TASK 1 2p

The noise in MEG experiments
High signal-to-noise ratio (SNR) is especially hard to achieve in MEG, as the measured fields are multiple orders of magnitude weaker than those generated by the surroundings. In this task we'll look into the most basic methods that exist for reducing the noise.
Basic calculus

A) Write a short answer
In the context of an MEG experiment, how can the following sources of noise be most efficiently mitigated?

i) Earth's magnetic field
ii) Power line interference
iii) Intrinsic noise of SQUID sensors
iv) Background brain activity

Give a unique answer for each problem.

B) Calculate and explain
A MEG experiment is conducted to measure a stimulus-evoked response whose amplitude is known to be 100 fT (in one channel). The measurement system has sensors with intrinsic white noise with amplitude 4.0 $\mathrm{fT}/\sqrt{\mathrm{Hz}}$, and the background brain activity ("brain noise") is assumed to be zero-mean noise with power spectral density $S(f) = 3000/f\ \mathrm{fT}^2/\mathrm{Hz}$. The frequency band of interest is 1–40 Hz. All noise sources can be assumed to be uncorrelated.

Signal-to-noise ratio (SNR) is defined as the ratio between the signal mean and the noise standard deviation:

$$\mathrm{SNR} = \frac{\mu}{\sigma}.$$

(i) In this task, the standard deviation of the noise σ_{noise} equals the noise amplitude, i.e. its RMS value. Show why

$$\sigma_{\text{noise}} = \mathrm{RMS}_{\text{noise}}.$$

(ii) Averaging multiple trials can improve the signal-to-noise ratio. How many trials must be averaged to reach $\mathrm{SNR} = 8.0$?

Hint: With repeated measurements, we sample multiple times from a distribution that describes the noise. What is the relationship between the standard error and the standard deviation?

(iii) Does the intrinsic sensor noise significantly reduce the signal-to-noise ratio in this case?

The evoked response can be assumed to be fully resilient to stimulus repetition, i.e. its amplitude remains constant across trials.

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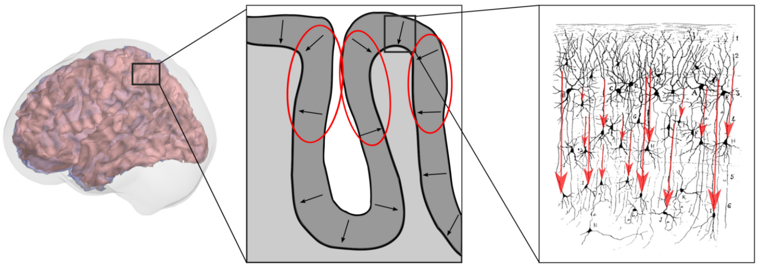
TASK 2 2p

Dipole localization

The signal measured by EEG and MEG mostly stems from postsynaptic potentials from pyramidal cells in the cortex. These pyramidal cells are oriented such that their apical dendrites point in the direction normal to the cortical surface, as in the Figure below. Some brain activity can be modeled as a single current dipole, since if only a small patch of cortex is active, it will look like a dipole when observed from far enough away.



The magnetic field $\vec{B}$ of a current dipole $\vec{Q}$ at the point $\vec{r}_0$ within a homogeneous conducting sphere (centered at the origin) at the point $\vec{r}$ outside the sphere is (Sarvas, 1987)

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi F^2}(F\vec{Q} \times \vec{r}_0 - \vec{Q} \times \vec{r}_0 \cdot \vec{r}\nabla F), \tag{1}$$

where

$$\begin{aligned} F &= a(ra + r^2 - \vec{r}_0 \cdot \vec{r}), & a &= |\vec{a}|, \\ \nabla F &= (r^{-1}a^2 + a^{-1}\vec{a} \cdot \vec{r} + 2a + 2r)\vec{r} - (a + 2r + a^{-1}\vec{a} \cdot \vec{r})\vec{r}_0, & r &= |\vec{r}|. \\ \vec{a} &= \vec{r} - \vec{r}_0, \end{aligned} \tag{2}$$

The formula doesn't include a dependence on the sphere radius, the only thing that matters is that $\vec{r}$ is outside the sphere, and $\vec{r}_0$ is inside. Also, notice that the formula does not include a dependence on the electrical conductivity of the sphere. In fact, the conductivity does not matter as long as it is homogeneous. Furthermore, adding concentric layers of different conductivities (e.g. brain, skull, scalp) does not change the equation either. As we move from a spherical conductor to more realistic geometries, this is no longer the case. However, the head is still at least somewhat spherical, and the take-away messages here still have some value.

A) Explain

What happens if the dipole is radially directed, i.e. pointing in the same direction as $\vec{r}_0$? What does this mean for MEG and its sensitivity to different parts of the cortex (i.e. its sensitivity pattern)?

B) Implement code & compute

Implement the above formula in MATLAB or Python. Consider a spherical head with a radius of 10 cm. If there is a single current dipole at $\vec{r}_0 = (-7.5, 0, 0)$ cm with dipole moment $\vec{Q} = (0, 1, 0) \cdot 50$ nAm, what is the measured magnetic field $\vec{B}$ at

i) the position $\vec{r}_1 = (-10.1, 0, 2.7)$ cm?
ii) the position $\vec{r}_2 = (-12.0, 0, 3.2)$ cm?

$\vec{r}_1$ roughly corresponds to the measurement distance of optically-pumped magnetometers (OPMs), while $\vec{r}_2$ corresponds to that of SQUID magnetometers, which have more thermal insulation.

Plot the magnitudes of the magnetic field component normals to the surface of the sphere on a spherical surface at radius 11 cm, i.e. just outside the sphere. Return your code as well as the plot.

TASK 3 2p

Derivation of minimum norm solution

Many medical imaging problems are formulated as linear systems

$$\mathbf{y} = \mathbf{L}\,\mathbf{x}, \tag{3}$$

where $\mathbf{y}$ is the measurement vector (signal values of all measurement channels), $\mathbf{L}$ is the lead field, or the system matrix, and $\mathbf{x}$ is the amplitudes of all potential sources producing the observed signals. Here m represents the number of measurements (sensors) and n represents the number of sources (current dipoles):

$$\mathbf{y} \in \mathbb{R}^{m \times 1}$$
$$\mathbf{L} \in \mathbb{R}^{m \times n}$$
$$\mathbf{x} \in \mathbb{R}^{n \times 1}$$

When there are more unknowns than equations ($n > m$), there is no unique solution to this problem. *Linear algebra, Matrix computations*

A) Derive

One widely used way to find a source estimate $\tilde{\mathbf{x}}$ for eq. 3 is to minimize the L^2-norm of the estimate. Using Lagrange multipliers, find $\tilde{\mathbf{x}}$ that minimizes

$$\tilde{x}^T\,\tilde{x} \tag{4}$$

subject to

$$\mathbf{y} - \mathbf{L}\tilde{\mathbf{x}} = 0 \tag{5}$$

Note that $\mathbf{L}$ is not a square matrix. λ has size $\mathbb{R}^{1 \times m}$.

B) Write a short answer

The solution derived in **A)** is usually not well behaving due to singularities and noise in the measurement. Tikhonov regularization is classical method of regularizing this kind of ill-posed problems. What does Tikhonov regularization do, and why is it used?

C) Write a short answer

In an *overdetermined* case the solution becomes of form

$$\tilde{x} = (\mathbf{L^T}\,\mathbf{L})^{-1}\mathbf{L^T}\mathbf{y}. \tag{6}$$

What is the difference between underdetermined and overdetermined problems?

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