# Calculating the Complex Cepstrum Without Phase Unwrapping or Integration

J. B. BEDNAR AND T. L. WATT

*Abstract*—**The complex cepstrum, as developed by Oppenheim [4] and Oppenheim, Schafer, and Stockham [5], originated from the pioneering paper of Bogert, Healy, and Tukey [3]. It is a nonlinear transformation from one signal space into another which converts two superposed signals into their sum. The cepstrum is usually computed by inverse Fourier transformation of the frequency domain complex logarithm of the original signal. The phase term or imaginary part of this complex logarithm presents a computational difficulty which is sometimes approached by integration of the derivative of the phase spectrum. An excellent approach to phase computation is the adaptive integration methodology as proposed by Tribolet [10]. Unfortunately, this technique can be computationally intensive. This note describes a simple technique for efficient computation of the complex cepstrum which requires no phase unwrapping or integration. The procedure is given in both one and two dimensions.**

## Introduction

The complex cepstrum arose as an extension of the ideas of Bogert, Healey, and Tukey [3]. In its original form, the cepstrum was computed from the logarithm of the spectrum. The complex cepstrum, as defined by Oppenheim *et al.* [5], is the inverse discrete Fourier transform of the frequency domain complex logarithm of the given time series.

Calculation of the complex cepstrum has received considerable attention in the literature. The most comprehensive study of the hardest part of this calculation, namely phase unwrapping, appears to be that of Tribolet [10, 11]. These works represent three years of comprehensive research and are excellent tutorials on the basic computational problems in homomorphic analysis and processing. The major contribution of Tribolet's work, as far as computation of the cepstrum is concerned, is the adaptive integration algorithm for estimating the phase.

This paper develops a new algorithm for calculation of the complex cepstrum which does not require phase unwrapping or integration. This new method seems to offer analysis avenues which are not available under older schemes. It is based entirely on the simple observation that it is computationally more efficient to calculate $n$ times the complex cepstrum, i.e., differentiations in frequency, than it is to compute the complex cepstrum itself. Frequency differentiation in the homomorphic content has been used by Schroeder [8] to provide direct relations between the cepstrum and the predictor coefficients. It does not appear to have been used to calculate the cepstra of time series which are not also "inverse filters."

## Calculating the Cepstrum

Let $x(n)$ be a sequence of numbers representing digitized real data. The complex cepstrum $\hat{x}(n)$ of $x(n)$ is defined to be the inverse $z$-transform of the complex logarithm

$$\log X(z) = \ln |X(z)| + jP(z) \tag{1}$$

of the $z$-transform

$$X(z) = \sum_{n=0}^{n=\infty} x(n)z^n = X_r(z) + jX_i(z) \tag{2}$$

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of $x(n)$. Here $j$ is the square root of $-1$, $P(z)$ is the phase function associated with $X(z)$, and

$$X(z) = |X(z)| \exp [jP(z)]. \tag{3}$$

The functions $X_r$ and $X_i$ are the real and imaginary parts of $X(z)$, respectively.

Using the standard substitution $z = \exp [-j2\pi f]$ and the frequency differentiation theorem, one notes that the $z$-transform of $n\hat{x}(n)$ is

$$j\hat{x}(f) = \sum_{n=-\infty}^{n=+\infty} n\hat{x}(n)z^n = -P'(f) + j(\ln |X(f)|)' \tag{4}$$

where

$$P'(f) = (X_r'(f)X_i(f) - X_i'(f)X_r(f))/|X(f)|^2 \tag{5}$$

and

$$(\ln |X(f)|)' = (X_r'(f)X_r(f) + X_i'(f)X_i(f))/|X(f)|^2. \tag{6}$$

In these equations, $'$ designates differentiation. To complete the algorithm, one notices that when (4) is inverse transformed to get $n\hat{x}(n)$, one has

$$n\hat{x}(n)|_{n=0} = \text{avg } [-P'(f)] \tag{7}$$

instead of the required

$$n\hat{x}(n)|_{n=0} = \text{avg } [\ln |X(f)|] \tag{8}$$

as desired. One also needs to recognize the right-hand side of (7) as the equivalent of a circular shift in the original time domain. Retention of $n\hat{x}|_{n=0}$ is essential for computation of the inverse cepstrum of $x(n)$.

