# Complex Analysis

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## Contents

# 1 De Moivre's Theorem

Using the property of exponentiation $\left(a^b\right)^c = a^{bc}$, we can see that $\left(e^{i\theta}\right)^n = e^{in\theta}$.
Using Euler's formula we can deduce that

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta), \quad n \in \mathbb{Z}$$

# 2 Nth Roots of Units

We can extend De Moivre's Theorem for the integers powers or any complex number, rather than the ones on the unit circle ($r = 1$).

$$(r\left(\cos(\theta) + i\sin(\theta)\right))^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right), \quad n \in \mathbb{Z}$$

The nth roots of 1 are the solutions to

$$x^n = 1$$

for a given $n$. We might write 1 as a complex number

$$x^n = \cos(0) + i\sin(0)$$

Comparing this to our extended De Moivre's theorem

$$\cos(0) + i\sin(0) = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

We can see that

$$\begin{aligned} r^n &= 1 \\ n\theta &= 0 \end{aligned}$$

As long as $n \neq 0$

$$\begin{aligned} r &= 1 \\ \theta &= 0 \end{aligned}$$

By plugging these values into

$$x^n = \left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^n$$

we get that $x = 1$.

However we could also write 1 as

$$\cos(2k\pi) + i\sin(2k\pi), \quad k \in \mathbb{Z}$$

We would then get that

$$
\begin{aligned}
r^n &= 1 \\
n\theta &= 2k\pi
\end{aligned}
$$

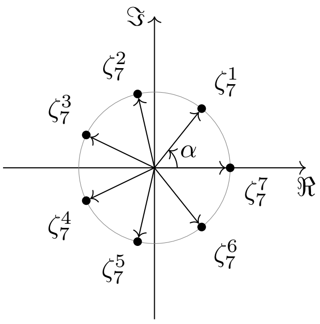
When solving for $x$ again we get

$$
\begin{aligned}
x^n &= \left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^n \\
&= \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)^n
\end{aligned}
$$

concluding that

$$
x = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)
$$

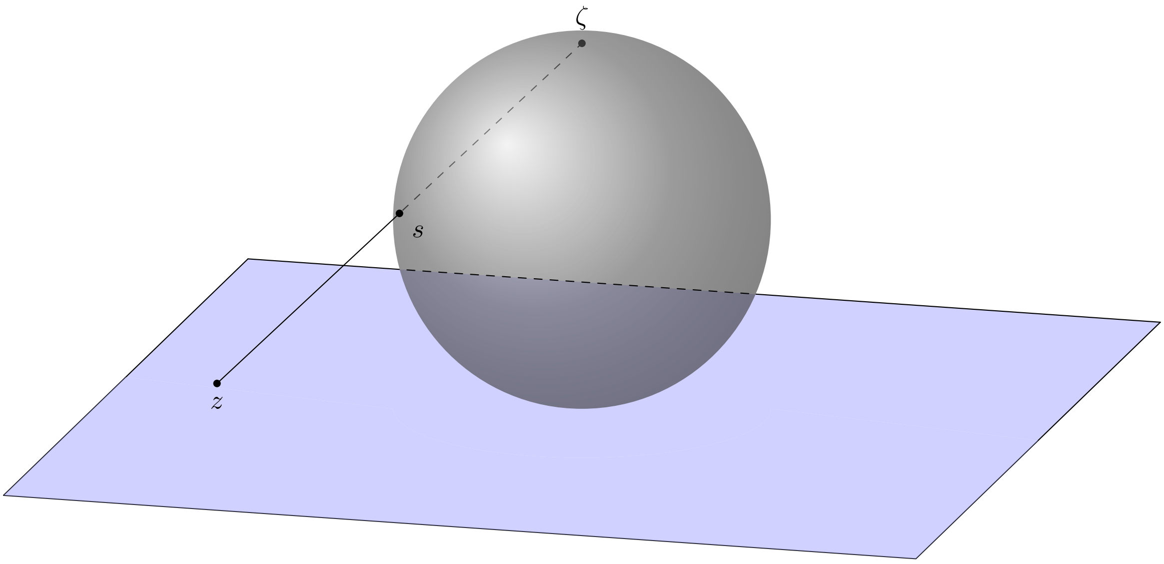
This gives us a solution for each $k$, however the solutions are redundant for $k \geq n$. In fact, the roots of unity of $n$ are $n$ distinct solutions (points on the unit circle).




The roots of units have the same angle $\alpha = \frac{2\pi}{n}$ between eachother.
The first root of unit counter-clockwise is denoted $\zeta_n$ because each subsequent root is a power of $\zeta_n$. In this case, $\zeta_7$.

# 3 Riemann Spheres

A Riemann sphere is a unit sphere used to represent the complex plane using stereographic projection.




The Riemann sphere lays on the complex plane. A complex number is represented by the intersection between the sphere and a ray starting from the topmost point of the sphere and intersecting with the given complex number on the complex plane.

