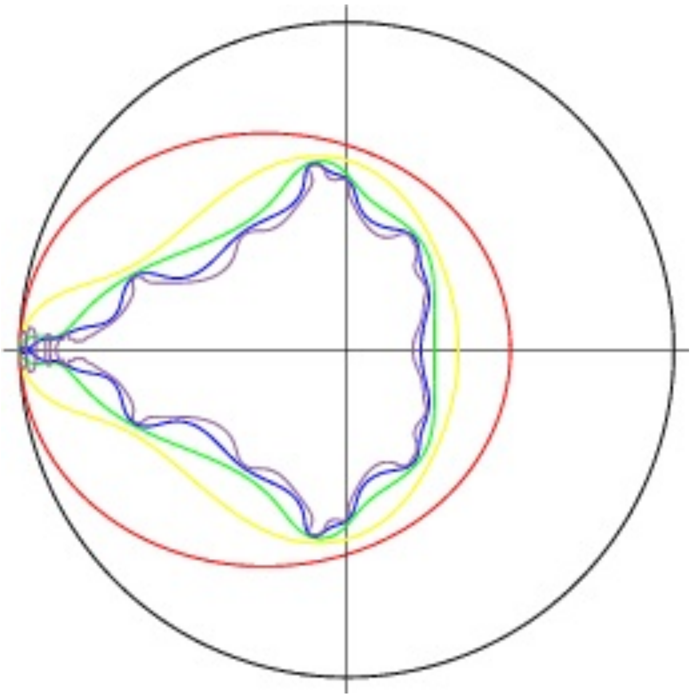
CALCULUS I
DIFFERENTIAL CALCULUS



Star Edition

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Preface

The first thought that quite understandably comes to mind when seeing these lecture notes is “Why?”. Indeed, there is a multitude of calculus books on the market. You have the good, the bad and the ugly. They come in hardcover or paperback and range from affordable over pricey to downright budget-sinking. And that is just for the English speaking market. Do I honestly believe that none of these books is any good? Of course not. The main motivation for me to write these notes is to give a very critical thinking of calculus ideas. Every student would expect a continuation of the nature of mathematics (calculation before thinking) at A'level schools, these lecture notes bring in a different approach to your mathematics - thinking before calculations. Furthermore, the student should not be intimidated by the sheer volume and the fact that the memorization of about 300 formulas - we have only 4 or 5 formulas to play with.

For every new concept, a few examples are discussed to give some intuitive understanding of this new concept. In most cases, if a theorem is stated, it comes with a proof, yet the proof itself is not a necessity to comprehend what follows after ”thm”. So proofs could be skipped during a first reading. ”First reading” of course implies that there has to be a second, and a third and so on.

None of your favorite subjects (and mine) have been skipped: however a lot of personal efforts will be required in going through all available examples and exercises for deeper understanding of differential calculus. And that is only the tip of a gargantuan iceberg. This is unfortunate but not tragic; further exploration is needed and nobody ever mastered mathematics by reading mathematics. Mathematics is mastered by doing mathematics.

At this point, I would enjoy thanking my proofreaders Ismail G Mirumbe and John M Kitayimbwa. Without them, a couple very nonsensical claims would still have been around on these pages. I would also like to thank Michael K Nganda, Juma Kasozi and Wilber G Naigambi for giving me the freedom in this program that enabled me to express my own peculiar tastes in calculus. But also for the use of their tutorial and test sheets to build up this volume. Finally, I feel indebted to my family for their friendly hospitality with which they have let me invade our house and spread calculus books, software manuals and empty bottles all over the place.

Happy Thinking and Calculations,
David Ddumba Walakira,
Makerere (Kampala),
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Instructions

These are lecture notes, and not a text book nor a reference book. There will be little or no extra classroom lecture notes, and therefore you will be required to keep summarizing within this book as the instructors take you through this wonderful piece of differential calculus.

You are expected to hand in all exercises from each chapter before the beginning of a new chapter. Exercise are indicated and numbered. Sections 1.9, Section 2.5, Section 3.5 and Section 4.6 will be fundamental before any test and final evaluation - the sections require you to think, question issues, challenge ideas, generate solutions to problems.

You are expected to indicate the "tricks" on every example you go through for easy future revision towards the examination.

Write in this book, it is your note book too.

Contents

Chapter 1

Functions

1.1 Interval Notation

In the definitions of Open, closed and Half closed intervals we shall assume that a and b are real numbers such that $a < b$.

1.1.1 Open Intervals

An Open Interval, written in interval notation as (a, b), is defined as the set of all numbers x such that $a < x < b$: where $a < b$ that is x lies between a and b. This can be represented in set-builder notation as $(a, b) = \{x : a < x < b\}$. The points a and b are themselves not included.

Example 1.1.1 *Express the following open intervals in set-builder notation:*

(a) $(2, 4)$

(b) $(-1, 3)$

These open intervals can be expressed in set-builder notation as:

(a) $(2, 4) = \{x : 2 < x < 4\}$

(b) $(-1, 3) = \{x : -1 < x < 3\}$

1.1.2 Closed Intervals

A Closed Interval, written in interval notation as $[a, b]$ is defined as the set of all numbers x such that $a \leq x \leq b$. Closed intervals can be represented in set-builder notation as: $[a, b] = \{x : a \leq x \leq b\}$, that is x lies between and includes a and b.

Example 1.1.2 *Express the following closed intervals in set builder notation.*

(a) $[m, n]$

(b) $[-1, 3]$

These closed intervals can be expressed in set builder notation as:

(a) $[m, n] = \{x : m \leq x \leq n\}$

(b) $[-1, 3] = \{x : -1 \leq x \leq 3\}$

1.1.3 Half-open Intervals

There are two types of half open intervals. The left half open interval $(a, b]$ and the right half open interval $[a, b)$.

The left half open interval is defined as the set of numbers x such that $a < x \leq b$. This can be represented in set-builder notation as

$$(a, b] = \{x : a < x \leq b\}$$

where $a < b$.

The right half open interval is defined as the set of number x such that $a \leq x < b$, where $a < b$. This can be represented in set-builder notation as

$$[a, b) = \{x : a \leq x < b\}$$

Example 1.1.3 *Express the following half open intervals in set-builder notation.*

(a) $(-2, 5]$

(b) $[-a, a)$

The half open intervals above can be expressed in set builder notation as:

(a) $(-2, 5] = \{x : -2 < x \leq 5\}$

(b) $[-a, a) = \{x : -a \leq x < a\}$

1.2 Infinite Intervals

We have four types of infinite intervals.

1.2.1 Open Infinite Intervals (a, ∞) or $(-\infty, b)$

Let a be a real number. The Open Infinite Interval (a, ∞) is the set of numbers x such that $a < x < \infty$. This is represented in set - builder notation by $(a, \infty) = \{x : a < x < \infty\}$. This can be displayed on the real line graph by the space between a and infinity but not including the point a.

Similarly, the open interval $(-\infty, b)$, is the set of numbers x such that $-\infty < x < b$. This is represented in set builder notation by $(-\infty, b) = \{x : -\infty < x < b)$. On the real line graph, this can be displayed by the space between negative infinity and b, but not including b.

1.2.2 A closed-infinite interval $[a, \infty)$ or $(-\infty, b]$

These are the set of numbers x such that $a \leq x < \infty$. This is expressed in set-builder notation by $[a, \infty) = \{x : a \leq x < \infty\}$. This can be displayed on the real line graph by the space between a and infinity including the point a.

Similarly, the closed interval $(-\infty, b]$ is the set of numbers x such that $-\infty < x \leq b$.

1.2.3 An Infinite-Infinite Interval $(-\infty, \infty)$.

This is the set of numbers x such that $-\infty < x < \infty$. This is represented in set builder notation by $(-\infty, \infty) = \{x : -\infty < x < \infty\}$. This can be displayed on the number line by all this points between negative infinity and positive infinity. This represents the entire number line, which is both open and closed.

In otherwords,

(a) left-bounded and right-unbounded:

 (i) left-open: $(a, \infty) = \{x \,|\, x > a\}$

 (ii) left-closed: $[a, \infty) = \{x \,|\, x \geq a\}$

(b) left-unbounded and right-bounded:

 (i) right-open: $(-\infty, b) = \{x \,|\, x < b\}$

 (ii) right-closed: $(-\infty, b] = \{x \,|\, x \leq b\}$

(c) unbounded at both ends: $(-\infty, +\infty) = \mathbb{R}$

Example 1.2.1 *Express the following intervals in set - builder notation.*

(a) $(2, 4)$ *(b)* $[2, 4)$ *(c)* $(-4, 3]$

The above intervals can be expressed in set-builder notation as

(a) $(2, 4) = \{x : 2 < x < 4\}$

(b) $[2, 4) = \{x : 2 \leq x < 4\}$

(c) $(-4, 3] = \{x : -4 < x \leq 3\}$

Example 1.2.2 *Write the following sets in interval notation:*

(a) $A = \{x : -\infty < x < 2\}$

(b) $B = \{x : -1 < x < 4\}$

(c) $C = \{x : -\infty < x \leq 3\}$

(d) $D = \{x : 2 \leq x < \infty\}$

These sets can be represented in interval notation as follows:-

(a) The set of points do not include 2*, but continues to the left of* 2 *up to negative infinity, that is* $(-\infty, 2)$.

(b) The set of points that lies between -1 *and* -4*, that is,* $(-1, 4)$.

(c) The set of points that includes 3 *and continues to the left of* 3 *up to negative infinity, that is,* $(-\infty, 3]$

(d) The set of points that includes 2 *and continues to the right of* 2 *up to infinity that is,* $[2, \infty)$.

1.3 Inequalities

The inequality $a < b$ means $b - a$ is positive. The inequality $a \leq b$ means that either $a = b$ or $a < b$.

The second inequality is the most misunderstood by beginning students. Many students are quite happy with $3 < 5$ but fear $3 \leq 5$ or $4 \geq 4$ may be incorrect. All the three statements are correct.

Theorem 1.3.1 Let x, y, z and c be real numbers. Then

(i) Exactly one of the following holds: $x = y$, or $x < y$ or $x > y$

(ii) If $x < y$ and $y < z$, then $x < z$. (Transitive property)

(iii) If $x < y$, then $x + c < y + c$ and $x - c < y - c$.

(iv) If $x < y$ and $c > 0$, then $cx < cy$ and $\frac{x}{c} < \frac{y}{c}$.

(v) If $x < y$ and $c < 0$, then $cx > cy$ and $\frac{x}{c} > \frac{y}{c}$.

These properties of inequalities are used to solve inequalities.

Note 1.3.1 Note that multiplying by a negative number reverses the direction of the inequality. The above properties can be proved. Properties (ii) and (v) are proved for you. The other proofs are left as an exercise.

Example 1.3.1 *Prove that if $a < b$ and $b < c$ then $a < c$.*
We need to recall that $a < b$ means $b - a$ is positive and $b < c$ means $c - b$ is positive. The sum of two positive numbers is a positive number. Thus $(b - a) + (c - b) = c - a$ is positive and therefore $a < c$.

1.3.1 Solving Inequalities

The process of finding a solution requires that the inequality be written in a form that shows the values of the variable that ascertain the truth of the initial inequality. Inequalities may involve sums differences, products and quotients.

Example 1.3.2 Solve the inequality $2x - 9 > -3$.

Add $+9$ to both sides of the inequality.
$2x - 9 + 9 > -3 + 9$; (using property of inequalities: if $a < b$ then $a + c < b + c$)
gives $2x > 6$.
Multiply both sides of the inequality by $\frac{1}{2}$ to obtain the coefficient of x as 1 :
$\frac{1}{2} \cdot 2x > 6 \cdot \frac{1}{2}$; (Using property of inequalities: if $a > b$ and $c > 0$ then $ac > bc$) from which $x > 3$.

Example 1.3.3 Solve the inequality $2(3 + 2x) - 4 > 7x$.

Distribute 2 to eliminate the brackets $6 + 4x - 4 > 7x$.
Combine the like terms on the left hand side (LHS) of the inequality. $4x + 2 > 7x$
Add $-4x + 4x + 2 > 7x - 4x$; (property of inequalities: if $a > b$ then $a + c > b + c$) then $2 > 3x$.
Multiply by $\frac{1}{3}$ to get a coefficient of $x = 1$. That is, $2 \cdot \frac{1}{3} > \frac{1}{3} \cdot 3x$ from which $\frac{2}{3} > x$.

Example 1.3.4 Solve the inequality $x^2 - 7x + 12 > 0$. This is a quadratic inequality.

We find the need to factor the inequality as $(x - 4)(x - 3) > 0$. We want the values of x that make $(x - 4)(x - 3)$ positive. Lets look at case 1
- (both numbers are positive). When $(x - 4) > 0$ and $(x - 3) > 0$. If $x - 4 > 0$ then $x > 4$. If $x - 3 > 0$ then $x > 3$.
We choose a solution that satisfies both inequalities. We select

$$x > 4$$

Case 2: (both numbers negative); $(x - 4) < 0$ and $(x - 3) < 0$.
If $x - 4 < 0$ then $x < 4$.
If $x - 3 < 0$ then $x < 3$.

The solution of this case is: All x such that $x < 4$ and $x < 3$. Both inequalities will be satisfied by $x < 3$

From case 1 we have $x > 4$ and from case 2 we have $x < 3$. The solution is therefore $x < 3$ or $x > 4$

Example 1.3.5 Solve

$$\frac{4}{x + 1} < 2.$$

Multiply both sides of the inequality by $(x + 1)$ to eliminate the fraction. $(x + 1)$ could be a positive $(x + 1) > 0$, or negative that is, $(x + 1) < 0$. We therefore consider both cases.
Case 1: $(x + 1) > 0$ that is, $x > -1$.

Multiply both sides of $\frac{4}{x+1} < 2$ by a positive number $(x+1) > 0$, gives

$$\begin{aligned} 4 &< 2(x+1) \\ 4 &< 2x+2 \\ 2 &< 2x \\ 1 &< x \end{aligned}$$

The solution must satisfy both $x > -1$ and $1 < x$.
The intersection gives the solution as $x > 1$.
Case 2: $(x+1) < 0$ that is, $x < -1$.
Multiplying both sides of $\frac{4}{x+1} < 2$ by a negative number $(x+1) < 0$ gives

$$\begin{aligned} 4 &> 2(x+1) \\ 4 &> 2x+2 \\ 2 &> 2x \\ 1 &> x \end{aligned}$$

The solution must satisfy both $x < -1$ and $1 > x$.
The solution is $x < -1$. From the case 1 we have the solution $x > 1$ and in case 2 we have the solution $x < -1$. The whole solution is therefore $x > 1$ or $x < -1$.

Example 1.3.6 *Prove that if $a < b$ then $a + c < b + c$.*
$a < b$ means $b - a$ is positive. But $b - a$ may be rewritten as $(b + c) - (a + c)$. This is a positive number. Hence $a + c < b + c$.

Example 1.3.7 Solve $3 + x < 8$.

Add -3 to both sides of the inequality to obtain $x < 5$. Therefore the solution set is $\{x \in \mathbb{R} : x < 5\}$ which is the same as $(-\infty, 5)$.

