Riemann Hypothesis

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Abstract

This document contains the main concepts about the Riemann Hypothesis and some derivations of the formulas and series used.

Contents

1 Zeta function

1.1 Definition

The zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1$$

1.2 Euler product

The zeta function can be represented as an Euler product.
We will start by using the first prime number: 2, and multiply both sides by 2^{-s}.

$$\zeta(s)\frac{1}{2^s} = \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

We then subtract the second definition from the first one, such that

$$\zeta(s) - \zeta(s)\frac{1}{2^s} = \sum_{n=1}^{\infty} \left[\frac{1}{n^s}\right] - \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^s}\right]$$
$$\zeta(s)\left(1 - \frac{1}{2^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \neq 2k, k \in \mathbb{Z}$$

Here we are excluding the multiples of 2 from the series.

If we do the same with the next prime number, which is 3, we get

$$\zeta(s)\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \neq 2k, n \neq 3k, k \in \mathbb{Z}$$

We can repeat this process with every prime number.
Eventually, we will exclude every nth-term to sum as we use every prime number, except for n=1.

$$\zeta(s)\prod_{p\in P}^{\infty} 1 - \frac{1}{p^s} = \frac{1}{1^s} = 1$$

Finally, we get the identity

$$\zeta(s) = \prod_{p\in P}^{\infty} \frac{1}{1 - p^{-s}}$$

2 Analytic continuation

2.1 Zeta function for positive Re(s)

We have seen that the classical zeta function definition only converges for $Re(s) > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1$$

We can use the eta function $\eta(s)$, which is defined for $Re(s) > 0 \setminus \{1\}$, to analytically extend the zeta function domain to $Re(s) > 0 \setminus \{1\}$.
The eta function is a Dirichlet series defined as

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad Re(s) > 0 \setminus \{1\}$$

We start by splitting the zeta function into two distinct series, one for n even and the other one for n odd. The index for the even series will be $2n$, while the odd one will use $2n - 1$ as the index.

$$\zeta(s) = \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^s}\right] + \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^s}\right]$$

We do the same thing with the eta function.
Notice that $(-1)^n$ is 1 when n is even and -1 when n is odd.

$$\eta(s) = \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^s}\right] - \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^s}\right]$$

We subtract these two definition from eachother

$$\begin{aligned}
\zeta(s) - \eta(s) &= 2\sum_{n=1}^{\infty} \frac{1}{(2n)^s} \\
&= 2^{1-s} \sum_{n=1}^{\infty} \frac{1}{k^s} \\
&= 2^{1-s}\zeta(s) \\
\frac{1}{1-2^{1-s}}\eta(s) &= \zeta(s)
\end{aligned}$$

We finally get

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad Re(s) > 0 \setminus \{1\}$$

This series can be used to compute value of the zeta function along the critical strip $0 < Re(s) < 1$.

2.2 Zeta function for negative Re(s)

2.3 Zeta function for s=0

2.4 Zeta function for s=1

The zeta function is holomorphic everywhere except for a pole at $s = 1$ with residue 1.
This is the only value of the complex plane that cannot be evaluated through analytic continuation.

3 Zeroes of the zeta function

3.1 Trivial zeroes

Considering the functional equation and analytic continuation of the zeta function

$$\zeta(s) = 2^s \pi^{s-1} sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

We can notice that the term $sin\left(\frac{\pi s}{2}\right)$ equals 0 when s is a multiple of 2.
However, the gamma function has a pole for every negative integer, this constrains our zeroes to be less or equal to 1.
Furthermore, the zeta function has a pole at $s = 1$, excluding the value $s = 0$ from the zeroes, leaving $\{-2; -4; -6; \cdots\}$

$$\zeta(2k) = 0, \quad k \in \mathbb{Z}^-$$

These zeroes are called trivial zeroes because they are not relevant to the Riemann hypothesis.

3.2 Non-trivial zeroes

The Riemann hypothesis states that every non-trivial zero lies on the critical line $Re(s) = \frac{1}{2}$.

4 Prime-counting function

4.1 Properties of the prime-counting function

The prime-counting function $\pi(x)$ is defined as the number of primes less or equals than x.

