Lawler 1980

- He implicitly shows that the probability of the LERW is the same as that of the Laplacian RW.
  - In particular, he shows that it is the same as the CONDITIONED probability of a ~~path~~ RW with "taboo region" equal to the already-visited points $\equiv$ Laplacian RW:
    - Indeed the Laplacian RW is a RW where the probability is given by solving the (discrete) Laplace equation with zero boundary condition on the already-present path and 1 at some other point.

Q?? In his 1980 paper, Lawler shows this correspondence in $d \geq 3$.
Does it hold also for $d \leq 2$?? I think that in a later paper he says yes, with some slight adjustments.

- Let's follow its derivation.
- First, define the RW and some other useful quantities.
- On a square lattice $\mathbb{Z}^d$, with $\{e_1, \dots, e_{2d}\}$ the unit vectors, we consider the random variables $Y_1, Y_2, \dots$ defined on the probability space $(\Omega, \mathcal{B}, P)$ with uniform distribution:

$$P\{Y_i = e_j\} = \frac{1}{2d} \quad , j = 1, \dots, 2d$$

$2d$ ← # of links at each point

DEF

SIMPLE RANDOM WALK S defined as:

$$S(n, \omega) := \sum_{i=1}^{n} Y_i(\omega) \quad \text{with } S(0,\omega) = \underset{\text{Origin}}{0} \in \mathbb{Z}^d \quad \forall \omega \in \Omega$$

the extracted "random number"

- Now, let's define the transition probability function for S:

$$p_k(x,y) := P_x\{S(k)=y\} \equiv P\{S(k+n)=y \mid S(n)=x\} \qquad (P\{S(n)=x\}>0)$$

i.e. the probability that, given the RW at point x at the time n, we get a RW passing through y at time $k+n$.

= P_x means: probability with S starting from x

N.B. This process is independent from ~~n~~ n (i.e. No Memory, not interested in the past)

- We also set

$$p_k := p_k(0,0) = p_k(x,x) \qquad \forall x \in \mathbb{Z}^d$$

i.e. the probability to come back to an already-visited point and therefore form a loop

- Now, for any set $A \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, let

$$E^A(x) := P_x\{S(n)\notin A \;\forall n>0\} = P\{S(n)\notin A\;\forall n>0 \mid S(0)=x\} = \frac{P\{S(n)\notin A\;\forall n>0,\, S(0)=x\}}{P\{S(0)=x\}}$$

i.e. the probability that a RW starting from x avoids the set A at any time after zero (It could start in A: $S(0)=x\in A$, but then leave it for good)

- Now assume $A\subset\mathbb{Z}^d$ s.t. $E^A(0)>0$, i.e. there is a finite probability that a RW starting from the origin never visits A.

= Now call $\Omega^A \subset \Omega$ (THIS IS A COLLECTION OF ω's) the set of those RW that avoid A, and let P^A be the conditioned probability:

$$P^A(B) := \frac{P(B)}{P(\Omega^A)} \qquad \forall B \subset \Omega^A$$

i.e. the Conditioned probability to get a subset B of Random Walks that avoid A ($B\subset\Omega^A$), provided that we get a RW avoiding A

↳ Then

DEF

RW WITH TABOO SET A: S^A ~ Laplacian RW

= It is a Markov Process (i.e. Ignorant/independent of the past) on Ω^A with stationary transition probabilities:

$$P^A(x,x+e_i) := P^A\{S^A(n+1)=x+e_i \mid S^A(n)=x\}$$

INDEPENDENT FROM n

i.e. the probability to go from x to $x+e_i$ respecting the Taboo set A

NB ~

NB Even if he doesn't say it, this is essentially the LAPLACIAN RW ! LAPLACIA RW (LRW)

- Let's Manipulate $P^A(x,x+e_i)$, using the definition of P^A: i.e. Conditioned Probability

$$P^A(x,x+e_i) = P\{S(n+1)=x+e_i \mid S(n)=x,\ \overbrace{S(j)\notin A\;\forall j\geq 1}^{\text{We assume that the RW avoids } A}\}$$

Def of Conditioned probability

$$= \frac{P\{S(n+1)=x+e_i,\ S(n)=x,\ S(j)\notin A\;\forall j\geq1\}/P\{S(n)=x\}}{P\{S(n)=x,\ S(j)\notin A\;\forall j\geq 1\}/P\{S(n)=x\}}$$

