

# Set Theory

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# 1 Definitions

## 1.1 Set

A *set* is a collection of unordered elements.

## 1.2 Cardinality

The *cardinality* of a set  $A$ , denoted  $|A|$ , is the amount of elements it contains.

## 1.3 Subset

If  $A$  and  $B$  are sets, then  $A$  is a *subset* of  $B$  ( $A \subseteq B$ ), if all the elements of  $A$  are also in  $B$ .

For every set  $A$ ,  $A \subseteq A$ .

## 1.4 Proper Subset

Given two sets  $A$  and  $B$ , if  $A \subseteq B$  but  $A \neq B$ , then  $A$  is a *proper* (or *strict*) subset of  $B$

$$A \subset B$$

## 1.5 Empty Set

The empty set  $\emptyset$  is a subset of all other sets.

$$|\emptyset| = 0$$

For every set  $A$

$$\emptyset \subseteq A$$

## 1.6 Power Set

If  $B$  is a set, then the *power set*  $\mathcal{P}(B)$  is defined as the set of all subsets of  $B$

$$\mathcal{P}(B) = \{A \mid A \subseteq B\}$$

Note that  $B \in \mathcal{P}(B)$ .

The cardinality of  $\mathcal{P}(A)$  is given by

$$|\mathcal{P}(A)| = 2^{|A|}$$

## 1.7 Union

If  $A$  and  $B$  are sets, then their *union* is

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

## 1.8 Intersection

If  $A$  and  $B$  are sets, then their *intersection* is

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

## 1.9 Difference

If  $A$  and  $B$  are sets, then their *difference* is

$$A \setminus B = \{x \mid x \in A \wedge x \notin B \vee x \in B \wedge x \notin A\}$$

Note that

$$A \setminus B = B \setminus A \iff A = B$$

## 1.10 Subset in terms of relationships

$$A \subseteq B \iff A \cup B = B \iff A \cap B = A \iff A \setminus B = \emptyset$$

## 1.11 Disjoint Sets

If  $A$  and  $B$  are sets and  $A \cap B = \emptyset$ , then  $A$  and  $B$  are disjoint sets.

## 1.12 Cartesian Product

If  $A$  and  $B$  are sets, then their *cartesian product* is

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$$

which is the set of all possible *ordered pairs*.

More generally, given  $n$  sets  $A_1, A_2, \dots, A_n$ , their cartesian product  $A_1 \times A_2 \times \dots \times A_n$  is the set of ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  with  $a_i \in A_i$ .

## 1.13 Cartesian Power

Given a set  $A$ ,  $A^n = \underbrace{A \times A \times \dots \times A}_n$ .

The  $n$ -dimensional plane of real numbers is a cartesian power  $\mathbb{R}^n$ .

## 1.14 Complement

If  $A$  is a set, its *complement* is

$$\bar{A} = \{x \mid x \notin A\}$$

## 1.15 Binary Relation

If  $A$  and  $B$  are sets, a function  $f : A \rightarrow B$  defines a *binary relation*  $R$

$$R = \{(a, b) \mid f(a) = b\}$$

Note that  $R \subseteq A \times B$

## 1.16 Homogeneous Relation

A *homogeneous relation* on a set  $S$  is a binary relation from a  $A$  to  $A$ .

## 1.17 Reflexive relation

A homogeneous relation  $R$  on a set  $A$  is *reflexive* iff

$$\forall a \in A, (a, a) \in R$$

### 1.18 Symmetric relation

A homogeneous relation  $R$  on a set  $A$  is *symmetric* iff

$$\forall (a, b) \in R, (b, a) \in R$$

### 1.19 Transitive relation

A homogeneous relation  $R$  on a set  $A$  is *transitive*

$$\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$$

### 1.20 Equivalence relation

An *equivalence relation* is a homogeneous relation  $\sim$  on a set  $A$  that is

1. *Reflexive*:  $\forall a \in A, a \sim a$
2. *Symmetric*:  $\forall a, b \in A, a \sim b \iff b \sim a$
3. *Transitive*:  $\forall a, b, c \in A, a \sim b \wedge b \sim c \implies a \sim c$

### 1.21 Equivalence class

Let  $\sim$  be an equivalence relation on a set  $A$ . Given an element  $a \in A$ , the equivalence class of  $a$ , is defined as

$$[a]_{\sim} = \{x \in A \mid a \sim x\}$$

By the symmetric property we have  $a \in [a]_{\sim}$ .