The complex cepstrum of a time series can then be computed through the following steps.

1) Fourier transform $x(n)$ and $nx(n)$ for $X_r(f)$, $X_i(f)$, $X_r'(f)$ and $X_i'(f)$.
2) Compute $j\hat{X}(f)$ via equations (4)–(6).
3) Compute avg $[\ln |X(f)|]$.
4) Inverse Fourier transform $j\hat{X}(f)$.
5) Divide $n\hat{x}(n)$ by $n$ when $n \neq 0$.
6) Save $\hat{x}(0)$ and replace the value there by the value computed in (3).

Tests indicate that this methodology is as accurate as phase unwrapping for computation of $\hat{x}(n)$.

The notation ofthe preceeding sections can be simplified by observing that the discrete Fourier transform $F\{n\hat{x}(n)\}$ can be expressed as

$$F\{n\hat{x}(n)\} = \overline{F\{x(n)\}}F\{nx(n)\}/|F\{x(n)\}|^2 \tag{9}$$

so that

$$|F\{x(n)\}|^2F\{n\hat{x}(n)\} = \overline{F\{x(n)\}}F\{nx(n)\}. \tag{10}$$

The algorithm for computation of $n\hat{x}(n)$ can therefore be rephrased as follows.

1*) Fourier transform $x(n)$ and $nx(n)$ to obtain $F\{x(n)\}$ and $F\{nx(n)\}$.
2*) Multiply $\overline{F\{x(n)\}}$ and $F\{nx(n)\}$.
3*) Divide the result in (2*) by $|F\{x(n)\}|^2$.
4*) Inverse Fourier transform and follow (5) and (6) in the previous statement of the algorithm.

Extension of the one-dimensional algorithm to 2 and higher dimensions is immediate. One need only observe that (10) has the forms

$$|F\{x(n, m)\}|^2F\{n\hat{x}(n, m)\} = \overline{F\{x(n, m)\}}F\{nx(n, m)\} \tag{11}$$

or

$$|F\{x(n, m)\}|^2F\{m\hat{x}(n, m)\} = \overline{F\{x(n, m)\}}F\{mx(n, m)\} \tag{12}$$

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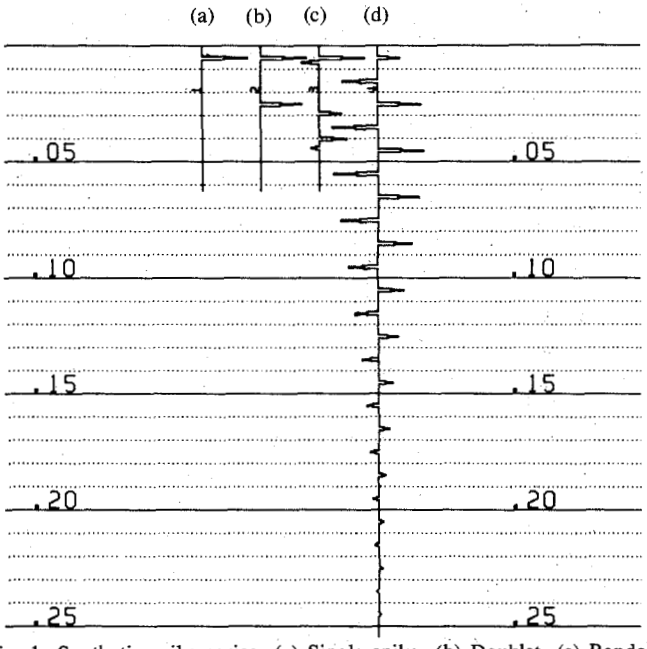


Fig. 1. Synthetic spike series. (a) Single spike. (b) Doublet. (c) Random series. (d) Reverberant series.

and then choose the form which makes most sense for the data format one is dealing with.

