Limits

Paolo Bettelini

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1 Definition

A limit is usually used to describe the behavior of a function as its argument approaches a given value. The limit towards a certain value c within a function can be be approached both from the right and from the left. The limit in a general sense exists if the value approached from both sides is the same and well-defined. We define the limit of x approaching c from the left within the function $f(x)$ as

$$\lim_{x \to c^-} f(x)$$

We define the limit of x approaching c from the right within function $f(x)$ as

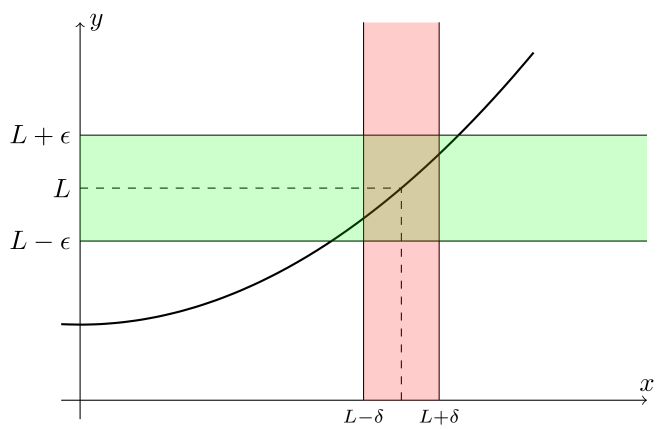
$$\lim_{x \to c^+} f(x)$$

We define the limit of x approaching c within function $f(x)$ as

$$\lim_{x \to c} f(x)$$

Formally, given a function $f : D \to \mathbb{R}$ the limit $L = \lim_{x \to c} f(x)$ exists if given an arbitrary small $\epsilon > 0$ there is another number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ when } 0 < |x - c| < \delta$$



This means that for any x in the red region $0 < |x - c| < \delta$ or $|x - c| \in (0; \delta)$, the function at that point will lie in the yellow region. This value is closer to L than either $L + \epsilon$ or $L - \epsilon$

$$|f(x) - L| < \epsilon$$

Notice that this defintion does not require f to be defined at c, but rather just around c.

We can also use this definition for limits from the right and from the left.

The right-hand limit $L = \lim_{x \to c^+} f(x)$ exists if for any arbitrary small $\epsilon > 0$ there is some $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ when } 0 < x - c < \delta$$

The left-hand limit $L = \lim_{x \to c^-} f(x)$ exists if for any arbitrary small $\epsilon > 0$ there is some $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ when } -\delta < x - c < 0$$

2 Infinite Limits and Limits and Infinity

The limit

$$\lim_{x \to c} f(x) = \infty$$

diverges to ∞ iff we can make it arbitrarily large for all x sufficiently close to c, without actually letting $x = a$. In other words iff

$$\forall M \in \mathbb{R} \exists \delta > 0 | f(x) > M \text{ when } 0 < |x - a| < \delta, x \neq a$$

meaning that we can shrink the region around the limit such that its value (expect when $x = a$) will always be greater than any number.

The same applies for the limit

$$\lim_{x \to c} f(x) = -\infty$$

where it diverges to $-\infty$ when

$$\forall M \in \mathbb{R} \exists \delta > 0 | f(x) < M \text{ when } 0 < |x - a| < \delta, x \neq a$$

These functions present a vertical asymptote at $x = a$.

Limits can approach values that are ∞ or $-\infty$. If the limit converges they will have an horizontal asymptote at $y = L$.

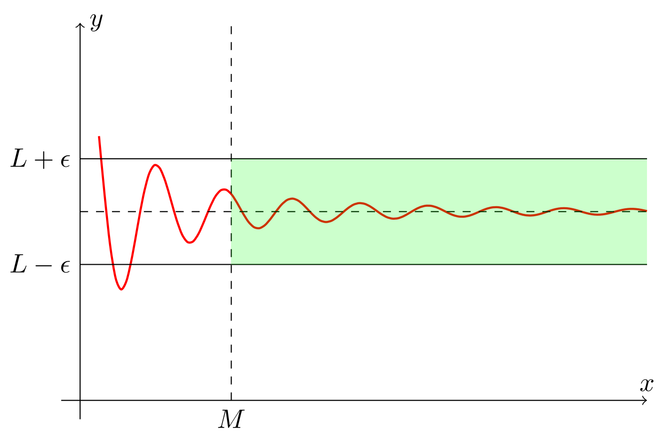
$$\lim_{x \to \infty} f(x) = L \quad \lim_{x \to -\infty} f(x) = L$$

The limit

$$\lim_{x \to \infty} f(x) = L$$

converges to L iff for every $\epsilon > 0$ there exists a $M > 0$ such that

$$|f(x) - L| < \epsilon \text{ when } x > M$$



Likewise, the limit

$$\lim_{x \to -\infty} f(x) = L$$

converges to L iff for every $\epsilon > 0$ there exists a $M > 0$ such that

$$|f(x) - L| < \epsilon \text{ when } x < M$$

Limits at infinities may also diverge to infinities

$$\lim_{x\to\infty} = \infty \text{ iff } \forall N \exists M > 0 | f(x) > N, x > M$$
$$\lim_{x\to\infty} = -\infty \text{ iff } \forall N \exists M > 0 | f(x) < N, x > M$$
$$\lim_{x\to-\infty} = \infty \text{ iff } \forall N \exists M > 0 | f(x) > N, x < M$$
$$\lim_{x\to-\infty} = -\infty \text{ iff } \forall N \exists M > 0 | f(x) < N, x < M$$

3 Properties

If the limit exists

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$$
$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$$
$$\lim_{x\to c} f(x) \pm g(x) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$$
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

4 Squeeze Theorem

Let $h(x)$, $f(x)$ and $g(x)$ be three functions such that $h(x) \leq f(x) \leq g(x)$.
If

$$\lim_{x\to x_0} g(x) = f(x) = L$$

then

$$\lim_{x\to x_0} f(x) = L$$