

Set Theory

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Contents

1	Definitions	2
1.1	Set	2
1.2	Cardinality	2
1.3	Subset	2
1.4	Proper Subset	2
1.5	Empty Set	2
1.6	Power Set	2
1.7	Union	2
1.8	Intersection	2
1.9	Difference	3
1.10	Subset in terms of relationships	3
1.11	Disjoint Sets	3
1.12	Cartesian Product	3
1.13	Cartesian Power	3
1.14	Complement	3
1.15	Binary Relation	3
1.16	Injection	3
1.17	Surjectivity	4
1.18	Bijectivity	4
1.19	Homogeneous Relation	4
1.20	Reflexive relation	4
1.21	Symmetric relation	4
1.22	Transitive relation	4
1.23	Equivalence relation	4
1.24	Equivalence class	4
1.25	Partition of a set	5
1.26	Preorder	5
1.27	Partial order	5
1.28	Total order	5
1.29	Greatest element	5
1.30	Least element	5
1.31	Maximal element	5
1.32	Minimal element	5

1 Definitions

1.1 Set

A *set* is a collection of unordered elements.

1.2 Cardinality

The *cardinality* of a set A , denoted $|A|$, is the amount of elements it contains.

1.3 Subset

If A and B are sets, then A is a *subset* of B ($A \subseteq B$), if all the elements of A are also in B .

For every set A , $A \subseteq A$.

1.4 Proper Subset

Given two sets A and B , if $A \subseteq B$ but $A \neq B$, then A is a *proper* (or *strict*) subset of B

$$A \subset B$$

1.5 Empty Set

The empty set \emptyset is a subset of all other sets.

$$|\emptyset| = 0$$

For every set A

$$\emptyset \subseteq A$$

1.6 Power Set

If B is a set, then the *power set* $\mathcal{P}(B)$ is defined as the set of all subsets of B

$$\mathcal{P}(B) = \{A \mid A \subseteq B\}$$

Note that $B \in \mathcal{P}(B)$.

The cardinality of $\mathcal{P}(A)$ is given by

$$|\mathcal{P}(A)| = 2^{|A|}$$

1.7 Union

If A and B are sets, then their *union* is

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

1.8 Intersection

If A and B are sets, then their *intersection* is

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

1.9 Difference

If A and B are sets, then their *difference* is

$$A \setminus B = \{x \mid x \in A \wedge x \notin B \vee x \in B \wedge x \notin A\}$$

Note that

$$A \setminus B = B \setminus A \iff A = B$$

1.10 Subset in terms of relationships

$$A \subseteq B \iff A \cup B = B \iff A \cap B = A \iff A \setminus B = \emptyset$$

1.11 Disjoint Sets

If A and B are sets and $A \cap B = \emptyset$, then A and B are disjoint sets.

1.12 Cartesian Product

If A and B are sets, then their *cartesian product* is

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$$

which is the set of all possible *ordered pairs*.

More generally, given n sets A_1, A_2, \dots, A_n , their cartesian product $A_1 \times A_2 \times \dots \times A_n$ is the set of ordered n -tuples (a_1, a_2, \dots, a_n) with $a_i \in A_i$.

1.13 Cartesian Power

Given a set A , $A^n = \underbrace{A \times A \times \dots \times A}_n$.

The n -dimensional plane of real numbers is a cartesian power \mathbb{R}^n .

1.14 Complement

If A is a set, its *complement* is

$$\bar{A} = \{x \mid x \notin A\}$$

1.15 Binary Relation

If A and B are sets, a function $f : A \rightarrow B$ defines a *binary relation* R

$$R = \{(a, b) \mid f(a) = b\}$$

Note that $R \subseteq A \times B$

1.16 Injection

A function $f : A \rightarrow B$ is *injective* iff

$$\forall a, b \in A, f(a) = f(b) \implies a = b$$

1.17 Surjectivity

A function $f : A \rightarrow B$ is *surjective* iff

$$\forall b \in B \exists a \mid f(a) = b$$

1.18 Bijectivity

A function $f : A \rightarrow B$ is *bijective* iff it has a one-to-one correspondence between each element of A and B . Every bijection is both surjective and injective.