Def of $E^A(x)$ with $n\to 0$

$$= \frac{{}_{①}P\{S(n+1)=x+e_i,\ S(j)\notin A\;\forall j\geq1 \mid S(n)=x\}/{}_{②}P\{S(n)=x\}\cdot\dfrac{P\{S(n+1)=x+e_i\}}{{}_{③}P\{S(n+1)=x+e_i\}}}{E^A(x)}$$

Since: $P_{②}P_{③} = P\{S(n)=x, S(n+1)=x+e_i\}$

$\dfrac{P_{①}}{P_{②}\cdot P_{③}} = E^A(x+e_i)$

$P\{S(n+1)=x+e_i\} = \frac{1}{2d}$

$$= \frac{E^A(x+e_i)}{(2d)\,E^A(x)} \qquad (\text{w/ } x+e_i\notin A)$$

Composite probability: We can choose from $(2d)$ options: $\{e_i\}_{i=1,\dots,2d}$ ↳ $\frac{1}{2d}$

- Then we define by Induction:

$$P_k^A(x,y) := \sum_{z\in\mathbb{Z}^d} P^A(x,z)\,P_{k-1}^A(z,y)$$

← all points in $\mathbb{Z}^d$

≡ Sum over all possible trajectories ~ Path integral with Boundary conditions: (i) start at x (ii) end at y } in k steps (iii) Avoid A

I THINK w/ $P_1^A = P^A$

e.g.

$k=1)$ $P_2^A(x,y) = \sum_{z\in\mathbb{Z}^d} P^A(x,z)\,P^A(z,y)$

Since $P^A(x,z) \propto \delta(|x-z|=1) \Rightarrow P_2^A(x,y) \propto \delta(|x-y|=2)$

$k=2)$ $P_3^A(x,y) = \sum_{z\in\mathbb{Z}^d} P^A(x,z)\,P_2^A(z,y) \propto \delta(|x-y|=3)$

etc.

↳ Then, again by Induction I can show:

$$p_K^A(x,y) = \frac{E^A(y)}{E^A(x)} P\{S(n+K)=y,\ S(j)\notin A\ \forall j=n+1,\dots,n+K \mid S(n)=x\}$$

PROOF: $K=2)\ p_2^A(x,y) = \sum_z p^A(x,z)p^A(z,y) = \sum_z \frac{\delta^A(|x-z|=1)}{2d}\frac{\cancel{E^A(z)}}{E^A(x)}\frac{\delta^A(|z-y|=1)}{2d}\frac{E^A(y)}{\cancel{E^A(z)}} = \frac{E^A(y)}{E^A(x)}\boxed{\frac{\delta^A(|x-y|=2)}{(2d)^2}}$ Prob. that after K=2 steps from x we reach y, Avoiding A

$K=1)\quad p_1^A(x,y) = p^A(x,y) = \delta^A(|x-y|=1)\frac{E^A(y=x+e_i)}{(2d)\,E^A(x)}$

and $\frac{\delta^A(|x-y|=1)}{2d}$ is the probability that among the 2d neighboring vertices, the path chooses y (Always avoiding A)

$$\Rightarrow \frac{\delta^A(|x-y|=1)}{2d} = P\{S(n+1)=y,\ S(j)\notin A\ \forall j=n+1 \mid S(n)=x\}$$
$$\equiv P_x\{S(1)=y,\ S(j)\notin A\ \forall j=1\}$$

$\Rightarrow$ Valid for $K=1$ ✓

- Now: assuming it's valid for K, let's show that it holds for K+1:

↳ $p_{K+1}^A(x,y) \overset{\text{def}}{=} \sum_{z\in\mathbb{Z}^d} p^A(x,z)\,p_K^A(z,y)$

$$= \sum_{z\in\mathbb{Z}^d} p^A(x,z)\frac{E^A(y)}{E^A(z)} P\{S(n+K)=y,\ S(j)\notin A\ \forall j=n+1,\dots,n+K \mid S(n)=z\}$$

$$= \sum_{z\in\mathbb{Z}^d} \frac{\cancel{E^A(z=x+e_i)}}{2d\ E^A(x)}\,\delta^A(|x-z|=1)\frac{E^A(y)}{\cancel{E^A(z)}} \underbrace{P\{S(n+K)=y,\ S(j)\notin A\ \forall j=n+1,\dots,n+K \mid S(n)=z\}}_{\text{def}}$$

