

# Riemann Hypothesis

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## **Abstract**

This document contains the main concepts about the Riemann Hypothesis and some derivations of the formulas and series used.

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# 1 Zeta function

## 1.1 Definition

The zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re}(s) > 1$$

## 1.2 Euler product

The zeta function can be represented as an Euler product.

We will start by using the first prime number: 2, and multiply both sides by  $2^{-s}$ .

$$\zeta(s) \frac{1}{2^s} = \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

We then subtract the second definition from the first one, such that

$$\begin{aligned} \zeta(s) - \zeta(s) \frac{1}{2^s} &= \sum_{n=1}^{\infty} \left[ \frac{1}{n^s} \right] - \sum_{n=1}^{\infty} \left[ \frac{1}{(2n)^s} \right] \\ \zeta(s) \left( 1 - \frac{1}{2^s} \right) &= \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \neq 2k, k \in \mathbb{Z} \end{aligned}$$

Here we are excluding the multiples of 2 from the series.

If we do the same with the next prime number, which is 3, we get

$$\zeta(s) \left( 1 - \frac{1}{2^s} \right) \left( 1 - \frac{1}{3^s} \right) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \neq 2k, n \neq 3k, k \in \mathbb{Z}$$

We can repeat this process with every prime number.

Eventually, we will exclude every nth-term to sum as we use every prime number, except for n=1.

$$\zeta(s) \prod_{p \in P} \left( 1 - \frac{1}{p^s} \right) = \frac{1}{1^s} = 1$$

Finally, we get the identity

$$\zeta(s) = \prod_{p \in P} \frac{1}{1 - p^{-s}}$$

## 2 Analytic continuation

### 2.1 Zeta function for positive $\text{Re}(s)$

We have seen that the classical zeta function definition only converges for  $\text{Re}(s) > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re}(s) > 1$$

We can use the eta function  $\eta(s)$ , which is defined for  $\text{Re}(s) > 0 \setminus \{1\}$ , to analytically extend the zeta function domain to  $\text{Re}(s) > 0 \setminus \{1\}$ .

The eta function is a Dirichlet series defined as

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \text{Re}(s) > 0 \setminus \{1\}$$

We start by splitting the zeta function into two distinct series, one for  $n$  even and the other one for  $n$  odd. The index for the even series will be  $2n$ , while the odd one will use  $2n-1$  as the index.

$$\zeta(s) = \sum_{n=1}^{\infty} \left[ \frac{1}{(2n)^s} \right] + \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)^s} \right]$$

We do the same thing with the eta function.

Notice that  $(-1)^n$  is 1 when  $n$  is even and -1 when  $n$  is odd.

$$\eta(s) = \sum_{n=1}^{\infty} \left[ \frac{1}{(2n)^s} \right] - \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)^s} \right]$$

We subtract these two definition from eachother

$$\begin{aligned} \zeta(s) - \eta(s) &= 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s} \\ &= 2^{1-s} \sum_{n=1}^{\infty} \frac{1}{k^s} \\ &= 2^{1-s} \zeta(s) \\ \frac{1}{1 - 2^{1-s}} \eta(s) &= \zeta(s) \end{aligned}$$

We finally get

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \text{Re}(s) > 0 \setminus \{1\}$$

This series can be used to compute value of the zeta function along the critical strip  $0 < \text{Re}(s) < 1$ .

### 2.2 Zeta function for negative $\text{Re}(s)$

### 2.3 Zeta function for $s=0$

## 2.4 Zeta function for $s=1$

The zeta function is holomorphic everywhere except for a pole at  $s = 1$  with residue 1. This is the only value of the complex plane that cannot be evaluated through analytic continuation.

$$\lim_{s \rightarrow 1} \zeta(s) = \lim_{s \rightarrow 1} \frac{s-1}{1-s^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

This is the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$$

so

$$\lim_{s \rightarrow 1} \frac{s-1}{1-s^{1-s}} \ln(2) = \frac{\ln(2)}{\ln(2) \cdot 2^{1-s}} = \frac{1}{2^{1-s}} = 1$$

## 3 Zeroes of the zeta function

### 3.1 Trivial zeroes

Considering the functional equation and analytic continuation of the zeta function

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

We can notice that the term  $\sin\left(\frac{\pi s}{2}\right)$  equals 0 when  $s$  is a multiple of 2.

However, the gamma function has a pole for every negative integer, this constrains our zeroes to be less or equal to 1.

