

## 1 Plane

A plane can be uniquely represented by its normal vector  $\vec{n}$  and a point on the plane  $P_0$ .

To describe the plane using an equation, we can consider an arbitrary point  $P = (x, y, z)$  on the plane. There is always a 90 degrees angle between the normal vector and the vector from  $P_0$  to  $P$  (i.e., their dot product is zero)

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

By plugging in the values for  $\vec{n}$  and  $P_0$  we get an equation in the form

$$Ax + By + Cz + D = 0$$

## 2 Vector-Valued Function

A vector-valued function is a function of a real parameter which returns a vector

$$r(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$$

## 3 Tangent Vector Vector-Valued Function

Given a vector-valued function

$$r(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$$

where  $f$ ,  $g$  and  $h$  are differentiable, then the Tangent vector to the curve is given by

$$r'(t) = \begin{pmatrix} f'(t) \\ g'(t) \\ h'(t) \end{pmatrix}$$

## 4 Curve length

The length of the curve between  $a$  and  $b$  is the integral from  $a$  to  $b$  of  $\sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$ .

## 5 Curvature

The tangent vector is given by

$$\vec{T} = \frac{d\vec{r}}{ds}$$

TODO: a smooth curve has  $r'(t)$  continuous and  $r'(t) \neq 0$

The curvature of a curve is given by

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

where  $\vec{T}$  is the unit tangent and  $s$  is the arc length. By the chain rule this can also be written as

$$\left\| \frac{d\vec{T}}{dt} \frac{dt}{ds} \right\| = \frac{1}{\|\vec{v}\|} \left\| \frac{d\vec{T}}{dt} \right\|$$

where  $\vec{v}$  is the velocity vector.

## 6 Principle unit normal

When  $\kappa \neq 0$

$$\vec{N} = \frac{1}{\kappa} \frac{d\vec{T}}{ds}$$

The curvature tells how much the curve is curving while the principle unit normal tells in which direction it is curving (normal to the curvature).

## 7 Vectors describing motion

Given a vector-valued curve  $r(t)$ , the motion can be described by

- the tangent vector (derivative of the position with respect to the arclength parameter), which the direction in which the curve is going.
- the normal vector captures the way in which the tangent vector is itself changing.
- Binormal vector is the cross product, which represents the vector normal to the plane created by the tangent vector and the normal vector. That is, it captures the torsion of said plane.

$$\begin{aligned}\vec{T} &= \frac{d\vec{r}}{ds} \\ \vec{N} &= \frac{d\vec{T}}{ds} \\ \vec{B} &= \vec{T} \times \vec{N}\end{aligned}$$

The change in the binormal vector is given by

$$\begin{aligned}\frac{d\vec{B}}{ds} &= \frac{d(\vec{T} \times \vec{N})}{ds} = \frac{d\vec{T}}{ds} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds} \\ &= \kappa \vec{N} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds} \\ &= \vec{T} \times \frac{d\vec{N}}{ds}\end{aligned}$$

Clearly,  $\frac{d\vec{B}}{ds}$  is orthogonal to both  $\vec{B}$  and  $\vec{T}$  and thus it is parallel to  $\vec{N}$ . Indeed,

$$\begin{aligned}\tau \vec{N} &= -\frac{d\vec{B}}{ds} \\ \tau &= -\frac{d\vec{B}}{ds} \cdot \vec{N}\end{aligned}$$

note that the minus sign is by convention. This value is called torsion.

## 8 Acceleration

The acceleration has both a tangential and normal component.

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d|\vec{v}|}{dt} \vec{T} + |\vec{v}| \frac{d\vec{T}}{dt} \\ &= \frac{d|\vec{v}|}{dt} \vec{T} + |\vec{v}| \left| \frac{d\vec{T}}{dt} \right| \vec{N} \\ &= \frac{d|\vec{v}|}{dt} \vec{T} + \kappa |\vec{v}|^2 \vec{N}\end{aligned}$$

## 9 Exercises

### 9.1 Open set 1

Prove that the set  $A = \{(x, y) \in \mathbb{R}^2 \mid 2 < x^2 + y^2 < 4\}$  is open.

