Financial Statistics: Time Series, Forecasting, Mean Reversion, and High Frequency Data
FINM 33170 and STAT 33910
Class # 1: Introduction

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Outline

Welcome to Fin Math 33170/Stat 33910

Financial Statistics: Time Series, Forecasting, Mean Reversion, and High Frequency Data
FINM 33170 and STAT 33910

- Instructor: Per Mykland
- Teaching assistants:
  - Ahmed Bou-Rabee
  - Yi Wang
- Further and updated information:
- http://www.stat.uchicago.edu/∼mykland/33170S21/index.html

Motivation: The short run and the long run

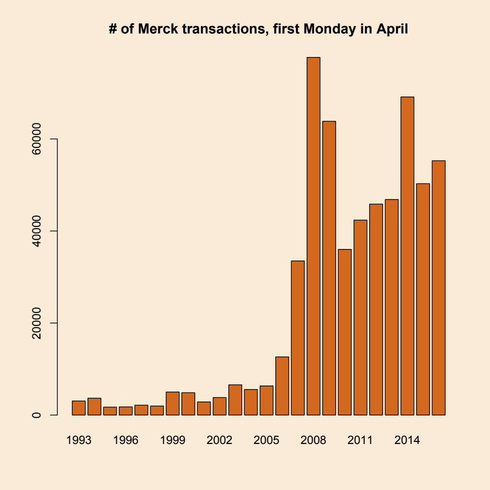
The Short Run and the Long Run

- Short run: high frequency data (time measured in seconds or less)
  - Volatilities (short run variances), covariances, regresion, ANOVA, factor analysis, leverage effect
  - market microstructure
  - “Low latency" trading
  - The shortest of frequencies: the price jump
- Long run: days, months, year, centuries
  - Time series, forecasting
  - Mean reversion and momentum, cointegration
- No grand unified theory (yet), but...
  - ... maybe the day will come
  - Meanwhile: Useful both separately and in combination
- For whom is this course?
  - Private sector: The Trader, the Risk manager, the Investor
  - Public sector: The Regulator, the Central Bank
  - The Observers: The Academic, the Journalist

High Frequency Data

- financial prices, volumes, number of trades, order time
- Intra-day:
  - transactions tick-by-tick, from TAQ, Reuters, CME
  - quotes - bid, ask - same sources
  - limit order books, harder to get but more information
  - stocks, bonds, futures, currencies, ...
  - low latency data

Evolution of Data Size per Day



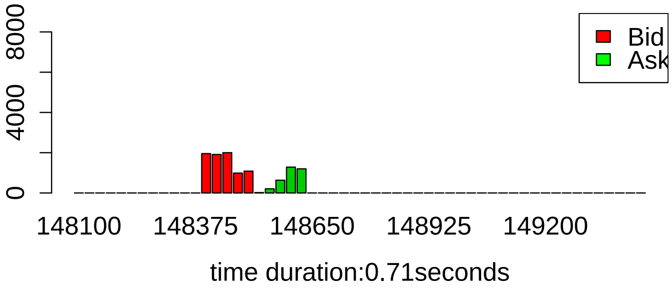
Note: Merck represents a medium-density data. A liquid stock has more than 200,000 trades per day.

Intraday Trading: almost time continuous

	Time	Size	Price
Apple April 2, 2012 Number of Trades 102,986	9:00:05.897	100	601.740
	9:00:11.257	100	601.700
	9:00:11.340	100	601.730
	9:00:12.190	100	601.700
	9:00:12.393	500	601.700
	9:00:12.807	200	601.700
	9:00:13.060	100	601.700
	9:00:13.460	100	601.650
	9:00:14.240	100	601.700
	9:00:14.913	100	601.700
	9:00:14.913	200	601.700
	9:00:15.310	100	601.700
	9:00:18.380	100	601.530
	⋮	⋮	⋮

Observation times are : (1) up to milli-seconds per trade, (2) non-equidistant, (3) could be endogenous.

Snapshot of Limit Order Book for E-mini S&P 500



Snapshot from May 1, 2007: horizontal line shows the five best bid prices (red) and five best ask prices (green), while vertical line shows the volume of each quote.

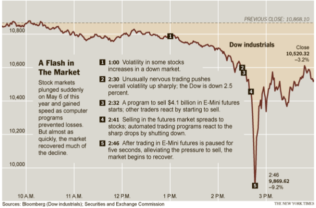
High Dimension

- Equity Cross Section: over 4000 stocks are traded at NYSE. Each day, NYSE has about 1 billion shares being traded.
- Options: contracts with varying excise prices, contracts with varying maturity times
- Order Book: varying depth

Price movement almost path-continuous, but . . .

