Project 3: Contagion in financial networks

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1 Introduction

In this short report, we present our analysis of the third exam proposal, studying and simulating a contagion in financial networks. The code implementation for the simulations is available at `https://github.com/PaoloPellizzoni/ContagionFinancialNetworks`.

2 Contagion Model Description

Following "Contagion in financial networks" by Gai & Kapadia [1] (from now on: the reference paper), we consider a network in which financial intermediaries are linked together by their claims on each other, and model it with a *oriented* and *weighted* graph in which each node represents a bank and the edges represents interbank exposures.
Two degrees are associated with each node: an *in-degree*, i.e. the number of links that point into the node, and an *out-degree*, which is the number pointing out. Incoming links to a node reflect the interbank assets of that bank (i.e. money owed to the bank by a counterparty) whereas outgoing links, by contrast, correspond to its interbank liabilities.
We make the assumption that the total interbank asset position of every bank is evenly distributed over each of its incoming links and is independent of the number of links the bank has. Since every interbank asset is another bank's liability, interbank liabilities are endogenously determined by the network links and interbank assets.
Besides the interbank links, each financial intermediary has also illiquid external assets, such as mortgages, and another form of liability given by customer deposits.
The node i of the graph is thus characterized by the following quantities:

- A_i^{IB} = interbank assets (equals 0 if a bank has no incoming links),
- A_i^M = illiquid external assets,
- L_i^{IB} = interbank liabilities,
- D_i = customer deposits liability,
- $\{j_{i,1}, ..., j_{i,n_i}\}$ set of the n_i incoming links,
- $\{k_{i,1}, ..., k_{i,m_i}\}$ set of the m_i outgoing links.

Making a zero recovery assumption, namely that, when a linked bank defaults, bank i loses all of its interbank assets held against that bank, the condition for bank i to be solvent and not make the collapse spread further is given by the following:

$$(1-\phi_i)A_i^{IB} + qA_i^M - L_i^{IB} - D_i > 0 \tag{1}$$

where ϕ_i the fraction of banks with obligations to bank i that have defaulted, and q is a factor that may be less than 1 in the event of asset sales by banks in default.
The solvency condition can be rewritten as:

$$\phi_i < \frac{K_i - (1-q)A_i^M}{A_i^{IB}} \quad \text{for } A^{IB} \neq 0 \tag{2}$$

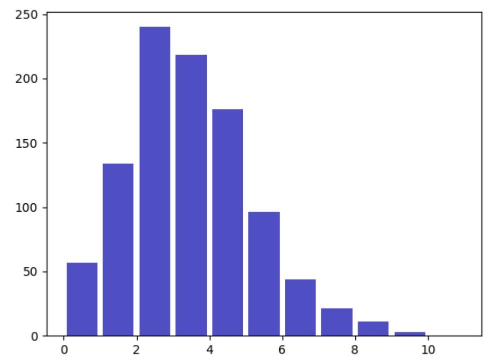
where $K_i = A_i^{IB} + A_i^M - L_i^{IB} - D_i$ is the capital buffer of the bank, i.e. the difference between the value of its assets and liabilities.
Initially, all banks in the network are supposed to be solvent and the network perturbation starts with the default of a single bank.
Since linked banks each lose a fraction $1/n_i$ of their interbank assets when a single counterparty defaults, recalling equation (2), it is clear that the only way the default can spread is if there is a neighbouring bank for which

$$\frac{K_i - (1-q)A_i^M}{A_i^{IB}} < \frac{1}{n_i} \tag{3}$$

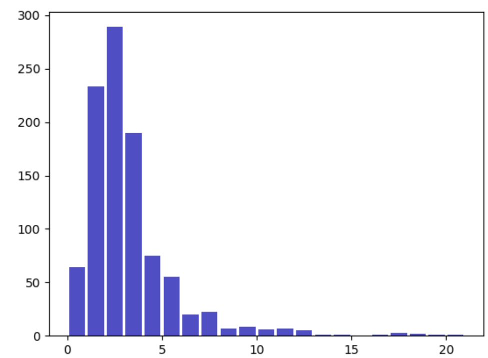
The spread of the contagion continues until all banks directly linked to the collapsed cluster are safe.

