1 Plane

A plane can be uniquely represented by its normal vector $\vec{n}$ and a point on the plane P_0.

To describe the plane using an equation, we can consider an arbitrary point $P = (x, y, z)$ on the plane. There is always a 90 degrees angle between the normal vector and the vector from P_0 to P (i.e., their dot product is zero)

$$\vec{n} \cdot \overrightarrow{P_0 P} = 0$$

By plugging in the values for $\vec{n}$ and P_0 we get an equation in the form

$$Ax + By + Cz + D = 0$$

2 Vector-Valued Function

A vector-valued function is a function of a real parameter which returns a vector

$$r(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$$

3 Tangent Vector Vector-Valued Function

Given a vector-valued function

$$r(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$$

where f, g and h are differentiable, then the Tangent vector to the curve is given b

$$r'(t) = \begin{pmatrix} f'(t) \\ g'(t) \\ h'(t) \end{pmatrix}$$

4 Curve length

The length of the curve beteen a and b is the integral from a to b of $\sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$.

5 Curvature

The tangent vector is given by

$$\vec{T} = \frac{d\vec{r}}{ds}$$

TODO: a smooth curve has $r'(t)$ continuous and $r'(t) \neq 0$

The curvature of a curve is given by

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

where $\vec{T}$ is the unit tangent and s is the arc length. By the chain rule this can also be written as

$$\left\| \frac{d\vec{T}}{dt} \frac{dt}{ds} \right\| = \frac{1}{\|\vec{v}\|} \left\| \frac{d\vec{T}}{dt} \right\|$$

where $\vec{v}$ is the velocity vector.

6 Principle unit normal

When $\kappa \neq 0$

$$\vec{N} = \frac{1}{k}\frac{d\vec{r}}{ds}$$

The curvature tells how much the curve is curving while the principle unit normal tells in which direction it is curving (normal to the curvature).

7 Vectors describing motion

Given a vector-valued curve $r(t)$, the motion can be described by

- the tangent vector (derivative of the position with respect to the arclength parameter), which the direction in which the curve is going.
- the normal vector captures the way in which the tangent vector is itself changing.
- Binormal vector is the cross product, which represents the vector normal to the plane created by the tangent vector and the normal vector. That is, it captures the torsion of said plane.

$$\begin{aligned}
\vec{T} &= \frac{d\vec{r}}{ds} \\
\vec{N} &= \frac{d\vec{T}}{ds} \\
\vec{B} &= \vec{T} \times \vec{N}
\end{aligned}$$

The change in the binormal vector is given by

$$\begin{aligned}
\frac{d\vec{B}}{ds} = \frac{d(\vec{T} \times \vec{N})}{ds} &= \frac{d\vec{T}}{ds} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds} \\
&= \kappa\vec{N} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds} \\
&= \vec{T} \times \frac{d\vec{N}}{ds}
\end{aligned}$$

Clearly, $\frac{d\vec{B}}{ds}$ is orthogonal to both $\vec{B}$ and $\vec{T}$ and thus it is parallel to $\vec{N}$. Indeed,

$$\begin{aligned}
\tau\vec{N} &= -\frac{d\vec{B}}{ds} \\
\tau &= -\frac{d\vec{B}}{ds} \cdot \vec{N}
\end{aligned}$$

note that the minus sign is by convention. This value is called torsion.

8 Acceleration

The acceleration has both a tangential and normal component.

$$\begin{aligned}
\vec{a} = \frac{d\vec{v}}{dt} &= \frac{d|\vec{v}|}{dt}\vec{T} + |\vec{v}|\frac{d\vec{T}}{dt} \\
&= \frac{d|\vec{v}|}{dt}\vec{T} + |\vec{v}|\left|\frac{d\vec{T}}{dt}\right|\vec{N} \\
&= \frac{d|\vec{v}|}{dt}\vec{T} + \kappa|\vec{v}|^2\vec{N}
\end{aligned}$$

9 Limits

Limits in multivariable calculus can be approached by infinite directions. A limit exists iff it is equal from all the directions.

