Riemann Hypothesis

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Abstract

This document contains the main concepts about the Riemann Hypothesis and some derivations of the formulas and series used.

Contents

1 Zeta function

1.1 Definition

The zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1$$

1.2 Euler product

The zeta function can be represented as an Euler product.
We will start by using the first prime number: 2, and multiply both sides by 2^{-s}.

$$\zeta(s)\frac{1}{2^s} = \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

We then subtract the second definition from the first one, such that

$$\begin{aligned}\zeta(s) - \zeta(s)\frac{1}{2^s} &= \sum_{n=1}^{\infty} \left[\frac{1}{n^s}\right] - \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^s}\right] \\ \zeta(s)\left(1 - \frac{1}{2^s}\right) &= \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \neq 2k, k \in \mathbb{Z}\end{aligned}$$

Here we are excluding the multiples of 2 from the series.

If we do the same with the next prime number, which is 3, we get

$$\zeta(s)\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \neq 2k, n \neq 3k, k \in \mathbb{Z}$$

We can repeat this process with every prime number.
Eventually, we will exclude every nth-term to sum as we use every prime number, except for n=1.

$$\zeta(s) \prod_{p \in P}^{\infty} 1 - \frac{1}{p^s} = \frac{1}{1^s} = 1$$

Finally, we get the identity

$$\zeta(s) = \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}}$$

2 Analytic continuation

2.1 Zeta function for positive Re (s)

We have seen that the classical zeta function definition only converges for $Re(s) > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1$$

We can use the eta function $\eta(s)$, which is defined for $Re(s) > 0 \setminus \{1\}$, to analytically extend the zeta function domain to $Re(s) > 0 \setminus \{1\}$.
The eta function is a Dirichlet series defined as

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad Re(s) > 0 \setminus \{1\}$$

We start by splitting the zeta function into two distinct series, one for n even and the other one for n odd. The index for the even series will be $2n$, while the odd one will use $2n-1$ as the index.

$$\zeta(s) = \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^s}\right] + \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^s}\right]$$

We do the same thing with the eta function.
Notice that $(-1)^n$ is 1 when n is even and -1 when n is odd.

$$\eta(s) = \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^s}\right] - \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^s}\right]$$

We subtract these two definition from eachother

$$\begin{aligned}
\zeta(s) - \eta(s) &= 2\sum_{n=1}^{\infty} \frac{1}{(2n)^s} \\
&= 2^{1-s} \sum_{n=1}^{\infty} \frac{1}{k^s} \\
&= 2^{1-s}\zeta(s) \\
\frac{1}{1-2^{1-s}}\eta(s) &= \zeta(s)
\end{aligned}$$

We finally get

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad Re(s) > 0 \setminus \{1\}$$

This series can be used to compute value of the zeta function along the critical strip $0 < Re(s) < 1$.

2.2 Zeta function for negative Re (s)

2.3 Zeta function for s=0

2.4 Zeta function for s=1

The zeta function is holomorphic everywhere except for a pole at $s = 1$ with residue 1.
This is the only value of the complex plane that cannot be evaluated through analytic continuation.

$$\lim_{s\to 1}\zeta(s) = \lim_{s\to 1}\frac{s-1}{1-s^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$$

This is the alternating harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \ln(2)$$

so

$$\lim_{s\to 1}\frac{s-1}{1-s^{1-s}}\ln(2) = \frac{\ln(2)}{\ln(2)\cdot 2^{1-s}} = \frac{1}{2^{1-s}} = 1$$

3 Zeroes of the zeta function

3.1 Trivial zeroes

Considering the functional equation and analytic continuation of the zeta function

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

We can notice that the term $\sin\left(\frac{\pi s}{2}\right)$ equals 0 when s is a multiple of 2.
However, the gamma function has a pole for every negative integer, this constrains our zeroes to be less or equal to 1.
Furthermore, the zeta function has a pole at $s = 1$, excluding the value $s = 0$ from the zeroes, leaving $\{-2; -4; -6; \cdots\}$

$$\zeta(2k) = 0, \quad k \in \mathbb{Z}^-$$

These zeroes are called trivial zeroes because they are not relevant to the Riemann hypothesis.

3.2 Non-trivial zeroes

The Riemann hypothesis states that every non-trivial zero lies on the critical line $Re(s) = \frac{1}{2}$.

