Complex Analysis

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Contents

1 De Moivre's Theorem

Using the property of exponentiation $\left(a^b\right)^c = a^{bc}$, we can see that $\left(e^{i\theta}\right)^n = e^{in\theta}$.
Using Euler's formula we can deduce that

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta), \quad n \in \mathbb{Z}$$

2 Nth Roots of Units

We can extend De Moivre's Theorem for the integers powers or any complex number, rather than the ones on the unit circle ($r = 1$).

$$\left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right), \quad n \in \mathbb{Z}$$

The nth roots of 1 are the solutions to

$$x^n = 1$$

for a given n. We might write 1 as a complex number

$$x^n = \cos(0) + i\sin(0)$$

Comparing this to our extended De Moivre's theorem

$$\cos(0) + i\sin(0) = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

We can see that

$$\begin{aligned} r^n &= 1 \\ n\theta &= 0 \end{aligned}$$

As long as $n \neq 0$

$$\begin{aligned} r &= 1 \\ \theta &= 0 \end{aligned}$$

By plugging these values into

$$x^n = \left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^n$$

we get that $x = 1$.

However we could also write 1 as

$$\cos(2k\pi) + i\sin(2k\pi), \quad k \in \mathbb{Z}$$

We would then get that

$$
\begin{aligned}
r^n &= 1 \\
n\theta &= 2k\pi
\end{aligned}
$$

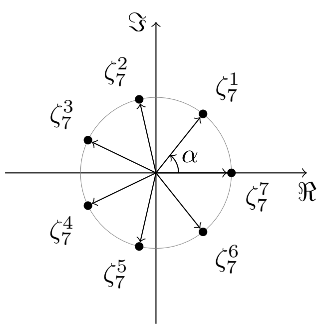
When solving for x again we get

$$
\begin{aligned}
x^n &= \left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^n \\
&= \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)^n
\end{aligned}
$$

concluding that

$$
x = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)
$$

This gives us a solution for each k, however the solutions are redundant for $k \geq n$. In fact, the roots of unity of n are n distinct solutions (points on the unit circle).



The roots of units have the same angle $\alpha = \frac{2\pi}{n}$ between eachother.
The first root of unit counter-clockwise is denoted ζ_n because each subsequent root is a power of ζ_n. In this case, ζ_7.