A-Level Further Maths (Edexcel) - Unit tests 2024-25 model answers

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Chapter 1

Series - CP Chapter 2

1.1 Question 1

a. **(5 marks)** Prove that

$$\sum_{r=1}^{n} \frac{3}{r(r+1)} = \frac{an}{n+1}, \ n \in \mathbb{Z}$$

where a is a constant to be found.

b. **(1 mark)** Find the value of $\sum_{r=1}^{50} \frac{3}{r(r+1)}$, giving your answer as an exact fraction.

c. **(4 marks)** Find an expression in its simplest form for

$$\sum_{r=n}^{2n} \frac{3}{r(r+1)}$$

Solution

a We notice that

$$\sum_{r=1}^{n} \frac{3}{r(r+1)} = 3\sum_{r=1}^{n} \frac{1}{r(r+1)} \tag{1.1}$$

First, we need to express $\frac{1}{r(r+1)}$ in partial fractions.

$$\frac{1}{r(r+1)} = \frac{A}{r} + \frac{B}{r+1}$$

for some $A, B \in \mathbb{R}$.
By multiplying both sides by the common denominator $r(r+1)$, we have:

$$1 = A(r+1) + Br$$

For $r = 0$, we get $A = 1$ and for $r = -1$, $B = -1$.

$$\therefore \qquad \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

So, from Equation 1.1 we have

$$\begin{aligned}
\textstyle\sum_{r=1}^{n} \frac{3}{r(r+1)} &= 3\textstyle\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) \\
&= \begin{array}{lcll} 3 & - & \frac{3}{2} & (r=1) \\ +\frac{3}{2} & - & 1 & (r=2) \\ +1 & - & \frac{3}{4} & (r=3) \\ \vdots & & & \\ +\frac{3}{n-1} & - & \frac{3}{n} & (r=n-1) \\ +\frac{3}{n} & - & \frac{3}{n+1} & (r=n) \end{array} \\
&= 3 - \frac{3}{n+1}
\end{aligned}$$

Thus, by simplifying

$$\sum_{r=1}^{n} \frac{3}{r(r+1)} = \frac{3n}{n+1}, \quad \forall n \in \mathbb{Z} \tag{1.2}$$

Therefore, $a = 3$.

b By substituting $n = 50$ in Equation 1.2

$$\sum_{r=1}^{50} \frac{3}{r(r+1)} = \frac{3 \times 50}{50+1} = \frac{3 \times 50}{51} = \frac{3 \times 50}{3 \times 17} = \frac{50}{17}$$

c We need to make the following observation about the range of values for r:

$$\sum_{r=n}^{2n} \frac{3}{r(r+1)} = \sum_{r=1}^{2n} \frac{3}{r(r+1)} - \sum_{r=1}^{n-1} \frac{3}{r(r+1)}$$

By substituting the appropriate values in Equation 1.2

$$
\begin{aligned}
\sum_{r=n}^{2n} \frac{3}{r(r+1)} &= \frac{3 \times 2n}{2n+1} - \frac{3(n-1)}{n-1+1} \\
&= \frac{6n}{2n+1} - \frac{3n-3}{n} \\
&= \frac{6n^2 - (3n-3)(2n+1)}{n(2n+1)} \\
&= \frac{6n^2 - (6n^2 - 3n - 3)}{n(2n+1)} \\
&= \frac{3n+3}{n(2n+1)} \\
&= \frac{3(n+1)}{n(2n+1)}
\end{aligned}
$$

1.2 Question 2

a. **(2 marks)** Simplify

$$r^2\ (r+1)^2 - (r-1)^2\ r^2$$

b. **(3 marks)** Use the method of differences to show that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Solution

a

$$
\begin{aligned}
r^2(r+1)^2 - (r-1)^2r^2 &= r^2(r^2+2r+1) - (r^2-2r+1)r^2 \\
&= r^4 + 2r^3 + r^2 - (r^4 - 2r^3 + r^2) \\
&= r^4 + 2r^3 + r^2 - r^4 + 2r^3 - r^2 \\
&= 4r^3
\end{aligned}
$$

b From part a. we can deduce that

$$r^3 = \frac{1}{4}(r^2(r+1)^2 - (r-1)^2r^2)$$

Therefore,

$$\begin{aligned}
\textstyle\sum_{r=1}^{n} r^3 \quad &= \quad \tfrac{1}{4}\textstyle\sum_{r=1}^{n}\left(r^2(r+1)^2-(r-1)^2r^2\right) \\
= \quad & 1 \quad - \quad 0 \qquad (r=1) \\
& +\frac{2^2\times 3^2}{4} \quad - \quad 1 \qquad (r=2) \\
& +\frac{3^2\times 4^2}{4} \quad - \quad \frac{2^2\times 3^2}{4} \qquad (r=3) \\
& \vdots \\
& +\frac{(n-1)^2n^2}{4} \quad - \quad \frac{(n-2)^2(n-1)^2}{4} \qquad (r=n-1) \\
& +\frac{n^2(n+1)^2}{4} \quad - \quad \frac{(n-1)^2n^2}{4} \qquad (r=n) \\
= \quad & \frac{n^2(n+1)^2}{4}
\end{aligned}$$