Example 1.3.8 Solve $x^2 + x + 1 > 7$.

Add -7 to both sides to obtain

$$\begin{aligned} x^2 + x - 6 &> 0 \\ \text{i.e. } (x+3)(x-2) &> 0 \end{aligned}$$

But when is a product of two real numbers positive?

(i) $(x+3) > 0$ **and** $(x-2) > 0$. Then $x > -3$ **and** $x > 2$, implying that $x > 2$, OR

(ii) $(x+3) < 0$ **and** $(x-2) < 0$. Then $x < -3$ **and** $x < 2$, implying $x < -3$.

Therefore the solution set of the inequality is the set $(-\infty, -3) \cup (2, \infty)$.

Example 1.3.9 Solve the problem

$$\frac{x^2 - 5x + 4}{(x+2)} < 0$$

or

$$\frac{(x-1)(x-4)}{(x+2)} < 0$$

In this case there are several ways in which this inequality could be less than or equal to zero. (that is, Negative). These are:

(i) $(x-1)$ is positive; $(x-4)$ is positive while $x+2$ is negative.

(ii) $(x-1)$ is positive; $(x-4)$ is negative while $(x+2)$ is positive.

(iii) $(x-1)$ is negative; $(x-4)$ is positive while $(x+2)$ is positive.

(iv) $(x-1)$ is negative; $(x-4)$ is negative while $(x+2)$ is negative.

The solution of this type of inequality involves considering each of the cases (i) to (iv) above. The final solution is obtained by combining the results from the cases, that have solutions.

Example 1.3.10 Solve

$$\frac{(x-1)(x-4)}{(x+2)} < 0$$

Consider Case 1: If

$$\frac{\text{(positive) (positive)}}{\text{(negative)}}$$

Then: $(x-1) > 0$ [that is, $x > 1$] **and** $(x-4) > 0$ [that is, $x > 4$] **and** $(x+2) < 0$ [that is, $x < -2$] A number can not simultaneously be greater than 4 and less than -2. There is therefore no solution in this case.

Case 2:

$$\frac{\text{(positive) (negative)}}{\text{(positive)}}$$

Then

$$\begin{aligned} (x-1) > 0 &\quad \text{that is, } x > 1 \text{ and} \\ (x-4) < 0 &\quad \text{that is, } x < 4 \text{ and} \\ (x+2) > 0 &\quad \text{that is, } x > -2 \end{aligned}$$

The solution is $1 < x < 4$.

Case 3: If

$$\frac{\text{(negative) (positive)}}{\text{(Positive)}}.$$

Then

$$\begin{aligned} (x-1) < 0 &\quad \text{that is, } x < 1 \text{ and} \\ (x-4) > 0 &\quad \text{that is, } x > 4 \text{ and} \\ (x+2) > 0 &\quad \text{that is, } x > -2 \end{aligned}$$

The solution must satisfy all these inequalities $x < 1; x > 4; x > -2$. A number cannot be less than one and at the same time greater than 4. There is no solution in this case.

Case 4: If

$$\frac{\text{(negative) (negative)}}{\text{(negative)}}.$$

Then

$$\begin{aligned} (x-1) < 0 &\quad \text{that is, } x < 1, \text{ and} \\ (x-4) < 0 &\quad \text{that is, } x < 4, \text{ and} \\ (x+2) < 0 &\quad \text{that is, } x < -2 \end{aligned}$$

The solution that satisfies all inequalities is $x < -2$.

The solution of the problem is obtained by combining the cases that produced solutions in this case, cases 2 and 4. Therefore the solution is

$$x < -2 \text{ or } 1 < x < 4$$

Example 1.3.11 Another group of inequalities are of the form $(x^3 + 2x^2 - 3x) > 0$ or $x(x + 3)(x - 1) > 0$. In this case there are several ways of producing a positive product. These could be:

1. x is positive; $(x + 3)$ is positive and $(x - 1)$ positive
2. x is positive; $(x + 3)$ is negative and $(x - 1)$ is negative
3. x is negative; $(x + 3)$ is negative and $(x - 1)$ is positive.
4. x is negative $(x + 3)$ is positive and $(x - 1)$ is negative.

Example 1.3.12 Solve

$$x(x + 3)(x - 1) > 0$$

Consider case 1: If (positive) (positive) (positive)
Then $x > 0$ and $(x + 3) > 0$ and $(x - 1) > 0$. Hence: $x > 0$ and $x > -3$ and $x > 1$. This yields the solution $x > 1$.

Case 2: If (positive) (negative) (negative)
Then $x > 0$ and $(x + 3) < 0$ and $(x - 1) < 0$ Hence: $x > 0$ and $x < -3$ and $x < 1$.
These inequalities have no solution.

Case 3: If (negative) (negative) (positive)
Then $x < 0$ and $(x + 3) < 0$ and $(x - 1) > 0$ Hence: $x < 0$ and $x < -3$ and $x > 1$
These inequalities have no solution.

Case 4: If (negative) (positive) negative)
Then $x < 0$ and $(x + 3) > 0$ and $(x - 1) < 0$ Hence: $x < 0$ and $x > -3$ and $x < 1$. The solution is $-3 < x < 0$ Thus combining the solutions in cases 1 and 2. The solution to the problem is $x > 1$ or $-3 < x < 0$.

Example 1.3.13 Solve the inequality

$$\frac{(x - 1)(x + 2)}{(x - 3)} < 0$$

Example 1.3.14 Solve the inequality

$$(x + 2)(x - 3) > 0$$

To be positive, both terms have to be positive,

$$\begin{aligned} (x + 2) &> 0 \quad \Rightarrow x > -2 \quad \text{and} \\ (x - 3) &> 0 \quad \Rightarrow x > 3 \end{aligned}$$

$$x > 3 \equiv (3, \infty)$$

"**Or**", both terms are negative

$$\begin{aligned} (x+2) &< 0 \quad \Rightarrow x < -2 \quad \text{and} \\ (x-3) &< 0 \quad \Rightarrow x < 3 \end{aligned}$$

$$x < -2 \equiv (-\infty, -2)$$

Thus the general solution from both options is

$$(-\infty, -2) \cup (3, \infty)$$

Example 1.3.15 Solve the inequality: $(x-1)(x-2)(x-5) \leq 0$
If one is negative and two are positive, or all negative. If have $\leq$ stick to $\leq$, and if $<$, stick to $<$ signs.

$$(-\infty, 1] \cup [2, 5]$$

Exercise 1.1 Solve for x.

1. $2x + 3 < 9$
2. $-4 \leq 2(x+2) < 10$
3. $(x-3)(x+1) < 0$
4. $(x+7)(2x-4) > 0$
5. $x^2 + x < 0$
6. $x^2 + x + 7 > 19$
7. $x^4 - 9x^2 < 0$

1.4 Absolute values in calculus

Definition 1.4.1 The absolute value of a real number x is denoted by $|x|$ and is given by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The absolute value of a real number x is a measure of how far the real number x is from 0, the origin of the number line. For this reason it is always a positive quantity and sometimes it is referred to as the magnitude of the number.

Definition 1.4.2 The absolute value of a real number x is defined as $|x| = \sqrt{x^2}$.
(Note that $\sqrt{x}$ stands for the positive square root of x). The distance between two real numbers a and b is the number $|a-b| = |b-a|$.

Example 1.4.1 Simplify the norm (absolute value) $|x-5|$

$$|x-5| = \begin{cases} (x-5)\ , & x \geq 5 \\ \\ -(x-5)\ , & x < 5 \end{cases}$$

Example 1.4.2 Solve the inequality $|x+3| = 4$. We know that

$$|x+3| = \begin{cases} (x+3)\ , & x \geq -3 \\ \\ -(x+3)\ , & x < -3 \end{cases}$$

$x + 3 = 4, \Rightarrow \ x = 1$ or $-(x+3) = 4, \Rightarrow \ x = -7$

Theorem 1.4.1 *The inequality $|x| \leq a$ is equivalent to the double inequality $-a \leq x \leq a$. The inequality $|y| \geq b$ holds for $y \leq -b$ or $y \geq b$.*

Theorem 1.4.2 Properties of absolute value functions: *For any real numbers x and y,*

(a) $|x| = |-x|$, (c) $|x| \geq 0$, (e) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$,

(b) $|x|^2 = x^2$, (d) $|xy| = |x||y|$, (f) $|x+y| \leq |x| + |y|$.

Note 1.4.1 The following statements are NOT true. $|x+y| = |x| + |y|$ or $|x-y| = |x| - |y|$ for any numbers $x, y \in \mathbb{R}$. For example, if $x = 3$ and $y = -2$, the two are violated.

Note 1.4.2 Think of absolute as in two parts.

Example 1.4.3 Solve the inequality

$$|x+2| < 3$$

Method I:

$$\begin{aligned} |x+2| < 3 &\Rightarrow -3 < x+2 < 3 \\ x+2 < 3 &\Rightarrow x < 1 \\ x+2 > -3 &\Rightarrow x > -5 \end{aligned}$$

$$x \in (-5, 1)$$

Method II: Splitting the absolute into two, this is the more general one, and more reliable.

$$|x+2| < 3$$

$$\begin{array}{ccccc} & (x+2) & & 3 & : \; x \geq -2 \\ \text{or} & & < & & \\ & -(x+2) & & 3 & : \; x < -2 \end{array}$$

$$\begin{aligned} (x+2) < 3 \;&\Rightarrow\; x < 1 \;\Rightarrow\; -2 \leq x < 1 \\ \text{or } -(x+2) < 3 \;&\Rightarrow\; x > -5 \;\Rightarrow\; -5 < x < -2 \end{aligned}$$

$$-2 \leq x < 1 \text{ or } -5 < x < -2 \;\Rightarrow\; x \in (-5, 1)$$

Example 1.4.4 Solve the inequality

$$|2x+3| \geq 8$$

Splitting the absolute into two for $|2x+3| \geq 8$

$$\begin{array}{ccccc} & (2x+3) & & 8 & : \; x \geq -\dfrac{3}{2} \\ \text{or} & & \geq & & \\ & -(2x+3) & & 8 & : \; x < -\dfrac{3}{2} \end{array}$$

$$\begin{aligned} (2x+3) \geq 8 \;&\Rightarrow\; x \geq \frac{5}{2} \\ \text{or } -(2x+3) \geq 8 \;&\Rightarrow\; x \leq -\frac{11}{2} \end{aligned}$$

$$\left(-\infty, -\frac{11}{2}\right] \cup \left[\frac{5}{2}, \infty\right)$$

Example 1.4.5 Solve the inequality

$$|-2x-4| \geq 9$$

$$\left(-\infty, -\frac{13}{2}\right] \cup \left[\frac{5}{2}, \infty\right)$$

Example 1.4.6 Solve the inequality

$$x+2 < |x^2-4|$$

This can be rewritten as $|x^2-4| > x+2$
Splitting the absolute into two for $|x^2-4| > x+2$

$$\begin{array}{ccccc} & (x^2-4) & & x+2 & : \ x \in (-\infty,-2] \cup [2,\infty) \\ \text{or} & & > & & \\ & -(x^2-4) & & x+2 & : \ x \in (-2,2) \end{array}$$

$$\begin{array}{rcccccc} (x^2-4) > x+2 & \Rightarrow & x^2-x-6>0 & \Rightarrow & (x+2)(x-3)>0 & \Rightarrow & (-\infty,-2)\cup(3,\infty) \\ -(x^2-4) > x+2 & \Rightarrow & x^2+x-2<0 & \Rightarrow & (x-1)(x+2)<0 & \Rightarrow & (-2,1) \end{array}$$

$$\begin{array}{l} [\{(-\infty,-2]\cup[2,\infty)\} \cap \{(-\infty,-3)\cup(3,\infty)\}] \\ \text{or} \\ [\{(-2,2)\cap(-2,1)\}] \end{array}$$

$$\Rightarrow \quad (-\infty,-2)\cup(-2,1)\cup(3,\infty)$$

Example 1.4.7 Solve the equation

$$4|2x-1|-2=10$$

$$\begin{aligned} 4|2x-1|-2 &= 10 \\ 4|2x-1| &= 12 \\ |2x-1| &= 3 \end{aligned}$$

Splitting the absolute into two for $|2x-1|=3$

$$\begin{array}{ccccc} & (2x-1) & & 3 & : \ x \geq \frac{1}{2} \\ \text{or} & & = & & \\ & -(2x-1) & & 3 & : \ x < \frac{1}{2} \end{array}$$

$$\begin{array}{rcl} (2x-1)=3 & \Rightarrow & x=2 \\ -(2x-1)=3 & \Rightarrow & x=-1 \end{array}$$

$$x=-1, \ x=2$$

Example 1.4.8 Solve for the inequality

$$|3x+1| < 2|x-6|$$

Whenever we have absolutes, we represent it as into two parts,

$$\begin{array}{ccc} (3x+1)_{x\geq -\frac{1}{3}} & & 2(x-6)_{x\geq 6} \\ \text{or} & < & \\ -(3x+1)_{x<-\frac{1}{3}} & & -2(x-6)_{x<6} \end{array}$$

Taking all possible combinations of the terms,

$$\begin{aligned}
[(3x+1)] < [2(x-6)] \quad & \Rightarrow x < -13 : \text{ for region } x \geq 6 \;\Rightarrow\; \text{no solution} \\
[(3x+1)] < [-2(x-6)] \quad & \Rightarrow\; x < \frac{11}{5} \;:\; \text{in region } x \in [-\frac{1}{3}, 6) \;\Rightarrow\; x \in \left[-\frac{1}{3}, \frac{11}{5}\right) \\
[-(3x+1)] < [2(x-6)] \quad & \Rightarrow\; x > \frac{11}{5} \;:\; \text{in no region} \;\Rightarrow\; \text{no solution} \\
[-(3x+1)] < [-2(x-6)] \quad & \Rightarrow x > -13 : \text{(for region) } x < -\frac{1}{3} \;\Rightarrow\; x \in \left(-13, -\frac{1}{3}\right)
\end{aligned}$$

$$x \in \left[-\frac{1}{3}, \frac{11}{5}\right) \cup \left(-13, -\frac{1}{3}\right)$$

Example 1.4.9 Solve the equation

$$|x-2| + |x+5| \leq 0$$

Splitting the absolutes into two for;

$$\begin{array}{cccc} (x-2)_{x\geq 2} & & (x+5)_{x\geq -5} & \\ & + & & \leq 0 \\ -(x-2)_{x<2} & & -(x+5)_{x<-5} & \end{array}$$

Taking all possible combinations of the terms,

$$\begin{aligned}
[(x-2)] + [(x+5)] \;\; & \leq 0 \;\; \Rightarrow x \leq -\frac{3}{2} : \text{ in region } x \geq 2 \;\Rightarrow\; \text{no solution} \\
[(x-2)] + [-(x+5)] \;\; & \leq 0 \;\; \Rightarrow\; \text{no solution} \;:\; \text{in no region} \;\Rightarrow\; \text{no solution} \\
[-(x-2)] + [(x+5)] \;\; & \leq 0 \;\; \Rightarrow\; \text{no solution} \;:\; \text{in region } x \in [-5, 2) \;\Rightarrow\; \text{no solution} \\
[-(x-2)] + [-(x+5)] \;\; & \leq 0 \;\; \Rightarrow x \geq -\frac{3}{2} : \text{ in region (for) } x < -5 \;\Rightarrow\; \text{no solution}
\end{aligned}$$

No solution for the problem above.

