Series

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1 Divergence and convergence

An infinite series converges if the limit of its partial sum sequence also converges, otherwise it diverges.

2 Properties

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$$

3 Covergence theorem

Theorem. *If* $\sum a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$

Proof. Consider the partial sum

$$s_n = \sum_{k=1}^{n} a_k$$

The sequence a_n can now be expressed as

$$a_n = s_n - s_{n-1}$$

Since $\sum a_n$ converges, $\lim_{n\to\infty} s_n = L$ for L finite.
The limit $\lim_{n\to\infty} s_{n-1} = L$ because $n-1 \to \infty$ as $n \to \infty$. This implies the following

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - s_{n-1} = L - L = 0$$

□

4 Divergence test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.

5 Absolute and conditional convergence

A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. This is a stronger type of convergence. Every absolutely convergent series is also convergent.

A series that is convergent but not absolutely convergent is called conditionally convergent.

6 Riemann rearrangement theorem

If a series is conditionally convergent, then its terms can be rearranged such that the series converges to any $r \in \mathbb{R}$ or such that it diverges (to infinity or no finite value). If the series is absolutely convergent then any rearrangement of its terms will converge to the same value.