3 Contagion Simulation

To simulate the spread of the financial contagion, we start by generating the interbank network. In the first moment, we use an Erdős–Rényi random graph as a baseline, in which each edge is simply present with independent probability p and absent with probability $1-p$. If N is the total number of nodes, the average degree is therefore $z = (N-1)p$. The graphs are generated using the algorithm described in [2].
Later, we try a scale-free network, one whose degree distribution follows a power law, at least asymptotically. This means the fraction $P(k)$ of nodes in the network having k connections to other nodes goes for large values of k as $P(k) \sim k^{-\gamma}$. We make sure the two networks have the same average degree for comparison purposes. The scale-free networks are generated using a preferential attachment mechanism as described by Barabasi and Albert [3]. This algorithm produces undirected graphs with degree sequence $P(k) \sim k^{-3}$, $k \geq z/2$. To make the graph directed, each edge is assigned a direction with equal probability in a postprocessing step. As shown in Figure 1, the out-degree distribution of the scale-free graph presents a much longer tail.
In the previous section, we assumed that the interbank asset A_i^{IB} is independent of the degree of the node i: to fulfill this assumption, we draw the interbank assets of each bank from the same distribution. We choose this distribution to be Gaussian with arbitrary mean value 10 and unitary standard deviation. For each bank, the value sampled from the distribution is evenly allocated over its incoming links, as described in the previous section.



(a)



(b)

Figure 1: a) Out-degree distribution for Erdos-Reny graph. ($z = 3$) b) Out-degree distribution for Barabasi-Albert graph ($z = 3$)

Following the reference paper, the external asset is then fixed to four times the interbank one $A_i^M = 4A_i^{IB}$, so that it represents the 80% of total assets.
After that, interbank liabilities are symmetrically determined by the previously fixed quantities using the fact that a bank's interbank asset is another bank's liability.
Finally, we want to fix the ratio between capital buffer and total assets to a certain percentage c. Therefore, we set the deposit D_i to the value that fulfills this assumption.
In every simulation, the number of banks N is set to 1000 and to keep matters simple, we also choose $q = 1$ in equation (2).
After setting up the network, we start the simulation by shocking one bank at random, wiping out all of its external assets, and setting it to be defaulted on all of its interbank liabilities. As a result, neighbouring banks may also default if their capital buffer is insufficient to cover their loss on interbank assets, so we then check the solvent condition (2) for all the out-neighbours and continue in this way to propagate the contagion until there are no new nodes that turn to default.
In the experiments that follow, we draw 100 realizations of the network for each tuple of (z, c). The value of the average connectivity z can assume all integers in the range 0 to 19 including extremes, while the capital buffer c varies from 0% to 9% including extremes with a step size of 1. In each of these draws, since we are only interested in the likelihood and conditional spread of system-wide contagion, we exclude very small outbreaks of default. Thus, when calculating the probability and conditional spread of contagion, we only count episodes in which over 5 per cent of the system defaults.
For each type of graph, we reproduce Figure 6 of the reference paper, which illustrates how changes in the average degree and capital buffers jointly affect the expected number of defaults in the system. Since this diagram does not take into account the probability of contagion, rare but high-impact events appear in the flat region as the expected number of defaults in these cases is low.

4 Experimental Results: Erdös–Rényi

We report the experimental results for the Erdős–Rényi random graph, analyzing both the situation in which the first bank to default is chosen at random and the case in which the bank to default is one with the highest out-degree. This type of graph is a useful benchmark for comparison with the scale-free network we will use later.
In Figure 2, we have the surface of the expected percentage of defaults when shocking the first bank at random, therefore reproducing exactly the experimental settings of the reference paper:

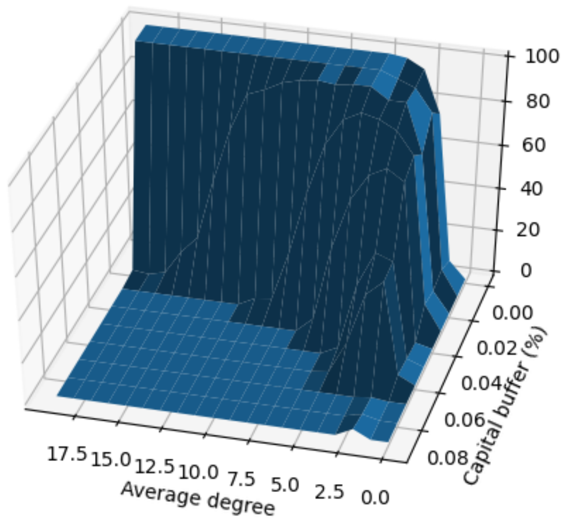


Figure 2: Connectivity, capital buffers and the expected percentage of systemic defaults for the Erdős–Rényi graph: the first bank to default is chosen at random.

