Lecture 4: Mean Variance Analysis and the Capital Asset Pricing Model

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Lecture Plan

- Mean Variance Analysis
  - Investment opportunity set with multiple risky assets
  - Mutual Fund Separation Theorem
  - Finding the tangency portfolio
- The Capital Asset Pricing Model
  - Derivation
  - What is risk in the CAPM?
  - Estimating betas
  - Does the CAPM work?

Learning Objectives

Understand ...

- how to construct the investment opportunity set for multiple risky assets
- the difference between the investment opportunity set and the efficient frontier
- the mutual fund separation theorem
- how to find the tangency portfolio
- the derivation of the CAPM
- the key message of the CAPM
- the applications of the CAPM

Mean Variance Analysis

Understanding Asset Allocation

- What we have done so far:
  - One risky asset, one riskless asset
  - Compute mean and variance of portfolios of risky assets
  - Find minimum variance portfolio
- We now study more general cases:
  1. Two risky assets, no riskless asset
  2. Multiple risky assets, no riskless asset
  3. Multiple risky assets and riskless asset
- These steps help us understand and derive the Capital Asset Pricing Model (CAPM)

Three Steps of Asset Allocation

Three Step Process

1. For each asset, forecast the key quantities we need:
    - expected returns
    - standard deviations
    - covariances
2. Map out the full set of investment opportunities attainable by combining the basic securities
    - use an expected return-standard deviation graph
3. Out of all investment opportunities, choose the portfolio that offers the highest utility

Case One: Two Risky Assets, No Riskless Asset (I)

Problem: how to allocate our money between a risky bond index (B) and a stock index (S)?

- Step One. We need to forecast some quantities to be able to compute a strategy's expected return and variance
  - Suppose we allocate w% to the stock index and the rest, $(1-w)$%, to the bond index
  - The expected return and variance of such a portfolio are:

$$\begin{aligned}
\mathrm{E}[\tilde{r}_p] &= (1-w)\mathrm{E}(\tilde{r}_B) + w\mathrm{E}(\tilde{r}_S) \\
\mathrm{var}(\tilde{r}_p) &= (1-w)\sigma_B^2 + w^2\sigma_S^2 + 2w(1-w)\sigma_{B,S}
\end{aligned}$$

- The basic parameters we need are therefore:
  1. Expected returns for basic securities
  2. Variances for each of the basic securities
  3. Covariance between bonds and stocks

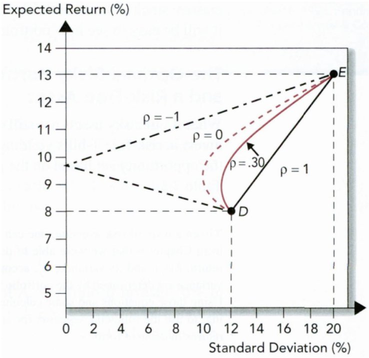
Case One: Two Risky Assets, No Riskless Asset (II)

- Step Two. Find the investment opportunity set
- What combinations of expected return and standard deviation are available to us?
- What we learned about diversification:
  - If we have a riskless asset and a risky asset, all combinations lie on a straight line in the E-S diagram
  - If we have two risky assets that are not perfectly positively correlated, the variance of the portfolio is less than the variance of the individual securities
  - We get diversification benefits: better risk/return opportunities
  - Assets should be less than perfectly correlated because news that affect one asset often do not affect others
    - If Goldman Sachs is investigated by the SEC, Kraft Foods is unaffected

Case One: Two Risky Assets, No Riskless Asset (III)

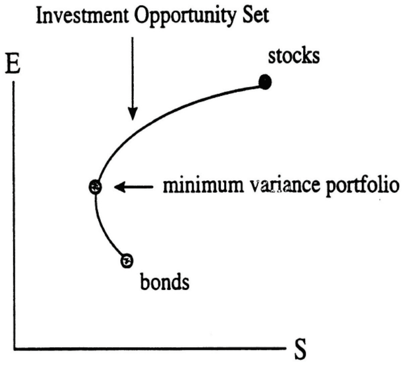
Visualizing Diversification

- The shape of the investment opportunity in the *E*-*S* diagram depends on the correlation between the two risky assets



Case One: Two Risky Assets, No Riskless Asset (IV)

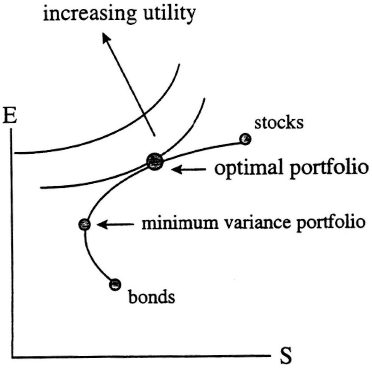
- In our case with two risky assets, the investment opportunity set might look like this:



- This is the idea of diversification (Markowitz 1952)

Case One: Two Risky Assets, No Riskless Asset (V)

- Step Three. Choose the strategy that yields the highest utility
  - Drag indifference curves down toward the investment opportunity set
  - If you like high expected returns and dislike high standard deviation, the indifference curves have the convex shape
  - The first point of contact is the optimal portfolio:



What Makes Assets Attractive?

What characteristics make assets attractive?

- The first two characteristics are obvious:
  1. High expected return
  2. Low variance

- The third characteristic is the new lesson:
  3. Low covariance $\rightarrow$ an asset is attractive if it has low or negative covariance with the assets you already own

     You can include such an asset in your portfolio to reduce risk

Case Two: Multiple Risky Assets, No Riskless Asset (I)

- Step One. Now we need to estimate lots of parameters
  - Suppose we have N assets, and we choose portfolio weights $w_1, w_2, \ldots, w_N$
  - The portfolio's expected return and variance are:

$$\begin{aligned} \mathrm{E}(\tilde{r}_p) &= w_1 \mathrm{E}(\tilde{r}_1) + w_2 \mathrm{E}(\tilde{r}_2) + \cdots + w_N \mathrm{E}(\tilde{r}_N) \\ \mathrm{var}(\tilde{r}_p) &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + \cdots + w_N^2 \sigma_N^2 \\ &\quad + 2 w_1 w_2 \sigma_{1,2} + \cdots + 2 w_{N-1} w_N \sigma_{N-1,N} \end{aligned}$$

  - Thus, with N assets, we need
    1. each asset's expected return (N parameters)
    2. each asset's variance (N parameters)
    3. covariance between each pair of assets ($\frac{N(N-1)}{2}$ parameters!)

