Elliptic Curve Criptography

Paolo Bettelini

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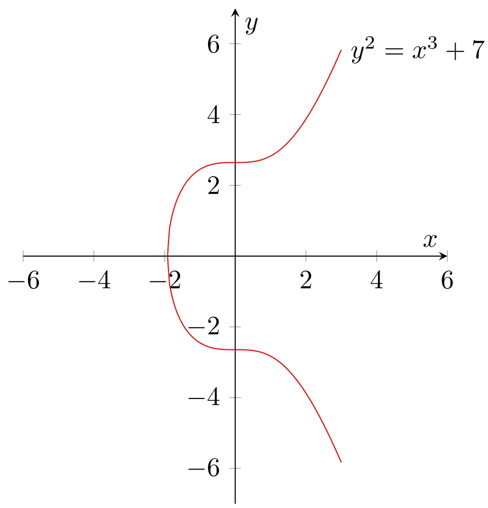
1 Elliptic Curves

1.1 Definition

An elliptic curve E is a set of points such that

$$E = \{(x, y) \mid y^2 = x^3 + ax + b\} \cup \{O\}, \quad 4a^3 + 27b^2 \neq 0$$



Where O is a point at infinity.

The elliptic curve is symmetrical about the x-axis.
The opposite of a point P is its reflection $-P$.

The coefficients a, b be part of

- $\mathbb{R}$ Real numbers
- $\mathbb{Q}$ Rational numbers
- $\mathbb{C}$ Complex numbers
- $\mathbb{Z}/p\mathbb{Z}$ Finite field

1.2 Addition

Given two points $P, Q \in E$ we can describe a unique third point.
We take the line that intersects P and Q, the opposite of the third intersection with the curve is out point.

$$P + Q = -R$$

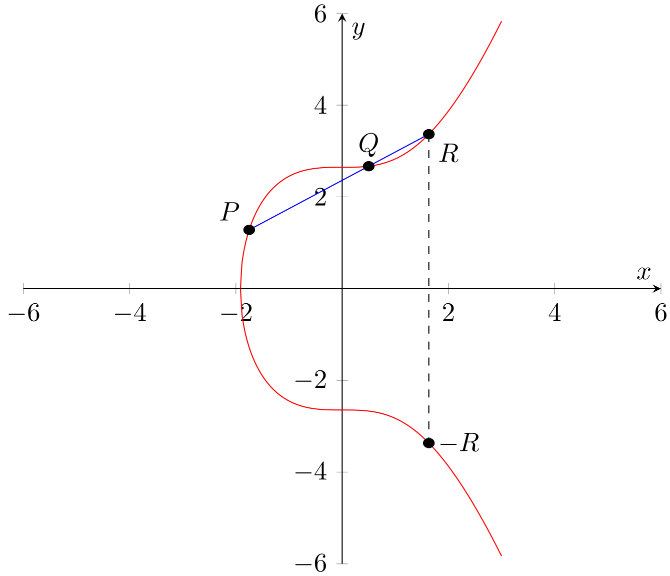
If $P = Q$, the intersection line will be given by the tangent at that point.
If $P = -Q$, $P + Q = O$.
If $P = -P$ (inflection point, the concavity of the curve changes) $R = P$, $P + P = -P = P$.
We consider $-O$ to be O.

The intersection line $mx + q$ is given by

$$m = \frac{P_y - Q_y}{P_y - Q_x}$$

and

$$q = P_y - mP_x$$



1.3 Scalar Multiplication

Given a point $P \in E$, multiplying kP where $k \in \mathbb{Z}$ is equivalent to adding P to itself k times. Computing $2P$ is the equivalent of $P + P$ which can be calculated as $P + Q = -R$.

2 Discrete logarithm problem

2.1 Definition

Given an elliptic curve E and a point $P \in E$, we consider

$$kP = Q, \quad k \in \mathbb{Z}$$

Given the value of P and Q it is a hard problem to find the value of k.
We can use many ECC multiplication algorithms to compute kP, but reversing this operation for big values of k is not an easy task.

Furthermore, given k_1 and k_2, we notice that

$$\begin{aligned} k_1(k_2P) &= k_2(k_1P) \\ &= (k_1 + k_2)P \end{aligned}$$

It does not matter if we first multiply P by k_1 and then k_2 or viceversa, P will always be multiplied $k_1 + k_2$ times.

2.2 Diffie–Hellman

We can use the scalar multiplication function with the Diffie–Hellman method.

We define an elliptic curve E over a finite field $\mathbb{F}_p$.

The *client* and the *server* publicly establish the domain parameters

- p The field that the curve is defined over $\mathbb{F}_p$ (mod p).
- a The parameter a of the elliptic curve equation.
- b The parameter b of the elliptic curve equation.
- G The generator, a fixed point $G \in E$.
- n The prime order of G, the smallest prime such that $kG = O$.
  In order words, the number of points that can be generated multiplying G with itself.
- h The cofactor, the ratio between the amount of points in E and the prime order of G.
  Ideally we would want $h = 1$, which means that the points are well-distribuited.

We then proceed with the Diffie–Hellman key exchange method.

1. The *client* generates a private key $k_c \in \mathbb{Z}$ such that $1 \leq k_c \leq n - 1$.
2. The *server* generates a private key $k_s \in \mathbb{Z}$ such that $1 \leq k_s \leq n - 1$.
3. The *client* computes a public key $P_c = k_cG$.
4. The *server* computes a public key $P_s = k_sG$.
5. The two parts publicly exchange the public keys.
6. The *client* computes $R = k_cP_s$.
7. The *server* computes $R = k_cP_s$.

Now both parts share the same secret point $R \in E$.