Integration

Paolo Bettelini

Contents

1 Indefinite Integrals

1.1 Definition

Given a function $f(x)$, an **anti-derivative** or **primitive** is any function $F(x)$ such that

$$\frac{dF}{dx} = f(x)$$

The operator to find a primitive function is called the **indefinite integral**

$$\int f(x)\,dx = F(x) + C, \quad C \in \mathbb{R}$$

The function to integrate (integrand) is delimited by the integral symbol $\int$ and a differential of the variable of integration dx.
A function has infinitely many primitives, hence the $+C$ term. This essentially means that the derivative of a function is the same when the function is shifted up or down, the rate of change is the same. By reversing the process we don't know the up or down shift of the original function.

$$f(x) = \int \frac{df}{dx}\,dx + C$$

for some specific C.

1.2 Properties

If k is a constant

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

1.3 Substitution Rule

Given an integral in the form

$$\int f(g(x))g'(x)\,dx$$

Let

$$u = g(x)$$

The differential of u is then

$$du = g'(x)dx$$

meaning that we can rewrite the integral as

$$\int f(u)\,du = F(u) + C = F(g(x)) + C$$

1.4 Integration By Parts

Starting from the product rule

$$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

if we integrate both parts we get

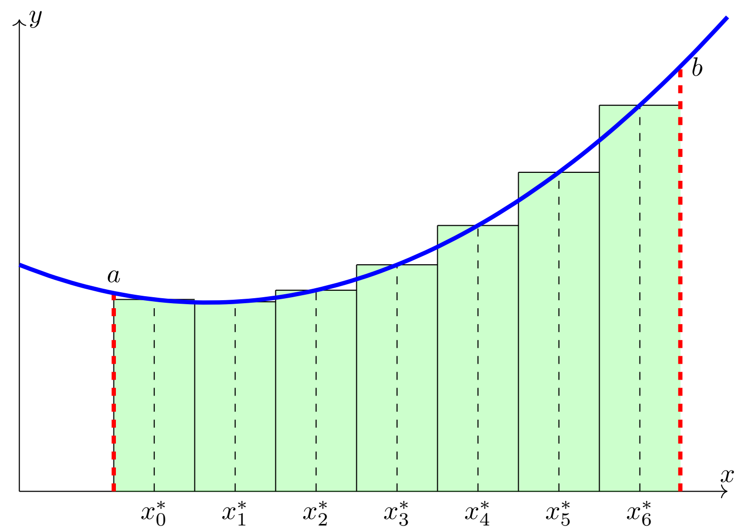
$$\begin{aligned} f(x)g(x) + C &= \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx \\ \int f(x)g'(x)\,dx &= f(x)g(x) + C - \int f'(x)g(x)\,dx \end{aligned}$$

Since the indefinite integral of $f'(x)g(x)$ is equal to some function plus an arbitrary constant, we can ignore the $+C$ term.

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

2 Definite Integrals

2.1 Area Problem



We want to find the signed area between $f(x)$ and the x-axis between the interval $[a; b]$.
One way to do it would be by dividing the area into n rectangles, each of width

$$\Delta x = \frac{b-a}{n}$$

The height of each triangle is given by $f(x_k^*)$. The area under the curve is approximately

$$A \approx \sum_{k=0}^{n-1} f(x_k^*)\Delta x$$

Notice that the position of x_k^* within the base of each rectangle controls the type of the approximation of the area under the curve. By moving x_k^* within the base we may achieve an approximation by abundance or defect. The type of approximation does not matter when we let $n \to \infty$. As the amount of rectangles approaches infinity, the approximation approaches the exact value of the area.

$$A = \lim_{n\to\infty} \sum_{k=0}^{n} f(x_k^*)\Delta x$$

2.2 Definition

Given a function $f(x)$ continuous on the interval $[a; b]$, we divide the interval into n rectangles of width $\Delta x = \frac{b-a}{n}$ and height $f(x_k^*)$. The definitive interval of $f(x)$ from a to b is

$$\int\limits_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n} f(x_k^*)\Delta x$$

2.3 Properties

$$\int\limits_a^b f(x)\,dx = -\int\limits_b^a f(x)\,dx$$

If k is constant

$$\int\limits_a^b cf(x)\,dx = c\int\limits_a^b f(x)\,dx$$

$$\int\limits_a^b f(x) \pm g(x)\,dx = \int\limits_a^b f(x)\,dx \pm \int\limits_a^b g(x)\,dx$$

$$\int\limits_a^a f(x)\,dx = 0$$

2.4 Fundamental Theorem of Calculus

A primitive function $F(x)$ (with $C = 0$, so passing through the origin) represents the area from 0 to x of $f(x)$.