       Example: with $N = 100$ assets, we need 4950 covariances

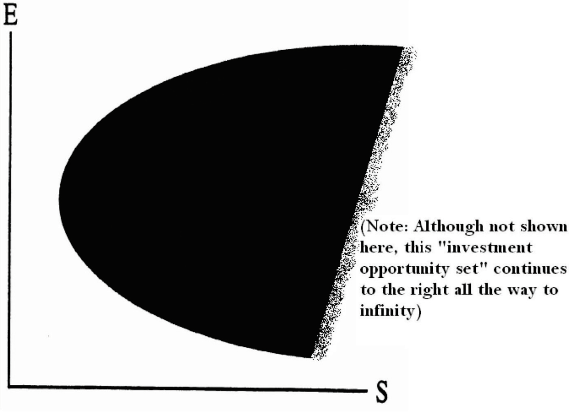
Case Two: Multiple Risky Assets, No Riskless Asset (II)

- Step Two. What does the investment opportunity set look like with multiple risky assets?
- With more than two risky assets, the investment opportunity set becomes an area rather than a curve
  - **Note:** adding more assets can never reduce the size of the investment opportunity set
  - At worst, the new assets are completely useless, and we can just ignore them
  - In the best case scenario, they extend the efficient frontier in the northwest direction, offering even more diversification
- We look for assets that have low covariance with our current holdings

  $\rightarrow$ International assets? Different asset classes?

Case Two: Multiple Risky Assets, No Riskless Asset (III)

- With multiple risky assets, the investment opportunity set has the same shape as with just two risky assets (parabola)
- However, we can now reach all the interior points as well

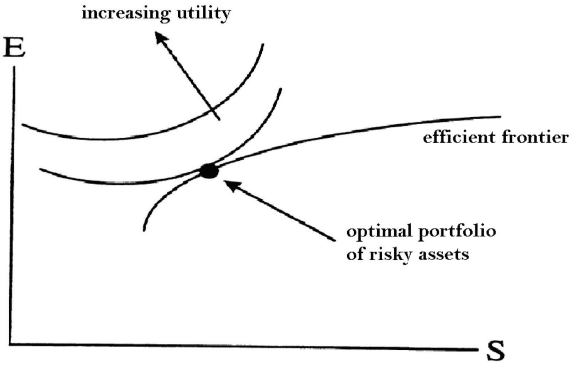


Case Two: Multiple Risky Assets, No Riskless Asset (IV)

- We can again ignore most of this investment opportunity set
- Many possible portfolios are mean-variance dominated
  - Another portfolio is just as risky but promises higher return
- Which portfolios remain if we remove all dominated portfolios?
  - Move as far north as possible from any feasible portfolio
  - For each level of risk, only keep the portfolio with the highest expected return
- This leaves us the northwest edge of the initial feasible area
- This "frontier" has its own name: the **efficient frontier** (of risky assets)

Case Two: Multiple Risky Assets, No Riskless Asset (V)

- Step Three. Choose the strategy that yields the highest utility
  - This step is the same as with it is with just two risky assets, no riskless asset
  - Drag the indifference curves towards the efficient frontier



Mutual Fund Separation Theorem

- We know the general shape of the investment opportunity set of risky assets, but how can we find it?
- We have a convenient theorem—a mutual fund or two-fund separation theorem—that tells us the answer:

Mutual Fund Separation Theorem

All mean-variance efficient portfolios are combinations of <u>any</u> two mean-variance efficient portfolios

- If we can find any two frontier portfolios, we can generate ALL other efficient frontier portfolios by mixing these two portfolios
- Two efficient portfolios are easy to compute:
    1. Minimum Variance Portfolio that we found last week
    2. Tangency Portfolio that we'll compute shortly

Case Three: Multiple Risky Assets and Riskless Asset (I)

- Step One. We now have N risky assets (as before) and a riskless asset
- Step Two. We can continue directly from the second case
  - We already know what portfolios can be constructed from the N risky assets
  - When we add another asset to the problem, the efficient frontier expands; the riskless asset is no exception
  - Since the riskless asset lies, by definition, beyond the efficient frontier of risky assets, the efficient frontier expands

    $\rightarrow$ A riskless asset itself is a mean-variance efficient asset
  - We can combine the riskless asset with any feasible risky portfolio
    - In a E-S graph, all combinations lie on a straight line between the riskless asset and the risky portfolio
  - What is the "best" risky portfolio that we should combine the riskless asset with?