## Remarks

The cepstral computation technique discussed above is similar in spirit but not identical to the popular integration method in Oppenheim and Schaefer [6]. The latter method requires estimation of the phase derivative exactly as described above. This estimate must then be numerically integrated to complete the calculation. By estimating the derivative of the log-amplitude as well as the phase spectrum and then performing frequency domain integration by dividing in time, no numerical integration or phase unwrapping is necessary to obtain $\hat{x}(n)$ from $x(n)$. Although this basic idea was used by Schroeder [8], the first computational exploitation of it seems to have been presented by Bednar and Watt [2]. A similar approach to the cepstrum was discussed by S. Treitel and R. Wang in an upublished AMOCO company report. In the authors view the importance of the algorithm lies not in its computational but its analysis potential. For example, equations (10)–(12) express the cepstrum as a solution to a standard least squares problem. In matrix notation, one must solve a Toeplitz equation whose right hand side is the crosscorrelation of the signal with $n$ times the signal. Thus, the cepstrum is seen to be a solution to the problem of finding that set of coefficients which do the best job of convolutionally transforming the original signal to its point-by-point product with the time index $n$. Use of the analysis potential is further supported by Schroeder's work and will become clearer in subsequent papers.

## Examples

It has been conjectured that the preceeding method for calculating the complex cepstrum produces a very aliased cepstrum. In numerous tests of this algorithm, the authors have found no evidence of this conjecture. A second question concerns the effect of zeros on or near the unit circle. Even though the cepstrum itself is undefined when unit circle zeros are present, the algorithm seems to perform well in the presence of such zeros.

To see how well the methodology for estimating $\hat{x}(n)$ compares with the more popular approach, a data set consisting of several synthetic traces was constructed by adding various levels of band-limited noise to each of four spike series. As displayed in Fig. 1, these spike series consist of a single spike (Fig. 1, trace a), a doublet (Fig. 1, trace b), a series of five randomly placed spikes (Fig. 1, trace c), and a reverberant series (Fig. 1, trace d). A 10–70 Hz

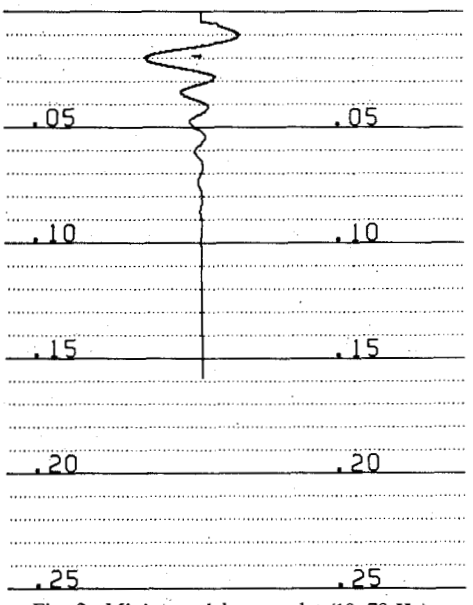


Fig. 2. Minimum delay wavelet (10–70 Hz).

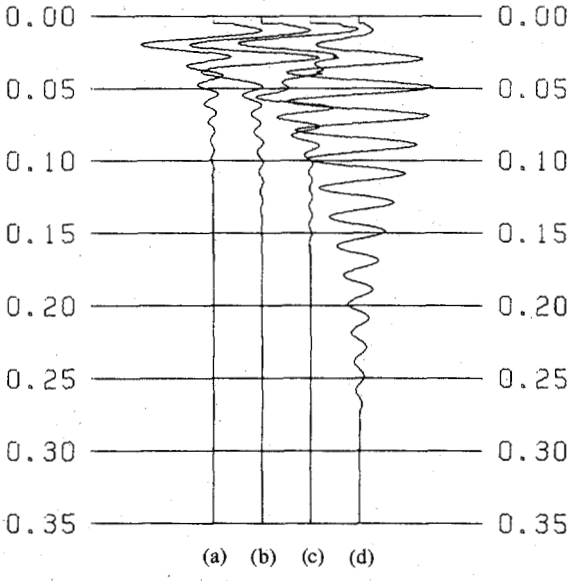


Fig. 3. Traces resulting from convolution of minimum delay wavelet of Fig. 2. (a) Single spike. (b) Doublet. (c) Random series. (d) Reverberant series of Fig. 1.

minimum phase wavelet, sampled at 1 ms intervals (Fig. 2), was then convolved with each of the traces in Fig. 1 to produce Fig. 3. White noise, limited to the 10–70 Hz band by a linear phase filter, was then added to each of the traces of Fig. 3 to get traces with signal-to-noise ratios of $\infty$, 10, 5, 2.5, 1.25 and 0.625. These traces are displayed in Fig. 4 and are grouped according to the type of spike series used to generate the trace.