"def" of $\frac{\delta^A}{2d}$

$$\overset{?}{=} \frac{E^A(y)}{E^A(x)} \underbrace{\sum_{z\in\mathbb{Z}^d} P_x\{S(1)=z,\ S(j)\notin A\ \forall j=1\}\cdot P_z\{S(K)=y,\ S(j)\notin A\ \forall j=1,\dots,K\}}$$

We basically add an extra step (at the beginning): $x\xrightarrow{1}z$ then $z\xrightarrow{K}y$, always avoiding A

$$= \frac{E^A(y)}{E^A(x)} P_x\{S(K+1)=y,\ S(j)\notin A\ \forall j=1,\dots,K+1\}$$ ⊠

NB

NB As a consequence, if $A\subset B$, i.e. B is a larger region we want to avoid

$$\Rightarrow p_K^A(x,x) \geq p_K^B(x,x) \quad \forall x\in\mathbb{Z}^d$$

Simply because the probability to avoid a larger region is smaller (and $\frac{E^A(x)}{E^A(x)}=1$)

- And in Particular, since the Simple RW is Just a Random Walk with the empty set as the Taboo set:

$$p_K^A(x,x) \leq p_K(0,0) \quad \forall x\in\mathbb{Z}^d$$

$\equiv$ i.e. from x to x

i.e. any RW with any Taboo Set (different from the Empty Set) is less likely to return to it's Origin than the unrestricted, Simple RW $\Rightarrow$ if $p_K(0,0)=0$ i.e. TRANSIENT, then $p_K^A(x,x)=0$ TRANSIENT

- Now, we define the loop-erasing procedure (Chronological Loop-Erasure) to create the LERW.

There are several ways in which one can define this procedure. Let's follow the one proposed by Lawler:

- Assume S a RW on $\mathbb{Z}^d$ with $d\geq 3$ — RWs are TRANSIENT $\equiv$ Probability to come back to the origin less than 1 (i.e. it is Possible that it never comes back)

WHAT ABOUT $d\leq 2$? in $d\leq 2$ the RW is RECURSIVE, i.e. it will come back to the origin with probability 1 (WE NEED INFINITE TIME TO DO THIS)

- Let

$$\tau_1(\omega) := \min\{j \text{ s.t. } \exists i<j \text{ with } S(i,\omega)=S(j,\omega)\}$$

i.e. consider the smallest time at which the RW visits a point it has already visited ($\equiv$ forms a loop)

- and let

$$\varphi_1(\omega) := \text{that time } i<\tau_1(\omega) \text{ for which } S(i,\omega)=S(\tau_1(\omega),\omega),$$

i.e. the time at which the RW first visited that point

- Then set

$$\hat{S}_1(n,\omega) := \begin{cases} S(n,\omega) & ,n\leq\varphi_1(\omega) \\ S(n+(\tau_1(\omega)-\varphi_1(\omega)),\omega) & ,n>\varphi_1(\omega)\end{cases}$$

i.e. $\hat{S}_1$ is the same RW as S up to time $\varphi_1(\omega)$ (when it visited the "knot" of a loop for the first time), then it skips the loop (between times $\varphi_1(\omega)$ and $\tau_1(\omega)$), and finally it is S again

S

= Now we iterate this process: ~~until there is no loop left.~~ THERE IS NO STOPPING CONDITION! it takes $k\to\infty$

Namely, for $k>1$:

$$\begin{cases} \tau_k(\omega) := \min\{ j \text{ s.t. } \exists\, i<j \text{ with } \hat S_{k-1}(i,\omega) = \hat S_{k-1}(j,\omega) \} \\ \varphi_k(\omega) := \text{that } i<\tau_k(\omega) \text{ for which } \hat S_{k-1}(i,\omega) = \hat S_{k-1}(\tau_k(\omega),\omega) \end{cases}$$

and

$$\hat S_k(n,\omega) := \begin{cases} \hat S_{k-1}(n,\omega) & n \le \varphi_k(\omega) \\ \hat S_{k-1}\big(n+(\tau_k(\omega)-\varphi_k(\omega)),\omega\big) & n > \varphi_k(\omega) \end{cases}$$

= Finally: $\boxed{\hat S \equiv LE(S)}$

$$\boxed{\hat S(n,\omega) := \lim_{k\to\infty} \hat S_k(n,\omega)}$$

LERW

Well defined because for $d\geq 3$, $S(n)$ approaches infinity with Prob.$=1$ (almost surely) (it is Transient) and thus $\hat S$ is Not empty. In $d\le 2$ there would be nothing left.