Case Three: Multiple Risky Assets and Riskless Asset (II)

- If we combine the riskless asset with, say, the MV portfolio, we get some new, interesting portfolios
  - But we can do better
- We should increase the slope of the line that connects the riskless asset to a risky portfolio as much as we can
  - As the line steepens, we reach better and better portfolios
  - We cannot go any further when we just barely touch the investment opportunity set of risky assets
  - When the line is as steep as it can be, the risky portfolio that we touch is known as the tangency portfolio
- In the case with multiple risky assets and a riskless asset, we again need two efficient portfolios to use the mutual fund separation theorem. These are
  1. The riskless asset
  2. The tangency portfolio

Case Three: Multiple Risky Assets and Riskless Asset (III)

- The line that connects the riskless asset with the tangency portfolio is the Capital Allocation Line (CAL)

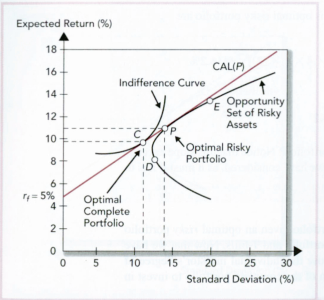


FIGURE 7.8 Determination of the optimal complete portfolio

- Step Three. We find the optimal portfolio by dragging the indifference curves toward the CAL
  - The optimal complete portfolio is the portfolio we want to hold

Finding the Tangency Portfolio (I)

- We find the tangency portfolio using methods very similar to the ones we used to find the minimum variance portfolio
  - For the MVP, we shifted the portfolio towards the asset that had the lowest covariance with our portfolio and away from the asset that had the highest covariance
  - MVP had the same covariance with all individual assets
  - We follow a similar strategy to find the tangency portfolio
    - However, we now face two conflicting objectives: (1) increase expected return and (2) lower variance
  - We want to find a portfolio of risky assets that maximizes the slope of the CAL
  - This slope is called the Sharpe ratio:

$$\text{Sharpe Ratio}(p) = \frac{\mathrm{E}[\tilde{r}_p] - r_f}{\sigma_p}$$

- Tangency portfolio maximizes the Sharpe ratio

Finding the Tangency Portfolio (II)

- We maximize the Sharpe ratio by starting from some trial portfolio
- We then consider what would happen to the portfolio if we were to tilt it more toward one of the risky assets
  - What happens to its expected return (over the riskless asset), $\mathrm{E}(\tilde{r}_p) - r_f$?
  - What happens to its standard deviation, σ_p?
- We find the tangency portfolio by shifting toward assets with high "marginal" Sharpe ratios:

$$\text{“Marginal” Sharpe Ratio}(j) = \frac{\mathrm{E}[\tilde{r}_j] - r_f}{\sigma_{j,p}},$$

where $\sigma_{j,p}$ is asset j's covariance with the current portfolio.

- If we shift toward an asset with a higher MSR, and away from an asset with a lower MSR, we increase the portfolio's Sharpe ratio

Finding the Tangency Portfolio (III)

- The portfolio's Sharpe ratio can be improved until the ratios are the same:

$$\frac{\mathrm{E}[\tilde{r}_1] - r_f}{\sigma_{1,p}} = \frac{\mathrm{E}[\tilde{r}_2] - r_f}{\sigma_{2,p}} = \cdots = \frac{\mathrm{E}[\tilde{r}_N] - r_f}{\sigma_{N,p}}$$

- This tangency portfolio condition is very important

  $\rightarrow$ We derive the **Capital Asset Pricing Model** from this condition

- Different methods for finding the tangency portfolio:
  1. Set up a spreadsheet that computes each stock's "marginal" Sharpe ratio. Change portfolio weights manually to increase the portfolio's Sharpe ratio. Stop when all ratios are equal
  2. Set up a spreadsheet that computes a portfolio's Sharpe ratio and maximize it using a Solver

Finding the Tangency Portfolio (IV)

3. Solve the problem analytically. With two stocks, the optimal investment in stock 1 is

$$w_1^* = \frac{(\mathrm{E}(\tilde{r}_1) - r_f)\sigma_2^2 - (\mathrm{E}(\tilde{r}_2) - r_f)\sigma_{1,2}}{(\mathrm{E}(\tilde{r}_1) - r_f)\sigma_2^2 + (\mathrm{E}(\tilde{r}_2) - r_f)\sigma_1^2 - (\mathrm{E}(\tilde{r}_1) + \mathrm{E}(\tilde{r}_2) - 2r_f)\sigma_{1,2}}$$

4. Write down the tangency portfolio condition as a system of linear equations and solve them by hand

Finding the Tangency Portfolio: Example (I)

Example: Finding the Tangency Portfolio

We have the following information about Dell, Microsoft and Google:

Stock	Expected Return	Covariance Matrix: Dell	Covariance Matrix: Microsoft	Covariance Matrix: Google
Dell	15%	0.032	0.016	0
Microsoft	17%	0.016	0.032	0.016
Google	17%	0	0.016	0.040

The risk-free rate is 6%.

- Find the tangency portfolio and its Sharpe ratio

Finding the Tangency Portfolio: Example (II)

- If we have the tangency portfolio, all marginal risk-return ratios have to be equal to some number C:

$$\frac{\mathrm{E}[\tilde{r}_j] - r_f}{\sigma_{j,p}} = C$$

  If we multiply both sides by $\sigma_{j,p}$ and then repeat this for every asset,

$$\begin{aligned} C \times \sigma_{1,p} &= \mathrm{E}[\tilde{r}_1] - r_f \\ C \times \sigma_{2,p} &= \mathrm{E}[\tilde{r}_2] - r_f \\ \dots & \quad \dots \\ C \times \sigma_{N,p} &= \mathrm{E}[\tilde{r}_N] - r_f \end{aligned}$$

- We also have one more equation: $w_1 + w_2 + \cdots + w_N = 1$
- However, we can just set $C = 1$, solve for $w_1, w_2, \dots, w_N$, and then correct for $C = 1$ by scaling the weights to sum up to 1

Finding the Tangency Portfolio: Example (III)

- Because $\sigma_{1,p} = w_1\sigma_1^2 + w_2\sigma_{1,2} + \cdots + w_N\sigma_{1,N}$, we solve

$$\begin{array}{ccccccc} 0.032w_1 & + & 0.016w_2 & & & = & 0.15 - 0.06 \\ 0.016w_1 & + & 0.032w_2 & + & 0.016w_3 & = & 0.17 - 0.06 \\ & & 0.016w_2 & + & 0.040w_3 & = & 0.17 - 0.06 \end{array}$$

