

Category Theory

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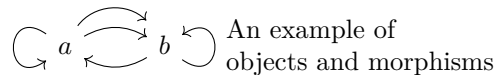
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1 Category

A category consists of *objects* and *morphism* or *arrows*.

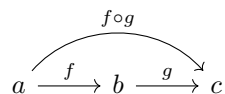
An arrow has a beginning and an ending, and it goes from one object to another.

Objects serve the purpose of marking the beginning and ending of a morphism.



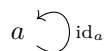
1.1 Composition

Composition is a property that says that if there is an arrow from a to b , and an arrow from b to c , there must exist an arrow from a to c .

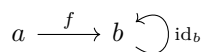


1.2 Identity

For every object there is an identity arrow.



The composition of an arrow with an identity is the arrow itself



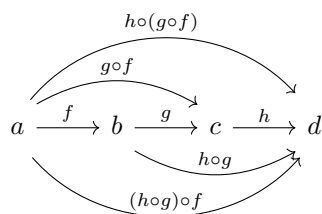
$$f \circ \text{id}_b = f$$

and also vice versa

$$\text{id}_b \circ f = f$$

1.3 Associativity

Compositions have the associative property



$$h \circ (g \circ f) = (h \circ g) \circ f$$

2 Homomorphisms

A homomorphism is a map between two structures of the same type.

2.1 Epimorphisms

An epimorphism is a **surjective** morphism.

We can define surjectivity only in terms of morphisms.

Consider a morphism $f : a \rightarrow b$ which maps elements of a onto b . Let's also define the morphisms g_1 and g_2 which map elements from b to c . The domain of g_1 and g_2 is the codomain of f . These two functions act as f for object in the image of f , but may map objects differently for objects in the codomain of f but outside the image of f . If the morphism is surjective, hence if the codomain and the image of f are the same, then g_1 and g_2 will always act as f .

$$a \xrightarrow{f} b \xrightarrow[g_2]{g_1} c$$

formally,

$$\forall c \forall g_1, g_2 : b \rightarrow c, g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

An epimorphism is labelled with \twoheadrightarrow .

2.2 Monomorphisms

An epimorphism is an **injective** morphism.

$$c \xrightarrow[g_2]{g_1} a \xrightarrow{f} b$$

A morphism $f : a \rightarrow b$ is a monomorphism if

$$\forall c \forall g_1, g_2 : c \rightarrow a, f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

A monomorphism is labelled with \hookrightarrow .

2.3 Isomorphisms

An isomorphism is a **bijective** morphism (mono and epic, but not every mono and epic is an isomorphism).

A morphism $f : a \rightarrow b$ is invertible if there is a function g that goes from b to a

$$b : b \rightarrow a$$

such that

$$g \circ f = \text{id}_b$$

$$f \circ g = \text{id}_a$$

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ & \xleftarrow{g} & \end{array}$$

An isomorphism is labelled with $\xrightarrow{\sim}$.

2.4 Homomorphism sets

A hom-set is a set of all morphisms between a pair of objects. It is denoted as

$$\begin{aligned} C(a, b) \\ \text{Hom}_C(a, b) \\ \text{Hom}(a, b) \end{aligned}$$

3 Types of elements

3.1 Void

The void element is equivalent to the logical **false**. It is impossible to construct.

3.2 Singleton

A singleton is a single empty tuple element and is equivalent to the logical **true**. It can be constructed from nothing.

4 Types of categories

4.1 Thin categories

A thin category is a category in which each pair of objects has either 0 or 1 morphism. Every hom-set of a thin category has either 1 or 0 elements.

4.1.1 Order categories

An order category is a thin category where morphisms represent relationships.

For example, here we have an equality relationship

$$a \xrightarrow{\leq} b$$

The relationship must be reflexive since there must exist an identity morphism.

$$\leq \bigcirc_a$$

4.2 Monoid

A monoid is a category with a single object.