§2. Regularization & Renormalization

§2.1 UV & IR divergence

IR: for massless propagator;

$$\frac{i}{q^2+i\epsilon} \quad \text{when } q\to 0$$

→ KNL theorm

⇒ Resolution of Exp. equipment

UV: loop integral when:

$$\mathcal{M} \propto \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(k_0-k)^2+m^2-i\epsilon}$$

$$\underset{k\to\infty}{\sim} \int \frac{d^4k}{k^4} \sim \int_0^\infty \frac{dk}{k} \sim \infty$$

$$F(q^2) \longrightarrow \infty$$

Not only QFT; (CED)

☆: Any field theory with infinite degree of freedom, or equivalently, any theory without a momentum UV cutoff, UV divergence must occur!

$$p\nearrow\infty \Leftrightarrow \Delta x \searrow_0 \Leftrightarrow DOF \nearrow \infty$$

1> There is no reason that the field has independent DOF in every spacetime point

2> It's unsafe to directly generalize the path integral of QM to infinity.

⇒ Introduce a UV momentum cutoff Λ!
(or a minimum resolution of Spacetime $\frac{1}{\Lambda}$)

REGULARIZATION:

Fabricate a dependency on cutoff Λ.

$$\mathcal{L} \Rightarrow \mathcal{L}(\Lambda)$$

① Every theory has it's scope of application

② Λ contains our ignorance of physics on extremely high erenge level

③ Low energy modes shouldn't be strongly correlated with high energy modes.

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ⓝ: It gives us a way to deal with divengency: quantify it, and hide it.

Some regularization scheme

1> Pauli–Villars

$$\Delta_F \to \frac{1}{k^2-m^2+i\epsilon} - \underbrace{\frac{1}{k^2-\Lambda^2+i\epsilon}}_{\text{gohst, have no effect on low energy}}$$

✗: break gauge symmetry

2> Cutoff

$$\int_0^\infty dp \to \int_0^\Lambda dp$$

✗ break Lorentzian symmetry

3> Dimentional

integral in $d=4-\epsilon$ spacetime

4> Lattice

Assuming the field is defined on Lattice.

✗ break Poincáre symmetry

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All of the regularization scheme:

by correcting false assumption, the infinite form is eliminated and replaced by finite term but with different form in different scheme

$\sim\Lambda^2$, $\sim \log\Lambda$, ...

——— we believe a good theory should decouple the L and H energy modes, as long as they are far enough apart, so we can (almost) arbitrarily concoct a sufficiently UV physics.

Meanwhile, when $\Lambda\to\infty$, the finite term becomes infinite again.

§2.2 Renormalization

Now we have $\mathcal{M}_T(g_0, m_0, \Lambda)$

but experimentalists have $\mathcal{M}_E(g, m)$

Targets:

① match $\mathcal{M}_T$ & $\mathcal{M}_E$ (LSZ)

② Restore the symmetry broken by regularization

③ Let theoretical prediction be independent of Λ in the limit of $\frac{\mu}{\Lambda}\to 0$

(the insensitive part is physical, and as the energy scale μ gets closer to Λ, the theory becomes more and more invalid.)

$$\frac{\mu}{\Lambda}\sim 0 \xleftrightarrow[\text{decoupled}]{} \frac{\mu}{\Lambda}\sim 1$$

$$\Delta x \sim \frac{1}{\Lambda} \longleftrightarrow \Delta x \sim 0$$

☆. The Λ exists but has no effect on Low μ scale physics

To achieve Target ① ②

1> mass & field strength renormalization.

In interaction field theory

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_0\partial_\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{g_0}{4!}\phi_0^4$$

$$\tilde{G}(p^2) \simeq \frac{i}{p^2-m_0^2-\Sigma(p^2)+i\epsilon} \simeq \frac{iZ_\phi}{p^2-m^2+i\epsilon}$$

we wish the theory is described by real physical mass m, and the residue be 1.

