Robust Causal Learning for the Estimation of Average Treatment Effects

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I. Proofs

We present the theoretical proofs of Theorems and Corollaries given in the main paper.

Proof of Theorem 1.
Given the nuisance parameters $\varrho = (\mathscr{g}^i, a_i)$ and the true nuisance parameters $\rho = (g^i, \pi^i)$, we find out the RCL score $\psi^i(W,\vartheta,\varrho)$ w.r.t. the nuisance parameters $\varrho = (\mathscr{g}^i, a_i)$ which can be used to construct the estimators of the causal parameter $\theta^i := \mathbb{E}\left[g^i(\mathbf{Z})\right]$. We try an ansatz of $\psi^i(W,\vartheta,\varrho)$ such that

$$\psi^i(W,\vartheta,\varrho) = \vartheta - \mathscr{g}^i(\mathbf{Z}) - (Y^i - \mathscr{g}^i(\mathbf{Z}))A(D,\mathbf{Z};a_i), \tag{1}$$

where

$$\begin{aligned} A(D,\mathbf{Z};a_i) = &\bar{b}_r\left[\mathbf{1}_{\{D=d^i\}} - a_i(\mathbf{Z})\right]^r \\ &+\sum_{q=1}^{k-1} b_q\big(\left[\mathbf{1}_{\{D=d^i\}} - a_i(\mathbf{Z})\right]^q - \mathbb{E}\left[(\nu^i)^q \mid \mathbf{Z}\right]\big). \end{aligned} \tag{2}$$

Here, the coefficients $b_1,\cdots,b_{k-1},\bar{b}_r$ depend on $\mathbf{Z}$ and ν^i only. Using the ansatz, we notice that $\psi^i(W,\vartheta,\varrho)$ satisfies condition 1, i.e., $\mathbb{E}\left[\psi^i(W,\vartheta,\varrho)\mid_{\vartheta=\theta^i,\ \varrho=\rho}\right] = 0$. Indeed, we have

$$\begin{aligned} &\mathbb{E}\left[\psi^i(W,\vartheta,\varrho)\mid_{\vartheta=\theta^i,\ \varrho=\rho}\right] \\ =&\mathbb{E}\left[\theta^i - g^i(\mathbf{Z}) - (Y^i - g^i(\mathbf{Z}))A(D,\mathbf{Z};\pi^i)\right] \\ =&\mathbb{E}\left[\theta^i - g^i(\mathbf{Z})\right] - \mathbb{E}\left[(Y^i - g^i(\mathbf{Z}))A(D,\mathbf{Z};\pi^i)\right] \\ =&-\mathbb{E}\left[\xi^i \times A(D,\mathbf{Z};\pi^i)\right] \\ =&-\mathbb{E}\left[\mathbb{E}\left[\xi^i \times A(D,\mathbf{Z};\pi^i)\mid D,\mathbf{Z}\right]\right] \\ =&-\mathbb{E}\left[A(D,\mathbf{Z};\pi^i)\mathbb{E}\left[\xi^i \mid D,\mathbf{Z}\right]\right] \\ =&-\mathbb{E}\left[A(D,\mathbf{Z};\pi^i)\mathbb{E}\left[\xi^i \mid \mathbf{Z}\right]\right] = 0. \end{aligned}$$

The second last equality comes from the fact that $A(D,\mathbf{Z};\pi^i)$ is a function of $(D,\mathbf{Z})$. The last equality comes from the fact that $(\xi^i \perp\!\!\!\perp D) \mid \mathbf{Z}$. Now, we aim to find out the coefficients of $b_1,\cdots,b_{k-1},\bar{b}_r$ such that the score (1) satisfies the k^{th} score. Indeed, we need to have $\mathbb{E}\left[\partial^{\alpha_1}_{\mathscr{g}^i}\partial^{\alpha_2}_{a_i}\psi^i(W,\vartheta,\varrho)\mid_{\vartheta=\theta^i,\ \varrho=\rho}\mid \mathbf{Z}\right] = 0$ for all α_1 and α_2 which are non-negative integers such that $1 \le \alpha_1 + \alpha_2 \le k$. Since $\partial^{\alpha_1}_{\mathscr{g}^i}\partial^{\alpha_2}_{a_i}\psi^i(W,\vartheta,\varrho) = 0$ when $\alpha_1 \ge 2$, we only need to solve the coefficients $b_1,\cdots,b_{k-1},\bar{b}_r$ from

$$\begin{cases} 0 = \mathbb{E}\left[\partial^k_{a_i}\psi^i(W,\vartheta,\varrho)\mid_{\vartheta=\theta^i,\ \varrho=\rho}\mid \mathbf{Z}\right], & \text{(3a)} \\ 0 = \mathbb{E}\left[\partial^1_{\mathscr{g}^i}\partial^q_{a_i}\psi^i(W,\vartheta,\varrho)\mid_{\vartheta=\theta^i,\ \varrho=\rho}\mid \mathbf{Z}\right] & \text{(3b)} \end{cases}$$

$\forall q = 0,\cdots,k-1$. However, (3a) always holds since

$$\begin{aligned} &\mathbb{E}\left[\partial^k_{a_i}\psi^i(W,\vartheta,\varrho)\mid_{\vartheta=\theta^i,\ \varrho=\rho}\mid \mathbf{Z}\right] \\ =&\mathbb{E}\left[(Y^i - g^i(\mathbf{Z}))\times\partial^k_{a_i}A(D,\mathbf{Z};a_i)\mid_{a_i=\pi^i}\mid \mathbf{Z}\right] \\ =&\mathbb{E}\left[\mathbb{E}\left[(Y^i - g^i(\mathbf{Z}))\times\partial^k_{a_i}A(D,\mathbf{Z};a_i)\mid_{a_i=\pi^i}\mid D,\mathbf{Z}\right]\mid \mathbf{Z}\right] \\ =&\mathbb{E}\left[\partial^k_{a_i}A(D,\mathbf{Z};a_i)\mid_{a_i=\pi^i}\mathbb{E}\left[(Y^i - g^i(\mathbf{Z}))\mid D,\mathbf{Z}\right]\mid \mathbf{Z}\right] \\ =&\mathbb{E}\left[\partial^k_{a_i}A(D,\mathbf{Z};a_i)\mid_{a_i=\pi^i}\mathbb{E}\left[\xi^i\mid \mathbf{Z}\right]\mid \mathbf{Z}\right] = 0. \end{aligned}$$

Consequently, we need to find out the coefficients $b_1,\cdots,b_{k-1},\bar{b}_r$ from

$$\mathbb{E}\left[\partial^1_{\mathscr{g}^i}\partial^q_{a_i}\psi^i(W,\vartheta,\varrho)\mid_{\vartheta=\theta^i,\ \varrho=\rho}\mid \mathbf{Z}\right] = 0 \tag{3b}$$

$\forall q = 0,\cdots,k-1$. From (3b), there are k equations and we need to solve the k unknowns $b_1,\cdots,b_{k-1},\bar{b}_r$ from the k equations. Generally, the k unknowns could be solved uniquely.

To start with, we compute $\partial^1_{\mathscr{g}^i}\partial^q_{a_i}\psi^i(W,\vartheta,\varrho)$ for $q = 0,\cdots,k-1$. Note that

$$\partial^1_{\mathscr{g}^i}\partial^q_{a_i}\psi^i(W,\vartheta,\varrho) = -1 + A(D,\mathbf{Z};a_i)$$

when $q = 0$ and

$$\begin{aligned} \partial^1_{\mathscr{g}^i}\partial^q_{a_i}\psi^i(W,\vartheta,\varrho) = &\bar{b}_r\frac{r!(-1)^q[\mathbf{1}_{\{D=d^i\}} - a_i(\mathbf{Z})]^{r-q}}{(r-q)!} \\ &+\sum_{u=q}^{k-1} b_u\frac{u!(-1)^q[\mathbf{1}_{\{D=d^i\}} - a_i(\mathbf{Z})]^{u-q}}{(u-q)!} \end{aligned}$$

when $1 \le q \le k-1$. Consequently, we need to solve for $b_1,\cdots,b_{k-1}$ and $\bar{b}_r$ simultaneously from

$$1 = \mathbb{E}\left[A(D,\mathbf{Z};\pi^i)\mid \mathbf{Z}\right] \tag{4a}$$

and

$$\begin{aligned} 0 = &\mathbb{E}\left[\bar{b}_r\frac{r!(-1)^q[\mathbf{1}_{\{D=d^i\}} - \pi^i(\mathbf{Z})]^{r-q}}{(r-q)!}\mid \mathbf{Z}\right] \\ &+\mathbb{E}\left[\sum_{u=q}^{k-1} b_u\frac{u!(-1)^q[\mathbf{1}_{\{D=d^i\}} - \pi^i(\mathbf{Z})]^{u-q}}{(u-q)!}\mid \mathbf{Z}\right]. \end{aligned} \tag{4b}$$

