Lecture 3: Introduction to Asset Allocation and Portfolio Mathematics

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Lecture Plan

- The Basic Allocation Problem
- Utility Functions
- Three-Stage Analysis
- Optimal Portfolio
- Portfolio Mathematics:
  1. Expected return
  2. Variance
  3. Covariance
- Diversification and the Minimum Variance Portfolio (MVP)

Learning Objectives

Understand ...

- how investors trade off risk and returns
- what the capital allocation line is
- what the optimal portfolio is given investment opportunities and utility function
- how to calculate expected return, variance, and covariance of a portfolio
- the concept of diversification
- how to calculate the minimum variance portfolio

The Basic Allocation Problem (I)

- You want to split your money between a riskless security (e.g., T-Bills) and a risky asset (e.g., S&P 500 index)
  - Suppose your investment horizon is one year
  - We will work with nominal returns for simplicity, although we should use real returns
- Evaluate your investment possibilities (opportunities):
  - A one year T-Bill would give you a return of 5.5% for sure
  - Suppose that the S&P 500 index is currently at 1,650
  - After some thought, you decide that the S&P could go to:
    - 1,911 points with probability 0.7 (a return of 15.8%) or
    - 1,477 points with probability 0.3 (a return of -10.5%)

The Basic Allocation Problem (II)

- This means that the S&P 500 index has an expected return of

$$\mathrm{E}[\tilde{r}_{\mathrm{mkt}}] = 0.7(0.158) + 0.3(-0.105) = 7.91\%$$

and a standard deviation of

$$\begin{aligned} \sigma_{\mathrm{mkt}} &= \sqrt{0.7(0.158 - 0.0791)^2 + 0.3(-0.105 - 0.0791)^2} \\ &= \sqrt{0.0145} = 12.1\% \end{aligned}$$

We can summarize these investment opportunities as follows:

Investment Opportunities

Asset	Expected Return	Standard Deviation
T-Bill	5.5%	0%
S&P 500 Index	7.9%	12.1%

The Basic Allocation Problem (III)

Question

How should we choose between a T-Bills-only investment and an investment only in the S&P 500 index?

- The stock index has a higher expected return (good) but a higher standard deviation or "risk" as well (bad)
- We use utility functions to make investment choices
  - Utility functions tell us how we balance good and bad aspects of different investments
  - Let E be the expected return and σ the standard deviation
  - Utility function $U(E, \sigma)$ represents how good you feel about an investment $\rightarrow$ it gives you a "score of happiness"

The Basic Allocation Problem (IV)

- We will use a *quadratic utility function*, $U(E, \sigma) = E - \gamma\sigma^2$
  - This function is increasing in E: you like higher expected return...
  - ...and decreasing in σ: you don't like risk
- γ is the coefficient of risk aversion, i.e., how risk affects you
  - Higher γ means more risk averse
  - If $\gamma = 0$, you don't care about risk

See Bodie, Kane, and Marcus (10th edition, pp. 170-175) for a discussion on utility functions

The Basic Allocation Problem (V)

- Use the utility function $U(E, \sigma) = E - \gamma\sigma^2$ to make the investment decisions
- Let $\gamma = 2$ and choose between T-Bills and the index:

Utility Computations with $U(E, \sigma) = E - \gamma\sigma^2$

Investment	**Utility**		
T-Bills	$0.055 - 2 \times (0)^2$	$=$	0.055
S&P 500 Index	$0.0791 - 2 \times (0.121)^2$	$=$	0.0498

- Since T-Bills give a higher utility score than the S&P 500 Index, an investor with $\gamma = 2$ should go for the T-Bill
- In reality, we do not have to choose just one: we can invest w% in T-Bills and $(1 - w)$% in the index

The Basic Allocation Problem (VI)

Question: Invest 40% in T-Bills, 60% in S&P 500 Index

How much "utility" would an investor with $\gamma = 2$ get if he/she invested 40% in T-Bills and 60% in S&P 500 index?

- If we can compute the expected return, E_{60}, and the standard deviation, σ_{60}, of this strategy, we can plug these numbers into $U(E, \sigma)$
- We can then compare the value with what we got by putting 100% into T-Bills
- How do we compute E_{60} and σ_{60}?

