

# Differential Equations

Paolo Bettelini

## Contents

<b>1</b>	<b>Definition</b>	<b>2</b>
<b>2</b>	<b>Slope Field</b>	<b>2</b>
<b>3</b>	<b>Euler's Method</b>	<b>2</b>

# 1 Definition

Differential equations are equations where the solution is a function or a set of functions.

## 2 First-Order Differential Equations

A first-order differential equation is a differential equation in the form

$$y'(t) = f(t, y(t))$$

where  $f$  is given.

The equation is said to be *linear* iff  $f$  is linear on the second argument.

$$y'(t) = a(t)y(t) + b(t)$$

The equation is also said to be *constant* iff  $a$  and  $b$  are also constant.

### 2.1 Linear Constant Coefficients Differential Equations

**Theorem.** *The general solution to the constant differential equation*

$$y' = ay + b, \quad a \neq 0$$

*is given by*

$$y(t) = Ce^{at} - \frac{b}{a}, \quad C \in \mathbb{R}$$

*Proof.* Let's first consider the case when  $b = 0$ ,

$$y' = ay$$

We divide both sides by  $y$  and simplify

$$\frac{y'}{y} = a \implies \ln|y'| = a \implies \ln|y| = at + c_0$$

concluding that

$$y = \pm e^{at+c_0} = \pm e^{c_0} \cdot e^{at} = Ce^{at}$$

Now let's consider  $b \in \mathbb{R}$

$$y' = a \left( y + \frac{b}{a} \right) \implies \left( y + \frac{b}{a} \right)' = a \left( y + \frac{b}{a} \right)$$

Note that  $\frac{d}{dx} \left( \frac{b}{a} \right) = 0$

Denoting  $\tilde{y} = y + \frac{b}{a}$ , we have

$$\tilde{y}' = a\tilde{y}$$

which has solution  $Ce^{at}$ , hence

$$\begin{aligned} y + \frac{b}{a} &= Ce^{at} \\ y &= Ce^{at} - \frac{b}{a} \end{aligned}$$

■

It is important to note that we solved the equation by turning it into a total derivative, which is simple to integrate ( $\ln |y|' = a$ ). This function is called a *potential function* ( $\psi$ ) and it's how the equation is transformed into a total derivative

$$y' = ay + b \rightarrow \psi(t, y(t))' = 0$$

In this case

$$\psi = \ln |y| - at$$

**The Integrating Factor Method** The integrating factor method is a method for solving linear differential equations.

We will prove the theorem again using this method.

*Proof.* We choose an integrating factor to be a function  $\mu$  such that

$$\mu' = -a\mu$$

By solving this differential equation we get

$$\frac{\mu'}{\mu} = -a \implies \ln |\mu| = -at + C \implies \mu(t) = Ce^{-at}$$

Now we multiply the equation by  $\mu$

$$\begin{aligned} y' - ay &= b \\ y'\mu - \mu ay &= b\mu \\ y'\mu + \mu'y &= b\mu \\ (\mu y)' &= \mu b \end{aligned}$$

Now choosing  $C = 1$

$$\begin{aligned} (e^{-at}y)' &= be^{-at} \\ (e^{-at}y)' &= \left(-\frac{b}{a}e^{-at}\right)' \\ \left(e^{-at}y + \frac{b}{a}e^{-at}\right)' &= 0 \\ \left[\left(\frac{b}{a} + y\right)e^{-at}\right]' &= 0 \end{aligned}$$

Now the differential equation is a total derivative of the potential function, which in this case is

$$\psi(t, y) = \left(\frac{b}{a} + y\right)e^{-at}$$

This is easy to integrate

$$\left(\frac{b}{a} + y\right)e^{-at} = C \implies y = Ce^{at} - \frac{b}{a}$$

■

## 2.2 Linear Constant Coefficients Differential Equations with Initial Point

We want to constraint the equation such that it has an unique solution rather than infinite solutions.

$$y' = ay + b, \quad y(t_0) = y_0$$

**Theorem.** *The genera solution to the ordinary constant differential equation*

$$y' = ay + b,$$

*with a given point*

$$y(t_0) = y_0$$

*is given by*

$$y(t) = \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a}, \quad a \neq 0$$

*Proof.* Starting from the general solution of a constant ordinary differential equation

$$y(t_0) = y_0 = Ce^{at_0} - \frac{b}{a}$$

meaning that

$$C = \left(y_0 + \frac{b}{a}\right) e^{-at_0}$$

this constraints out result to

$$\begin{aligned} y(t) &= Ce^{at} - \frac{b}{a} \\ &= \left(y_0 + \frac{b}{a}\right) e^{-at_0} e^{at} - \frac{b}{a} \\ &= \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a} \end{aligned}$$

■

## 2.3 Linear Variable Coefficients Differential Equations

An ordinary linear differential equation with variable coefficients is defined as

$$y' = a(t)y(t) + b(t)$$

The solution to the linear equation with constant coefficients still applies if  $\frac{b}{a}$  is constant.