In Figure 3, we have instead the surface for the expected percentage of defaults when shocking the bank with the maximum out-degree. We expect it to have a similar behaviour to the random case due to the not so pronounced difference between the maximum and the average out-degrees for a random graph, but it should increase the probability of a contagion since the selected bank is (one of) the heart(s) of the financial system.

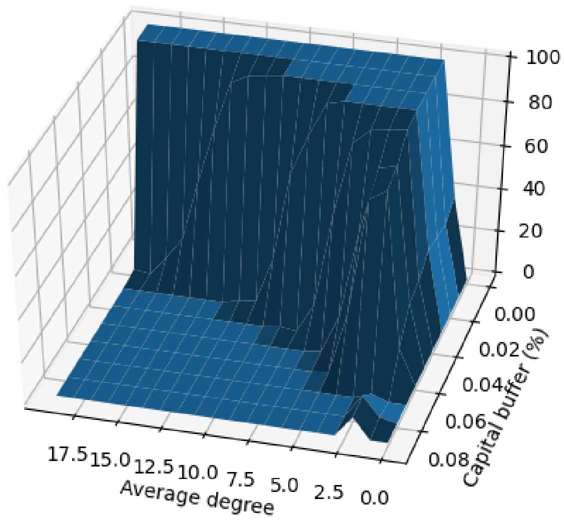


Figure 3: Connectivity, capital buffers and the expected percentage of systemic defaults for the Erdős–Rényi graph: the first bank to default is the one with maximum out-degree.

5 Experimental Results: Scale-free Network

In this section, we report the experimental results for the scale-free network, analyzing again both the situation in which the first bank to default is chosen at random and the case in which the bank to default is one with the highest out-degree. The distinctive feature of a scale-free network is the existence of nodes with very different degrees, and in particular the existence of hubs with a large number of connections.

The network generated in this way should better represent a real financial network, since we have both very central banks and peripheral ones with a low number of connections. The capital buffer and average degree ranges remain the same to the previous case simulations to help with comparison. In Figure 4, we have the surface of the expected percentage of defaults when shocking the first bank at random. Under the scale-free assumption of the degree distribution, this bank could be a peripheral one having almost no impact on the system, as well as the core of the financial system. On the other side, in Figure 5, we have the surface for the expected percentage of defaults when shocking the bank with the maximum out-degree. This time, contrarily to the experiments with Erdős–Rényi, we expect to see a very different behaviour to the random shocking case, since the difference between the highest degree banks and the others is definitely more significant than in the Erdős–Rényi case, where the average degree of all nodes is the same.

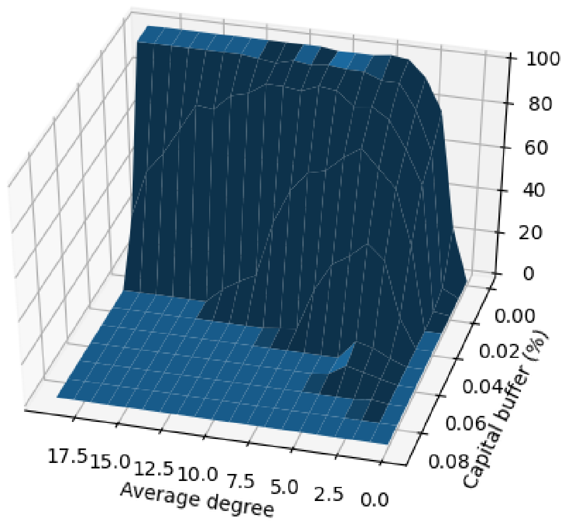


Figure 4: Connectivity, capital buffers and the expected percentage of systemic defaults for the scale-free graph: the first bank to default is chosen at random.