Example 1.4.10 Solve the equation

$$|2x-2| = x+1$$

$$x = 3, \; x = \frac{1}{3}$$

Example 1.4.11 Solve for $|x-5| \leq 2$

$$|x-5| = \begin{cases} (x-5)\,, & x \geq 5 \\ -(x-5)\,, & x < 5 \end{cases}$$

(i) $(x-5) \leq 2$ and $x \geq 5, \quad \Rightarrow \quad x \leq 7$ and $x \geq 5, \quad \Rightarrow \quad 5 \leq x \leq 7$

(ii) $-(x-5) \leq 2$ and $x < 5, \quad \Rightarrow \quad x \geq 3$ and $x < 5, \quad \Rightarrow \quad 3 \leq x \leq 4$

(iii) From (i) and (ii), (the union), we realise $3 \leq x \leq 7$

Alternatively, $|x-5| \leq 2 \Rightarrow \quad -2 \leq (x-5) \leq 2 \;\Rightarrow\; 3 \leq x \leq 7$ by adding a 5 everywhere

Example 1.4.12 Solve for $|x+3| \leq 4$

$$|x+3| = \begin{cases} (x+3)\ , & x \geq -3 \\ \\ -(x+3)\ , & x < -3 \end{cases}$$

(i) $(x+3) \leq 4$ and $x \geq -3, \quad \Rightarrow \quad x \leq 1$ and $x \geq -3, \quad \Rightarrow \quad -3 \leq x \leq 1$

(ii) $-(x+3) \leq 4$ and $x < -3, \quad \Rightarrow \quad x \geq -7$ and $x < -3, \quad \Rightarrow \quad -7 \leq x < -3$

(iii) From (i) and (ii), (the union), we realise $-7 \leq x \leq 1$

Alternatively, $|x+3| \leq 4 \quad \Rightarrow \quad -4 \leq (x+3) \leq 4$

$$-7 \leq x \leq 1$$

Example 1.4.13 Solve $|3-2x| \geq 1$.

$$|3-2x| = \begin{cases} (3-2x)\ , & x \leq \frac{3}{2} \\ \\ -(3-2x)\ , & x > \frac{3}{2} \end{cases}$$

(i) $(3-2x) \geq 1$ and $x \leq \frac{3}{2}, \quad \Rightarrow \quad x \leq 1$ and $x \leq \frac{3}{2}, \quad \Rightarrow \quad x \leq 1$

(ii) $-(3-2x) \geq 1$ and $x < \frac{3}{2}, \quad \Rightarrow \quad x \geq 2$ and $x > \frac{3}{2}, \quad \Rightarrow \quad x \geq 2$

(iii) From (i) and (ii), (the union), we realise $x \in (-\infty, 1] \cup [2, \infty)$

Alternatively, Either $3-2x \geq 1$ or $3-2x \leq -1$. That is, either $x \leq 1$ or $x \geq 2$.

$$x \in (-\infty, 1] \cup [2, \infty)$$

Example 1.4.14 Solve $|2x+3| < 6$

$$-\frac{9}{2} < x < \frac{3}{2}$$

Example 1.4.15 Solve $|2x-3| > 5$.

$$x < -1 \text{ or } x > 4$$

Example 1.4.16 Find the absolute-value inequality statement that corresponds to the inequality

$$-2 < x < 4$$

I first look at the endpoints. Negative two and four are six units apart. Half of six is three. So I want to adjust this inequality so that it relates to -3 and 3, instead of to -2 and 4. To accomplish this, I will adjust the ends by subtracting 1 from all three "sides":

$$\begin{aligned} -2 &< x < 4 \\ -2-1 &< x-1 < 4-1 \\ -3 &< x-1 < 3 \end{aligned}$$

Since the last line above is in the "less than" format, the absolute-value inequality will be of the form "absolute value of something is less than 3". I can convert this nicely to

$$|x-1|<3$$

Example 1.4.17 Find the absolute-value inequality statement that corresponds to the inequalities

$$x<19 \text{ or } x>24$$

I first look at the endpoints. Nineteen and 24 are five units apart. Half of five is 2.5. So I want to adjust the inequality so it relates to -2.5 and 2.5, instead of relating to 19 and 24. Since $19-(-2.5)=21.5$ and $24-2.5=21.5$, I need to subtract 21.5 all around:

$$\begin{aligned} x<19 &\text{ or } x>24 \\ x-21.5<19-21.5 &\text{ or } x-21.5>24-21.5 \\ x-21.5<-2.5 &\text{ or } x-21.5>2.5 \end{aligned}$$

Since the last line above is the "greater than" format, the absolute-value inequality will be of the form "absolute value of something is greater than or equal to 2.5". I can convert this nicely to:

$$|x-21.5|>2.5$$

Exercise 1.2 Solve for x in the following:

1. $2x+7>4x-5$
2. $-4\leq 2(x+2)<12$
3. $(x+8)(4x-6)>0$
4. $x^2+x<0$
5. $|x-8|=2$
6. $|x+4|-|x-1|<4$
7. $|x+2|+|x-5|\geq 10.$

1.4.1 Sketching absolute functions

Example 1.4.18 Graph the curve

$$f(x)=\frac{|x+2|x^2}{|x|}$$

Now its not only one point to think of, but now both $x=-2$ and $x=0$. We need to have different functions for

$$\begin{aligned} x&<-2 \\ -2\leq x&\leq 0 \\ x&>0 \end{aligned}$$

$$f(x)=\begin{cases} \frac{-(x+2)x^2}{-(x)}, & \text{if } x<-2 \\ \frac{(x+2)x^2}{-(x)}, & \text{if } -2\leq x\leq 0 \\ \frac{(x+2)x^2}{(x)}, & \text{if } x>0 \end{cases}$$

and sketch those function in the different ranges.

Example 1.4.19 Sketch the graph

$$f(x) = |x-2||x-4|$$

Example 1.4.20 Graph the function

$$f(x) = \frac{x\left(|4x-3| - |x+6| + |x|\right)}{3|x|}$$

Hence or otherwise, solve the inequality

$$\frac{x\left(|4x-3| - |x+6| + |x|\right)}{3|x|} > 0$$

Example 1.4.21 Let

$$g(x) = \begin{cases} x+2\ , & x \geq 0 \\ x-2\ , & x < 0 \end{cases}$$

Sketch the graph of $g(x)$.

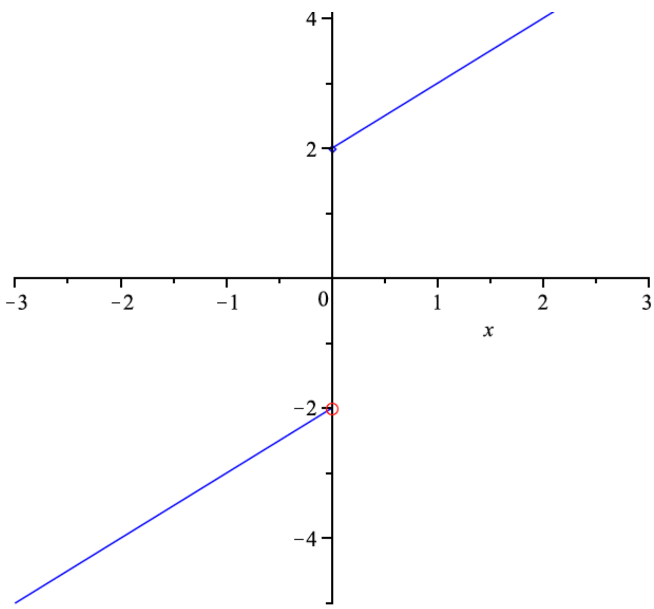


Figure 1.1: A piecewise function sketch

Does $\lim_{x \to 0} g(x)$ exist? Justify your answer.

1.4.2 Some Proofs

Proof of Definition 1.4.2

Since $a^2 = (+a)^2 = (-a)^2$, the numbers $+a$ and $-a$ are square roots of a^2. If $a \geq 0$, then $+a$ is the nonnegative square root of a^2, and if $a < 0$, then $-a$ is the nonnegative square root of a^2. Since $\sqrt{a^2}$ denotes the nonnegative square root of a^2, it follows that

$$\begin{aligned} \sqrt{a^2} &= +a \text{ if } a \geq 0 \\ \sqrt{a^2} &= -a \text{ if } a < 0 \end{aligned}$$

That is, $\sqrt{a^2} = |a|$.

Proof of Theorem 1.4.2 (a)

From Definition 1.4.2

$$|a| = \sqrt{a^2} = \sqrt{(-a)^2} = |-a|$$

Proof of Theorem 1.4.2 (d), the Multiplicativeness property

From Definition 1.4.2 and a basic property of square roots,

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a}\sqrt{b} = |a||b|$$

The result can be extended to three or more factors. More precisely, for any n real numbers, $a1, a_2, \ldots, a_n$, it follows that

$$|a_1a_2\cdots a_n| = |a_1||a_2|\cdots|a_n|$$

In the special case where $a1, a_2, \ldots, a_n$ have the same value, a, it follows that

$$|a^n| = |a|^n$$

Remark 1.4.1 Properties of absolute value often involve a *proof by cases.*

Example 1.4.22 For all $a \in \mathbb{R}$

$$-|a| \leq a \leq |a|. \tag{1.1}$$

Also

$$-|a| \leq -a \leq |a|. \tag{1.2}$$

Proof. If $a \geq 0$, then $|a| = a$, and

$$-|a| \leq 0 \leq a = |a|.$$

If $a \leq 0$, then $-|a| = a$, and

$$-|a| = a \leq 0 \leq |a|.$$

The second statement is shown in a similar manner, or by multiplying each expression in the first statement by -1.

Proof of Theorem 1.4.2 (f), the Triangular inequality

$\forall\ a, b\ \in \mathbb{R}$

$$a \leq |a| \tag{1.3}$$
$$b \leq |b| \tag{1.4}$$

adding equations (1.3) and (1.4) we get

$$(a + b) \leq |a| + |b| \tag{1.5}$$

Similarly,

$$-a \leq |a| \tag{1.6}$$
$$-b \leq |b| \tag{1.7}$$

(equality if $a = 0$ or $b = 0$) adding equations (1.6) and (1.7) we get

$$-(a + b) \leq |a| + |b| \tag{1.8}$$

From (1.5) and (1.8) and the absolute value definition (1.4.1) of a plus and minus

$$|a + b| \leq |a| + |b|$$

Exercise 1.3 For $a, b \in \mathbb{R}$, show that

1. $|a - b| \leq |a| + |b|$
2. $|a + b| \geq |a| - |b|$
3. $|a - b| \geq |a| - |b|$

Example 1.4.23 For $a, b\ \in \mathbb{R}$,

$$||a| - |b|| \leq |a - b| \tag{1.9}$$

By the Triagular inequality,

$$|a| = |a - b + b| \leq |a - b| + |b| \quad \Rightarrow \quad |a| - |b| \leq |a - b|$$

Interchanging the roles of a and b, and noting that $|a| = |-a|$, gives $|b| - |a| \leq |a - b|$. Multiplying this inequality by -1 and combining these we have

$$-|a - b| \leq |a| - |b| \leq |a - b|$$

and this is the required result.

Example 1.4.24 Let $a, b\ \in \mathbb{R}$, prove that

$$\frac{a^2 + b^2}{2} \geq ab$$

The proof is

$$\begin{aligned}(a - b)^2 &\geq 0\\ a^2 - 2ab + b^2 &\geq 0\\ \frac{a^2 + b^2}{2} &\geq ab\end{aligned}$$

Example 1.4.25 Similarly, for $x, y \in \mathbb{R}$ the (Arithmetic Mean - Geometric Mean) property

$$\sqrt{xy} \leq \frac{x+y}{2} \tag{1.10}$$

the proof is to let $x = a^2, y = b^2$ for Example 1.4.24

Example 1.4.26 Let $a, b \in \mathbb{R}_+$, prove that

$$a < b \text{ iff } a^2 < b^2$$

$\Rightarrow$ Multiplying through $a < b$ by a and b to get

$$a^2 < ab \tag{1.11}$$
$$ab < b^2 \tag{1.12}$$

From (1.11) and (1.12) we get $a^2 < b^2$
$\Leftarrow$ from $a^2 < b^2$

$$a^2 < b^2 \quad \Leftrightarrow \quad a^2 - b^2 < 0 \quad \Leftrightarrow \quad (a-b)(a+b) < 0 \tag{1.13}$$

Since $a, b \in \mathbb{R}_+$, $(a+b) > 0$ then from equation 1.13, $(a-b) < 0 \Rightarrow a < b$

Example 1.4.27 *Describe* $\{x \in \mathbb{R} : |5x - 3| > 4\}$

$$\{x \in \mathbb{R} : |5x-3| > 4\} = \left(-\infty, -\frac{1}{5}\right) \cup \left(\frac{7}{5}, \infty\right)$$

Exercise 1.4 *Describe* $\{x \in \mathbb{R} : |x + 3| < 1\}$

Exercise 1.5 *Describe the set* $\{x \in \mathbb{R} : 1 \leq x \leq 3\}$ *using the absolute value function.*

Proof of Theorem 1.4.2 (c), the Non-negativity property

We have three possibilities:

1. If $a = 0$, then $|a| = a = 0$.

2. If $a > 0$, then $|a| = a > 0$.

3. If $a < 0$, then $|a| = -a$ and $-a > 0$ imply $|a| > 0$.

Other important properties of the absolute value include:

(i) $|(|a|)| = |a|$, the Idempotence property (the absolute value of the absolute value is the absolute value).

(ii) $|a| = 0 \;\Leftrightarrow\; a = 0$, the Positive-definiteness property.

(iii) $|a - b| = 0 \;\Leftrightarrow\; a = b$, the Identity of indiscernibles (equivalent to positive-definiteness) property.

Proof of Theorem 1.4.2 (a), the evenness (reflection symmetry of the graph)

Using Definition 1.4.1

$$|-x| = \begin{cases} -x\,, & -x \geq 0 \\ \\ x\,, & -x < 0 \end{cases} = \begin{cases} -x\,, & x \leq 0 \\ \\ x\,, & x > 0 \end{cases} = \begin{cases} x\,, & x > 0 \\ \\ -x\,, & x \leq 0 \end{cases} \qquad = \qquad |x|$$

Equality is always at $|\star| = 0$.

We can also use the product property $|ab| = |a||b|$

$$|-x| = |-1||x| = 1|x| = |x|$$

Remark 1.4.2 Theorem 1.4.2 (e) is the preservation of division property.

1.5 Functions

Definition 1.5.1 *A function is a set of ordered pairs of elements such that not two ordered pairs of a set have the same first element.*

Functions can be considered as mappings. A function is a special kind of relation in which all ordered pairs have unique first elements. Every element of the first set is assigned to an element of the second set. Usually subsets of the set of real numbers $\mathbb{R}$ are considered.