10 Differentiability

Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a function. If the partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_m}$ exist and are continuous in an open region R, then f is differentiable in R.

11 Exercises

11.1 Open set 1

Prove that the set $A = \{(x, y) \in \mathbb{R}^2 \,|\, 2 < x^2 + y^2 < 4\}$ is open.

Let $p = (x, y)$ where $p \in A$. The set A is open iff $\exists\epsilon > 0 \,|\, B_\epsilon(p) \subset A$. Let $d = \sqrt{x^2 + y^2}$. For a radius $\epsilon \leq \min(d - \sqrt{2}, d - \sqrt{4})$, the open ball $B_\epsilon(p) \subset A$.

11.2 Norm 1

TODO

11.3 Countable set

TODO

11.4 Open set 2

Prove that the set $A = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 < y < x\}$ is open.

TODO

11.5 Sequence 1

Consider the sequence $\{x_k\}$ in $\mathbb{R}^2$ defined by

$$x_k = \left(\sin\left(\frac{\pi k}{2}\right), \frac{(-1)^k}{\sqrt{k}}\right)$$

for each $k \in \mathbb{N}^*$. Determine whether $\{x_k\}$ is bounded, and if so, find a convergent subsequence and identity its limit.

The sinusoidal part of the pair of the sequence is bounded because $-1 \leq \sin\theta \leq 1$. The other part has a numerator oscillating between 1 and -1, and the denominator goes from 1 to $+\infty$ in the limit. Thus, the sequence is absolutely decreasing and $-1 \leq \{x_k\} \leq \frac{1}{\sqrt{2}}$. We now notice that

$$\sin\left(\frac{\pi k}{2}\right) = \begin{cases} 1 \text{ or } -1, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

By considering the subsequence where k is even we get a converging sequence

$$\lim_{k\to\infty} \{x_{2k}\} = (0, 0)$$

11.6 Level curves

TODO: the level curve is the curve on the horizontal plane intersecting a 2 variables function

Find the equation of the level curve of the function $f(x, y)$ that passes through the given point p.

1. $f(x, y) = 16 - x^2 - y^2$ and $p = (2\sqrt{2}, \sqrt{2})$;
2. $f(x, y) = \sqrt{x^2 - 1}$ and $p = (1, 0)$;
3. $f(x, y) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}}$ and $p = (0, 1)$.

1. $f(2\sqrt{2}, \sqrt{2}) = 6$, so the height of the plane is 6. By plugging $z = 6$ in we get $6 = 16 - x^2 - y^2$ and thus the level curve is given by $10 = x^2 + y^2$;
2. $f(1, 0) = 0$, so the height of the plane is 0. By plugging $z = 0$ in we get $0 = \sqrt{x^2 - 1}$ and thus the level curve is given by $x^2 = 1$, that is the lines $x = 1$ and $x = -1$.

3. We first solve the integral
$$\int_x^y \frac{d\theta}{\sqrt{1-\theta^2}} = \arcsin(y) - \arcsin(x)$$
Now we can evaluate $f(0,1) = \arcsin(1) - \arcsin(0) = \frac{\pi}{2}$, so the height of the plane is $\frac{\pi}{2}$. By plugging $z = \frac{\pi}{2}$ in we get
$$\begin{aligned}\frac{\pi}{2} &= \arcsin(y) - \arcsin(x)\\ y &= \sin\left(\frac{\pi}{2} + \arcsin(x)\right)\\ y &= \sqrt{1-x^2}\end{aligned}$$
with $x < 0$.

11.7 Limits 1

Solve the limit $\lim_{(x,y)\to(2,1)} \frac{x^2-3y}{x+2y^2}$

$$\lim_{(x,y)\to(2,1)} \frac{x^2-3y}{x+2y^2} = \frac{2^2-3}{2+2} = \frac{1}{4}$$

11.8 Limits 2

Solve the limit $\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2}$

By using polar coordinate $x = r\cos\theta$ and $y = r\sin\theta$, and thus $x^2+y^2 = r^2$, therefore

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r)}{r} = 1$$

11.9 Limits 3

Solve the limit $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

11.10 Limits 4

Solve the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^4}$