- Flash Crash on May 6 2010: All major US stock indices plunged and rebounded within about 30 minutes. Dow Jones Industrial Average plunged 998.5 points (about 9%), most within minutes.
- Twitter Flash Crash on Tuesday April 23, 2013: Dow quickly plunged 140 points (about 1%) after a false tweet. The S&P 500 lost $121 billion of its value within minutes.
- 2017 Gold Flash Crash: On Monday June 26, 2017, around 2 billion dollars worth (1.85 millions oz) of Gold futures were sold in the early morning, which triggered the price suddenly plunge by $18 an ounce (1.6%) before bouncing back $10 an ounce a minute later.
- Swiss Franc: On 15 January 2015, the SNB scrapped the peg to the Euro

Figure: Intraday Sudden Price Movement



(a) 2010 Crash of 2:45pm

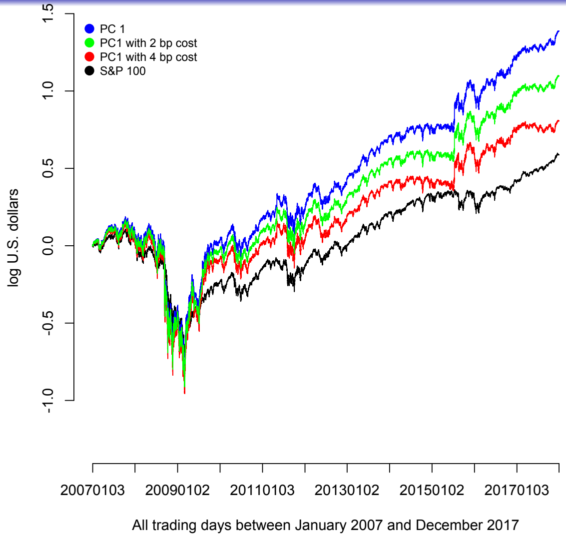
(b) 2013 Twitter Crash

Left: (a) On May 6 2010: All major US stock indices plunged and rebounded within about 30 minutes. Dow Jones Industrial Average plunged 998.5 points (about 9%), most within minutes. Graph source: NYT. Right: (b) On Tuesday April 23, 2013: Dow quickly plunged 140 points (about 1%) after a false tweet. The S&P 500 lost $121 billion of its value within minutes. Graph source: CNN money

Example of Short Run Meets Long Run

- Principal Component Analysis (PCA): Find investment weights based on 2500 seconds of data
- S&P 100 index: Value Weighted. Based on long run considerations

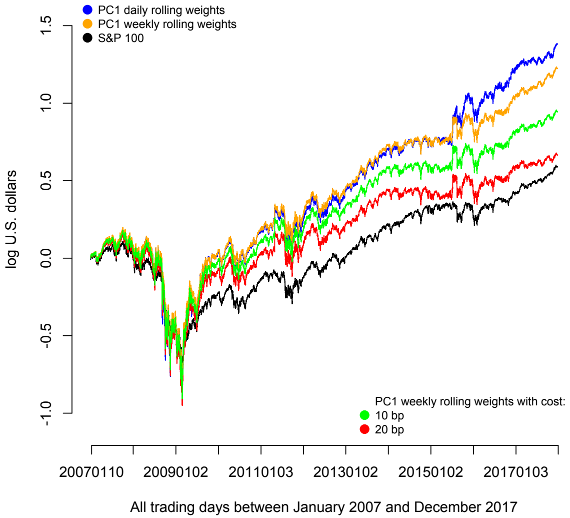
Unsupervised Learning in Intraday Data: PC1 Portfolio vs. S&P 100: 1 Day Rolling Mean Eigenvector



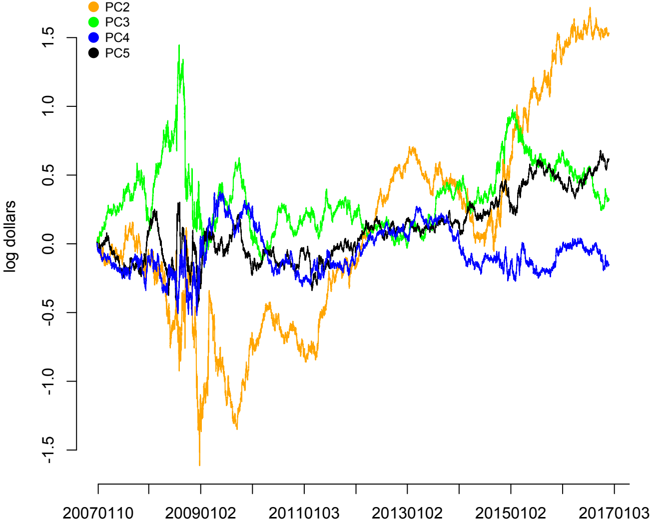
Why rolling Mean?

