# Euler's Formula

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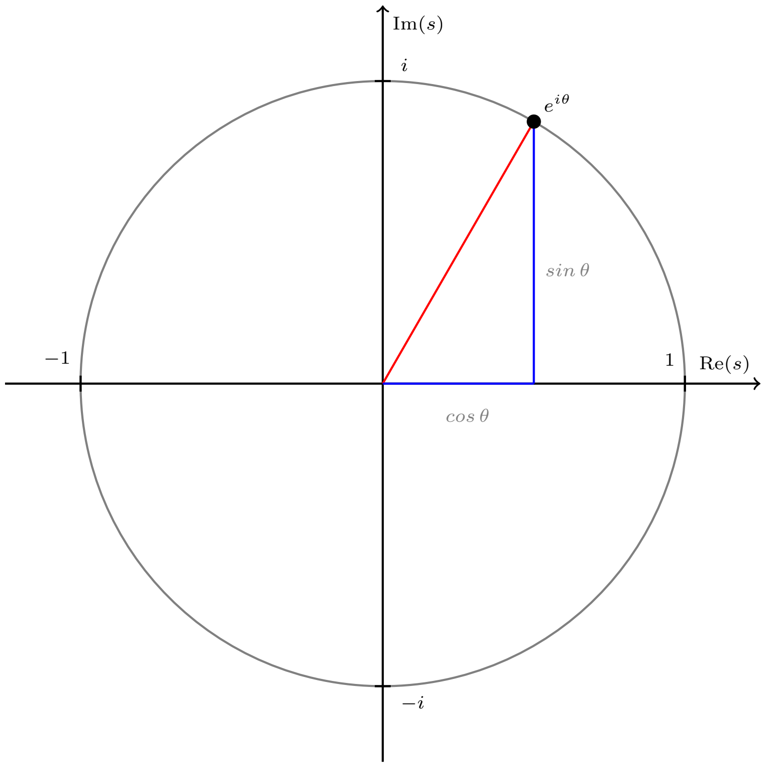
# 1 Definition

Euler's formula states that for every $x \in \mathbb{R}$

$$e^{ix} = cos\, x + i\, sin\, x$$

We can represent the formula on the complex plane




We can notice that $\left|e^{ix}\right| = 1$ since $|e^{i\theta}| = cos^2\theta + sin^2\theta = 1$

# 2 Proof

To understand this identity we must first look at the Taylor series of some functions.
These functions are all equal to their Taylor series, so we can easily compute their Maclaurin series.

## 2.1 Exponential function

The first thing we can notice is that the derivative of $f(x) = e^x$ is $e^x$ itself, so

$$f^{(n)}(x) = e^x, \quad n \in \mathbb{Z}$$

which means that for every term of the Maclaurin series

$$f^{(n)}(a) = f^{(n)}(0) = e^0 = 1$$

The exponential function can then be expressed as the series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

## 2.2 Sine function

Here's a table of the derivatives of $f(x) = sin(x)$ and their value at $a = 0$

$$\begin{cases} f^{(0)}(x) = sin(x), & f^{(0)}(0) = 0 \\ f^{(1)}(x) = cos(x), & f^{(1)}(0) = 1 \\ f^{(2)}(x) = -sin(x), & f^{(2)}(0) = 0 \\ f^{(3)}(x) = -cos(x), & f^{(3)}(0) = -1 \end{cases}$$

The next derivative is $f^{(4)}(x) = sin(x)$ making the sequence start over.
We can notice that every even n-th derivative will multiply the term by 0.
We will use $2n + 1$ to skip all the even integers.
We can also notice that the sign of the term is given by $(-1)^n$ since we only consider odd indexes.

$$\begin{aligned} sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^{2n+1}(-1)^n}{(2n+1))!} \end{aligned}$$

## 2.3 Cosine function

Here's a table of the derivatives of $f(x) = cos(x)$ and their value at $a = 0$

$$\begin{cases} f^{(0)}(x) = cos(x), & f^{(0)}(0) = 1 \\ f^{(1)}(x) = -sin(x), & f^{(1)}(0) = 0 \\ f^{(2)}(x) = -cos(x), & f^{(2)}(0) = -1 \\ f^{(3)}(x) = sin(x), & f^{(3)}(0) = 0 \end{cases}$$

The next derivative is $f^{(4)}(x) = cos(x)$ making the sequence start over.
We can notice that every odd n-th derivative will multiply the term by 0.
We will use $2n$ to skip all the odd integers.
We can also notice that the sign of the term is given by $(-1)^n$ since we only consider even indexes.

$$\begin{aligned} cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^n}{(2n)!} \end{aligned}$$

## 2.4 Conclusion

Given the Taylor series for the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we use $ix$ instead of $x$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

The imaginary number $i$ has some amazing property when it comes to exponentiation.

$$\begin{cases} i^0 = +1 \\ i^1 = +i \\ i^2 = -1 \\ i^3 = -i \end{cases} \qquad \begin{cases} i^4 = +1 \\ i^5 = +i \\ i^6 = -1 \\ i^7 = -i \end{cases} \qquad \cdots$$

We can use these properties to simplify the $e^{ix}$ Taylor series

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

We notice that the two terms correspond to the sine and cosine Taylor series

$$e^{ix} = cos\,x + i\,sin\,x$$