Portfolio Return (I)

- Consider a strategy that invests $w\%$ in S&P 500 and the rest, $1 - w\%$ in T-Bills
  - Notation: $\tilde{r}_{\mathrm{mkt}}$ is return on the S&P 500-only strategy, r_f is the return on the riskless T-Bill, and $\tilde{r}_p$ is the portfolio return
  - The return on the portfolio is a weighted average of the two returns:

$$\tilde{r}_p = w\tilde{r}_{\mathrm{mkt}} + (1 - w)r_f$$

  - **Note:** This weighted-average idea is general. It also applies when dealing with many risky assets.
- Example. If we have three risky assets (named 1, 2, and 3), and we invest w_1, w_2, and the rest, $1 - w_1 - w_2$ in these three assets, the portfolio return $\tilde{r}_p$ is

$$\tilde{r}_p = w_1\tilde{r}_1 + w_2\tilde{r}_2 + (1 - w_1 - w_2)\tilde{r}_3$$

Portfolio Return (II)

- The expected return is always easy to compute
- If the portfolio return is

$$\tilde{r}_p = w\tilde{r}_{\text{mkt}} + (1-w)r_f$$

The expected portfolio return is

$$\mathrm{E}[\tilde{r}_p] = w\mathrm{E}[\tilde{r}_{\text{mkt}}] + (1-w)r_f$$

- The portfolio variance is more difficult to compute with many risky assets
  - We would need to worry about correlations between assets
- However, with just one risky and one riskless asset,

$$\text{var}(\tilde{r}_p) = \text{var}(w\tilde{r}_{\text{mkt}} + (1-w)r_f) = \text{var}(w\tilde{r}_{\text{mkt}}) = w^2\sigma^2_{\text{mkt}}$$

The Basic Allocation Problem (VII)

- Going back to the 60-40 strategy, the expected return and standard deviation are:

$$\begin{aligned} \mathrm{E}[\tilde{r}_p] &= w\mathrm{E}[\tilde{r}_{\mathrm{mkt}}] + (1-w)r_f = 0.6(0.0791) + 0.4(0.055) = 7.0\% \\ \mathrm{SD}(\tilde{r}_p) &= \sqrt{w^2\sigma^2_{\mathrm{mkt}}} = \sqrt{(0.6)^2(0.121)^2} = 7.26\% \end{aligned}$$

- We can now evaluate how good the 60-40 strategy is:

$$U(E,\sigma) = U(0.07, 0.0726) = 0.07 - 2 \times (0.0726)^2 = 0.0595$$

- The utility from investing in T-Bills alone was 0.055
- The new strategy yields higher utility

  $\rightarrow$ We prefer it over both all-T-Bills and all-S&P 500 strategies

The Basic Allocation Problem (VIII)

- We found a strategy that is better (gives higher utility) than a strategy that invests everything into T-Bills or everything into the S&P 500
- However, this may not be the best strategy we can get
- Why not try different values of w?
  - What if we invest 10%, 20%, ... into the S&P 500?
  - In fact, we could try all values $0 \leq w \leq 1$
  - But why stop there?
    - We could also borrow or lend money and invest more than 100% in S&P 500, if that yields the highest utility

      $\rightarrow$ w could even be negative or greater than 100%
- What is the main goal of asset allocation?

The Basic Allocation Problem (IX)

Main Goal of Asset Allocation

The main goal of asset allocation is to find the optimal portfolio:

- Pick w to maximize your utility—i.e., one that makes you as happy as possible
- We'll call this value of w the optimal portfolio weight

- We can solve the problem analytically for intuition on (optimal) portfolio choice
- Formally, we want to maximize utility with respect to w:

$$\max_w \{U(E, \sigma)\} = \max_w \left\{ \underbrace{w\mathrm{E}[\tilde{r}_{\mathrm{mkt}}] + (1 - w)r_f}_{\text{Expected Return}} - \underbrace{\gamma \left(w^2 \sigma^2_{\mathrm{mkt}} \right)}_{\text{Penalty for Risk}} \right\}$$

The Basic Allocation Problem (X)

- We solve a problem like this by taking the derivative with respect to w, and setting it equal to zero:

$$\mathrm{E}[\tilde{r}_{\mathrm{mkt}}] - r_f - 2\gamma w \sigma^2_{\mathrm{mkt}} = 0$$

  This is "the first-order condition for optimality"
- We can solve for optimal w from this condition:

$$w^* = \frac{\mathrm{E}(\tilde{r}_{\mathrm{mkt}}) - r_f}{2\gamma\sigma^2_{\mathrm{mkt}}}$$

- In our example with $\gamma = 2$:

$$w^* = \frac{\mathrm{E}[\tilde{r}_{\mathrm{mkt}}] - r_f}{2\gamma\sigma^2_{\mathrm{mkt}}} = \frac{0.0791 - 0.055}{2 \times 2 \times (0.121)^2} = 0.41$$

- A strategy that invests 41% in the S&P 500 index and the rest (59%) in T-Bills is the best strategy for us

The Basic Allocation Problem (XI)

- Consider the optimal-weight formula:

$$w^* = \frac{\mathrm{E}[\tilde{r}_{\mathrm{mkt}}] - r_f}{2\gamma\sigma^2_{\mathrm{mkt}}}$$

- Keeping everything else fixed,
  1. You invest more in stocks if the risk premium, $\mathrm{E}[\tilde{r}_{\mathrm{mkt}}] - r_f$, increases
  2. You invest less in stocks if they become riskier, $\sigma_{\mathrm{mkt}} \uparrow$
  3. You invest less in stocks if you become more risk averse, $\gamma \uparrow$
- If we have a more risk averse investor, for example, an investor with $\gamma = 4$, the investment in S&P 500 decreases
  - A more risk averse investor wants to take less risk
  - If we repeat the computation for this investor, we find $w^* = 21\%$, down from 41% for $\gamma = 2$ investor

Three-Stage Analysis (I)

What are the steps we take when constructing the optimal portfolio?

