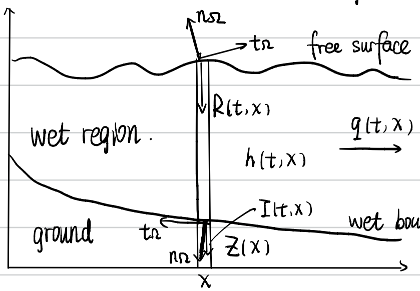
St. Venant model for overland flows with precipitation and recharge.

Navier-Stokes equations with infiltration and recharge.



local height. topography of channel bed.

$\underbrace{h(x,t)}:=\underbrace{H(x,t)}-\underbrace{Z(x)}$

absolute height of surface.

$\vec{n}_\Omega$: normal vector of the surface.

$\vec{t}_\Omega$: tangent vector ...

for entropy balance considerations. there exists $c\in\mathbb{R}$ such that $\int_{\mathbb{R}}(Z(x)-c)\,dx$ is finite.

wet region at time $t\in[0,T]$ $\Omega(t):=\{(x,z)\in\mathbb{R}^2: Z(x)<z<H(t,x)\}$

global counterpart $\Omega ::= \bigcup_{0\le t\le T}\Omega(t)$

wet boundary $\mathcal{B}=\{(x,Z):x\in\mathbb{R}\}$. free surface $\mathcal{F}=\{(t,x,H):t>0,x\in\mathbb{R}\}$.

viscous flow $\vec{u}$, on space-time domain Ω. by 2D incompressible Navier-Stokes equation

$$\nabla\cdot(\rho_0\vec{u}^T)=0.$$

$$\frac{\partial}{\partial t}[\rho_0\vec{u}]+\nabla\cdot[\rho_0\vec{u}\otimes\vec{u}]-\nabla\cdot\vec{\sigma}[\vec{u}]-\rho_0\vec{F}=0.$$

$\begin{cases}(\nabla\cdot\vec{A})_i=\sum_j\partial_j A_i^j, & i,j=x,z\\ \text{tensor product } \vec{u}\otimes\vec{u}=\vec{u}\,\vec{u}^T\end{cases}$

$\vec{u}=(u,v)$ velocity field. ρ_0 density of fluid. $\vec{F}=(0,-g)$ external force (gravity here)

stress tensor $\vec{\sigma}[\vec{u}]=\begin{bmatrix}-p+2\mu\partial_x u & \mu(\partial_z u+\partial_x v)\\ \mu(\partial_z u+\partial_x v) & -p+2\mu\partial_z v\end{bmatrix}$

p: pressure.

$\mu>0$: dynamic viscosity.

for this case $\vec{A}=A_x\hat{x}+A_z\hat{z}$.

$$\nabla\cdot\vec{A}=[\frac{\partial}{\partial x}A_x+\frac{\partial}{\partial z}(A_x\hat{x}\hat{z})]\hat{x}+[\frac{\partial}{\partial z}A_z+\frac{\partial}{\partial x}(A_z\hat{z}\hat{x})]\hat{z}.$$

if $\hat{x}\hat{z}=0$. $=(\frac{\partial}{\partial x}A_x)\hat{x}+(\frac{\partial}{\partial z}A_z)\hat{z}$.

indicator function.

$\vec{\Phi}(t,x,z):=1_{\Omega(t)}(x,z)=1_{[Z(x)\le z\le H(x,t)]}$ for $t,x,z\in\mathbb{R}$.

satisfies indicator transport equation.

$$\partial_t\vec{\Phi}+\partial_x(\vec{\Phi}u)+\partial_z(\vec{\Phi}v)=0.$$

wet boundary (effect of infiltration here)

1. On wet boundary, topography is assumed to be rough.

friction: considering Navier boundary condition: $(\vec{\sigma}[\vec{u}]\, n_\Omega)\cdot t_\Omega = -\rho_0 (k(\vec{u}) + k_-(I))\vec{u}\cdot t_\Omega$ on $\mathcal{B}$.

by general kinematic friction law on the channel bed: $k(\vec{\xi}) := (C_{lam} + C_{tur}|\vec{\xi}|)$, for all $\vec{\xi} \in R^2$.