Let  $b \in [a]_{\sim}$ , meaning  $a \sim b$ .  $\forall c \in [b]_{\sim}$ , meaning  $b \sim c$ , we have  $a \sim c$  by the transitive property. Thus,  $c \in [a]_{\sim}$  and  $[b]_{\sim} \subseteq [a]_{\sim}$ . By the symmetric property we also have  $b \sim a$ ,  $\forall d \in [a]_{\sim}$ , meaning  $a \sim d$ , we have  $b \sim d$  by the transitive property. Thus,  $d \in [b]_{\sim}$  and  $[a]_{\sim} \subseteq [b]_{\sim}$ . Hence,

$$b \in [a]_{\sim} \iff [a]_{\sim} = [b]_{\sim}$$

This means that every element of an equivalence class has the same equivalence class. Thus, if two classes share an element they are the same

$$[a]_{\sim} \cap [b]_{\sim} \neq \emptyset \implies [a]_{\sim} = [b]_{\sim}$$

### 1.22 Partition of a set

Given a set  $A$ , a *partition of a set*  $P = \{C_i\}$  is a collection of non-empty subsets of  $A$  such that  $\bigcup_{E \in P} E = P$  and  $C_i \cap C_j = \emptyset, i \neq j$ . In other words the sets  $C_i$  contain every element of  $A$  exactly once.

Given an equivalence relationship  $\sim$  of a set  $A$ , the set of its equivalence classes form a partition of  $A$ .

### 1.23 Preorder

A *preorder* is a homogeneous relation  $\leq$  on a set  $A$  with the following properties:

1. *Reflexive*:  $\forall a \in A, a \leq a$
2. *Transitive*:  $\forall a, b, c \in A, a \leq b \wedge b \leq c \implies a \leq c$

### 1.24 Partial order

A *partial order* is a homogeneous relation  $\leq$  on a set  $A$  with the following properties:

1. *Reflexive*:  $\forall a \in A, a \leq a$
2. *Transitive*:  $\forall a, b, c \in A, a \leq b \wedge b \leq c \implies a \leq c$
3. *Antisymmetric*:  $\forall a, b \in A, a \leq b \wedge b \leq a \implies a = b$

### 1.25 Total order

A *total order* is a homogeneous relation  $\leq$  on a set  $A$  with the following properties:

1. *Reflexive*:  $\forall a \in A, a \leq a$
2. *Transitive*:  $\forall a, b, c \in A, a \leq b \wedge b \leq c \implies a \leq c$
3. *Antisymmetric*:  $\forall a, b \in A, a \leq b \wedge b \leq a \implies a = b$
4. *Strongly connected* (or *total*):  $\forall a, b \in A, a \leq b \vee b \leq a$

A total order is a partial order where any two elements are comparable.

### 1.26 Greatest element

Given a partial order on a set  $A$ , an element  $g$  is a *greatest element* if  $\forall a \in A, a \leq g$ .

### 1.27 Least element

Given a partial order on a set  $A$ , an element  $g$  is a *least element* if  $\forall a \in A, g \leq a$ .

### 1.28 Maximal element

Given a partial order on a set  $A$ , an element  $g \in A$  that is a greatest element is a *maximal element*.

### 1.29 Minimal element

Given a partial order on a set  $A$ , an element  $g \in A$  that is a least element is a *minimal element*.