Set Theory

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Contents

1 Definitions

1.1 Cardinality

Definition Cardinality
The *cardinality* of a set A, denoted $|A|$, is the amount of elements it contians.

1.2 Subset

Definition Subset
If A and B are sets, then A is a *subset* of B ($A \subseteq B$), if all the elements of A are also in B.

For every set A, $A \subseteq A$.

1.3 Proper Subset

Definition Proper Subset
Given two sets A and B, if $A \subseteq B$ but $A \neq B$, then A is a *proper* (or *strict*) subset of B

$$A \subset B$$

1.4 Empty Set

Definition Proper Subset
The empty set $\emptyset$ is a subset of all other sets.

$$|\emptyset| = 0$$

For every set A

$$\emptyset \subseteq A$$

1.5 Power Set

Definition Proper Subset
If B is a set, then the *power set* $\mathcal{P}(B)$ is defined as the set of all subsets of B

$$\mathcal{P}(B) = \{A \,|\, A \subseteq B\}$$

Note that $B \in \mathcal{P}(B)$.

Theorem Cardinality of the power set
The cardinality of $\mathcal{P}(A)$ is given by

$$|\mathcal{P}(A)| = 2^{|A|}$$

1.6 Union

Definition Union

If A and B are sets, then their *union* is

$$A \cup B = \{x \,|\, x \in A \vee x \in B\}$$

1.7 Intersection

Definition Intersection

If A and B are sets, then their *intersection* is

$$A \cap B = \{x \,|\, x \in A \wedge x \in B\}$$

1.8 Difference

Definition Intersection

If A and B are sets, then their *difference* is

$$A \backslash B = \{x \,|\, x \in A \wedge x \notin B\}$$

Note that

$$A \backslash B = B \backslash A \iff A = B$$

1.9 Subset in terms of relationships

Corollary Subset in terms of relationships

$$A \subseteq B \iff A \cup B = B \iff A \cap B = A \iff A \backslash B = \emptyset$$

1.10 Disjoint Sets

Definition Disjoint Sets

If A and B are sets and $A \cap B = \emptyset$, then A and B are *disjoint sets.*

1.11 Cartesian Product

Definition Cartesian Product

If A and B are sets, then their *cartesian product* is

$$A \times B = \{(x, y) \,|\, x \in A \wedge y \in B\}$$

which is the set of all possible *ordered pairs.*

More generally, given n sets $A_1, A_2, \ldots, A_2$, their cartesian product $A_1 \times A_2 \times \cdots \times A_n$ is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ with $a_i \in A_i$.

1.12 Cartesian Power

Definition Cartesian Power

Given a set A, $A^n = \underbrace{A \times A \times \cdots \times A}_{n}$.

The n-dimensional plane of real numbers is a cartesian power $\mathbb{R}^n$.

1.13 Disjoint union

Definition Disjoint union

Given sets $A_{i \in I}$, their disjoint union is

$$\bigsqcup_{i \in I} A_i = \bigcup_{i \in I} \{(x, i) \mid x \in A_i\}$$

which consists of prdered pairs where the second element is the index of the set.

1.14 Complement

Definition Complement

If A is a set, its *complement* is

$$\bar{A} = \{x \mid x \notin A\}$$

1.15 Binary Relation

Definition Binary Relation

If A and B are sets, a function $f : A \to B$ defines a *binary relation* R

$$R = \{(a, b) \mid f(a) = b\}$$

Note that $R \subseteq A \times B$

1.16 Homogeneous Relation

Definition Homogeneous Relation

A *homogeneous relation* on a set S is a binary relation from a A to A.

