Differential Equations

Paolo Bettelini

Contents

1 Definition

Differential equations are equations where the solution is a function or a set of functions.

2 First-Order Differential Equations

A first-order differential equation is a differential equation in the form

$$y'(t) = f(t, y(t))$$

where f is given.

The equation is said to be *linear* iff f is linear on the second argument.

$$y'(t) = a(t)y(t) + b(t)$$

The equation is also said to be *constant* iff a and b are also constant.

2.1 Constant Linear Differential Equations

Theorem. *The general solution to the constant differential equation*

$$y' = ay + b, \quad a \neq 0$$

is

$$y(t) = Ce^{at} - \frac{b}{a}, \quad C \in \mathbb{R}$$

Proof. Let's first consider the case when $b = 0$,

$$y' = ay$$

We divide both sides by y and simplify

$$\frac{y'}{y} = a \implies \ln|y|' = a \implies \ln|y| = at + c_0$$

concluding that

$$y = \pm e^{at+c_0} = \pm e^{c_0} \cdot e^{at} = Ce^{at}$$

Now let's consider $b \in \mathbb{R}$

$$y' = a\left(y + \frac{b}{a}\right) \implies \left(y + \frac{b}{a}\right)' = a\left(y + \frac{b}{a}\right)$$

Note that $\frac{d}{dx}\left(\frac{b}{a}\right) = 0$
Denoting $\tilde{y} = y + \frac{b}{a}$, we have

$$\tilde{y} = a\tilde{y}$$

which has solution Ce^{at}, hence

$$y + \frac{b}{a} = Ce^{at}$$
$$y = Ce^{at} - \frac{b}{a}$$

∎

It is important to note that we solved the equation by turning it into a total derivative, which is simple to integrate ($\ln|y|' = a$). This function is called a *potential function* (ψ) and it's how the equation is transformed into a total derivative

$$y' = ay + b \to \psi(t, y(t))' = 0$$

In this case

$$\psi = \ln|y| - at$$

The Integrating Factor Method The integrating factor method is a method for solving linear differential equations. The integrating factor is a function μ such that

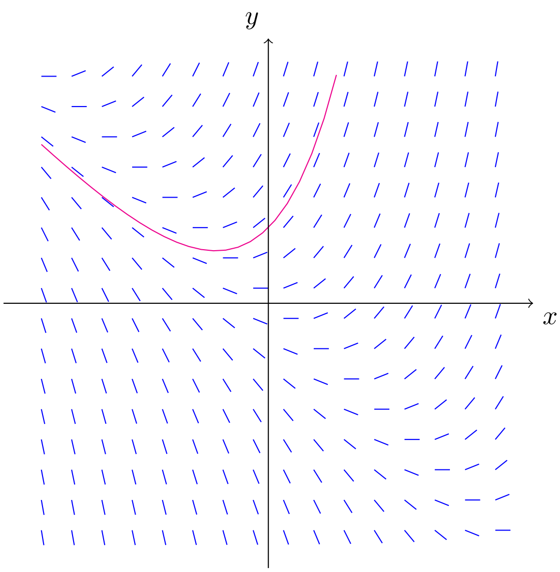
$$\mu' = -a\mu$$

By solving this differential equation we get

$$\frac{\mu'}{\mu} = -a \implies \ln|\mu| = -at + C \implies \mu = Ce^{at}$$

3 Slope Field

A slope field or directional field is a field to visualize solutions to a first-order differential equation.



Slope field of $\frac{dy}{dx} = x + y$.

This field is obtained by picking points on the plane.
For each point (x, y) we know that the slope $(\frac{dy}{dx})$ is $x+y$.
This means that if a solution passes through (x, y), then its slope is $x + y$.
The red curve shows a solution.

4 Euler's Method

Euler's method is a technique for solving a first-order differential equation numerically given a point of the solution.
Starting at the known solution point A_0, we take small steps the direction of the slope field. As the length of the steps $s \to 0$ we approach the solution to the equation.
The angle of the slope is given by

$$\theta = \tan\left(\frac{dy}{dx}\right)$$

so each step gives the sequence of points

$$A_n = A_{n-1} \cdot s\left(\cos(\theta), \sin(\theta)\right)$$