1 Curvature

The tangent vector is given by

$$\vec{T} = \frac{d\vec{r}}{ds}$$

TODO: a smooth curve has $r'(t)$ continuous and $r'(t) \neq 0$

The curvature of a curve is given by

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

where $\vec{T}$ is the unit tangent and s is the arc length. By the chain rule this can also be written as

$$\left\| \frac{d\vec{T}}{dt} \frac{dt}{ds} \right\| = \frac{1}{\|\vec{v}\|} \left\| \frac{d\vec{T}}{dt} \right\|$$

where $\vec{v}$ is the velocity vector.

2 Principle unit normal

When $\kappa \neq 0$

$$\vec{N} = \frac{1}{k} \frac{d\vec{r}}{ds}$$

The curvature tells how much the curve is curving while the principle unit normal tells in which direction it is curving (normal to the curvature).

3 Vectors describing motion

Given a vector-valued curve $r(t)$, the motion can be described by

- the tangent vector (derivative of the position with respect to the arclength parameter), which the direction in which the curve is going.
- the normal vector captures the way in which the tangent vector is itself changing.
- Binormal vector is the cross product, which represents the vector normal to the plane created by the tangent vector and the normal vector. That is, it captures the torsion of said plane.

$$\begin{aligned} \vec{T} &= \frac{d\vec{r}}{ds} \\ \vec{N} &= \frac{d\vec{T}}{ds} \\ \vec{B} &= \vec{T} \times \vec{N} \end{aligned}$$

The change in the binormal vector is given by

$$\begin{aligned} \frac{d\vec{B}}{ds} = \frac{d(\vec{T} \times \vec{N})}{ds} &= \frac{d\vec{T}}{ds} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds} \\ &= \kappa\vec{N} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds} \\ &= \vec{T} \times \frac{d\vec{N}}{ds} \end{aligned}$$

Clearly, $\frac{d\vec{B}}{ds}$ is orthogonal to both $\vec{B}$ and $\vec{T}$ and thus it is parallel to $\vec{N}$. Indeed,

$$\tau\vec{N} = -\frac{d\vec{B}}{ds}$$
$$\tau = -\frac{d\vec{B}}{ds}\cdot\vec{N}$$

note that the minus sign is by convention. This value is called torsion.

4 Acceleration

The acceleration has both a tangential and normal component.

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d|\vec{v}|}{dt}\vec{T} + |\vec{v}|\frac{d\vec{T}}{dt} \\ &= \frac{d|\vec{v}|}{dt}\vec{T} + |\vec{v}|\left|\frac{d\vec{T}}{dt}\right|\vec{N} \\ &= \frac{d|\vec{v}|}{dt}\vec{T} + \kappa|\vec{v}|^2\vec{N}\end{aligned}$$

5 Differentiability

Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a function. If the partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_m}$ exist and are continuous in an open region R, then f is differentiable in R.

6 Exercises

6.1 Open set 1

Prove that the set $A = \{(x, y) \in \mathbb{R}^2 \,|\, 2 < x^2 + y^2 < 4\}$ is open.

Let $p = (x, y)$ where $p \in A$. The set A is open iff $\exists \epsilon > 0 \,|\, B_\epsilon(p) \subset A$. Let $d = \sqrt{x^2 + y^2}$. For a radius $\epsilon \leq \min(d - \sqrt{2}, d - \sqrt{4})$, the open ball $B_\epsilon(p) \subset A$.

6.2 Norm 1

TODO

6.3 Countable set

TODO

6.4 Open set 2

Prove that the set $A = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 < y < x\}$ is open.

TODO

6.5 Sequence 1

Consider the sequence $\{x_k\}$ in $\mathbb{R}^2$ defined by

$$x_k = \left(\sin\left(\frac{\pi k}{2}\right), \frac{(-1)^k}{\sqrt{k}}\right)$$

for each $k \in \mathbb{N}^*$. Determine whether $\{x_k\}$ is bounded, and if so, find a convergent subsequence and identity its limit.

The sinusoidal part of the pair of the sequence is bounded because $-1 \leq \sin\theta \leq 1$. The other part has a numerator oscillating between 1 and -1, and the denominator goes from 1 to $+\infty$ in the limit. Thus, the sequence is absolutely decreasing and $-1 \leq \{x_k\} \leq \frac{1}{\sqrt{2}}$. We now notice that

$$\sin\left(\frac{\pi k}{2}\right) = \begin{cases} 1 \text{ or } -1, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

By considering the subsequence where k is even we get a converging sequence

$$\lim_{k\to\infty} \{x_{2k}\} = (0, 0)$$

6.6 Induction

Prove $n! > n^2$ for $n \geq 4$.

The base case is $4! = 24 > 4^2 = 16$.

The induction step is to prove $n! > n^2 \implies (n+1)! > (n+1)^2$. Note that $(n+1)! = (n+1)n!$. Since $n! > n^2$, then

$$\begin{aligned} n!(n+1) &> n^2(n+1) \\ n!(n+1) &> n^3 + n^2 \end{aligned}$$

Since $n \geq 4$, $n^3 + n^2 > (n+1)^2 = n^2 + 2n + 1$. Thus, by the transitive property, $(n+1)! > (n+1)^2$.

6.7 Sheet 6

Let $f(x, y, z) = yxz$ be a function and let $\vec{a} = (1, -1, 2)$ be a point.

a. Find the directional derivative of f at $\vec{a}$ along the vector $\vec{v} = (2, -1, 2)^T$ and along the unit vector $\vec{e} = \frac{1}{3}(1, -2, 2)^T$.

The directional derivative is given by $\nabla f(\vec{a}) \cdot \vec{v} = -8$ and $\nabla f(\vec{e}) \cdot \vec{v} = -\frac{8}{3}$.

b. Let $\vec{u}$ be a unit vector expressed in spherical coordinates, that is,

$$\vec{u} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))^T$$

Calculate the slope of f at $\vec{a}$ along the vector $\vec{u}$ as a function of (θ, ϕ).

The directional derivative is given by $\nabla f(\vec{a}) \cdot \vec{u} = -2\sin\theta\cos\phi + 2\sin\theta\sin\phi - \cos\theta$.

c. TODO. Guardare quando le parziali sono zero ...

$$\frac{\partial f}{\partial \theta} = -2\cos\theta\cos\phi + 2\cos\theta\sin\phi + \sin\theta$$

and

$$\begin{aligned}
\frac{\partial f}{\partial \theta} &= 0 \\
-\sin\theta &= 2\cos\theta(-\cos\phi + \sin\phi) \\
\tan\theta &= 2\cos\phi - \sin\phi \\
\theta &= \arctan(2\cos\phi - 2\sin\phi)
\end{aligned}$$