Category Theory

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1 Category

A category consists of *objects* and *morphism* or *arrows*.
An arrow has a beginning and an ending, and it goes from one object to another.
Objects serve the purpose of marking the beginning and ending of a morphism.

$$\begin{tikzcd} a \arrow[loop left] \arrow[r, bend left] \arrow[r, bend left=10] & b \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

An example of
objects and morphisms

The set of morphisms of a category C is denoted $\hom(C)$ and the set of objects $\mathrm{ob}(C)$.

1.1 Composition or Transitivity

Composition is a property that says that if there is an arrow from a to b, and an arrow from b to c, there must exist an arrow from a to c.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend left, "f \circ g"] & b \arrow[r, "g"] & c \end{tikzcd}$$

1.2 Identity or Reflexivity

For every object there is an identity arrow.

$$\begin{tikzcd} a \arrow[loop right, "\mathrm{id}_a"] \end{tikzcd}$$

The composition of an arrow with an identity is the arrow itself

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[loop right, "\mathrm{id}_b"] \end{tikzcd}$$

$$f \circ \mathrm{id}_b = f$$

and also vice versa

$$\mathrm{id}_b \circ f = f$$

1.3 Associativity

Compositions have the associative property

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] \arrow[rrr, bend left=50, "h \circ (g \circ f)"] \arrow[rrr, bend right=50, "(h \circ g) \circ f"'] & b \arrow[r, "g"] \arrow[rr, bend right, "h \circ g"'] & c \arrow[r, "h"] & d \end{tikzcd}$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

2 Homomorphisms

A homomorphism is a map between two structures of the same type.

2.1 Epimorphisms

An epimorphism is a **surjective** morphism.

We can define surjectivity only in terms of morphisms.
Consider a morphism $f : a \to b$ which maps elements of a onto b. Let's also define the morphisms g_1 and g_2 which map elements from b to c. The domain of g_1 and g_2 is the codomain of f. These two functions act as f for object in the image of f, but may map objects differently for objects in the codomain of f but outside the image of f. If the morphism is surjective, hence if the codomain and the image of f are the same, then g_1 and g_2 will always act as f.

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & c \end{tikzcd}$$

formally,

$$\forall c \forall g_1, g_2 : b \to c, g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

An epimorphism is labelled with $\twoheadrightarrow$.

2.2 Monomorphisms

An epimorphism is an **injective** morphism.

$$\begin{tikzcd} c \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & a \arrow[r, "f"] & b \end{tikzcd}$$

A morphism $f : a \to b$ is a monomorphism if

$$\forall c \forall g_1, g_2 : c \to a, f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

A monomorphism is labelled with $\hookrightarrow$.

2.3 Isomorphisms

An isomorphism is a **bijective** morphism (mono and epic, but not every mono and epic is an isomorphism).

A morphism $f : a \to b$ is invertible if there is a function g that goes from b to a

$$b : \quad b \to a$$

such that

$$g \circ f = \mathrm{id}_b$$
$$f \circ g = \mathrm{id}_a$$

$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "g"] \end{tikzcd}$$

An isomorphism is labelled with $\xrightarrow{\sim}$.

2.4 Homomorphism sets

A hom-set is a set of all morphisms from an object to another of a category C. It is denoted as

$$\begin{aligned}&C(a,b)\\&\mathrm{Hom}_C(a,b)\\&\mathrm{Hom}(a,b)\end{aligned}$$

Note

$$\mathrm{Hom}(a,b) \neq \mathrm{Hom}(b,a)$$

3 Types of elements

3.1 Initial objects

An initial object I is an object if for every object X in a category C there exists exactly one morphism $I \to X$.

$$\forall X \in \mathrm{ob}(C)\, \exists_{=1} f : I \to X$$

3.2 Terminal objects

An terminal object T is an object if for every object X in a category C there exists exactly one morphism $X \to T$.

$$\forall X \in \mathrm{ob}(C)\, \exists_{=1} f : X \to T$$

3.3 Void

The void element is equivalent to the logical **false**. It is impossible to construct. Thus, functions that take void as an argument are impossible to call. It is impossible to map a set of the empty set, because any object canot have an image.