- We find $w_1 = 2.215$, $w_2 = 1.193$, and $w_3 = 2.273$
- The actual tangency portfolio weights are proportional to these numbers:

$$\begin{aligned} w_1(\mathrm{T}) &= \frac{2.215}{2.215 + 1.193 + 2.273} = 39\% \\ w_2(\mathrm{T}) &= \frac{1.193}{2.215 + 1.193 + 2.273} = 21\% \\ w_3(\mathrm{T}) &= \frac{2.273}{2.215 + 1.193 + 2.273} = 40\% \end{aligned}$$

Finding the Tangency Portfolio: Example (IV)

- The expected return of the tangency portfolios

$$\mathrm{E}[\tilde{r}_T] = 0.39 \times 15\% + 0.21 \times 17\% + 0.4 \times 17\% = 16.22\%$$

- The portfolio's variance is

$$\begin{aligned}\mathrm{var}(\tilde{r}_T) &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + \cdots + 2w_2w_3\sigma_{2,3} \\ &= (0.39)^2(0.032) + (0.21)^2(0.032) + \cdots + 2(0.21)(0.4)(0.016) \\ &= \underline{0.01799}\end{aligned}$$

$\rightarrow$ The portfolio's standard deviation is $\sigma_T = \sqrt{0.01799} = 13.41\%$

- The tangency portfolio's Sharpe ratio is thus

$$\text{Sharpe Ratio}(T) = \frac{\mathrm{E}[\tilde{r}_T] - r_f}{\sigma_T} = \frac{16.22\% - 6\%}{13.41\%} = 0.762$$

Remarks on Mean-Variance Analysis

- We have used a framework called the mean-variance analysis
  - The investor is assumed to care only about the mean and variance of his or her investment
- We have not justified this assumption
  - Why should people act this way?
- Very difficult to know what your utility function is
- The success of the framework depends on the *quality of the inputs* (i.e., expected returns, variances, and covariances)
  - We <u>cannot</u> use these tools to examine 8,000 stocks (the inputs are so difficult to estimate)
  - But we <u>can</u> examine optimal allocations between asset classes: e.g., domestic and foreign equity, emerging markets, bonds, and real estate

The Capital Asset Pricing Model

The Fundamental Question

The Fundamental Question about Assets

Why do some securities earn higher returns on average than others?

- Finance theory says that this is "because of risk," but...
  1. What is risk?
  2. How much extra expected return do we require to bear extra risk?
- The CAPM answers these questions, and has many important applications:
  - Valuation
    - How much is a company worth?
    - Is a stock under- or overvalued?
  - Mutual fund performance evaluation
  - Company's investment decision making (required rate of return)

Separation Principle (I)

- Assumptions:
  1. Assume that people use mean-variance analysis
  2. Assume everyone uses the same values as the inputs
- As a consequence:
  - Everyone will hold the same combination of risky assets, regardless of their level of risk aversion
  - This combination is the tangency portfolio
  - Of course, more risk-averse people will hold more of the riskless asset and less of the tangency portfolio of risky assets (i.e., they are to the left on the CAL)
    - But they will hold the risky assets in the same proportions (Separation Principle)

Separation Principle (II)

Example: Separation Principle

Suppose there are only two risky assets:

1. An index of utility stocks
2. An index of technology stocks

- The tangency portfolio is 40% utility stocks and 60% technology stocks
- Someone not very risk-averse could thus hold precisely 40% in utilities and 60% in technology, and 0% in the riskless asset
- Someone more risk-averse will mainly hold the riskless asset, say 90%, but will then hold 4% in utilities, 6% in technology
  - The ratio of the risky asset holdings is constant, 4 : 6
  - This ratio is the same across <u>all</u> investors

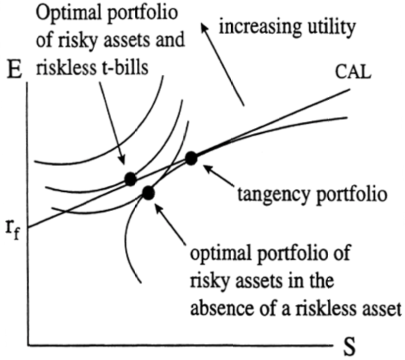
Separation Principle (III)

- Another way of looking at the Separation Principle is:
  - An investment advisor should recommend the same portfolio of risky assets to all clients
- This was not the case in the 50s and 60s
- "Interior decorating" was the norm
- Even today, the investment advice often conflicts with the separation principle
- How can we explain this diversity of advice?
  - *One response:* Investors may have preferences for dividend yield, tax considerations, political issues, differences in labor income risk etc.
  - *Related question:* Why are there so many mutual funds?

Deriving the CAPM (I)

- We found that the optimal portfolio of risky assets is the tangency portfolio. . .

  . . . and that every investor chooses some combination of the riskless asset and the tangency portfolio



Deriving the CAPM (II)

- The following condition had to hold for the tangency portfolio for any two assets i and j:

$$\frac{\mathrm{E}[\tilde{r}_i] - r_f}{\mathrm{cov}(\tilde{r}_i, \tilde{r}_T)} = \frac{\mathrm{E}[\tilde{r}_j] - r_f}{\mathrm{cov}(\tilde{r}_j, \tilde{r}_T)}$$

- This condition must hold not only for all assets but, by extensions, for all risky portfolios as well
- Thus it must also hold of the tangency portfolio itself:

$$\frac{\mathrm{E}[\tilde{r}_T] - r_f}{\mathrm{cov}(\tilde{r}_T, \tilde{r}_T)} = \frac{\mathrm{E}[\tilde{r}_i] - r_f}{\mathrm{cov}(\tilde{r}_i, \tilde{r}_T)}$$

- Set $\mathrm{cov}(\tilde{r}_T, \tilde{r}_T) = \mathrm{var}(\tilde{r}_T)$ and solving for $\mathrm{E}[\tilde{r}_i]$:

$$\mathrm{E}[\tilde{r}_i] = r_f + \frac{\mathrm{cov}(\tilde{r}_i, \tilde{r}_T)}{\mathrm{var}(\tilde{r}_T)} \left(\mathrm{E}[\tilde{r}_T] - r_f\right)$$