- 2500 seconds based estimators too variable relative to bias
- When trading: rolling mean $\implies$ less trading cost:
  - 9 period (one day) rolling mean means that only about $(1/9)^{\text{th}}$ of portfolio is updated every 2500 seconds
  - 45 period (one week) rolling mean means that only about $(1/45)^{\text{th}}$ of portfolio is updated every 2500 seconds
- Both phenomena documented by plots (a few slides later)
- Overnight position uses same weights as period 1 next day (based on data from trading periods ending at 4 pm on the preceeding day)

Allowing for higher Trading Cost in PC1: 5 Days Rolling Mean Eigenvector



Higer Order PC Portfolios



Plan for course

- Weave a little between short and long run
- Start with some background in statistics, and high frequency data
- Time series
- High frequency data

Statistical Background

Approaches to Data Analysis and Statistics

- Formal analysis
  - Testing: seeing whether a structure is present
  - Confidence intervals: setting error limits on estimates
  - Prediction intervals: setting error limits on future outcomes (risk management)
  - Bayesian methods
- Exploratory analysis
  - Finding good graphical representations, or descriptive statistics
  - “Inspirational” text: Edward R. Tufte: *The Visual Display of Quantitative Information*

Hypothesis Testing and Confidence Intervals: Review

Typical setting (discussion in scalar case):

- β is unknown parameter; $\hat{\beta}_n$ is estimator based on n observations
- For example, (x_i, Y_i), $i = 1, ..., n$, are observed, generated by
$$Y_i = x_i\beta + \epsilon_i,\ E(\epsilon) = 0 \qquad (1)$$
- $\hat{\sigma}_n^2$ is estimator of $\sigma_n^2 = \mathrm{Var}(\sqrt{\mathrm{n}}(\hat{\beta}_\mathrm{n} - \beta))$
- In setting (1), if the ϵ_i's are iid normal,
$\frac{\sqrt{n}(\hat{\beta}_n - \beta)}{\hat{\sigma}_n}$ has distribution t_{n-1}
- $1 - \alpha$ confidence interval:
    - Let $t_{n-1,\alpha}$ be such that $P(|T_{n-1}| > t_{n-1,\alpha}) = 1 - \alpha$
    - Interval: $\left|\frac{\sqrt{n}(\hat{\beta}_n - \beta)}{\hat{\sigma}_n}\right| \le t_{n-1,\alpha}$, or
    $\beta \in [\hat{\beta}_n - t_{n-1,\alpha}\hat{\sigma}_n/\sqrt{n}, \hat{\beta}_n + t_{n-1,\alpha}\hat{\sigma}_n/\sqrt{n}]$

Confidence Intervals

- Property of confidence interval:
  $$P(\beta \in \mathrm{CI}) = 1 - \alpha \text{ independently of } \beta, \mathrm{Var}(\epsilon)$$
- $1 - \alpha$ level **test** of $\beta = \beta_0$ (say, $\beta_0 = 0$):
  $$\text{accept } \mathrm{H}_0 : (\beta = \beta_0) \text{ if } \beta \in \mathrm{CI}, \text{ reject otherwise}$$
- p-value: the α which is such that the test is right on the border between acceptance or rejection
- Typical α's/p-values: 5% or smaller
- Purpose of test: determining presence or absence of certain structures. In the case of simple regression: whether Y depends on x

Theory faces reality

The above machinery depends on the following assumption:

The law of $\frac{\sqrt{n}(\hat{\beta}_n - \beta)}{\hat{\sigma}_n}$ *is independent of the unknown parameters*

This is true only in rare circumstances.

For example, it is true in regression when the errors ϵ_i are i.i.d. normally distributed, but it is not true for general distributions of ϵ_i

What to do?

Asymptotics

The theory is saved by the following “meta-theorem”:
Central Limit “Theorem” (CLT). *Under a variety of regularly conditions, the law of* $\frac{\sqrt{n}(\hat{\beta}_n-\beta)}{\hat{\sigma}_n}$ *approaches N(0,1) as* $n \to \infty$*. To be precise:*

$$P\left(\frac{\sqrt{n}(\hat{\beta}_n-\beta)}{\hat{\sigma}_n} \leq z\right) \to P\left(\mathrm{N}(0,1) \leq z\right) \text{ as } n \to \infty$$

Convergence in law

We say that random variable Z_n *converges in law* to random variable Z, or

$$Z_n \xrightarrow{\mathcal{L}} Z$$

if $P(Z_n \leq z) \to P(Z \leq z)$ at all continuity points of the function $P(Z \leq z)$. For N(0,1), all points are continuity points

Approximate (“apymptotically accurate") CIs

An asymptotically accurate confidence interval is thus obtained by

- Let z_α be such that $P(|Z| > z_\alpha) = 1 - \alpha$ (z_α is the “$\alpha/2$ *upper quantile* of N(0,1)).
- Asymptotics based Confidence Interval:
  $\left|\frac{\sqrt{n}(\hat{\beta}_n - \beta)}{\hat{\sigma}_n}\right| \leq z_\alpha$, or $\beta \in \mathrm{CI}_n = [\hat{\beta}_n - z_\alpha \hat{\sigma}_n/\sqrt{n}, \hat{\beta}_n + z_\alpha \hat{\sigma}_n/\sqrt{n}]$
- Property of approximate CI:
  Under the conditions of the CLT,

$$P\left(\beta \in \mathrm{CI}_n\right) \to 1 - \alpha \text{ as } n \to \infty$$

