Taylor Series

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Contents

1 Definition

A function $f(x)$ can be approximated around a point a by a power series.
The polynomial must be centered around a, so the variable will be $x - a$ such as

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Our goal is to find the coefficients c_n.
We notice that $f(a) = c_0$, which is the only coefficient that does not multiply a variable.
If we take the derivative, the coefficient c_1 will lose its variable

$$c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

Now we have $f'(a) = c_1$.
We take the derivative of the polynomial again

$$2c_2 + 6c_3(x-a) + \cdots$$

This time we have

$$\begin{aligned} f''(a) &= 2c_2 \\ c_2 &= \frac{f''(a)}{2} \end{aligned}$$

And again

$$\begin{aligned} f'''(a) &= 6c_3 \\ c_3 &= \frac{f'''(a)}{6} \end{aligned}$$

More generally, to extract the coefficient c_n we take the n-th derivative of the function.
By the power rule, we know that

$$\frac{d^k}{dx^k}\left(x^k\right) = k!$$

For example

$$\begin{aligned} \frac{d^4}{dx^4}\left(x^4\right) &= \frac{d^3}{dx^3}\left(4x^3\right) \\ &= \frac{d^2}{dx^2}\left(4 \cdot 3x^2\right) \\ &= \frac{d}{dx}\left(4 \cdot 3 \cdot 2x\right) \\ &= 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 4! \end{aligned}$$

This brings us to

$$c_n = \frac{f^{(n)}(a)}{n!}$$

The Taylor series of $f(x)$ around the point a is defined as

$$\sum_{n=0}^{\infty} \frac{(x-a)^n f^{(n)}(a)}{n!}$$

2 Divergence