A function involves two sets and a rule of correspondence between them. The rule of correspondence specifies how to pair the elements of the one set with those in the other.

A function is a relation in which for each member of the first set there is a single corresponding member of the second set. A rule exists between two quantities.

A function has to satisfy The Vertical Line Test.

1.5.1 Domain and Range as a set of Ordered Pairs.

The domain (D) is the set of all values that the first elements, in the ordered pair can take. The Range (R) is the set of all values that the second elements, in the ordered pair, can take.

Example 1.5.1 The following table defines a function.

x	-3	5	6	7
y	-2	8	9	5

(a) List the ordered pairs of the function

(b) State the domain of the function

(c) State the range of the function

(a) The ordered pairs of the function are: $(-3, -2); (5, 8); (6, 9) \text{and} (7, 5)$.

(b) The Domain are the set of the first elements of the ordered pairs. $D = \{-3, 5, 6, 7\}$

(c) The Range are the set of the second elements of the ordered pairs. $R = \{-2, 8, 9, 5\}$

Example 1.5.2 State the domain and range of the function $f = \{(-2, 4)(-1, 4)(0, 4)(2, 4)\}$

The Domain $D = \{-2, -1, 0, 2\}$, are the first elements in the ordered pairs. The Range $R = \{4\}$, is the second element in the ordered pairs.

Example 1.5.3 A mapping defined by

$$\{(-1, 2), \quad (4, 6), \quad (5, 0), \quad (4, 3)\}$$

is not a function, since the first element 4 appears twice (more than once).

Note 1.5.1 A function can be given in the form of ordered pairs such as

$$\{(-1, 1), \quad (4, 11), \quad (5, 13), \quad (1, 5)\}$$

or by a formula i.e $f(x) = 2x + 3$

Definition 1.5.2 The domain of a function is the set of "input" or argument values for which the function is defined. That is, the function provides an "output" or "value" for each member of the domain.

Definition 1.5.3 The range is the codomain or the image of the function

Example 1.5.4 Determine the domain and range of the given function:

$$y = \frac{(x^2 + x - 2)}{(x^2 - x - 2)}$$

The domain is all the values that x is allowed to take on. The only problem I have with this function is that I need to be careful not to divide by zero. So the only values that x can not take on are those which would cause division by zero. So I'll set the denominator equal to zero and solve; my domain will be everything else.

$$\begin{aligned} x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x = 2 \text{ or } x = -1 & \end{aligned}$$

Then the domain is "all x not equal to -1 or 2".

Since the graph will eventually cover all possible values of y, then the range is "all real numbers".

Example 1.5.5 Determine the domain and range of the given function:

$$y = -\sqrt{-2x + 3}$$

The domain is all values that x can take on. The only problem I have with this function is that I cannot have a negative inside the square root. So I'll set the insides greater-than-or-equal-to zero, and solve. The result will be my domain:

$$\begin{aligned} -2x + 3 &\geq 0 \\ -2x &\geq -3 \\ 2x &\leq 3 \\ x &\leq \frac{3}{2} = 1.5 \end{aligned}$$

Then the domain is "all $x \leq \frac{3}{2}$".

The range requires a graph. You need to be careful when graphing radicals:The range is "$y < 0$".

Example 1.5.6 Determine the domain and range of the given function:

$$y = -x^4 + 4$$

This is just a garden-variety polynomial. There are no denominators (so no division-by-zero problems) and no radicals (so no square-root-of-a-negative problems). There are no problems with a polynomial. There are no values that I can't plug in for x. When I have a polynomial, the answer is always that the domain is "all x".

The range will vary from polynomial to polynomial, and they probably won't even ask, but when they do, I look at the picture: The graph goes only as high as y = 4, but it will go as high as I like. Then: The range is "all $y \geq 4$".

Example 1.5.7 Find the domain of function f defined by

$$f(x) = \frac{1}{(x-1)}$$

The domain x is such that $x \neq 1$, that is

$$x \in (-\infty, 1) \cup (1, +\infty)$$

Example 1.5.8 what is the domain of the function

$$f(x) = \sqrt{x^2 - 1} \quad ?$$

Well, what could go wrong here? No division is indicated at all, so there is no risk of dividing by 0. But we are taking a square root, so we must insist that $x^2 - 1 \geq 0$ to avoid having complex numbers come up. That is, a preliminary description of the 'domain' of this function is that it is the set of real numbers x so that $x^2 - 1 \geq 0$.

But we can be clearer than this: we know how to solve such inequalities. Often it's simplest to see what to *exclude* rather than *include*: here we want to *exclude* from the domain any numbers x so that $x^2 - 1 < 0$ from the domain.

We recognize that we can factor

$$x^2 - 1 = (x-1)(x+1) = (x-1)\ (x-(-1))$$

This is negative exactly on the interval $(-1, 1)$, so this is the interval we must prohibit in order to have just the domain of the function. That is, the domain is the union of two intervals:

$$(-\infty, -1] \cup [1, +\infty)$$

Exercise 1.6 Find the domain of the function

$$f(x) = \frac{x-2}{x^2 + x - 2}$$

That is, find the largest subset of the real line on which this formula can be evaluated meaningfully.

Exercise 1.7 Find the domain of the function

$$f(x) = \frac{x-2}{\sqrt{x^2 + x - 2}}$$

Exercise 1.8 Find the domain of the function

$$f(x) = \sqrt{x(x-1)(x+1)}$$

Example 1.5.9 Compute the domain of the fuction

$$|x - 3.2| + |x - 5.2| = 2$$

Note 1.5.2 The domain is where the function lives (are the x values), and the range is what the function can be (is the $f(x)$). To get domain, make sure the denominator is not equal to zero, and cannot have a square root of a negative number.

Example 1.5.10 Find the domain and range of the following function

(i)
$$f(x) = \frac{3}{x^2 - 1}$$
$$f(x) = \frac{3}{x^2 - 1} = \frac{3}{(x-1)(x+1)}$$
so function defined every where otherthan at $x = 1$ and $x = -1$. Thus the domain, $D_f = \mathbb{R} - \{-1, 1\}$, the domain are all possible solutions, which is all numbers, i.e $R_f = \mathbb{R}$

(ii) $f(x) = \sqrt{x-4}$, We know that we want only positive entries in the square root sign, that is, $x - 4 \geq 0$, meaning the domain is only values of x, such that $x \geq 4$

(iii) Find the domain of
$$f(x) = \sqrt{x^2 - 1}$$
$$(-\infty, -1] \cup [1, +\infty)$$

(iv)
$$f(x) = \sqrt{\frac{2x}{x+2}}$$
This is just a mix of the two conditions, make sure you don't have zero below, and no negative numbers in the square root. That is $x \neq -2$, and $\frac{2x}{x+2} \geq 0$, $x \geq -2$

Example 1.5.11 Find the domain and range of the function
$$f(x) = \sqrt{\frac{x(x-2)}{(4-3x)(6-2x)}}$$
The domain is
$$x \neq \frac{4}{3},\ x \neq 3,\ \frac{x(x-2)}{(4-3x)(6-2x)} \geq 0$$
The range is $\mathbb{R}^+$ since we have $^+\sqrt{}$. But if had $^-\sqrt{}$, then the range will be $\mathbb{R}^-$.

Remark 1.5.1 We have to first specify either positive or negative square root, otherwise if the range is both positive and negative square roots, it would mean, $f(x)$ is not a function, as it will fail the vertical line test and not a function.

Example 1.5.12 For the function h below, the domain D and range R are
$$h = \left\{(x, y) : y = \sqrt{x} + 1\right\} \quad , \quad D_h = \mathbb{R}^+ \ , \quad R_h = [1, \infty)$$

Example 1.5.13 Find the domain and range of values of the given function
$$f(x) = \frac{x}{|x|}$$
Domain $x \neq 0 \equiv \mathbb{R} - \{0\}$, Range $\{-1, 1\}$

Example 1.5.14 Let
$$f(x) = \frac{7x}{x^2 - 16} \quad , \quad g(x) = \sqrt{x}$$
Find $(f \circ g)(x)$ and give its domain.
$$f \circ g(x) = \frac{7\sqrt{x}}{x - 16} \ , \quad \text{Domain } := \mathbb{R}^+ - 16$$

1.5.2 Function Notation

The notation $f(x)$ represents the second element in the ordered pair that has x as its first element. This ordered pair can be represented as $(x, f(x))$. We read $f(x)$ as "f of x" or "f at x" since $f(x)$ gives the value of f at x. This relation can also be represented as:

$$x \Rightarrow f(x)$$

Often we replace $f(x)$ by y, also called the dependent variable, and write $y = f(x)$. With this notation, x is called the independent variable. For example $f(x) = 3x + 2$ may be written as $y = 3x + 2$.

1.5.3 Evaluation of Functions

The evaluation of a function is illustrated by examples. A rule describing a function can be represented by an equation or formula, a table or a graph, which are recorded as ordered pairs. The evaluation of a function is to find a number that is paired with a given number.

Example 1.5.15 To evaluate the function $f(x) = 3x+2$ at $x = 2$, we need the number $f(2)$. Since $f(x) = 3x + 2$, replace x by 2 in the expression for the function and simplify. Thus $f(2) = 3 \times 2 + 2 = 6 + 2 = 8$. Therefore the ordered pair is $(2, 8)$. We write $f(2) = 8$ or $2 \to 8$. This is the y value in the expression by $y = 3x + 2$ when $x = 2$.

Example 1.5.16 Let $f(x) = x^2 + 6x + 2$ Find the value of

(a) $f(0)$ (c) $f(1)$ (e) $f(n-2)$

(b) $f(-2)$ (d) $f(x+1)$

(a) $f(0) = 0^2 + 6(0) + 2 = 2$. Thus $0 \to 2$ or as an ordered pair, $(0, 2)$.

(b) $f(-2) = (-2)^2 + 6(-2) + 2 = -6$. Thus $-2 \to -6$ or as an ordered pair, $(-2, -6)$.

(c) $f(1) = 1^2 + 6(1) + 2 = 9$. Thus $1 \to 9$ or as an ordered pair, $(1, 9)$.

(d)

$$\begin{aligned} f(x+1) &= (x+1)^2 + 6(x+1) + 2 \\ &= (x^2 + 2x + 1 + 6x + 6 + 2) \\ &= x^2 + 8x + 9. \end{aligned}$$

Thus $(x + 1) \to x^2 + 8x + 9$ or as an ordered pair, $((x + 1),\ (x^2 + 8x + 9))$.

(e)

$$\begin{aligned} f(n-2) &= (n-2)^2 + 6(n-2) + 2 \\ &= n^2 - 4n + 4 + 6n - 12 + 2 \\ &= n^2 + 2n - 6 \text{ or as an ordered pair, } ((n-2), (n^2 + 2n - 6) \end{aligned}$$

Thus $(n - 2) \to n^2 + 2n - 6$

Example 1.5.17 Let $f(x) = 2x^2 - 1$. Let $h > 0$. Find the value of $\frac{f(x+h)-f(x)}{h}$.

$$\begin{aligned}
f(x+h) &= 2(x+h)^2 - 1 \\
&= 2(x^2 + 2hx + h^2) - 1 \\
&= 2x^2 + 4hx + 2h^2 - 1 \\
f(x+h) - f(x) &= (2x^2 + 4hx + 2h^2 - 1) - (2x^2 - 1) \\
&= 4hx + 2h^2 \\
\text{Therefore,} \quad \frac{f(x+h) - f(x)}{h} &= \frac{4hx + 2h^2}{h} \\
&= 4x + 2h
\end{aligned}$$

Example 1.5.18 Given the function $f(x) = 2x + 3$ and $a = 2$. Compute and simplify the value of

$$\frac{f(x) - f(a)}{x - a}$$

with $x \neq a$

Since $f(x) = 2x + 3$ and $a = 2$, $f(a) = f(2) = 7$. In addition $(x - a) = (x - 2)$ and therefore

$$\frac{f(x) - f(a)}{x - a} = \frac{2x + 3 - 7}{x - 2} = \frac{2x - 4}{x - 2} = 2\frac{(x - 2)}{(x - 2)} = 2$$

1.6 Types of functions

1.6.1 Equal Functions

Let $f_1(x)$ and $f_2(x)$ be functions that are defined on the same domain, D. If for each element x of $D, f_1(x) = f_2(x)$, then the two functions are equal and we write $f_1(x) = f_2(x)$. Two functions $f_1(x)$ and $f_2(x)$ are equal if and only if $f_1(x)$ and $f_2(x)$ have the same domains and $f_1(x) = f_2(x)$ for all x in this common domain.

Example 1.6.1 The functions

$$\begin{aligned} f(x) &= (x-2) \\ g(x) &= \frac{(x^2-5x+6)}{(x-3)} \end{aligned}$$

are not equal since $D_f \not\equiv D_g$, although they have the same values other than at $x = 3$.

Example 1.6.2 The functions

$$\begin{aligned} f(x) &= x+5 \\ g(x) &= x-1 \end{aligned}$$

Although $D_f \equiv D_g = \mathbb{R}$, the two functions are not equal since $f(x_0) \neq g(x_0)$ for some $x_0 \in \mathbb{R}$.

Example 1.6.3 Prove that the functions:

(a) $f_1(x) = 2x + 1$ and $f_2(x) = \frac{4x}{2} + 1$ are equal, where the domain for both functions is the set of real numbers R.

Let a be a real number. Then $f_1(a) = 2a + 1 = \frac{4}{2}a + 1 = \frac{4a}{2} + 1 = f_2(a)$. Since a was arbitrary, $f_1(x) = f_2(x)$ for all real numbers. The two functions are therefore equal.

(b) $f_1(x) = \sin x$ and $f_2(x) = \cos\left(\frac{\pi}{2} - x\right)$ are equal, where the domain of for both functions is the set of real numbers.

Let a be a real number. Then

$$f_2(a) = \cos\left(\frac{\pi}{2} - a\right) = \cos\left(\frac{\pi}{2}\right)\cos a + \sin\left(\frac{\pi}{2}\right)\sin a = (0)(\cos a) + (1)(\sin a) = \sin a$$

1.6.2 Identity Function

A function f that associates each member of the domain with itself is called the identity function. The identity function is defined by the equation $f : x \to x$ or $f(x) = x$. The domain and the range of the identity function is the set of real numbers.

The identity function is a special kind of one-to-one and onto function.

1.6.3 Constant Function

A function f that associates each real number in a set A the same fixed number k in the set B is called a constant function. The constant function is defined by the formula $f : x \to k$: $f(x) = k$(a single fixed number).

1.6.4 One-to-One Function

If f is a function of A into B is said to be one-to-one $(1-1)$ if no two elements of A correspond to the same element in B. Each element of the domain has a different image in the Range.

A function is one-to-one if and only if each element in the domain is mapped into a unique element of the co-domain (range).

A one to one function is a function in which every element in the range of the function corresponds with one and only one element in the domain.