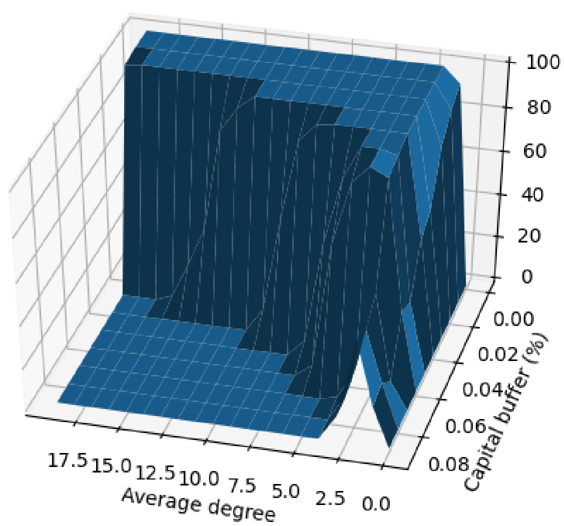


Figure 5: Connectivity, capital buffers and the expected percentage of systemic defaults for the scale-free graph: the first bank to default is the one with maximum out-degree.

6 Analysis Of The Results

First, we list some general properties that apply to each scenario, whereas in the second part, we focus more on comparing the results across different graph types.
Common behaviors that we can observe across all simulations include:

- fixed the value of average connectivity z, an erosion of capital buffers widens the contagion window.
- fixed a value of the capital buffer c, the extent of the contagion is non-monotonic in connectivity, occurs only within a certain window of z, and there is an upper phase transition after which a high connectivity drastically reduces the extent of expected contagion. The lower the capital buffer, however, the higher becomes the value needed for the transition.
- the joint changes show when capital buffers are eroded to critical levels, the level of contagion risk can increase extremely rapidly, passing from zero to high values very steeply along the curve in the (z, c) plane that marks this separation.

In Figure 6, we compare using heatmaps the four experiments we conducted. In this way, we can better appreciate the border that separates relatively "safe" configurations (rare but high-impact events appear in the flat region) from very risky ones. Of particular interest are the comparison between the two plots (a) and (b), as well as (b) and (d).

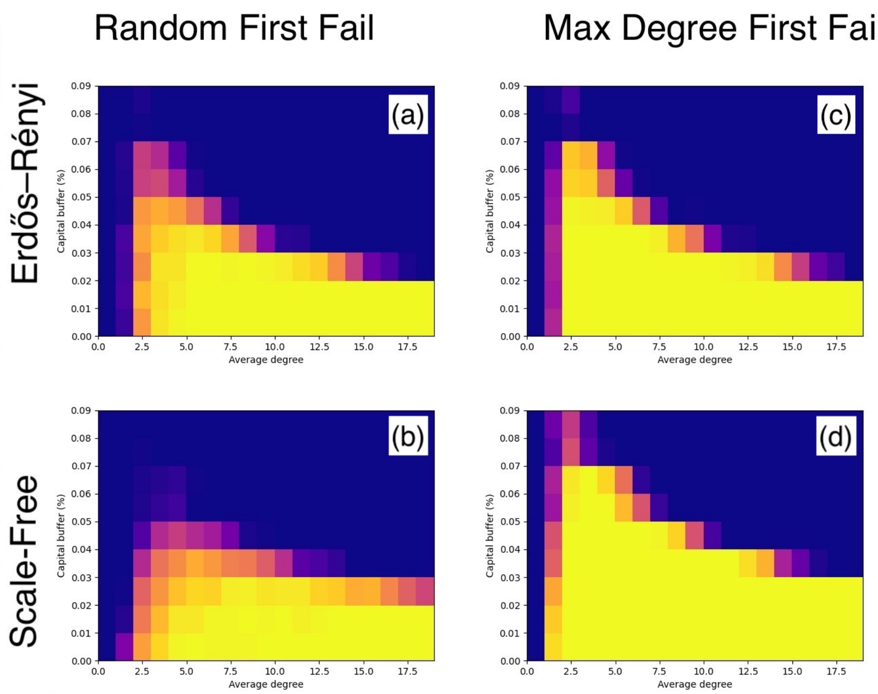
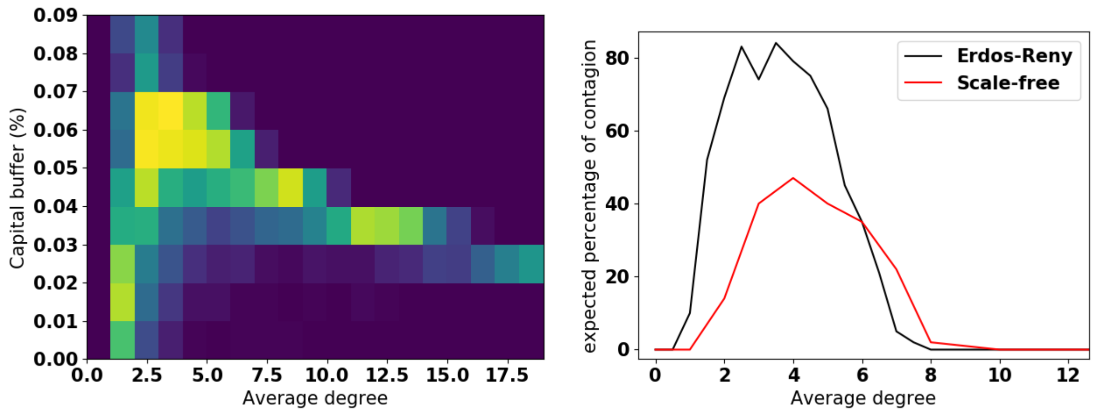