Let  $p = (x, y)$  where  $p \in A$ . The set  $A$  is open iff  $\exists \epsilon > 0 \mid B_\epsilon(p) \subset A$ . Let  $d = \sqrt{x^2 + y^2}$ . For a radius  $\epsilon \leq \min(d - \sqrt{2}, d - \sqrt{4})$ , the open ball  $B_\epsilon(p) \subset A$ .

### 9.2 Norm 1

TODO

### 9.3 Countable set

TODO

### 9.4 Open set 2

Prove that the set  $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 < y < x\}$  is open.

TODO

### 9.5 Sequence 1

Consider the sequence  $\{x_k\}$  in  $\mathbb{R}^2$  defined by

$$x_k = \left( \sin\left(\frac{\pi k}{2}\right), \frac{(-1)^k}{\sqrt{k}} \right)$$

for each  $k \in \mathbb{N}^*$ . Determine whether  $\{x_k\}$  is bounded, and if so, find a convergent subsequence and identify its limit.

The sinusoidal part of the pair of the sequence is bounded because  $-1 \leq \sin \theta \leq 1$ . The other part has a numerator oscillating between 1 and  $-1$ , and the denominator goes from 1 to  $+\infty$  in the limit. Thus, the sequence is absolutely decreasing and  $-1 \leq \{x_k\} \leq \frac{1}{\sqrt{2}}$ . We now notice that

$$\sin\left(\frac{\pi k}{2}\right) = \begin{cases} 1 \text{ or } -1, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

By considering the subsequence where  $k$  is even we get a converging sequence

$$\lim_{k \rightarrow \infty} \{x_{2k}\} = (0, 0)$$

### 9.6 Level curves

TODO: the level curve is the curve on the horizontal plane intersecting a 2 variables function

Find the equation of the level curve of the function  $f(x, y)$  that passes through the given point  $p$ .

1.  $f(x, y) = 16 - x^2 - y^2$  and  $p = (2\sqrt{2}, \sqrt{2})$ ;
  2.  $f(x, y) = \sqrt{x^2 - 1}$  and  $p = (1, 0)$ ;
  3.  $f(x, y) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}}$  and  $p = (0, 1)$ .
1.  $f(2\sqrt{2}, \sqrt{2}) = 6$ , so the height of the plane is 6. By plugging  $z = 6$  in we get  $6 = 16 - x^2 - y^2$  and thus the level curve is given by  $10 = x^2 + y^2$ ;
  2.  $f(1, 0) = 0$ , so the height of the plane is 0. By plugging  $z = 0$  in we get  $0 = \sqrt{x^2 - 1}$  and thus the level curve is given by  $x^2 = 1$ , that is the lines  $x = 1$  and  $x = -1$ .

3. We first solve the integral

$$\int_x^y \frac{d\theta}{\sqrt{1-\theta^2}} = \arcsin(y) - \arcsin(x)$$

Now we can evaluate  $f(0,1) = \arcsin(1) - \arcsin(0) = \frac{\pi}{2}$ , so the height of the plane is  $\frac{\pi}{2}$ . By plugging  $z = \frac{\pi}{2}$  in we get

$$\begin{aligned}\frac{\pi}{2} &= \arcsin(y) - \arcsin(x) \\ y &= \sin\left(\frac{\pi}{2} + \arcsin(x)\right) \\ y &= \sqrt{1-x^2}\end{aligned}$$

with  $x < 0$ .

### 9.7 Limits 1

Solve the limit  $\lim_{(x,y) \rightarrow (2,1)} \frac{x^2-3y}{x+2y^2}$

$$\lim_{(x,y) \rightarrow (2,1)} \frac{x^2-3y}{x+2y^2} = \frac{2^2-3}{2+2} = \frac{1}{4}$$

### 9.8 Limits 2

Solve the limit  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

By using polar coordinate  $x = r \cos \theta$  and  $y = r \sin \theta$ , and thus  $x^2 + y^2 = r^2$ , therefore

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r \rightarrow 0} \frac{\sin(r)}{r} = 1$$

### 9.9 Limits 3

Solve the limit  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^4}$

### 9.10 Limits 4

Solve the limit  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^2+y^4}$