- terminology:
  - $1 - \alpha$: *nominal level, nominal coverage probability*
  - $P\left(\beta \in \mathrm{CI}_n\right)$: *actual coverage probability*
- Test for $\beta = \beta_0$: check if β_0 is in CI

Relationship to exact small sample normal theory

- $$T_n \xrightarrow{\mathcal{L}} \mathrm{N}(0, 1)$$
- Therefore: $t_{n-1,\alpha} \to z_\alpha$ as $n \to \infty$, and so
- Actual coverage probability is asymptotically the same for $t-$ and $z-$ confidence intervals
- In practice, most people use $t-$intervals also for general (non-normal) regression data, since slightly more conservative ($t_{n-1,\alpha} > z_\alpha$)
- For non-regression data, people usually use normal intervals

Some background on the CLT: The simple case of a sum

Let $X_1, ..., X_n$ be iid random variables. Set

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Then the following main results are true:

- Law of large numbers: if $E|X| < \infty$: $\bar{X}_n \to E(X)$ ("almost surely", "in probability"): $\bar{X}_n$ is *consistent* for $E(X)$
- Central limit theorem: if $E(X^2) < \infty$: set $\sigma^2 = \mathrm{Var(X)} = \mathrm{E(X - E(X))^2} = \mathrm{E(X^2) - (EX)^2}$; we have

$$\sqrt{n}\left(\bar{X}_n - E(X)\right) \xrightarrow{\mathcal{L}} \mathrm{N}(0, \sigma^2) = \sigma \times \mathrm{N}(0, 1)$$

Application to estimation of the mean

Let $X_1, ..., X_n$ be iid random variables for which $E(X^2) < \infty$. In particular, $\beta = E(X)$ and $\sigma^2 = \mathrm{Var(X)}$ exist.

- Estimate β by $\hat{\beta}_n = \bar{X}_n$
- Estimate σ^2 by $\hat{\sigma}^2_n = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{1}{n-1}\left(\sum_{i=1}^n X_i^2 - n(\bar{X}_n)^2\right)$
- Note that by LLN:

$$\hat{\sigma}^2_n = \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^n X_i^2 - (\bar{X}_n)^2\right) \to E(X^2) - (EX)^2 = \sigma^2 \text{ as } n \to \infty$$

In other words, $\hat{\sigma}^2_n$ is consistent for σ^2

Slutsky's Theorem

Theorem *Let Z_n, U_n and V_n be sequences of random variables, so that $Z_n \xrightarrow{\mathcal{L}} Z$, $U_n \to u$, $V_n \to v$ as $n \to \infty$, where u and v are nonrandom. Then $U_n Z_n + V_n \xrightarrow{\mathcal{L}} uZ + v$.*

Consequence for estimation of the mean:

- $\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{\mathcal{L}} \sigma \times \mathrm{N}(0, 1)$
- $\hat{\sigma}_n^2 \to \sigma^2$

Therefore:

$$\begin{aligned}
\frac{\sqrt{n}(\hat{\beta}_n - \beta)}{\hat{\sigma}_n} &= \frac{1}{\hat{\sigma}_n} \times \sqrt{n}(\hat{\beta}_n - \beta) \\
&\xrightarrow{\mathcal{L}} \frac{1}{\sigma} \times \sigma \times \mathrm{N}(0, 1) \\
&= \mathrm{N}(0, 1)
\end{aligned}$$

Computational Illustration of the CLT

This will use our computer package:

- Splus (proprietray software)
- R (freeware)
- generically: R

R will also be used in other courses

Help with R:
- Get a book about it (such as Krause and Olson, or Venables and Ripley)
- the help command in R

We now open R, and get...

Open R

```
R version 3.3.2 (2016-10-31) - "Sincere Pumpkin
Patch" Copyright (C) 2016 The R Foundation for Stat
Computing
R is free software and comes with ABSOLUTELY NO
WARRANTY. You are welcome to redistribute it under
certain conditions. Type 'license()' or 'licence()'
for distribution details. Natural language support
but running in an English locale
R is a collaborative project with many contributors
Type 'contributors()' for more information and 'cit
on how to cite R or R packages in publications.

Type 'demo()' for some demos, 'help()' for on-line
help, or 'help.start()' for an HTML browser interfa
to help. Type 'q()' to quit R.
>
```

Example: Binomial Distribution

- X_i is 0 or 1
- $P(X_i = 1) = p, \; P(X_i = 0) = q = 1 - p$
- The distribution of $S_n = X_1 + ... + X_n$ is called the *binomial distribution* with parameters (n, p), or simply $b(n, p)$ (see Chapter 2.1.2 in Rice)
- $\beta = E(X_i) = 1 \times P(X_i = 1) = p$
- $\sigma^2 = \mathrm{Var(X_i)} = \mathrm{E(X_i^2){-}(EX_i)^2 = E(X_i){-}(EX_i)^2 = p{-}p^2 = pq}$
- The LLN and CLT hold for $\hat{\beta}_n = \bar{X}_n = S_n/n$

Let's see in R if this is true...

