Fourier Analysis

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1 Abstract

Mathematician and physicist Joseph Fourier determined that a function can be represented as a series of sines and cosines of different frequencies and different amplitudes. Fourier is notoriously known for having developed Fourier series and the Fourier transform.

2 Fourier Series vs Fourier Transform

Fourier Series
The Fourier series is the representation of a periodic function with a summation of sine and cosine waves of discrete frequencies. Each wave is weighted according to "how important" it is to represent the original function.
Fourier Series are often represented in two ways: trigonometric and exponential. They both work in the same way, but the exponential one is also defined on the complex plane and as we'll see, has a nicer, more elegant form.

Fourier Transform
The Fourier transform is an operation that transforms a signal from time-domain to a continuous frequency-domain. The function can be a generic, not necessarily period function $f(t)$. The output of the Fourier transform is a complex-valued function whose absolute value represents the magnitude of each frequency.

3 A Simple Example

Let's look at a simple example. We are going to derive the Fourier series of a function $f(x)$ defined as such:

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ +1 & \text{if } 0 < x < \pi \end{cases}$$

The period of this function is $T = 2\pi$. We can already simplify the $\frac{2\pi}{T}$ term, leaving us with

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

First, we need to find a_n. Simplifying $\frac{2\pi}{T}$ and $\frac{T}{2}$ we get

$$a_n = \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

Looking at the graph we notice that we can split the integral into two parts at $x = 0$. On the left part, the function is $-\cos(nx)$, while on the right part the function is $\cos(nx)$.

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{0} -\cos(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} \cos(nx)\, dx \\ &= -\frac{1}{\pi} \int_{-\pi}^{0} \cos(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} \cos(nx)\, dx \\ &= -\frac{1}{\pi} \left[\frac{\sin(xn)}{n} \right]_{-\pi}^{0} + \frac{1}{\pi} \left[\frac{\sin(xn)}{n} \right]_{0}^{-\pi} \\ &= -\frac{1}{\pi} \left[\frac{\sin(\pi n)}{n} \right] + \frac{1}{\pi} \left[\frac{\sin(\pi n)}{n} \right] \\ &= \left(\frac{1}{\pi} - \frac{1}{\pi} \right) \left[\frac{\sin(\pi n)}{n} \right] \\ &= 0 \end{aligned}$$

a_n is always going to be 0. (note). We can remove the $a_n \cos(nx)$ and $\frac{a_0}{2}$ terms from the series.

Now for b_n

$$b_n = \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Again, we split the integral into two parts

$$
\begin{aligned}
b_n &= -\frac{1}{\pi}\int\limits_{-\pi}^{0} \sin(nx)\,dx + \frac{1}{\pi}\int\limits_{0}^{\pi} \sin(nx)\,dx \\
&= -\frac{1}{\pi}\left[\frac{-\cos(xn)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-\cos(xn)}{n}\right]_{0}^{-\pi} \\
&= -\frac{1}{\pi}\left[-\frac{1}{n} + \frac{\cos(\pi n)}{n}\right] + \frac{1}{\pi}\left[\frac{-\cos(\pi n)}{n} + \frac{1}{n}\right] \\
&= -\frac{1}{\pi}\left[\frac{\cos(\pi n) - 1}{n}\right] + \frac{1}{\pi}\left[\frac{1 - \cos(\pi n)}{n}\right] \\
&= \frac{2}{\pi} \cdot \frac{1 - \cos(\pi n)}{n} \\
&= \frac{2 - 2\cos(\pi n)}{\pi n}
\end{aligned}
$$

Given b_n our series is now complete!

$$
f(x) = \sum_{n=1}^{\infty} \frac{2 - 2\cos(\pi n)}{\pi n} \cdot \sin(nx)
$$

We won't simplify this further, therefore this is our final result.
The effort pays off when we graph this function, as more terms are added, the function looks more and more like the original square wave.