C_{lam}: laminar friction factor. C_{tur}: turbulent friction factor.

2. the ground may absorb water (infiltration), inject water (recharge)

by permeable boundary condition: $\vec{u}(t,x,z)\cdot n_\Omega(x,z) = I(t,x)$ on $\mathcal{B}$.

infiltration function I. $I>0$ water into ground / $I<0$ water back to wet region.

3. $k_-(I)$ models friction effect when water that is recharging through the ground.

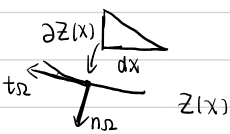
infiltration mixing friction law $k_-(I) = \alpha I_- = \alpha \max(0, -I)$

this friction only occurs when water enters the flow ($I<0$).

for simplicity, consider the function I to be a given piecewise linear function of space-time.

define Ω's normal, tangent vector on $\mathcal{B}$ by

$$t_\Omega(x, Z(x)) = \frac{(-1, -\partial_x Z(x))}{\sqrt{1+|\partial_x Z(x)|^2}}, \quad n_\Omega(x, Z(x)) = \frac{(\partial_x Z(x), -1)}{\sqrt{1+|\partial_x Z(x)|^2}}$$



condition 2. $$\vec{u}(t,x,z)\cdot n_\Omega(x,z) = (u, v)\frac{(\partial_x Z(x), -1)}{\sqrt{1+|\partial_x Z(x)|^2}} = \frac{u\partial_x Z(x) - v}{\sqrt{1+|\partial_x Z(x)|^2}} = I.$$

then $$v - u\partial_x Z(x) + I\sqrt{1+|\partial_x Z(x)|^2} = 0$$

condition 1. $$(\vec{\sigma}[\vec{u}]\, n_\Omega)\, t_\Omega = \left(\begin{bmatrix} -p+2\mu\partial_x u & \mu(\partial_z u + \partial_x v) \\ \mu(\partial_z u + \partial_x v) & -p+2\mu\partial_z v \end{bmatrix}(\partial_x Z(x), -1)\right)(-1, -\partial_x Z(x))\frac{1}{1+|\partial_x Z(x)|^2}$$

$$= \frac{\mu(\partial_z u + \partial_x v)(1-|\partial_x Z(x)|^2) - 2\mu(\partial_x u - \partial_z v)\partial_x Z(x)}{1+|\partial_x Z(x)|^2}.$$

$$\vec{u}\cdot t_\Omega = \frac{-u - v\partial_x Z(x)}{\sqrt{1+|\partial_x Z(x)|^2}}$$ then $$\frac{\mu(\partial_z u + \partial_x v)(1-|\partial_x Z(x)|^2) - 2\mu(\partial_x u - \partial_z v)\partial_x Z(x)}{\sqrt{1+|\partial_x Z(x)|^2}}$$

$$= \rho_0(k(u,v) + k_-(I))(u + v\partial_x Z(x))$$

free surface.

neglect all other meteorological phenomena (evaporation) and consider only addition of water (rainfall) runoff

Assume kinematic boundary condition: $\vec{u} n_\Omega = \frac{\partial_t H - R}{\sqrt{1+|\partial_x H|^2}}$ on $\mathcal{F}$.

$R(t,x)$ is the recharge rate due to rainfall

similarily. normal / tangent vector can be written as

$$t_\Omega = \frac{(1, \partial_x H(t,x))}{\sqrt{1+|\partial_x H(t,x)|^2}} \qquad n_\Omega = \frac{(-\partial_x H(t,x), 1)}{\sqrt{1+|\partial_x H(t,x)|^2}}.$$

then kinematic boundary condition. $\frac{-u\partial_x H(t,x) + v}{\sqrt{1+|\partial_x H|^2}} = \frac{\partial_t H - R}{\sqrt{1+|\partial_x H|^2}} \Rightarrow \partial_t H + u\partial_x H - v = R$ on $\mathcal{F}$

stress condition $(\vec{\sigma}[\vec{u}] n_\Omega)\cdot t_\Omega = -\rho_0 k_+(R)\vec{u}\cdot t_\Omega$

where $k_+(R) = \alpha R$ following surface mixing friction law.

