Differential Equations

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Contents

1 Definition

Differential equations are equations where the solution is a function or a set of functions.

1.1 Order

The *order* of a differential equation is the largest derivative present in the differential equation

1.2 Types

Ordinary differential equations are equations with only ordinary derivatives in them, whilst *partial differential equations* have partial derivatives in them.

1.3 Linearity

A differential equation is said to be linear if it can be written as

$$\sum_n a_n(t) \frac{d^n}{dt^n} y(t) = g(t)$$

where there are no products of the function $y(t)$ and its derivatives, $y(t)$ or its derivative do not occur to any power other than the first power and $y(t)$ or any of its derivative are composed with another function.

2 First-Order Differential Equations

A first-order differential equation is a differential equation in the form

$$y'(t) = f(t, y(t))$$

where f is given.

The equation is said to be *linear* if f is linear on the second argument.

$$y'(t) = a(t)y(t) + b(t)$$

The equation is also said to be *constant* if a and b are also constant.

2.1 Constant Coefficients Differential Equations

Theorem. *The general solution to the constant differential equation*

$$y' = ay + b, \quad a \neq 0$$

is given by

$$y(t) = Ce^{at} - \frac{b}{a}, \quad C \in \mathbb{R}$$

Proof. Let's first consider the case when $b = 0$,

$$y' = ay$$

We divide both sides by y and simplify

$$\frac{y'}{y} = a \implies \ln|y|' = a \implies \ln|y| = at + c_0$$

concluding that

$$y = \pm e^{at+c_0} = \pm e^{c_0} \cdot e^{at} = Ce^{at}$$

Now let's consider $b \in \mathbb{R}$

$$y' = a\left(y + \frac{b}{a}\right) \implies \left(y + \frac{b}{a}\right)' = a\left(y + \frac{b}{a}\right)$$

Note that $\frac{d}{dx}\left(\frac{b}{a}\right) = 0$
Denoting $\tilde{y} = y + \frac{b}{a}$, we have

$$\tilde{y} = a\tilde{y}$$

which has solution Ce^{at}, hence

$$y + \frac{b}{a} = Ce^{at}$$
$$y = Ce^{at} - \frac{b}{a}$$

∎

It is important to note that we solved the equation by turning it into a total derivative, which is simple to integrate ($\ln|y|' = a$). This function is called a *potential function* (ψ) and it's how the equation is transformed into a total derivative

$$y' = ay + b \to \psi(t, y(t))' = 0$$

In this case

$$\psi = \ln|y| - at$$

The Integrating Factor Method The integrating factor method is a method for solving linear differential equations.
We will prove the theorem again using this method.

Proof. We choose an integrating factor to be a function μ such that

$$\mu' = -a\mu$$

By solving this differential equation we get

$$\frac{\mu'}{\mu} = -a \implies \ln|\mu| = -at + C \implies \mu(t) = Ce^{-at}$$

Now we multiply the equation by μ

$$\begin{aligned} y' - ay &= b \\ y'\mu - \mu ay &= b\mu \\ y'\mu + \mu' y &= b\mu \\ (\mu y)' &= \mu b \end{aligned}$$

Now choosing $C = 1$

$$\begin{aligned} \left(e^{-at}y\right)' &= be^{-at} \\ \left(e^{-at}y\right)' &= \left(-\frac{b}{a}e^{-at}\right)' \\ \left(e^{-at}y + \frac{b}{a}e^{-at}\right)' &= 0 \\ \left[\left(\frac{b}{a} + y\right)e^{-at}\right]' &= 0 \end{aligned}$$

Now the differential equation is a total derivative of the potential function, which in this case in

$$\psi(t, y) = \left(\frac{b}{a} + y\right)e^{-at}$$

This is easy to integrate

$$\left(\frac{b}{a} + y\right)e^{-at} = C \implies y = Ce^{at} - \frac{b}{a}$$

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2.2 Constant Coefficients Differential Equations with Initial Point

We want to constraint the equation such that it has an unique solution rather than infinite solutions.

$$y' = ay + b, \quad y(t_0) = y_0$$

Theorem. *The general solution to the ordinary constant differential equation*

$$y' = ay + b,$$

with a given point

$$y(t_0) = y_0$$

is given by

$$y(t) = \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a}, \quad a \neq 0$$

Proof. Starting from the general solution of a constant ordinary differential equation

$$y(t_0) = y_0 = Ce^{at_0} - \frac{b}{a}$$

meaning that

$$C = \left(y_0 + \frac{b}{a}\right) e^{-at_0}$$

this constraints out result to

$$\begin{aligned} y(t) &= Ce^{at} - \frac{b}{a} \\ &= \left(y_0 + \frac{b}{a}\right) e^{-at_0} e^{at} - \frac{b}{a} \\ &= \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a} \end{aligned}$$

■

2.3 Linear Coefficients Differential Equations

An ordinary linear differential equation with variable coefficients is defined as

$$y' = a(t)y(t) + b(t)$$

Where a and b are continuous functions. The solution to the linear equation with constant coefficients still applies if $\frac{b}{a}$ is constant.

