

Complex Analysis

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1 De Moivre's Theorem

Using the property of exponentiation $(a^b)^c = a^{bc}$, we can see that $(e^{i\theta})^n = e^{in\theta}$. Using Euler's formula we can deduce that

$$(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta), \quad n \in \mathbb{Z}$$

2 Nth Roots of Units

We can extend De Moivre's Theorem for the integers powers or any complex number, rather than the ones on the unit circle ($r = 1$).

$$(r (\cos(\theta) + i \sin(\theta)))^n = r^n (\cos(n\theta) + i \sin(n\theta)), \quad n \in \mathbb{Z}$$

The nth roots of 1 are the solutions to

$$x^n = 1$$

for a given n . We might write 1 as a complex number

$$x^n = \cos(0) + i \sin(0)$$

Comparing this to our extended De Moivre's theorem

$$\cos(0) + i \sin(0) = r^n (\cos(n\theta) + i \sin(n\theta))$$

We can see that

$$\begin{aligned} r^n &= 1 \\ n\theta &= 0 \end{aligned}$$

As long as $n \neq 0$

$$\begin{aligned} r &= 1 \\ \theta &= 0 \end{aligned}$$

By plugging these values into

$$x^n = (r (\cos(\theta) + i \sin(\theta)))^n$$

we get that $x = 1$.

However we could also write 1 as

$$\cos(2k\pi) + i \sin(2k\pi), \quad k \in \mathbb{Z}$$

We would then get that

$$\begin{aligned} r^n &= 1 \\ n\theta &= 2k\pi \end{aligned}$$

When solving for x again we get

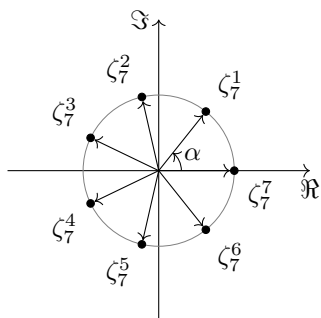
$$\begin{aligned} x^n &= (r(\cos(\theta) + i\sin(\theta)))^n \\ &= \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \right)^n \end{aligned}$$

concluding that

$$x = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

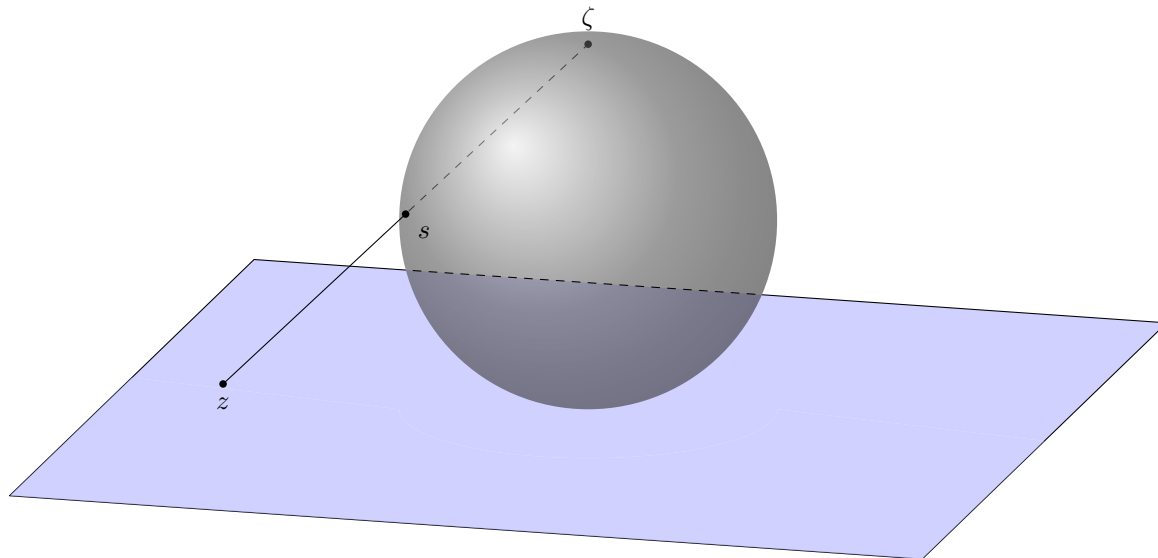
This gives us a solution for each k , however the solutions are redundant for $k \geq n$. In fact, the roots of unity of n are n distinct solutions (points on the unit circle).