1.17 Reflexive relation

Definition Reflexive relation

A homogeneous relation R on a set A is *reflexive* iff

$$\forall a \in A, (a, a) \in R$$

1.18 Symmetric relation

Definition Symmetric relation

A homogeneous relation R on a set A is *symmetric* iff

$$\forall (a, b) \in R, (b, a) \in R$$

1.19 Transitive relation

Definition Transitive relation

A homogeneous relation R on a set A is *transitive*

$$\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$$

1.20 Equivalence relation

Definition Equivalence relation

An *equivalence relation* is a homogeneous relation $\sim$ on a set A that is

1. *Reflexive*: $\forall a \in A, a \sim a$
2. *Symmetric*: $\forall a, b \in A, a \sim b \iff b \sim a$
3. *Transitive*: $\forall a, b, c \in A, a \sim b \wedge b \sim c \implies a \sim c$

1.21 Equivalence class

Definition Equivalence relation

Let $\sim$ be an equivalence relation on a set A. Given an element $a \in A$, the equivalence class of a, is defined as

$$[a]_\sim = \{x \in A \,|\, a \sim x\}$$

Theorem Shared element in equivalence class

Let $\sim$ be an equivalence relation on a set A and $a, b \in A$. Then,

$$b \in [a]_\sim \iff [a]_\sim = [b]_\sim$$

Proof Shared element in equivalence class

By the symmetric property we have $a \in [a]_\sim$. Let $b \in [a]_\sim$, meaning $a \sim b$. $\forall c \in [b]_\sim$, meaning $b \sim c$, we have $a \sim c$ by the transitive property. Thus, $c \in [a]_\sim$ and $[b]_\sim \subseteq [a]_\sim$. By the symmetric property we also have $b \sim a$, $\forall d \in [a]_\sim$, meaning $a \sim d$, we have $b \sim d$ by the transitive property. Thus, $d \in [b]_\sim$ and $[a]_\sim \subseteq [b]_\sim$. Hence,

$$b \in [a]_\sim \iff [a]_\sim = [b]_\sim$$

This means that every element of an equivalence class has the same equivalence class. Thus, if two classes share an element they are the same.

1.22 Partition of a set

Definition Partition of a set

Given a set A, a *partition of a set* $P = \{C_i\}_{i \in I}$ is a collection of non-empty subsets of A such that $\bigcup_{i \in I} C_i = P$ and $C_i \cap C_j = \emptyset, i \neq j$.

In other words the sets C_i contain every element of A exactly once.

Given an equivalence relationship $\sim$ of a set A, the set of its equivalence classes form a partition of A.

1.23 Preorder

Definition Preorder order

A *preorder* is a homogeneous relation $\leq$ on a set A with the following properties:

1. *Reflexive*: $\forall a \in A, a \leq a$
2. *Transitive*: $\forall a, b, c \in A, a \leq b \wedge b \leq c \implies a \leq c$

1.24 Partial order

Definition Partial order

A *partial order* is a homogeneous relation $\leq$ on a set A with the following properties:

1. *Reflexive*: $\forall a \in A, a \leq a$
2. *Transitive*: $\forall a, b, c \in A, a \leq b \wedge b \leq c \implies a \leq c$
3. *Antisymmetric*: $\forall a, b \in A, a \leq b \wedge b \leq a \implies a = b$

1.25 Total order

Definition Total order

A *total order* is a homogeneous relation $\leq$ on a set A with the following properties:

1. *Reflexive*: $\forall a \in A, a \leq a$
2. *Transitive*: $\forall a, b, c \in A, a \leq b \wedge b \leq c \implies a \leq c$
3. *Antisymmetric*: $\forall a, b \in A, a \leq b \wedge b \leq a \implies a = b$
4. *Strongly connected* (or *total*): $\forall a, b \in A, a \leq b \vee b \leq a$

A total order is a partial order where any two elements are comparable.

1.26 Greatest element

Definition Greatest element

Given a partial order on a set A, an element g is a *greatest element* if $\forall a \in A, a \leq g$.

1.27 Least element

Definition Least element

Given a partial order on a set A, an element g is a *least element* if $\forall a \in A, g \leq a$.

1.28 Maximal element

Definition Maximal element

Given a partial order on a set A, an element $g \in A$ that is a greatest element is a *maximal element*.

1.29 Minimal element

Definition Minimal element

Given a partial order on a set A, an element $g \in A$ that is a least element is a *minimal element*.