# 4 Subsets of the complex plane

## 4.1 Open Disk

An open disk $D_\delta(z_0)$ is the set of points with distance less than $\delta$ from $z_0$

$$D_\delta(z_0) = \{z \in \mathbb{C} \,|\, |z - z_0| < \delta\}$$

## 4.2 Closed Disk

A closed open disk $D_\delta(z_0)$ is the set of points with distance less than or equal to $\delta$ from $z_0$

$$\overline{D_\delta(z_0)} = \{z \in \mathbb{C} \,|\, |z - z_0| \leq \delta\}$$

## 4.3 Circle

A circle $C_\delta(z_0)$ is the set of points with distance equal to $\delta$ from $z_0$

$$C_\delta(z_0) = \{z \in \mathbb{C} \,|\, |z - z_0| = \delta\}$$

## 4.4 Interior point

$z$ is an interior point of $\Omega$ iff there is an open disk at $z$ whose point are in $\Omega$

$$\exists D_{r>0}(z) \subset \Omega$$

## 4.5 Boundary point

$z$ is a boundary point of $\Omega$ iff every open disk at $z$ contains points both in $\Omega$ and not in $\Omega$.

## 4.6 Exterior point

$z$ is an exterior point of $\Omega$ iff it is not a boundary point of an interior point.

## 4.7 Accumulation points

$z$ is an accumulation point or limit point of $\Omega$ if any $D_\delta(z)\backslash\{z\}$ always contains points of $\Omega$.
In order to always contain points of $\Omega$, $\Omega$ must have an infinite amount of points, since $\delta$ can be as little as we want.

## 4.8 Open sets

A set $\Omega$ is called open if all points in $\Omega$ are interior points of $\Omega$.

## 4.9 Closed sets

A set $\Omega$ is closed if every accumulation point of $\Omega$ is in $\Omega$.

## 4.10 Bounded Set

A set $\Omega$ is bounded iff

$$\exists M > 0 \,|\, \Omega \subset D_M(0)$$

In other words there must exist an $M > 0$ such that $\forall z \in \Omega : |z| < M$

## 4.11 Connected Set

An open set $\Omega$ is connected if it cannot be written as $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1 \cap \Omega_2 = \emptyset$. In other words any two points in $\Omega$ must be connectable by a continuous curve where all the points of the curve are also in $\Omega$.

# 5 Differentiability

## 5.1 Derivative

Let $f(z)$ be a complex-valued function of a complex value which can be written as $f(x+iy) = u(x,y) + iv(x,y)$. Then

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

Note that $\Delta z$ can approach 0 from infinite directions. For the derivative to exist, the answer should not depend on how $\Delta z$ tends to 0.

## 5.2 Cauchy-Riemann Equations

Let us write $\Delta z = \Delta x + i\Delta y$.
We now compute $f'(z)$ by approaching $z$ from the horizontal direction ($\Delta y = 0$).

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta x \to 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x + iy) - f(x + iy)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(u(x + \Delta x, y) + iv(x + \Delta x, y)) - (u(x,y) + iv(x,y))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x + \Delta x, y) - v(x,y)}{\Delta x} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\end{aligned}$$

We now compute $f'(z)$ by approaching $z$ from the vertical direction ($\Delta x = 0$).

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta y \to 0} \frac{f(z + \Delta y) - f(z)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{f(x + iy + i\Delta y) - f(x + iy)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{(u(x, y + \Delta y) + iv(x, y + \Delta y)) - (u(x,y) + iv(x,y))}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y + \Delta y) - v(x,y)}{i\Delta y} \\
&= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}
\end{aligned}$$

We have found two different representations of $f'(z)$ in terms of the partial derivatives of $u$ and $v$.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

From this equality we can derive tha **Cauchy-Riemann equations**.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Thus, if $f'$ exists at the point $z$ the Cauchy-Riemann equations must hold at that point.

### 5.3 Sufficient condition

A complex function $f(z)$ is differentiable at a point $z$ iff

1. The Cauchy-Riemann equations hold at $z$.
2. $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x}$ are continuous.

## 6 Holomorphic functions

### 6.1 Definition

A function is holomorphic in $\Omega$ if it is complex differentiable in a neighbourhood of each point of $\Omega$.

In complex analysis, the holomorphic and analytic properties are equivalent.

### 6.2 Sufficient condition for analytic functions

A function $f(z)$ is analytic in a region $\Omega$ if it is differentable in a neighborhood of every point in $\Omega$.

### 6.3 Entire functions

A function $f(z)$ is entire if it is analytic on the complete complex plane $\mathbb{C}$.

# 7 Complex integration

## 7.1 Complex integrals

Let $f(t)$ be a complex-valued function of a real parameter $t$. Then we can decompose $f$ into its real and imaginary parts

$$\int\limits_a^b f(t)\,dt = \int\limits_a^b u(t)\,dt + i\int\limits_a^b v(t)\,dt$$

## 7.2 Contour integrals

Let $f(z)$ be a complex-valued function of a complex parameter $z$. When computing a definite integral we need a way to go from $z_0$ to $z_1$.

$$\int\limits_{z_0}^{z_1} f(z)\,dz$$

In order to compute this we need a continuous parametrised curve $z : [t_0; t_1] \to \mathbb{C}$ such that $z(t_0) = z_0$ and $z(t_1) = z_1$. Let $\Gamma$ be a smooth curve from $z_0$ to $z_1$, then

$$\int_\Gamma f(z)\,[z] = \int\limits_{t_0}^{t_1} f(z(t))z'(t)\,dt$$