Theory of Computation

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Contents

1 Fields of Study

1.1 Complexity Theory

Classify problems according to their degree of ”difficulty”.

1.2 Computability Theory

Classify problems as being solvable or unsolvable.

1.3 Automata Theory

Compare different computation models.

2 Alphabets and Languages

An *alphabet* is a finite set of *symbols*. For example: $\{a, b, c, \cdots, z\}$
The set $\{0, 1\}$ is the binary set. The empty string is denoted λ.
Note that $\lambda \neq \varnothing \neq \{\lambda\}$.
The length of a string w is denoted as $|w|$.

If Σ is an alphabet,

$$\Sigma_\lambda = \lambda \cup \Sigma$$

A set of strings is called a *language*.

3 Deterministic Finite Automaton

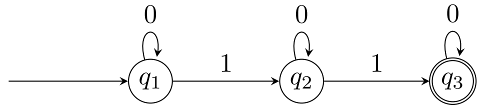
A deterministic finite automaton (DFA) is a state-machine which processes a string symbol by symbol from left to right. The automaton is in one of his *states* after processing a symbol. The machine might terminate in an *accept state* or not.

A DFA $M = (Q, \Sigma, \delta, q, F)$

- Q is a finite set of *states*
- Σ is an alphabet
- $\delta : Q \times \Sigma \to Q$ is the *transition function*
- q is an element of Q called the *start state*
- F is a subset of Q which contains the *accept states*

The transition function is the logical components, it determines in which state the machine will be after processing a symbol at any state.

The following automaton processes a binary string. The start state is q_1 and the only accept state is q_3. The program moves to the next state only if the symbol is 1, so it will reach q_3 only if the input string contains at least two 1s.



If a DFA is in a state r and it reads the symbol a, then it will uniquely switch to the state $\delta(r, a)$

The language of M, denoted $L(M)$ is the set of all accepted strings by M.

$$L(M) = \{w \in \Sigma^* \,|\, M \text{ accepts } w\}$$

4 Regular Operations

4.1 Union

If A and B are two languages over the same alphabet, the union of A and B is defined as

$$A \cup B = \{w \,|\, w \in A \vee w \in B\}$$

4.2 Concatenation

If A and B are two languages over the same alphabet, the concatenation of A and B is defined as

$$AB = \{ab \,|\, a \in A \wedge b \in B\}$$

4.3 Kleene star operator

The kleene star operator can be applied to alphabets or languages. It represent the union of all n-permutations of the set.
The set $\{0, 1\}^*$ is the set of all binary strings. If Σ is an alphabet, Σ^* is the set of all strings over Σ

$$\Sigma^* = \lambda \cup \bigcup_{n \in \mathbb{N}} \Sigma^n$$

4.4 Regular language

A language is regular if it can be expressed as a regular expression, or if an automaton that accepts said language exists.

4.4.1 Closure under union

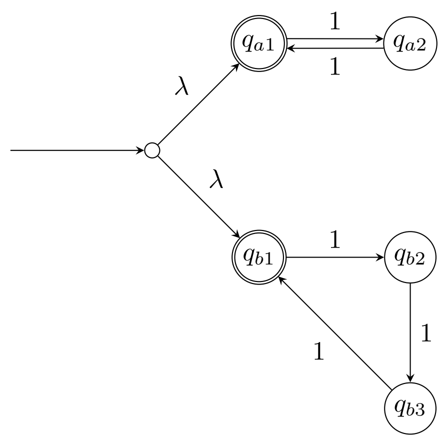
If A and B are two regular languages over the same alphabet Σ, there $A \cup B$ is also regular.

We can prove this by making an automaton that accepts both languages. Let's say that $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accepts A and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ accepts B. The automaton $M = (Q, \Sigma, \delta, q, F)$ must run M_1 and M_2 *simultaneously*, so any state must represent the current states of M_1 and M_2. This means that the states of M must represent any combination of state between M_1 and M_2, meaning $Q = Q_1 \times Q_2$. The transition function is now in the form $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$ where $a \in \Sigma$. The initial state is the state in Q which contains the initial state of M_1 and M_2, namely (q_1, q_2). Finally, the set of accept states is every tuple in Q_1 containing a state in F_2 or in Q_2 containing a state in F_1, namely $Q_1 \times F_2 \cup Q_2 \times F_1$.
We can conclude that $M = (Q_1 \times Q_2, \Sigma, \delta((r_1, r_2), a), (q_1, q_2), Q_1 \times F_2 \cup Q_2 \times F_1)$ accepts $A \cup B$ so $A \cup B$ is regular.

5 Nondeterministic Finite Automaton

Nondeterministic finite automata (NFA) are state-machines like DFAs but can change multiple states at a time by processing empty strings λ and when processing a symbol a may have multiple possible states to switch to. The NFA will choose the ”correct” switch in order to end in an accept state, if possible.

The following automaton where $\Sigma = \{1\}$ will end in an accept state if the input has length which is a multiple of 2 or 3.



The first switch is done by processing an empty string and the direction is chosen magically in order to end in an accept state.

A NFA is defined as $M = (Q, \Sigma, \delta, q, F)$ where

- Q is a finite set of *states*
- Σ is an alphabet
- $\delta : Q \times \Sigma_\lambda \to \mathcal{P}(Q)$ is the *transition function*
- q is an element of Q called the *start state*
- F is a subset of Q which contains the *accept states*

6 Equivalente of DFAs and NFAs

Anything that can be computed by a NFA can also be computed by a DFA and vice versa.

6.1 DFA to NFA conversion

Let $M = (Q, \Sigma, \delta, q, F)$ be a DFA. δ is not a transition function of a NFA, so we need to redefine it as δ'. Since δ cannot process λ, δ' it is defined as

$$\delta'(r, a) = \begin{cases} \delta(r, a) & x \neq \lambda \\ \emptyset & x = \lambda \end{cases}$$

where r is a state in Q and a is a symbol in Σ_λ.
We can conclude that $N = (Q, \Sigma, \delta', q, F)$.

6.2 NFA to DFA conversion

Let $N = (Q, \Sigma, \delta, q, F)$ be a NFA. The idea is to construct a DFA $M = (Q', \Sigma, \delta', q', F')$ that runs all the possible combinations that could be run by N at the same time. Any state of M is a set of states $R \in \mathcal{P}(Q)$, so we will say that $Q' = \mathcal{P}(Q)$. The set of accept states is any state R which contains an accept state of N

$$F' = \{R \in Q' \mid R \cap F \neq \emptyset\}$$

Let's assume that N does not execute any λ-transitions. q' would be $\{q\}$ and δ' would be

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$$

which is the union of all possible states N could switch to. Recall that for every $r \in R$, $\delta(r, a)$ is a set of all possible states to switch to.

Let's now remove the previous assumption. Now, M must also consider every state that could be reached by making zero or more λ-transitions. The λ-closure for a state r, $C_\lambda(r)$, is defined as the set of all possible states that can be reached from r by making zero or more λ-transitions. The λ-closure for a set of states R is defined as

$$C_\lambda(R) = \bigcup_{r \in R} C_\lambda(r)$$

The initial state q' is now given by $C_\lambda(q)$ and the transition function

$$\delta'(R, a) = \bigcup_{r \in R} C_\lambda(\delta(r, a))$$