From (4a), we have

$$\begin{aligned} 1 = &\bar{b}_r\mathbb{E}\left[\left(\mathbf{1}_{\{D=d^i\}} - \pi^i(\mathbf{Z})\right)^r\mid \mathbf{Z}\right] \\ &+\sum_{q=1}^{k-1} b_q\mathbb{E}\left[\left(\mathbf{1}_{\{D=d^i\}} - \pi^i(\mathbf{Z})\right)^q\mid \mathbf{Z}\right] \\ &-\sum_{q=1}^{k-1} b_q\mathbb{E}\left[\mathbb{E}\left[(\nu^i)^q\mid \mathbf{Z}\right]\mid \mathbf{Z}\right] \end{aligned}$$

Since $\mathbb{E}\left[\left(\mathbf{1}_{\{D=d^i\}}-\pi^i(\mathbf{Z})\right)^q \mid \mathbf{Z}\right] = \mathbb{E}\left[(\nu^i)^q \mid \mathbf{Z}\right]$ and $\mathbb{E}\left[\mathbb{E}\left[(\nu^i)^q \mid \mathbf{Z}\right] \mid \mathbf{Z}\right] = \mathbb{E}\left[(\nu^i)^q \mid \mathbf{Z}\right]$, we understand that $\mathbb{E}\left[\left(\mathbf{1}_{\{D=d^i\}}-\pi^i(\mathbf{Z})\right)^q \mid \mathbf{Z}\right] - \mathbb{E}\left[\mathbb{E}\left[(\nu^i)^q \mid \mathbf{Z}\right] \mid \mathbf{Z}\right] = 0$. The above equality can therefore be reduced as

$$\begin{aligned}\bar{b}_r\mathbb{E}\left[\left(\mathbf{1}_{\{D=d^i\}}-\pi^i(\mathbf{Z})\right)^r \mid \mathbf{Z}\right] &= 1\\ \Rightarrow \bar{b}_r\mathbb{E}\left[(\nu^i)^r \mid \mathbf{Z}\right] &= 1.\end{aligned}$$

Hence, we can solve for $\bar{b}_r$ such that

$$\bar{b}_r = \frac{1}{\mathbb{E}\left[(\nu^i)^r \mid \mathbf{Z}\right]},$$

It remains to find out $b_1, \cdots, b_{k-1}$ from (4b). Indeed, we can simplify (4b) as

$$\begin{aligned}&\bar{b}_r\mathbb{E}\left[\frac{r!(\nu^i)^{r-q}}{(r-q)!} \mid \mathbf{Z}\right] + \sum_{u=q}^{k-1} b_u\mathbb{E}\left[\frac{u!(\nu^i)^{u-q}}{(u-q)!} \mid \mathbf{Z}\right] = 0\\ &\Rightarrow \bar{b}_r\binom{r}{q}\mathbb{E}\left[(\nu^i)^{r-q} \mid \mathbf{Z}\right] + \sum_{u=q}^{k-1} b_u\binom{u}{q}\mathbb{E}\left[(\nu^i)^{u-q} \mid \mathbf{Z}\right] = 0\end{aligned} \tag{5}$$

$\forall 1 \le q \le k-1$. Now, we solve $b_1, \cdots, b_{k-1}$. We start with finding out b_{k-1}, followed by $b_{k-2},\ b_{k-3}, \cdots, b_1$. When $q = k-1$, (5) becomes

$$\begin{aligned}0 = \bar{b}_r&\binom{r}{k-1}\mathbb{E}\left[(\nu^i)^{r-k+1} \mid \mathbf{Z}\right]\\ &+ b_{k-1}\binom{k-1}{k-1}\mathbb{E}\left[(\nu^i)^{0} \mid \mathbf{Z}\right]\\ \Rightarrow b_{k-1} &= -\bar{b}_r\binom{r}{k-1}\mathbb{E}\left[(\nu^i)^{r-k+1} \mid \mathbf{Z}\right].\end{aligned}$$

Now, when $q = k-2$, (5) becomes

$$\begin{aligned}0 = \bar{b}_r&\binom{r}{k-2}\mathbb{E}\left[(\nu^i)^{r-k+2} \mid \mathbf{Z}\right]\\ &+ b_{k-1}\binom{k-1}{k-2}\mathbb{E}\left[(\nu^i)^{(k-1)-(k-2)} \mid \mathbf{Z}\right]\\ &+ b_{k-2}\mathbb{E}\left[(\nu^i)^{0} \mid \mathbf{Z}\right]\\ \Rightarrow b_{k-2} &= -b_{k-1}\binom{k-1}{k-2}\mathbb{E}\left[(\nu^i)^{1} \mid \mathbf{Z}\right]\\ &- \bar{b}_r\binom{r}{k-2}\mathbb{E}\left[(\nu^i)^{r-k+2} \mid \mathbf{Z}\right].\end{aligned}$$

Iteratively, supposing $b_{q+1}, \cdots, b_{k-1}$ are known and we want to find out what b_q is, we have to solve it from

$$\begin{aligned}0 = b_q&\mathbb{E}\left[(\nu^i)^{0} \mid \mathbf{Z}\right]\\ &+ \bar{b}_r\binom{r}{q}\mathbb{E}\left[(\nu^i)^{r-q} \mid \mathbf{Z}\right]\\ &+ \sum_{u=q+1}^{k-1} b_u\binom{u}{q}\mathbb{E}\left[(\nu^i)^{u-q} \mid \mathbf{Z}\right].\end{aligned}$$

We can obtain b_q from the above equation, which gives

$$\begin{aligned}b_q = &-\sum_{u=q+1}^{k-1} b_u\binom{u}{q}\mathbb{E}\left[(\nu^i)^{u-q} \mid \mathbf{Z}\right]\\ &- \bar{b}_r\binom{r}{q}\mathbb{E}\left[(\nu^i)^{r-q} \mid \mathbf{Z}\right]\\ \Rightarrow b_q = &-\sum_{u=1}^{k-1-q} b_{q+u}\binom{q+u}{q}\mathbb{E}\left[(\nu^i)^{u} \mid \mathbf{Z}\right]\\ &- \bar{b}_r\binom{r}{q}\mathbb{E}\left[(\nu^i)^{r-q} \mid \mathbf{Z}\right].\end{aligned}$$

The proof is completed. □

Proof of Corollary 2. We have discussed the way to obtain the estimator in the main paper. To facilitate our following proofs, we first define some notations. We denote $Y^{i;F}$ as the factual outcome if one receives d^i, i.e., $Y^{i;F} = Y^F = Y^i$ if $D = d^i$. We denote $Y^{i;CF}$ as the counterfactual outcome if one is not treated with d^i, i.e., $Y^{i;CF} = Y^i$ if $D \neq d^i$. Thus, for m^{th} individual, $\xi_m^{i;F} = Y_m^{i;F} - g^i(\mathbf{Z}_m)$ if $m \in \mathscr{I}$ and $\xi_m^{i;CF} = Y_m^{i;CF} - g^i(\mathbf{Z}_m)$ if $m \in \mathscr{I}^c$. □

First, we introduce proposition 1.

Proposition 1. *Given the covariates* $\mathbf{Z}$*, the random variable* $\xi_m^{i;F}$ *and* $\xi_{\bar{m}}^{i;CF}$ *are independent and identically distributed, i.e.,* $\xi_m^{i;F} \overset{d}{=} \xi_{\bar{m}}^{i;CF} \overset{d}{=} \xi^i$ *and* $\xi_m^{i;F} \perp\!\!\!\perp \xi_{\bar{m}}^{i;CF}$.

Proof.