For a $1-1$ function, If $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

Example 1.6.4 If $f(a) = f(b)$ implies that $a = b$, then f is $1-1$, show whether or not $g(x) = 3x - 2$ is one-on-one?

$$\begin{aligned} \text{see if } g(a) &= g(b) \text{ implies } a = b \\ 3a - 2 &= 3b - 2 \\ 3a &= 3b \\ a &= b \end{aligned}$$

Thus g is $1-1$.

Example 1.6.5 The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ is injective.

Example 1.6.6 The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is not injective, because (for example) $g(1) = 1 = g(-1)$. However, if g is redefined so that its domain is the non-negative real numbers $[0, +\infty)$, then g is injective.

A one-to-one function is one in which each x has only one y and each y has at most one x to form ordered pairs.

Example 1.6.7 Let the function $f : R \to R$ be defined by the equation $f(x) = 3x + 2$ where R is the set of real numbers. Each real number will be mapped onto a unique image by the function $f(x) = 3x + 2$. Hence f is a one-to-one function.

Example 1.6.8 Let the function $f : R \to R$ be defined by the formula $f(x) = 3x^2 + 2$ Verify whether or not f is a one-to-one function.

The negative values of R are mapped onto the same elements as the corresponding positive elements. For example, when $x = -2, f(-2) = 14$ and $f(2) = 14$. The images of two real numbers -2 and 2 are the same number equal to 14. It follows that f is NOT a one-to-one function.

Definition 1.6.1 A one to one function is also called an injective function.

Example 1.6.9 Which functions below are one to one ?

Function #1 $\{(2, 27), (3, 28), (4, 29), (5, 30)\}$

Function #2 $\{(11, 14), (12, 14), (16, 7), (18, 13)\}$

Function #3 $\{(3, 12), (4, 13), (6, 14), (8, 1)\}$

Relation #1 and Relation #3 are both one-to-one functions.

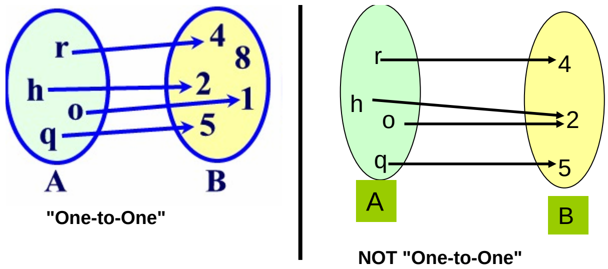


Figure 1.2: 1-1 and not 1-1 respectively

Example 1.6.10 A diagramatical example of $1-1$

Theorem 1.6.1 The Horizontal Line Test: If a function is one to one, then the function not only passes the vertical line test, but it also passes the horizontal line test.

Definition 1.6.2 The Horizontal Line Test : If a horizontal line only intersects with the graph of a function once, then this function is one-to-one. If a horizontal line intersects the graph of the function more than once, then this function is not one to one.

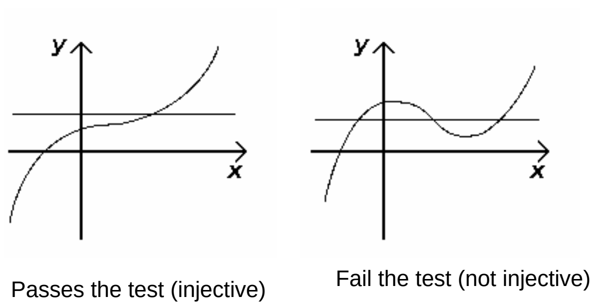


Figure 1.3: A 1-1 and not 1-1 by Horizontal test

Example 1.6.11 Which functions below are one to one ?

Function #1 $\{(2,1),(4,5),(6,7),(8,9)\}$, Function #2 $\{(3,4),(8,5),(6,7),(22,4)\}$

Function #3 $\{(-3,4),(21,-5),(0,0),(8,9)\}$, Function #4 $\{(9,19),(34,5),(6,17),(8,19)\}$

Relation #1 and Relation #3 are both one-to-one functions.

1.6.5 Many-to-One Function

If f is a relation of A into B. If more than one element of A is mapped into the same element of B then f is a many-to-one function.

1.6.6 Onto-Functions

If f is a function of A into B. Sometimes the range, $f(A)$ does not exhaust all the elements of the set B called the co-domain. The range is therefore a subset of the co-domain. If each member of the co-domain is an image of at least one member of A then f is a function of A ONTO B and is therefore an onto function.

Definition 1.6.3 A function f from A to B is called *onto* if for all b in B there is an a in A such that $f(a) = b$. All elements in B are used. Such functions are referred to as **surjective**.

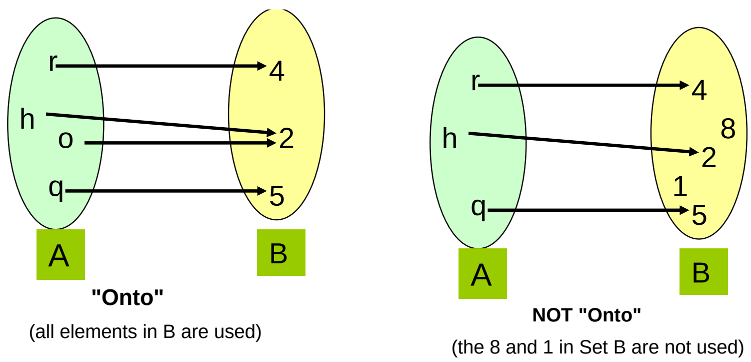


Figure 1.4: An "onto" and "Not onto" functions

Remark 1.6.1 A function is said to be onto if all in range is an image (is a result of the transformation - mapping)

Example 1.6.12 Is $f(x) = 3x - 4$ onto where $f : \mathbb{R} \to \mathbb{R}$?

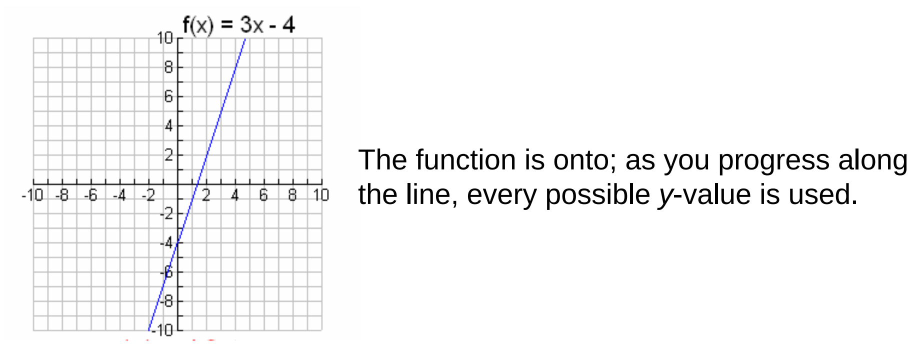


Figure 1.5: An onto function

Example 1.6.13 Is $g(x) = x^2 - 2$ onto where $g : \mathbb{R} \to \mathbb{R}$??

1.6.7 Bijective Function

A Bijective, is a function that is both $1-1$ and onto.

Example 1.6.14 Example (1.6.12) is $1-1$ and Surjective, thus Bijective.

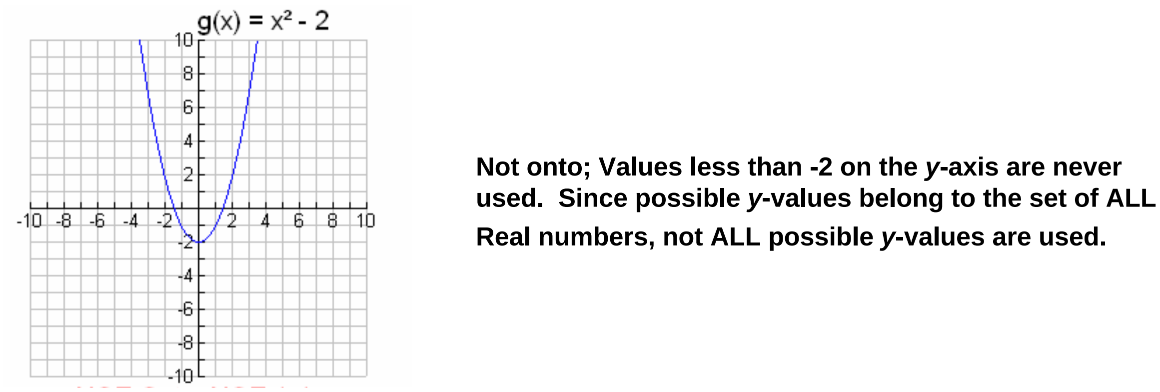


Figure 1.6: A Not onto function

Example 1.6.15 Example (1.6.13) is not $1-1$ and not "onto"

Definition 1.6.4 Bijections are functions that are both injective and surjective.

Example 1.6.16 Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = \sqrt{x}$. This function is an injection and a surjection and so it is also a bijection.

Example 1.6.17 Describe the four functions below as injective, onto or bijective.

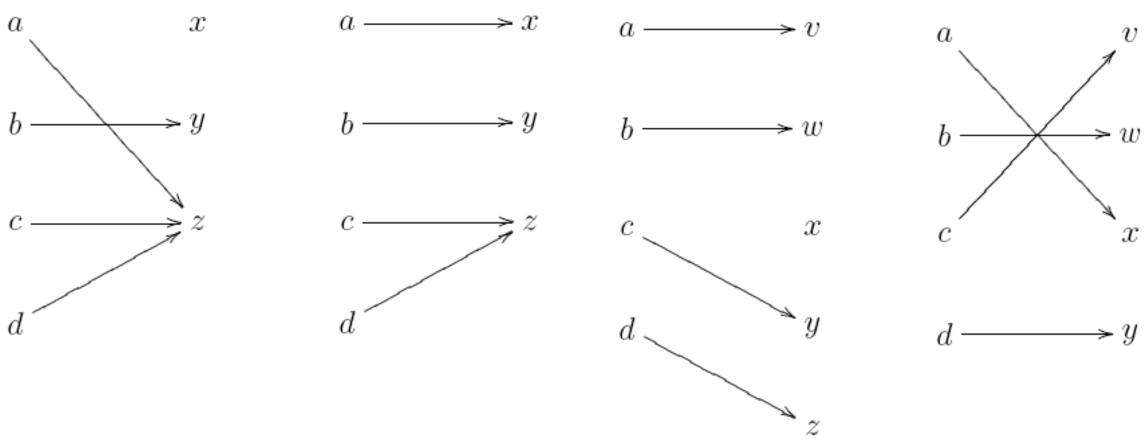


Figure 1.7: Mixed functions

(i) the function is neither injective nor surjective.

(ii) is a surjection, but not an injection

(iii) is an injection, but not a surjection.

(iv) is both a surjection and an injection, and therefore a bijection.

Example 1.6.18 Prove that the function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$ is injective.

Proof: Let $a, b \in \mathbb{N}$ be such that $f(a) = f(b)$. This implies $a^2 = b^2$ by the definition of f. Thus $a = b$ or $a = -b$. Since the domain of f is the set of natural numbers (positive integers), both a and b must be non-negative. Thus $a = b$. This shows

$$\forall a, \forall b, \ [f(a) = f(b) \Rightarrow a = b]$$

which shows f is injective.

Example 1.6.19 Prove that the function $g : \mathbb{N} \to \mathbb{N}$, defined by $g(n) = \frac{n}{3}$, is surjective.

Proof: Let $n \in \mathbb{N}$. Notice that $g(3n) = \frac{3n}{3} = n$.
Since $3n \in \mathbb{N}$, this shows n is in the range of g. Hence g is surjective.

Example 1.6.20 Prove that the function $g : \mathbb{N} \to \mathbb{N}$, defined by $g(n) = 0$, is not injective.

Proof: The numbers 1 and 2 are in the domain of g and are not equal, but $g(1) = g(2) = 0$. Thus g is not injective.

Example 1.6.21 Prove that the function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$, is not surjective.

Proof: The number 3 is an element of the codomain, $\mathbb{N}$. However, 3 is not the square of any integer. Therefore, there is no element of the domain that maps to the number 3, so f is not surjective.

1.6.8 Even Functions

A function f is called an even function if it maps the negative of x in the domain to the same image as x. This can be defined as

$$f(-x) = f(x) \tag{1.14}$$

for all x in the domain.

Example 1.6.22 The function $f(x) = x^2 + 2$ is even since

$$f(-x) = (-x)^2 + 2 = x^2 + 2 = f(x)$$

Exercise 1.9 Show that the function

(a) $f(x) = \cos x$ is even.

(b) $f(x) = x^2$ is even.

(c) $f(x) = -x^2$ is not even.

(d) $f(x) = (x + 1)^2$ is not even.

1.6.9 Odd Functions

A function f is called an <u>Odd</u> function if it maps the negative of x to the negative of $f(x)$ for all x in the domain. That is, if

$$f(-x) = -f(x) \tag{1.15}$$

Example 1.6.23 The function $f(x) = x^3 + x$ is odd since

$$f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x)$$

Example 1.6.24 Determine whether the function $f(x) = x^2 - x$ is even, odd or neither.

$$\begin{aligned} f(-x) &= (-x)^2 - (-x) \\ &= x^2 + x \\ &\neq f(x) \\ &\neq -f(x) \end{aligned}$$

Thus neither even nor odd.

Example 1.6.25 The function $f(x) = x^3 - x$ is an odd function since

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -\left[x^3 - x\right] = -f(x)$$

Exercise 1.10 Show that the function

(a) $f(x) = x^3$ is odd.

(b) $h(x) = x + 1$ is neither even nor odd.

(c) $f(x) = \sin x$ is an odd function.

(d) $g(x) = x^3 + 1$ is not an odd function.

(e) $h(x) = \frac{x}{x^2-1}$ is an odd function.

(f) $f(x) = x^4(4 + x^3)$ is neither even nor odd.

1.6.10 Increasing and Decreasing functions

Definition 1.6.5 A function $f(x)$ is said to be increasing if

$$\text{when } x_1 < x_2 \text{ then } f(x_1) \leq f(x_2)\ ,\ \forall x \in \mathbb{R}$$

Definition 1.6.6 A function $f(x)$ is said to be strictly increasing if

$$\text{when } x_1 < x_2 \text{ then } f(x_1) < f(x_2)\ ,\ \forall x \in \mathbb{R}$$

Definition 1.6.7 A function $f(x)$ is said to be decreasing if

$$\text{when } x_1 < x_2 \text{ then } f(x_1) \geq f(x_2)\ ,\ \forall x \in \mathbb{R}$$

Definition 1.6.8 A function $f(x)$ is said to be strictly decreasing if

$$\text{when } x_1 < x_2 \text{ then } f(x_1) > f(x_2)\ ,\ \forall x \in \mathbb{R}$$

Remark 1.6.2 A function is "increasing" when the y-value increases as the x-value increases. Similarly, a function is said to be decreasing if the y-value decreases as the x-value increases.

Example 1.6.26 For the function $f(x) = x^3 - 4x$, for x in the interval $[-1, 2]$

- the curve decreases in the interval $[-1, \approx 1.2]$
- the curve increases in the interval $[\approx 1.2, 2]$

Example 1.6.27 The function $f(x) = x + 1$ is an increasing function on $\mathbb{R}$.