- This equation links expected return to risk

Deriving the CAPM (III)

- This condition is not useful by itself
  - We need the tangency portfolio to get an asset's expected return
  - But to get the tangency portfolio, we need the expected return
- Separation principle completes the derivation of the CAPM:
  - Everyone holds the risky assets in the same proportions—everyone allocates between the riskless asset and the tangency portfolio
  - Because of this separation principle and because "supply of risky assets = demand for risky assets":

The Punch Line of the CAPM

The Tangency Portfolio is the Market Portfolio

  - Market portfolio = a portfolio of <u>all</u> risky assets (such as equity), weighted by their market capitalization

Deriving the CAPM (IV)

- We get the CAPM equation by acknowledging that the tangency portfolio is the market portfolio:

$$\begin{aligned} \mathrm{E}[\tilde{r}_i] &= r_f + \beta_i \left(\mathrm{E}[\tilde{r}_m] - r_f \right), \\ \text{where } \beta_i &= \frac{\mathrm{cov}(\tilde{r}_i, \tilde{r}_m)}{\mathrm{var}(\tilde{r}_m)} \end{aligned}$$

- **Note:** Because $T = M$, we sometimes refer to the CAL as the Capital Market Line (CML)

What is Risk?

What is the relevant measure of risk in a CAPM world?

- The relevant measure of risk is the covariance between an asset's return and the return on the market portfolio
- The fair compensation for risk is proportional to how much the market portfolio is expected to yield over the safe asset

Risk in the Capital Asset Pricing Model

- Why is risk covariance in the CAPM?
  - Covariance = marginal variance; covariance tells us how a portfolio's variance *changes* when we tilt it
  - Intuition: in economics, it is the *marginal cost* of goods that determines their prices, not their total or average cost
  - Likewise, the *marginal variance* determines the additional risk of an investment, and therefore its price (expressed as returns)
- We typically refer to $\frac{\mathrm{cov}(\tilde{r}_i, \tilde{r}_m)}{\mathrm{var}(\tilde{r}_m)}$ as β_i, because its a regression slope coefficient:

$$\tilde{r}_{i,t} = r_f + \beta_i \left(\tilde{r}_{m,t} - r_f\right) + \varepsilon_{i,t}$$

- **Note 1:** An asset's beta is negative if it covaries negatively with the market
- **Note 2:** A portfolio's beta is the weighted average of security betas

Why No Compensation for Diversifiable Risk?

- The CAPM says that you are only compensated for systematic risk... but why?
  - Suppose that you were compensated for idiosyncratic risk
  - Now, find assets that have only idiosyncratic risk and for which $\mathrm{E}[\tilde{r}_i] > r_f$
  - Form a diversified portfolio composed of these assets
  - Because these assets have only idiosyncratic risk, the portfolio variance goes to zero as the number of assets increases

    $\rightarrow$ You tend towards a riskless portfolio that yields $\mathrm{E}[\tilde{r}_p] > r_f$

- Such a portfolio is too good to be true: Buy this portfolio and borrow at the risk-free rate for a free lunch

  To rule out arbitrage, we <u>must</u> have $\mathrm{E}[\tilde{r}_i] = r_f$ for an asset with only idiosyncratic risk

Intuition behind the CAPM

- The Capital Asset Pricing Model says that
  - High-beta stocks are risky, and must therefore offer a higher return on average to compensate for the risk
- But *why* are high-beta stocks risky?
  - Because they pay up just when you need the money the least, i.e., when the entire market is doing well
  - To get you to hold this security, it must offer a high expected return
- Suppose an asset has very high σ_i^2 but zero covariance with the market
  - CAPM says you get expected return = riskless rate, i.e., no compensation for variance
  - If you were to hold only this security, you'd be in trouble
  - But if you are diversified, variance doesn't matter

    $\rightarrow$ You do not need compensation

Facts about the CAPM: Fact 1 (I)

Fact 1

Two assets that have the same covariance with the market must have the same expected return.

- For example, take Nokia and Apple, and suppose

$$\operatorname{cov}(\tilde{r}_{\text{Nokia}}, \tilde{r}_m) = \operatorname{cov}(\tilde{r}_{\text{Apple}}, \tilde{r}_m)$$

  but also that

$$\mathrm{E}[\tilde{r}_{\text{Nokia}}] > \mathrm{E}[\tilde{r}_{\text{Apple}}]$$

- We currently hold the market portfolio (weights: $w_1, \ldots, w_N$)
- Tilt the portfolio by making two trades:
  - Increase the share of Nokia in your portfolio ("asset 1") from w_1 to $w_1 + \delta$
  - Decrease the share in Apple ("asset 2") from w_2 to $w_2 - \delta$

Facts about the CAPM: Fact 1 (II)

- Your new portfolio (m^*) has a rate of return of

$$\begin{aligned}\tilde{r}_{m^*} &= (w_1+\delta)\tilde{r}_1 + (w_2-\delta)\tilde{r}_2 + w_3\tilde{r}_3 + \cdots \\ &= \tilde{r}_m + \delta(\tilde{r}_1 - \tilde{r}_2)\end{aligned}$$

- The expected return and variance of this portfolio are:

$$\begin{aligned}\mathrm{E}[\tilde{r}_{m^*}] &= \mathrm{E}[\tilde{r}_m] + \delta\left(\mathrm{E}[\tilde{r}_1] - \mathrm{E}[\tilde{r}_2]\right) > \mathrm{E}[\tilde{r}_m] \\ \mathrm{var}(\tilde{r}_{m^*}) &= \mathrm{var}(\tilde{r}_m) + \delta^2\mathrm{var}(\tilde{r}_1 - \tilde{r}_2) + 2\delta\mathrm{cov}(\tilde{r}_m, \tilde{r}_1 - \tilde{r}_2) \approx \mathrm{var}(\tilde{r}_m)\end{aligned}$$

