Series

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Contents

1 Divergence and convergence

An infinite series converges if the limit of its partial sum sequence also converges, otherwise it diverges.

2 Properties

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$$

3 Covergence theorem

Theorem. *If* $\sum a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$

Proof. Consider the partial sum

$$s_n = \sum_{k=1}^{n} a_k$$

The sequence a_n can now be expressed as

$$a_n = s_n - s_{n-1}$$

Since $\sum a_n$ converges, $\lim_{n\to\infty} s_n = L$ for L finite.
The limit $\lim_{n\to\infty} s_{n-1} = L$ because $n-1 \to \infty$ as $n \to \infty$. This implies the following

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - s_{n-1} = L - L = 0$$

□

4 Divergence test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.

5 Absolute and conditional convergence

A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. This is a stronger type of convergence. Every absolutely convergent series is also convergent.

A series that is convergent but not absolutely convergent is called conditionally convergent.

6 Riemann rearrangement theorem

If a series is conditionally convergent, then its terms can be rearranged such that the series converges to any $r \in \mathbb{R}$ or such that it diverges (to infinity or no finite value). If the series is absolutely convergent then any rearrangement of its terms will converge to the same value.

7 Geometric series

A geometric series is a series of the form

$$\sum_{n=0}^{\infty} r^n$$

meaning that the ratio between two adject terms is constant. This type of series converges for $|r| < 1$ and always converges absolutely.

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

8 Telescoping series

A telescoping series is a series where the terms in the partial sums cancel eachother, leaving a finite number of terms.

For example:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=0}^{\infty} \left[\frac{1}{n+1} - \frac{1}{n+2}\right] = \lim_{N\to\infty} \sum_{n=0}^{N} \left[\frac{1}{n+1} - \frac{1}{n+2}\right] \\
&= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} \\
&= \lim_{N\to\infty} 1 - \frac{1}{n+2} = 1
\end{aligned}$$

9 Harmonic series

The harmonic series is the following digergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

10 Integral test

Let $f(x)$ be a continuous function on $[k;\infty)$ such that it is decreasing and positive on the interval $[N;\infty)$ for some N.

$$\int\limits_{k}^{\infty} f(x)\,dx \text{ converges} \implies \sum_{n=k}^{\infty} f(n) \text{ converges}$$

and

$$\int\limits_{k}^{\infty} f(x)\,dx \text{ diverges} \implies \sum_{n=k}^{\infty} f(n) \text{ diverges}$$