

# Differentiation

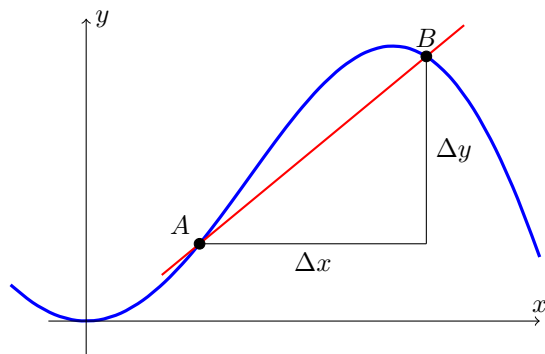
Paolo Bettelini

## Contents

<b>1</b>	<b>Definition</b>	<b>2</b>
1.1	Tangent . . . . .	2
1.2	Derivative . . . . .	2
<b>2</b>	<b>Chain Rule</b>	<b>3</b>
2.1	Definition . . . . .	3
2.2	Proof . . . . .	3
<b>3</b>	<b>Rules for differentiation</b>	<b>4</b>

# 1 Definition

## 1.1 Tangent



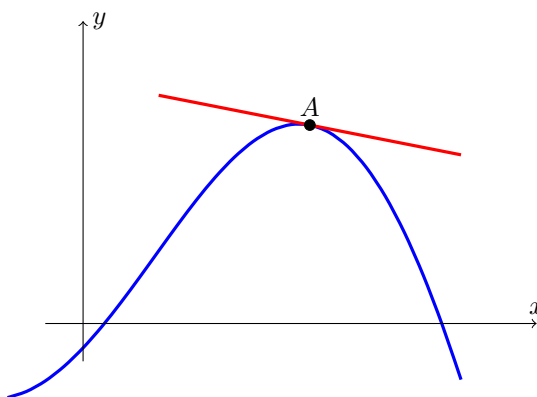
The mean slope of a function  $f$  between a point  $A$  and  $B$  is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(B) - f(A)}{B - A}$$

As we make  $A$  and  $B$  closer to each other,  $\Delta x$  decreases. As  $\Delta x$  decreases the mean slope is more representative of the rate of change of  $f$  in the interval  $[A; B]$ .

When  $\Delta x$  is infinitely small, we have the precise slope of a given point on the function. This slope is represented by the tangent line, which is parallel to the given point.

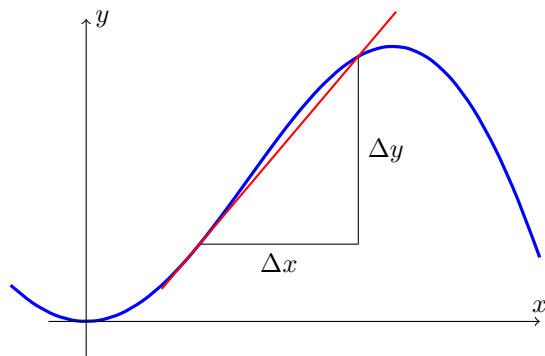
$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$



## 1.2 Derivative

The derivative of a function  $f(x)$  is another function  $f'(x)$  which represents the rate of change of  $f(x)$ . In other words,  $f'(x)$  represents the slope at each  $x$  of  $f(x)$ .

We define  $f'(x)$  by taking the limit of the slope for every  $x$ .



We define the derivative as

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

or

$$f'(x) = \lim_{h \rightarrow x} \frac{f(h) - f(x)}{h - x}$$

- If  $f'(a) > 0$ , then  $f(x)$  is increasing at  $x = a$
- If  $f'(a) < 0$ , then  $f(x)$  is decreasing at  $x = a$
- If  $f'(a) = 0$ , then  $f(x)$  is critical at  $x = a$  (changing from increase to decrease or from decrease to increase)

## 2 Chain Rule

### 2.1 Definition

If  $z$  depends on  $y$ , and  $y$  depends on  $x$ , then  $z$  also depends on  $x$ .

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

### 2.2 Proof

Assuming that  $z$  and  $y$  are differentiable in  $x$

$$\begin{aligned}\frac{dz}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x} \\ &= \left( \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta y} \right) \left( \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \right) \\ &= \left( \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta y} \right) \cdot \frac{dy}{dx}\end{aligned}$$

As  $\Delta x \rightarrow 0$  also  $\Delta y \rightarrow 0$ , so we can replace  $\Delta x$  with  $\Delta y$

$$\begin{aligned}\frac{dz}{dx} &= \left( \lim_{\Delta y \rightarrow 0} \frac{\Delta z}{\Delta y} \right) \cdot \frac{dy}{dx} \\ &= \frac{dz}{dy} \cdot \frac{dy}{dx}\end{aligned}$$

### 3 Rules for differentiation

$$\frac{d}{dx}(n) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{R}^*$$

$$\frac{d}{dx}(n \cdot f(x)) = n \frac{d}{dx}(f(x))$$

$$\frac{d}{dx}(f + g) = f' + g'$$

$$\frac{d}{dx}(f \cdot g) = g'f + gf'$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}(f^g) = f^g \left( \frac{f'g}{f} + g' \ln f \right)$$