- We have created a portfolio with a higher expected return without increasing its variance

  $\rightarrow$ The risk stays the same at the margin (i.e., for tiny δ)
- But if so, people will buy Nokia and sell Apple:
  - Nokia's price will rise $\rightarrow$ its expected return will fall
  - Apple's price will fall $\rightarrow$ its expected return will rise
  - Prices adjust until $\mathrm{E}[\tilde{r}_{\mathrm{Nokia}}] = \mathrm{E}[\tilde{r}_{\mathrm{Apple}}]$

Facts about the CAPM: Fact 1 (III)

- Similarly, if a stock's expected return differs from what CAPM predicts, we can trade the stock's tracking portfolio (TP)
- The tracking portfolio is a portfolio of the risk-free asset and the market, designed to track a stock's price movements
- For example, suppose

$$\mathrm{E}[\tilde{r}_{\mathrm{Nokia}}] > r_f + \beta_{\mathrm{Nokia}} \left(\mathrm{E}[\tilde{r}_m] - r_f\right)$$

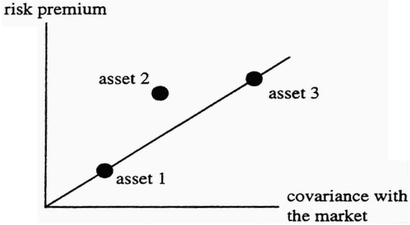
- To track Nokia, invest $(1 - \beta_{\mathrm{Nokia}})\%$ in the riskless asset and $\beta_{\mathrm{Nokia}}\%$ in the market portfolio:

$$\begin{aligned} \tilde{r}_{\mathrm{TP}} &= (1 - \beta_{\mathrm{Nokia}}) r_f + \beta_{\mathrm{Nokia}} \tilde{r}_m \\ \mathrm{E}(\tilde{r}_{\mathrm{TP}}) &= (1 - \beta_{\mathrm{Nokia}}) r_f + \beta_{\mathrm{Nokia}} \mathrm{E}(\tilde{r}_m) = r_f + \beta_{\mathrm{Nokia}} \left(\mathrm{E}[\tilde{r}_m] - r_f\right) \end{aligned}$$

- This portfolio tracks Nokia's movements (same beta)
  - It <u>must</u> have the same expected return as Nokia
  - If not, Nokia's price will adjust until expected returns equate

Facts about the CAPM: Fact 2

- The risk premium of an asset, $\mathrm{E}(\tilde{r}_i) - r_f$, depends *linearly* on the asset's covariance with the market portfolio
- Take three assets whose risk premia *do not* grow linearly with covariance:



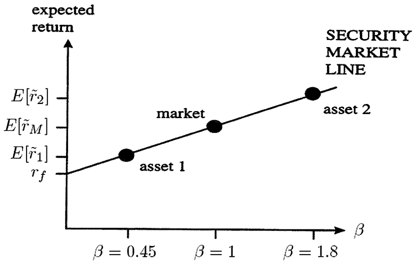
- Form a portfolio which invests a little more in asset 2 at the expense of assets 1 and 3
  - Expected return increases, variance stays the same
  - Investors will buy asset 2 and sells assets 1 and 3 until their expected returns fall into line

Security Market Line (I)

- We have established that

$$\underbrace{\mathrm{E}(\tilde{r}_i) - r_f}_{\text{risk premium}} = \underbrace{\beta_i\,(\mathrm{E}(\tilde{r}_m) - r_f)}_{\text{risk}\times\text{price of risk}}$$

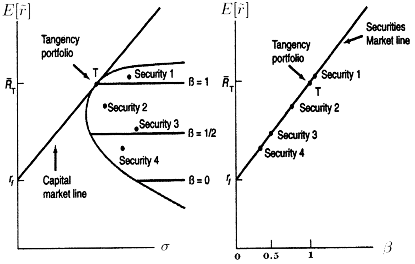
- Whereas the CAL is about the relation between expected returns and standard deviation, the Security Market Line (SML) is the relation between expected returns and β



- **Note:** <u>All</u> portfolios lie on the SML if the CAPM holds
  - By contrast, only efficient portfolios lie on the CAL

Security Market Line (II)

- All assets must lie on the Security Market Line
- What does this mean?
  - Investments with the same mean return can have different standard deviations (or variances)
  - However, they must have the same β
  - Again, the only relevant measure of risk for pricing securities is β, the measure of covariance
- The SML is <u>not</u> about the efficiency of portfolios—a portfolio can lie deep inside the efficient frontier and yet lie on the SML:



Estimating Beta

- The theory says

$$\beta_i = \frac{\mathrm{cov}(\tilde{r}_i, \tilde{r}_m)}{\mathrm{var}(\tilde{r}_m)}$$

- We usually run a regression to estimate β
  - Actual returns fluctuate randomly around expected returns

    $\rightarrow$ we can write the CAPM as a time-series regression of *excess* returns on *excess* market returns

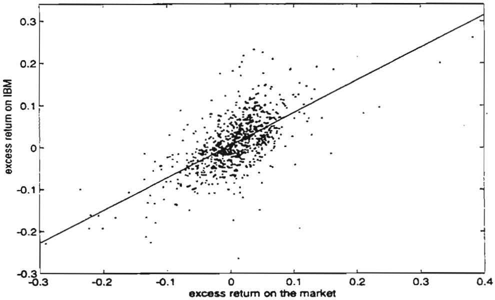
$$\tilde{r}_{i,t} - \tilde{r}_{f,t} = \alpha_i + \beta_i \left(\tilde{r}_{m,t} - \tilde{r}_{f,t} \right) + \tilde{\varepsilon}_{i,t}$$

    Note: The risk-free rate is allowed to change over time
  - The estimate of the coefficient β_i is the stock's beta
  - If the CAPM is right, α_i should be *zero*