4 Prime-counting function

4.1 Properties of the prime-counting function

The prime-counting function $\pi(x)$ is defined as the number of primes less or equals than x.

We can consider the difference between $\pi(x)$ of two consecutive integers

$$\pi(x) - \pi(x-1) = \begin{cases} 1, & \text{if } x \in P \\ 0, & \text{otherwise} \end{cases}$$

Given a series over all prime numbers, we can extend it to all integers and multiply each term by this difference.
The terms whose index is not a prime number will be multiplied by 0.

$$\sum_{p \in P}^{\infty} f(p) = \sum_{n=2}^{\infty} \left[\pi(n) - \pi(n-1)\right] f(n)$$

Here we start at 2 since there are no prime numbers less than 2.

4.2 Relationship with the zeta function

We have seen that the zeta function can be written as an Euler Product

$$\zeta(s) = \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}}$$

However, we need convert this product into a series in order to apply the identity of the last paragraph.
We can take the natural logarithm of both sides and use the multiplication property

$$\begin{aligned} \ln\left(\zeta(s)\right) &= \ln \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}} \\ &= \sum_{p \in P}^{\infty} \ln\left(\frac{1}{1 - p^{-s}}\right) \\ &= \sum_{p \in P}^{\infty} -\ln\left(1 - p^{-s}\right) \end{aligned}$$

Now we can apply the identity

$$\ln\left(\zeta(s)\right) = \sum_{n=2}^{\infty} -\ln\left(1 - n^{-s}\right)\left[\pi(n) - \pi(n-1)\right]$$

The next goal is to factor out $\pi(n)$

$$\begin{aligned} \ln\left(\zeta(s)\right) &= \sum_{n=2}^{\infty} \left[\pi(n-1)\ln\left(1 - n^{-s}\right)\right] - \sum_{n=2}^{\infty} \left[\pi(n)\ln\left(1 - n^{-s}\right)\right] \\ &= \sum_{n=2}^{\infty} \left[\pi(n)\ln\left(1 - (n+1)^{-s}\right)\right] - \sum_{n=2}^{\infty} \left[\pi(n)\ln\left(1 - n^{-s}\right)\right] \\ &= \sum_{n=2}^{\infty} \pi(n)\left[\ln\left(1 - (n+1)^{-s}\right) - \ln\left(1 - n^{-s}\right)\right] \end{aligned}$$

To simplify further more, we consider the derivative of the function $\ln(1-x^{-s})$.
Using the chain rule we get

$$\frac{d}{dx}\ln\left(1-x^{-s}\right)=\frac{s}{x(x^s-1)}$$

Therefore,

$$\ln\left(1-x^{-s}\right)=\int\frac{s}{x(x^s-1)}\,dx+C$$

Considering $f(x)=\ln(1-x^{-s})$, our series can be expressed as

$$\ln\left(\zeta(s)\right)=\sum_{n=2}^{\infty}\pi(n)\left[f(n+1)-f(n)\right]$$

which can be written as an integral from n to $n+1$

$$\begin{aligned}\ln\left(\zeta(s)\right)&=\sum_{n=2}^{\infty}\pi(n)\int\limits_{n}^{n+1}f'(x)\,dx\\&=\sum_{n=2}^{\infty}\pi(n)\int\limits_{n}^{n+1}\frac{s}{x(x^s-1)}\,dx\\&=\sum_{n=2}^{\infty}\int\limits_{n}^{n+1}\frac{s\pi(x)}{x(x^s-1)}\,dx\end{aligned}$$

Instead of taking the sum of each of these integrals (2 to 3, 3 to 4, ...), we can make a single integral

$$\ln\left(\zeta(s)\right)=s\int\limits_{2}^{\infty}\frac{\pi(x)}{x(x^2-1)}\,dx$$

4.3 Approximations

A pretty good approximation to $\pi(x)$ is

$$li(x) = \int\limits_0^x \frac{dt}{\ln\, t}$$

called the logarithmic integral function

4.4 Exact form

Riemann proved that the exact form the prime counting function is

$$\pi(x) = Re(x) - \sum_p R(x^p)$$

where

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} li(\sqrt[n]{x})$$

and $\mu(x)$ is the Möbius function.
p indexes every non-trivial zero of the zeta function.