$\xi_m^{i;F} \mid D_m = d^i, \mathbf{Z}_m \overset{\triangle}{=} \xi_m^i \mid D_m = d^i, \mathbf{Z}_m;$
Using ignorability assumption, we have
$\mathbb{E}\left[(\xi_m^i)^r \mid D_m = d^i, \mathbf{Z}_m\right] = \mathbb{E}\left[(\xi_m^i)^r \mid \mathbf{Z}_m\right] = \mathbb{E}\left[(\xi^i)^r \mid \mathbf{Z}\right].$
$\xi_{\bar{m}}^{i;CF} \mid D_{\bar{m}} \neq d^i, \mathbf{Z}_{\bar{m}} \overset{\triangle}{=} \xi_{\bar{m}}^i \mid D_{\bar{m}} \neq d^i, \mathbf{Z}_{\bar{m}};$
Using ignorability assumption, we have
$\mathbb{E}\left[(\xi_{\bar{m}}^i)^r \mid D_{\bar{m}} \neq d^i, \mathbf{Z}_{\bar{m}}\right] = \mathbb{E}\left[(\xi_{\bar{m}}^i)^r \mid \mathbf{Z}_{\bar{m}}\right] = \mathbb{E}\left[(\xi^i)^r \mid \mathbf{Z}\right].$
According to the moment generating function, we can conclude $\xi_m^{i;F} \overset{d}{=} \xi_{\bar{m}}^{i;CF} \overset{d}{=} \xi^i \mid \mathbf{Z}$. Using the SUTVA assumption, we have $\xi_m^{i;F} \perp\!\!\!\perp \xi_{\bar{m}}^{i;CF}$. □

For notational simplicity, let our RCL estimator $\hat{\theta}^i_{RCL}$ be $\hat{\theta}^i_N$.

$$\begin{aligned}\hat{\theta}^i_N = &\underbrace{\frac{1}{N}\sum_{m=1}^{N}\hat{g}^i(\mathbf{Z}_m)}_{(a)}\\ &+ \underbrace{\frac{1}{N}\sum_{m\in\mathscr{I}}(Y_m^{i;F} - \hat{g}^i(\mathbf{Z}_m))\hat{A}_m^i}_{(b)}\\ &+ \underbrace{\frac{1}{R}\sum_{u=1}^{R}\left[\frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}_{m,u}^{i;F}\hat{A}_m^i\right]}_{(c)},\end{aligned} \tag{6}$$

In the remaining sequel, we investigate the consistency of our RCL estimators based on the basics of orthogonal machine learning theory. In this supplementary note, we only state the assumptions on the nuisance parameters that are helpful in studying the consistency of our RCL estimators. Other assumptions concentrate on the conditions of the scores, including orthogonality, identifiability, non-degeneracy, smoothness, and the regularity of moments can be found in [1] and references therein.

Assumption 1. *Given that the nuisance parameters and the true nuisance parameters are* $(\hat{g}^i, \hat{\pi}^i)$ *and* (g^i, π^i). $S = \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}^2 : \|\boldsymbol{\alpha}\|_1 \leq k\}$, *we have*

1) $\mathbb{E}\big[\,|\hat{g}^i(\mathbf{Z}) - g^i(\mathbf{Z})|^{4\alpha_1}\,|\hat{\pi}^i(\mathbf{Z}) - \pi^i(\mathbf{Z})|^{4\alpha_2} \;\mid\; \hat{g}^i, \hat{\pi}^i\big] \xrightarrow{p} 0 \;\; \forall\, \boldsymbol{\alpha} \in S$
2) $N^{\frac{1}{2}}\sqrt{\mathbb{E}\big[\,|\hat{g}^i(\mathbf{Z}) - g^i(\mathbf{Z})|^{2\alpha_1}\,|\hat{\pi}^i(\mathbf{Z}) - \pi^i(\mathbf{Z})|^{2\alpha_2} \mid \hat{g}^i, \hat{\pi}^i\big]} \xrightarrow{p} 0 \;\forall\, \boldsymbol{\alpha} \in \{\boldsymbol{\alpha} \in \mathbb{N}^2 : \|\boldsymbol{\alpha}\|_1 \leq k+1\} \backslash S.$

We are ready to investigate the consistency of our RCL estimators. Before that, we define two quantities $\hat{\tilde{\theta}}^i_N$ and $\hat{\bar{\theta}}^i_N$. They are

$$\hat{\tilde{\theta}}^i_N = \frac{1}{N}\sum_{m=1}^{N}\hat{g}^i(\mathbf{Z}_m) + \frac{1}{N}\sum_{m\in\mathscr{I}}(Y^{i;F}_m - \hat{g}^i(\mathbf{Z}_m))\hat{A}^i_m + \frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}^{i;CF}_m\hat{A}^i_m, \tag{7}$$

$$\hat{\bar{\theta}}^i_N = \frac{1}{N}\sum_{m=1}^{N}\hat{g}^i(\mathbf{Z}_m) + \frac{1}{N}\sum_{m\in\mathscr{I}}(Y^{i;F}_m - \hat{g}^i(\mathbf{Z}_m))\hat{A}^i_m + \frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}^{i;F}_m\hat{A}^i_m. \tag{8}$$

We also define

$$\kappa^{i;F}_N = \frac{1}{N}\sum_{m\in\mathscr{I}^c}\xi^{i;F}_m A^i_m,\; \hat{\kappa}^{i;F}_N = \frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}^{i;F}_m\hat{A}^i_m,$$
$$\kappa^{i;CF}_N = \frac{1}{N}\sum_{m\in\mathscr{I}^c}\xi^{i;CF}_m A^i_m,\; \hat{\kappa}^{i;CF}_N = \frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}^{i;CF}_m\hat{A}^i_m,$$
$$\hat{\kappa}^{i;F}_{R,N} = \frac{1}{R}\sum_{u=1}^{R}\Big[\frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}^{i;F}_{m,u}\hat{A}^i_m\Big],$$
$$\kappa^{i;F}_{R,N} = \frac{1}{R}\sum_{u=1}^{R}\Big[\frac{1}{N}\sum_{m\in\mathscr{I}^c}\xi^{i;F}_{m,u}A^i_m\Big].$$

Then (7) and (8) can be rewritten as

$$\hat{\tilde{\theta}}^i_N = \frac{1}{N}\sum_{m=1}^{N}\hat{g}^i(\mathbf{Z}_m) + \frac{1}{N}\sum_{m\in\mathscr{I}}(Y^{i;F}_m - \hat{g}^i(\mathbf{Z}_m))\hat{A}^i_m + \hat{\kappa}^{i;CF}_N, \tag{7}$$

$$\hat{\bar{\theta}}^i_N = \frac{1}{N}\sum_{m=1}^{N}\hat{g}^i(\mathbf{Z}_m) + \frac{1}{N}\sum_{m\in\mathscr{I}}(Y^{i;F}_m - \hat{g}^i(\mathbf{Z}_m))\hat{A}^i_m + \hat{\kappa}^{i;F}_N. \tag{8}$$

for simplicity.

In addition, we have to use two lemmas and two propositions to study the consistency of $\hat{\theta}^i_N$. We state them with the proofs.

Lemma 2. *Given two sequences of random variables* $(X_N)_{N=1}^{\infty}$ *and* $(Y_N)_{N=1}^{\infty}$ *such that* $X_N \overset{d}{=} Y_N$. *If* $X_N \xrightarrow{p} c$ *for some constant* c, *then* $Y_N \xrightarrow{p} c$.

Proof. Let $f_{X_N}(\cdot)$ and $f_{Y_N}(\cdot)$ be the density functions of the random variables X_N and Y_N respectively. Since $X_N \overset{d}{=} Y_N$, $f_{X_N}(\cdot) = f_{Y_N}(\cdot)$. Hence, $\mathbb{P}\{|X_N - c| \geq \epsilon\} = \int_{|z-c|\geq\epsilon} f_{X_N}(z)dz = \int_{|z-c|\geq\epsilon} f_{Y_N}(z)dz = \mathbb{P}\{|Y_N - c| \geq \epsilon\}$. Consequently, $X_N \xrightarrow{p} c$ implies $Y_N \xrightarrow{p} c$. □

Lemma 3. *Given random variables* X, Y, E, Z. *If* $(X \overset{d}{=} Y) \mid Z$, $(X \perp\!\!\!\perp E) \mid Z$, $(Y \perp\!\!\!\perp E) \mid Z$, *then* $Xh(E,Z) \overset{d}{=} Yh(E,Z)$ *for any function* h.

