Vectorization of a matrix as a linear transformation

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Let $\mathbf{X}$ be a $M \times N$ matrix we want to vectorize, and let $\mathbf{e}_i$ be the i-th canonical basis vector for the N-dimensional space, that is $\mathbf{e}_i = [0, \dots, 0, 1, 0, \dots, 0]^T$ with a 1 in the i-th position and all other elements zero. Let $\mathbf{B}_i$ be a $(MN) \times M$ block matrix defined as follows

$$\mathbf{B}_i = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{I}_M \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

That is, $\mathbf{B}_i$ consists of N block matrixes of size $M \times M$, stacked column-wise, and all these matrices are all-zero except the i-th one, which is a $M \times M$ identity matrix $\mathbf{I}_M$.

Then the vectorized version of $\mathbf{X}$, denoted $\mathbf{x}$, can be expressed as follows

$$\mathbf{x} = vec(\mathbf{X}) = \sum_{i=1}^{N} \mathbf{B}_i \mathbf{X} \mathbf{e}_i$$

Multiplication of $\mathbf{X}$ by $\mathbf{e}_i$ extracts the i-th column, while multiplication by $\mathbf{B}_i$ puts it into the desired position in the final vector.