Step One: Inputs

- Make forecasts for the basic securities you have to choose from
  - In our case, we only had to forecast S&P 500's expected return and the standard deviation of returns
- This forecast step is the most critical—and the most difficult—step

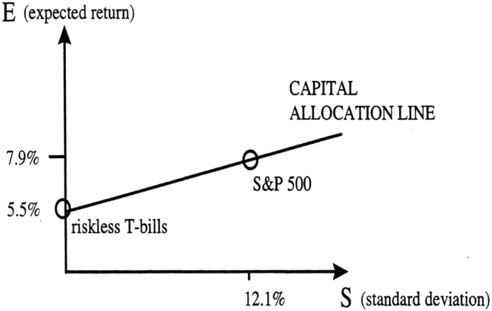
Three-Stage Analysis (II)

Step Two: Investment Opportunity Set

- Figure out the complete set of "feasible" risk-return combination
  - What (E, σ) pairs are available to you?
- If one asset is risky (S&P 500) and the other is riskless, the feasible investments lie on a straight line
- This line is called the Capital Allocation Line (CAL)
- These are all the risk-return combinations available to us
- This line is easy to draw: create a coordinate system with standard deviations on the x-axis and expected returns on the y-axis
  - Plot two points: the risky asset and the riskless asset
  - The CAL is a straight line that connects these two points

Three-Stage Analysis (III): Step Two continued

- In our example, we get the following graph:



Note: When we have multiple risky assets, the investment opportunity set won't be a straight line.

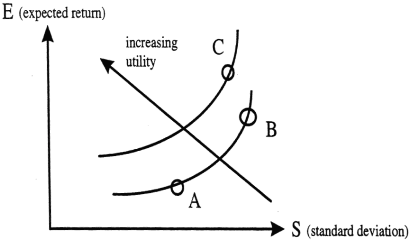
Three-Stage Analysis (IV)

Step Three: Find the Optimal Portfolio

- Find the one feasible portfolio that gives you the highest utility
- You find it in one of three ways:
  1. Try each possible strategy in turn, compute utility for each, and choose the one with the highest score
  2. Use a computer to search for the best strategy (e.g., Excel's Solver)
  3. Write down the maximization problem and solve it analytically
- We used the third approach to solve the one risky-one riskless asset problem

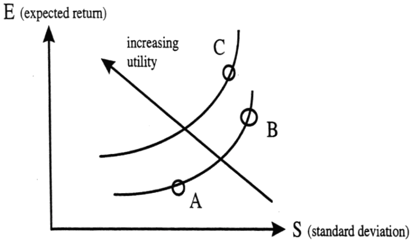
Three-Stage Analysis (V): Step Three continued

- Drawing a diagram about "how much utility" investor gets from each feasible investment can be useful



- The lines in this graph are indifference curves
- All portfolios that lie on the same curve give the same utility
  - That is, the investor is *indifferent* between these portfolios

Three-Stage Analysis (VI): Step Three continued



- Portfolios A and B have different expected returns and standard deviations
  - However, both give the investor precisely the same utility
  - Portfolio C gives higher utility than either A or B
- Utility is increasing towards the top-left corner
  - Investors prefer higher returns and lower risk

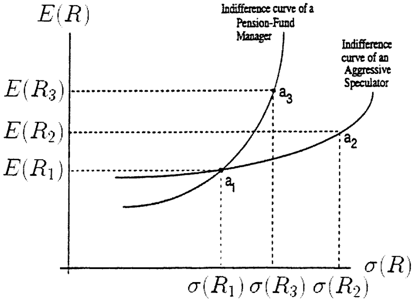
Indifference Curves (I)

- The level of risk aversion (the parameter γ) determines how steep the investor's indifference curves are:

  1. A pension fund manager, with high risk aversion, requires a large increase in the expected return to compensate for an increase in risk
  2. An aggressive speculator, with low risk aversion, requires a lower increase in the expected return to compensate for an increase in risk

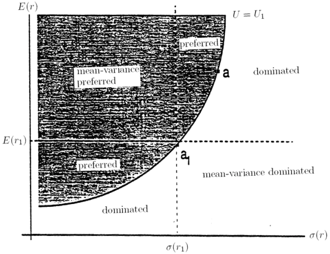
Indifference Curves (II)

