Complex Analysis

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1 De Moivre's Theorem

Using the property of exponentiation $\left(a^b\right)^c = a^{bc}$, we can see that $\left(e^{i\theta}\right)^n = e^{in\theta}$.
Using Euler's formula we can deduce that

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta), \quad n \in \mathbb{Z}$$

2 Nth Roots of Units

We can extend De Moivre's Theorem for the integers powers or any complex number, rather than the ones on the unit circle $(r = 1)$.

$$(r\,(\cos(\theta) + i\sin(\theta)))^n = r^n\,(\cos(n\theta) + i\sin(n\theta)), \quad n \in \mathbb{Z}$$

The nth roots of 1 are the solutions to

$$x^n = 1$$

for a given n. We might write 1 as a complex number

$$x^n = \cos(0) + i\sin(0)$$

Comparing this to our extended De Moivre's theorem

$$\cos(0) + i\sin(0) = r^n\,(\cos(n\theta) + i\sin(n\theta))$$

We can see that

$$r^n = 1$$
$$n\theta = 0$$

As long as $n \neq 0$

$$r = 1$$
$$\theta = 0$$

By plugging these values into

$$x^n = (r\,(\cos(\theta) + i\sin(\theta)))^n$$

we get that $x = 1$.

However we could also write 1 as

$$\cos(2k\pi) + i\sin(2k\pi), \quad k \in \mathbb{Z}$$
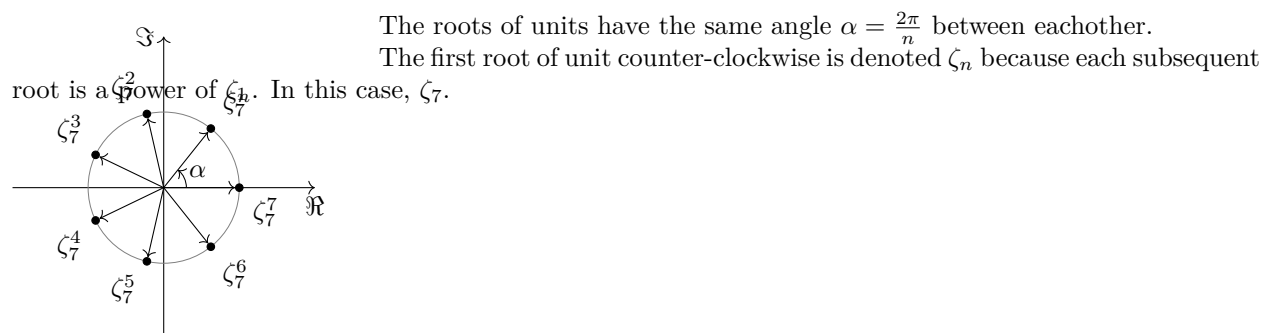
We would then get that

$$r^n = 1$$
$$n\theta = 2k\pi$$

When solving for x again we get

$$x^n = (r\left(\cos(\theta) + i\sin(\theta)\right))^n$$
$$= \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)^n$$

concluding that

$$x = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

This gives us a solution for each k, however the solutions are redundant for $k \geq n$. In fact, the roots of unity of n are n distinct solutions (points on the unit circle).

The roots of units have the same angle $\alpha = \frac{2\pi}{n}$ between eachother.
The first root of unit counter-clockwise is denoted ζ_n because each subsequent root is a power of ζ_n. In this case, ζ_7.

3 Riemann Spheres

A Riemann sphere is a unit sphere used to represent the complex plane using stereographic projection.

TODO | plot

The Riemann sphere lays on the complex plane. A complex number is represented by the intersection between the sphere and a ray starting from the topmost point of the sphere and intersecting with the given complex number on the complex plane.