Example 1.6.28 The function $f(x) = x^2$ is a increasing function on $\mathbb{R}^+$ and decreasing on $\mathbb{R}^-$. A sketch of the parabola is very useful.

1.7 Inverse of functions

An inverse of a relation that satisfies the definition of a function is called an inverse function.

The inverse of a function f is denoted by f^{-1}.

Note 1.7.1 An inverse **exists if** the function is one-to-one and onto.

Let A and B be any sets and f be a function from A into B, that is, $f : A \to B$. The inverse function maps elements in B into those in A, that is $f^{-1} : B \to A$. The domain of the function f is the range of f^{-1} and the range of f is the domain of f^{-1}.

Definition 1.7.1 Suppose $f : A \to B$ is a bijection. Then the inverse of f, denoted

$$f^{-1} : B \to A$$

is the function defined by the rule

$$f^{-1}(y) = x \text{ if and only if } f(x) = y$$

1.7.1 Finding the Inverse Function

If we are given a function that is one-to-one and onto then we can find its inverse. If a relation is defined by an equation, the inverse can be obtained by

1. interchanging x and y in the equation,
2. solving the new equation with y as the subject of the equation, that is, $y = \cdots$ and
3. replacing y by $f^{-1}(x)$.

This gives the inverse function.

Example 1.7.1 Find the inverse of the function $f(x) = 2x + 4$ if it exists.

This function is surely one-to-one and onto (verify this). It is therefore possible to find its inverse. The function $f(x) = 2x + 4$ can be written as $y = 2x + 4$. Now, interchanging x and y gives $x = 2y + 4$. Making y the subject of the equation gives $y = \frac{x-4}{2}$. The inverse function $f^{-1}(x) = \frac{x-4}{2} = \frac{1}{2}x - 2$.

Example 1.7.2 Suppose $f : \mathbb{R} - (2) \to \mathbb{R} - (1)$ is defined by

$$f(x) = \frac{x}{x-2}$$

Then the function

$$f^{-1}(x) = \frac{2x}{x-1}$$

is the inverse of f. Show.

Exercise 1.11 Find the inverse of the function $f : \mathbb{R} - (2) \to \mathbb{R} - (1)$ defined by

$$f(x) = \frac{x-1}{x+2}$$

Exercise 1.12 Find the inverse of the function $f : \mathbb{R} \to (-\infty, 1)$ defined by $f(x) = 1 - e^x$.

Theorem 1.7.1 Let A and B be any sets and the function $f : A \to B$ be one-to-one and onto (the inverse function exists) $(f^{-1}of) : A \to A$ is the identity function on A similarly the composite function $(fof^{-1}) : B \to B$ is the identity function on B.

Proof: Let $a \in A$. Then $(f^{-1}of)(a) = f^{-1}(f(a)) = f^{-1}(b)$ for some $b \in B$. Now we have $f(a) = b$ which implies that $f^{-1}(b) = a$. It follows that $(f^{-1}of)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$ and therefore that $(f^{-1}of) : A \to A$ is the identity function on A.

Similarly if $b \in B$, then $(fof^{-1})(b) = f(f^{-1}(b)) = f(a)$ for some a satisfying $f^{-1}(b) = a$ and $f(a) = b$. Hence $(fof^{-1})(b) = f(f^{-1}(b)) = f(a) = b$. Therefore (fof^{-1}) is the identity function on B.

Theorem 1.7.2 *If a function is a bijection, then its inverse is also a bijection.*

Example 1.7.3 The function $f(x) = x^2$ does not have an inverse, since it is not a bijection. Whenever required to compute an inverse, need to first check whether it is bijective.

Exercise 1.13 Compute the inverse of the bijective function $h(x) : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \frac{3x+2}{x-1}\ , & x \neq 1 \\ 3\ , & x = 1 \end{cases}$$

1.8 Operations of functions

1.8.1 Sums of functions

Given two functions $g(x), f(x)$, their sum denoted by $(f+g)(x)$ is defined as

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (g+f)(x) &= g(x)+f(x)\end{aligned}$$

Example 1.8.1 Given that $f(x)=3x$ and $g(x)=x^2$, then

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)=3x+x^2\\ (g+f)(x) &= g(x)+f(x)=x^2+3x\end{aligned}$$

Example 1.8.2 Given that $f(x)=2x$ and $g(x)=3x$, then

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)=2x+3x=5x\\ (g+f)(x) &= g(x)+f(x)=3x+2x=5x\end{aligned}$$

Example 1.8.3 Given $f_1=5x+2$ and $f_2=x^2+4$. Find $(f_1+f_2)(x)$.

$$\begin{aligned}(f_1+f_2)(x) &= f_1(x)+f_2(x)\\ &= (5x+2)+(x^2+4)\\ &= x^2+5x+6\end{aligned}$$

1.8.2 Difference of functions

Given two functions $g(x), f(x)$, their difference denoted by $(f-g)(x)$ is defined as

$$\begin{aligned}(f-g)(x) &= f(x)-g(x)\\ (g-f)(x) &= g(x)-f(x)\end{aligned}$$

Example 1.8.4 Given that $f(x)=3x$ and $g(x)=x^2$, then

$$\begin{aligned}(f-g)(x) &= f(x)-g(x)=3x-x^2\\ (g-f)(x) &= g(x)-f(x)=x^2-3x\end{aligned}$$

Example 1.8.5 Given that $f(x)=2x$ and $g(x)=3x$, then

$$\begin{aligned}(f-g)(x) &= f(x)-g(x)=2x-3x=-x\\ (g-f)(x) &= g(x)-f(x)=3x-2x=x\end{aligned}$$

Example 1.8.6 Let $f_1(x)=3x^2+5$ and $f_2(x)=\frac{1}{x}+2$. Evaluate $(f_1-f_2)(x)$.

$$\begin{aligned}(f_1-f_2)(x) &= f_1(x)-f_2(x)\\ &= \left[3x^2+5\right]-\left(\frac{1}{x}+2\right)\\ &= 3x^2-\frac{1}{x}+3\end{aligned}$$

1.8.3 Product of functions

Given two functions $g(x), f(x)$, their product denoted by $(f \cdot g)(x)$ or simply $(fg)(x)$ is defined as

$$\begin{aligned} (f \cdot g)(x) &= f(x) \cdot g(x) \\ (g \cdot f)(x) &= g(x) \cdot f(x) \end{aligned}$$

Example 1.8.7 Given that $f(x) = 3x$ and $g(x) = x^2$, then

$$\begin{aligned} (f \cdot g)(x) &= f(x) \cdot g(x) = 3x \cdot x^2 = 3x^3 \\ (g \cdot f)(x) &= g(x) \cdot f(x) = x^2 \cdot 3x = 3x^3 \end{aligned}$$

Example 1.8.8 Given that $f(x) = 2x$ and $g(x) = 3x$, then

$$\begin{aligned} (f \cdot g)(x) &= f(x) \cdot g(x) = 2x \cdot 3x = 6x^2 \\ (g \cdot f)(x) &= g(x) \cdot f(x) = 3x \cdot 2x = 6x^2 \end{aligned}$$

Example 1.8.9 If $f_1(x) = x^2 - 4$ and $f_2(x) = x - 2$ find $(f_1 \cdot f_2)(x)$.

$$\begin{aligned} (f_1 \cdot f_2)(x) &= f_1(x) \cdot f_2(x) \\ &= (x^2 - 4)(x - 2) \\ &= x^3 - 2x^2 - 4x + 8 \end{aligned}$$

1.8.4 Quotient of functions

Given two functions $g(x), f(x)$, their quotient denoted by $(f/g)(x)$ is defined as

$$\begin{aligned} \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\ ,\ g(x) \neq 0 \\ \left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)}\ ,\ f(x) \neq 0 \end{aligned}$$

Example 1.8.10 Given that $f(x) = 3x$ and $g(x) = x^2$, then

$$\begin{aligned} \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} = \frac{3x}{x^2} = \frac{3}{x}\ ,\ x \neq 0 \\ \left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} = \frac{x^2}{3x} = \frac{1}{3}x\ ,\ x \neq 0 \end{aligned}$$

Example 1.8.11 Given that $f(x) = 2x$ and $g(x) = 3x$, then

$$\begin{aligned} \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} = \frac{2x}{3x} = \frac{2}{3}\ ,\ x \neq 0 \\ \left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} = \frac{3x}{2x} = \frac{3}{2}\ ,\ x \neq 0 \end{aligned}$$

The quotient is sometimes referred to as a rational function.

Example 1.8.12 If $f_1(x) = x^2 - 4$ and $f_2(x) = x - 2$. Evaluate $(f_1/f_2)(x)$.

$$\begin{aligned} \left(\frac{f_1}{f_2}\right)(x) &= f_1(x) \div f_2(x) \\ &= (x^2 - 4) \div (x - 2) \\ &= \frac{(x+2)(x-2)}{(x-2)}\ ,\ x \neq 2 \\ &= (x + 2) \end{aligned}$$

1.8.5 Composite functions

Given two functions $g(x), f(x)$, their composite function denoted by $(f \circ g)(x)$ is defined as

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ (g \circ f)(x) &= g(f(x))\end{aligned}$$

Example 1.8.13 Given that $f(x) = 3x$ and $g(x) = x^2$, then

$$\begin{aligned}(f \circ g)(x) &= f\left(x^2\right) = 3x^2 \\ (g \circ f)(x) &= g(3x) = (3x)^2 = 9x^2\end{aligned}$$

Example 1.8.14 Given that $h(x) = 2x$ and $g(x) = 5x$, then

$$\begin{aligned}(h \circ g)(x) &= h(5x) = 2(5x) = 10x \\ (g \circ h)(x) &= g(2x) = 5(2x) = 10x\end{aligned}$$

Example 1.8.15 Let $f_1(x) = x^2+3x+2$ and $f_2(x) = x+3$. Compute and simplify (if possible) $(f_2 \circ f_1)(x)$ and $(f_1 \circ f_2)(x)$.

$$\begin{aligned}(f_2 \circ f_1)(x) &= f_2(f_1(x)) \\ &= f_2(x^2 + 3x + 2) \\ &= (x^2 + 3x + 2) + 3 \\ &= x^2 + 3x + 5\end{aligned}$$

To compute $(f_1 \circ f_2)(x)$, substitute f_2 into f_1

$$\begin{aligned}(f_1 \circ f_2)(x) = f_1(f_2(x)) & \\ &= f_1(x + 3) \\ &= (x + 3)^2 + 3(x + 3) + 2 \\ &= x^2 + 6x + 9 + 3x + 9 + 2 \\ &= x^2 + 9x + 20\end{aligned}$$

Note that in this case commutativity does not hold that is,

$$(f_2 \circ f_1)(x) \neq (f_1 \circ f_2)(x).$$

Example 1.8.16 Let $f_1(x) = \frac{1}{x}$ and $f_2(x) = x^2$. Compute and simplify (if possible) $(f_2 \circ f_1)(x)$ and $(f_1 \circ f_2)(x)$.

$$\begin{aligned}(f_2 \circ f_1)(x) = f_2(f_1(x)) & \\ &= f_2\left(\frac{1}{x}\right) \\ &= \frac{1}{x^2} \\ \text{whereas} \quad (f_1 \circ f_2)(x) &= f_1(f_2(x)) \\ &= f_1(x^2) \\ &= \frac{1}{x^2}\ , \ x \neq 0\end{aligned}$$

Note that in this case the composite functions commute, that is,

$$(f_2 \circ f_1)(x) = (f_1 \circ f_2)(x)$$

Theorem 1.8.1 *The composition of two injective functions is injective.*

1.8.6 Domain of Composite, Difference, addition and product of functions

Definition 1.8.1 If we denote a domain as D, we define the domains

(i) $D_{f+g} = D_{f-g} = D_{fg} = D_f \cap D_g$

(ii) $D_{f/g} = (D_f \cap D_g) \setminus \{x : g(x) = 0\}$

(iii) $D_{f \circ g} = \{x \in D_g : g(x) \in D_f\}$.
Since $g(x)$ is a domain of $f \circ g$, then domain of g is domain of the composite, where $g(x)$ is in domain of $f \circ g$.

Note 1.8.1 Finding the domain of a composite function consists of two steps:

Step 1. Find the domain of the "inside" (input) function. If there are any restrictions on the domain, keep them.

Step 2. Construct the composite function. Find the domain of this new function. If there are restrictions on this domain, add them to the restrictions from Step 1. If there is an overlap, use the more restrictive domain (or the intersection of the domains). The composite may also result in a domain unrelated to the domains of the original functions.

Example 1.8.17 Let function $f(x) = \sqrt{x+1}$, function $g(x) = \frac{1}{x}$, and function $h(x) = x + 3$. Find an equation defining each function and state the domain.

(1) $f + g$

$$f + g = \sqrt{x+1} + \frac{1}{x}$$
$$D_f : x \geq -1\ , \quad D_g : \ x \neq 0$$

The domain of the sum

$$\begin{aligned} D_{f+g} &\equiv D_f \cap D_g \\ &\equiv [-1, \infty) \cap \{\mathbb{R} - \{0\}\} \\ &\equiv [-1, \infty) - \{0\} \\ D_{f+g} &\equiv [-1, 0) \cup (0, +\infty) \end{aligned}$$

(2) $f - g$

$$f - g = \sqrt{x+1} - \frac{1}{x}$$
$$D_f : x \geq -1\ , \quad D_g : \ x \neq 0 \qquad D_{f-g} : \ [-1, 0) \cup (0, +\infty)$$

(3) $f \cdot g$

$$f \cdot g = \sqrt{x+1} \cdot \left(\frac{1}{x}\right) = \frac{\sqrt{x+1}}{x}$$
$$D_f : x \geq -1\ , \quad D_g : \ x \neq 0 \qquad D_{f \cdot g} : \ [-1, 0) \cup (0, +\infty)$$

(4) $\frac{f}{h}$

$$\frac{f}{h} = \frac{\sqrt{x+1}}{x+3}$$

$$D_f : x \geq -1\ , \quad D_h :\ \Re, \quad h(x) \neq 0\ \rightarrow\ x \neq -3 \qquad D_{f/h} :\ [-1, +\infty)$$

Example 1.8.18 Given

$$f(x) = x^2 + 2 \text{ and } \ g(x) = \sqrt{3-x}$$

The composite functions

$$\begin{aligned} f \circ g(x) = f\,(g(x)) &= \left(\sqrt{3-x}\right)^2 + 2 = 5 - x \\ g \circ f(x) = g\,(f(x)) &= \sqrt{3 - (x^2+2)} = \sqrt{1-x^2} \end{aligned}$$

The domain for $g(x) = \sqrt{3-x}$ is $x \leq 3$.
The domain for $f(g(x)) = 5 - x$ is all real numbers, but you must keep the domain of the inside function. So the domain for the composite function is also $x \leq 3$.

The domain for $f(x) = x^2 + 2$ is all real numbers.
The domain for the composite function $g(f(x)) = \sqrt{1-x^2}$ is $-1 \leq x \leq 1$. The input function $f(x)$ has no restrictions, so the domain of $g(f(x))$ is determined only by the composite function. So the domain is $-1 \leq x \leq 1$.

