1 Plane

A plane can be uniquely represented by its normal vector $\vec{n}$ and a point on the plane P_0.

To describe the plane using an equation, we can consider an arbitrary point $P = (x, y, z)$ on the plane. There is always a 90 degrees angle between the normal vector and the vector from P_0 to P (i.e., their dot product is zero)

$$\vec{n} \cdot \overrightarrow{P_0 P} = 0$$

By plugging in the values for $\vec{n}$ and P_0 we get an equation in the form

$$Ax + By + Cz + D = 0$$

2 Vector-Valued Function

A vector-valued function is a function of a real parameter which returns a vector

$$r(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$$

3 Tangent Vector Vector-Valued Function

Given a vector-valued function

$$r(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$$

where f, g and h are differentiable, then the Tangent vector to the curve is given b

$$r'(t) = \begin{pmatrix} f'(t) \\ g'(t) \\ h'(t) \end{pmatrix}$$

4 Curve length

The length of the curve beteen a and b is the integral from a to b of $\sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$.

5 Exercises

5.1 Open set 1

Prove that the set $A = \{(x, y) \in \mathbb{R}^2 \mid 2 < x^2 + y^2 < 4\}$ is open.

Let $p = (x, y)$ where $p \in A$. The set A is open iff $\exists \epsilon > 0 \mid B_\epsilon(p) \subset A$. Let $d = \sqrt{x^2 + y^2}$. For a radius $\epsilon \leq \min(d - \sqrt{2}, d - \sqrt{4})$, the open ball $B_\epsilon(p) \subset A$.

5.2 Norm 1

TODO

5.3 Countable set

TODO

5.4 Open set 2

Prove that the set $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 < y < x\}$ is open.

TODO

5.5 Sequence 1

Consider the sequence $\{x_k\}$ in $\mathbb{R}^2$ defined by

$$x_k = \left(\sin\left(\frac{\pi k}{2}\right), \frac{(-1)^k}{\sqrt{k}} \right)$$

for each $k \in \mathbb{N}^*$. Determine whether $\{x_k\}$ is bounded, and if so, find a convergent subsequence and identity its limit.

The sinusoidal part of the pair of the sequence is bounded because $-1 \leq \sin\theta \leq 1$. The other part has a numerator oscillating between 1 and -1, and the denominator goes from 1 to $+\infty$ in the limit. Thus, the sequence is absolutely decreasing and $-1 \leq \{x_k\} \leq \frac{1}{\sqrt{2}}$. We now notice that

$$\sin\left(\frac{\pi k}{2}\right) = \begin{cases} 1 \text{ or } -1, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

By considering the subsequence where k is even we get a converging sequence

$$\lim_{k\to\infty} \{x_{2k}\} = (0, 0)$$

5.6 Level curves

Find the equation of the level curve of the function $f(x, y)$ that passes through the given point p.

1. $f(x, y) = 16 - x^2 - y^2$ and $p = (2\sqrt{2}, \sqrt{2})$;
2. $f(x, y) = \sqrt{x^2 - 1}$ and $p = (1, 0)$;
3. $f(x, y) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}}$ and $p = (0, 1)$.

1. $f(2\sqrt{2}, \sqrt{2}) = 6$, so the height of the plane is 6. By plugging $z = 6$ in we get $6 = 16 - x^2 - y^2$ and thus the level curve is given by $10 = x^2 + y^2$;
2. $f(1, 0) = 0$, so the height of the plane is 0. By plugging $z = 0$ in we get $0 = \sqrt{x^2 - 1}$ and thus the level curve is given by $x^2 = 1$, that is the lines $x = 1$ and $x = -1$.

3. We first solve the integral
$$\int_x^y \frac{d\theta}{\sqrt{1-\theta^2}} = \arcsin(y) - \arcsin(x)$$
Now we can evaluate $f(0,1) = \arcsin(1) - \arcsin(0) = \frac{\pi}{2}$, so the height of the plane is $\frac{\pi}{2}$. By plugging $z = \frac{\pi}{2}$ in we get
$$\begin{aligned}\frac{\pi}{2} &= \arcsin(y) - \arcsin(x)\\ y &= \sin\left(\frac{\pi}{2} + \arcsin(x)\right)\\ y &= \sqrt{1-x^2}\end{aligned}$$
with $x < 0$.

5.7 Limits 1

Solve the limit $\lim_{(x,y)\to(2,1)} \frac{x^2-3y}{x+2y^2}$

$$\lim_{(x,y)\to(2,1)} \frac{x^2-3y}{x+2y^2} = \frac{2^2-3}{2+2} = \frac{1}{4}$$

5.8 Limits 2

Solve the limit $\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2}$

By using polar coordinate $x = r\cos\theta$ and $y = r\sin\theta$, and thus $x^2+y^2 = r^2$, therefore

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r)}{r} = 1$$

5.9 Limits 3

Solve the limit $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

5.10 Limits 4

Solve the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^4}$