

# Differentiation

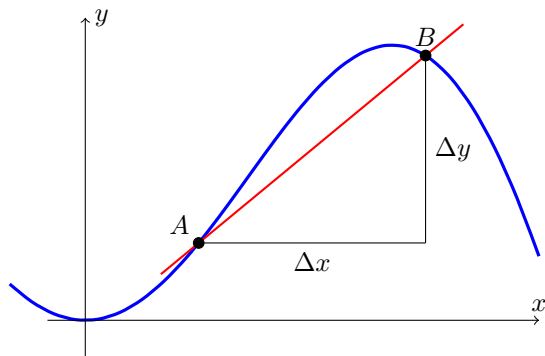
Paolo Bettelini

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# 1 Definition

## 1.1 Tangent



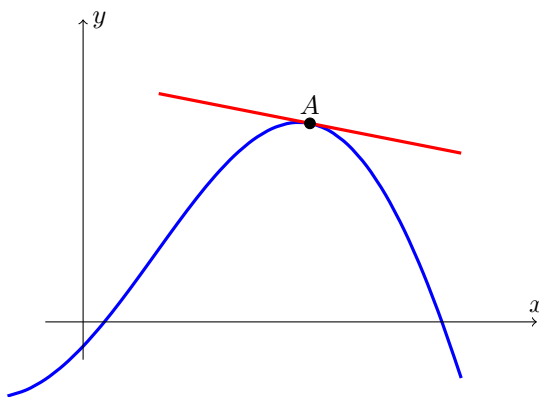
The mean slope of a function  $f$  between a point  $A$  and  $B$  is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(B) - f(A)}{B - A}$$

As we make  $A$  and  $B$  closer to each other,  $\Delta x$  decreases. As  $\Delta x$  decreases the mean slope is more representative of the rate of change of  $f$  in the interval  $[A; B]$ .

When  $\Delta x$  is infinitely small, we have the precise slope of a given point on the function. This slope is represented by the tangent line, which is parallel to the given point.

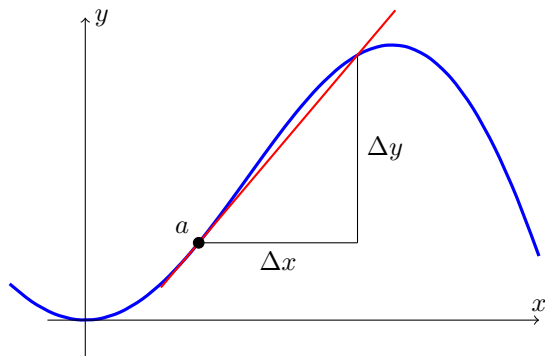
$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$



## 1.2 Derivative

The derivative of a function  $f(x)$  is another function  $f'(x)$  which represents the rate of change of  $f(x)$ . In other words,  $f'(x)$  represents the slope at each  $x$  of  $f(x)$ .

We define  $f'(x)$  by taking the limit of the slope for every  $x$ .



We define the derivative as

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

or

$$f'(x) = \lim_{h \rightarrow x} \frac{f(h) - f(x)}{x - h}$$

Using the derivative, the tangent line at  $x = a$  is given by

$$y = f'(a)(x - a) + f(a)$$

# 2 Interpretation

## 2.1 Rate of Growth

Since the derivative  $f'(x)$  represents the rate of change of  $f(x)$ , assuming that  $f(a)$  is defined.

- If  $f'(a) > 0$ , then  $f(x)$  is increasing at  $x = a$
- If  $f'(a) < 0$ , then  $f(x)$  is decreasing at  $x = a$
- If  $f'(a) = 0$ , then  $f(x)$  is critical at  $x = a$
- If  $f'(a)$  is not defined, then  $f(x)$  is critical at  $x = a$  (sharp corner)

A critical point is when the function is stationary.

A function increases on an interval  $I$  iff

$$\forall x_1, x_2 \in I f(x_1) < f(x_2)$$

and decreases iff

$$\forall x_1, x_2 \in I f(x_1) > f(x_2)$$

## 2.2 First Derivative Test

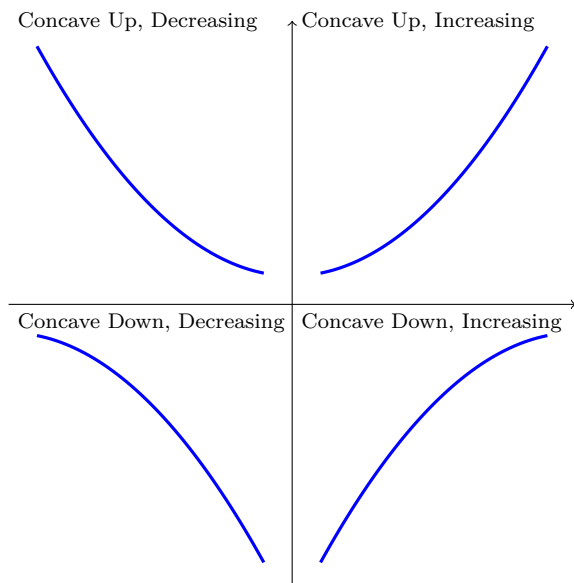
A critical point  $f'(c) = 0$  does not generally imply that  $x = c$  is a minimum or a maximum.

Let  $f(x)$  be critical at  $x = c$

- If  $f'(x) > 0$  to the left of  $x = c$  and  $f'(x) < 0$  to the right  $x = c$  is a maximum
- If  $f'(x) < 0$  to the left of  $x = c$  and  $f'(x) > 0$  to the right  $x = c$  is a minimum
- If  $f'(x)$  has the same sign on both sides of  $x = c$  then  $x = c$  is neither.