    We'll talk more about $\alpha_i = 0$ in a moment

Estimating Beta: Example (I)

- If we want to estimate IBM's beta, we need to regress IBM's excess returns on excess market returns
- We can visualize this regression by plotting IBM's excess returns against the market excess returns:



Estimating Beta: Example (II)

- The line in the figure is the following regression:

$$\tilde{r}_{i,t} - \tilde{r}_{f,t} = 0.005 + 0.7754\,(\tilde{r}_{m,t} - \tilde{r}_{f,t}) + \tilde{\varepsilon}_{i,t}$$

- So, our estimate of IBM's β is 0.7754
  - If the risk-free rate is 4% and the market's expected return 8%, IBM's expected return, according to the CAPM, is

$$\mathrm{E}[\tilde{r}_i] = r_f + \beta_i\,(\mathrm{E}[\tilde{r}_m] - r_f) = 4\% + 0.7754\,(8\% - 4\%) = 7.1\%$$

- **Note:** We need to worry about some estimation issues
  - Betas may change over time $\rightarrow$ do not use data that's too old
  - Five years of monthly data is reasonable

Implications of Non-Zero Alphas

- Look at the estimated CAPM regression and consider the meaning of α:

$$\bar{r}_i - \bar{r}_f = \hat{\alpha}_i + \hat{\beta}_i \left(\bar{r}_m - \bar{r}_f \right),$$

  where $\bar{r}_i$ denotes an average and $\hat{\alpha}_i$ is a parameter estimate
- Rewrite this equation in terms of $\hat{\alpha}_i$

$$\hat{\alpha}_i = \bar{r}_i - \left(\bar{r}_f + \hat{\beta}_i \left(\bar{r}_m - \bar{r}_f \right) \right)$$

- α_i is the difference between what we got and what we had expected to get according to the CAPM
- If $\alpha_i > 0$,
  1. The security's average return is higher than what is required for its level of risk

     $\rightarrow$ it *has been* attractive (CAPM may or may not be true)
  2. The security might be mispriced, or CAPM might be wrong!

Implications and Benefits of the CAPM

Implications of the CAPM

- The market portfolio is mean-variance efficient
- Only β differences should explain average return differences

Benefits of the CAPM

- Instead of doing mean-variance analysis with a large number of inputs…

  …we argue that we already know what the tangency portfolio is: the market portfolio
- To estimate a stock's expected return, we only need to estimate it's covariance with the market
  - Companies often use CAPM for this purpose

CAPM Applications: Discounted Cash Flows (DCF) I

- CAPM can be used to
  - Value companies, including IPO's
  - Evaluate projects/capital budgeting
  - Determine regulated utility prices
- Let's see how we can value a company, say IBM
  - We need to value the *whole* firm, not just its equity: debtholders also have a claim
- Start with the present value formula (or DCF formula)

$$P_t = \frac{E(C_{t+1})}{1+r} + \frac{E(C_{t+2})}{(1+r)^2} + \frac{E(C_{t+3})}{(1+r)^3} + \cdots$$

CAPM Applications: Discounted Cash Flows (DCF) II

- Rewrite this as

$$P_t = \frac{E(C_{t+1})}{1+r} + \frac{E(C_{t+2})}{(1+r)^2} + \cdots + \frac{E(C_{t+5})}{(1+r)^5} + \frac{\mathbf{T}_{t+5}}{(1+r)^5}$$

  where:

$$\begin{aligned} C_{t+s} &= \text{cash flow to capital at time } t+s \\ T_{t+5} &= \text{ terminal value of firm at time } t+5 \end{aligned}$$

- The **Terminal Value** T_{t+5} is typically defined by assuming that cash flows will grow at a constant rate g after $t+5$:

$$T_{t+5} = \frac{E(C_{t+5})(1+g)}{r-g}$$

  - Similar to the Gordon growth formula

CAPM Applications: Discounted Cash Flows (DCF) III

- We still need to determine the appropriate discount rate (or cost of capital) r
- Use CAPM and the **Weighted Average Cost of Capital (WACC)** method
- Since we are discounting cash flows to *total* capital, we use an *average* discount rate

$$r = r^D(1-\tau)\frac{D}{D+E} + r^E\frac{E}{D+E}$$

  - τ is the tax rate
  - D, E are the market value of debt and equity
  - r^D, r^E are the debt and equity cost of capital
- r^D and r^E can be computed using CAPM:

$$r^D = r_f + \beta^D (r_M - r_f)$$
$$r^E = r_f + \beta^E (r_M - r_f)$$

CAPM Applications: Discounted Cash Flows (DCF) IV

- **Example:** Use WACC to estimate the total cost of capital for IBM
- First, what numbers do you need to know?
- For IBM
  - (Corporate) tax rate $\tau = 35\%$
  - Leverage ratio $D/E = 0.8 \implies \frac{D}{D+E} = 0.44, \ \frac{E}{D+E} = 0.56$
  - Equity beta $\beta^E = 1.5$
  - Debt beta $\beta^D = 0.21$ from Fama & French (1993)

CAPM Applications: Discounted Cash Flows (DCF) V

- Assume a market equity premium of 7%:

$$r_M - r_f = 7\%$$

- Now we can compute r^D and r^E:

$$r^D = r_f + \beta^D \times (r_M - r_f) = 3\% + 0.21 \times 7\% = 4.47\%$$
$$r^E = r_f + \beta^E \times (r_M - r_f) = 3\% + 1.5 \times 7\% = 13.5\%$$

- Finally, r is given by the WACC formula:

$$\begin{aligned} r &= r^D (1-\tau) \frac{D}{D+E} + r^E \frac{E}{D+E} \\ r &= 4.47\% \times 65\% \times 0.44 + 13.5\% \times 0.56 \\ r &= 8.84\% \end{aligned}$$

