-assisted computation is intractable.
[2] You'll learn the formalism behind what you covered in single-variable calculus, as well as its extensions to multivariable calculus.

- To provide a strong foundation in linear algebra, including an awareness of the scientific problems which drove the development of this subject.
- To build a strong sense of community among students potentially interested in deeper study of mathematics.

Upon successful completion of Math 25a and 25b you will be prepared for any 100-level mathematics course which does not list some other 100-level mathematics course as prerequisite. *Remark:* If you complete Math 25a and 25b, you cannot earn credit for Math 101, 112, or 121.

Textbooks: Besides course notes from class meetings, here are several references that you may wish to have on-hand. All of them can be downloaded for free (see below).

- S. Axler, *Linear Algebra Done Right, 3rd Ed.*, Springer, New York, 2015.
- S. Treil, *Linear Algebra Done Wrong*, https://www.math.brown.edu/streil/papers/LADW/LADW.html
- R. Hammack, *Book of Proof*, https://richardhammack.github.io/BookOfProof/
- S. Abbott, *Understanding Analysis, 2nd Ed.*, Springer, New York, 2015.

Note: Harvard's library subscriptions allow you to download the electronic versions of Axler's and Abbott's books for free! If you prefer hard copies, Harvard has an arrangement with Springer so that you can purchase the "My Copy" versions of their texts at vastly reduced prices.

Resources Outside the Classroom: Outside of our regular class meetings, we're pleased to offer several ways to get extra support. First, you're encouraged to attend Wes's office hours. These tend to be lively discussions in which students can help clear up confusions regarding key concepts, cultivate their ability to write rigorous mathematical proofs, and seek support on problem set questions. Second, Math Night is a weekly event, held on Monday evenings 8:00-10:00pm (likely in one of the house dining halls). You can find friends in your math class or other classes, work on your problem sets, or get help from your Course Assistants (CAs), many of whom will be present. Also snacks are provided! Finally, our graduate TF and some CAs may hold office hours or problem sessions; once we know more about everyone's schedule for this semester, we will post details on the course website.

Problem Sets: There is no question that the best way to learn math is by *doing* math, and homework exercises are an essential part of any math course. We encourage you to form study groups with other students in the class so that you can discuss your work with each other; however, all work submitted must be written up individually. Make sure that even if you do work in groups, that you come away with the ability to explain everything you end up writing up in your homework.

There will be one problem set due each Wednesday at 5:00pm, to be submitted electronically via Canvas. Starting with the second problem set, we strongly prefer that you typeset your solutions using LaTeX. Assignments will be graded by CAs, and we will post solutions to the homework on the course website immediately after the submission deadline. Be sure to review the solutions so that you can learn from your work. In order for us to post solutions as soon as possible, and in light of the fact that falling behind in a math class is one of the most uncomfortable things you can do to yourself, homework must be turned in on time. Please do not try to persuade a CA to accept a late homework assignment, as this goes against our academic integrity policy. Homework *must* be submitted electronically via Canvas, and cannot be transmitted to CAs by other means such as email. Moreover, claiming to have submitted the "wrong version" of an assignment (or perhaps not all of the pages that you intended to submit) is never grounds for late submission of an assignment.

Grading and Exams: Your course grade will be based upon your performance on homework problem sets, a 30-minute in-class quiz on elementary proof techniques, two midterm exams, and a final exam. Because problem sets will have different weights, rather than "dropping" a lowest problem set score, your homework grade (converted to a 100-point scale) will be computed as follows:

$$100 \times \frac{\text{total homework points earned}}{0.95 \times (\text{total homework points possible})}.$$

The maximum possible homework score is approximately 105.26. We may, on rare occasions, offer an extra credit homework problem, giving you an opportunity to add to your total homework points earned. Importantly, notice that it is always to your advantage to submit at least *something* for every problem set.

Your course grade will be determined as follows:

Component	**Date**	**Percentage**
Homework	see course website	30%
Proof Techniques Quiz	Friday, September 20, 2024 in class	5%
Midterm I	Thursday, Oct 03, 2024 from 7:00–9:00pm	15% or 20%*
Midterm II	Thursday, Nov 14, 2024 from 7:00–9:00pm	15% or 20%*
Final Exam	Date/location TBD	30%

*Your highest midterm score will count for 20%, and your lowest midterm score will count for 15%.