Thus,

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

1.3 Question 3

a. **(3 marks)** Express in partial fractions

$$\frac{2}{(r+2)(r+3)(r+4)}$$

b. **(5 marks)** Show that

$$\sum_{r=1}^{n}\frac{2}{(r+2)(r+3)(r+4)} = \frac{n(n+b)}{c(n+3)(n+4)}$$

where b and c are constants to be found.

Solution

a

$$\frac{2}{(r+2)(r+3)(r+4)} = \frac{A}{r+2}+\frac{B}{r+3}+\frac{C}{r+4}$$

for some $A, B, C \in \mathbb{R}$. By multiplying both sides by the common denominator $(r+2)(r+3)(r+4)$, we have

$$2 = A(r+3)(r+4) + B(r+2)(r+4) + C(r+2)(r+3)$$

For $r=-2$, we get $A=1$, for $r=-3$, $B=-2$ and for $r=-4$, $C=1$.

$$\therefore \qquad \frac{2}{(r+2)(r+3)(r+4)} = \frac{1}{r+2}-\frac{2}{r+3}+\frac{1}{r+4} \qquad (1.3)$$

b Using the result Equation 1.3 from part a

$\sum_{r=1}^{n} \frac{2}{(r+2)(r+3)(r+4)} = \sum_{r=1}^{n} \frac{1}{r+2} - \frac{2}{r+3} + \frac{1}{r+4}$

$$\begin{array}{cccccccl}
= & \frac{1}{3} & - & \frac{2}{4} & + & \frac{1}{5} & (r=1) \\
+ & \frac{1}{4} & - & \frac{2}{5} & + & \frac{1}{6} & (r=2) \\
+ & \frac{1}{5} & - & \frac{2}{6} & + & \frac{1}{7} & (r=3) \\
+ & \frac{1}{6} & - & \frac{2}{7} & + & \frac{1}{8} & (r=4) \\
\vdots & & & & & & \\
+ & \frac{1}{n} & - & \frac{2}{n+1} & + & \frac{1}{n+2} & (r=n-2) \\
+ & \frac{1}{n+1} & - & \frac{2}{n+2} & + & \frac{1}{n+3} & (r=n-1) \\
+ & \frac{1}{n+2} & - & \frac{2}{n+3} & + & \frac{1}{n+5} & (r=n)
\end{array}$$

Thus,

$$\begin{aligned}
\sum_{r=1}^{n} \frac{2}{(r+2)(r+3)(r+4)} &= \frac{1}{2} - \frac{2}{4} + \frac{1}{4} + \frac{1}{n+3} - \frac{2}{n+4} + \frac{1}{n+4} \\
&= \frac{1}{12} - \frac{1}{n+3} + \frac{1}{n+4} \\
&= \frac{(n+3)(n+4) - 12(n+4) + 12(n+3)}{12(n+3)(n+4)} \\
&= \frac{n^2 + 7n + 12 - 12n - 48 + 12n + 36}{12(n+3)(n+4)} \\
&= \frac{n^2 + 7n}{12(n+3)(n+4)} \\
&= \frac{n(n+7)}{12(n+3)(n+4)}
\end{aligned}$$

Thus, $b = 7$ and $c = 12$.

1.4 Question 4

a. **(3 marks)** Use standard results to show that the first four terms of the series expansion of e^{2-x} in ascending powers of x can be expressed as

$$e^2\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\right)$$

b. **(3 marks)** Use standard results to obtain the first four non-zero terms of the series expansion of

$$sin(3x^2)$$

c. **(4 marks)** Use standard results to show that for all real values of x

$$\cosh^2 x \geq 1 + x^2$$

Solution

a First we notice that the requested expression has a factor of e^2. Because Maclaurin series approximate e^t as a polynomial series, it means that we need to factorise before applying the standard rule for e^t.
Using laws of indices, we get

$$e^{2-x} = e^2 e^{-x} \tag{1.4}$$

Now we apply the standard result for $e^t = \sum_{r=0}^{\infty} \frac{t^r}{r!}$, $\forall t \in \mathbb{R}$ for $t = -x$ to obtain the first four terms of the series expansion for e^{-x}.

$$\begin{aligned} e^{-x} &= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} \\ &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \end{aligned}$$

