

# Category Theory

Paolo Bettelini

## Contents

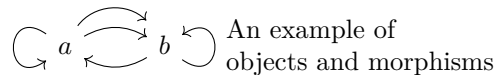
<b>1</b>	<b>Category</b>	<b>2</b>
1.1	Composition . . . . .	2
1.2	Identity . . . . .	2
1.3	Associativity . . . . .	2
<b>2</b>	<b>Homomorphisms</b>	<b>3</b>
2.1	Epimorphisms . . . . .	3
2.2	Monomorphisms . . . . .	3
2.3	Isomorphisms . . . . .	3
2.4	Homomorphism sets . . . . .	4
<b>3</b>	<b>Types of elements</b>	<b>4</b>
3.1	Void . . . . .	4
3.2	Singleton . . . . .	4
<b>4</b>	<b>Types of categories</b>	<b>4</b>
4.1	Thin categories . . . . .	4
4.2	Order categories . . . . .	4

# 1 Category

A category consists of *objects* and *morphism* or *arrows*.

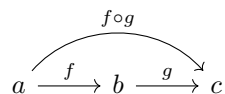
An arrow has a beginning and an ending, and it goes from one object to another.

Objects serve the purpose of marking the beginning and ending of a morphism.



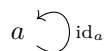
## 1.1 Composition

Composition is a property that says that if there is an arrow from  $a$  to  $b$ , and an arrow from  $b$  to  $c$ , there must exist an arrow from  $a$  to  $c$ .

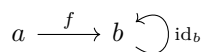


## 1.2 Identity

For every object there is an identity arrow.



The composition of an arrow with an identity is the arrow itself



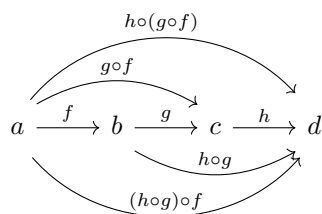
$$f \circ \text{id}_b = f$$

and also vice versa

$$\text{id}_b \circ f = f$$

## 1.3 Associativity

Compositions have the associative property



$$h \circ (g \circ f) = (h \circ g) \circ f$$

## 2 Homomorphisms

An Homomorphism is a map between two structures of the same type.

### 2.1 Epimorphisms

An epimorphism is a **surjective** morphism.

We can define surjectivity only in terms of morphisms.

Consider a morphism  $f : a \rightarrow b$  which maps elements of  $a$  onto  $b$ . Let's also define the morphisms  $g_1$  and  $g_2$  which map elements from  $b$  to  $c$ . The domain of  $g_1$  and  $g_2$  is the codomain of  $f$ . These two functions act as  $f$  for object in the image of  $f$ , but may map objects differently for objects in the codomain of  $f$  but outside the image of  $f$ . If the morphism is surjective, hence if the codomain and the image of  $f$  are the same, then  $g_1$  and  $g_2$  will always act as  $f$ .

$$a \xrightarrow{f} b \xrightarrow[g_2]{g_1} c$$

formally,

$$\forall c \forall g_1, g_2 : b \rightarrow c, g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

An epimorphism is labelled with  $\twoheadrightarrow$ .

### 2.2 Monomorphisms

An epimorphism is an **injective** morphism.

$$c \xrightarrow[g_2]{g_1} a \xrightarrow{f} b$$

A morphism  $f : a \rightarrow b$  is a monomorphism if

$$\forall c \forall g_1, g_2 : c \rightarrow a, f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

A monomorphism is labelled with  $\hookrightarrow$ .

### 2.3 Isomorphisms

An isomorphism is a **bijective** morphism (mono and epic, but not every mono and epic is an isomorphism).

A morphism  $f : a \rightarrow b$  is invertible if there is a function  $g$  that goes from  $b$  to  $a$

$$b : b \rightarrow a$$

such that

$$g \circ f = \text{id}_b$$

$$f \circ g = \text{id}_a$$

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ & \xleftarrow{g} & \end{array}$$

An isomorphism is labelled with  $\xrightarrow{\sim}$ .

## 2.4 Homomorphism sets

A hom-set is a set of all morphisms between a pair of objects. It is denoted as

$$\begin{aligned} C(a, b) \\ \text{Hom}_C(a, b) \\ \text{Hom}(a, b) \end{aligned}$$

## 3 Types of elements

### 3.1 Void

The void element is equivalent to the logical **false**. It is impossible to construct.

### 3.2 Singleton

A singleton is a single empty tuple element and is equivalent to the logical **true**. It can be constructed from nothing.

## 4 Types of categories

### 4.1 Thin categories

A thin category is a category in which each pair of objects has either 0 or 1 morphism. Every hom-set has either 1 or 0 elements.

### 4.2 Order categories

An order category is a thin category where morphisms represent relationships.

For example, here we have an equality relationship

$$a \xrightarrow{\leq} b$$

The relationship must be reflexive since there must be an identity morphism.

$$\leq \bigcirc_a$$