- Let's suppose we have these two investors, both starting from portfolio a_1
  - This portfolio has an expected return of $\mathrm{E}(\tilde{r}_1)$ and standard deviation of returns of $\sigma(\tilde{r}_1)$
  - Let's find portfolios a_2 and a_3 that are equally desirable to these two investors
  - We get a diagram like this:



Comparing Investments with Indifference Curves

- What type of investments would both investors prefer to portfolio a_1?
  - They prefer investments that lie on higher indifference curves
  - This leads us to some terminology: An investment is
    1. preferred if it lies on a higher indifference curve
    2. mean-variance preferred if it has both higher expected return and lower standard deviation
    3. dominated if it lies on a lower indifference curve
    4. mean-variance dominated if it has both lower expected return and higher standard deviation

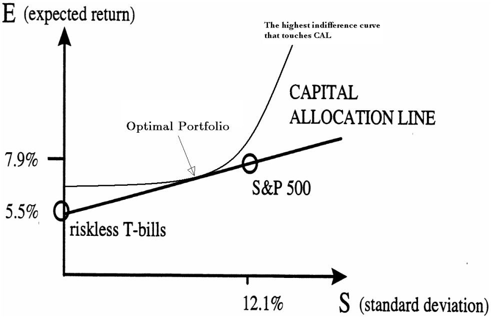


Capital Allocation Line and the Optimal Portfolio

- All risk averse investors prefer investments in the mean variance preferred-region compared to their initial portfolios
  - But one investor's "preferred" investment may be "dominated" for another investor
- We can use indifference curves together with the Capital Allocation Line to find the optimal portfolio
- To find the optimal portfolio, drag the indifference curve down until it just touches the Capital Allocation Line, CAL
  - Why? The CAL is a graphical presentation of all *feasible* investments
  - We want to pick the investment that is both (1) feasible and (2) gives the highest utility
  - When we find an indifference curve that just touches the CAL, we have accomplished our task

The Optimal Portfolio

- Find the highest indifference curve that just touches the CAL:



- The optimal portfolio is at the point of contact

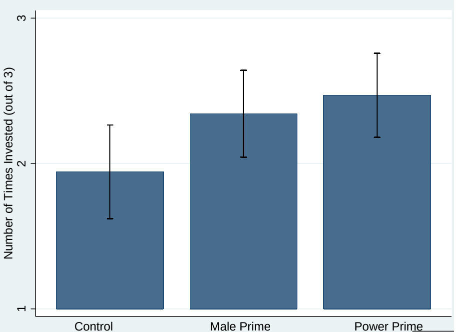
How do we measure Risk Tolerance/ Risk Aversion?

- Financial advisers, mutual fund managers: risk quizzes
  - A good investment opportunity just came along. But you have to borrow money to get it. Would you take out a loan?
- Academic research: risk elicitation task

Risk Preferences are manipulable I

- Threaten male identity in lab: male will invest more...

A. Times Invested out of 3 opportunities (Experiment 3)

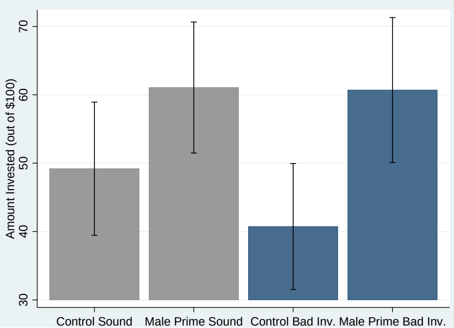


Source: D'Acunto (2014) Identity, Overconfidence, and Investment Decisions

Risk Preferences are manipulable III

- ... especially in negative NPV projects!

B. Dollars Invested in Delegated opportunities (Experiment 4)

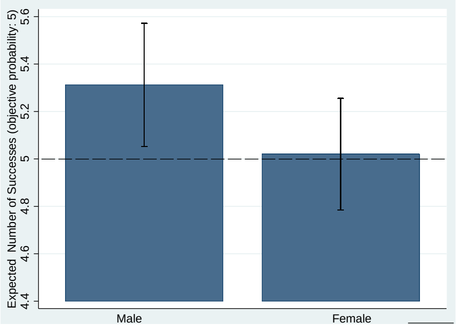


Source: D'Acunto (2014) Identity, Overconfidence, and Investment Decisions

Risk Preferences are manipulable III

- Channel: Better than average beliefs!

A. Subjective beliefs of succeeding in an investment which succeeds on average 5 out of 10 times



Source: D'Acunto (2014) Identity, Overconfidence, and Investment Decisions

Moving from Two Assets to N Assets

- We studied a very simple case:
  - The investor only had access to one risky asset (S&P 500 index) and the riskless asset (T-Bills)
- We want to study a more general problem:

How do you find the optimal portfolio when you have access to N different assets, such as the riskless asset and $N - 1$ stocks?