Proof. Define $f_Z(z)$ as the density function of Z, $f_{X|Z}(x|z)$ is the conditional density function of $X|Z$, $f_{Y|Z}(y|z)$ is the conditional density function of $Y|Z$, $f_{E|Z}(e|z)$ is the conditional density function of $E|Z$, $f_{X,E|Z}(x,e|z)$ is the conditional joint density function of $X, E|Z$, and $f_{Y,E|Z}(y,e|z)$ is the conditional joint density function of $Y, E|Z$. For a measurable set $\mathscr{A}$, we have

$$\begin{aligned}
&\mathbb{P}\{Xh(E,Z) \in \mathscr{A}\}\\
=&\int_{\Omega_Z}\left\{\iint_{\Omega_X\times\Omega_E}\mathbf{1}_{\{xh(e,z)\in\mathscr{A}\}}f_{X,E|Z}(x,e|z)dxde\right\}f_Z(z)dz\\
\overset{*}{=}&\int_{\Omega_Z}\left\{\iint_{\Omega_X\times\Omega_E}\mathbf{1}_{\{xh(e,z)\in\mathscr{A}\}}f_{X|Z}(x|z)f_{E|Z}(e|z)dxde\right\}f_Z(z)dz\\
\overset{\triangle}{=}&\int_{\Omega_Z}\left\{\iint_{\Omega_Y\times\Omega_E}\mathbf{1}_{\{yh(e,z)\in\mathscr{A}\}}f_{Y|Z}(y|z)f_{E|Z}(e|z)dyde\right\}f_Z(z)dz\\
\overset{\square}{=}&\int_{\Omega_Z}\left\{\iint_{\Omega_Y\times\Omega_E}\mathbf{1}_{\{yh(e,z)\in\mathscr{A}\}}f_{Y,E|Z}(y,e|z)dyde\right\}f_Z(z)dz\\
=&\mathbb{P}\{Yh(E,Z) \in \mathscr{A}\}
\end{aligned}$$

$*$ holds since $(X \perp\!\!\!\perp E) \mid Z$, $\triangle$ holds since $(X \overset{d}{=} Y) \mid Z$, and $\square$ holds since $(Y \perp\!\!\!\perp E) \mid Z$. □

Proposition 4. *Given two i.i.d. sequences* $(\xi^{i;F}_m)$ *and* $(\xi^{i;CF}_m)$. *Suppose* $(\xi^{i;F}_m \overset{d}{=} \xi^{i;CF}_{\bar{m}}) \mid \mathbf{Z}$ *and* $(\xi^{i;F}_m \perp\!\!\!\perp \xi^{i;CF}_{\bar{m}}) \mid \mathbf{Z}$ *for any* m *and* $\bar{m}$. *Furthermore, suppose* $\mathbb{E}\left[(\xi^{i;F}_m)^2 \mid \mathbf{Z}\right]$ *and* $(A^i_m)^2$ *exist such that* $\mathbb{E}\left[(A^i_m)^2\mathbb{E}\left[(\xi^{i;F}_m)^2 \mid \mathbf{Z}\right]\right]$ *is finite for all* m. *We have* $\kappa^{i;CF}_N - \kappa^{i;F}_N \xrightarrow{p} 0$ *when* $N \to \infty$.

Proof. $\forall\epsilon > 0$, we consider $\mathbb{P}\left\{\left|\kappa^{i;CF}_N - \kappa^{i;F}_N\right| \geq \epsilon\right\}$. Indeed, we have

$$\begin{aligned}
&\mathbb{P}\left\{\left|\kappa^{i;CF}_N - \kappa^{i;F}_N\right| \geq \epsilon\right\} \leq \frac{\mathbb{E}\left[\left(\kappa^{i;CF}_N - \kappa^{i;F}_N\right)^2\right]}{\epsilon^2}\\
&= \frac{\frac{1}{N^2}\mathbb{E}\left[\left(\sum_{m\in\mathscr{I}^c}(\xi^{i;CF}_m - \xi^{i;F}_m)A^i_m\right)^2\right]}{\epsilon^2}.
\end{aligned}$$

Denoting $\xi_m^{i;CF}-\xi_m^{i;F}$ as Ξ_m^i, we have

$$
\begin{aligned}
&\mathbb{E}\big[\big(\sum_{m\in\mathscr{I}^c}(\xi_m^{i;CF}-\xi_m^{i;F})A_m^i\big)^2\big]=\mathbb{E}\big[\big(\sum_{m\in\mathscr{I}^c}\Xi_m^i A_m^i\big)^2\big]\\
=&\mathbb{E}\big[\sum_{m,\bar{m}\in\mathscr{I}^c}\Xi_m^i A_m^i\Xi_{\bar{m}}^i A_{\bar{m}}^i\big]=\sum_{m,\bar{m}\in\mathscr{I}^c}\mathbb{E}\left[\Xi_m^i A_m^i\Xi_{\bar{m}}^i A_{\bar{m}}^i\right]\\
&+\sum_{\substack{m,\bar{m}\in\mathscr{I}^c\\ m\neq\bar{m}}}\mathbb{E}\left[A_m^i A_{\bar{m}}^i\mathbb{E}\left[\Xi_m^i\Xi_{\bar{m}}^i\mid D,\mathbf{Z}\right]\right]\\
=&\sum_{m\in\mathscr{I}^c}\mathbb{E}\left[(A_m^i)^2\mathbb{E}\left[(\Xi_m^i)^2\mid D,\mathbf{Z}\right]\right]\\
&+\sum_{\substack{m,\bar{m}\in\mathscr{I}^c\\ m\neq\bar{m}}}\mathbb{E}\left[A_m^i A_{\bar{m}}^i\mathbb{E}\left[\Xi_m^i\mid D,\mathbf{Z}\right]\mathbb{E}\left[\Xi_{\bar{m}}^i\mid D,\mathbf{Z}\right]\right]\\
=&\sum_{m\in\mathscr{I}^c}\mathbb{E}\left[(A_m^i)^2\mathbb{E}\left[(\Xi_m^i)^2\mid \mathbf{Z}\right]\right]=2\sum_{m\in\mathscr{I}^c}\mathbb{E}\left[(A_m^i)^2\mathbb{E}\left[(\xi_m^{i;F})^2\mid \mathbf{Z}\right]\right]\\
\leq&2N\mathbb{E}\left[(A^i)^2\mathbb{E}\left[(\xi^{i;F})^2\mid \mathbf{Z}\right]\right].
\end{aligned}
$$

The last equality follows from

$$
\begin{aligned}
&\mathbb{E}\big[(\Xi_m^i)^2\mid\mathbf{Z}\big]=\mathbb{E}\big[(\xi_m^{i;CF}-\xi_m^{i;F})^2\mid\mathbf{Z}\big]\\
=&\mathbb{E}\big[(\xi_m^{i;CF})^2\mid\mathbf{Z}\big]-2\mathbb{E}\big[\xi_m^{i;F}\xi_m^{i;CF}\mid\mathbf{Z}\big]+\mathbb{E}\big[(\xi_m^{i;F})^2\mid\mathbf{Z}\big]\\
=&\mathbb{E}\big[(\xi_m^{i;CF})^2\mid\mathbf{Z}\big]-2\mathbb{E}\big[\xi_m^{i;F}\mid\mathbf{Z}\big]\mathbb{E}\big[\xi_m^{i;CF}\mid\mathbf{Z}\big]+\mathbb{E}\big[(\xi_m^{i;F})^2\mid\mathbf{Z}\big]\\
=&\mathbb{E}\big[(\xi_m^{i;CF})^2\mid\mathbf{Z}\big]+\mathbb{E}\big[(\xi_m^{i;F})^2\mid\mathbf{Z}\big]=2\mathbb{E}\big[(\xi_m^{i;F})^2\mid\mathbf{Z}\big].
\end{aligned}
$$

As a consequence, we have

$$
\begin{aligned}
\mathbb{P}\left\{\left|\kappa_N^{i;CF}-\kappa_N^{i;F}\right|\geq\epsilon\right\}\leq&\frac{2N\mathbb{E}\left[(A^i)^2\mathbb{E}\left[(\xi^{i;F})^2\mid\mathbf{Z}\right]\right]}{N^2\epsilon^2}\\
=&\frac{2\mathbb{E}\left[(A^i)^2\mathbb{E}\left[(\xi^{i;F})^2\mid\mathbf{Z}\right]\right]}{N\epsilon^2}\to 0
\end{aligned}
$$

when $N\to\infty$. As a result, we have $\kappa_N^{i;CF}-\kappa_N^{i;F}\xrightarrow{p}0$. The proof is completed. □