Furthermore, the zeta function has a pole at  $s = 1$ , excluding the value  $s = 0$  from the zeroes, leaving  $\{-2; -4; -6; \dots\}$

$$\zeta(2k) = 0, \quad k \in \mathbb{Z}^-$$

These zeroes are called trivial zeroes because they are not relevant to the Riemann hypothesis.

### 3.2 Non-trivial zeroes

The Riemann hypothesis states that every non-trivial zero lies on the critical line  $Re(s) = \frac{1}{2}$ .

## 4 Prime-counting function

### 4.1 Properties of the prime-counting function

The prime-counting function  $\pi(x)$  is defined as the number of primes less or equals than  $x$ .

We can consider the difference between  $\pi(x)$  of two consecutive integers

$$\pi(x) - \pi(x-1) = \begin{cases} 1, & \text{if } x \in P \\ 0, & \text{otherwise} \end{cases}$$

Given a series over all prime numbers, we can extend it to all integers and multiply each term by this difference.

The terms whose index is not a prime number will be multiplied by 0.

$$\sum_{p \in P}^{\infty} a_k = \sum_{n=2}^{\infty} [\pi(n) - \pi(n-1)] a_n$$

Here we start at 2 since there are no prime numbers less than 2.

### 4.2 Relationship with the zeta function

We have seen that the zeta function can be written as an Euler Product

$$\zeta(s) = \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}}$$

However, we need convert this product into a series in order to apply the identity of the last paragraph.

We can take the natural logarithm of both sides and use the multiplication property

$$\begin{aligned} \ln(\zeta(s)) &= \ln \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}} \\ &= \sum_{p \in P}^{\infty} \ln \left( \frac{1}{1 - p^{-s}} \right) \\ &= \sum_{p \in P}^{\infty} -\ln(1 - p^{-s}) \end{aligned}$$

Now we can apply the identity

$$\ln(\zeta(s)) = \sum_{n=2}^{\infty} -\ln(1 - n^{-s}) [\pi(n) - \pi(n-1)]$$

The next goal is to factor out  $\pi(n)$

$$\begin{aligned} \ln(\zeta(s)) &= \sum_{n=2}^{\infty} [\pi(n-1) \ln(1 - n^{-s})] - \sum_{n=2}^{\infty} [\pi(n) \ln(1 - n^{-s})] \\ &= \sum_{n=2}^{\infty} [\pi(n) \ln(1 - (n+1)^{-s})] - \sum_{n=2}^{\infty} [\pi(n) \ln(1 - n^{-s})] \\ &= \sum_{n=2}^{\infty} \pi(n) [\ln(1 - (n+1)^{-s}) - \ln(1 - n^{-s})] \end{aligned}$$

To simplify further more, we consider the derivative of the function  $\ln(1 - x^{-s})$ .  
Using the chain rule we get

$$\frac{d}{dx} \ln(1 - x^{-s}) = \frac{s}{x(x^s - 1)}$$

Therefore,

$$\ln(1 - x^{-s}) = \int \frac{s}{x(x^s - 1)} dx + C$$

Considering  $f(x) = \ln(1 - x^{-s})$ , our series can be expressed as

$$\ln(\zeta(s)) = \sum_{n=2}^{\infty} \pi(n) [f(n+1) - f(n)]$$

which can be written as an integral from  $n$  to  $n+1$

$$\begin{aligned} \ln(\zeta(s)) &= \sum_{n=2}^{\infty} \pi(n) \int_n^{n+1} f'(x) dx \\ &= \sum_{n=2}^{\infty} \pi(n) \int_n^{n+1} \frac{s}{x(x^s - 1)} dx \\ &= \sum_{n=2}^{\infty} \int_n^{n+1} \frac{s\pi(x)}{x(x^s - 1)} dx \end{aligned}$$

Instead of taking the sum of each of these integrals (2 to 3, 3 to 4, ...), we can make a single integral

$$\ln(\zeta(s)) = s \int_2^{\infty} \frac{\pi(x)}{x(x^s - 1)} dx$$

### 4.3 Approximations

A pretty good approximation to  $\pi(x)$  is

$$li(x) = \int_0^x \frac{dt}{\ln t}$$

called the logarithmic integral function

### 4.4 Exact form

Riemann proved that the exact form the prime counting function is

$$\pi(x) = Re(x) - \sum_p R(x^p)$$

where

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} li(\sqrt[n]{x})$$

and  $\mu(x)$  is the Möbius function.

Also,  $p$  indexes every zero of the zeta function.