⇒ redefine $\phi_0 = \sqrt{Z_\phi}\,\phi_r$

$$\mathcal{L} = \frac{1}{2}Z_\phi\partial^\mu\phi_r\partial_\mu\phi_r + \frac{1}{2}Z_\phi m_0^2\phi_r^2 - \frac{g_0}{4!}Z_\phi^2\phi_r^4$$

⇒ using $m = m_0 - \delta m$ to express $\mathcal{L}$:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_r\partial_\mu\phi_r + \frac{1}{2}m^2\phi_r^2 \quad \left(-\frac{g}{4!}\phi_r^4\right) \quad \text{— main term}$$

$$+\frac{1}{2}\underbrace{(Z_\phi-1)}_{\delta_\phi}\partial^\mu\phi_r\partial_\mu\phi_r + \frac{1}{2}\underbrace{(Z_\phi m_0^2-m^2)}_{\delta_{m^2}}\phi_r^2 \quad \left(-\frac{\delta_g}{4!}\phi_r^4\right) \quad \text{— counter term}$$

Now recall target ③:

Main term has no UV divergency because m, g are defined by real physics; the UV divergencies are absorbed in the definition of counter term

↳ Using non-physical parameters to offset non-physical cutoff

RENORMALIZATION:

In actual calculations it is simplest just to say that from the loop terms $\Pi^*_{LOOP}(q^2)$ we must subtract a first-order polynomial in q^2 with coefficients chosen so that the difference satisfies Eqs. (10.3.17) and (10.3.18). As we shall see, this subtraction procedure incidentally cancels the infinities that arise from the momentum space integrals in Π^*_{LOOP}. However, as this discussion should make clear, *the renormalization of masses and fields has nothing directly to do with the presence of infinities, and would be necessary even in a theory in which all momentum space integrals were convergent.*

—— S. Weinberg

✗ note that the "divergent part" and "finite part" of a divergency is NOT unique; so we can choose different renormalization schemes, as long as it's self-consistent.

different schemes: OS, MS, MOM ...

? How to define a self-consistent scheme? (OS)

⇔ How to choose a set of δ

↳ renormalization condition

$$\text{eg. } \phi^4 \begin{cases} \Sigma(m^2)=0 \\ \frac{d}{dp^2}\Sigma(p^2)\big|_{p^2=m^2} = 1 \\ \Gamma^{(4)}(q=0) = g \end{cases}$$

§2.3 renormalizability

for ϕ^4 we only considered the divergencies in $\Gamma^{(2)}$ & $\Gamma^{(4)}$, but there are still higher-order such as $\Gamma^{(6)}$ MAY has divergency,

⇒ In priciple we need infinitely many independent parameters to offset infinitely many divergencies.

↳ Such a theory loses ability of prediction

LUCKILY ϕ^4, QED, QCD, EW are all renormalizable

RENORMALIZABLE:

Divergence can be completely canceled by introducing only a limited number of parameters (on Low energy level)

EFT

$$S[\phi] = S[\phi_L + \phi_H]$$

how to extract S_Λ?

$$e^{iS_\Lambda[\phi_L]} = \int D\phi_H\, e^{iS[\phi_H+\phi_L]}$$

$$S_\Lambda = \int d^4x\,\mathcal{L} = \int d^4x \sum_i g_i\langle O_i\rangle$$

$$[\mathcal{L}] = [\Lambda]^4 \Rightarrow [g_i] = \frac{[\Lambda]^4}{[O_i]} \equiv 4-\delta^i$$

$$\Rightarrow g_i = \frac{c_i}{\Lambda^{\delta_i-4}}$$

thus in the Low scale μ we concerned, O_i contribution

$$\int d^4x\, g_i\langle O_i\rangle \sim c_i\left(\frac{\mu}{\Lambda}\right)^{\delta^i-4}$$

① $\delta^i>4$: irrelative / nonrenormalizable

② $\delta^i=4$: marginal / renormalizable

③ $\delta^i<4$: relative / super-renormalizable

☆ Low energy physics depends on high level mainly via relative and marginal operators; only when considering some small correction there will be contribution from irrelative operators.

Given a low energy effect on Λ^{-r} order, even if high-energy physics contains infinite operators, there are only limited number of them contributes to it, since others are all suppressed by $\frac{\mu}{\Lambda}$.

① Λ in EFT represents the scale of $\langle O_i\rangle$
It's a "soft" limit, not the "hard"-cutoff

② In EFT, a theory no longer has to be a renormalizable theory, the irrelative ops can also provide information about high-level

③ Despite infinite number of couplings in EFT, It still has ability of prediction