Example 1.8.19 Find $f \circ g$ and $g \circ f$ and the domain of each, where

$$f(x) = \frac{1-x}{3x} \quad \text{and} \quad g(x) = \frac{1}{1+3x}$$

$f \circ g$: Step 1. What is the domain of the inside function $g(x)$ $\qquad$ $x \neq -\frac{1}{3}$ Keep this!!
Step 2. The composite

$$f(g(x)) = \frac{1 - \left(\frac{1}{1+3x}\right)}{3\left(\frac{1}{1+3x}\right)} = \frac{\frac{3x}{1+3x}}{\frac{3}{1+3x}} = x$$

This function puts no additional restrictions on the domain, so the composite domain is $x \neq -\frac{1}{3}$.

$g \circ f$: Step 1. What is the domain of the inside function $f(x)$? $\qquad$ $x \neq 0$ Keep this!!
Step 2. The composite

$$g(f(x)) = \frac{1}{1 + 3\frac{1-x}{3x}} = \frac{1}{\frac{1}{x}} = x$$

This function puts <u>no additional</u> restrictions on the domain, so the composite domain is $x \neq 0$.

Example 1.8.20 Find $f \circ g$ and $g \circ f$ and the domain of each, where

$$f(x) = \frac{3x}{x-1} \quad \text{and} \quad g(x) = \frac{2}{x}$$

$f \circ g$: Step 1. What is the domain of the inside function $g(x)$ $\qquad$ $x \neq 0$ Keep this!!
Step 2. The composite

$$f(g(x)) = \frac{3\left(\frac{2}{x}\right)}{\left(\frac{2}{x}\right) - 1} = \frac{\frac{6}{x}}{\frac{2-x}{x}} = \frac{6}{2-x} \qquad \text{Domain :} \quad x \neq 2$$

Combine this domain with the domain from Step 1: the composite domain is $x \neq 0$ and $x \neq 2$

$g \circ f$: Step 1. What is the domain of the inside function $f(x)$? $x \neq 1$ Keep this!!
Step 2. The composite

$$g(f(x)) = \frac{2}{\frac{3x}{x-1}} = \frac{2(x-1)}{3x} \quad \text{Domain : } \quad x \neq 0$$

Combine this domain with the domain from Step 1: the composite domain is $x \neq 1$ and $x \neq 0$

Example 1.8.21 Find $f \circ g$ and $g \circ f$ and the domain of each, where

$$f(x) = \sqrt{x-2} \text{ and } g(x) = \sqrt{x^2-1}$$

$f \circ g$: Step 1. What is the domain of the inside function $g(x)$ $x \geq 1$ or $x \leq -1$ Keep this!!
Step 2. The composite

$$f(g(x)) = \sqrt{\sqrt{x^2-1}-2}$$

The domain of this function is where

$$\sqrt{x^2-1} \geq 2 \Rightarrow x^2 - 1 \geq 4 \Rightarrow x^2 \geq 5 \Rightarrow x \geq \sqrt{5} \text{ or } x \leq -\sqrt{5}$$

This function has a more restrictive domain than $g(x)$, so the composite domain is

$$D_{f \circ g} := x \geq \sqrt{5} \text{ or } x \leq -\sqrt{5}$$

We can also show that

$$D_{g \circ f} := x \geq 3$$

Exercise 1.14 Find $f \circ g$ and $g \circ f$ and the domain of each for the following functions.

1. $f(x) = x + 3$ $g(x) = \sqrt{9 - x^2}$

$$\begin{aligned} f \circ g(x) &= \sqrt{9 - x^2} + 3 \quad && \text{Domain : } \quad -3 \leq x \leq 3 \\ g \circ f(x) &= \sqrt{-x^2 - 6x} \quad && \text{Domain : } \quad -6 \leq x \leq 0 \end{aligned}$$

2. $f(x) = \sqrt{x+3}$ $g(x) = 2x - 5$

$$\begin{aligned} f \circ g(x) &= \sqrt{2x-2} \quad && \text{Domain : } \quad x \geq 1 \\ g \circ f(x) &= 2\sqrt{x+3} - 5 \quad && \text{Domain : } \quad x \geq -3 \end{aligned}$$

3. $f(x) = \frac{-3}{x}$ $g(x) = \frac{x}{x-2}$

$$\begin{aligned} f \circ g(x) &= -\frac{3(x-2)}{x} \quad && \text{Domain : } \quad x \neq 2 \text{ and } x \neq 0 \\ g \circ f(x) &= \frac{3}{3+2x} \quad && \text{Domain : } \quad x \neq 0 \text{ and } x \neq -3/2 \end{aligned}$$

4. $f(x) = x^2 + 2$ $g(x) = \sqrt{x-5}$

$$\begin{aligned} f \circ g(x) &= x - 3 \quad && \text{Domain : } \quad x \geq 5 \\ g \circ f(x) &= \sqrt{x^2 - 3} \quad && \text{Domain : } \quad x \geq \sqrt{3} \text{ or } x \leq -\sqrt{3} \end{aligned}$$

5. $f(x) = \frac{2}{x-3}$ $g(x) = \frac{5}{x+2}$

$$f \circ g(x) = -\frac{2(x+2)}{3x+1} \qquad \text{Domain}: \quad x \neq -2 \text{ and } \quad x \neq -1/3$$
$$g \circ f(x) = \frac{5(x-3)}{2x-4} \qquad \text{Domain}: \quad x \neq 3 \text{ and } \quad x \neq 2$$

Example 1.8.22 Let $f(x) = \sqrt{x+3}$ and $g(x) = \sqrt{16-x^2}$, find

(i) $D_f = [-3, \infty)$

(ii) $D_g = [-4, 4]$

(iii) $D_f \cap D_g = [-3, \infty) \cap [-4, 4] = [-3, 4]$

(iv) $D_{f+g} = D_{f-g} = D_{fg} = [-3, 4]$

(v) $D_{f/g} = (D_f \cap D_g) \setminus \{x : g(x) = 0\} = [-3, 4] \setminus \{-4, 4\} = [-3, 4)$

Example 1.8.23 Given the functions $f(x) = \sqrt{x}$ and $g(x) = x^2 + 5$

(i) $(f \circ g)(x) = f(g(x)) = f(x^2 + 5) = \sqrt{x^2 + 5}$

(ii) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 + 5 = x + 5$

(iii) $D_{f \circ g} = \{x \in D_g : g(x) \in D_f\}$. Since $D_g = \mathbb{R}$ and $g(x) \in D_f$, so $D_{f \circ g} = \mathbb{R}$

(iv) $D_{g \circ f} = \{x \in D_f : f(x) \in D_g\}$. Since $D_f = x \geq 0 = \mathbb{R}^+ = [0, \infty)$ and $f(x) \in D_g$, so $D_{f \circ g} = [0, \infty)$

Example 1.8.24 Find $(fog)(-2)$ given

$$f(x) = -3x + 2, \; g(x) = |x - 4|$$

Ans= -16 [Note that we only take $|x - 4| = -(x - 4)$ since $x = -2 < 4$]

Example 1.8.25 Find $(fog)(x)$ and the domain of fog, given

$$f(x) = \frac{(x-1)}{(x+2)}, \quad g(x) = \frac{(x+1)}{(x-2)}$$

Ans: $(fog)(x) = 3/(3x - 3)$. The domain of fog is: $(-\infty, 1) \cup (1, 2) \cup (2, +\infty)$

Example 1.8.26 Let function $f(x) = \sqrt{x+1}$, function $g(x) = \frac{1}{x}$, and function $h(x) = x + 3$. Find $g \circ f \circ h$ and state the domain.

$$g \circ f \circ h(x) = \frac{1}{\sqrt{x+4}} \quad ; \quad x > -4$$

1.8.7 Other functions

Definition 1.8.2 A function is said to be **piecewise** defined if its defined by applying different formulas to the different parts of its domain.

Example 1.8.27 An example of a piecewise function in t,

$$g(t) = \begin{cases} t-1\ , & t \leq -3 \\ 2t^3\ , & -3 < t \leq 9 \\ 4-6t, & t > 9 \end{cases}$$

Example 1.8.28 An example of a piecewise function in x,

$$f(x) = \begin{cases} x^2+1\ , & x \geq 0 \\ -x-9\ , & x < 0 \end{cases}$$

Definition 1.8.3 A function f is said to be a **power function** if it has the form $f(x) = x^n$, for some $n \in \mathbb{N}$.

Definition 1.8.4 A function f is said to be a **quadratic function** if it can be written in the form

$$f(x) = Ax^2 + Bx + C$$

Definition 1.8.5 A **polynomial function** f is one that can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are constants and n is a positive integer.
This integer n is then called the *degree of the polynomial*, provided $a_n \neq 0$.

Definition 1.8.6 A function g is said to be a **rational function** if it is of the type

$$g(x) = \frac{f(x)}{h(x)}$$

where both f and h are polynomial functions.

Definition 1.8.7 A function f is said to be **periodic**, with period T, if

$$f(x+nT) = f(x)$$

for all x in its domain and for $n = 1, 2, \ldots$

That means the function repeats itself as the rate of T intervals of x. For example the function $f(x) = \sin x$ is periodic with period 2π since $\sin(x + 2n\pi) = \sin x$ for all n = 1, 2, . .

Definition 1.8.8 A function f is said to be strictly **monotonic increasing** in the interval (a, b) if, for all pairs of numbers x_1, x_2 in (a, b),

$$f(x_1) < f(x_2) \text{ when } x_1 < x_2$$

Definition 1.8.9 The **floor function** or **greatest integer function** is the function defined as follows: for a real number x, the floor of x, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x.

The floor of x is sometimes referred to as the integer part or integral value of x. (Is the integer just below - on the left)

Example 1.8.29

(i) $\lfloor 5 \rfloor = 5$

(ii) $\lfloor 1.78 \rfloor = 1$

(iii) $\lfloor 0 \rfloor = 0$

(iv) $\lfloor -1.6 \rfloor = -2$

(v) $\lfloor -3.5 \rfloor = -4$

(vi) $\lfloor -99.9 \rfloor = -100$

Example 1.8.30

$$\lfloor 1.7 \rfloor = 1, \ \lfloor \pi \rfloor = 3 \text{ and } \lfloor -3.2 \rfloor = -4$$

The floor of x satisfies the following inequality.

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

Definition 1.8.10 The **ceiling function** or **smallest integer function** is the function defined as follows: for a real number x, the ceiling of x, denoted $\lceil x \rceil$, is the smallest integer not less than x. (The integer just above - on the right)

Example 1.8.31

(i) $\lceil 5 \rceil = 5$

(ii) $\lceil 1.78 \rceil = 2$

(iii) $\lceil 0 \rceil = 0$

(iv) $\lceil -1.6 \rceil = -1$

(v) $\lceil -3.5 \rceil = -3$

(vi) $\lceil -99.9 \rceil = -99$

Example 1.8.32

$$\lceil 1.7 \rceil = 2, \ \lceil \pi \rceil = 4 \text{ and } \lceil -3.2 \rceil = -3$$

The ceiling of x satisfies the following inequality.

$$x \leq \lceil x \rceil < x + 1$$

Definition 1.8.11 The **sign function** is the function, denoted *sgn*, defined as

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Note that for any real number $x = \operatorname{sgn}(x)|x|$. So if $x \neq 0$,

$$\operatorname{sgn}(x) = \frac{x}{|x|}$$

The sgn function is usually referred to as the *signum* function.

Example 1.8.33 Graph the curve

$$f(x) = \frac{|x-3|}{|x-1|}\ , \ x \in \mathbb{R}$$

Note that, we cannot solve and play around any mathematical expression with norms (absolute, modulus), so we remove it by considering the positive and negative options for each modulus

$$\begin{aligned}
f_1(x) &= \frac{+(x-3)}{+(x-1)} : \ x \geq 3, x > 1 \ \Rightarrow \ f_1(x) = \frac{(x-3)}{(x-1)} : \ x \in [3, \infty) \\
f_2(x) &= \frac{+(x-3)}{-(x-1)} : \ x \geq 3, x < 1 \ \Rightarrow \ f_2(x) = -\frac{(x-3)}{(x-1)} : \ x \ \texttt{DNE} \\
f_3(x) &= \frac{-(x-3)}{+(x-1)} : \ x < 3, x > 1 \ \Rightarrow \ f_3(x) = -\frac{(x-3)}{(x-1)} : \ x \in (1, 3) \\
f_4(x) &= \frac{-(x-3)}{-(x-1)} : \ x < 3, x < 1 \ \Rightarrow \ f_4(x) = \frac{(x-3)}{(x-1)} : \ x \in (-\infty, 1)
\end{aligned}$$

We sketch each function f_1, f_2, f_3, f_4 within its region/domain. For example, for

$$f_1(x) = \frac{(x-3)}{(x-1)} : \; x \in [3, \infty) \quad \Rightarrow \quad \begin{array}{|c|c|c|c|c|} x & 3 & 4 & 5 & 6 \\ \hline f_1(x) & 0 & \frac{1}{3} & \frac{2}{4} & \frac{3}{5} \end{array}$$

Sketching the functions f_1, f_2, f_3, f_4 will result into the graph

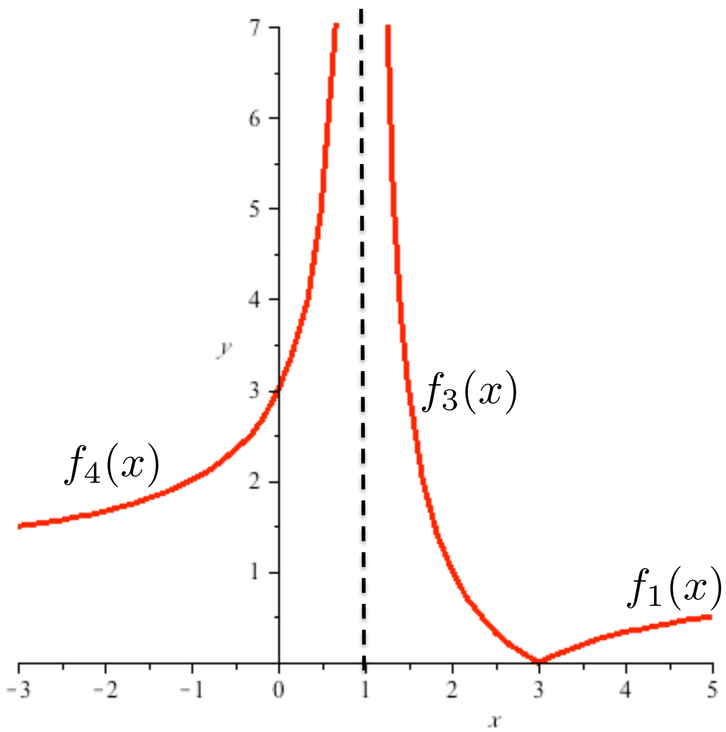


Figure 1.8: A graph of $f(x) = \frac{|x-3|}{|x-1|}$

Exercise 1.15 Graph the curve

$$f(x) = \frac{|x+6|}{|x+2|}$$

Exercise 1.16 Graph the curve

$$f(x) = \frac{|x+3| - 4|x-1|}{|x-2|}$$

Exercise 1.17 Compute the domain of the function $f(x)$ where

$$f(x) = \frac{a\sqrt{x+b}}{x+c}$$

where $a, b, c \; \in \mathbb{R}$

1.9 Chapter Examples

1. Find the inverse function of $g(x) = |1 - 2x|$; $x > 1$ and determine it's domain.

$$g(x) = \begin{cases} 1 - 2x, & x \leq \frac{1}{2} \\ 2x - 1, & x > \frac{1}{2} \end{cases}$$

   Hence for $x > 1$, $g(x) = |1 - 2x| = 2x - 1$ and $g^{-1}(x) = \frac{x+1}{2}$ with $D_g = \mathbb{R}$

2. Solve

$$\frac{x^2 - 1}{x^2 + 1} > 0$$

   The denominator will never be negative due to the square below. So the product of two numerator terms $(x - 1)(x + 1)$ to be positive.

