Differential Equations

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1 Definition

Differential equations are equations where the solution is a function or a set of functions.

2 First-Order Differential Equations

A first-order differential equation is a differential equation in the form

$$y'(t) = f(t, y(t))$$

where f is given.

The equation is said to be *linear* iff f is linear on the second argument.

$$y'(t) = a(t)y(t) + b(t)$$

The equation is also said to be *constant* iff a and b are also constant.

2.1 Linear Constant Coefficients Differential Equations

Theorem. *The general solution to the constant differential equation*

$$y' = ay + b, \quad a \neq 0$$

is given by

$$y(t) = Ce^{at} - \frac{b}{a}, \quad C \in \mathbb{R}$$

Proof. Let's first consider the case when $b = 0$,

$$y' = ay$$

We divide both sides by y and simplify

$$\frac{y'}{y} = a \implies \ln|y|' = a \implies \ln|y| = at + c_0$$

concluding that

$$y = \pm e^{at+c_0} = \pm e^{c_0} \cdot e^{at} = Ce^{at}$$

Now let's consider $b \in \mathbb{R}$

$$y' = a\left(y + \frac{b}{a}\right) \implies \left(y + \frac{b}{a}\right)' = a\left(y + \frac{b}{a}\right)$$

Note that $\frac{d}{dx}\left(\frac{b}{a}\right) = 0$
Denoting $\tilde{y} = y + \frac{b}{a}$, we have

$$\tilde{y} = a\tilde{y}$$

which has solution Ce^{at}, hence

$$y + \frac{b}{a} = Ce^{at}$$
$$y = Ce^{at} - \frac{b}{a}$$

■

It is important to note that we solved the equation by turning it into a total derivative, which is simple to integrate ($\ln|y|' = a$). This function is called a *potential function* (ψ) and it's how the equation is transformed into a total derivative

$$y' = ay + b \to \psi(t, y(t))' = 0$$

In this case

$$\psi = \ln|y| - at$$

The Integrating Factor Method The integrating factor method is a method for solving linear differential equations.
We will prove the theorem again using this method.

Proof. We choose an integrating factor to be a function μ such that

$$\mu' = -a\mu$$

By solving this differential equation we get

$$\frac{\mu'}{\mu} = -a \implies \ln|\mu| = -at + C \implies \mu(t) = Ce^{-at}$$

Now we multiply the equation by μ

$$\begin{aligned} y' - ay &= b \\ y'\mu - \mu a y &= b\mu \\ y'\mu + \mu' y &= b\mu \\ (\mu y)' &= \mu b \end{aligned}$$

Now choosing $C = 1$

$$\begin{aligned} \left(e^{-at}y\right)' &= be^{-at} \\ \left(e^{-at}y\right)' &= \left(-\frac{b}{a}e^{-at}\right)' \\ \left(e^{-at}y + \frac{b}{a}e^{-at}\right)' &= 0 \\ \left[\left(\frac{b}{a} + y\right)e^{-at}\right]' &= 0 \end{aligned}$$

Now the differential equation is a total derivative of the potential function, which in this case in

$$\psi(t, y) = \left(\frac{b}{a} + y\right)e^{-at}$$

This is easy to integrate

$$\left(\frac{b}{a} + y\right)e^{-at} = C \implies y = Ce^{at} - \frac{b}{a}$$

■

2.2 Linear Constant Coefficients Differential Equations with Initial Point

We want to constraint the equation such that it has an unique solution rather than infinite solutions.

$$y' = ay + b, \quad y(t_0) = y_0$$

Theorem. *The genera solution to the ordinary constant differential equation*

$$y' = ay + b,$$

with a given point

$$y(t_0) = y_0$$

is given by

$$y(t) = \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a}, \quad a \neq 0$$

Proof. Starting from the general solution of a constant ordinary differential equation

$$y(t_0) = y_0 = Ce^{at_0} - \frac{b}{a}$$

meaning that

$$C = \left(y_0 + \frac{b}{a}\right) e^{-at_0}$$

this constraints out result to

$$\begin{aligned} y(t) &= Ce^{at} - \frac{b}{a} \\ &= \left(y_0 + \frac{b}{a}\right) e^{-at_0} e^{at} - \frac{b}{a} \\ &= \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a} \end{aligned}$$

∎

2.3 Linear Variable Coefficients Differential Equations

An ordinary linear differential equation with variable coefficients is defined as

$$y' = a(t)y(t) + b(t)$$

The solution to the linear equation with constant coefficients still applies if $\frac{b}{a}$ is constant.