NB $\hat S$ is a Non-Markov Process: the distribution for each step is dependent on the entire past history. !

Indeed, we cannot know if a piece of the RW will be kept until we have finished the process, because it could come back to itself, thus forming a loop that needs to be erased.

= However, what is Remarkable, is that this has the same statistics as the Laplacian RW, which is instead Markovian (as we will later show) !

?? Is it TRUE ?? NO, Laplacian RWs are Non-Markovian too (Faraday Cages (e.g.)) !

NB LERWs are self-avoiding by Construction

• Now, we want to compute the transition probability for a LERW.

• Let $\Gamma = [x_0, x_1, \dots, x_n]$ be a n-step self-avoiding path, $\neq$ Walk (a walk has a specific statistic)

i.e. an Ordered (WITHOUT STATISTICS) set of points in $\mathbb{Z}^d$ s.t.:

$$\begin{cases} x_0 = 0 \\ \|x_i - x_{i-1}\| = 1 & \forall\, 0< i\le n \\ x_i \neq x_j & \text{for } i\neq j \end{cases}$$

• We want to compute:

$$\boxed{P\{ \hat S(n+1) = x_n+e_i \mid \underbrace{[\hat S(0),\dots,\hat S(n)] = [x_0,\dots,x_n]}_{\equiv LERW} \}}$$

i.e. the transition prob. that at the next step $(n+1)$, $\hat S$ will be at x_n+e_i, provided that at each previous step it was a self-avoiding path $[x_0,\dots,x_n]$

• Let's start by setting

(2D: we would need to place a cut off, otherwise $\exists\, \tilde k$ s.t. $S(\tilde k)=S(k)$ with Prob.$=1$)

$$B_k^\Gamma := \{ \omega \text{ s.t. } [\hat S(0,\omega),\dots,\hat S(n,\omega)] = [x_0,\dots,x_n],\ S(k,\omega) = \hat S(n,\omega) \text{ and } S(\tilde k,\omega) \neq S(k,\omega) \text{ for } \tilde k>k \}$$

i.e. the set of all random events $\omega\in\Omega$ s.t. we have a (RW S whose loop erasure is a) LERW up to time n, and which is the same as $\Gamma=[x_0,\dots,x_n]$; and s.t. at time k the RW S overlaps $\hat S(n)$, but, at any later time $\tilde k>k$ never again (the k is the last time that S visits x_n)

• Now, let's define a RW T on B_k^Γ:

$$\boxed{T(j) := S(j+k) - S(k)}$$

i.e. a RW normalized in such a way that $T(0)=0$ (starts from the origin) and that later it never comes back there (since in B_k^Γ: $S(\tilde k,\omega)\neq S(k,\omega)$ for $\tilde k>k$) NOR to any previous points of $\hat S$ as it is shown later on

NB

NB In order for ω to be in B_K^Γ, two conditions must be met:

(1) If one loop-erases S for the first K steps, one must get the n-step LERW $\hat{S} = [x_0, \dots, x_n]$; $\boxed{\hat{S} = LE(S)}$

(2) $\forall m > K \quad S(m,\omega) \notin \{x_0, \dots, x_n\}$, i.e. S will never come back on its ~~...~~ first n-steps for any later time $m > K$.

This is so because:

- $S(m,\omega) = x_n$ for $m > K$ is impossible by definition of B_K^Γ;
- And if $S(m,\omega) = x_j$ for $m > K$ and $j < n$, then the loop-erasure would eliminate the loop containing $S(K,\omega) = x_n$. BECAUSE THE LOOP ERASURE IS TAKEN TO INFINITY
  This would imply $\hat{S}(n,\omega) \neq x_n$ because $S(m,\omega) \neq x_n \ \forall m > K$.
  Therefore, this $\omega \notin B_K^\Gamma$.