To find out about the binomial distribution

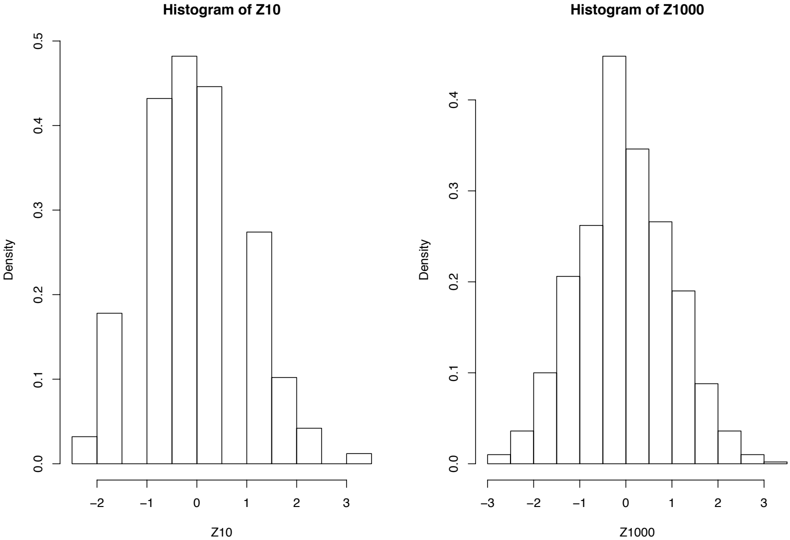
```
> help(rbinom)
[...]
Usage:
```

- `dbinom(x, size, prob, log = FALSE)`
- `pbinom(q, size, prob, lower.tail = TRUE, log.p = FALSE)`
- `qbinom(p, size, prob, lower.tail = TRUE, log.p = FALSE)`
- `rbinom(n, size, prob)`

```
[...]
```

```
M<-1000 # number of simulations
n<-10
p<-.33
S<- rbinom(M,n,p) #“help” is wrong
Xbar<-S/n
sigma<- sqrt(p*(1-p))
Z10 <- sqrt(n)*(Xbar -p)/sigma
par(mfrow=c(1,2)) # check this command out!!!
hist(Z10,freq=F)
# try again with a larger n
n<-1000
S<- rbinom(M,n,p) #“help” is wrong
Xbar<-S/n
Z1000 <- sqrt(n)*(Xbar -p)/sigma
hist(Z1000,freq=F)
```

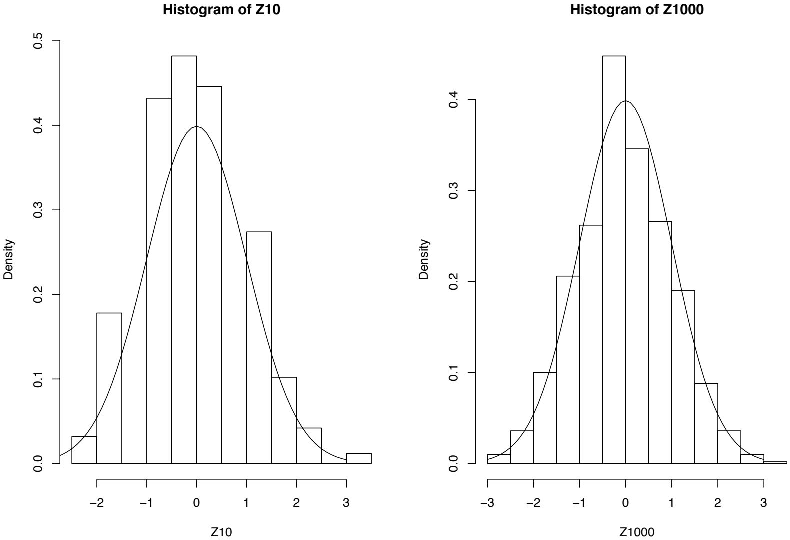
The distribution of Z_n stabilizes



Superimposing the normal distribution on the histogram

```
par(mfrow=c(1,2))
hist(Z10,freq=F)
# compare to normal distribution
xpoints <- c(-30:30)/10
density <- dnorm(xpoints,mean=0,sd=1)
lines(xpoints,density)
# try again with larger n hist(Z1000,freq=F)
lines(xpoints,density)
```

Normal curve superimposed on histograms



More General Quantities than the Mean

- For many (but not all) well chosen estimators, $\sqrt{n}(\hat{\beta}_n - \beta)$ has a CLT, i.e.,

$$\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{\mathcal{L}} \mathrm{N}(0, \sigma^2)$$

