Sequences

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1 Definition

A sequence denotes a series of indexed values. A sequence may be written as

$$\{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

If $\lim_{n\to\infty} a_n$ exists and is finite we say that the sequence is *convergent*. If $\lim_{n\to\infty} a_n$ doesn't exist or is infinite we say that the sequence *diverges*.

If $f(x)$ is a function such that $f(n) = a_n$

$$\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n = L$$

1.1 Properties

if $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$

2 Squeeze Theorem

If $a_n \leq c_n \leq b_n$ for sufficiently large $n > N$ for some N and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$ then $\lim_{n\to\infty} c_n = L$

3 Absolute Value

Note the following

$$-|a_n| \leq a_n \leq |a_n|$$

Then, if we assume

$$\lim_{n\to\infty} (-|a_n|) = -\lim_{n\to\infty} |a_n| = 0$$

by the Squeeze Theorem we get

$$\lim_{n\to\infty} a_n = 0$$

We conclude that if $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

4 Exponential sequence

The sequence $\{a^n\}_{n=0}^{\infty}$ converges if $-1 < r \leq 1$

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } -1 < a < 1 \\ 1 & \text{if } a = 1 \end{cases}$$

5 Convergence of even and odd indexes

Theorem. *If* $\lim_{n\to\infty} a_{2n} = L$ *and* $\lim_{n\to\infty} a_{2n+1} = L$ *then* $\{a_n\}$ *is convergent and* $\lim_{n\to\infty} a_n = L$*.*

Proof. Let $\epsilon > 0$.
Since $\lim_{n\to\infty} a_{2n} = L$ there exists an N_1 such that if $n > N_1$ then

$$|a_{2n} - L| < \epsilon$$

Also, since $\lim_{n\to\infty} a_{2n+1} = L$ there exists an N_2 such that if $n > N_2$ then

$$|a_{2n+1} - L| < \epsilon$$

Let $N = \max(2N_1, 2N_2 + 1)$ and let $n > N$. Then either $a_n = a_{2k}$ for some $k > N_1$ or $a_n = a_{2k+1}$ for some $k > N_2$. Either way we have

$$|a_n - L| < \epsilon$$

which satisfies the convergence of $\{a_n\}$. □

6 Properties of a sequence

6.1 Increasing

A sequence is *increasing* if $\forall n : a_n < a_{n+1}$.

6.2 Decreasing

A sequence is *decreasing* if $\forall n : a_n > a_{n+1}$.

6.3 Monotonic

If $\{a_n\}$ is increasing or decrasing it is also called *monotonic*.

6.4 Bounded below

If there exists a number m such that $\forall n : m \leq a_n$ the sequence is *bounded below* by a lower bound.

6.5 Bounded above

If there exists a number m such that $\forall n : m \geq a_n$ the sequence is *bounded above* by an upper bound.

6.6 Bounded

If the sequence is both bounded above and bounded below it is said *bounded*.

7 Convergence of bounded and monotonic sequences

If a_n is bounded and monotonic then it is convergent.