Integration

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1 Indefinite Integrals

1.1 Definition

Given a function $f(x)$, an **anti-derivative** or **primitive** is any function $F(x)$ such that

$$\frac{dF}{dx} = f(x)$$

The operator to find a primitive function is called the **indefinite integral**

$$\int f(x)\,dx = F(x) + C, \quad C \in \mathbb{R}$$

The function to integrate (integrand) is delimited by the integral symbol $\int$ and a differential of the variable of integration dx.
A function has infinitely many primitives, hence the $+C$ term. This essentially means that the derivative of a function is the same when the function is shifted up or down, the rate of change is the same. By reversing the process we don't know the up or down shift of the original function.

$$f(x) = \int \frac{df}{dx}\,dx + C$$

for some specific C.

1.2 Properties

If k is a constant

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

1.3 Substitution Rule

Given an integral in the form

$$\int f(g(x))g'(x)\,dx$$

Let

$$u = g(x)$$

The differential of u is then

$$du = g'(x)dx$$

meaning that we can rewrite the integral as

$$\int f(u)\,du = F(u) + C = F(g(x)) + C$$

1.4 Integration By Parts

Starting from the product rule

$$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

if we integrate both parts we get

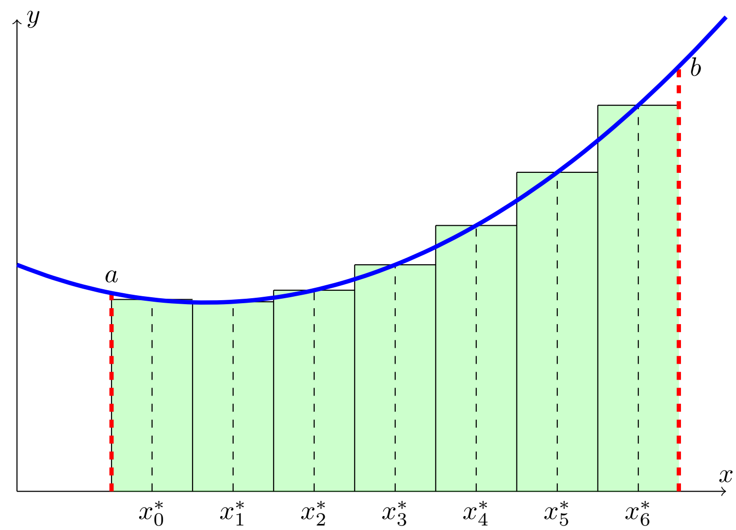
$$f(x)g(x) + C = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$
$$\int f(x)g'(x)\,dx = f(x)g(x) + C - \int f'(x)g(x)\,dx$$

Since the indefinite integral of $f'(x)g(x)$ is equal to some function plus an arbitrary constant, we can ignore the $+C$ term.

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

2 Definite Integrals

2.1 Area Problem



We want to find the signed area between $f(x)$ and the x-axis between the interval $[a;b]$.
One way to do it would be by dividing the area into n rectangles, each of width

$$\Delta x = \frac{b-a}{n}$$

The height of each triangle is given by $f(x_k^*)$. The area under the curve is approximately

$$A \approx \sum_{k=0}^{n-1} f(x_k^*)\Delta x$$

Notice that the position of x_k^* within the base of each rectangle controls the type of the approximation of the area under the curve. By moving x_k^* within the base we may achieve an approximation by abundance or defect. The type of approximation does not matter when we let $n \to \infty$. As the amount of rectangles approaches infinity, the approximation approaches the exact value of the area.

$$A = \lim_{n\to\infty} \sum_{k=0}^{n} f(x_k^*)\Delta x$$

2.2 Definition

Given a function $f(x)$ continuous on the interval $[a;b]$, we divide the interval into n rectangles of width $\Delta x = \frac{b-a}{n}$ and height $f(x_k^*)$. The definitive interval of $f(x)$ from a to b is

$$\int\limits_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n} f(x_k^*)\Delta x$$

2.3 Properties

$$\int\limits_a^b f(x)\,dx = -\int\limits_b^a f(x)\,dx$$

If k is constant

$$\int\limits_a^b cf(x)\,dx = c\int\limits_a^b f(x)\,dx$$

$$\int\limits_a^b f(x) \pm g(x)\,dx = \int\limits_a^b f(x)\,dx \pm \int\limits_a^b g(x)\,dx$$

$$\int\limits_a^a f(x)\,dx = 0$$

2.4 Fundamental Theorem of Calculus

If $f(x)$ is continuous on $I = [a;b]$,

$$g(x) \int\limits_{a}^{x} f(t)\,dt$$

is also continuous on I and

$$g'(x) = f(x)$$

or in other words

$$\frac{d}{dx} \int a\,dx f(t)[t] = f(x)$$