- (We shall discuss how to find estimators next time)
- If the have a consistent estimator $\hat{\sigma}_n$ for σ, Slutsky's Theorem then yields that

$$\frac{\sqrt{n}(\hat{\beta}_n - \beta)}{\hat{\sigma}_n} \xrightarrow{\mathcal{L}} \mathrm{N}(0, 1)$$

- We can then use the normal disribution to set confidence intervals, tests

Similar results apply in the vector case

How High Frequency Data differ from Low Frequency Data

Volatility Estimation without Microstructure:

Parametric and Non-parametric Approaches

Models for Inference in High Frequency Data

- Natural to use same model as in quantitative finance: the Itô process:

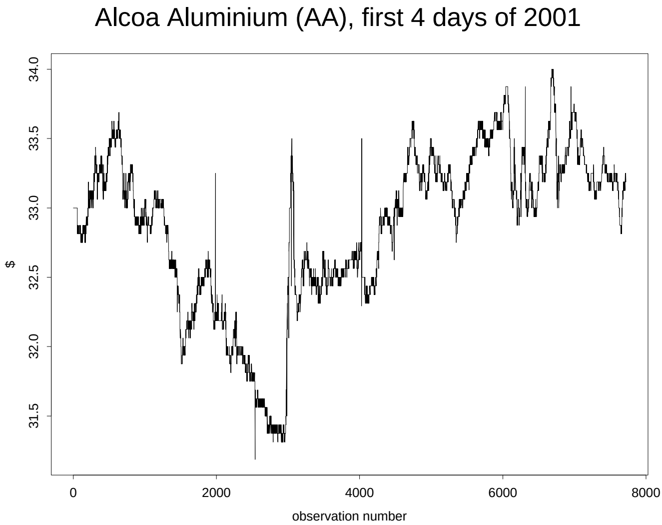
$$\text{log securities price:} \quad X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s$$

  B_t is Brownian motion; μ_t and σ_t can be random processes
- Model can also include jumps

High frequency data formalism:

- Up to several transcations per second, sampling times $0 = t_0 < t_1 < \ldots < t_n = T$
- Typical time period of analysis $[0, T]$: one day (or 5 minutes)
- Can also combine results for several days
- Typical asymptotics as $n \to \infty$, T fixed

Model vs. Data: An Intra-day Time Series



Brownian motion and Geometric Brownian motion

$$X_t = \log S_t = \text{ the logarithm of the stock price } S_t \text{ at time } t.$$

The Geometric Brownian motion (GBM) model is now that

$$X_t = X_0 + \mu t + \sigma W_t,$$

where μ and σ are constants, W_t is a *Brownian Motion (BM)*.

The process $(W_t)_{0 \leq t \leq T}$ is a Brownian motion provided
(1) $W_0 = 0$ ("time zero" is an arbitrary reference time);
(2) $t \to W_t$ is a continuous function of t;
(3) W has independent increments: if $t > s > u > v$, then $W_t - W_s$ is independent of $W_u - W_v$;
(4) for $t > s$, $W_t - W_s$ is normal with mean zero and variance $t - s$ (N(0,t-s)).

Estimation in the GBM model: Parametric inference

- Equal spacing: $t_{n,i} = i\Delta t_n = iT/n$
- Observations: $X_{t_{n,i}}$, or $\Delta X_{t_{n,i+1}} = X_{t_{n,i+1}} - X_{t_{n,i}}$, $i = 0, ..., n-1$
- The $\Delta X_{t_{n,i+1}}$ are iid with law $\mathrm{N}(\mu\Delta t_n, \sigma^2\Delta t_n)$.
- Natural estimators are:

$$\hat{\mu}_n = \frac{1}{n\Delta t_n}\sum_{i=0}^{n-1}\Delta X_{t_{n,i+1}} = (X_T - X_0)/T \text{ MLE and UMVU}$$

$$\hat{\sigma}^2_{n,MLE} = \frac{1}{n\Delta t_n}\sum_{i=0}^{n-1}(\Delta X_{t_{n,i+1}} - \overline{\Delta X}_{t_n})^2 \text{ MLE; or}$$

$$\hat{\sigma}^2_{n,UMVU} = \frac{1}{(n-1)\Delta t_n}\sum_{i=0}^{n-1}(\Delta X_{t_{n,i+1}} - \overline{\Delta X}_{t_n})^2 \text{ UMVU.}$$

Behavior of parametric estimators: $\hat{\mu}$

- μ *cannot be consistently estimated* for fixed T
- $\hat{\mu}_n$ does not depend on n, but only on T, X_0, X_T
- If $T \to \infty$, then μ *can* be estimated consistently: $(X_T - X_0)/T \xrightarrow{p} \mu$ as $T \to \infty$. This is because $\mathrm{Var}((X_T - X_0)/T) = \sigma^2/T \to 0$.

