§1.2 LSZ reduction formula

For 2-point $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$.

$$\int d^4x\, e^{ip\cdot x}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle \underset{p^2\to m^2}{\sim} \frac{iZ}{p^2-m^2+i\epsilon}$$

For higher point correlation? e.g. $(m+n)$-point correlation

$$\left(\prod_{i=1}^{n}\int d^4x_i e^{ip_i\cdot x_i}\right)\left(\prod_{j=1}^{m}\int d^4y_j e^{-ik_j\cdot y_j}\right)\langle\Omega|T\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)|\Omega\rangle$$

① integrate over first variable

$$\int d^4x_1 e^{ip_1\cdot x_1}\langle\ \rangle = \int_{-\infty}^{+\infty}dx_1^0\, e^{ip_1^0x_1^0}\int d^3x_1\, e^{i\vec{p}_1\cdot\vec{x}_1}\langle\ \rangle$$

$$= \Big(\underbrace{\int_{-\infty}^{T_-}dx_1^0}_{\mathrm{I}} + \underbrace{\int_{T_-}^{T_+}dx_1^0}_{\mathrm{II}} + \underbrace{\int_{T_+}^{+\infty}dx_1^0}_{\mathrm{III}}\Big) e^{ip_1^0x_1^0}\int d^3x_1\, e^{i\vec{p}_1\cdot\vec{x}_1}\langle\ \rangle$$

for $\forall i,j$. set $T_- < x_i^0, y_j^0 < T_+$, then the integral will be analytic in the region II. Considering region III: (insert $\mathbb{1}_q$)

$$\int_{T_+}^{+\infty}dx_1^0\, e^{ip_1^0x_1^0}\int d^3x_1 e^{i\vec{p}_1\cdot\vec{x}_1}\sum_\lambda\int\frac{d^3q}{(2\pi)^3}\frac{1}{2E_q(\lambda)}\langle\Omega|\phi(x_1)|\lambda_q\rangle\langle\lambda_q|\cdots|\Omega\rangle$$

$$= - - - - - - - - - - \langle\Omega|\phi(0)|\lambda_0\rangle e^{-iq\cdot x}\langle\lambda_p|\cdots|\Omega\rangle\Big|_{q^0=E_q}$$

$$= \sum_\lambda\int_{T_+}^{+\infty}dx_1^0\int\frac{d^3q}{(2\pi)^3}\frac{e^{i(p_1^0-q^0)x_1^0}}{2E_q(\lambda)}\langle\Omega|\phi(0)|\lambda_0\rangle(2\pi)^3\delta^{(3)}(p_1-q)\langle\lambda_p|\cdots|\Omega\rangle\Big|_{q^0=E_q}$$

$$= \sum_\lambda\frac{1}{2E_{p_1}(\lambda)}\frac{i\,e^{i(p_1^0-E_{p_1}(\lambda)+i\epsilon)T_+}}{p_1^0-E_{p_1}(\lambda)+i\epsilon}\langle\Omega|\phi(0)|\lambda_0\rangle\langle\lambda_p|\cdots|\Omega\rangle$$

recall that

$$\frac{i}{p^2-m^2+i\epsilon} = \frac{1}{2E_p}\left(\frac{1}{p^0-E_p+i\epsilon}-\frac{1}{p^0+E_p-i\epsilon}\right)$$

1) if $m_\lambda \sim$ multi-particle state, then there is a branch cut

2) if $m_\lambda \sim$ 1-particle state then there is a pole

at $p_1^0 \sim E_{p_1} = \sqrt{\vec{p}_1^2+m^2}$

$$\int_{\mathrm{III}}d^4x_1\, e^{ip_1\cdot x_1}\langle\Omega|T\phi(x_1)\cdots|\Omega\rangle \underset{p_1^0\to E_{p_1}}{\sim} \frac{i\sqrt{Z_i}}{p_1^2-m_1^2+i\epsilon}\langle\vec{p}|T\cdots|\Omega\rangle$$

Similarly

$$\int_{\mathrm{I}}d^4x_1\, e^{ip_1\cdot x_1}\langle\Omega|T\phi(x_1)\cdots|\Omega\rangle \underset{p_1^0\to -E_{p_1}}{\sim} \langle\Omega|T\cdots|-\vec{p}\rangle\frac{i\sqrt{Z_i}}{p_1^2-m_1^2+i\epsilon}$$

