

Group Theory

Paolo Bettelini

Contents

1	Groups	2
1.1	Cayley tables	2
1.2	Definition	2
1.3	Proof of uniqueness of the identity element	2
1.4	Proof of uniqueness of the inverse element	2
1.5	Cancellation laws	3
1.6	Inverse of Product	3
2	Subgroups	4
2.1	Definition	4
2.2	One-Step Subgroup Test	4
2.3	The centralizer subgroup	4
2.4	The conjugate subgroup	4
3	Center of a group	5

1 Groups

1.1 Cayley tables

A binary operation \circ on a finite set G can be visualized using a *Cayley table*.

Example: $G = \{0, 1\}$ and $\circ \equiv$ multiplication.

\circ	0	1
0	0	0
1	0	1

1.2 Definition

Definition Monoid

A *monoid* (G, \circ) is a tuple containing a set G and a binary operation $\circ: G \times G \rightarrow G$.

The relation must satisfy the following properties

1. **Associativity:** $\forall a, b, c \in G, a \circ (b \circ c) = (a \circ b) \circ c$
2. **Identity:** $\exists e \mid \forall a \in G, ea = ae = a$

The operation \circ between a and b may be written as $a \circ b$ or just ab .

Definition Group

A *group* (G, \circ) is a tuple containing a set G and a binary operation $\circ: G \times G \rightarrow G$.

The relation must satisfy the following properties

1. **Associativity:** $\forall a, b, c \in G, a \circ (b \circ c) = (a \circ b) \circ c$
2. **Identity:** $\exists e \mid \forall a \in G, ea = ae = a$
3. **Inverse:** $\forall a \in G, \exists a^{-1} \in G \mid a^{-1}a = aa^{-1} = e$

A group is a monoid with inverse elements.

1.3 Proof of uniqueness of the identity element

Theorem Uniqueness of the inverse element

If e is an identity element of a group, then it is unique.

Proof Uniqueness of the identity element

Suppose there is more than one identity element, e_1 and e_2 .

$$\begin{aligned} e_1 &= e_1 \circ e_2 && \text{since } e_2 \text{ is an identity} \\ &= e_2 && \text{since } e_1 \text{ is an identity} \end{aligned}$$

Thus, e_1 and e_2 must be the same. This reasoning can be extended to when we may suppose to have n identity elements.

1.4 Proof of uniqueness of the inverse element

Theorem Uniqueness of the inverse element

If a^{-1} is an inverse of a in a group, then it is unique.

Proof Uniqueness of the inverse element

Suppose we have $a \in G$ with inverses b and c .

$$\begin{aligned} b &= b \circ e = b \circ (a \circ c) \\ (b \circ a)c &= e \circ c \\ &= c \end{aligned}$$

Thus, b and c must be the same. This reasoning can be extended to when we may suppose to have n inverses of a .

1.5 Cancellation laws

Theorem Right cancellation law

$$ba = ca \implies b = c$$

Theorem Left cancellation law

$$ab = ac \implies b = c$$

1.6 Inverse of Product

Theorem Inverse of Product

Consider a group (G, \circ) . For any two elements $a, b \in G$

$$(a \circ b)^{-1} = b^{-1} \circ a^{-1}$$

Proof Inverse of Product

We start by noticing that by associativity we have

$$\begin{aligned} (a \circ b) \circ (b^{-1} \circ a^{-1}) &= a \circ (b \circ b^{-1}) \circ a^{-1} \\ &= a \circ e \circ a^{-1} \\ &= a \circ a^{-1} \\ &= e \end{aligned}$$

This implies that $(a \circ b)$ is the inverse of $(b^{-1} \circ a^{-1})$. Since $(a \circ b) \circ (a \circ b)^{-1} = e$ we have

$$(a \circ b) \circ (b^{-1} \circ a^{-1}) = e = (a \circ b) \circ (a \circ b)^{-1}$$

We can clearly see that $(b^{-1} \circ a^{-1}) = (a \circ b)^{-1}$.