- We need some new tools to compute this portfolio
- Surprisingly, we'll find that
  1. The Capital Allocation Line still exists and that
  2. All investors pick a portfolio that lies on this line

Portfolio Mathematics

Portfolio Mathematics

- We want to understand asset allocation:
  - How do we compare investments and how do we choose the portfolio that is the best for us
- To study the more general problem, we need to understand the statistics used to describe portfolios
- We study the following before moving forward with the portfolio choice problem:
  1. Expected returns
  2. Covariances and correlations
  3. Interpreting covariance as marginal variance
- We'll also find something called the minimum variance portfolio (MVP)

Expected Returns

- Suppose we can invest in N assets
  - The realized returns on these assets are $\tilde{r}_1$, $\tilde{r}_2$, ..., $\tilde{r}_N$
  - We invest w_1 in the first asset, w_2 in the second asset, and so forth
  - These are portfolio weights, so they sum up to one: $w_1 + \cdots + w_N = 1$
- The realized portfolio return is

$$\tilde{r}_p = w_1 \tilde{r}_1 + w_2 \tilde{r}_2 + \cdots + w_N \tilde{r}_N$$

and the expected return is thus

$$\mathrm{E}(\tilde{r}_p) = w_1 \mathrm{E}(\tilde{r}_1) + w_2 \mathrm{E}(\tilde{r}_2) + \cdots + w_N \mathrm{E}(\tilde{r}_N)$$

$\rightarrow$ *the expected return of a portfolio is the weighted average of individual assets' expected returns*

Covariances and Correlations (I)

- Covariances and correlations both measure the degree to which two variables, such as stocks, move together
- If the covariance (or the correlation) between two stocks is positive, $\mathrm{cov}(\tilde{r}_1, \tilde{r}_2) > 0$, they tend to move together
  - If Stock A goes up, Stock B usually also goes up
- If the covariance (or the correlation) is negative, $\mathrm{cov}(\tilde{r}_1, \tilde{r}_2) < 0$, the stocks tend to move in opposite directions
- Covariance is very similar to variance
  - Variance = the expected squared deviation around the mean
  - Covariance = the expected product of deviations around the means:

$$\mathrm{cov}(\tilde{r}_1, \tilde{r}_2) = \sigma_{1,2} = \mathrm{E}[(\tilde{r}_1 - \mathrm{E}(\tilde{r}_1))(\tilde{r}_2 - \mathrm{E}(\tilde{r}_2))]$$

Covariances and Correlations (II)

Example: Covariance

There are three possible states of the world that may be realized in 2011. You have views about

1. How probable these states are and
2. How IBM and Google will perform in these states

Here is a summary of your beliefs:

	State of the Nature (probability)		
Stock	State 1 ($p = 0.8$)	State 2 ($p = 0.1$)	State 3 ($p = 0.1$)
IBM	20%	−10%	−40%
Google	25%	−60%	−10%

What is the covariance between IBM and Google?

Covariances and Correlations (III)

- Answer. The expected returns, based on these data, are:

$$\begin{aligned} \mathrm{E}(\tilde{r}_{\mathrm{IBM}}) &= 11\% \\ \mathrm{E}(\tilde{r}_{\mathrm{Google}}) &= 13\% \end{aligned}$$

- We can then apply the covariance formula:

$$\begin{aligned} \mathrm{cov}(\tilde{r}_{\mathrm{IBM}}, \tilde{r}_{\mathrm{Google}}) &= \mathrm{E}[(\tilde{r}_{\mathrm{IBM}} - \mathrm{E}(\tilde{r}_{\mathrm{IBM}}))(\tilde{r}_{\mathrm{Google}} - \mathrm{E}(\tilde{r}_{\mathrm{Google}}))] \\ &= 0.8(20\% - 11\%)(25\% - 13\%) \\ &\quad + 0.1(-10\% - 11\%)(-60\% - 13\%) \\ &\quad + 0.1(-40\% - 11\%)(-10\% - 13\%) \\ &= 0.0357 \end{aligned}$$

Covariances and Correlations (IV)

- Correlation between two returns is the covariance divided by the product of their standard deviations:

$$\mathrm{corr}(\tilde{r}_1, \tilde{r}_2) = \rho_{1,2} = \frac{\mathrm{cov}(\tilde{r}_1, \tilde{r}_2)}{\mathrm{SD}(\tilde{r}_1)\mathrm{SD}(\tilde{r}_2)} = \frac{\sigma_{1,2}}{\sigma_1 \sigma_2}$$

