

Fourier Analysis

Paolo Bettelini

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1 Introduction

A function $f(x)$ is periodic if there is a positive number T (the period of f) such that

$$f(x + nT) = f(x) \quad \forall x \in D_f, n \in \mathbb{Z}$$

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$$f(x) = h + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right)$$

1.1 The h

We take the integral over the period $[t_0; t_0 + T]$ on both sides

$$\int_{t_0}^{t_0+T} f(x) dx = \int_{t_0}^{t_0+T} h dx + \sum_{n=1}^{\infty} \left[\int_{t_0}^{t_0+T} a_n \cos\left(\frac{2\pi nx}{T}\right) dx + \int_{t_0}^{t_0+T} b_n \sin\left(\frac{2\pi nx}{T}\right) dx \right]$$

if you think about it, the integral over a period interval of a function such as $\sin(x)$ or $\cos(x)$ is 0.

If we consider $\sin(w_n x)$ or $\cos(w_n x)$ the function will make more full cycles in the span of the period T , all of which yield an area of 0.

$$\begin{aligned} \int_{t_0}^{t_0+T} f(x) dx &= \int_{t_0}^{t_0+T} h dx \\ &= h \int_{t_0}^{t_0+T} dx \\ &= h[x]_{t_0}^{t_0+T} \end{aligned}$$

concluding that

$$h = \frac{1}{T} \int_{t_0}^{t_0+T} f(x) dx$$

1.2 a_n

1.3 b_n

2 Complex form

3 An Example

Let's look at a simple example. We are going to derive the Fourier series of a $f(x)$ defined as such:

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ +1 & \text{if } 0 < x < \pi \end{cases}$$

The period of this function is $T = 2\pi$. We can already simplify the $\frac{2\pi}{T}$ term, leaving us with

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

First, we need to find a_n . Simplifying $\frac{2\pi}{T}$ and $\frac{T}{2}$ we get

$$a_n = \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

Looking at the graph we notice that we can split the integral into two parts at $x = 0$. On the left part, the function is $-\cos(nx)$, while on the right part the function is $\cos(nx)$.

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^0 -\cos(nx) dx + \frac{1}{\pi} \int_0^{\pi} \cos(nx) dx \\ &= -\frac{1}{\pi} \int_{-\pi}^0 \cos(nx) dx + \frac{1}{\pi} \int_0^{\pi} \cos(nx) dx \\ &= -\frac{1}{\pi} \left[\frac{\sin(xn)}{n} \right]_{-\pi}^0 + \frac{1}{\pi} \left[\frac{\sin(xn)}{n} \right]_0^{\pi} \\ &= -\frac{1}{\pi} \left[\frac{\sin(\pi n)}{n} \right] + \frac{1}{\pi} \left[\frac{\sin(\pi n)}{n} \right] \\ &= \left(\frac{1}{\pi} - \frac{1}{\pi} \right) \left[\frac{\sin(\pi n)}{n} \right] \\ &= 0 \end{aligned}$$

a_n is always going to be 0. (note). We can remove the $a_n \cos(nx)$ and $\frac{a_0}{2}$ terms from the series.

Now for b_n

$$b_n = \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Again, we split the integral into two parts

$$\begin{aligned}
b_n &= -\frac{1}{\pi} \int_{-\pi}^0 \sin(nx) dx + \frac{1}{\pi} \int_0^{\pi} \sin(nx) dx \\
&= -\frac{1}{\pi} \left[\frac{-\cos(xn)}{n} \right]_{-\pi}^0 + \frac{1}{\pi} \left[\frac{-\cos(xn)}{n} \right]_0^{\pi} \\
&= -\frac{1}{\pi} \left[-\frac{1}{n} + \frac{\cos(\pi n)}{n} \right] + \frac{1}{\pi} \left[\frac{-\cos(\pi n)}{n} + \frac{1}{n} \right] \\
&= -\frac{1}{\pi} \left[\frac{\cos(\pi n) - 1}{n} \right] + \frac{1}{\pi} \left[\frac{1 - \cos(\pi n)}{n} \right] \\
&= \frac{2}{\pi} \cdot \frac{1 - \cos(\pi n)}{n} \\
&= \frac{2 - 2 \cos(\pi n)}{\pi n}
\end{aligned}$$

Given b_n our series is now complete!

$$f(x) = \sum_{n=1}^{\infty} \frac{2 - 2 \cos(\pi n)}{\pi n} \cdot \sin(nx)$$

We won't simplify this further, therefore this is our final result.

The effort pays off when we graph this function, as more terms are added, the function looks more and more like the original square wave.