Again, in 2D we simply use a cut-off that represents the time up to which we loop-erase S. then we would consider $m > K$ but smaller than the cut-off

- Now, let's see ~~...~~ what (1) and (2) imply on T:

(1): Since $\{S(i)$ with $i = 0, \dots, K\}$ and $\{S(j+K) - S(K)$ with $j = 1, 2, \dots\}$ are independent sets of random variables, this condition has no implication on the RW T_j;

the Random walk

T, which is "ANOTHER class of RW"

BECAUSE RWs are Markov Processes, and so $S(j+K)$ knows nothing of $S(K)$

(2): This condition is exactly the definition of a RW starting at x_n with taboo set $\{x_0, \dots, x_n\}$, which is "its previous trail."

$\Rightarrow T$ acts on B_K^Γ as a RW with taboo set $\{x_0, \dots, x_n\}$,

$$\Rightarrow \quad T = S^{\{x_0, \dots, x_n\}}$$

"LRW !! & THIS MEANS THAT IT WON'T ERASE ITS NEXT T(1) STEP !!

NB

NB All of this is actually independent on the K one chooses

$\hookrightarrow$ We can make slight changes:

$$B^\Gamma := \{\omega \text{ s.t. } [\hat{S}(0,\omega), \dots, \hat{S}(n,\omega)] = [x_0, \dots, x_n]\}$$

i.e. the set of ω s.t. we get a LERW with trail $[x_0, \dots, x_n]$

and on B^Γ define

$$T(j,\omega) := S(j + K_n(\omega), \omega) - S(K_n(\omega), \omega)$$

w/ $K_n(\omega) := \max\{K \text{ s.t. } S(K,\omega) = \hat{S}(n,\omega)\}$

i.e. the last time S ~~...~~ visited $\hat{S}(n,\omega)$

2D: we would take the max up to the cut-off

The $S(\tilde{K},\omega) \neq S(K,\omega)$ $\forall \tilde{K} > K$ condition in B_K^Γ has been replaced by the def. of $K_n(\omega)$

$\Rightarrow$ Again, T is a RW with Taboo set $\{x_0, \dots, x_n\}$!

- Now, we use the transition prob. for RWs with a taboo set to compute the transition prob. for $\hat{S}$, which is what we were looking for:

recall: $$P^A(x, x+e_i) = \frac{E^A(x+e_i)}{(2d)\, E^A(x)}$$

= If we set $A_n = \{x_0, \dots, x_n\}$:

This is a little sketchy (*) !

$$= P\{T(1) = x + e_i\} =$$

$$P\{\hat{S}(n+1) = x + e_i \mid [\hat{S}(0), \dots, \hat{S}(n)] = [x_0, \dots, x_n]\} = P^{A_n}(x, x+e_i)$$

$$= \frac{E^{A_n}(x+e_i)}{(2d)\, E^{A_n}(x)}$$

NB

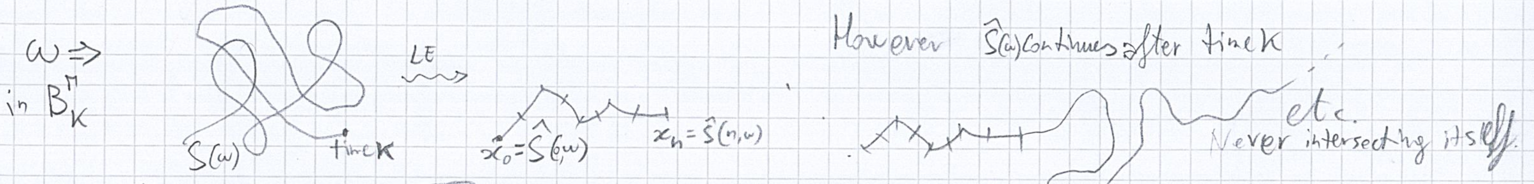
NB This completes the proof/computation of P for the LERW $\hat{S}$, but at the same time it shows that

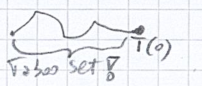
$$\boxed{LERW \equiv LRW}$$!

By construction, T is a LRW, At each step, one should define a new T, i.e. "update" its taboo set

In fact a fixed T could form a loop later on, but never in its taboo set

(*) This is TRUE because: given a ω, the LERW is completely FIXED:



- On B_K^Γ, i.e. for these ω's that give that $\hat{S}$, we take: $T(j) := S(j+K) - S(K) \equiv$ 

which gives the LERW $\hat{S}$ without its first n points after loop erasure.

- The thing is: I just consider its next step: $\boxed{T(1)}$. By construction it will not form a loop AND it will not be cancelled by any future loop because it would mean that it came back on its TABOO SET !