- Correlations are *always* between -1 and 1
  - If $\mathrm{corr}(\tilde{r}_1, \tilde{r}_2) = 1$, returns are perfectly positively correlated
  - If $\mathrm{corr}(\tilde{r}_1, \tilde{r}_2) = -1$, returns are perfectly negatively correlated
- The covariance between IBM and Google was 0.0357
  - In the same data, the standard deviations are 19.2% and 26.5%, respectively
  - The correlation between IBM and Google is thus

$$\mathrm{corr}(\tilde{r}_{\mathrm{IBM}}, \tilde{r}_{\mathrm{Google}}) = \frac{\sigma_{\mathrm{IBM,Google}}}{\sigma_{\mathrm{IBM}} \sigma_{\mathrm{Google}}} = \frac{0.0357}{0.192 \times 0.265} = 0.7$$

Portfolio Variance (I)

- We now have the tools for analyzing portfolios
- Consider a portfolio of two stocks
  - The portfolio return is a weighted average of the individual stock returns
  - From statistics, the variance of the portfolio return is

$$\mathrm{var}(\tilde{r}_p) = \mathrm{var}(w_1\tilde{r}_1 + w_2\tilde{r}_2) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{1,2}$$

  - We would only have the first two terms if the stocks moved independently of each other
  - The covariance term accounts for the comovement in returns

Portfolio Variance (II)

Example: Portfolio Variance

Stock 1 has a standard deviation of returns of 30% and stock 2 has a standard deviation of returns of 20%. The covariance between the stocks is 0.0002.

- What is the variance of an equal-weighted portfolio?

- **Answer.** The portfolio weights are $w_1 = w_2 = 0.5$
- The variance is then

$$\begin{aligned}\mathrm{var}(\tilde{r}_p) &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{1,2} \\ &= (0.5)^2(0.3)^2 + (0.5)^2(0.2)^2 + 2(0.5)(0.5)(0.0002) \\ &= 0.0326\end{aligned}$$

- The standard deviation is $\sigma_p = \sqrt{0.0326} = 18.1\%$

Portfolio Variance (III)

- We can rewrite the covariance term using the definition of correlation

$$\rho_{1,2} = \frac{\sigma_{1,2}}{\sigma_1 \sigma_2} \quad \Rightarrow \quad \sigma_{1,2} = \rho_{1,2}\sigma_1\sigma_2$$

- We can thus rewrite the two-stock portfolio variance as

$$\begin{aligned} \mathrm{var}(\tilde{r}_p) &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{1,2} \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2 \end{aligned}$$

Diversification (I)

- We can use

$$\mathrm{var}(\tilde{r}_p) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2$$

to understand the principle of diversification.

Example: Diversification

Suppose both stock variances are 0.04 and we construct an equal-weighted portfolio. What is the variance of this portfolio, as a function of the correlation between the stocks, $\rho_{1,2}$?

Diversification (II)

- Answer. We can use the two-stock variance formula and clean it up:

$$
\begin{aligned}
\mathrm{var}(\tilde{r}_p) &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2 \\
&= (0.5)^2(0.04) + (0.5)^2(0.04) + 2(0.5)(0.5)\rho_{1,2}\sqrt{0.04}\sqrt{0.04} \\
&= 0.02 + 0.02\rho_{1,2}
\end{aligned}
$$

- This formula has an important lesson:
  - Unless the two stocks are perfectly positively correlated, the portfolio's variance is always strictly lower than 0.04
  - That is, as long as $\rho_{1,2} < 1$, we get diversification benefits.

Diversification (III)

- The portfolio variance formula shows that the lower the correlation, the lower the variance of the portfolio

Diversification

- Unless the stocks in the portfolio track each other perfectly, some of the individual variations cancel out
- The portfolio variance is thus lower than the (weighted) sum of individual variances

- We have the following formula for an N-stock portfolio:

$$\begin{aligned}\mathrm{var}(\tilde{r}_p) &= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{i,j} \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + \cdots + w_N^2\sigma_N^2 \\ &\quad + 2w_1 w_2 \sigma_{1,2} + 2w_1 w_3 \sigma_{1,3} + \cdots + 2w_{N-1} w_N \sigma_{N-1,N}\end{aligned}$$

Diversification (IV)

- It may be useful to think about this formula by looking at a covariance matrix
- A covariance matrix is a symmetric table that lists all covariances:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \sigma_{1,3} \\ \sigma_{1,2} & \sigma_2^2 & \sigma_{2,3} \\ \sigma_{1,3} & \sigma_{2,3} & \sigma_3^2 \end{bmatrix}$$

- Now do two multiplication operations:
  1. Multiply every number on the first row by w_1, every number on the second row by w_2, and so forth
  2. Multiply every number in the first column by w_1, every number in the second column by w_2, and so forth

Diversification (V)

