Differentiation

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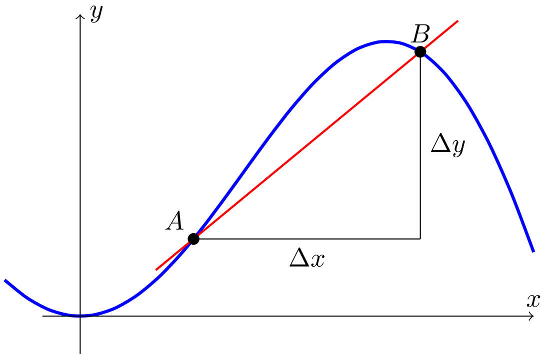
1 Definition

1.1 Tangent



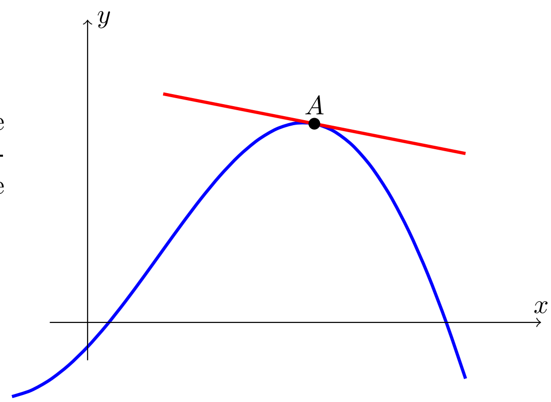
The mean slope of a function f between a point A and B is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(B) - f(A)}{B - A}$$

As we make A and B closer to eachother, Δx decreases. As Δx decreases the mean slope is more representative of the rate of change of f in the interval $[A; B]$.

When Δx is infinitely small, we have the precise slope of a given point on the function. This slope is represented by the tangent line, which is parallel to the given point.

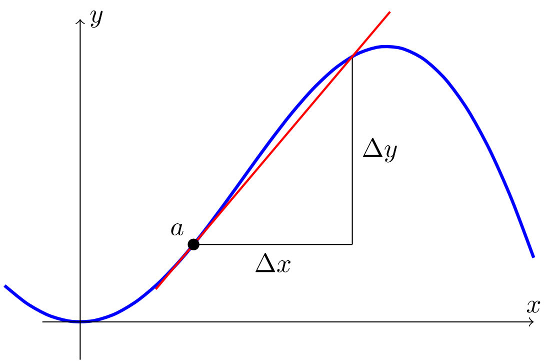
$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$$



1.2 Derivative

The derivative of a function $f(x)$ is another function $f'(x)$ which represents the rate of change of $f(x)$. In other words, $f'(x)$ represents the slope at each x of $f(x)$.

We define $f'(x)$ by taking the limit of the slope for every x.



We define the derivative as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

or

$$f'(x) = \lim_{h \to x} \frac{f(h) - f(x)}{x - h}$$

Using the derivative, the tangent line at $x = a$ is given by

$$y = f'(a)(x - a) + f(a)$$

2 Interpretation

Since the derivative $f'(x)$ represents the rate of change of $f(x)$

- If $f'(a) > 0$, then $f(x)$ is increasing at $x = a$
- If $f'(a) < 0$, then $f(x)$ is decreasing at $x = a$
- If $f'(a) = 0$, then $f(x)$ is critical at $x = a$ (changing from increase to decrease or from decrease to increase)

3 Rules for differentiation

$$\frac{d}{dx}(n) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in \mathbb{R}^*$$

$$\frac{d}{dx}\left(n \cdot f(x)\right) = n\frac{d}{dx}\left(f(x)\right)$$

$$\frac{d}{dx}(f+g) = f' + g'$$

Product Rule

$$\frac{d}{dx}(f \cdot g) = g'f + gf'$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

Chain Rule

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}(f^g) = f^g\left(\frac{f'g}{f} + g' \ln f\right)$$

4 Chain Rule

4.1 Definition

If z depends on y, and y depends on x, then z also depends on x.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

which is equivalent to

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

4.2 Proof

Assuming that z and y are differentiable in x

$$\begin{aligned}
\frac{dz}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x} \\
&= \left(\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y} \right) \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \right) \\
&= \left(\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y} \right) \cdot \frac{dy}{dx}
\end{aligned}$$

As $\Delta x \to 0$ also $\Delta y \to 0$, so we can replace Δx with Δy

$$\begin{aligned}
\frac{dz}{dx} &= \left(\lim_{\Delta y \to 0} \frac{\Delta z}{\Delta y} \right) \cdot \frac{dy}{dx} \\
&= \frac{dz}{dy} \cdot \frac{dy}{dx}
\end{aligned}$$