Thus, from Equation 1.1

$$e^{2-x} = e^2 \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\right)$$

b We use the standard results $\sin t = \sum_{r=0}^{\infty} (-1)^r \frac{t^{2r+1}}{(2r+1)!}$, $\forall t \in \mathbb{R}$ for $t = 3x^2$, to find the first four non-zero terms.

$$\begin{aligned} \sin 3x^2 &= 3x^2 - \frac{(3x^2)^3}{3!} + \frac{(3x^2)^5}{5!} - \frac{(3x^2)^7}{7!} \\ &= 3x^2 - \frac{3^3}{3!}x^6 + \frac{3^5}{5!}x^10 - \frac{3^7}{7!}x^14 \\ &= 3x^2 - \frac{9}{2}x^6 + \frac{81}{40}x^10 - \frac{243}{560}x^14 \end{aligned}$$

c

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \tag{1.5}$$

We need to consider the standard result for e^t.

$$\begin{aligned} e^x &= 1 + 2x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \quad \ldots \\ e^{-x} &= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(x)^4}{4!} \quad \ldots \\ &= 1 - x + \frac{(x)^2}{2!} - \frac{(x)^3}{3!} + \frac{(x)^4}{4!} \quad \ldots \\ \therefore \qquad e^x + e^{-x} &= 2 + 2\frac{(x)^2}{2!} + 2\frac{(x)^4}{4!} + \quad \ldots \end{aligned}$$

By substituting this result in Equation 1.5, we get

$$\cosh x = \frac{1}{2}\left(2 + 2\frac{(x)^2}{2!} + 2\frac{(x)^4}{4!} + \quad ...\right)$$
$$= 1 + \frac{(x)^2}{2!} + \frac{(x)^4}{4!} + \quad ...$$

The residual(you don't need to know how it's called)

$$\frac{(x)^4}{4!} + \quad ... \geq 0, \quad \forall x \in \mathbb{R}$$

as all terms are of even degree with positive coefficients.
Therefore,

$$\cosh x \geq 1 + \frac{x^2}{2}$$

So,

$$\cosh^2 x \geq \left(1 + \frac{x^2}{2}\right)^2$$
$$\cosh^2 x \geq 1 + 2\frac{x^2}{2} + \frac{x^4}{4}$$
$$\geq 1 + x^2$$

as $\frac{x^4}{4} \geq 0$, $\forall x \in \mathbb{R}$.
Alternative method: This method is slightly longer but it requires less planning.
First,

$$\cosh^2 x = \left(\frac{1}{2}(e^x + e^{-x})\right)^2$$
$$= \frac{1}{4}(e^x + e^{-x})^2$$
$$= \frac{1}{4}(e^{2x} + 2e^x e^{-x} + e^{-2x})$$
$$= \frac{1}{4}(e^{2x} + 2 + e^{-2x})$$

Thus,

$$\cosh^2 x = \frac{1}{2} + \frac{1}{4}(e^{2x} + e^{-2x}) \tag{1.6}$$

Then we need to consider the standard result for e^t.
[**Note:** to prove the inequality $\forall x \in \mathbb{R}$ using this method, we need to consider the Maclaurin series of $\cosh^2 x$. An approximation is not enough, because we do not know if the approximation underestimates or overestimates the true values.]

$$
\begin{aligned}
e^{2x} &= 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} \quad ... \\
e^{-2x} &= 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(2x)^4}{4!} \quad ... \\
&= 1 - 2x + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} \quad ... \\
\therefore \qquad e^{2x} + e^{-2x} &= 2 + 2\frac{(2x)^2}{2!} + 2\frac{(2x)^4}{4!} + \quad ...
\end{aligned}
$$

By substituting this result in Equation 1.6, we get

$$
\begin{aligned}
\cosh^2 x &= \frac{1}{2} + \frac{1}{4}\left(2 + 2\frac{(2x)^2}{2!} + 2\frac{(2x)^4}{4!} + \quad ...\right) \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\frac{(2x)^2}{2!} + \frac{1}{2}\frac{(2x)^4}{4!} + \quad ... \\
&= 1 + x^2 + \frac{2}{3}x^4 + \quad ...
\end{aligned}
$$

The residual(you don't need to know how it's called)

$$
\frac{2}{3}x^4 + \quad ... \geq 0, \quad \forall x \in \mathbb{R}
$$

as all terms are of even degree with positive coefficients.
Therefore,

$$
\cosh^2 x \geq 1 + x^2
$$

1.5 Question 5

a. **(5 marks)** Show that the series expansion of $ln\left(\frac{1+3x}{1-2x}\right)$ in ascending powers of x, up to and including the term in x^4, is

$$
\frac{5x^2}{2} + \frac{35x^3}{3} - \frac{65x^4}{4}
$$

b. **(1 mark)** State the range of values of x for which the answer to part **a** is valid.