- You'll get the following table:

$$\begin{bmatrix} w_1^2\sigma_1^2 & w_1 w_2\sigma_{1,2} & w_1 w_3\sigma_{1,3} \\ w_1 w_2\sigma_{1,2} & w_2^2\sigma_2^2 & w_2 w_3\sigma_{2,3} \\ w_1 w_3\sigma_{2,3} & w_2 w_3\sigma_{2,3} & w_3^2\sigma_3^2 \end{bmatrix}$$

- The sum of these numbers is the portfolio variance; here, it is

$$\mathrm{var}(\tilde{r}_p) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1 w_2\sigma_{1,2} + 2w_1 w_3\sigma_{1,3} + 2w_2 w_3\sigma_{2,3}$$

A Technical Note: Matrix Algebra

The reason this works is that, if $\mathbf{w}$ is a $N \times 1$ vector of portfolio weights, $\tilde{\mathbf{r}}$ is a $N \times 1$ vector of realized returns, and Σ is the $N \times N$ covariance matrix, we get from statistics that

$$\mathrm{var}(\tilde{r}_p) = \mathrm{var}(\mathbf{w}'\tilde{\mathbf{r}}) = \mathbf{w}'\Sigma\mathbf{w}$$

Portfolio Variance: Example (I)

Example: Computing Portfolio Variance

Suppose we invest in three stocks: stocks 1, 2, and 3.

- These stocks' standard deviations are 20%, 30%, and 50%, respectively
- The correlations are as follows: $\rho_{1,2} = 0.3$, $\rho_{1,3} = 0.5$, and $\rho_{2,3} = -0.1$

What is the standard deviation of a portfolio that invests 40% in stock 1, 30% in stock 2, and 30% in stock 3?

Portfolio Variance: Example (II)

- The covariance matrix is:

$$\Sigma = \begin{bmatrix} 0.04 & 0.018 & 0.05 \\ 0.018 & 0.09 & -0.015 \\ 0.05 & -0.015 & 0.25 \end{bmatrix}$$

- Using either the variance formula directly, or the multiplication approach, we find that

$$\begin{aligned} \mathrm{var}(\tilde{r}_p) &= (0.4)^2(0.04) + (0.3)^2(0.09) + (0.3)^2(0.25) \\ &\quad + 2(0.4)(0.3)(0.018) + 2(0.4)(0.3)(0.05) \\ &\quad + 2(0.3)(0.3)(-0.015) = \underline{0.05062} \end{aligned}$$

- The portfolio's standard deviation is thus

$$\sigma_p = \sqrt{0.05062} = 22.5\%$$

Covariance between a Portfolio and an Individual Asset

- Consider a portfolio that invests in assets 1, 2, ..., N
  - The portfolio weights are w_1, w_2, ..., w_N
- **Question:** What is the covariance between this portfolio and asset j?
- The formal computation is

$$
\begin{aligned}
\mathrm{cov}(\tilde{r}_p, \tilde{r}_j) &= \mathrm{cov}(w_1\tilde{r}_1 + w_2\tilde{r}_2 + \cdots + w_N\tilde{r}_N, \tilde{r}_j) \\
&= w_1\mathrm{cov}(\tilde{r}_1, \tilde{r}_j) + w_2\mathrm{cov}(\tilde{r}_2, \tilde{r}_j) + \cdots + w_N\mathrm{cov}(\tilde{r}_N, \tilde{r}_j) \\
&= w_1\sigma_{1,j} + w_2\sigma_{2,j} + \cdots + w_j\sigma_j^2 + \cdots + w_N\sigma_{N,j}
\end{aligned}
$$

- **Note:** This formula says that we compute each asset's covariance against asset j and take the weighted average of these covariances

Covariances as Marginal Variances

Question: What happens to the variance of a portfolio if we tilt it a little bit toward asset j?

Result: Covariances as Marginal Variances

For small changes in portfolio weights, a stock's covariance with the portfolio is the only relevant influence on the portfolio's variance

1. If a stock is positively correlated with a portfolio, a tilt toward this stock increases the portfolio's variance
2. If a stock is negatively correlated with a portfolio, a tilt toward this stock decreases the portfolio's variance

Minimum Variance Portfolio (I)

- Based on the previous analysis and viewing covariances as marginal variances, how can we lower a portfolio's variance?
- First, we find two stocks with different covariances with the current portfolio
- We then:
    1. Give a bit more weight to the stock with the lower covariance and
    2. Give a bit less weight to the stock with the higher covariance
- Because only covariances matter, the portfolio's variance decreases
- We repeat these steps until we achieve the smallest possible variance

Minimum Variance Portfolio (II)

- When do we stop?
  - We stop when every stock's covariance with the portfolio is the same
  - At this point no more variance-lowering adjustments exist
  - We have arrived at the minimum variance portfolio