Figure 6: Heatmaps of expected percentage of systemic defaults for the Erdős–Rényi and scale-free graphs. The first bank to default is chosen at random or the one with maximum out-degree.

Looking at (a) and (b), we see the extent of a systemic collapse is reduced in the region delimited by an average degree of 2 and 6, and a capital buffer above 5: in the scale-free network, this value of c seems to offer a higher protection if a random bank collapses.
Looking at (b) and (d) instead, we observe that in a scale-free network the contagion extension conditional on the initial failure of the most connected bank is significantly increased. This is

because a system with scale-free topology is characterized by a few hubs and many less connected nodes, and the failure of one of these hubs potentially has an impact on many other nodes. However, in the case of the failure of a random bank, the probability that the selected bank is highly connected is quite small. We can better visualize the increment by looking at the difference between the two heat maps, plotted in Figure 7(a). By taking the cross-section corresponding to a 4 % capital buffer, we can see that the scale-free network presents in general a lower expected spreading of the default.



(a) (b)

Figure 7: a) Heatmaps difference between random first fail and max degree first fail in the scale-free network. b) Comparison between Erdős–Rényi and Scale-free network of the expected percentage of contagion for 4% capital buffer.

From the solvent condition written in the form (3) we can foresee the behavior of the boundary of the phase transition. Indeed, considering that we set $q = 1$ and $A_i^{IB} = (A_i^{IB} + A_i^{M})/5$, the condition (3) will be on average satisfied when

$$c_{pt} = \frac{K}{5A^{IB}} > \frac{1}{5z} \tag{4}$$

This behavior can be seen especially in the Erdős–Rényi graph where large fluctuations of the degree are less likely, as we show in figure 8.

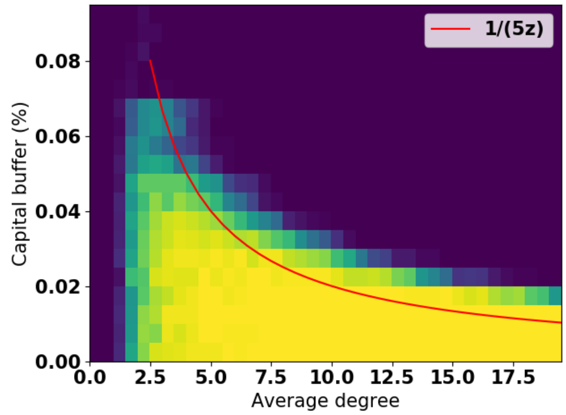
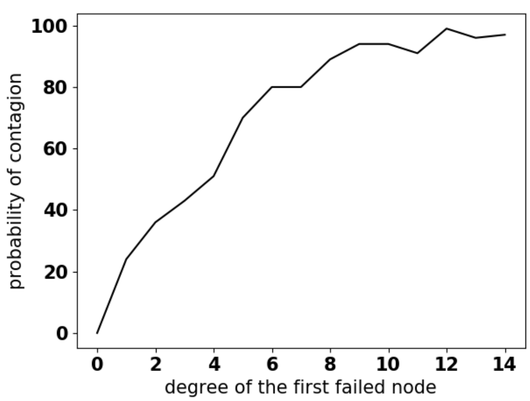
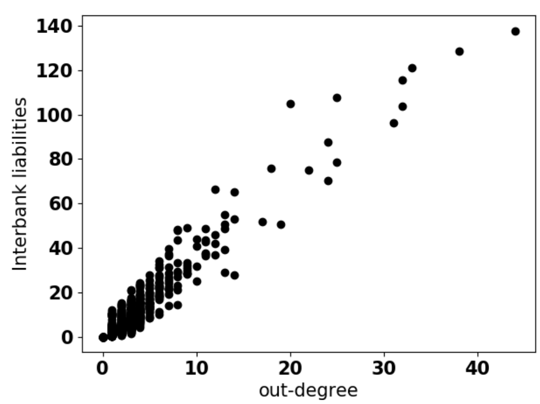


