Linear Algebra

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1 Definitions

1.1 Vectors and Vector Spaces

- The set V is called a *vector space* over field F when the *vector addition* and *scalar multiplications* operations satisfy the following properties (for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in F$)
  - **Closure property for addition**: $\mathbf{u} + \mathbf{v} \in V$
  - **Associativity of addition**: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
  - **Commutativity of addition**: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - **Zero of addition**: There is an element $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
  - **Inverse element of addition**: For every $\mathbf{u} \in V$ there exists an element $-\mathbf{u} \in V$, called the additive inverse of $\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
  - **Closure of multiplication**: $a\mathbf{u} \in V$ for all $a \in F$
  - **Compatibility of multiplication with field**: $(ab)\mathbf{u} = a(b\mathbf{u})$
  - **Identity element of multiplication**: There is an element $\mathbf{1} \in V$ such that $\mathbf{1u} = \mathbf{u}$
  - **Distributivity of multiplication with respect to addition**: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
  - **Distributivity of multiplication with respect to field addition**: $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
- A *linear combination* is $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$
- The *span* of $\mathbf{u}$ and $\mathbf{y}$ is the set of **all** *linear combinations* ($a, b \in \mathbb{R}$):

$$a\mathbf{u} + b\mathbf{v}$$

- When $\mathbf{u}$ is a linear combination of the others basis vectors, then it's *linearly dependent*
- If $\mathbf{u} \neq a\mathbf{v} + b\mathbf{w}$ for all values of a, b then $\mathbf{u}$ is linearly independent
- The basis $\mathbf{B}$ of a *vector space* V is a set of *linearly independent* vectors that *span* the full space

1.2 Linear Transformations and Matrices

2 Linear Algebra Intuition

Notes from the 3B1B video series.

2.1 Vectors and Vector Spaces (Chap 1 & 2)

- Vectors $\mathbf{x} = (x_1, \ldots, x_n)$ can be viewed interchangeably as **ordered lists** or as objects with length and direction, i.e. an arrow
- Vectors are points in L.A., sometimes it's easier to think of them as arrows, sometimes as points (when you have many), sometimes as lists (when you're programming)
- Vectors live in a **vector space** V that has certain properties

- In Linear Algebra (L.A.) vectors always start at the origin $\mathbf{0} = (0, \ldots, 0)$
- Vectors are almost always represented as column vectors
- Vectors have a sum $+$ and a scalar multiplication $\cdot$
- Vector sum can be viewed as a translation to the vector space, similarly to how real numbers translate the $\mathbb{R}$
- Scalar multiplication can be viewed as *scaling* the vector by the magnitude of the scalar
- Any vector $\mathbf{x}$ can be viewed as the *linear combination* of the basis vectors (on a given space). e.g. for $\mathbb{R}^2$
$$\mathbf{x} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} = (3, -2)$$
- When you think of a vector space, you have to think about the basis of it $\mathbf{B}$, in the previous example $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ *span* $\mathbb{R}^2$. This means that any time you write a vector, it implicitly depends on the choice o f basis you're using!
- In higher dimensions, the *span* is more important because it creates *subspaces*. e.g. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$, i.e. a plane that cuts through 3D space

2.2 Linear Transformations and Matrices (Chap 3 & 4)

- A *linear transformation* can be seen as a function that takes an input and returns one of the same class. In this case, vectors in a given vector space. The transformation however, implies *movement*, in this case, the vector space
- In the case, given a matrix A, its multiplication scales all the space itself! However, it does satisfy certain special properties: a) all lines remain lines and b) the origin remains fixed. I.e. grid lines remain parallel and evenly spaced. e.g. rotations around the origin, translations, etc.
- An easy way to think about a linear transformation (matrix) A is to take each column and associate it with each basis vectors e.g. $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ for 2D. Each basis vector will land on each column of A fully specifying the transformation

3 Useful Properties

3.1 Matrices

- $2x2$ **matrices** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - **Inverse** If $\det(A) = ad - cd \neq 0 \Rightarrow A^{-1} = \dfrac{1}{ad - cd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$