Definition: Minimum Variance Portfolio

The Minimum Variance Portfolio (MVP) is the portfolio of N risky assets that has the lowest possible variance

- **Note:** We do not include the risk-free asset in this analysis; we search for a portfolio of *risky* assets with the lowest possible variance

  $\rightarrow$ We sometimes emphasize this point by calling the MVP the "minimum variance portfolio *of risky assets*"

Minimum Variance Portfolio (III)

- We search for the MVP for two reasons:
  1. When confronted with multiple risky assets, we need the MVP to fully characterize the *efficient* set of risky portfolios
  2. The MVP illustrates the power of diversification; it summarizes information about the covariance structure
- Many ways for finding the minimum variance portfolio:
  1. Set up a spreadsheet to compute a portfolio's variance for a set of portfolio weights. Use a solver to minimize portfolio variance by changing portfolio weights
  2. Set up a spreadsheet that computes a portfolio's covariance against each stock. Adjust the portfolio weights to equate all covariances.
  3. "All covariances are the same" is the first-order condition of optimality for the problem of minimizing portfolio variance. This condition can be solved for the portfolio weights.

Minimum Variance Portfolio: Example (I)

Example: Finding the Minimum Variance Portfolio

- Stock 1 has a standard deviation of returns of 20%; Stock 2's standard deviation is 40%
- The correlation between the stocks is 0.3

How much does the minimum variance portfolio invest in stock 1?

- Let's solve this problem by hand
- We start from the "all covariances are the same" condition
- The covariances of stocks 1 and 2 against a portfolio are

$$\begin{aligned}\sigma_{1,p} &= w_1\sigma_1^2 + w_2\sigma_{1,2} = C,\\ \sigma_{2,p} &= w_1\sigma_{1,2} + w_2\sigma_2^2 = C,\end{aligned}$$

where C is some constant.

Minimum Variance Portfolio: Example (II)

- We have three unknowns (w_1, w_2, and C) but just two equations
- The missing equation: $w_1 + w_2 = 1$
- We solve for w_1 and w_2 in two ways:
  1. Substitute in $w_2 = 1 - w_1$ and solve for w_1 and C
  2. *Assume* that $C = 1$ and solve for w_1 and w_2; after finding w_1 and w_2, scale these numbers so that they sum up to 1
- If we take the second approach—which is the easier way when $N > 2$—the equations become

$$
\begin{aligned}
w_1\sigma_1^2 + w_2\sigma_{1,2} &= 1 \\
w_1\sigma_{1,2} + w_2\sigma_2^2 &= 1
\end{aligned}
$$

Minimum Variance Portfolio: Example (III)

- Substituting in the numbers for our problem,

$$0.04w_1 + 0.024w_2 = 1$$
$$0.024w_1 + 0.16w_2 = 1$$

- If we multiply the first equation by (-0.6) (why?) and sum the equations together, we get one equation with one unknown
- Solving for w_2 and then for w_1, we find $w_1 = 23.352$ and $w_2 = 2.747$
- The actual MVP weights are proportional to these numbers:

$$\begin{aligned} w_1(\mathrm{MVP}) &= \frac{23.352}{23.352 + 2.747} = 89.47\% \\ w_2(\mathrm{MVP}) &= \frac{2.747}{23.352 + 2.747} = 10.53\% \end{aligned}$$

Minimum Variance Portfolio: Example (IV)

- If we solve the original equations,

$$\begin{aligned} w_1\sigma_1^2 + w_2\sigma_{1,2} &= C, \\ w_1\sigma_{1,2} + w_2\sigma_2^2 &= C, \end{aligned}$$

analytically, using $w_1 + w_2 = 1$, we find that

$$\begin{aligned} w_1(\mathrm{MVP}) &= \frac{\sigma_2^2 - \sigma_{1,2}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}} \\ w_2(\mathrm{MVP}) &= 1 - w_1(\mathrm{MVP}) \end{aligned}$$

A Technical Note: MVP and Matrix Algebra

A portfolio's variance is $\mathbf{w}'\Sigma\mathbf{w}$ with $\mathbf{w}'\mathbf{1} = 1$, where $\mathbf{1}$ is a $N \times 1$ vector of ones. Taking the first-order condition and solving for $\mathbf{w}$:

$$\mathbf{w}(\mathrm{MVP}) = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}.$$

What We Learned Today

- We learned that investors use utility functions to trade of risk and return
- We can represent utility function by indifference curves to compare different investments
- We saw that the tangency portfolio gives the highest utility among the set of attainable investments
- We learned that the portfolio return is the weighted sum of the returns of the portfolio constituents
- We derived that the portfolio variance is lower than the weighted sum of the variances unless returns are perfectly correlated $-->$ the benefits of **diversification**!!