# Limits

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# 1 Definition

A limit is usually used to describe the behavior of a function as its argument approaches a given value. The limit towards a certain value $c$ within a function can be be approached both from the right and from the left. The limit in a general sense exists if the value approached from both sides is the same and well-defined. We define the limit of $x$ approaching $c$ from the left within the function $f(x)$ as

$$\lim_{x \to c^-} f(x)$$

We define the limit of $x$ approaching $c$ from the right within function $f(x)$ as

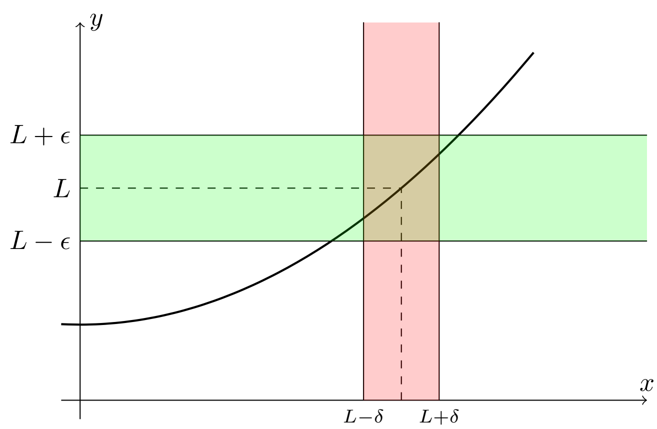
$$\lim_{x \to c^+} f(x)$$

We define the limit of $x$ approaching $c$ within function $f(x)$ as

$$\lim_{x \to c} f(x)$$

Formally, given a function $f : D \to \mathbb{R}$ the limit $L = \lim_{x \to c} f(x)$ exists if given an arbitrary small $\epsilon > 0$ there is another number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ when } 0 < |x - c| < \delta$$




This means that for any $x$ in the red region $0 < |x - c| < \delta$ or $|x - c| \in (0; \delta)$, the function at that point will lie in the yellow region. This value is closer to $L$ than either $L + \epsilon$ or $L - \epsilon$

$$|f(x) - L| < \epsilon$$

Notice that this defintion does not require $f$ to be defined at $c$, but rather just around $c$.

We can also use this definition for limits from the right and from the left.

The right-hand limit $L = \lim_{x \to c^+} f(x)$ exists if for any arbitrary small $\epsilon > 0$ there is some $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ when } 0 < x - c < \delta$$

The left-hand limit $L = \lim_{x \to c^-} f(x)$ exists if for any arbitrary small $\epsilon > 0$ there is some $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ when } -\delta < x - c < 0$$

## 2 Properties

If the limit exists

$$\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x))$$

## 3 Continuity

A function $f$ is continuous at a point $c$ iff

$$\lim_{c_0 \to c^+} f(c_0) = \lim_{c_0 \to c^-} f(c_0) = f(c)$$

A function $f$ is continuous on an interval $[a; b]$ iff it is continuous at each point $c \in [a; b]$

$$\forall c \in [a; b], \lim_{c_0 \to c^+} f(c_0) = \lim_{c_0 \to c^-} f(c_0) = f(c)$$

## 4 Intermediate value theorem

A function $f$ continuous on an interval $[a; b]$ will take every value in the interval $[f(a); f(b)]$.

## 5 Bolzano's Theorem

If $f(x)$ is continuous on $[a; b]$ and $f(a) < f(b)$ then there is a root.

$$f(a) < f(b) \implies \exists c \in [a; b] \mid f(c) = 0$$

## 6 Squeeze Theorem

Let $h(x)$, $f(x)$ and $g(x)$ be three functions such that $h(x) \leq f(x) \leq g(x)$.
If

$$\lim_{x \to x_0} g(x) = f(x) = L$$

then

$$\lim_{x \to x_0} f(x) = L$$