this can be written as

$$(\vec{\sigma}[\vec{u}] n_\Omega)\cdot t_\Omega = \left(\begin{bmatrix} -p+2u\partial_x u & u(\partial_z u+\partial_x v) \\ u(\partial_z u+\partial_x v) & -p+2u\partial_z v \end{bmatrix}\begin{pmatrix} -\partial_x H \\ 1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ \partial_x H \end{pmatrix}\frac{1}{1+|\partial_x H|^2}$$

$$= \frac{u(\partial_z u+\partial_x v)(1-|\partial_x H|^2) - 2u(\partial_x u - \partial_z v)\partial_x H}{1+|\partial_x H|^2}$$

$$-\rho_0 k_+(R)\vec{u}\cdot t_\Omega = -\rho_0 k_+(R)\frac{u+v\partial_x H}{\sqrt{1+|\partial_x H|^2}}$$

then $\frac{u(\partial_z u+\partial_x v)(1-|\partial_x H|^2) - 2u(\partial_x u - \partial_z v)\partial_x H}{\sqrt{1+|\partial_x H|^2}} = -\rho_0 k_+(R)(u+v\partial_x H)$

St. Venant system.

Dimensionless Navier-Stokes equations

assume 1. water height is small with respect to the horizontal length of the domain.

2. vertical variations in velocity are small compared to the horizontal variations.

which can be written as $\varepsilon := \frac{D}{L} = \frac{V}{U} \ll 1$

D: water height. L: domain length. V: vertical velocity U: horizontal velocity

then. time scale $T = \frac{L}{U} = \frac{D}{V}$

choose pressure scale: $P := \rho_0 U^2$

Define L. U as finite constants. then $D = \varepsilon L$, $V = \varepsilon U$.

introduce dimensionless quantities of time $\tilde{t}$. space $(\tilde{x}, \tilde{z})$. pressure $\tilde{p}$. velocity field $(\tilde{u}, \tilde{v})$

by
$$\left\{\begin{array}{ll} \tilde{t} = \frac{t}{T} & \tilde{p}(\tilde{x}, \tilde{t}, \tilde{z}) = \frac{p(x,t,z)}{P} \\ \tilde{x} = \frac{x}{L} & \tilde{u}(\tilde{x}, \tilde{t}, \tilde{z}) = \frac{u(x,t,z)}{U} \\ \tilde{z} = \frac{z}{D} = \frac{z}{\varepsilon L} & \tilde{v}(\tilde{x}, \tilde{t}, \tilde{z}) = \frac{v(x,t,z)}{V} = \frac{v(x,t,z)}{\varepsilon U} \end{array}\right\}$$

rescale laminar and turbulent friction factor as $C_{lam,0} = \frac{C_{lam}}{V} = \frac{C_{lam}}{\varepsilon U}$. $C_{tur,0} = \frac{C_{tur}}{\varepsilon}$

infiltration rate and rainfall rate rescales as $\tilde{I}(\tilde{t}, \tilde{x}) = \frac{I(t,x)}{V}$ $\tilde{R}(\tilde{t}, \tilde{x}) = \frac{R(t,x)}{V}$

non-dimensional numbers Froude's number $Fro := U/\sqrt{gD}$

Reynold's number $Rey := \rho_0 UL/\mu$.

consider asymptotic setting $Rey^{-1} = \varepsilon \mu_0$. μ_0 for viscosity.

then dimensionless incompressible Navier-Stokes system.

$$\nabla \cdot \vec{\tilde{u}} = 0$$

$$\partial_{\tilde{t}} \tilde{u} + \partial_{\tilde{x}}[\tilde{u}^2] + \partial_{\tilde{z}}[\tilde{u}\tilde{v}] + \partial_{\tilde{x}} \tilde{p} = \partial_{\tilde{z}}[\frac{\mu_0}{\varepsilon} \partial_{\tilde{z}} \tilde{u}] + \rho_1$$

$$\partial_{\tilde{z}} \tilde{p} = -\frac{1}{Fro^2} + \rho_2$$

where $\rho_1 = \varepsilon \mu_0 (2 \partial_{\tilde{u}} \tilde{x}\tilde{x} + \partial_{\tilde{v}} \tilde{z}\tilde{x})$

