Riemann Hypothesis

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1 Zeta function

1.1 Definition

The zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 0$$

1.2 Euler product

The zeta function can be represented as an Euler product.
We will start by using the first prime number: 2.

$$\zeta(s)\frac{1}{2^s} = \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

We then subtract the second definition from the first one, such that

$$\zeta(s) - \zeta(s)\frac{1}{2^s} = \sum_{n=1}^{\infty} \left[\frac{1}{n^s}\right] - \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^s}\right]$$

$$\zeta(s)\left(1 - \frac{1}{2^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \neq 2k, k \in \mathbb{Z}$$

Here we are excluding the multiples of 2 from the series.

If we do the same with the next prime number, which is 3, we get

$$\zeta(s)\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad n \neq 2k, n \neq 3k, k \in \mathbb{Z}$$

We can repeat this process with every prime number.
Eventually, we will exclude every nth-term to sum as we use every prime number, except for n=1.

$$\zeta(s)\prod_{p\in P}^{\infty} 1 - \frac{1}{p^s} = \frac{1}{1^s} = 1$$

Finally, we get the identity

$$\zeta(s) = \prod_{p\in P}^{\infty} \frac{1}{1 - p^{-s}}$$

2 Analytic continuation

2.1 Zeta function for positive Re(s)

We have seen that the classical zeta function definition only converges for $Re(s) > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1$$

We can use the eta function $\eta(s)$, which is defined for $Re(s) > 0 \setminus \{1\}$, to analytically extend the zeta function domain to $Re(s) > 0 \setminus \{1\}$.
The eta function is a Dirichlet series defined as

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad Re(s) > 0 \setminus \{1\}$$

We start by splitting the zeta function into two distinct series, one for n even and the other one for n odd. The index for the even series will be $2n$, while the odd one will use $2n - 1$ as the index.

$$\zeta(s) = \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^s} \right] + \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^s} \right]$$

We do the same thing with the eta function.
Notice that $(-1)^n$ is 1 when n is even and -1 when n is odd.

$$\eta(s) = \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^s} \right] - \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^s} \right]$$

We subtract these two definition from eachother

$$\zeta(s) - \eta(s) = 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$
$$= 2^{1-s} \sum_{n=1}^{\infty} \frac{1}{k^s}$$
$$\zeta(s) - \eta(s) = 2^{1-s} \zeta(s)$$
$$\frac{1}{1 - 2^{1-s}} \eta(s) = \zeta(s)$$

We finally get

$$\zeta(s) = \frac{1}{1 - s^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad Re(s) > 0 \setminus \{1\}$$

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3 Prime-counting function

3.1 Properties of the prime-counting function

The prime-counting function $\pi(x)$ is defined as the number of primes less or equals than x.

We can consider the difference between $\pi(x)$ of two consecutive integers

$$\pi(x) - \pi(x-1) = \begin{cases} 1, & \text{if } x \in P \\ 0, & \text{otherwise} \end{cases}$$

Given a series over all prime numbers, we can extend it to all integers and multiply each term by this difference.
The terms whose index is not a prime number will be multiplied by 0.

$$\sum_{p \in P}^{\infty} a_k = \sum_{n=2}^{\infty} \left[\pi(n) - \pi(n-1)\right] a_n$$

Here we start at 2 since there are no prime numbers less than 2.

3.2 Relationship with the zeta function

We have seen that the zeta function can be written as an Euler Product

$$\zeta(s) = \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}}$$

However, we need convert this product into a series in order to apply the identity of the last paragraph.
We can take the natural logarithm of both sides and use the multiplication property

$$ln\left(\zeta(s)\right) = ln \prod_{p \in P}^{\infty} \frac{1}{1 - p^{-s}}$$
$$= \sum_{p \in P}^{\infty} ln\left(\frac{1}{1 - p^{-s}}\right)$$
$$= \sum_{p \in P}^{\infty} -ln\left(1 - p^{-s}\right)$$

Now we can apply the identity

$$ln\left(\zeta(s)\right) = \sum_{n=2}^{\infty} -ln\left(1 - n^{-s}\right) \left[\pi(n) - \pi(n-1)\right]$$

The next goal is to factor out $\pi(n)$

$$ln\left(\zeta(s)\right) = \sum_{n=2}^{\infty} \left[\pi(n-1)ln\left(1-n^{-s}\right)\right] - \sum_{n=2}^{\infty} \left[\pi(n)ln\left(1-n^{-s}\right)\right]$$

$$= \sum_{n=2}^{\infty} \left[\pi(n)ln\left(1-(n+1)^{-s}\right)\right] - \sum_{n=2}^{\infty} \left[\pi(n)ln\left(1-n^{-s}\right)\right]$$

$$= \sum_{n=2}^{\infty} \pi(n) \left[ln\left(1-(n+1)^{-s}\right) - ln\left(1-n^{-s}\right)\right]$$

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