

# Complex Numbers

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# 1 Imaginary unit

## 1.1 Definition

The imaginary unit or imaginary number  $i$  is a solution to the quadratic equation  $x^2 = -1$  and is defined as

$$i^2 = -1$$

The equation  $x^2 = -1$  has two solutions:  $i$  and  $-i$ , however, there is not any algebraic difference between these two solutions.

## 1.2 Properties

The imaginary number  $i$  has some amazing properties when it comes to exponentiation.

$$\left\{ \begin{array}{l} i^0 = +1 \\ i^1 = +i \\ i^2 = -1 \\ i^3 = -i \end{array} \right. \quad \left\{ \begin{array}{l} i^4 = +1 \\ i^5 = +i \\ i^6 = -1 \\ i^7 = -i \end{array} \right. \quad \dots$$

The multiplicative inverse of  $i$  is  $-i$ .

$$\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{i^2} = -i$$

## 2 Complex Numbers

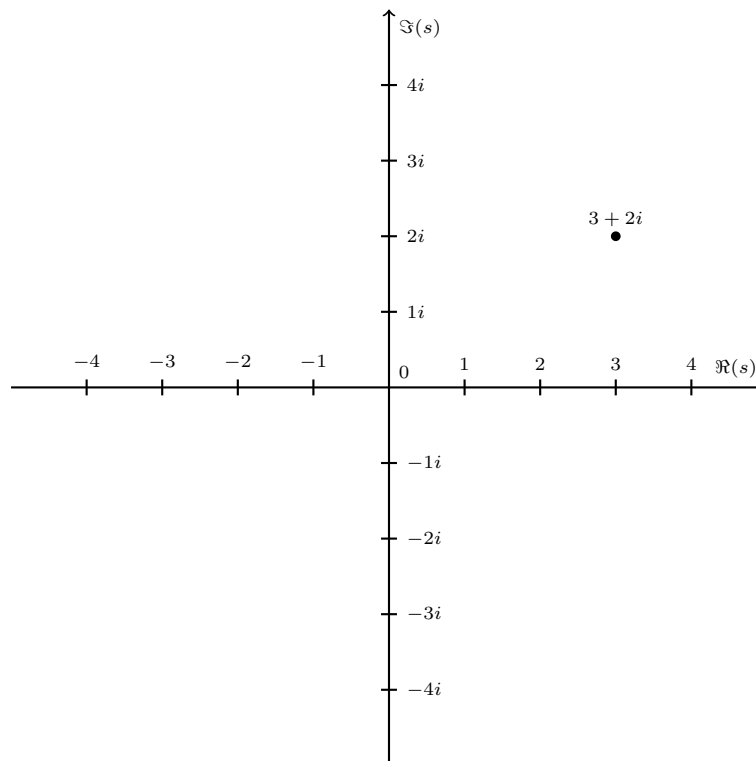
### 2.1 Definition

Complex numbers are numbers in the form  $a + bi$ , where  $a, b \in \mathbb{R}$  and  $i$  is the imaginary unit. This set of numbers is called  $\mathbb{C}$ .

Since every number  $n \in \mathbb{R}$  can be represented as a complex number in the form  $n + 0i$ ,  $\mathbb{R} \subset \mathbb{C}$ .

### 2.2 Complex plane

We can represent each complex number on a plane (Argand plane), where the horizontal axis represent the real numbers  $\mathbb{R}$  and the vertical axis represents every scalar multiple of the imaginary unit  $i$ .



### 2.3 Operations

#### 2.3.1 Addition

$$(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$$

#### 2.3.2 Subtraction

$$(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$$

#### 2.3.3 Multiplication

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - db) + (ad + bc)i$$

### 2.3.4 Division

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2i^2} \\ &= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i\end{aligned}$$

### 2.3.5 Real part

The real part of a complex number  $s$  is denoted as  $\text{Re}(s)$  or  $\Re(s)$ .

$$\text{Re}(a + bi) = a$$

### 2.3.6 Imaginary part

The imaginary part of a complex number  $s$  is denoted as  $\text{Im}(s)$  or  $\Im(s)$ .

$$\text{Im}(a + bi) = b$$

### 2.3.7 Absolute value

The absolute value (or module) of a complex number is its distance from the origin.

$$|a + bi| = \sqrt{a^2 + b^2}$$

### 2.3.8 Conjugate

The complex conjugate of a number  $s = a + bi$  is denoted as  $s^*$  or  $\bar{s}$ . It is defined as

$$\overline{a + bi} = a - bi$$

Geometrically,  $s^*$  is the reflection about the real axis in the complex plane.

We also have the following trivial properties.

$$\begin{aligned}\overline{\bar{s}} &= s \\ \text{Re}(\bar{s}) &= \text{Re}(s) \\ \text{Im}(\bar{s}) &= -\text{Im}(s)\end{aligned}$$

### 2.3.9 Argument

The argument of a complex number is the angle formed with the x-axis in the complex plane

$$\arg(a + bi) = \arctan\left(\frac{b}{a}\right)$$

### 2.3.10 Axiomatic definition

A complex number is a tuple  $(a, b)$  where  $a \in \mathbb{R}$  and  $b \in \mathbb{R}$ .

**equality**

$$(a, b) = (c, d) \implies a = c \wedge b = d$$

**Addition**

$$(a, b) + (c, d) = (a + c, b + d)$$

**Multiplication**

$$(a, b) \cdot (c, d) = (ac - db, ad + bc)$$
$$m(a, b) = (ma, mb)$$

If  $z_1, z_2, z_3 \in \mathbb{C}$ .

1.  $z_1 + z_2$  and  $z_1 z_2$  are also in  $\mathbb{C}$
2.  $z_1 + z_2 = z_2 + z_1$
3.  $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
4.  $z_1 z_2 = z_2 z_1$
5.  $z_1(z_2 z_3) = (z_1 z_2) z_3$
6.  $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
7.  $z_1 + 0 = z_1$
8.  $z_1 \cdot 1 = z_1$
9.  $\exists! z \mid z + z_1 = 0$
10.  $\exists! z \mid z \cdot z_1 = 1$

## 2.4 Trigonometric form

Any complex number can be represented in a trigonometric form

$$a + bi = r(\cos \theta + i \sin \theta)$$

where  $r$  is the absolute value and  $\theta$  is the argument.