Fig. 5 shows the cepstra of the data of Fig. 4 as computed by the Tribolet algorithm while Fig. 6 shows the cepstra as computed by the $nx(n)$ method. Both algorithms used radix two fast Fourier transforms. The transform length was chosen to be the least power of two greater than twice the length of the input trace. The resulting cepstra are virtually identical. Time domain exponential weighting can effectively handle computational problems associated with unit circle zeros. When compared with traditional frequency domain integration methods, the greater computational efficiency of the $nx(n)$ algorithm makes it an attractive choice in large data volume situations.

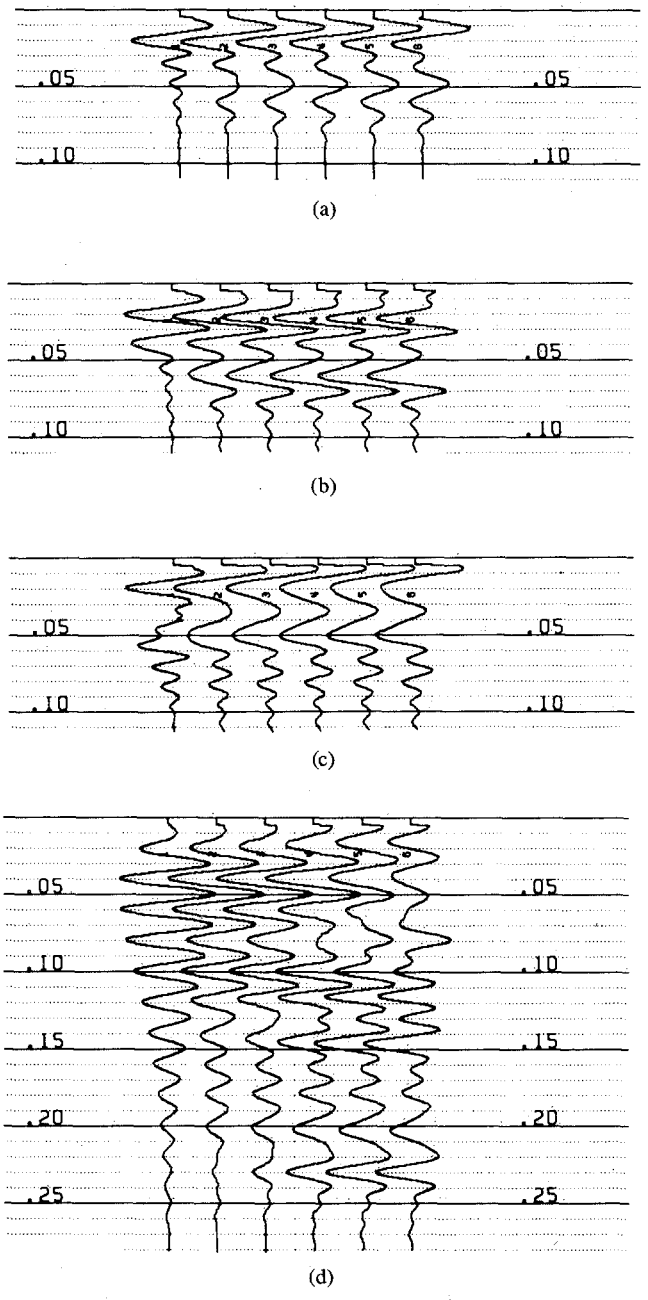


Fig. 4. Synthetic data obtained by adding 10–70 Hz noise to each of the traces of Fig. 3. Traces in (a) are from single spike, (b) from doublet, (c) from random series, and (d) from reverberant series.