Proposition 5. *Suppose that, conditioning on* $\mathbf{Z}$, $\xi_{m,u}^{i;F}$ *are i.i.d. of* $\xi_m^{i;F}$ *and* $\xi_{m,u}^{i;F}$ *are i.i.d. of* $\xi_{m,\bar{u}}^{i;F}$ $\forall u,\ \bar{u}\in\{1,2,\cdots,R\}$. *We have*

$$
\kappa_N^{i;F}-\kappa_{R,N}^{i;F}\xrightarrow{p}0\quad\textit{when}\quad N\to\infty.
$$

Proof.
Write

$$
\begin{aligned}
\kappa_{R,N}^{i;F}=&\frac{1}{R}\sum_{u=1}^{R}\big[\frac{1}{N}\sum_{m\in\mathscr{I}^c}\xi_{m,u}^{i;F}A_m^i\big]\\
=&\frac{1}{N}\sum_{m\in\mathscr{I}^c}\left(\frac{1}{R}\sum_{u=1}^{R}\xi_{m,u}^{i;F}\right)A_m^i=\frac{1}{N}\sum_{m\in\mathscr{I}^c}\mathscr{E}_m^i A_m^i
\end{aligned}
$$

$\forall\epsilon>0$, we have

$$
\mathbb{P}\left\{\left|\kappa_N^{i;F}-\kappa_{R,N}^{i;F}\right|\geq\epsilon\right\}\leq\frac{\mathbb{E}\left[\left(\frac{1}{N}\sum_{m\in\mathscr{I}^c}\left[\mathscr{E}_m^i-\xi_m^{i;F}\right]A_m^i\right)^2\right]}{\epsilon^2}.
$$

Considering $\mathbb{E}\left[\left(\frac{1}{N}\sum_{m\in\mathscr{I}^c}\left[\mathscr{E}_m^i-\xi_m^{i;F}\right]A_m^i\right)^2\right]$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{N}\sum_{m\in\mathscr{I}^c}\left[\mathscr{E}_m^i-\xi_m^{i;F}\right]A_m^i\right)^2\right]\\
=&\frac{1}{N^2}\sum_{m,\bar{m}\in\mathscr{I}^c}\mathbb{E}\left[\left(\mathscr{E}_m^i-\xi_m^{i;F}\right)\left(\mathscr{E}_{\bar{m}}^i-\xi_{\bar{m}}^{i;F}\right)A_{\bar{m}}^i A_m^i\right]\\
=&\frac{1}{N^2}\sum_{m\in\mathscr{I}^c}\mathbb{E}\left[\left(\mathscr{E}_m^i-\xi_m^{i;F}\right)^2(A_m^i)^2\right]\\
&+\frac{1}{N^2}\sum_{\substack{m,\bar{m}\in\mathscr{I}^c\\ m\neq\bar{m}}}\mathbb{E}\left[\left(\mathscr{E}_m^i-\xi_m^{i;F}\right)\left(\mathscr{E}_{\bar{m}}^i-\xi_{\bar{m}}^{i;F}\right)A_{\bar{m}}^i A_m^i\right]\\
=&\frac{1}{N^2}\sum_{m\in\mathscr{I}^c}\mathbb{E}\left[\left(\mathscr{E}_m^i-\xi_m^{i;F}\right)^2(A_m^i)^2\right]\\
&+\frac{1}{N^2}\sum_{\substack{m,\bar{m}\in\mathscr{I}^c\\ m\neq\bar{m}}}\mathbb{E}\left[A_{\bar{m}}^i A_m^i\mathbb{E}\left[\left(\mathscr{E}_m^i-\xi_m^{i;F}\right)\left(\mathscr{E}_{\bar{m}}^i-\xi_{\bar{m}}^{i;F}\right)\mid D,\mathbf{Z}\right]\right]\\
=&\frac{1}{N^2}\sum_{m\in\mathscr{I}^c}\mathbb{E}\left[\left(\mathscr{E}_m^i-\xi_m^{i;F}\right)^2(A_m^i)^2\right]\\
&+\frac{1}{N^2}\sum_{\substack{m,\bar{m}\in\mathscr{I}^c\\ m\neq\bar{m}}}\mathbb{E}\left[A_{\bar{m}}^i A_m^i\mathbb{E}\left[\left(\mathscr{E}_m^i-\xi_m^{i;F}\right)\mid\mathbf{Z}\right]\mathbb{E}\left[\left(\mathscr{E}_{\bar{m}}^i-\xi_{\bar{m}}^{i;F}\right)\mid\mathbf{Z}\right]\right]\\
=&\frac{1}{N^2}\sum_{m\in\mathscr{I}^c}\mathbb{E}\left[\left(\mathscr{E}_m^i-\xi_m^{i;F}\right)^2(A_m^i)^2\right].
\end{aligned}
$$

The last equality in the above derivation follows from the fact that, conditioning on $\mathbf{Z}$, $\xi_{m,u}^{i;F}$ are i.i.d. of $\xi_m^{i;F}$ for any $u\in\{1,2,\cdots,R\}$. Indeed, we have $\mathbb{E}\left[\xi_{m,u}^{i;F}\mid\mathbf{Z}\right]=\mathbb{E}\left[\xi_u^{i;F}\mid\mathbf{Z}\right]$ for any m and $u\in\{1,2,\cdots,R\}$. Consequently, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\mathscr{E}_m^i-\xi_m^{i;F}\right)\mid\mathbf{Z}\right]=\mathbb{E}\left[\left(\frac{1}{R}\sum_{u=1}^{R}\xi_{m,u}^{i;F}-\xi_m^{i;F}\right)\mid\mathbf{Z}\right]\\
=&\frac{1}{R}\sum_{u=1}^{R}\mathbb{E}\left[\xi_{m,u}^{i;F}\mid\mathbf{Z}\right]-\mathbb{E}\left[\xi_m^{i;F}\mid\mathbf{Z}\right]=\frac{1}{R}\sum_{u=1}^{R}\mathbb{E}\left[\xi_m^{i;F}\mid\mathbf{Z}\right]-\mathbb{E}\left[\xi_m^{i;F}\mid\mathbf{Z}\right]\\
=&\mathbb{E}\left[\xi_m^{i;F}\mid\mathbf{Z}\right]-\mathbb{E}\left[\xi_m^{i;F}\mid\mathbf{Z}\right]=0.
\end{aligned}
$$

In addition, we simplify the quantity $\mathbb{E}\left[\left(\mathscr{E}_m^i-\xi_m^{i;F}\right)^2\right]$. Note that $\mathscr{E}_m^i-\xi_m^{i;F}=\frac{1}{R}\sum_{u=1}^{R}[\xi_{m,u}^{i;F}-\xi_m^{i;F}]$. We therefore have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{u=1}^{R}\left[\xi_{m,u}^{i;F}-\xi_m^{i;F}\right]\right)^2\right]\\
=&\sum_{u,\bar{u}=1}^{R}\mathbb{E}\left[\left(\xi_{m,u}^{i;F}-\xi_m^{i;F}\right)\left(\xi_{m,\bar{u}}^{i;F}-\xi_m^{i;F}\right)\right]\\
=&\sum_{u,\bar{u}=1}^{R}\left\{\mathbb{E}\left[\xi_{m,u}^{i;F}\xi_{m,\bar{u}}^{i;F}\right]-\mathbb{E}\left[\xi_{m,u}^{i;F}\xi_m^{i;F}\right]\right.\\
&\left.-\mathbb{E}\left[\xi_m^{i;F}\xi_{m,\bar{u}}^{i;F}\right]+\mathbb{E}\left[\xi_m^{i;F}\xi_m^{i;F}\right]\right\}\\
=&\sum_{u=1}^{R}\mathbb{E}\left[(\xi_{m,u}^{i;F})^2\right]-2R\sum_{u=1}^{R}\mathbb{E}\left[\xi_{m,u}^{i;F}\xi_m^{i;F}\right]+R^2\mathbb{E}\left[(\xi_m^{i;F})^2\right]\\
&+\sum_{\substack{u,\bar{u}=1\\ u\neq\bar{u}}}^{R}\mathbb{E}\left[\xi_{m,u}^{i;F}\xi_{m,\bar{u}}^{i;F}\right]=\left[R^2+R\right]\mathbb{E}\left[\left(\xi_m^{i;F}\right)^2\right].
\end{aligned}
$$