Let $f(x)$ be continuous on the interval $I = [a; b]$ and $F(x)$ any primitive of $f(x)$, then $F(x)$ is also continuous on I and

$$\int_0^x f(t)\,dt = F(x)$$

and

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This also means that

$$f(x) = \int_0^x f'(t)\,dt$$

or

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

When the upper or lower limit is not constant,

$$\begin{aligned}\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt &= \frac{d}{dx}\left[F(u(x)) - F(v(x))\right] \\ &= u'(x)f(u(x)) - v'(x)f(v(x))\end{aligned}$$

3 Average Function Value

The average value of a continuous function $f(x)$ over the interval $[a; b]$ is given by

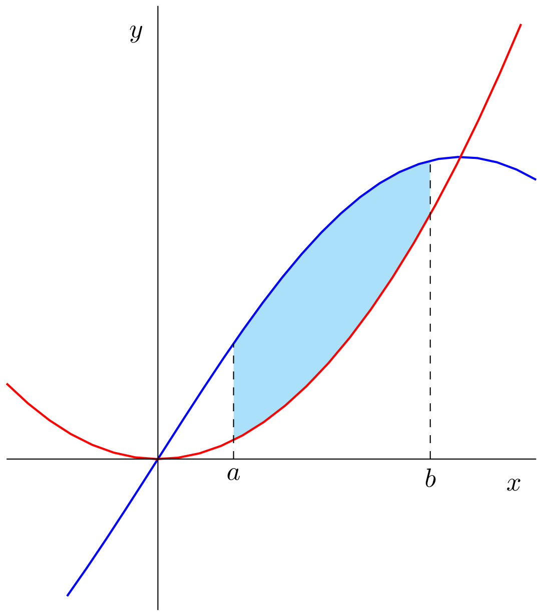
$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

4 Mean Value Theorem

If $f(x)$ is a continuous function on $I = [a; b]$, $f(x)$ will at some point in I reach its average value, meaning

$$\exists c \mid f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

5 Area Between Functions

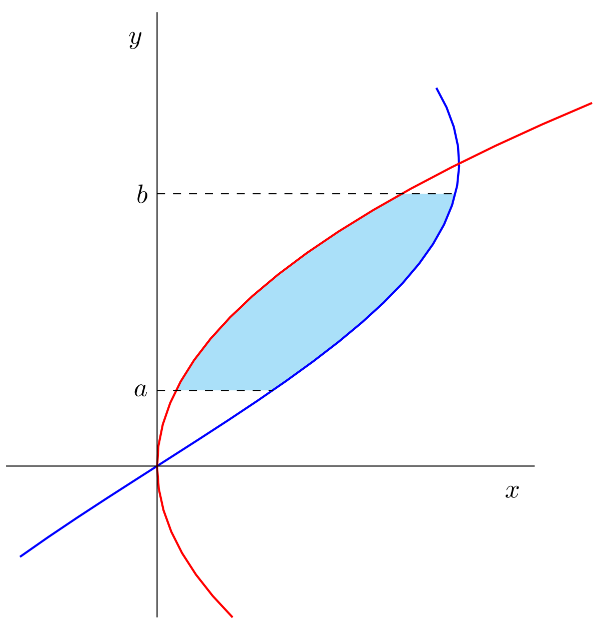


Given a function $y = f(x)$ and $y = g(x)$, the area enclosed by the two functions in the interval $I = [a; b]$ is given by

