Riemann Hypothesis

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1 Zeta function

1.1 Definition

The zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 0$$

1.2 Euler product

The zeta function can be represented as an Euler product.
We will start by using the first prime number: 2.

$$\zeta(s)\frac{1}{2^s} = \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

We then subtract the second definition from the first one, such that

$$\begin{aligned} \zeta(s) - \zeta(s)\frac{1}{2^s} &= \sum_{n=1}^{\infty} \left[\frac{1}{n^s}\right] - \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^s}\right] \\ \zeta(s)\left(1 - \frac{1}{2^s}\right) &= \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \neq 2k, k \in \mathbb{Z} \end{aligned}$$

Here we are excluding the multiples of 2 from the series.

If we do the same with the next prime number, which is 3, we get

$$\zeta(s)\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \neq 2k_1, n \neq 3k_2, k \in \mathbb{Z}$$

We can repeat this process with every prime number.
Eventually, we will exclude every nth-term to sum as we use every prime number, except for n=1.

$$\zeta(s) \prod_{p \in P}^{\infty} 1 - \frac{1}{p^s} = \frac{1}{1^s} = 1$$

Finally, we get the identity

$$\zeta(s) = \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}}$$

2 Prime-counting function

2.1 Properties of the prime-counting function

The prime-counting function $\pi(x)$ is defined as the number of primes less or equals than x.

We can consider the difference between $\pi(x)$ of two consecutive integers

$$\pi(x) - \pi(x-1) = \begin{cases} 1, & \text{if } x \in P \\ 0, & \text{otherwise} \end{cases}$$

Given a series over all prime numbers, we can extend it to all integers and multiply each term by this difference.
The terms whose index is not a prime number will be multiplied by 0.

$$\sum_{p \in P}^{\infty} a_k = \sum_{n=2}^{\infty} \left[\pi(n) - \pi(n-1)\right] a_n$$

Here we start at 2 since there are no prime numbers less than 2.

2.2 Relationship with the zeta function

We have seen that the zeta function can be written as an Euler Product

$$\zeta(s) = \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}}$$

However, we need convert this product into a series in order to apply the identity of the last paragraph.
We can take the natural logarithm of both sides and use the multiplication property

$$\begin{aligned} ln\left(\zeta(s)\right) &= ln \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}} \\ &= \sum_{p \in P}^{\infty} ln\left(\frac{1}{1 - p^{-s}}\right) \\ &= \sum_{p \in P}^{\infty} -ln\left(1 - p^{-s}\right) \end{aligned}$$

Now we can apply the identity

$$ln\left(\zeta(s)\right) = \sum_{n=2}^{\infty} -ln\left(1 - n^{-s}\right)\left[\pi(n) - \pi(n-1)\right]$$

The next goal is to factor out $\pi(n)$

$$
\begin{aligned}
ln\left(\zeta(s)\right) &= \sum_{n=2}^{\infty}\left[\pi(n-1)ln\left(1-n^{-s}\right)\right] - \sum_{n=2}^{\infty}\left[\pi(n)ln\left(1-n^{-s}\right)\right] \\
&= \sum_{n=2}^{\infty}\left[\pi(n)ln\left(1-(n+1)^{-s}\right)\right] - \sum_{n=2}^{\infty}\left[\pi(n)ln\left(1-n^{-s}\right)\right] \\
&= \sum_{n=2}^{\infty}\pi(n)\left[ln\left(1-(n+1)^{-s}\right) - ln\left(1-n^{-s}\right)\right]
\end{aligned}
$$

[...]