When we calculate your final grade at the end of the course, we will calculate a score on a 0-100 point scale using the scores that you have obtained during the course, and using the percentages given above. Your course grade will then be obtained using this table:

Numerical cutoff	**Corresponding letter**
92.5	A
89.5	A-
86.5	B+
82.5	B
79.5	B-
76.5	C+
72.5	C
69.5	C-
59.5	D
below 59.5	E

Academic Integrity: Discussion and the exchange of ideas are essential to doing academic work. For assignments in this course, you are encouraged to consult with your classmates as you work on problem sets. However, after discussions with peers (or course instructional staff such as tutors, TF/TAs, course assistants), make sure that you can work through the problem yourself and ensure that any answers you submit for evaluation are the result of your own efforts. In addition, you must cite any books, articles, websites, lectures, etc that have helped you with your work using appropriate citation practices. Similarly, you must list the names of students with whom you have collaborated on problem sets.

During exams, all forms of communication (written, verbal, electronic, etc.) with non-course staff members is forbidden. Use of electronic devices during an exam is also forbidden.

Generative artificial intelligence (GenAI) tools such as ChatGPT can sometimes help you explore a concept, but beware—these tools can give answers that sound confident regardless of whether they are correct. You should never ask such tools to solve your homework problems.

Redistribution of any of our course materials (class notes, readings, homework assignments, exams, project materials, and so on) is strictly forbidden unless written permission from your course head is granted.

Disabilities Requiring Accommodation: If you need accommodation or assistance for a documented disability, please contact the course head as soon as you can so that we can make the necessary arrangements. Please note that accommodations for extended time on homework assignments and/or use of electronic devices during exams (including phones, calculators, or computers) do **NOT** apply to this course. Refer to

https://dao.fas.harvard.edu

for more information.

Syllabus[3] for Math 25a, Fall 2024

Date	Due	Math Topics
Wed 09/04		Course orientation, sets, notational conventions
Fri 09/06		Logic, proofs and counterexamples
Mon 09/09		Linear systems, Gaussian elimination, RREF
Wed 09/11	PS01	Relations, equivalence relations, and functions
Fri 09/13		Number systems, countability, completeness
Mon 09/16		Fields and vector spaces
Wed 09/18	PS02	Subspaces, sums and direct sums of subspaces
Fri 09/20	Proof Tech. Quiz	Basis and dimension
Mon 09/23		Linear independence
Wed 09/25	PS03	Algebra and geometry of linear transformations
Fri 09/27		Linear transformations continued; kernel and image
Mon 09/30		Rank-nullity theorem; matrices revisited
Wed 10/02	PS04	Review
Thu 10/03	**Exam 1**	**7:00pm–9:00pm in Room TBD**
Fri 10/04		Matrix multiplication and inversion
Mon 10/07		Isomorphic vector spaces, $GL_n(\mathbb{F})$, change of basis
Wed 10/09	PS05	Determinants and their properties
Fri 10/11		More properties of determinants; transpose
Wed 10/16	PS06	Eigenvalues and eigenvectors
Fri 10/18		Eigenvalues and eigenvectors, diagonalizability
Mon 10/21		Diagonalizability and systems of DEs
Wed 10/23	PS07	Systems of DEs; discrete-time systems
Fri 10/25		Norms and inner products
Mon 10/28		Orthonormal bases, Gram-Schmidt, Riesz theorem
Wed 10/30	PS08	Orthogonal complements, projections, optimization
Fri 11/01		Self-adjoint and normal operators
Mon 11/04		Spectral theorem
Wed 11/06	PS09	Positive operators and isometries
Fri 11/08		Singular value decomposition
Mon 11/11		Operator norms, image of the unit ball
Wed 11/13	PS10	Review
Thu 11/14	**Exam 2**	**7:00-9:00pm in Room TBD**
Fri 11/15		Generalized eigenvectors, nilpotent operators
Mon 11/18		Decomposition of operators
Wed 11/20	PS11	Minimal polynomials; Cayley-Hamilton theorem
Fri 11/22		Jordan normal form
Mon 11/25		Complexification, real canonical form
Mon 12/02		Metric spaces, completeness revisited
Wed 12/04	PS12	Connectedness and compactness; prelude to analysis
12/05–12/10		Reading period
TBD	**Final Exam**	**Date, time and location TBD**

[3]The dates listed here are approximate, so don't worry if there are times when we're a bit ahead/behind what the syllabus indicates.