#[Q1]

$$m = \begin{bmatrix} F_A \\ F_B \end{bmatrix} \qquad c = \begin{bmatrix} x_A \\ F \end{bmatrix}$$

- For open loop :

We set the matrix, $G \Rightarrow \quad c = Gm$

Therefore, $G = \begin{bmatrix} \left(\frac{\partial y_1}{\partial m_1}\right)_{m_2=0} & \left(\frac{\partial y_1}{\partial m_2}\right)_{m_1=0} \\ \left(\frac{\partial y_2}{\partial m_1}\right)_{m_2=0} & \left(\frac{\partial y_2}{\partial m_2}\right)_{m_1=0} \end{bmatrix} = \begin{bmatrix} \frac{F_B}{(F_A+F_B)^2} & \frac{-F_A}{(F_A+F_B)^2} \\ 1 & 1 \end{bmatrix}$

- For close loop :

We can only make proposition and assume they still obey $y = Gm$.

Here, we can calculate the $\hat{g}_{11}$.

To calculate $\hat{g}_{11}$, we need to find $\left(\frac{\partial y_1}{\partial m_1}\right)_{y_2=0}$

$$\xrightarrow[y = Gm]{\text{sub into}} \quad \begin{cases} y_1 = g_{11} m_1 + g_{12} m_2 \\ 0 = g_{21} m_1 + g_{22} m_2 \end{cases}$$

$$\Rightarrow \begin{cases} -\left(g_{21} / g_{22} \right) m_1 = m_2 \\ \quad \hookrightarrow \; y_1 = g_{11} m_1 + g_{22} \left(\dfrac{-g_{21}}{g_{22}} \right) m_1 \end{cases}$$

$$\Rightarrow \quad y_1 = \left(g_{11} - \frac{g_{12} g_{21}}{g_{22}} \right) m_1$$

$$\Rightarrow \quad \hat{g}_{11} = g_{11} - \frac{g_{12} g_{21}}{g_{22}}$$

Now we can calculate one element of RGA

$$\lambda_{11} = \frac{\text{open loop relative gain}}{\text{close loop} \;\; \text{″}} = \frac{g_{11}}{\hat{g}_{11}}$$

$$= \frac{1}{1 - \dfrac{g_{12} g_{21}}{g_{11} g_{22}}} = \frac{1}{1 - \dfrac{\left(\dfrac{-F_A}{(F_A + F_B)^2} \right)}{\left(\dfrac{F_B}{(F_A + F_B)^2} \right)}} = \frac{1}{1 + \dfrac{F_A}{F_B}}$$

And we know that $\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & 1 - \lambda_{11} \\ 1 - \lambda_{11} & \lambda_{11} \end{bmatrix}$

Therefore, RGA is : $\Lambda = \begin{bmatrix} \dfrac{F_B}{F} & \dfrac{F_A}{F} \\ \dfrac{F_A}{F} & \dfrac{F_B}{F} \end{bmatrix}$

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[Q2]

As the Q1, we follow the same process.

- As open loop : $y = Gm$

$$G = \begin{bmatrix} \frac{2}{3s+1} & \frac{5}{s+1} \\ \frac{4}{s+1} & \frac{1}{2s+1} \end{bmatrix}$$

- For close loop :

$$\begin{cases} y_1 = g_{11} m_1 + g_{12} m_2 \\ 0 = g_{21} m_1 + g_{22} m_2 \end{cases}$$

$$y_1 = \left(g_{11} - \frac{g_{12} g_{21}}{g_{22}} \right) m_1$$

$$\hat{g}_{11} = \left(g_{11} - \frac{g_{12} g_{21}}{g_{22}} \right)$$

$$\lambda_{11} = \frac{g_{11}}{\hat{g}_{11}} = \frac{1}{1 - \left(\dfrac{g_{12}\, g_{21}}{g_{11}\, g_{22}} \right)}$$

$$= \frac{1}{1 - \left(\dfrac{\left(\frac{5}{s+1}\right)\left(\frac{4}{s+1}\right)}{\left(\frac{2}{3s+1}\right)\left(\frac{1}{2s+1}\right)} \right)} = \frac{1}{1 - 10\left(\dfrac{(3s+1)(2s+1)}{(s+1)(s+1)} \right)}$$

$$= \frac{s^2 + 2s + 1}{s^2 + 2s + 1 - 10(6s^2 + 5s + 1)} = \frac{s^2 + 2s + 1}{-59s^2 - 58s - 9}$$

$$1 - \lambda_{11} = \frac{-59s^2 - 58s - 9 - (s^2 + 2s + 1)}{-59s^2 - 58s - 9} = \frac{-60s^2 - 60s - 10}{-59s^2 - 58s - 9}$$

$$\Lambda = \begin{bmatrix} \lambda_{11} & 1 - \lambda_{11} \\ 1 - \lambda_{11} & \lambda_{11} \end{bmatrix}$$

[Q3]

To decouple, we start from $y = Gm$

Now we want to change m to another controlled var, u.

$m = \square u$, here we set $\square = \begin{bmatrix} 1 & d_{12} \\ d_{21} & 1 \end{bmatrix}$

for simplicity.

directly effect

ORF $\Rightarrow$ $y = G\square u$

if we see $G\square$ as a matrix $\hookrightarrow$ $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} gd_{11} & gd_{12} \\ gd_{21} & gd_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\Rightarrow$ What we expect is that $\begin{cases} y_1 \leftarrow u_1 \\ y_2 \leftarrow u_2 \end{cases}$ (1,1/2,2 pairing)

therefore,

$y_1 = gd_{11} u_1 + gd_{12} u_2$ $\rightsquigarrow$ $gd_{12} = 0$

$y_2 = gd_{21} u_1 + gd_{22} u_2$ $\qquad$ $gd_{21} = 0$

$$\Rightarrow GD = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} 1 & d_{12} \\ d_{21} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (g_{11} + g_{12} d_{21}) & (g_{11} d_{12} + g_{12}) \\ (g_{21} + g_{22} d_{21}) & (g_{21} d_{12} + g_{22}) \end{bmatrix}$$

$$gd_{12} = g_{11} d_{12} + g_{12} = 0 \qquad d_{12} = \left(\frac{-g_{12}}{g_{11}} \right)$$

$$gd_{21} = g_{21} + g_{22} d_{21} = 0 \qquad d_{21} = \left(\frac{-g_{21}}{g_{22}} \right)$$

$$D = \begin{bmatrix} 1 & \left(\frac{-g_{12}}{g_{11}} \right) \\ \left(\frac{-g_{21}}{g_{22}} \right) & 1 \end{bmatrix}$$