$\rho_2 = \varepsilon \mu_0 (\partial_{\tilde{x}\tilde{z}} \tilde{u} + \varepsilon^2 \partial_{\tilde{x}\tilde{x}} \tilde{v} + 2 \partial_{\tilde{z}\tilde{z}} \tilde{v}) - \varepsilon^2 (\partial_{\tilde{t}} \tilde{v} + \partial_{\tilde{x}}[\tilde{u}\tilde{v}] + \partial_{\tilde{z}}[\tilde{v}^2])$

Assuming $\tilde{u}$ has bounded second derivatives $\rho_1 = O(\varepsilon)$. $\rho_2 = O(\varepsilon)$

On wet boundary $\mathcal{B}$. by rescaling relations and $\frac{\partial Z}{\partial x} = \frac{\varepsilon L}{L}\frac{\partial \tilde{Z}}{\partial \tilde{x}} = \varepsilon \partial_{\tilde{x}} \tilde{Z}$

Navier boundary condition

$$\left[\frac{\partial_{\tilde{z}} \tilde{u}}{\varepsilon Rey}\right]_B = \left(\frac{C_{lam}}{U}\tilde{u} + C_{tur}(|\tilde{u}| + \varepsilon|\tilde{v}|)\tilde{u} + \varepsilon k_-(\tilde{I})\tilde{u}\right)\frac{\sqrt{1+\varepsilon^2(\partial_{\tilde{x}}\tilde{Z})^2}}{1-\varepsilon^2(\partial_{\tilde{x}}\tilde{Z})^2}$$

$$\underbrace{+\varepsilon^2\partial_{\tilde{x}}\tilde{Z}\left(\frac{C_{lam}}{U}\tilde{v} + C_{tur}(|\tilde{u}| + \varepsilon|\tilde{v}|)\tilde{v} + \varepsilon k_-(\tilde{I})\tilde{v}\right)\frac{\sqrt{1+\varepsilon^2(\partial_{\tilde{x}}\tilde{Z})^2}}{1-\varepsilon^2(\partial_{\tilde{x}}\tilde{Z})^2}}_{} \quad O(\varepsilon^2)$$

$$\underbrace{-\frac{\varepsilon}{Rey}\left(\partial_{\tilde{x}}\tilde{v} + \frac{2\partial_{\tilde{x}}\tilde{Z}(\partial_{\tilde{z}}\tilde{v} - \partial_{\tilde{x}}\tilde{u})}{1-\varepsilon^2(\partial_{\tilde{x}}\tilde{Z})^2}\right)}_{} \quad O\left(\frac{\varepsilon}{Rey}\right)$$

Applying non-dimensional friction factors.

$$\left[\frac{\partial_{\tilde{z}} \tilde{u}}{\varepsilon Rey}\right]_B = \varepsilon\left(C_{lam,\varepsilon}\tilde{u} + C_{tur,\varepsilon}(|\tilde{u}| + \varepsilon|\tilde{v}|)\tilde{u} + k_-(\tilde{I})\tilde{u}\right)\frac{\sqrt{1+\varepsilon^2(\partial_{\tilde{x}}\tilde{Z})^2}}{1-\varepsilon^2(\partial_{\tilde{x}}\tilde{Z})^2} + O(\varepsilon^2)$$

$$= \varepsilon\left(C_{lam,\varepsilon}\tilde{u} + C_{tur,\varepsilon}|\tilde{u}|\tilde{u} + k_-(\tilde{I})\tilde{u}\right) + O(\varepsilon^2)$$

$$= \varepsilon\left(k_0(\tilde{u}) + k_-(\tilde{I})\right)\tilde{u} + O(\varepsilon^2)$$

with asymptotic friction law $\begin{cases} k_0(\xi) = C_{lam,0} + C_{tur,0}|\xi| & \text{for } \xi \in R. \\ k_-(\tilde{I}) = \alpha \tilde{I}_- = -\alpha \min(0, \tilde{I}) & \text{for } \alpha \in R. \end{cases}$

permeable boundary condition. $\tilde{v} = \tilde{u}\partial_x Z - I\sqrt{1+\varepsilon^2(\partial_x Z)^2} = \tilde{u}\partial_x Z - I + O(\varepsilon)$

On free surface $\mathcal{F}$.

kinematic boundary condition. $\partial_{\tilde{t}}\tilde{H} + u\partial_{\tilde{x}}\tilde{H} - \tilde{v} = \tilde{R}$

Navier boundary condition. $\left[\frac{\partial_{\tilde{z}}\tilde{u}}{\varepsilon Rey}\right]_{\mathcal{F}} = -\varepsilon k_+(\tilde{R})\tilde{u} + O(\varepsilon^2)$

where $k_+(\tilde{R}) = \alpha\tilde{R}$ for $\alpha \in R$.

