

# Category Theory

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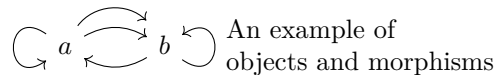
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# 1 Category

A category consists of *objects* and *morphism* or *arrows*.

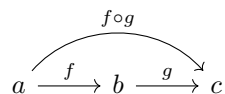
An arrow has a beginning and an ending, and it goes from one object to another.

Objects serve the purpose of marking the beginning and ending of a morphism.



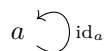
## 1.1 Composition

Composition is a property that says that if there is an arrow from  $a$  to  $b$ , and an arrow from  $b$  to  $c$ , there must exist an arrow from  $a$  to  $c$ .

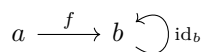


## 1.2 Identity

For every object there is an identity arrow.



The composition of an arrow with an identity is the arrow itself



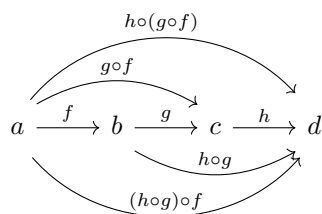
$$f \circ \text{id}_b = f$$

and also vice versa

$$\text{id}_b \circ f = f$$

## 1.3 Associativity

Compositions have the associative property



$$h \circ (g \circ f) = (h \circ g) \circ f$$

## 2 Homomorphism

An Homomorphism is a map between two structures of the same type.

### 2.1 Isomorphisms

A function  $f$  going from  $a$  to  $b$

$$f : a \rightarrow b$$

is invertible if there is a function  $g$  that goes from  $b$  to  $a$

$$g : b \rightarrow a$$

such that

$$g \circ f = \text{id}_b$$

$$f \circ g = \text{id}_a$$

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ & \xleftarrow{g} & \end{array}$$

This is a *bijective* homomorphism and it's called isomorphism. An isomorphism is labelled  $\xrightarrow{\sim}$ .

### 2.2 Epimorphisms

An epimorphism is a **surjective** morphism.

We can define surjectivity only in terms of morphisms.

Consider a morphism  $f : a \rightarrow b$  which maps elements of  $a$  onto  $b$ . Let's also define the morphisms  $g_1$  and  $g_2$  which map elements from  $b$  to  $c$ . The domain of  $g_1$  and  $g_2$  is the codomain of  $f$ . These two functions act as  $f$  for object in the image of  $f$ , but may map objects differently for objects in the codomain of  $f$  but outside the image of  $f$ . If the morphism is surjective, hence if the codomain and the image of  $f$  are the same, then  $g_1$  and  $g_2$  will always act as  $f$ .

$$a \xrightarrow{f} b \xrightarrow[g_2]{g_1} c$$

formally,

$$\forall c \forall g_1, g_2 : b \rightarrow c, g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

### 2.3 Monomorphisms