In general, we have

$$(a_1 \circ a_2 \circ \dots \circ a_n)^{-1} = a_n^{-1} \circ \dots \circ a_2^{-1} \circ a_1^{-1}$$

2 Subgroups

2.1 Definition

Definition Subgroups

Given an algebraic structure $g = (G, \circ)$ and a group $h = (H, \circ)$, h is a subgroup of g ($g \leq h$) if $H \subseteq G$.

2.2 One-Step Subgroup Test

Theorem One-Step Subgroup Test

Let (G, \circ) be a group and let $H \subseteq G$ where $\emptyset \neq H$.

Then (H, \circ) is a subgroup of $(G, \circ) \iff \forall a, b \in H, a \circ b^{-1} \in H$.

Proof One-Step Subgroup Test

(\implies): Assume $(H, \circ) \leq (G, \circ)$. The properties of a group directly infer $\forall a, b \in H, a \circ b^{-1} \in H$

(\impliedby): Assume $\forall a, b \in H, a \circ b^{-1} \in H$

- **Identity:** let $a = b$, then $a \circ a^{-1}H \implies e \in H$.
- **Inverse:** Let $k \in H$, $a = e$ and $b = k$. $a \circ b^{-1} = e \circ k^{-1} \implies k^{-1} \in H$.
- **Closure:** Let $m, n \in H \implies n^{-1} \in H$ and let $a = m$ and $b = n^{-1}$. $a \circ b^{-1} = a \circ (b^{-1})^{-1} = a \circ b$. This implies $a, b \in H$.

2.3 The centralizer subgroup

Definition The centralizer subgroup

Let $H \leq G$ be groups and define

$$C_G(H) = \{g \in G \mid \forall h \in H, gh = hg\}$$

as the centralizer of H .

This is the set of all elements of G such that they commute with every element of H .

Theorem

Let $H \leq G$, then $C_G(H) \leq G$.

Proof

Suppose $a, b \in C_G(H)$. We want to show $ab^{-1} \in C_G(H)$.

Note that the condition $gh = hg \iff hg^{-1} = g^{-1}h$.

Consider the expression $(ab^{-1})h = a(b^{-1}h) = ahb^{-1} = h(ab^{-1})$. This means that $ab^{-1} \in C_G(H)$ and thus in H .

2.4 The conjugate subgroup

Definition The conjugate subgroup

Let $H \leq G$ be groups and define

$$g^{-1}Hg = \{g^{-1}hg \mid h \in H\}$$

as the conjugate subgroup.

Theorem

Let $H \leq G$, then $g^{-1}Hg \leq G$.

Proof

Suppose $a, b \in g^{-1}Hg$. We want to show $ab^{-1} \in g^{-1}Hg$.

Note that $a = g^{-1}h_1g$ and $b = g^{-1}h_2g$ for some $h_1, h_2 \in H$.

This means that $ab^{-1} = a(g^{-1}h_2g)^{-1} = a(g^{-1}h_2^{-1}g) = g^{-1}h_1gg^{-1}h_2^{-1}g = g^{-1}(h_1h_2)g \in g^{-1}Hg$ because $h_1h_2 \in H$.

3 Center of a group

Definition Center of a group

Let G be a group. The center of the group G is defined as

$$Z(G) = \{g \in G \mid \forall x \in G, gx = xg\}$$

This is the set of all elements that commute with every other element. The condition $gx = xg$ is also sometimes expressed as $gxg^{-1} = x$.

Theorem

Let G be a group, then $Z(G) \leq G$.

Proof

Assume $a, b \in Z(G)$ meaning $a = gag^{-1}$ and $b = gbg^{-1}$ for any $g \in G$.

We want to show $ab^{-1} \in Z(G)$. $ab^{-1} = (gag^{-1})(gbg^{-1})^{-1} = gag^{-1}gb^{-1}g^{-1} = gab^{-1}g^{-1}$ which is precisely the requirement to be in $Z(G)$.