Euler's Formula

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1 Definition

Euler's formula states that for every $x \in \mathbb{R}$

$$e^{ix} = cos\,x + i\,sin\,x$$

2 Proof

To understand this identity we must first look at the Taylor series of some functions.

2.1 Sine function

2.2 Cosine function

2.3 Exponential function

2.4 Conclusion

Given the Taylor series for the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we plug in ix instead of x:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(xi)^n}{n!}$$

The imaginary number i has some amazing property when it comes to exponentiation.

$$\begin{cases} i^0 = +1 \\ i^1 = +i \\ i^2 = -1 \\ i^3 = -i \end{cases} \qquad \begin{cases} i^4 = +1 \\ i^5 = +i \\ i^6 = -1 \\ i^7 = -i \end{cases} \qquad \dots$$

We can use these properties to simplify the e^{ix} Taylor series

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \end{aligned}$$

We notice that the two terms correspond to the sine and cosine Taylor series

$$e^{ix} = cos\, x + i\, sin\, x$$