In the category of sets, the singleton is the only terminal initial object, as there exists a unique function (the empty function) from the empty set into any other given set.

3.4 Singleton

A singleton is a set with one element (e.g. $\{0\}$) and is equivalent to the logical **true**. It can be constructed from nothing.

In the category of sets, any singleton is a terminal object, as there exists a unique function (the constant function) from any given set into the singleton.

4 Types of categories

4.1 Thin categories

A thin category is a category in which each pair of objects has either 0 or 1 morphism. Every hom-set of a thin category has either 1 or 0 elements.

4.1.1 Order categories

An order category is a thin category where morphisms represent relationships.

For example, here we have an equality relationship

$$a \xrightarrow{\leq} b$$

The relationship must be reflexive since there must exist an identity morphsim.

$$\begin{tikzcd} a \arrow[loop left, "\leq"] \end{tikzcd}$$

It must also be composable and associative. The relations may for example be preorders, partial orders or total orders.

4.2 Monoid

A monoid is a category with a single object. It is equivalent to a set closed under an associative binary operation.

4.3 Kleisli category

4.4 Monad

4.5 Opposite Category

Given a category C we can defined an opposite category C^{op} where the objects are the same and the morphisms are inverted.

$$\begin{tikzcd} a \arrow[d, "f"'] \arrow[dr, "g \circ f"] & \\ b \arrow[r, "g"'] & c \end{tikzcd} \qquad \begin{tikzcd} a & \\ b \arrow[u, "f^{\mathrm{op}}"] & c \arrow[l, "g^{\mathrm{op}}"] \arrow[ul, "f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}"'] \end{tikzcd}$$

5 Universal Construction

Universal construction is used to define objects in terms of their relationships up to a unique isomorphism. This means that the objects that satisfy the same universal property are isomorphic. Its purpose is to define an object without knowledge about the object itself, but rather just the morphisms.

6 Product

The categorical product represents many operations such as the cartesian product of sets. A product of two objects a and b has the following morphisms:

1. $p : a \times b \to a$, which returns the first value of the ordered pair
2. $q : a \times b \to b$, which returns the second value of the ordered pair

In the following diagram c is the actual product and c' is a candidate object for the caridnal product. We rank a candidate c_i higher than c_j if there is a morphism $m : c_i \to c_j$.

$$
\begin{tikzcd}
 & c' \arrow[d, "m"] \arrow[ddl, bend right, "p'"'] \arrow[ddr, bend left, "q'"] & \\
 & c \arrow[dl, "p"] \arrow[dr, "q"'] & \\
a & & b
\end{tikzcd}
$$

$$
\begin{aligned}
p' &= p \circ m \\
q' &= q \circ m
\end{aligned}
$$

Whenever m is *flawed*, such as when it loses information or does not preserve structure, we discard c_i.

The universal property is that for any other product c' with morphisms p' and q', s a unique morphism

$$m : c' \to c$$

such that

$$
\begin{aligned}
p \circ m &= p' \\
q \circ m &= q'
\end{aligned}
$$

6.1 Coproduct

The coproduct represents operations such as the disjoint union of sets. The coproduct is the product when the morphisms are inverted. The coproduct of two objects a and b has the following morphisms:

1. $p : a \to a \sqcup b$
2. $q : b \to a \sqcup b$

Let $c = a \sqcup b$.

$$\begin{tikzcd}
 & c' & \\
 & c \arrow[u, "m"'] & \\
a \arrow[ur, "p"'] \arrow[uur, bend left, "p'"] & & b \arrow[ul, "q"] \arrow[uul, bend right, "q'"']
\end{tikzcd}$$

$$p' = m \circ p$$
$$q' = m \circ q$$

The universal property is that for any other object c' with morphisms $p' : a \to c'$ and $q' : b \to c'$, there exists a unique morphism $m : c \to c'$ such that

$$p' = m \circ p$$
$$q' = m \circ q$$