We can consider the difference between $\pi(x)$ of two consecutive integers

$$\pi(x) - \pi(x-1) = \begin{cases} 1, & \text{if } x \in P \\ 0, & \text{otherwise} \end{cases}$$

Given a series over all prime numbers, we can extend it to all integers and multiply each term by this difference.
The terms whose index is not a prime number will be multiplied by 0.

$$\sum_{p \in P}^{\infty} a_k = \sum_{n=2}^{\infty} [\pi(n) - \pi(n-1)]\, a_n$$

Here we start at 2 since there are no prime numbers less than 2.

4.2 Relationship with the zeta function

We have seen that the zeta function can be written as an Euler Product

$$\zeta(s) = \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}}$$

However, we need convert this product into a series in order to apply the identity of the last paragraph.
We can take the natural logarithm of both sides and use the multiplication property

$$\begin{aligned} \ln(\zeta(s)) &= \ln \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}} \\ &= \sum_{p \in P}^{\infty} \ln\left(\frac{1}{1 - p^{-s}}\right) \\ &= \sum_{p \in P}^{\infty} - \ln\left(1 - p^{-s}\right) \end{aligned}$$

Now we can apply the identity

$$\ln(\zeta(s)) = \sum_{n=2}^{\infty} - \ln\left(1 - n^{-s}\right) [\pi(n) - \pi(n-1)]$$

The next goal is to factor out $\pi(n)$

$$\begin{aligned} \ln(\zeta(s)) &= \sum_{n=2}^{\infty} \left[\pi(n-1) \ln\left(1 - n^{-s}\right)\right] - \sum_{n=2}^{\infty} \left[\pi(n) \ln\left(1 - n^{-s}\right)\right] \\ &= \sum_{n=2}^{\infty} \left[\pi(n) \ln\left(1 - (n+1)^{-s}\right)\right] - \sum_{n=2}^{\infty} \left[\pi(n) \ln\left(1 - n^{-s}\right)\right] \\ &= \sum_{n=2}^{\infty} \pi(n) \left[\ln\left(1 - (n+1)^{-s}\right) - \ln\left(1 - n^{-s}\right)\right] \end{aligned}$$

To simplify further more, we consider the derivative of the function $\ln(1 - x^{-s})$.
Using the chain rule we get

$$\frac{d}{dx} \ln\left(1 - x^{-s}\right) = \frac{s}{x(x^s - 1)}$$

Therefore,

$$\ln\left(1 - x^{-s}\right) = \int \frac{s}{x(x^s - 1)}\, dx + C$$

Considering $f(x) = \ln(1 - x^{-s})$, our series can be expressed as

$$\ln\left(\zeta(s)\right) = \sum_{n=2}^{\infty} \pi(n) \left[f(n+1) - f(n)\right]$$

which can be written as an integral from n to $n + 1$

$$\begin{aligned}
\ln\left(\zeta(s)\right) &= \sum_{n=2}^{\infty} \pi(n) \int_{n}^{n+1} f'(x)\, dx \\
&= \sum_{n=2}^{\infty} \pi(n) \int_{n}^{n+1} \frac{s}{x(x^s - 1)}\, dx \\
&= \sum_{n=2}^{\infty} \int_{n}^{n+1} \frac{s\pi(x)}{x(x^s - 1)}\, dx
\end{aligned}$$

Instead of taking the sum of each of these integrals (2 to 3, 3 to 4, ...), we can make a single integral

$$\ln\left(\zeta(s)\right) = s \int_{2}^{\infty} \frac{\pi(x)}{x(x^2 - 1)}\, dx$$

4.3 Approximations

A pretty good approximation to $\pi(x)$ is

$$li(x) = \int\limits_0^x \frac{dt}{\ln\, t}$$

called the logarithmic integral function

4.4 Exact form

Riemann proved that the exact form the prime counting function is

$$\pi(x) = Re(x) - \sum_p R(x^p)$$

where

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} li(\sqrt[n]{x})$$

and $\mu(x)$ is the Möbius function.
Also, p indexes every zero of the zeta function.