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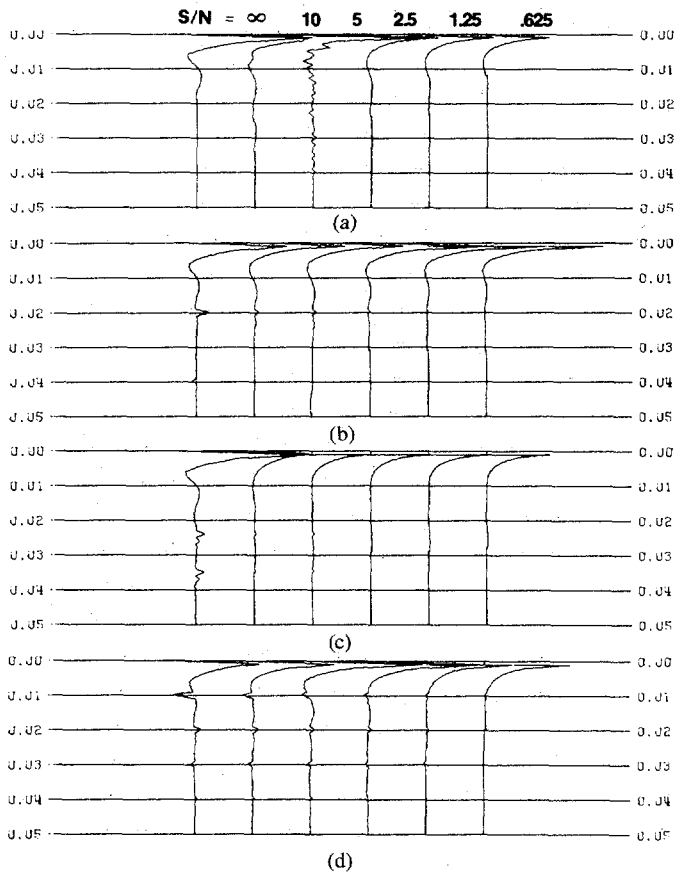


Fig. 5. Cepstrum by phase unwrapping via Tribolet algorithm. Traces in (a) are of single spike data, (b) doublet data, (c) random series data, and (d) reverberant data..

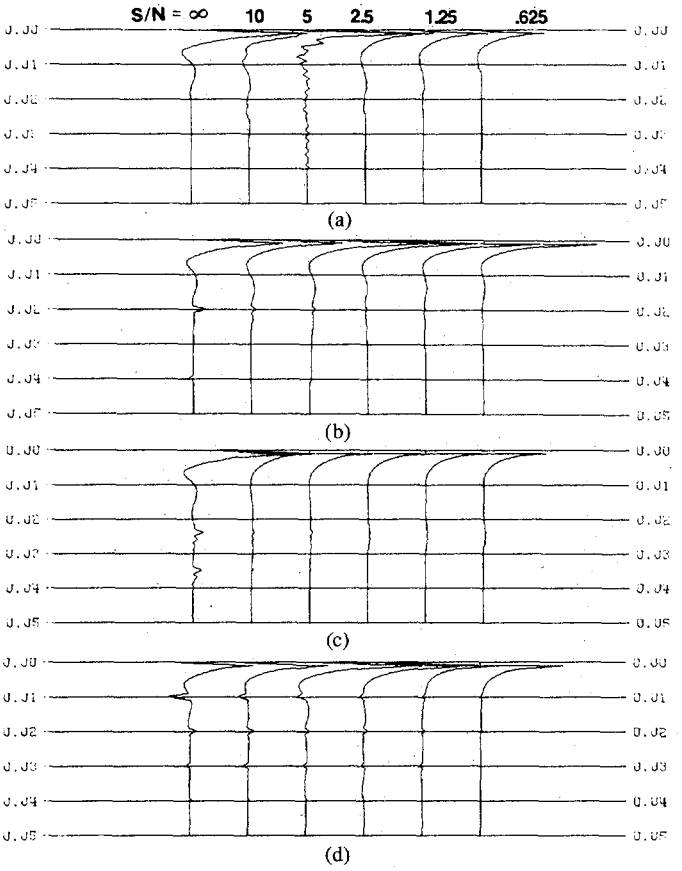


Fig. 6. Cepstrums as computed by $n\hat{x}(n)$ method. Traces in (a) from single spike data, (b) from doublet data, (c) random series, and (d) from reverberant series.