Theorem. *The general solution to the differential equation*

$$y' = a(t)y(t) + b(t)$$

is given by

$$y(t) = Ce^{A(t)} + e^{A(t)} \int e^{-A(t)} b(t)\, dt$$

where $A(t) = \int a\, dt$, $c \in \mathbb{R}$ and a and b are continuous.

Proof. Let's start by letting $b(t) = 0$

$$y' = ay \implies \frac{y'}{y} = a \implies \ln|y|' = a \implies \ln|y| = \int a\, dt$$

concluding that

$$y = \pm e^{A+c_0} = \pm e^{c_0} \cdot e^A = Ce^A$$

where $A = \int a\,dt$.
As previosuly, we choose an integrating factor μ such that

$$-a\mu = \mu'$$

By solving this differential equation we get

$$\frac{\mu'}{\mu} = -a \implies \ln|\mu| = -A + C \implies \mu(t) = Ce^{-A}$$

And by choosing $C = 1$ we have

$$\mu(t) = e^{-A(t)}$$

Now we multiply our equation by the integrating factor

$$\begin{aligned}
y' - ay &= b \\
y'\mu - a\mu y &= \mu b \\
y'\mu + \mu' y &= \mu b \\
(y\mu)' &= \mu b \\
\left(e^{-A(t)}y\right)' &= e^{-A(t)}b \\
e^{-A(t)}y &= \int e^{-A(t)}b\,dt + C \\
y(t) &= Ce^{A(t)} + e^{A(t)}\int e^{-A(t)}b\,dt
\end{aligned}$$

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2.4 Linear Coefficients Differential Equations with Initial Point

A linear differential equation with variable coefficients and an initial point is given by

$$y'(t) = a(t)y(t) + b(t), \quad y(t_0) = y_0$$

Theorem. *The general solution to the differential equation*

$$y' = a(t)y(t) + b(t)$$

where a and b are continuous functions, with a given point

$$y(t_0) = y_0$$

is given by

$$y(t) = y_0 e^{A(t)} + e^{A(t)} \int_{t_0}^{t} e^{-A(s)} b(s)\,ds$$

Proof. ■

2.5 Bernoulli Equation

The Bernoulli equation has the form

$$y' = p(t)y + q(t)y^n$$

Theorem. *The general solution of the Bernoulli equation is the general solution of the linear equation*

$$v' = -(n-1)p(t)v - (n-1)q(t)$$

where

$$v = \frac{1}{y^{n-1}}$$

Proof. They idea is to transform this equation into a simplier linear first-order equation.
Start by dividing both sides by y^n

$$\frac{y'}{y^n} = \frac{p(t)}{y^{n-1}} + q(t)$$

Let

$$v = y^{-(n-1)}, \quad v' = -(n-1)y^{-n}y'$$

Thus

$$-\frac{v'}{n-1} = \frac{y'(t)}{y^n(t)}$$

By substituting we get

$$\begin{aligned} -\frac{v'}{n-1} &= p(t)v + q(t) \\ v' &= -(n-1)p(t)v - (n-1)g(t) \end{aligned}$$

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2.6 Separable equations

Separable equations are equations that can be solved by integrating both sides. This doesn't generally work with first-order linear equations. A separable equation has the form

$$h(y)y' = g(t)$$

Theorem. *A separable differential equation has an implicit solution*

$$H(y(t)) = G(t) + C$$

where

$$H(y) = \int h(s)\,ds, \quad G(t) = \int g(t)\,dt$$

Proof. Start by integrating both sides of the equation

$$\int h(y(t))y'(t)\,dt = \int g(t)\,dt + C$$

Now substitute for

$$s = y(t), \quad ds = y'(t)\,dt$$

meaning

$$\int h(s)\,ds = \int g(t)\,dt$$

which could be written as

$$H(y) = G(t) + C$$

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2.7 Exact equations

Consider a differential equation with the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Then, the equation is *exact* if there exist a continuously differentiable function $\Psi(x,y)$ such that

$$\frac{\partial \Psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = N(x,y)$$