The roots of unity have the same angle $\alpha = \frac{2\pi}{n}$ between each other. The first root of unity counter-clockwise is denoted ζ_n because each subsequent root is a power of ζ_n . In this case, ζ_7 .



3 Riemann Spheres

A Riemann sphere is a unit sphere used to represent the complex plane using stereographic projection.



The Riemann sphere lies on the complex plane. A complex number is represented by the intersection between the sphere and a ray starting from the topmost point of the sphere and intersecting with the given complex number on the complex plane.

4 Subsets of the complex plane

4.1 Open Disk

An open disk $D_\delta(z_0)$ is the set of points with distance less than δ from z_0

$$D_\delta(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \delta\}$$

4.2 Closed Disk

A closed open disk $D_\delta(z_0)$ is the set of points with distance less than or equal to δ from z_0

$$\overline{D_\delta(z_0)} = \{z \in \mathbb{C} \mid |z - z_0| \leq \delta\}$$

4.3 Circle

A circle $C_\delta(z_0)$ is the set of points with distance equal to δ from z_0

$$C_\delta(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = \delta\}$$

4.4 Interior point

z is an interior point of Ω iff there is an open disk at z whose point are in Ω

$$\exists D_{r>0}(z) \subset \Omega$$

4.5 Boundary point

z is a boundary point of Ω iff every open disk at z contains points both in Ω and not in Ω .

4.6 Exterior point

z is an exterior point of Ω iff it is not a boundary point of an interior point.

4.7 Accumulation points

z is an accumulation point or limit point of Ω if any $D_\delta(z) \setminus \{z\}$ always contains points of Ω .

In order to always contain points of Ω , Ω must have an infinite amount of points, since δ can be as little as we want.

4.8 Open sets

A set Ω is called open iff all points in Ω are interior points of Ω .

4.9 Closed sets

A set Ω is closed if every accumulation point of Ω is in Ω .

4.10 Bounded Set

A set Ω is bounded iff

$$\exists M > 0 \mid \Omega \subset D_M(0)$$

In other words there must exist an $M > 0$ such that $\forall z \in \Omega : |z| < M$

4.11 Connected Set

An open set Ω is connected iff it cannot be written as $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1 \cap \Omega_2 = \emptyset$. In other words any two points in Ω must be connectable by a continuous curve where all the points of the curve are also in Ω .

5 Differentiability and analyticity

Let $f(z)$ be a complex-valued function of a complex value which can be written as $f(x+iy) = u(x, y) + iv(x, y)$. Then

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

Note that Δz can approach 0 from infinite directions. For the derivative to exist, the answer should not depend on how Δz tends to 0. Let us write $\Delta z = \Delta x + i\Delta y$.

We now compute $f'(z)$ by approaching z from the horizontal direction ($\Delta y = 0$).

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \rightarrow 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x + iy) - f(x + iy)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(u(x + \Delta x, y) + iv(x + \Delta x, y)) - (u(x, y) + iv(x, y))}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \end{aligned}$$

We now compute $f'(z)$ by approaching z from the vertical direction ($\Delta x = 0$).

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \rightarrow 0} \frac{f(z + \Delta y) - f(z)}{i\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{f(x + iy + i\Delta y) - f(x + iy)}{i\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{(u(x, y + \Delta y) + iv(x, y + \Delta y)) - (u(x, y) + iv(x, y))}{i\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y} \\ &= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \end{aligned}$$

We have found two different representations of $f'(z)$ in terms of the partial derivatives of u and v .

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

From this equality we can derive the **Cauchy-Riemann equations**.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$