**Theorem.** *The generation solution to the differential equation*

$$y' = a(t)y(t) + b(t)$$

*is given by*

$$y(t) = Ce^{A(t)} + e^{A(t)} \int e^{-A(t)} b(t) dt$$

where  $A(t) = \int a dt$ ,  $c \in \mathbb{R}$  and  $a$  and  $b$  are continuous.

*Proof.* Let's start by letting  $b(t) = 0$

$$y' = ay \implies \frac{y'}{y} = a \implies \ln |y|' = a \implies \ln |y| = \int a \, dt$$

concluding that

$$y = \pm e^{A+c_0} = \pm e^{c_0} \cdot e^A = C e^A$$

where  $A = \int a \, dt$ .

As previously, we choose an integrating factor  $\mu$  such that

$$-a\mu = \mu'$$

By solving this differential equation we get

$$\frac{\mu'}{\mu} = -a \implies \ln |\mu| = -A + C \implies \mu(t) = C e^{-A}$$

And by choosing  $C = 1$  we have

$$\mu(t) = e^{-A(t)}$$

Now we multiply our equation by the integrating factor

$$\begin{aligned} y' - ay &= b \\ y'\mu - a\mu y &= \mu b \\ y'\mu + \mu' y &= \mu b \\ (y\mu)' &= \mu b \\ \left(e^{-A(t)} y\right)' &= e^{-A(t)} b \\ e^{-A(t)} y &= \int e^{-A(t)} b \, dt + C \\ y(t) &= C e^{A(t)} + e^{A(t)} \int e^{-A(t)} b \, dt \end{aligned}$$

■

## 2.4 Linear Variable Coefficients Differential Equations with Initial Point

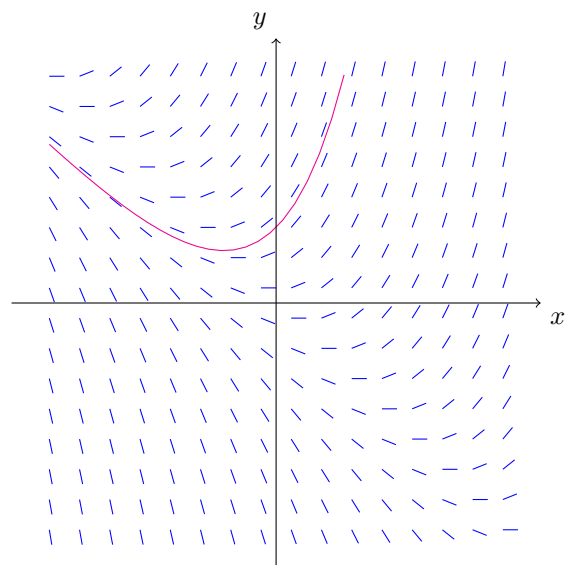
A linear differential equation with variable coefficients and an initial point is given by

$$y'(t) = a(t)y(t) + b(t), \quad y(t_0) = y_0$$

**Theorem.** *TODO*

### 3 Slope Field

A slope field or directional field is a field to visualize solutions to a first-order differential equation.



Slope field of  $\frac{dy}{dx} = x + y$ .

This field is obtained by picking points on the plane. For each point  $(x, y)$  we know that the slope ( $\frac{dy}{dx}$ ) is  $x + y$ . This means that if a solution passes through  $(x, y)$ , then its slope is  $x + y$ . The red curve shows a solution.

### 4 Euler's Method

Euler's method is a technique for solving a first-order differential equation numerically given a point of the solution.

Starting at the known solution point  $A_0$ , we take small steps the direction of the slope field. As the length of the steps  $s \rightarrow 0$  we approach the solution to the equation.

The angle of the slope is given by

$$\theta = \tan\left(\frac{dy}{dx}\right)$$

so each step gives the sequence of points

$$A_n = A_{n-1} \cdot s(\cos(\theta), \sin(\theta))$$