CAPM Applications: Performance Evaluation

- Suppose an investment strategy has had a high average return over many years. Is it due to true ability, or to the investor's taking more risk (hence higher expected return)?
- CAPM gives a way to decide: regress excess strategy/portfolio returns on excess market returns

$$\tilde{r}_{P,t} - \tilde{r}_{f,t} = \alpha_P + \beta_P(\tilde{r}_{M,t} - \tilde{r}_{f,t}) + \tilde{\epsilon}_{P,t}$$

- If $\alpha_P > 0$, this is evidence that the investor has stock-picking ability (assuming that CAPM holds)
- Why? Take expectations of both sides to get

$$\alpha_P = E_P - ((r_f + \beta_P(E_M - r_f))$$

- So α_P is the amount by which the portfolio return has exceeded its fair rate of return
- α_P is also known as **Jensen's alpha**, and it's often used to evaluate mutual funds

Does the CAPM Work? (I)

- Recommended Reading: "Does the Capital Asset Pricing Model Work?" (CoursePack)
- Average rates of return differ widely from asset to asset
  - Example: stocks versus bonds
- In theory, these differences should be due to differences in risk
  - Riskier assets should have higher *average* returns to compensate us
- Problem is, what is risk exactly?
  - The CAPM claims, risk = beta
  - Beta should explain all differences in average returns
- How do we test the CAPM?
  - Take a large number of stocks over some time period (say, 1975–1979) and estimate their betas
  - Compute each stock's average return over the following years (say, 1980–1984)
  - Do average returns line up with betas?

Does the CAPM Work? (II)

- The results are mixed:
  - Higher beta stocks do earn higher returns on average, but not that much higher
  - Low beta stocks have higher returns than the formula predicts
  - High beta stocks have lower returns than the formula predicts
- Also, a number of stock characteristics explain average returns better than beta
- Fama and French (1992) find even weaker results:
  - First, assign stocks into bins based on two characteristics:
    1. Firm size (market capitalization)
    2. Book value of equity-to-market value of equity ratio
  - Then, plot returns against betas within each bin

    $\rightarrow$ <u>No</u> relationship between β and returns!
  - Fama and French declared beta as being dead

Does the CAPM Work? (III)

- There is an important problem in testing the CAPM:
  - Roll (1977): CAPM is untestable because the *true* market portfolio is not observable
  - This is the Roll's critique or the Joint Hypothesis Problem
  - For example, if we use S&P 500 index as a proxy for the market index, we can only test for S&P 500's mean-variance efficiency
  - We therefore never know if the CAPM is wrong or if our market proxy is just bad
  - We need good estimates of risk-free return, beta, market risk premium, as well as the market portfolio
- Forty years later, CAPM is still an active area of research
  - For example: what if betas vary over time with business conditions?
- **Note:** No matter what we think, CAPM is still widely used

Resurrecting the CAPM I

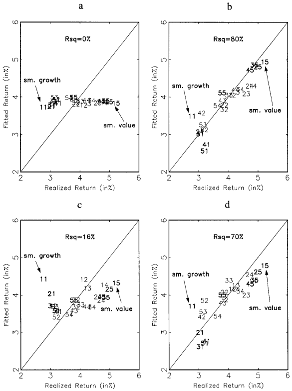


FIG. 1.—Realized vs. fitted returns: 25 Fama-French portfolios: *a*, CAPM; *b*, Fama-French; *c*, consumption CAPM; *d*, consumpion CAPM scaled. The figure shows the pricing errors for each of the 25 Fama-French portfolios for the four models. Each two-digit number represents one portfolio. The first digit refers to the size quintiles (1 indicating the smallest firms, 5 the largest), and the second digit refers to book-to-market quintiles (1 indicating the portfolio with the lowest book-to-market ratio, 5 with the highest). The pricing errors are generated using the Fama-MacBeth regressions in table 3 below. The scaling variable is $\widehat{cay}$.

Source: Lettau and Ludvigson (2001) Resurrecting the (C)CAPM: A Test When Risk Premia Are Time-Varying

Resurrecting the CAPM II

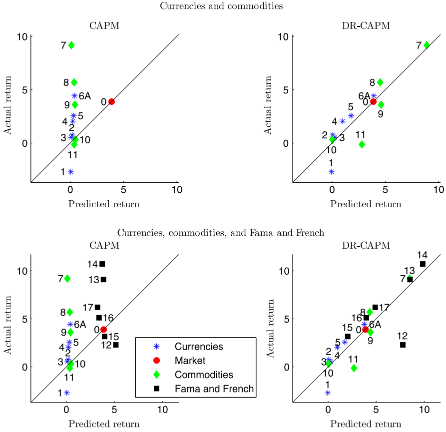


Fig. 8. Model performance: currencies, equities, and commodities.
Depicted are annualized mean excess returns versus the predicted excess returns in percent for the unconditional capital asset pricing model (CAPM) in the left panels and the downside risk CAPM (DR-CAPM) in the right panels for six currency portfolios (1 to 6A), monthly re-sampled based on the interest rate differential with the US, five commodity futures portfolios monthly re-sampled based on basis (7 to 11) as well as the six Fama and French portfolios sorted on size and book-to-market (12 to 17). The market excess return is included as a test asset (0). High inflation countries in the last currency portfolio are excluded. A country is considered to have high inflation if it has an annualized monthly inflation of 10% higher than US inflation. The sample period is January 1974 to December 2008 for a total of 420 observations.

Source: Lettau, Maggiori, Weber (2014) Conditional Risk Premia

What We Learned Today

- We studied the case of multiple risky assets and derived the investment opportunity set
- We saw that all investors hold the tangency portfolio
- The tangency portfolio has the highest Sharpe ratio
- Market equilibrium implies that the market portfolio is efficient
- We studied applications of the CAPM and the potential pitfalls

Additional (Self-)Study Problems

- BKM 10^{th} edition end of chapter problems 4, 8, 9, 17, 18, 19
- BKM 10^{th} edition end of chapter CFA problems 4, 5, 7, 12