Complex Numbers

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Contents

1 Imaginary unit

1.1 Definition

The imaginary unit or imaginary number i is a solution to the quadratic equation $x^2 = -1$ and is defined as

$$i^2 = -1$$

The equation $x^2 = -1$ has two solutions: i and $-i$, however, there is not any algebraic difference between these two solutions.

1.2 Properties

The imaginary number i has some amazing properties when it comes to exponentiation.

$$\begin{cases} i^0 = +1 \\ i^1 = +i \\ i^2 = -1 \\ i^3 = -i \end{cases} \qquad \begin{cases} i^4 = +1 \\ i^5 = +i \\ i^6 = -1 \\ i^7 = -i \end{cases} \qquad \cdots$$

The multiplicative inverse of i is $-i$.

$$\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{i^2} = -i$$

2 Complex Numbers

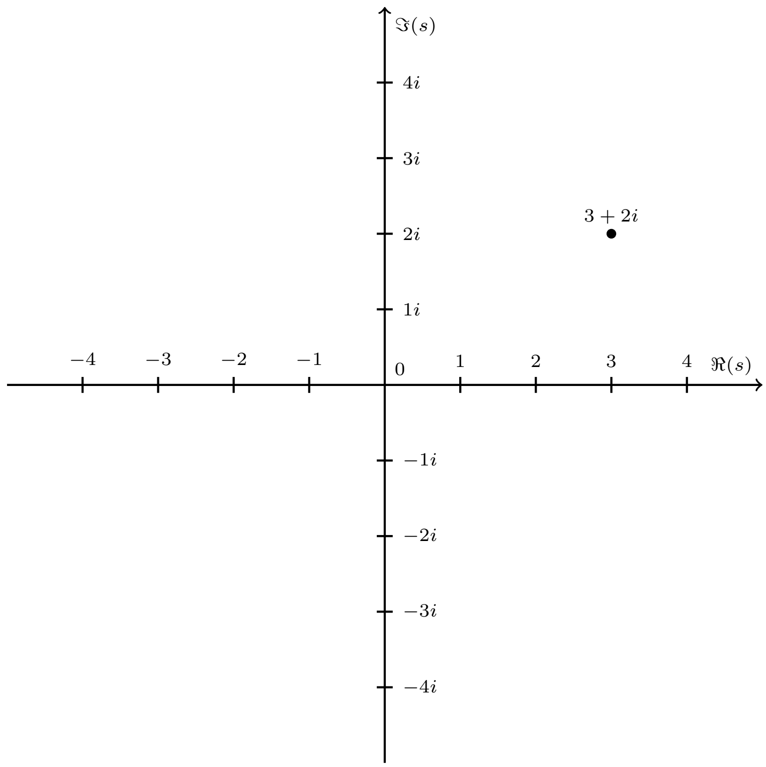
2.1 Definition

Complex numbers are numbers in the form $a + bi$, where $a, b \in \mathbb{R}$ and i is the imaginary unit.
This set of numbers is called $\mathbb{C}$.

Since every number $n \in \mathbb{R}$ can be reprsented as a complex number in the form $n + 0i$, $\mathbb{R} \subset \mathbb{C}$.

2.2 Complex plane

We can represent each complex number on a plane, where the horizontal axis represent the real numbers $\mathbb{R}$ and the vertical axis represents every scalar multiple of the imaginary unit i.



2.3 Operations

2.3.1 Real part

The real part of a complex number s is denoted by $\mathrm{Re}(s)$ or $\Re(s)$.

$$\mathrm{Re}(a + bi) = a$$

2.3.2 Imaginary part

The imaginary part of a complex number s is denoted by $\mathrm{Im}(s)$ or $\Im(s)$.

$$\mathrm{Im}(a + bi) = b$$

2.3.3 Absolute value

The absolute value of a complex number is its distance from the origin.

$$|a + bi| = \sqrt{a^2 + b^2}$$

2.3.4 Conjugate

The complex conjugate of a number $s = a + bi$ is denoted as s^* or $\overline{s}$. It is defined as

$$\overline{a + bi} = a - bi$$

Geometrically, s^* is the reflection about the real axis in the complex plane.

We also have the following trivial properties.

$$\begin{aligned}
\overline{\overline{s}} &= s \\
\mathrm{Re}(\overline{s}) &= \mathrm{Re}(s) \\
\mathrm{Im}(\overline{s}) &= -\mathrm{Im}(s)
\end{aligned}$$