We justify the last equality. The last equality follows from the fact that, conditioning on $\mathbf{Z}$, $\xi_{m,u}^{i;F}$ are i.i.d. of $\xi_m^{i;F}$ and $\xi_{m,u}^{i;F}$ are i.i.d. of $\xi_{m,\bar{u}}^{i;F}$ for any $u,\ \bar{u} \in \{1, 2, \cdots, R\}$. Indeed, under the given fact, we have

$$\begin{aligned} &\mathbb{E}\left[\xi_{m,u}^{i;F}\xi_m^{i;F}\right] = \mathbb{E}\left[\mathbb{E}\left[\xi_{m,u}^{i;F}\xi_m^{i;F} \mid D, \mathbf{Z}\right]\right] \\ =&\mathbb{E}\left[\mathbb{E}\left[\xi_{m,u}^{i;F}D, \mid \mathbf{Z}\right]\mathbb{E}\left[\xi_m^{i;F} \mid D, \mathbf{Z}\right]\right] = 0 \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}\left[\xi_{m,u}^{i;F}\xi_{m,\bar{u}}^{i;F}\right] = \mathbb{E}\left[\mathbb{E}\left[\xi_{m,u}^{i;F}\xi_{m,\bar{u}}^{i;F} \mid D, \mathbf{Z}\right]\right] \\ =&\mathbb{E}\left[\mathbb{E}\left[\xi_{m,u}^{i;F} \mid D, \mathbf{Z}\right]\mathbb{E}\left[\xi_{m,\bar{u}}^{i;F} \mid D, \mathbf{Z}\right]\right] = 0. \end{aligned}$$

Consequently, we have

$$\mathbb{E}\left[\left(\mathscr{E}_m^i - \xi_m^{i;F}\right)^2\right] = \left(1 + \frac{1}{R}\right)\mathbb{E}\left[\left(\xi_m^{i;F}\right)^2\right].$$

Thus, we have

$$\begin{aligned} \mathbb{P}\left\{\left|\kappa_N^{i;F} - \kappa_{R,N}^{i;F}\right| \geq \epsilon\right\} &\leq \frac{\frac{1}{N^2}\sum_{m\in\mathscr{I}^c}\left(1+\frac{1}{R}\right)\mathbb{E}\left[\left(\xi_m^{i;F}\right)^2\right]}{\epsilon^2} \\ &\leq \frac{\left(1+\frac{1}{R}\right)\mathbb{E}\left[\left(\xi^{i;F}\right)^2\right]}{N\epsilon^2}. \end{aligned} \tag{9}$$

We notice that no matter we set $R \to \infty$ followed by $N \to \infty$ or vice versa, or we fix R but let $N \to \infty$, we see that $\mathbb{P}\left\{\left|\kappa_N^{i;F} - \kappa_{R,N}^{i;F}\right| \geq \epsilon\right\} \to 0$. ☐

Now, we are ready to investigate if the estimator $\hat{\theta}_N^i$ is a consistent estimator of θ^i. Our goal is to show that $\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\theta}_N^i - \theta^i\right| \geq \epsilon\right\} \xrightarrow{p} 0$.

Proof. $\forall \epsilon > 0$, we have

$$\begin{aligned} &\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\theta}_N^i - \theta^i\right| \geq \epsilon\right\} \\ =&\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\theta}_N^i - \hat{\bar{\theta}}_N^i + \hat{\bar{\theta}}_N^i - \theta^i\right| \geq \epsilon\right\} \\ \leq&\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\theta}_N^i - \hat{\bar{\theta}}_N^i\right| \geq \frac{\epsilon}{2}\right\} + \mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\bar{\theta}}_N^i - \theta^i\right| \geq \frac{\epsilon}{2}\right\}. \end{aligned}$$

Since $\left(\xi^{i;F} \overset{d}{=} \xi^{i;CF}\right) \mid \mathbf{Z}$ and $\left(\hat{\xi}^{i;F} \overset{d}{=} \hat{\xi}^{i;CF}\right) \mid \mathbf{Z}$, we have $\hat{\bar{\theta}}_N^i \overset{d}{=} \hat{\bar{\theta}}_N^i$ by Lemma 3. Moreover, we know that $\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\bar{\theta}}_N^i - \theta^i\right| \geq \frac{\epsilon}{2}\right\} \xrightarrow{p} 0$ under the assumptions given in [1]. Together with the fact that $\hat{\bar{\theta}}_N^i \overset{d}{=} \hat{\bar{\theta}}_N^i$, we have $\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\bar{\theta}}_N^i - \theta^i\right| \geq \frac{\epsilon}{2}\right\} \xrightarrow{p} 0$ by Lemma 2. We turn to consider the quantity $\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\theta}_N^i - \hat{\bar{\theta}}_N^i\right| \geq \frac{\epsilon}{2}\right\}$, and we aim to show that $\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\theta}_N^i - \hat{\bar{\theta}}_N^i\right| \geq \frac{\epsilon}{2}\right\} \xrightarrow{p} 0$. Notice that

$$\begin{aligned} &\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\theta}_N^i - \hat{\bar{\theta}}_N^i\right| \geq \frac{\epsilon}{2}\right\} = \mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\kappa}_{R,N}^{i;F} - \hat{\kappa}_N^{i;F}\right| \geq \frac{\epsilon}{2}\right\} \\ \leq&\underbrace{\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\kappa}_{R,N}^{i;F} - \kappa_{R,N}^{i;F}\right| \geq \frac{\epsilon}{8}\right\}}_{(a)} + \underbrace{\mathbb{P}_{\hat{\rho}}\left\{\left|\kappa_{R,N}^{i;F} - \kappa_N^{i;F}\right| \geq \frac{\epsilon}{8}\right\}}_{(b)} \\ &+ \underbrace{\mathbb{P}_{\hat{\rho}}\left\{\left|\kappa_N^{i;F} - \kappa_N^{i;CF}\right| \geq \frac{\epsilon}{8}\right\}}_{(c)} + \underbrace{\mathbb{P}_{\hat{\rho}}\left\{\left|\kappa_N^{i;CF} - \hat{\kappa}_N^{i;F}\right| \geq \frac{\epsilon}{8}\right\}}_{(d)}. \end{aligned} \tag{10}$$

Note that $\left|\kappa_{R,N}^{i;F} - \kappa_N^{i;F}\right|$ and $\left|\kappa_N^{i;F} - \kappa_N^{i;CF}\right|$ do not incorporate any terms related to the estimated function $\hat{\rho}$. From Proposition 4 and Proposition 5, we conclude that (10b) and (10c) converge to 0 in probability respectively. It remains to show the convergence of $\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\kappa}_{R,N}^{i;F} - \kappa_{R,N}^{i;F}\right| \geq \frac{\epsilon}{8}\right\}$ and $\mathbb{P}_{\hat{\rho}}\left\{\left|\kappa_N^{i;CF} - \hat{\kappa}_N^{i;F}\right| \geq \frac{\epsilon}{8}\right\}$. Consider (10d) first. Since

$$\begin{aligned} &\left|\kappa_N^{i;CF} - \hat{\kappa}_N^{i;F}\right| = \left|\frac{1}{N}\sum_{m\in\mathscr{I}^c}\xi_m^{i;CF}A_m^i - \frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}_m^{i;F}\hat{A}_m^i\right| \\ \leq&\underbrace{\left|\frac{1}{N}\sum_{m\in\mathscr{I}^c}(\xi_m^{i;CF}A_m^i - \xi_m^{i;F}A_m^i)\right|}_{\Gamma_1} \\ &+\underbrace{\left|\frac{1}{N}\sum_{m\in\mathscr{I}^c}(\xi_m^{i;F}A_m^i - \hat{\xi}_m^{i;F}A_m^i)\right|}_{\Gamma_2} + \underbrace{\left|\frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}_m^{i;F}(A_m^i - \hat{A}_m^i)\right|}_{\Gamma_3}, \end{aligned}$$

(10d) is bounded above by

$$\underbrace{\mathbb{P}_{\hat{\rho}}\left\{\Gamma_1 \geq \frac{\epsilon}{24}\right\}}_{(a)} + \underbrace{\mathbb{P}_{\hat{\rho}}\left\{\Gamma_2 \geq \frac{\epsilon}{24}\right\}}_{(b)} + \underbrace{\mathbb{P}_{\hat{\rho}}\left\{\Gamma_3 \geq \frac{\epsilon}{24}\right\}}_{(c)}. \tag{11}$$

(11a) converges to 0 in probability due to Proposition 4. We study the quantities (11b) and (11c).