Theorem. *The general solution to the differential equation*

$$y' = a(t)y(t) + b(t)$$

is given by

$$y(t) = Ce^{A(t)} + e^{A(t)} \int e^{-A(t)} b(t)\, dt$$

where $A(t) = \int a\, dt$*,* $c \in \mathbb{R}$ *and* a *and* b *are continuous.*

Proof. Let's start by letting $b(t) = 0$

$$y' = ay \implies \frac{y'}{y} = a \implies \ln|y|' = a \implies \ln|y| = \int a\,dt$$

concluding that

$$y = \pm e^{A+c_0} = \pm e^{c_0} \cdot e^A = Ce^A$$

where $A = \int a\,dt$.
As previosuly, we choose an integrating factor μ such that

$$-a\mu = \mu'$$

By solving this differential equation we get

$$\frac{\mu'}{\mu} = -a \implies \ln|\mu| = -A + C \implies \mu(t) = Ce^{-A}$$

And by choosing $C = 1$ we have

$$\mu(t) = e^{-A(t)}$$

Now we multiply our equation by the integrating factor

$$\begin{aligned}
y' - ay &= b \\
y'\mu - a\mu y &= \mu b \\
y'\mu + \mu' y &= \mu b \\
(y\mu)' &= \mu b \\
\left(e^{-A(t)}y\right)' &= e^{-A(t)}b \\
e^{-A(t)}y &= \int e^{-A(t)}b\,dt + C \\
y(t) &= Ce^{A(t)} + e^{A(t)}\int e^{-A(t)}b\,dt
\end{aligned}$$

∎

2.4 Linear Variable Coefficients Differential Equations with Initial Point

A linear differential equation with variable coefficients and an initial point is given by

$$y'(t) = a(t)y(t) + b(t), \quad y(t_0) = y_0$$

Theorem. *TODO*

2.5 Bernoulli Equation

The Bernoulli equation has the form

$$y' = p(t)y + q(t)y^n$$

Theorem. *The general solution of the Bernoulli equation is the general solution of the linear equation*

$$v' = -(n-1)p(t)v - (n-1)q(t)$$

where

$$v = \frac{1}{y^{n-1}}$$

Proof. They idea is to transform this equation into a simplier linear first-order equation.
Start by dividing both sides by y^n

$$\frac{y'}{y^n} = \frac{p(t)}{y^{n-1}} + q(t)$$

Let

$$v = y^{-(n-1)}, \quad v' = -(n-1)y^{-n}y'$$

Thus

$$-\frac{v'}{n-1} = \frac{y'(t)}{y^n(t)}$$

By substituting we get

$$\begin{aligned} -\frac{v'}{n-1} &= p(t)v + q(t) \\ v' &= -(n-1)p(t)v - (n-1)g(t) \end{aligned}$$

∎

2.6 Separable equations

Separable equations are equations that can be solved by integrating both sides. This doesn't generally work with first-order linear wquations. A separable equation has the form

$$h(h)h' = g(t)$$

Theorem. *A separable differential equation has solutions where*

$$H(y(t)) = G(t) + C$$

where

$$H(y) = \int h(s)\, ds, \quad G(t) = \int g(t)\, dt$$

Proof. Start by integrating both sides of the equation

$$\int h(y(t))y'(t)\, dt = \int g(t)\, dt + C$$

Now substitute for

$$s = y(t), \quad ds = y'(t)\, dt$$

meaning

$$\int h(s)\, ds = \int g(t)\, dt$$

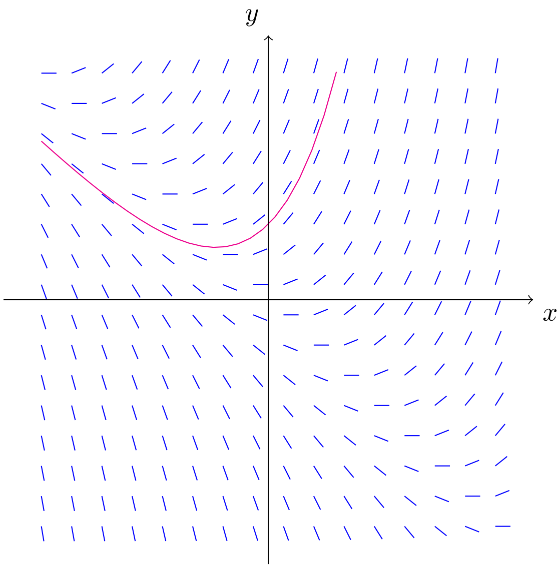
which could be written as

$$H(y) = G(t) + C$$

∎

3 Slope Field

A slope field or directional field is a field to visualize solutions to a first-order differential equation.



Slope field of $\frac{dy}{dx} = x + y$.

This field is obtained by picking points on the plane.
For each point (x, y) we know that the slope $(\frac{dy}{dx})$ is $x+y$.
This means that if a solution passes through (x, y), then its slope is $x + y$.
The red curve shows a solution.

4 Euler's Method

Euler's method is a technique for solving a first-order differential equation numerically given a point of the solution.
Starting at the known solution point A_0, we take small steps the direction of the slope field. As the length of the steps $s \to 0$ we approach the solution to the equation.
The angle of the slope is given by

$$\theta = \tan\left(\frac{dy}{dx}\right)$$

so each step gives the sequence of points

$$A_n = A_{n-1} \cdot s\,(\cos(\theta), \sin(\theta))$$