Behavior of parametric estimators: Consistency of $\hat{\sigma}$

- σ^2 *can* be estimated consistently for fixed T, as $n \to \infty$: $\hat{\sigma}_n^2 \xrightarrow{p} \sigma^2$ as $n \to \infty$.
- Set $U_{n,i} = \Delta X_{t_{n,i}}/(\sigma \Delta t_n^{1/2})$, then: Then the $U_{n,i}$ are iid with distribution $N((\mu/\sigma)\Delta t_n^{1/2}, 1)$. Set $\bar{U}_{n,\cdot} = n^{-1}\sum_{i=0}^{n-1} U_{n,i}$.
- From considerations for normal random variables:

$$\sum_{i=0}^{n-1}(U_{n,i} - \bar{U}_{n,\cdot})^2$$

is χ^2 distributed with $n-1$ df (and independent of $\bar{U}_{n,\cdot}$)

- For the UMVU estimator,

$$\hat{\sigma}_n^2 = \sigma^2 \Delta t_n \frac{1}{(n-1)\Delta t_n} \sum_{i=0}^{n-1}(U_{n,i} - \bar{U}_{n,\cdot})^2 \stackrel{\mathcal{L}}{=} \sigma^2 \frac{\chi^2_{n-1}}{n-1}$$

Behavior of parametric estimators: Consistency of $\hat{\sigma}$ (cont'd)

- $\hat{\sigma}_n^2 = \sigma^2 \frac{\chi^2_{n-1}}{n-1}$
- It follows that

$$E(\hat{\sigma}_n^2) = \sigma^2 \text{ and } \mathrm{Var}(\hat{\sigma}_n^2) = \frac{2\sigma^4}{n-1},$$

  since $E\chi^2_m = m$ and $\mathrm{Var}(\chi^2_\mathrm{m}) = 2\mathrm{m}$.
- Hence $\hat{\sigma}_n^2$ is consistent for σ^2: $\hat{\sigma}_n^2 \to \sigma^2$ in probability as $n \to \infty$.

Behavior of parametric estimators: Asymptotic normality of $\hat{\sigma}$

- $\hat{\sigma}_n^2 = \sigma^2 \frac{\chi^2_{n-1}}{n-1}$
- Since χ^2_{n-1} is the sum of $n-1$ iid χ^2_1 random variables, by the central limit theorem we have the following convergence in law:

$$\frac{\chi^2_{n-1} - E\chi^2_{n-1}}{\sqrt{\mathrm{Var}(\chi^2_{\mathrm{n-1}})}} = \frac{\chi^2_{n-1} - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{\mathcal{L}} \mathrm{N}(0,1),$$

- and so

$$n^{1/2}(\hat{\sigma}_n^2 - \sigma^2) \sim (n-1)^{1/2}(\hat{\sigma}_n^2 - \sigma^2)$$
$$\stackrel{\mathcal{L}}{=} \sqrt{2}\sigma^2 \frac{\chi^2_{n-1} - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{\mathcal{L}} \sigma^2 N(0,2) = N(0, 2\sigma^4).$$

Confidence intervals for σ

- $n^{1/2}(\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{\mathcal{L}} \sigma^2 N(0,2) = N(0, 2\sigma^4)$
- Intervals: $\sigma^2 = \hat{\sigma}_n^2 \pm 1.96 \times \frac{\sqrt{2}\hat{\sigma}_n^2}{\sqrt{n}}$ is asymptotic 95 % confidence interval for σ^2.
- Since $\hat{\sigma}^2_{n,MLE} = \frac{n-1}{n}\hat{\sigma}^2_{n,UMVU}$, the same asymptotics applies to the MLE

Non-Centered Estimators

$$\hat{\sigma}^2_{n,nocenter} = \frac{1}{n\Delta t_n}\sum_{i=0}^{n-1}(\Delta X_{t_{n,i+1}})^2.$$

For MLE version of $\hat{\sigma}_n$:

$$\begin{aligned}
\hat{\sigma}^2_{n,MLE} &= \frac{1}{n\Delta t_n}\sum_{i=0}^{n-1}(\Delta X_{t_{n,i+1}} - \overline{\Delta X}_{t_n})^2 \\
&= \frac{1}{n\Delta t_n}\left(\sum_{i=0}^{n-1}(\Delta X_{t_{n,i+1}})^2 - n(\overline{\Delta X}_{t_n})^2\right) \\
&= \hat{\sigma}^2_{n,nocenter} - \Delta t_n \hat{\mu}^2_n \\
&= \hat{\sigma}^2_{n,nocenter} - \frac{T}{n}\hat{\mu}^2_n
\end{aligned}$$

Hence: $n^{1/2}\left(\hat{\sigma}^2_{n,MLE} - \hat{\sigma}^2_{n,nocenter}\right) \xrightarrow{p} 0.$

The Classical Nonparametric Case: Integrated Volatility (IV)