(11b) can be further bounded. If N^c is the size of $\mathfrak{D}_i^c \cap \mathscr{I}$, then we have

$$\begin{aligned} \Gamma_2 =& \left|\frac{1}{N}\sum_{m\in\mathscr{I}^c}(\xi_m^{i;F}A_m^i - \hat{\xi}_m^{i;F}A_m^i)\right| \\ \leq& \underbrace{\left|\frac{1}{N}\sum_{m\in\mathscr{I}^c}\xi_m^{i;F}A_m^i - \frac{N^c}{N}\mathbb{E}_{\hat{\rho}}\left[\xi^{i;F}A^i\right]\right|}_{\Gamma_{2;1}} \\ &+ \underbrace{\left|\frac{N^c}{N}\mathbb{E}_{\hat{\rho}}\left[\xi^{i;F}A^i\right] - \frac{N^c}{N}\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}A^i\right]\right|}_{\Gamma_{2;2}} \\ &+ \underbrace{\left|\frac{N^c}{N}\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}A^i\right] - \frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}_m^{i;F}A_m^i\right|}_{\Gamma_{2;3}}, \end{aligned}$$

we see that (11b) can be further bounded by

$$\underbrace{\mathbb{P}_{\hat{\rho}}\left\{\Gamma_{2;1} \geq \frac{\epsilon}{72}\right\}}_{(a)} + \underbrace{\mathbb{P}_{\hat{\rho}}\left\{\Gamma_{2;2} \geq \frac{\epsilon}{72}\right\}}_{(b)} + \underbrace{\mathbb{P}_{\hat{\rho}}\left\{\Gamma_{2;3} \geq \frac{\epsilon}{72}\right\}}_{(c)}. \tag{12}$$

We investigate if (12a), (12b), and (12c) converge to 0 in probability. We consider (12a) first. Recall the assumptions that $(\xi^{i;F} \perp\!\!\!\perp D) \mid \mathbf{Z}$, $\left(\xi^{i;F} \overset{d}{=} \xi^{i;CF}\right) \mid \mathbf{Z}$ and $(\xi^{i;CF} \perp\!\!\!\perp D) \mid \mathbf{Z}$, we have $\frac{1}{N}\sum_{m\in\mathscr{I}^c}\xi_m^{i;F}A_m^i \overset{d}{=} \frac{1}{N}\sum_{m\in\mathscr{I}^c}\xi_m^{i;CF}A_m^i$ by Lemma 3.

Since $\Gamma_{2;1}=\frac{N^c}{N}\left|\frac{1}{N^c}\sum_{m\in\mathscr{I}^c}\xi_m^{i;F}A_m^i-\mathbb{E}_{\hat{\rho}}\left[\xi^{i;F}A^i\right]\right|$, we have

$$
\begin{aligned}
&\mathbb{P}_{\hat{\rho}}\left\{\Gamma_{2;1}\geq\frac{\epsilon}{72}\right\}\\
=&\mathbb{P}_{\hat{\rho}}\left\{\left|\frac{1}{N^c}\sum_{m\in\mathscr{I}^c}\xi_m^{i;F}A_m^i-\mathbb{E}_{\hat{\rho}}\left[\xi^{i;F}A^i\right]\right|\geq\frac{\epsilon}{72}\cdot\frac{N}{N^c}\right\}\\
\leq&\mathbb{P}_{\hat{\rho}}\left\{\left|\frac{1}{N^c}\sum_{m\in\mathscr{I}^c}\xi_m^{i;F}A_m^i-\mathbb{E}_{\hat{\rho}}\left[\xi^{i;F}A^i\right]\right|\geq\frac{\epsilon}{72}\right\}\\
\leq&\frac{\mathbb{E}_{\hat{\rho}}\left[\left|\frac{1}{N^c}\sum_{m\in\mathscr{I}^c}\xi_m^{i;F}A_m^i-\mathbb{E}_{\hat{\rho}}\left[\xi^{i;F}A^i\right]\right|^2\right]}{\left(\frac{\epsilon}{72}\right)^2}.
\end{aligned}
$$

Consider $\mathbb{E}_{\hat{\rho}}\left[\left|\frac{1}{N^c}\sum_{m\in\mathscr{I}^c}\xi_m^{i;F}A_m^i-\mathbb{E}_{\hat{\rho}}\left[\xi^{i;F}A^i\right]\right|^2\right]$. Note that

$$
\begin{aligned}
&\mathbb{E}_{\hat{\rho}}\left[\left|\frac{1}{N^c}\sum_{m\in\mathscr{I}^c}\xi_m^{i;F}A_m^i-\mathbb{E}_{\hat{\rho}}\left[\xi^{i;F}A^i\right]\right|^2\right]\\
&\quad+\frac{1}{(N^c)^2}\sum_{\substack{m,\bar{m}\in\mathscr{I}^c\\ m\neq\bar{m}}}\mathbb{E}_{\hat{\rho}}\left[(\xi_m^{i;F}A_m^i-\mathbb{E}_{\hat{\rho}}[\xi^{i;F}A^i])(\xi_{\bar{m}}^{i;F}A_{\bar{m}}^i-\mathbb{E}_{\hat{\rho}}[\xi^{i;F}A^i])\right]\\
=&\frac{1}{(N^c)^2}\sum_{m\in\mathscr{I}^c}\mathbb{E}_{\hat{\rho}}\left[\left(A_m^i\right)^2\mathbb{E}_{\hat{\rho}}\left[\left(\xi_m^{i;F}\right)^2\mid D,\mathbf{Z}\right]\right]\\
&\quad+\frac{1}{(N^c)^2}\sum_{\substack{m,\bar{m}\in\mathscr{I}^c\\ m\neq\bar{m}}}\mathbb{E}_{\hat{\rho}}\left[A_m^iA_{\bar{m}}^i\mathbb{E}_{\hat{\rho}}\left[\xi_m^{i;F}\mid D,\mathbf{Z}\right]\mathbb{E}_{\hat{\rho}}\left[\xi_{\bar{m}}^{i;F}\mid D,\mathbf{Z}\right]\right]\\
=&\frac{1}{(N^c)^2}\sum_{m\in\mathscr{I}^c}\mathbb{E}_{\hat{\rho}}\left[\left(A_m^i\right)^2\mathbb{E}_{\hat{\rho}}\left[\left(\xi_m^{i;F}\right)^2\mid\mathbf{Z}\right]\right]\\
=&\frac{1}{N^c}\mathbb{E}_{\hat{\rho}}\left[\left(A^i\right)^2\mathbb{E}_{\hat{\rho}}\left[\left(\xi^{i;F}\right)^2\mid\mathbf{Z}\right]\right].
\end{aligned}
$$

Since A^i and $\xi^{i;F}$ do not include the estimated nuisance parameters, $\mathbb{E}_{\hat{\rho}}\left[\left(A^i\right)^2\mathbb{E}_{\hat{\rho}}\left[\left(\xi^{i;F}\right)^2\mid\mathbf{Z}\right]\right]$ is a constant. Moreover, note that $N^c\to\infty$ when $N\to\infty$, we have

$$
\mathbb{P}_{\hat{\rho}}\left\{\Gamma_{2;1}\geq\frac{\epsilon}{72}\right\}\leq\frac{72^2\,\mathbb{E}_{\hat{\rho}}\left[\left(A^i\right)^2\mathbb{E}_{\hat{\rho}}\left[\left(\xi^{i;F}\right)^2\mid\mathbf{Z}\right]\right]}{\epsilon^2N^c}\xrightarrow{p}0.
$$