Figure 8: Phase transition boundary behavior.

We want to further investigate the importance of the choice of the first node to fail. To do that, we fixed the average out-degree and the capital buffer, respectively to $z = 3$ and $c = 4\%$ and we studied how the contagion probability is affected by the degree of the first chosen node. The result is shown in figure 9 (a). As can be expected, the probability increases with the out-degree of the bank, with a rapid initial growth that saturates at high degrees. A similar result can be expected if instead we focus on the amount of interbank liability of the node. Indeed, even if the assets were set independently of the in-degree, the out-degree and the interbank liability show a clear correlation, as can be visualized by figure 9 (b).



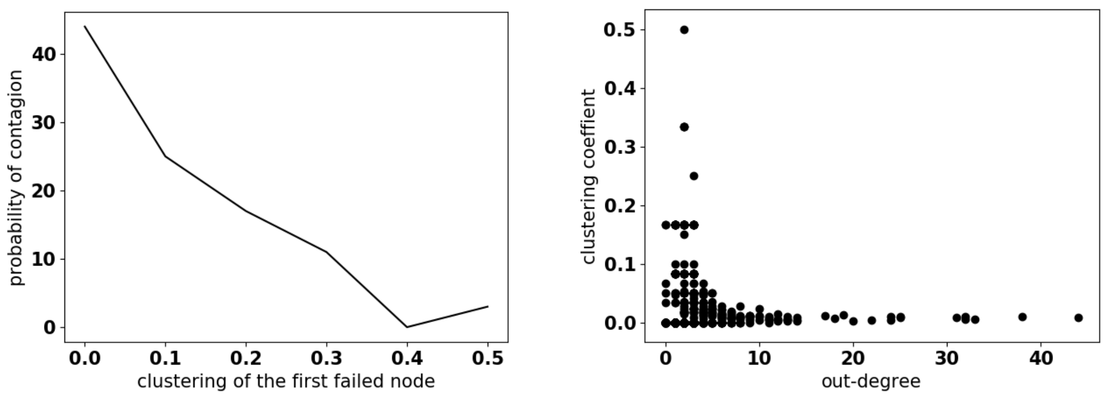
(a)



(b)

Figure 9: a) Contagion probability vs out-degree of the first failed banck. b) correlation between out-degree and liabilities

A second parameter we considered the clustering coefficient of the initial node. The presence of triangular connections may increase the chances of contagion. Indeed, if a neighbour of the first node defaults and a third bank is neighbour of both, than is more likely that it also defaults. However, what we observe is a decreasing probability as we select the initial node with higher clustering coefficient, as shown in figure 10 (a). That can be explained by the fact that in a sparse network only the more isolated nodes can manage to saturate many of the possible triangles to which they can belong. Indeed, the nodes with higher clustering coefficient have a low out-degree, as shown in figure 10 (b).



(a)

(b)

Figure 10: a) Contagion probability vs clustering coefficient of the first failed bank. b) correlation between out-degree and clustering coefficient

7 Conclusion

Our simulation reproduces a behavior similar to the one observed in [1]. Indeed, we observe that in the region of low connectivity and low capital buffer, the chances of a systemic crisis are higher. Increasing the connectivity, or the capital buffer, we transition to a robust-yet-fragile behavior in which the probability of contagion drops down lowering the expected number of defaults. Quantitative values observed in the simulation are sensitive simulation parameters that need to be tuned to obtain realistic data. The overall behavior described above, however, is quite general and is shown in both the network structures we considered.

References

[1] Sujit Kapadia Prasanna Gai. Contagion in financial networks. 2010.

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[3] Albert-Laszlo Barabasi and Reka Albert. Emergence of scaling in random networks. *Science*, 286(5439):509–512, Oct 1999.