In scattering, region I, III represents in, out state respectively

② integrate other variables (assuming a $n\to m$ scattering)

$$\left(\prod_{i=1}^{n}\int d^4x_i\, e^{ip_i\cdot x_i}\right)\left(\prod_{j=1}^{m}\int d^4y_j\, e^{-ik_j\cdot y_j}\right)\langle\Omega|T\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)|\Omega\rangle$$

$$\underset{\substack{\forall p_i^0\to E_{p_i}\\ \forall k_i^0\to E_{k_i}}}{\sim}\left(\prod_{i=1}^{n}\frac{i\sqrt{Z_i}}{p_i^2-m_i^2+i\epsilon}\right)\left(\prod_{j=1}^{n}\frac{i\sqrt{Z_j}}{k_j^2-m_j^2+i\epsilon}\right)\underbrace{{}_{\text{out}}\langle\vec{p}_1\cdots\vec{p}_n|\vec{k}_1\cdots\vec{k}_m\rangle_{\text{in}}}_{=S_{\beta\alpha}=\langle f|\hat{S}|i\rangle}$$

LSZ reduction formula

✱ 1) All momentums are on-shell

2) Only considering the leading singularity.

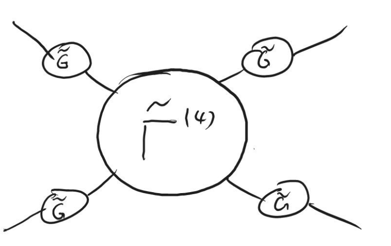
§1.3 Feynman diagram of $S_{\beta\alpha}$

rewrite the LSZ formula

$$\tilde{G}^{(m+n)} = \left(\prod_{i=1}^{m+n}\tilde{G}^{(1)}\right)\tilde{\Gamma}^{(m+n)}$$

$$= \left(\prod_{i=1}^{m+n}\frac{\tilde{G}^{(1)}}{\sqrt{Z_i}}\right)S_{\beta\alpha}$$

$\tilde{G}^{(4)} =$



$\tilde{\Gamma}$ is called the amputated part of $\tilde{G}$

thus to compute the $S_{\beta\alpha}$:

① compute the n-point correlation function $\langle\Omega|T\phi(x)\cdots|\Omega\rangle$

② Let all the out-legs on-shell $p^2\to m^2$

③ Drop all the 2-point correlation of out-legs, obtain $\tilde{\Gamma}$

④ Multiply field-strength renormalization factors $\sqrt{Z}$ of each out-leg.

then obtain $S_{\beta\alpha}$

Now the question is how to compute the 2-point $\tilde{G}$ in an interaction theory.

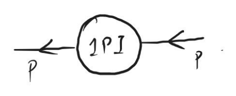
Def: 1-particle irreducible (1PI) diagram:

Any diagram that can not be split to two by cutting one line

Then the total 2-point correlation can be written as the sum of an infinite series of 1PIs and a free line:

denote, ($\Sigma(p^2)$ called self-energy)

$-i\Sigma(p^2) =$



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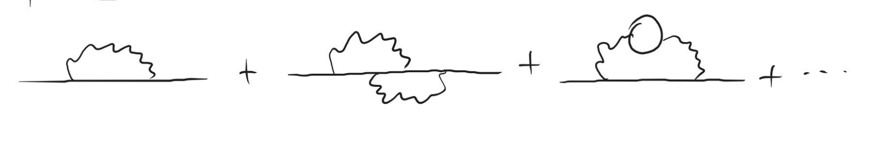
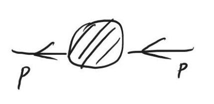


Then

$\tilde{G}(p^2) =$



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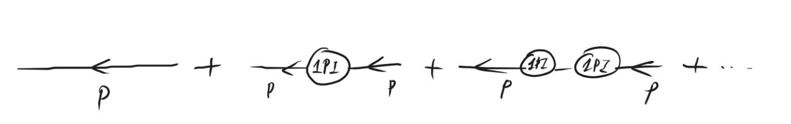

$$= \frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}(-i\Sigma(p^2))\frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}(-i\Sigma(p^2))\frac{i}{p^2-m_0^2}(-i\Sigma(p^2))\frac{i}{p^2-m_0^2} + \cdots$$

$$\boxed{\tilde{G}(p^2) = \frac{i}{p^2-m_0^2-\Sigma(p^2)} = \frac{iZ}{p^2-m^2}} \qquad m_0^2 = m^2-\Sigma(m^2)$$

Self-energy correction (mass renormalization)

Summary