$$A = \int_a^b f(x) - g(x)\, dx$$

assuming that $f(x) \geq g(x)$ when $x \in I$.
Note that $A \geq 0$.
If $f(x) < g(x)$ for some $x \in I$ this formula won't work. However, it is still possible to split the integral into multiple integrals at every point where $f(x) - g(x)$ changes sign. To remove the sign contraint we could say

$$A = \int_a^b |f(x) - g(x)|\, dx$$

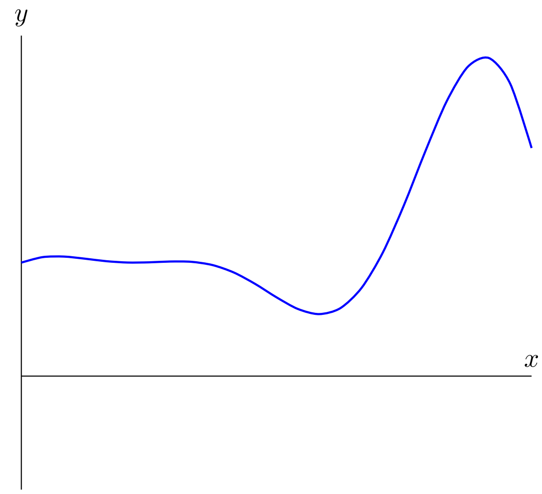


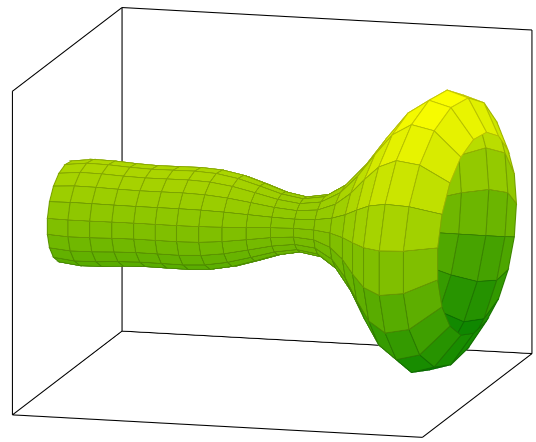
The same thing applies when we have functions in the form $x = f(y)$ and $x = g(y)$ and we want to find the areas enclosed by the functions when $y \in [d; c]$

$$A = \int_d^c |f(y) - g(y)|\, dy$$

6 Volume of solid revolutions

A solid of revolution is a solid made by rotating about an axis a function on an interval $[a; b]$.





6.1 Simple revolutions

Let's start by assuming a rotation aroud the x-axis of a function $y = f(x)$ on $[a; b]$. we divide the interval into n subsections of width $\Delta x = \frac{b-a}{n}$. For each section we then choose a point x_k^*. Every subsection is the height of a disk. The volume is therefore given by the sum of volumes of every disk.

$$V \approx \sum_{k=0}^{n} A(x_k^*)\Delta x$$

where $A(x_k^*)$ is the area of the base of the disk of the k-th subsection.
Since we are summing over disks the area of the base is given by

$$A = \pi \cdot [f(x)]^2$$

If we divide the interval into ever smaller sections, the volume becomes exact

$$V = \lim_{n\to\infty} \sum_{k=0}^{n} A(x_k^*)\Delta x$$
$$= \int_a^b A(x)\, dx$$

The same approach works for a rotation around the y-axis.

$$V = \int_a^b A(y)\, dy$$

where $A = \pi \cdot [f(y)]^2$

6.2 Hollow volumes

When the volume is obtained by rotating an area bounded by two functions, the volume is partially hollow. This can also happen with a single function. In this case we can obtain the volume as **outer volume** − **inner volume** or by summing a set of annuluses rather than disks. Using annuluses,

$$A = \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2 \right]$$

When we have disks, inner radius = 0. More complex shapes may require other area functions.

6.3 Other axis

When we are rotating a function about $x = a$ or $y = a$ we need to change the area function accordingly, therefore usually adjusting the **outer radius** and **inner radius** variables.

6.4 Method of cylinders

Sometimes it is tricky or impossible to integrate the volume using disks or annuluses. We can also cover the volume by using the area of an empty cylinder. In this case we would have

$$A = 2\pi r h$$

where r is the radius of the cylinder ad h is the height of the cylinder.