First order approximation of the dimensionless Navier-Stokes equations

hydrostatic approximation of the dimensionless Navier-Stokes system

$$\partial_x u_\varepsilon + \partial_z v_\varepsilon = 0.$$

$$\partial_t u_\varepsilon + \partial_x[u_\varepsilon^2] + \partial_z[u_\varepsilon v_\varepsilon] + \partial_x p_\varepsilon = \partial_z[\frac{\mu_0}{\varepsilon}\partial_z u_\varepsilon]$$

$$\partial_z p_\varepsilon = -\frac{1}{Fr_0^2}$$

boundary conditions $[\frac{\mu_0}{\varepsilon}\partial_z u_\varepsilon] = (k_0(u_\varepsilon) + k_-(I))u_\varepsilon$, $v_\varepsilon = u_\varepsilon \partial_x Z - I$ on $\mathcal{B}$.

$[\frac{\mu_0}{\varepsilon}\partial_z u_\varepsilon] = -k_+(R)u_\varepsilon$, $\partial_t H + u_\varepsilon \partial_x H - v_\varepsilon = R$ on $\mathcal{F}$.

$(u_\varepsilon, v_\varepsilon, p_\varepsilon)$ solution to this system.

integrating on z from $[z, H(t,x)]$ $p_\varepsilon(t,x,H) - p_\varepsilon(t,x,z) = -\frac{1}{Fr_0^2}(H(t,x) - z)$

assume pressure at free surface $\mathcal{F}$ be constant $p_\varepsilon(t,x,H) = p_c \rightarrow$

$$p_\varepsilon(t,x,z) = \frac{1}{Fr_0^2}(H(t,x) - z) + p_c.$$

motion. $u_\varepsilon(t,x,z) = u_0(t,x) + O(\varepsilon)$ for some function $u_0 = u_0(t,x)$

as a consequence of $\partial_z[\mu_0 \partial_z u_\varepsilon] = O(\varepsilon)$ for $z \in (Z(x), H(t,x))$

with $\mu_0 \partial_z u_\varepsilon|_{z=Z(x)} = O(\varepsilon)$, $\mu_0 \partial_z u_\varepsilon|_{z=H(t,x)} = O(\varepsilon)$

mean speed $\langle u_\varepsilon(t,x)\rangle = \frac{1}{h(t,x)}\int_{Z(x)}^{H(t,x)} u_\varepsilon(t,x,z)\,dz$.

approximations. $u_\varepsilon(t,x,z) = \langle u_\varepsilon(t,x)\rangle + O(\varepsilon)$

$$\langle u_\varepsilon(t,x,z)^2\rangle = \langle u_\varepsilon(t,x)\rangle^2 + O(\varepsilon)$$

St. Venant system with recharge.

integrate indicator transport equation

$$0=\int_{Z(x)}^{H(t,x)} \partial_t \phi(t,x,z)+\partial_x[\phi u_\varepsilon]+\partial_z[\phi v_\varepsilon]\,dz$$

$$=\partial_t h+\partial_x q-[\partial_t H+u_\varepsilon\partial_x H-v_\varepsilon|_{z=H(t,x)}]+[u_\varepsilon\partial_x Z-v_\varepsilon]_{z=Z(x)}$$

discharge $q(t,x):=\langle u_\varepsilon(t,x)\rangle h(t,x)$

mass-balance equation. $\partial_t h+\partial_x q=S$ where $S=R-I$ R: recharge rate. I: infiltration rate.

