Vectors

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Contents

1 Definition

A vector is a geometric object that has a direction and a magnitude.
Vectors don't have an origin point and they can be represented in any position on the space space.

A vector can be expressed with its components

$$\vec{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For an n-dimensional vector

$$\vec{a} = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{pmatrix}, \quad \vec{a} \in \mathbb{R}^n$$

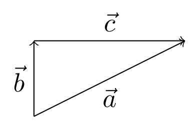
2 Addition

Vectors can be added together. This operation is commutative.

$$\vec{a} + \vec{b} = \vec{c}$$

2.1 Graphically

This plot shows how two vectors are added together in a two-dimensional space.



2.2 Using its components

The respective components of the vectors can be added to perform vector addition.

$$\vec{a} + \vec{b} = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{pmatrix} + \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_n \end{pmatrix} = \begin{pmatrix} \vec{a}_1 + \vec{b}_1 \\ \vec{a}_2 + \vec{b}_2 \\ \vdots \\ \vec{a}_n + \vec{b}_n \end{pmatrix}$$

3 Scalar Product

A vector $\vec{a}$ can be multiplied by a scalar value k

$$k \cdot \vec{a} = \begin{pmatrix} k\vec{a}_1 \\ k\vec{a}_2 \\ \vdots \\ k\vec{a}_n \end{pmatrix}, \quad k \in \mathbb{R}$$

4 Linear Combination

4.1 Definition

A linear combination is a sum of two or more vectors, each with a coefficient.

$$\vec{c} = a \cdot \vec{a} + b \cdot \vec{b}, \quad a, b \in \mathbb{R}$$

4.2 Span

The *span* of a set of vectors, is the set of every possible linear combination.

4.3 Linear dependency

Whenever you can remove a vector from a set of vectors without reducing the span, the vectors are *linearly dependent*. One of the vectors of this set can be expressed as a linear combination of the other vectors in the span.

If every vector in the set extends the span, they are *linearly independent*.

5 Magnitude

To find the magnitude (or length) of a vector $\vec{a}$ we can apply Pythagorean theorem.

$$||\vec{a}|| = \sqrt{{a_x}^2 + {a_y}^2}$$

For an n-dimensional vector.

$$||\vec{a}|| = \sqrt{{a_1}^2 + {a_1}^2 + \cdots + {a_n}^2}, \quad \vec{a} \in \mathbb{R}^n$$

6 Vector given points

We can find a vector given two points A and B.

$$A(A_x; A_y)$$
$$B(B_x; B_y)$$

The vector with direction going from A to B is given by

$$\begin{pmatrix} B_x - A_x \\ B_y - A_y \end{pmatrix}$$

For $\mathbb{R}^n$

$$\begin{pmatrix} B_1 - A_1 \\ B_2 - A_2 \\ \vdots \\ B_n - A_n \end{pmatrix}$$

7 Unitary Vector

A unitary vector (or normalized vector) is a vector of magnitude 1.

$$||\hat{a}|| = 1$$

To normalize a vector it is sufficient to divide its components by its magnitude

$$\frac{\vec{a}}{||\vec{a}||} = \hat{a}$$

$\vec{a}$ can't be the *null* vector.

8 Basis

A set of vectors in a vector space is called a *basis* if every element of the vector space can be expressed as a linear combination of the *basis vectors*.

This means that the span of a basis is always every possible vector.

$$\mathcal{B} = \left\{\vec{b_1}, \vec{b_2}, \cdots, \vec{b_n}\right\}, \quad \forall \vec{b} \in \mathcal{B}, \vec{b} \in \mathbb{R}^n$$

In other words a *basis* of a vector space is a set of linearly independent vectors that span the full vector space.

8.1 Orthonormal basis

An Orthonormal basis in a basis in which each vector is orthogonal to eachother and every vector is unitary.

Given a basis $\{\vec{i}, \vec{j}\}$

$$\vec{v} = a \cdot \hat{i} + b \cdot \hat{j}, \quad a, b \in \mathbb{R}$$

1. $\hat{i}$ is perpendicular to $\hat{j}$
2. $\hat{i}$ and $\hat{j}$ are unitary
3. $\hat{i} \neq a \cdot \hat{j}, \quad a \in \mathbb{R}$