c. **(4 marks)** By choosing a suitable value for x, use the expansion from part **a** to obtain an estimate for the value of $ln\frac{1}{2}$. Give your answer to 3 decimal places.

d. **(2 marks)** Write down the first four terms of the series expansion for $ln\sqrt{\frac{1+3x}{1-2x}}$

Solution

a From the laws of logarithms

$$\ln\left(\frac{1+3x}{1-2x}\right) = \ln(1+3x) + \ln(1-2x) \tag{1.7}$$

From the standard result $\ln(1+t)\sum_{r=0}^{\infty}(-1)^{r+1}\frac{t^r}{r}$, for $-1 < t \leq 1$, for $t = 3x$ and $t = -2x$ we've got

$$\begin{aligned}
\ln(1+3x) &= 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} \\
&= 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 \\
\ln(1-2x) &= -2x - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} \\
&= -2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \frac{(2x)^4}{4} \\
&= -2x - 2x^2 - \frac{8}{3}x^3 - \frac{16}{4}x^4
\end{aligned}$$

up to and including x^4.
Thus, from Equation 1.7

$$\ln\left(\frac{1+3x}{1-2x}\right) \approx 5x - \frac{5}{2}x^2 + \frac{35}{3}x^3 - \frac{65}{4}x^4 \tag{1.8}$$

b We require the approximation to be valid for both the expansion of $\ln(1+3x)$ and $\ln(1-2x)$. Therefore, Equation 1.8 is valid for

$$-\frac{1}{3} < x \leq -\frac{1}{3}$$

c First we need to find the value of x that satisfies the equality

$$\begin{aligned}
\frac{1+3x}{1-2x} &= \frac{1}{2} \\
2(1+3x) &= 1-2x \\
2+6x &= 1-2x \\
8x &= -1 \\
x &= -\frac{1}{8}
\end{aligned}$$

By substituting $x = -\frac{1}{8}$ in Equation 1.8

$$\begin{aligned}
\ln\left(\frac{1}{2}\right) &\approx 5\left(-\frac{1}{8}\right) - \frac{5}{2}\left(-\frac{1}{8}\right)^2 + \frac{35}{3}\left(-\frac{1}{8}\right)^3 - \frac{65}{4}\left(-\frac{1}{8}\right)^4 \\
&= -0.6908162435 \\
&= 0.691 \quad \text{(3 d.p.)}
\end{aligned}$$

d By the laws of logarithms

$$\ln\left(\sqrt{\frac{1+3x}{1-2x}}\right) = \left(\ln\left(\frac{1+3x}{1-2x}\right)\right)^{\frac{1}{2}}$$
$$= \frac{1}{2}\ln\left(\sqrt{\frac{1+3x}{1-2x}}\right)$$
$$\approx \frac{5}{2}x - \frac{5}{4}x^2 + \frac{35}{6}x^3 - \frac{65}{8}x^4$$

1.6 Question 6

(from Q2 Paper 1 November 2021)

a. **(2 marks)** Use the Maclaurin series expansion for $\cos x$ to determine the series expansion of $\cos^2 \frac{x}{3}$ in ascending powers of x,up to and including the term x^4.

   Give each term in simplest form.

b. **(3 marks)** Use the answer to part a and calculus to find an approximation, to 5 decimal places, for

$$\int_{\pi/6}^{\pi/2} \left(\frac{1}{x}\cos^2\left(\frac{x}{3}\right)\right) dx$$

Given that $\int \frac{1}{x} dx = \ln x + c$.

Solution

a To simplify the calculations, we can consider the double angle formula

$$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$$

Using the standard results for the expansion of $\cos t$

$$\cos^2 t = \frac{1}{2}(1 + 1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} \quad ...)$$
$$= 1 - t^2 + \frac{1}{3}t^4 \quad ...$$

By substituting $t = \frac{x}{3}$, we get

$$\cos^2\left(\frac{x}{3}\right) = 1 - \left(\frac{x}{3}\right)^2 + \frac{1}{3}\left(\frac{x}{3}\right)^4$$
$$= 1 - \frac{x^2}{9} + \frac{x^4}{243}$$

up to and including x^4.
b From part a

$$
\begin{aligned}
\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{x}\cos^2\left(\frac{x}{3}\right)\,dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{x}\left(1-\frac{x^2}{9}+\frac{x^4}{243}\right)\,dx \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{x}-\frac{x}{9}+\frac{x^3}{243}\,dx \\
&= \left[\ln x-\frac{x^2}{18}+\frac{x^4}{972}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= 0.98295
\end{aligned}
$$