$$\int_{Z(x)}^{H(t,x)} \partial_t u_\varepsilon+\partial_x[u_\varepsilon^2]+\partial_z[u_\varepsilon v_\varepsilon]+\partial_x p_\varepsilon\,dz=\partial_t q+\partial_x\left[\frac{q^2}{h}+\frac{h^2}{2Fr_0^2}\right]+\frac{h}{Fr_0^2}\partial_x Z.$$

$$-[(\partial_t H+u_\varepsilon\partial_x H-v_\varepsilon)u_\varepsilon]_{(t,x,H)}+[(u_\varepsilon\partial_x Z-v_\varepsilon)u_\varepsilon]_{(t,x,Z(x))}$$

$$=\partial_t q+\partial_x\left[\frac{q^2}{h}+\frac{h^2}{2Fr_0^2}\right]+\frac{h}{Fr_0^2}\partial_x Z-R[u_\varepsilon]_{(t,x,H)}+I[u_\varepsilon]_{(t,x,Z)}$$

$$=\partial_t q+\partial_x\left[\frac{q^2}{h}+\frac{h^2}{2Fr_0^2}\right]+\frac{h}{Fr_0^2}\partial_x Z-S\frac{q}{h}$$

$$\int_{Z(x)}^{H(t,x)}\partial_z\left[\frac{u_0}{\varepsilon}\partial_z u_\varepsilon\right]dz=\left[\frac{u_0}{\varepsilon}\partial_z u_\varepsilon\right]_{z=H(t,x)}-\left[\frac{u_0}{\varepsilon}\partial_z u_\varepsilon\right]_{z=Z(x)}$$

$$=-\left(k_+(R)+k_-(I)+k_0\left(\frac{q}{h}\right)\right)\frac{q}{h}.$$

then St. Venant system with recharge.

$$\partial_t h+\partial_x q=S=R-I.$$

$$\partial_t q+\partial_x\left[\frac{q^2}{h}+g\frac{h^2}{2}\right]=-gh\partial_x Z+S\frac{q}{h}-\left(k_+(R)+k_-(I)+k_0\left(\frac{q}{h}\right)\right)\frac{q}{h}.\quad \text{where } q=hu$$

$$=-gh\left(\partial_x Z+\frac{(k_+(R)+k_-(I)+k_0(u))u}{gh}\right)+Su.$$

Numerical method.

Well balanced schemes.

equilibrium state / lake-at-rest $h+Z = \text{constant}$ and $u=0$.

filling-the-lake state $\partial_t h = R$ and $u=0$

kinetic function. kinetic averaging weight function χ. kinetic density function M.

$$\chi(w) = \chi(-w) > 0. \quad \int \chi(w)\,dw = 1. \quad \int w^2 \chi(w)\,dw = \frac{g}{2}$$

$$M(t,x,\xi) = \sqrt{h(t,x)}\,\chi\left(\frac{\xi - u(t,x)}{\sqrt{h(t,x)}}\right)$$ density of particles with speed ξ at (t,x)

macroscopic-microscopic relations.

if $h(t,x) > 0$ at (t,x)
$$\int_R \begin{pmatrix} 1 \\ \xi \\ \xi^2 \end{pmatrix} M(t,x,\xi)\,d\xi = \begin{pmatrix} h(t,x) \\ h(t,x)u(t,x) \\ h(t,x)u(t,x)^2 + \frac{g}{2}h(t,x)^2 \end{pmatrix}$$

define nonlinear flux integral operator
$$\hat{W}[h(t,\cdot),u(t,\cdot)](x) = Z(x) + \int_0^x \frac{(k_+(R)+k_-(I)+k_0(u))u}{gh}(t,s)\,ds$$

then. St. Venant system can be rewritten as
$$\partial_t h + \partial_x[hu] = S$$
$$\partial_t[hu] + \partial_x\left[hu^2 + \frac{gh^2}{2}\right] = -gh\partial_x \hat{W}[h,u] + Su$$

auxilliary microscopic velocity variable ξ

density M follows $\partial_t M + \xi\partial_x M - g\partial_x \hat{W}\left[\langle M\rangle_0, \frac{\langle M\rangle_1}{\langle M\rangle_0}\right]\partial_\xi M + \frac{SM}{\langle M\rangle_0} = Q$ $\quad$ M for Gibbs equilibrium.