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# Some Relations Between the Various Criteria for Autoregressive Model Order Determination

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*Abstract*—**The problem of model order determination in autoregressive (AR) modeling of a given time series is considered. It is shown analytically that the model order selected using the final prediction error (FPE) criterion never exceeds the model order selected using the Akaike information criterion (AIC). It is further shown that under certain conditions, the model order selected using the criterion autoregressive transfer function (CAT) never exceeds the model order selected using the FPE criterion. These relations give some indication to what should be the preferred criterion, at least for the extreme cases when the model order is likely to be underestimated or overestimated.**

## I. Introduction

Fitting an autoregressive (AR) model to a given time series is a fundamental important problem in linear prediction, system identification, and spectral analysis. Since the optimal choice of model order (i.e., the number of AR parameters) is generally unknown, it is usually necessary in practice to postulate several model orders. Based on these, one can then compute some error criterion that indicates which model order to choose.

Several criteria have been proposed as subjective bases for selection of the AR model order. Akaike [1], [2] suggested choosing a model order so that the average error due to estimating the AR coefficients and the innovation for the one-step prediction is minimized. This criterion, called the final prediction error (FPE), is given by

$$\mathrm{FPE}(m) = \frac{N+m}{N-m} P_m \tag{1}$$

where $P_m$ is the residual squared error for an $m$th order filter, and $N$ is the number of available data samples. The order $m$ is selected to be that value for which FPE($m$) is minimized. Alternatively, Akaike ([3]–[6]) suggested choosing a model order so that the log likelihood of the innovation variance is minimized. This criterion, called the Akaike information criterion (AIC), is given by

$$\mathrm{AIC}(m) = 1n\, P_m + 2m/N. \tag{2}$$

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Again, the order $m$ selected is the one for which AIC($m$) is minimum. A third method was proposed by Parzen ([7]–[9]) and is known as the criterion autoregressive transfer (CAT) function. The model order is selected to be the one for which the estimate of the difference of the mean-square errors between the true prediction error filter, which exactly gives the innovation, and the estimated filter is a minimum. Parzen showed that this difference can be calculated, without explicitly knowing the exact filter (which may be of infinite length), by

$$\mathrm{CAT}(m) = \frac{1}{N}\sum_{n=1}^{m}\frac{N-n}{N\cdot P_n} - \frac{N-m}{N\cdot P_m}. \tag{3}$$

Again, $m$ is chosen to minimize CAT($m$).

The results obtained by utilizing the various criteria have been mixed. For simulated AR processes it has been concluded that both FPE and AIC work fairly well (e.g., [10]–[12]). However, when processing short realizations of the data, the order selected tends to be too low (e.g., [13]–[15]). When processing long realizations of the data, the order selected tends to be too high (e.g., [16]). Application of the FPE criterion to ARMA processes is discussed in [17]. It is shown that the order of the AR model fit to the ARMA data tends to increase with the data length. Application of the FPE and AIC to harmonic processes in additive noise is discussed in [10], [14], and [18]. It is shown that both criteria tend to underestimate the order if the SNR is high. To add to that convention, it has also been observed that the order selected critically depends on the manner in which the AR parameters are estimated (e.g., [10], [14]).

In this paper we derive several relations between the various criteria. These relations are completely independent of he actual nature of the data or the manner in which the AR parameters are estimated. The results obtained can be used to indicate which is the preferred criterion, at least for the extreme cases when the model order is likely to be underestimated or overestimated.

## II. Relations Between FPE, AIC, and CAT

The results derived in this section are based on the following two lemmas.

*Lemma 1:*

Given that

$$\mathrm{AIC}(k) < \mathrm{AIC}(k+i) \tag{4}$$

then

$$\mathrm{FPE}(k) < \mathrm{FPE}(k+i) \tag{5}$$

for all combinations of $k$ and $i$ such that $0 \le k \le N-1$ and $1 \le i \le N-k-1$.

*Proof:*

Substituting (2) into (4) and following some straightforward algebra manipulations, one obtains

$$\frac{P_k - P_{k+i}}{P_k} < 1 - e^{-2i/N}. \tag{6}$$

Similarly, substituting (1) into (5), one obtains

$$\frac{P_k - P_{k+i}}{P_k} < \frac{2Ni}{(N-k)(N+k+i)}. \tag{7}$$

Thus, to prove the lemma we must show that (6) implies (7). Since the same term appears on the left-hand side (l.h.s.) of (6) and (7), we only have to show that

$$\frac{2Ni}{(N-k)(N+k+i)} \ge 1 - e^{-2i/N} \tag{8}$$

for all $k$ and $i$ such that $0 \le k \le N-1$ and $1 \le i \le N-k-1$. Now since

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