We can then rewrite the differential equation as

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} = 0$$

Using the multi variable chain rule it can be reduced to

$$\frac{d}{dx}\left(\Psi(x, y(x))\right) = 0$$

We can clearly see that here the derivative is equal to 0, meaning that the function must be a constant. This gives us an implicit solution

$$\Psi(x,y) = C$$

2.8 Homogeneous equations

A *homogenous* equation has the form

$$\frac{dy}{dx} = F(\frac{y}{x})$$

We use the substitution

$$v = \frac{y}{x}$$

Note that

$$\begin{aligned} y' &= (xv)' = v + yv' \\ &= F(v) \end{aligned}$$

We then have

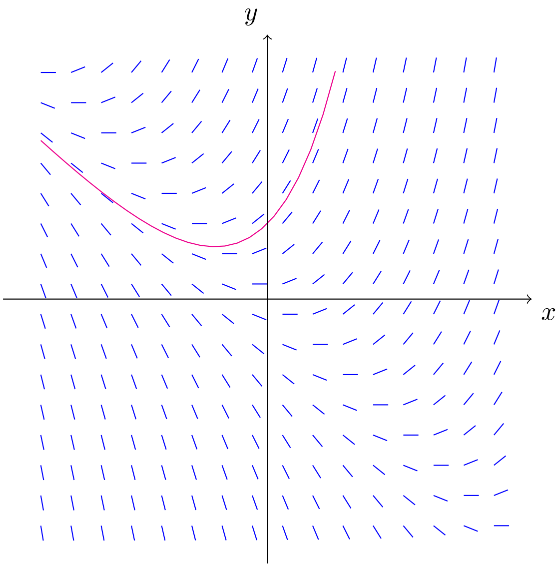
$$\begin{aligned} v + xv' &= F(v) \\ xv' &= F(v) - v \\ \frac{v'}{F(v) - v} &= \frac{1}{x} \end{aligned}$$

This is a separable equation. Thus, an implicit solution is given by

$$\int \frac{1}{F(v) - v} \, dv = \ln|x| + C$$

2.9 Slope Field

A slope field or directional field is a field to visualize solutions to a first-order differential equation.



Slope field of $\frac{dy}{dx} = x + y$.

This field is obtained by picking points on the plane. For each point (x, y) we know that the slope $(\frac{dy}{dx})$ is $x + y$.
This means that if a solution passes through (x, y), then its slope is $x + y$.
The red curve shows a solution.

2.10 Euler's Method

Euler's method is a technique for solving a first-order differential equation numerically given a point of the solution.
Starting at the known solution point A_0, we take small steps the direction of the slope field. As the length of the steps $s \to 0$ we approach the solution to the equation.

3 Second-Order Differential Equations

A second-order differential equation has the form

$$y''(t) + a(t)y'(t) + b(t)y(t) = f(t)$$

if $f(t) = 0$ then the equation is said to be *homogeneous*.

4 Laplace Transform

Given a piecewise continuous function $f(t)$, the Laplace transform is defined as

$$\mathcal{L}\{f(t)\} = \int\limits_0^\infty e^{-st} f(t)\, dt = F(s)$$

4.1 Properties of the Laplace Transform

It is easy to see that given $f(t)$ and $g(t)$

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

for any constants a and b.

4.2 Inverse Laplace Transform

The Inverse Laplace Transform is defined as

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

4.3 Properties of the Inverse Laplace Transform

Given the Laplace transforms $F(s)$ and $G(s)$

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$$

for any constants a and b.

4.4 Heaviside function

A consider a function in the form $H(t-c)f(t-c)$ where H is the Heaviside step function, meaning $f(t)$ is shifted by c and is 0 for $t < c$.

$$\begin{aligned}
\mathcal{L}\{H(t-c)f(t-c)\} &= \int\limits_0^\infty e^{-st} H(t-c) f(t-c)\, dt \\
&= \int\limits_c^\infty e^{-st} f(t-c)\, dt
\end{aligned}$$

Now substitue for $u = t - c$

$$\begin{aligned}
\int\limits_0^\infty e^{-s(u+c)} f(u)\, du &= \int\limits_0^\infty e^{-su} e^{-cs} f(u)\, du \\
&= e^{-cs} \int\limits_0^\infty e^{-su} f(u)\, du
\end{aligned}$$

Concluding that

$$\mathcal{L}\{H(t-c)f(t-c)\} = e^{-cs}F(s) \quad \text{and} \quad \mathcal{L}^{-1}\{e^{-cs}F(s)\} = H(t-c)f(t-c)$$

4.5 Laplace Transform of derivatives

Let f', f'', $\cdots$, $f^{(n-1)}$ be continuous functions and $f^{(n)}$ a piecewise continuous functions. Then,

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - \sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$$

4.6 Solving Initial value problems

Given an initial value problem differential equation, we can apply the Laplace transform on both sides of the equation. After applying the initial conditions $(y(0), y'(0), \cdots)$, the differential equation is transformed into an algebraic equation.

Note that if the initial conditions are not expressed as $y(0), y'(0), \cdots$, we need to first make a change of variable.

We can then isolate $\mathcal{L}\{y\}$ in the equation and get

$$\mathcal{L}\{y\} = M$$

Then apply the inverse Laplace transform to solve the equation

$$y = \mathcal{L}^{-1}\{M\}$$