where for $m = 0,1,\cdots$. $\langle M\rangle_m = \int_R \xi^m M(\cdot,\cdot,\xi)\,d\xi$

$Q(t,x,\xi)$ collision term $\quad \langle Q\rangle_m = 0. \quad m = 0,1.$

relation between (h,u) and (M,Q) $\Rightarrow$ $h = \langle M\rangle_0$. $hu = \langle M\rangle_1$

Discretisation and kinetic fluxes

kinetic scheme for standard St. Venant system.

integral the kinetic equation over domain of interest, the vector of unknown. $U_i^n = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M_i^n(\xi)\, d\xi = \begin{pmatrix} h_i^n \\ h_i^n u_i^n \end{pmatrix}$

for cell $C_i \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ $\quad x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$ $\qquad$ piecewise function of $Z(x)$

$\Delta x = len(C_i)$, $\quad x_i = i\Delta x$. $\qquad \bar{Z}(x) = \sum_i Z_i 1_{C_i}(x)$, where $Z_i = \frac{1}{\Delta x}\int_{C_i} Z(x)\,dx$.

time step Δt, $\quad t_n = n\Delta t$. (i for space, n for time)

numerical scheme $\quad U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) + \Delta t \begin{pmatrix} S_i^n \\ S_i^n u_i^n \end{pmatrix}$

S_i^n: a discretisation of the combined rain and infiltration terms.

time step $\quad \Delta t = \frac{CFL \cdot \Delta x}{\max_i(|u_i^n| + \sqrt{2gh_i^n})}$ $\qquad CFL \in (0,1]$ be a constant.

numerical flux $\quad F_{i\pm\frac{1}{2}}^n = \int_{\mathbb{R}} \xi \begin{pmatrix} 1 \\ \xi \end{pmatrix} M_{i\pm\frac{1}{2}}^{\mp}(\xi)\, d\xi$

$$\begin{cases} M_{i+\frac{1}{2}}^- = M_i^n(\xi) 1_{\xi>0} + M_{i+\frac{1}{2}}^n(\xi) 1_{\xi<0} \\ M_{i-\frac{1}{2}}^+ = M_i^n(\xi) 1_{\xi<0} + M_{i-\frac{1}{2}}^n(\xi) 1_{\xi>0} \end{cases}$$

where $\quad M_{i\pm\frac{1}{2}}^n = M_i^n(-\xi) 1_{|\xi|^2 \le 2g\Delta W_{i\pm\frac{1}{2}}^n} + M_{i\pm 1}^n\left(\mp\sqrt{|\xi|^2 - 2g\Delta W_{i\pm\frac{1}{2}}^n}\right) 1_{|\xi|^2 \ge 2g\Delta W_{i\pm\frac{1}{2}}^n}$

where $\quad \Delta W_{i+\frac{1}{2}}^n = Z_{i+1}(t_n) - Z_i(t_n)$, $\quad \Delta W_{i-\frac{1}{2}}^n = Z_{i-1}(t_n) - Z_i(t_n)$

where $\quad Z_i(t) = 1_{[C_i]}(x) \frac{1}{\Delta x}\int_{C_i} Z(t,x)\,dx$

semi-discretised kinetic energy $\quad M_i^n(\xi) = \sqrt{h_i^n}\, \chi\left(\frac{\xi - u_i^n}{\sqrt{h_i^n}}\right)$

where Barrenblatt kinetic weighting function $\chi(w) = \frac{1}{\pi g}\sqrt{(2g - w^2)_+}$ for w.

entropy describe as energy $E(h,u,z) = \frac{hu^2+gh^2}{2} + ghZ$.

entropy flux $\psi(h,u,Z,S) = (E(h,u,Z) + \frac{1}{2}gh^2)u - g\Theta$

where $\Theta(t,x) = \int_0^x S(t,s)\, Z(s)\, ds$ for $t>0$, $x \in R$.

satisfy entropy production relation

$$\partial_t E + \partial_x \psi = S\left(\frac{u^2}{2} + gh\right) - \left(k_+(R) + k_-(I) + k_0(u)\right) u^2.$$

entropy inequality $\partial_t E + \nabla \cdot [u(E + \frac{1}{2}gh^2)] \leq 0$. for smooth solutions inequality becomes equality.