Now, we consider (12b). Indeed, we have

$$
\begin{aligned}
&\mathbb{P}_{\hat{\rho}}\left\{\Gamma_{2;2}\geq\frac{\epsilon}{72}\right\}\\
=&\mathbb{P}_{\hat{\rho}}\left\{\left|\mathbb{E}_{\hat{\rho}}\left[\xi^{i;F}A^i\right]-\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}A^i\right]\right|\geq\frac{\epsilon}{72}\cdot\frac{N}{N^c}\right\}\\
\leq&\mathbb{P}_{\hat{\rho}}\left\{\left|\mathbb{E}_{\hat{\rho}}\left[(\xi^{i;F}-\hat{\xi}^{i;F})A^i\right]\right|\geq\frac{\epsilon}{72}\right\}\\
\leq&\frac{72^2\left\{\mathbb{E}_{\hat{\rho}}\left[(\xi^{i;F}-\hat{\xi}^{i;F})A^i\right]\right\}^2}{\epsilon^2}\\
\leq&\frac{72^2\left\{\mathbb{E}_{\hat{\rho}}\left[(\xi^{i;F}-\hat{\xi}^{i;F})^{4q}\right]\right\}^{\frac{1}{2q}}\left\{\mathbb{E}_{\hat{\rho}}\left[(A^i)^{\frac{4q}{4q-1}}\right]\right\}^{2-\frac{1}{2q}}}{\epsilon^2}\xrightarrow{p}0.
\end{aligned}
$$

Here, the last inequality follows from the Hölders inequality, while the convergence holds $\forall q\in\{1,2,\ldots,k\}$ according to Assumption 1.5 of [1]. Finally, we consider (12c). We can rewrite $\Gamma_{2;3}$ as

$$
\Gamma_{2;3}=\frac{N^c}{N}\left|\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}A^i\right]-\frac{1}{N^c}\sum_{m\in\mathscr{I}^c}\hat{\xi}_m^{i;F}A_m^i\right|.
$$

Now, we have

$$
\begin{aligned}
&\mathbb{P}_{\hat{\rho}}\left\{\Gamma_{2;3}\geq\frac{\epsilon}{72}\right\}\\
\leq&\mathbb{P}_{\hat{\rho}}\left\{\left|\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}A^i\right]-\frac{1}{N^c}\sum_{m\in\mathscr{I}^c}\hat{\xi}_m^{i;F}A_m^i\right|\geq\frac{\epsilon}{72}\right\}\\
\leq&\frac{72^2\,\mathbb{E}_{\hat{\rho}}\left[\left\{\sum_{m\in\mathscr{I}^c}\left(\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}A^i\right]-\hat{\xi}_m^{i;F}A_m^i\right)\right\}^2\right]}{\epsilon^2\left(N^c\right)^2}\\
=&\frac{72^2\sum_{m\in\mathscr{I}^c}\mathbb{E}_{\hat{\rho}}\left[\left(\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}A^i\right]-\hat{\xi}_m^{i;F}A_m^i\right)^2\right]}{\epsilon^2\left(N^c\right)^2}\\
&+\frac{72^2\sum_{\substack{m,\bar{m}\in\mathscr{I}^c\\ m\neq\bar{m}}}\mathbb{E}_{\hat{\rho}}[(\hat{\xi}_m^{i;F}-\xi_m^{i;F})A_m^i]\mathbb{E}_{\hat{\rho}}[(\hat{\xi}_{\bar{m}}^{i;F}-\xi_{\bar{m}}^{i;F})A_{\bar{m}}^i]}{\epsilon^2\left(N^c\right)^2}\\
&-2\frac{72^2\sum_{\substack{m,\bar{m}\in\mathscr{I}^c\\ m<\bar{m}}}\mathbb{E}_{\hat{\rho}}[(\hat{\xi}_m^{i;F}-\xi_m^{i;F})A_m^i]\mathbb{E}_{\hat{\rho}}[(\hat{\xi}^{i;F}-\xi^{i;F})A^i]}{\epsilon^2\left(N^c\right)^2}\\
&+\frac{72^2\sum_{\substack{m,\bar{m}\in\mathscr{I}^c\\ m\neq\bar{m}}}\left\{\mathbb{E}_{\hat{\rho}}\left[\left(\hat{\xi}^{i;F}-\xi^{i;F}\right)A^i\right]\right\}^2}{\epsilon^2\left(N^c\right)^2}.
\end{aligned}
$$

Using Assumption 1.5 of [1], we can conclude that $\mathbb{P}_{\hat{\rho}}\left\{\Gamma_{2;3}\geq\frac{\epsilon}{72}\right\}\xrightarrow{p}0$. Next, we come to bound (11c). Again, we denote N^c as the size of $\mathfrak{D}_i^c\cap\mathscr{I}$. Since

$$
\begin{aligned}
\Gamma_3=&\left|\frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}_m^{i;F}A_m^i-\frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}_m^{i;F}\hat{A}_m^i\right|\\
\leq&\left|\frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}_m^{i;F}A_m^i-\frac{N^c}{N}\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}A^i\right]\right|\\
&+\left|\frac{N^c}{N}\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}A^i\right]-\frac{N^c}{N}\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}\hat{A}^i\right]\right|\\
&+\left|\frac{N^c}{N}\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}\hat{A}^i\right]-\frac{1}{N}\sum_{m\in\mathscr{I}^c}\hat{\xi}_m^{i;F}\hat{A}_m^i\right|\\
=&\frac{N^c}{N}\underbrace{\left|\frac{1}{N^c}\sum_{m\in\mathscr{I}^c}\hat{\xi}_m^{i;F}A_m^i-\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}A^i\right]\right|}_{\Gamma_{3;1}}\\
&+\frac{N^c}{N}\underbrace{\left|\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}A^i\right]-\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}\hat{A}^i\right]\right|}_{\Gamma_{3;2}}\\
&+\frac{N^c}{N}\underbrace{\left|\mathbb{E}_{\hat{\rho}}\left[\hat{\xi}^{i;F}\hat{A}^i\right]-\frac{1}{N^c}\sum_{m\in\mathscr{I}^c}\hat{\xi}_m^{i;F}\hat{A}_m^i\right|}_{\Gamma_{3;3}},
\end{aligned}
$$

we see that (11c) can be further bounded by

$$
\underbrace{\mathbb{P}_{\hat{\rho}}\left\{\Gamma_{3;1}\geq\frac{\epsilon}{72}\right\}}_{(a)}+\underbrace{\mathbb{P}_{\hat{\rho}}\left\{\Gamma_{3;2}\geq\frac{\epsilon}{72}\right\}}_{(b)}+\underbrace{\mathbb{P}_{\hat{\rho}}\left\{\Gamma_{3;3}\geq\frac{\epsilon}{72}\right\}}_{(c)}. \tag{13}
$$

Similarly, we can prove that (13a) and (13c) converge to 0 in probability when $N\to\infty$ using the arguments in proving that (12a) and (12c) converge to 0. As a result, the quantity (10d) converges to 0 in probability when $N\to\infty$.

Lastly, we turn to consider the quantity (10a). In fact, we have

$$\begin{aligned}
&\mathbb{P}_{\hat{\rho}}\left\{\left|\hat{\kappa}^{i;F}_{R,N}-\kappa^{i;F}_{R,N}\right|\geq\frac{\epsilon}{8}\right\}\\
\leq&\mathbb{P}_{\hat{\rho}}\left\{\left|\frac{1}{N}\sum_{m\in\mathscr{I}^c}\frac{1}{R}\sum_{u=1}^{R}\left(\hat{\xi}^{i;F}_{m,u}\right)\left(\hat{A}^i_m-A^i_m\right)\right|\geq\frac{\epsilon}{16}\right\} \qquad \text{(14a)}\\
&+\mathbb{P}_{\hat{\rho}}\left\{\left|\frac{1}{N}\sum_{m\in\mathscr{I}^c}A^i_m\frac{1}{R}\sum_{u=1}^{R}\left[\left(\hat{\xi}^{i;F}_{m,u}-\xi^{i;F}_{m,u}\right)\right]\right|\geq\frac{\epsilon}{16}\right\}. \qquad \text{(14b)}
\end{aligned}$$

We can argue that (14a) converges to 0 in probability as $N\to\infty$ using similar arguments when we prove that (11b) converges to 0 in probability. Simultaneously, we can argue (14b) converges to 0 in probability as $N\to\infty$ using similar arguments when we prove that (11c) converges to 0 in probability. Consequently, we have $\hat{\kappa}^{i;F}_{R,N}-\kappa^{i;F}_{R,N}$ converges to 0 in probability.

The proof is completed. $\square$

References

[1] L. Mackey, V. Syrgkanis, and I. Zadik, "Orthogonal machine learning: Power and limitations," in *International Conference on Machine Learning*. PMLR, 2018, pp. 3375–3383.