1.19 Homogeneous Relation

A *homogeneous relation* on a set S is a binary relation from a A to A .

1.20 Reflexive relation

A homogeneous relation R on a set A is *reflexive* iff

$$\forall a \in A, (a, a) \in R$$

1.21 Symmetric relation

A homogeneous relation R on a set A is *symmetric* iff

$$\forall (a, b) \in R, (b, a) \in R$$

1.22 Transitive relation

A homogeneous relation R on a set A is *transitive*

$$\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$$

1.23 Equivalence relation

An *equivalence relation* is a homogeneous relation \sim on a set A that is

1. *Reflexive*: $\forall a \in A, a \sim a$
2. *Symmetric*: $\forall a, b \in A, a \sim b \iff b \sim a$
3. *Transitive*: $\forall a, b, c \in A, a \sim b \wedge b \sim c \implies a \sim c$

1.24 Equivalence class

Let \sim be an equivalence relation on a set A . Given an element $a \in A$, the equivalence class of a , is defined as

$$[a]_{\sim} = \{x \in A \mid a \sim x\}$$

By the symmetric property we have $a \in [a]_{\sim}$.

Let $b \in [a]_{\sim}$, meaning $a \sim b$. $\forall c \in [b]_{\sim}$, meaning $b \sim c$, we have $a \sim c$ by the transitive property. Thus, $c \in [a]_{\sim}$ and $[b]_{\sim} \subseteq [a]_{\sim}$. By the symmetric property we also have $b \sim a$, $\forall d \in [a]_{\sim}$, meaning $a \sim d$, we have $b \sim d$ by the transitive property. Thus, $d \in [b]_{\sim}$ and $[a]_{\sim} \subseteq [b]_{\sim}$. Hence,

$$b \in [a]_{\sim} \iff [a]_{\sim} = [b]_{\sim}$$

This means that every element of an equivalence class has the same equivalence class. Thus, if two classes share an element they are the same

$$[a]_{\sim} \cap [b]_{\sim} \neq \emptyset \implies [a]_{\sim} = [b]_{\sim}$$

1.25 Partition of a set

Given a set A , a *partition of a set* $P = \{C_i\}$ is a collection of non-empty subsets of A such that $\bigcup_{E \in P} E = P$ and $C_i \cap C_j = \emptyset, i \neq j$. In other words the sets C_i contain every element of A exactly once.

Given an equivalence relationship \sim of a set A , the set of its equivalence classes form a partition of A .

1.26 Preorder

A *preorder* is a homogeneous relation \leq on a set A with the following properties:

1. *Reflexive*: $\forall a \in A, a \leq a$
2. *Transitive*: $\forall a, b, c \in A, a \leq b \wedge b \leq c \implies a \leq c$

1.27 Partial order

A *partial order* is a homogeneous relation \leq on a set A with the following properties:

1. *Reflexive*: $\forall a \in A, a \leq a$
2. *Transitive*: $\forall a, b, c \in A, a \leq b \wedge b \leq c \implies a \leq c$
3. *Antisymmetric*: $\forall a, b \in A, a \leq b \wedge b \leq a \implies a = b$

1.28 Total order

A *total order* is a homogeneous relation \leq on a set A with the following properties:

1. *Reflexive*: $\forall a \in A, a \leq a$
2. *Transitive*: $\forall a, b, c \in A, a \leq b \wedge b \leq c \implies a \leq c$
3. *Antisymmetric*: $\forall a, b \in A, a \leq b \wedge b \leq a \implies a = b$
4. *Strongly connected* (or *total*): $\forall a, b \in A, a \leq b \vee b \leq a$

A total order is a partial order where any two elements are comparable.

1.29 Greatest element

Given a partial order on a set A , an element g is a *greatest element* if $\forall a \in A, a \leq g$.

1.30 Least element

Given a partial order on a set A , an element g is a *least element* if $\forall a \in A, g \leq a$.

1.31 Maximal element

Given a partial order on a set A , an element $g \in A$ that is a greatest element is a *maximal element*.

1.32 Minimal element

Given a partial order on a set A , an element $g \in A$ that is a least element is a *minimal element*.