- Classical target: Integrated volatility:
  $\langle X, X \rangle = \int_0^T \sigma_t^2 dt = \lim_{\Delta t \to \infty} \sum_{t_{i+1} \leq T} (X_{t_{i+1}} - X_{t_i})^2$
- Purpose of Estimating IV
  - Asset management, portfolio optimization
  - Options hedging
  - Risk management
  - Model dynamics
  - Prediction interval based hedging of options

The Classical case: Realized Volatility as Measure of Integrated Volatility

High frequency data: up to several transactions per second
Chance to estimate $\langle X, X\rangle_T$ very accurately

Usual estimator: $$[X, X]_T = \sum_{t_{i+1} \leq T} (X_{t_{i+1}} - X_{t_i})^2$$ "realized volatility"

- a standard way to measure volatility (Andersen, Bollerslev, many others)
- consistent: $[X, X]_T \xrightarrow{p} \langle X, X\rangle_T$ as $\Delta t \to 0$ (stoch calc)
- asymptotically mixed normal, variance $\frac{2T}{n} \int_0^T \sigma_t^4 dt$ (Barndorff-Nielsen & Shephard, Jacod & Protter, Mykland & Zhang)
- can estimate variance by $\frac{2}{3}[X, X, X, X]_T$ (Barndorff-Nielsen & Shephard)

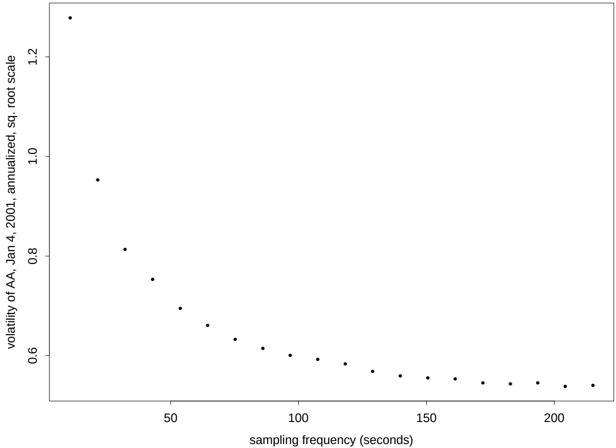
Other Quantities that can be Estimated in Data from One Day

- Other powers of volatility: $\int_0^T \sigma_t^p dt$
- Leverage effect: $\langle \sigma^2, X \rangle_T$, or corresponding correlation, relates to skewness and volatility risk
- Volatility of volatility $\langle \sigma^2, \sigma^2 \rangle_T$, related to kurtosis
- Regression of one process on another, integrated alphas and betas, ANOVA, related to systematic risk, options hedging, and model testing
- Same quantities, but instantaneously
- Nonparametric trading strategies
- Liquidity; time to execution; dark pools

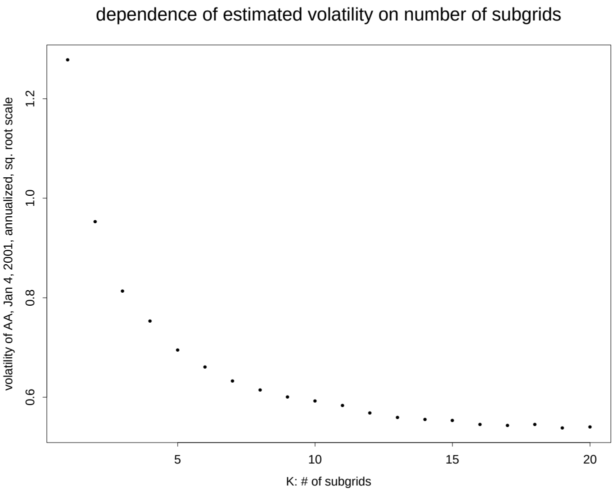
Inference in the presence of market microstructure

RV as One Samples More Frequently

dependence of estimated volatility on sampling frequency



RV vs Sampling Interval



The Failure of Realized Volatility

The realized volatility methodology does not work so well in ultra high frequency data

- In real data, when $\Delta t \to 0$, $[X, X]_T$ does not converge
- Theory usually illustrated with $\Delta t = $ 5-15 minutes

Why? Our candidate explanation:

- existence of microstructure (e.g. bid-ask spread, strategic trading, limited market depth, price impact, discrete price changes, ...)
- transaction as measurement device

The realized volatility methods form part of a more general theory which can also handle noise

Microstructure Noise & Hidden Semimartingale model

- observed log stock price: $Y_{t_i} = X_{t_i} + \epsilon_i$
- X_t is latent log price, semimartingale, say, Ito process

$$dX_t = \mu_t dt + \sigma_t dB_t$$

  B_t is Brownian motion; μ_t and σ_t can be random processes
- Model in X can also include jumps
- Microstructure ϵ_i is iid, or fast mixing

Challenges

- Dependent noise ϵ_i can have short run dependence
- Irregular spacings
- Endogenous times
- Epps effect
- Time scrambling
- Relationship to trading

To be continued