A function may also change sign when it is undefined.

## 2.3 Concavity



Functions may present **concavity**

- $f(x)$  is **concave up** on an interval  $I$  if all of the tangents on  $I$  are below the graph.
- $f(x)$  is **concave down** on an interval  $I$  if all of the tangents on  $I$  are above the graph.
- $f''(x) > 0$  for all  $x$  in some interval  $I$  then  $f(x)$  is concave up on  $I$
- $f''(x) < 0$  for all  $x$  in some interval  $I$  then  $f(x)$  is concave down on  $I$

This works because when the function is concave up, it increases or decreases more and more. So  $f'(x)$  tells us that  $f(x)$  is increasing or decreasing, and  $f''(x)$  tells us the rate at which the increment is increasing or the decrease is decrementing. The same goes for when the function is concave down.

An **inflection point** is a point where the function is continuous and the concavity at that point changes. Hence, when  $f''(x)$  changes sign we have an inflection point.

## 2.4 Second Derivative Test

Suppose that  $x = c$  is a critical point of  $f(x)$  such that  $f'(x) = 0$  and that  $f''(x)$  is continuous around  $x = c$ .

- If  $f''(x) < 0$  then  $x = c$  is a maximum.
- If  $f''(x) > 0$  then  $x = c$  is a minimum.
- If  $f''(x) = 0$  then  $x = c$  could be a maximum, minimum or neither.

## 3 Absolute Extrema

When looking for an absolute extrema in a function  $f(x)$ , asking when  $f'(x) = 0$  is not enough since the function may not be continuous and have a maxima at a discontinuity point.

## 4 Rules for differentiation

$$\frac{d}{dx}(n) = 0$$

### Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{R}^*$$

$$\frac{d}{dx}(n \cdot f(x)) = n \frac{d}{dx}(f(x))$$

$$\frac{d}{dx}(f + g) = f' + g'$$

### Product Rule

$$\frac{d}{dx}(f \cdot g) = g'f + gf'$$

### Quotient Rule

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

### Chain Rule

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}(f^g) = f^g \left( \frac{f'g}{f} + g' \ln f \right)$$

## 5 L'Hôpital Rule

Given the function  $f(x)$  and  $g(x)$  which are differentiable in an open interval  $I$  except possibly at  $x = c$ , if

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0 \text{ or } \pm \infty, \quad g(x \in I) \neq 0$$

then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

if such limit exists.

## 6 Intermediate value Theorem

A function  $f$  continuous on an interval  $[a; b]$  will take every value in the interval  $[f(a); f(b)]$ .

## 7 Bolzano's Theorem

If  $f(x)$  is continuous on  $[a; b]$  and  $f(a) \cdot f(b) < 0$  then there is a root.

$$f(a) \cdot f(b) < 0 \implies \exists c \in [a; b] \mid f(c) = 0$$

## 8 Weierstrass Theorem

If  $f(x)$  is continuous in  $[a; b]$  then the function will have a maxima and a minima.

## 9 Rolle's Theorem

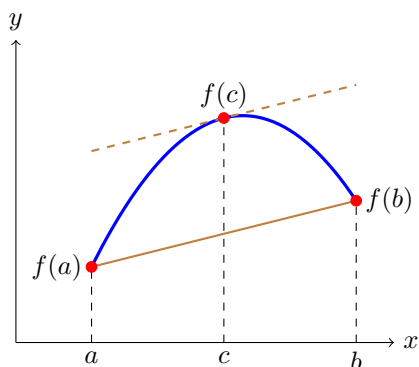
Suppose that  $f(x)$  is continuous on  $[a; b]$  and differentiable on  $(a; b)$ .

$$f(a) = f(b) \implies \exists c \mid f'(c) = 0, \quad a < c < b$$

## 10 Mean Value Theorem

Suppose  $f(x)$  is a function continuous on  $[a; b]$  and differentiable on  $(a; b)$ , there there exist a number  $c$  such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}, \quad a < c < b$$



The mean value on the interval can be represented by the **secant** line. What this means is that the interval contains a point whose tangent is equal to the secant.

Note that if  $f(a) = f(b)$  this is Rolle's theorem.

## 11 Chain Rule

### 11.1 Definition

If  $z$  depends on  $y$ , and  $y$  depends on  $x$ , then  $z$  also depends on  $x$ .

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

which is equivalent to

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

### 11.2 Proof

Assuming that  $z$  and  $y$  are differentiable in  $x$

$$\begin{aligned}\frac{dz}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x} \\ &= \left( \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta y} \right) \left( \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \right) \\ &= \left( \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta y} \right) \cdot \frac{dy}{dx}\end{aligned}$$

As  $\Delta x \rightarrow 0$  also  $\Delta y \rightarrow 0$ , so we can replace  $\Delta x$  with  $\Delta y$

$$\begin{aligned}\frac{dz}{dx} &= \left( \lim_{\Delta y \rightarrow 0} \frac{\Delta z}{\Delta y} \right) \cdot \frac{dy}{dx} \\ &= \frac{dz}{dy} \cdot \frac{dy}{dx}\end{aligned}$$

## 12 Differentials

Given a function  $y = f(x)$  we call  $dy$  and  $dx$  differentials and their relationship is

$$dy = f'(x)dx$$

If we are given just  $f(x)$  then the differentials would be  $df$  and  $dx$

$$df = f'(x)dx$$