Complex Numbers

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Contents

1 Imaginary unit

1.1 Definition

The imaginary unit or imaginary number i is a solution to the quadratic equation $x^2 = -1$ and is defined as

$$i^2 = -1$$

The equation $x^2 = -1$ has two solutions: i and $-i$, however, there is not any algebraic difference between these two solutions.

1.2 Properties

The imaginary number i has some amazing properties when it comes to exponentiation.

$$\begin{cases} i^0 = +1 \\ i^1 = +i \\ i^2 = -1 \\ i^3 = -i \end{cases} \qquad \begin{cases} i^4 = +1 \\ i^5 = +i \\ i^6 = -1 \\ i^7 = -i \end{cases} \qquad \cdots$$

The multiplicative inverse of i is $-i$.

$$\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{i^2} = -i$$

2 Complex Numbers

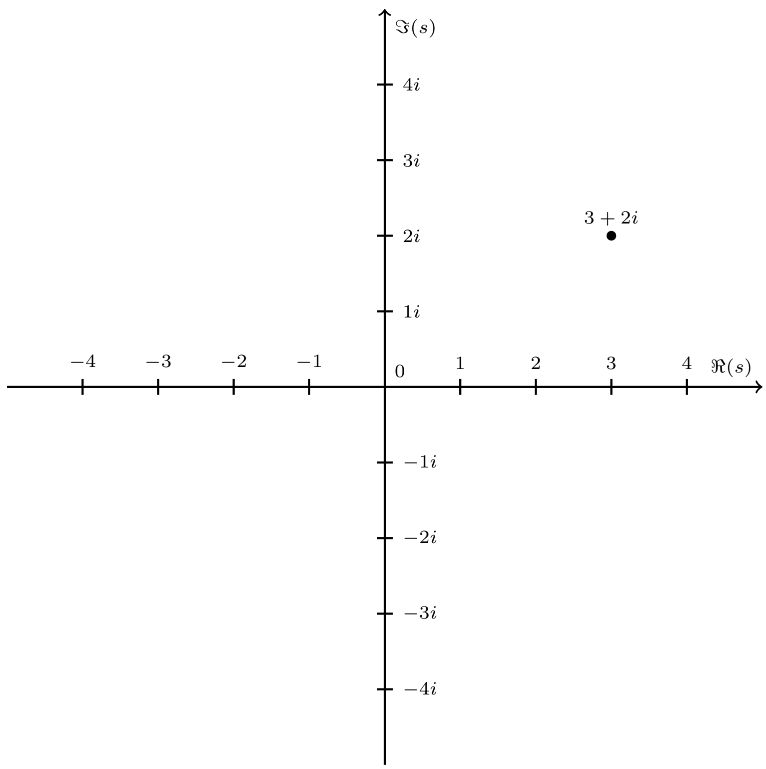
2.1 Definition

Complex numbers are numbers in the form $a + bi$, where $a, b \in \mathbb{R}$ and i is the imaginary unit.
This set of numbers is called $\mathbb{C}$.

Since every number $n \in \mathbb{R}$ can be represented as a complex number in the form $n + 0i$, $\mathbb{R} \subset \mathbb{C}$.

2.2 Complex plane

We can represent each complex number on a plane (Argand plane), where the horizontal axis represent the real numbers $\mathbb{R}$ and the vertical axis represents every scalar multiple of the imaginary unit i.



2.3 Operations

2.3.1 Addition

$$(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$$

2.3.2 Subtraction

$$(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$$

2.3.3 Multiplication

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - db) + (ad + bc)i$$

2.3.4 Division

$$\begin{aligned}\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} &= \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2} \\ &= \frac{ac+bd+(bc-ad)i}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i\end{aligned}$$

2.3.5 Real part

The real part of a complex number s is denoted as $\mathrm{Re}(s)$ or $\Re(s)$.

$$\mathrm{Re}(a+bi) = a$$

2.3.6 Imaginary part

The imaginary part of a complex number s is denoted as $\mathrm{Im}(s)$ or $\Im(s)$.

$$\mathrm{Im}(a+bi) = b$$

2.3.7 Absolute value

The absolute value (or module) of a complex number is its distance from the origin.

$$|a+bi| = \sqrt{a^2+b^2}$$

2.3.8 Conjugate

The complex conjugate of a number $s = a+bi$ is denoted as s^* or $\overline{s}$. It is defined as

$$\overline{a+bi} = a - bi$$

Geometrically, s^* is the reflection about the real axis in the complex plane.

We also have the following trivial properties.

$$\begin{aligned}\overline{\overline{s}} &= s \\ \mathrm{Re}(\overline{s}) &= \mathrm{Re}(s) \\ \mathrm{Im}(\overline{s}) &= -\mathrm{Im}(s)\end{aligned}$$

2.3.9 Argument

The argument of a complex number is the angle formed with the x-axis in the complex plane

$$\arg(a+bi) = \arctan\left(\frac{b}{a}\right)$$

2.3.10 Axiomatic definition

A complex number is a tuple (a,b) where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

equality

$$(a,b) = (c,d) \implies a = c \land b = d$$

Addition

$$(a,b) + (c,d) = (a+c, b+d)$$

Multiplication

$$(a,b) \cdot (c,d) = (ac - db, ad + bc)$$
$$m(a,b) = (ma, mb)$$

If $z_1, z_2, z_3 \in \mathbb{C}$.

1. $z_1 + z_2$ and $z_1 z_2$ are also in $\mathbb{C}$
2. $z_1 + z_2 = z_2 + z_1$
3. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
4. $z_1 z_2 = z_2 z_1$
5. $z_1(z_2 z - 3) = (z_1 z_2) z_3$
6. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
7. $z_1 + 0 = z_1$
8. $z_1 \cdot 1 = z_1$
9. $\exists! z \,|\, z + z_1 = 0$
10. $\exists! z \,|\, z \cdot z_1 = 1$

2.4 Trigonometric form

Any complex number can be represented in a trigonometric form

$$a + bi = r(\cos\theta + i\sin\theta)$$

where r is the absolute value and θ is the argument.

2.5 Vector form

Any complex number $a + bi$ can be represented by a vector (a,b).

Scalar product The scalar product between $z_1 = a + bi$ and $z_2 = c + di$ is given by

$$z_1 \circ z_2 = |z_1|\,|z_2|\cos\theta = ac + bd = \Re(z_1^* z_2) = \frac{1}{2}(z_1^* z_2 + z_1 z_2^*)$$

where θ is the angle formed by the two vectors.

Vector product The vector product between $z_1 = a + bi$ and $z_2 = c + di$ is given by

$$z_1 \times z_2 = |z_1|\,|z_2| \sin\theta = ad - cb = \Im(z_1^* z_2) = \frac{1}{2i}(z_1^* z_2 + z_1 z_2^*)$$

where θ is the angle formed by the two vectors.

We can see that

$$z_1^* z_2 = (z_1 \circ z_2) + i(z_1 \times z_2)$$

2.6 Complex conjugate coordinates

Since for any complex number $z = a + bi$

$$a = \frac{1}{2}(z + z^*)$$
$$b = \frac{1}{2i}(z - z^*)$$

z can also be represented by the conjugate coordinates (z, z^*).