3. Find the domain of the function

$$|x - 3.2| + |x - 5.2| = 2$$

   With equality, we solve the equation for the domain.

4. *True* or *False*? If a horizontal line intersects the graph of the equation $y = f(x)$ in more than one point, the equation $f(x)$ cannot determine y as a function of x. Explain.

5. *True* or *False*?

   (a) If

$$\begin{aligned} g(x) &= x^2 + 4x + 4 \\ &\text{and} \\ f(x) &= \frac{(x^2 - 4)(x + 2)}{x - 2} \end{aligned}$$

   then $f(x) = g(x)$

   (b) If

$$\begin{aligned} g(x) &= x + 3 \\ &\text{and} \\ f(x) &= \frac{x^2 - 9}{x - 3} \end{aligned}$$

   then $f(x) = g(x)$

   `False, since` $f(-3) \neq g(-3)$, `domain not same`

   (c) If

$$\begin{aligned} g(x) &= x + 3, x \neq 3 \\ &\text{and} \\ f(x) &= \frac{x^2 - 9}{x - 3}, x \neq 3 \end{aligned}$$

   then $f(x) = g(x)$

   `Now this is True as both functions are compared at other points not` $x = 3$

(d) If

$$\begin{aligned} g(x) &= x+3, x \neq -3 \\ &\text{and} \\ f(x) &= \frac{x^2-9}{x-3} \end{aligned}$$

then $f(x) = g(x)$

`Again True as the condition` $x \neq -3$`is automatically/mathematically implied with` $f(x$

(e) If

$$f(x) = \begin{cases} x^2; & \text{if } x < 6, \\ 6x; & \text{if } x > 6, \end{cases}$$

then $f(x)$ is continuous at $x = 6$.

`False as` $f(6)$ `not defined`

6. Suppose that f is a function that is defined for all real numbers. Which of the following conditions assures that f has an inverse function?

(A) The function is periodic

(B) The graph of f is symmetric with respect to the y-axis

(C) The graph of f is concave up

(D) The function f is strictly increasing function

(E) The function f is continuous

`Which are not one to one? and why?`

7. If the solutions of $f(x) = 0$ are -1 and 2, then the solutions of $f\left(\frac{x}{2}\right) = 0$ are

(A) -1 and 2 (C) $-\frac{3}{2}$ and $\frac{3}{2}$ (E) -2 and 4

(B) $-\frac{1}{2}$ and $\frac{5}{2}$ (D) $-\frac{1}{2}$ and 1

8. *True* or *False*?

(a) If $f(x) \cdot c = g(x) \cdot d$, for $c, d \in \mathbb{R}\ \forall x$, then $\frac{f(x)}{g(x)} = \frac{d}{c}$

`False because there is no condition for denominators`

(b) If $f(x) \cdot c = g(x) \cdot d$, for $c \neq 0, d \in \mathbb{R}\ \forall x$, then $\frac{f(x)}{g(x)} = \frac{d}{c}$

`False because there is no condition for` $g(x)$

(c) If $f(x) \cdot c = g(x) \cdot d$, for $c \neq 0, d \in \mathbb{R}\ \forall x$ and $g(x) \neq 0$ then $\frac{f(x)}{g(x)} = \frac{d}{c}$

`True, no mathematical condition is violated`

9. Prove that:

$$|u+v| \leq |u| + |v|; \quad u, v \in \mathbb{R}$$

Hence, determine $|a+b-7|$ if $|a-3| \leq 0.003$ and $|b-4| \leq 0.004$; $\quad a, b \in \mathbb{R}$.

`Sum the last two inequalities`

10. *True* or *False*? An inverse $f^{-1}(x)$ exists if and only if the function $f(x)$ is bijective.

    `"if and only if", is not the same as "if"`

11. Give reasons why each of the following is/is not a function:
    (*a*) $y = 5x$, (*b*) $y = 7$, (*c*) $x = 5$, (*d*) $x = 3y$, (*e*) $|y| = x$.

12. If $f''(x) - f'(x) - 2f(x) = 0$, $f'(0) = 0$, and $f(0) = 2$, then $f(1) =$

    (A) $e^2 + e^{-1}$ (B) 1 (C) 0 (D) e^2 (E) $2e^{-1}$

13. If $f(x) = x^3 + 3x^2 + 4x + 5$ and $g(x) = 5$, then $g(f(x)) =$

    (A) $5x^2 + 15x + 25$
    (B) $5x^3 + 15x^2 + 20x + 25$
    (C) 1125
    (D) 225
    (E) 5

14. What is the domain of the function f given by $f(x) = \frac{\sqrt{x^2-4}}{x-3}$?

    (A) $\{x : x \neq 3\}$
    (B) $\{x : |x| \leq 2\}$
    (C) $\{x : |x| \geq 2\}$
    (D) $\{x : |x| \geq 2 \text{ and } x \neq 3\}$
    (E) $\{x : x \geq 2 \text{ and } x \neq 3\}$

15. For the pollution levels in Lake Nalubaale has been modeled by the equation

    $$y = x^{\frac{2}{3}}(6 - x)^2$$

    Find all the extremas/ stationary point/ turning points for the equation.

16. At Nasser Graphics Ltd, the average starting salary for a new graphics designer is UShs. $376,000/=$. But the actual salary of a new graphics designer could differ from the average by as much as UShs. $25,900/=$

    (a) Write the absolute value inequality to describe this situation
    (b) Solve the inequality to find the range of the starting salaries

17. Determine the domain and range of

    (a) $f(x) = \frac{3}{\sqrt{6-2|x|}}$ (b) $g(x) = \sqrt{\frac{x(x-2)}{6-2x}}$ (c) $h(x) = \sqrt{\frac{x(x-2)}{|4-3x|(6-2x)}}$

    Is $g(x)$ an even function?. Is $h(x)$ an odd function?

18. Let $S(x) = \sqrt{x}$ and $H(x) = x + 1$, show that $(H(S(x)))^2 = H(x) + 2S(x)$

19. Given a function, $S = \{(a, b),\ (b, a),\ (c, d),\ (e, f),\ (g, h),\ (i, j),\ (c, d)\}$, does S^{-1} exist. Explain your answer.

20. Graph the function

    $$f(x) = \frac{x|4x-3|}{3|x+6|} \quad x \neq -6$$

    Hence or otherwise, solve the inequality

    $$\frac{x|4x-3|}{3|x+6|} > 0$$

21. Use sketch diagrams (or graphs) to explain each of the statements in parts (a) and (b) below:
    (a) $\lim_{x \to a} f(x)$ exists
    (b) $f(x)$ is continuous at $x = a \in \mathbb{R}$

22. Determine whether $f(x)$ is even or odd or neither and sketch the graph.
    (a) $f(x) = x^2$
    (b) $f(x) = 2 - x^2$
    (c) $f(x) = x^3$
    (d) $f(x) = 1 - x^3$
    (e) $f(x) = x^3 + x$
    (f) $f(x) = 2x^4 + x^2$
    (g) $f(x) = |x| + 2$
    (h) $f(x) = \cos x$
    (i) $f(x) = x^2 + x$

    In exercises $23 - 31$, use the functions

$$f(x) = 3x + 1\ , \quad g(x) = x^3\ , \qquad h(x) = \sqrt{x}$$

    to form the indicated composite function.

23. $f \circ g$
24. $f \circ h$
25. $g \circ f$
26. $h \circ g$
27. $h\,(f(x))$
28. $h\,(f(g(x)))$
29. $f\,(g(h(x)))$
30. $f \circ (g \circ h)$
31. $g\,(h(f(x)))$

32. Given the function $f(x) = x^2$ with domain $\{x\ ; -1 \le x \le 3\}$ and the function $g(x) = 2x + 6$ find
    (a) the domain of $f \circ g$
    (b) the range of $f \circ g$
    (c) the domain of $g \circ f$
    (d) the range of $g \circ f$

33. Find two nonconstant functions f and g so that the composite function $f \circ g$ is constant.

34. For the functions $f(x) = 2x + 1$, $\ g(x) = \sqrt{x}$, graph the functions

$$f + g,\ f - g,\ fg,\ f/g,\ g/f,\ f \circ g,\ g \circ f$$

    Determine the domain and range of each function.

35. (a) Given the function $f(x) = |x|$ with domain $[-1, 3]$ and the function $g(x) = 2x + 6$, find the domain of the composite function $f \circ g$.
    (b) Find the values of x satisfying the inequality

$$\frac{2x + 3}{4x - 5} < -4$$

36. Determine the domain of

$$f(x) = \sqrt{(2 - x)}$$

    A. $0 \le x \le 2$
    B. $x \ge 2$
    C. $\infty < x \le 2$
    D. $-2 \le x \le 2$

E. None of the above.

`The solution is` $-\infty < x \leq 2$`, 36E, but not` $x < 2$ `neither` $(-\infty, 2)$ `nor` $[2, -\infty)$

37. Given that

$$f(x) = |1 - 2x|, \; x > 1$$

Determine $f^{-1}(x)$.

`Expand absolute, use function for` $x > 1$ `and drop the other piece, to have`

$$f^{-1}(x) = \frac{x+1}{2}$$

38. The functions

$$\begin{aligned} f(x) &= \frac{x}{x+2} \\ g(x) &= x^2 + 1 \end{aligned}$$

are not equal since $D_f \not\equiv D_g$. That is $\mathbb{R} - \{-2\} \not\equiv \mathbb{R}$

39.

40.

Chapter 2

Limits of functions

Introduction

In this lecture, we try to analyze the behavior of a function around and near a point. Examining the function as it approaches a point, from the left and from the right.

2.1 Informal definition of a limit of a function

Definition 2.1.1 We say that a number L is a limit of a function $f(x)$ as x approaches a number $'a'$ from either direction if $\lim_{x\to a} f(x)$ from one direction is the same as $\lim_{x\to a} f(x)$ from the other direction. The statement can be abbreviated and written as,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x)$$

$$\lim_{x\to a} f(x) = L \qquad (2.1)$$

The words either direction in the statement are important.

Note 2.1.1 the number L depends only on the behavior of $f(x)$ near $x = a$ but not on the functional value $f(a)$. In fact we shall later learn that L may exist and yet $f(a)$ is not defined. This is an important property of limits of functions. Later you will learn that for functions called **continuous** functions,

$$\lim_{x\to a} f(x) = f(a)$$

This is not true for any other function.

Example 2.1.1 Determine the limit of the function

$$f(x) = \begin{cases} x^2+1\ , & x<1 \\ 2x\ , & x \geq 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(x^2+1) = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1}(2x) = 2$$

Since

$$\lim_{x\to 1^-} f(x) = 2 = \lim_{x\to 1^+} f(x)$$

The limit exists

Example 2.1.2 Establish the limit of the function

$$f(x) = \begin{cases} \frac{x-3}{x^2}\ , & x \geq 3 \\ 5x+7\ , & x<3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3}(5x+7) = 22 \quad \text{and} \quad \lim_{x\to 3^+} f(x) = \lim_{x\to 3}\left(\frac{x-3}{x^2}\right) = 0$$

Since

$$\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$$

The limit DNE

Example 2.1.3 Given

$$f(x) = \begin{cases} x^2+k & x<1 \\ \\ x^3 & x \geq 1 \end{cases}$$

Find the value of k such that $\lim\limits_{x\to 1} f(x)$ exists and find it.

$$\begin{aligned} \text{Since } \lim_{x\to 1^-} f(x) &= \lim_{x\to 1}(x^2+k) \\ &= 1^2+k \\ &= k+1 \\ \text{and } \lim_{x\to 1^+} f(x) &= \lim_{x\to 1} x^3 = 1^3 = 1 \end{aligned}$$

But for the $\lim_{x\to 1} f(x)$ to exist

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^+} f(x) \\ \text{i.e , } k+1 &= 1 \\ \Rightarrow k &= 1-1 = 0 \end{aligned}$$

and when $k=0$, then $\lim\limits_{x\to 1} f(x) = 1$

Example 2.1.4 Show that the limit of the function $h(x) = |c|$ doesn't exist at $x \to 0$.

Since

$$|c| = \begin{cases} c\ , & c \geq 0 \\ -c\ , & c < 0 \end{cases}$$

$$\begin{aligned} \text{Since } \lim_{x \to 0^-} h(x) &= \lim_{x \to 0}(-c) = -c \\ \text{and } \lim_{x \to 0^+} h(x) &= \lim_{x \to 0}(c) = c \end{aligned}$$

But for the $\lim_{x \to 0} h(x)$ to exist

$$\begin{aligned} \lim_{x \to 0^-} h(x) &= \lim_{x \to 0^+} h(x) \\ \text{but },\ -c &\neq c \end{aligned}$$

thus limit does not exist (DNE).

Example 2.1.5 Determine the limit of the piecewise function

$$f(x) = \begin{cases} 2x^2 - 9\ , & x < 3 \\ 3x\ , & x \geq 3 \end{cases}$$

From left of 3*, the function, the road is* $f(x) = 2x^2 - 9$*, and from the right of* 3*, we use* $f(x) = 3x$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3}(2x^2 - 9) = 9 \quad \text{and} \quad \lim_{x \to 3^+} f(x) = \lim_{x \to 3}(3x) = 9$$

Since

$$\lim_{x \to 3^-} f(x) = 9 = \lim_{x \to 3^+} f(x)$$

The limit exists

Example 2.1.6 Establish the limit of the function

$$f(x) = \begin{cases} 2x - 3\ , & x \geq 2 \\ \frac{2x}{4+x^2}\ , & x < 2 \end{cases}$$

From left of 2*, the function, the road is* $f = \frac{2x}{4+x^2}$*, and from the right of* 2*, we use* $f = 2x - 3$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2}\left(\frac{2x}{4 + x^2}\right) = \frac{1}{2} \quad \text{and} \quad \lim_{x \to 2^+} f(x) = \lim_{x \to 2}(2x - 3) = 1$$

Since

$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$$

The limit DNE

Example 2.1.7 Investigate the limit of the following function

$$f(x) = \begin{cases} x^2 + 1\ , & x > 0 \\ 4x\ , & x \leq 0 \end{cases}$$

From left of 0*, the function, the road is* $f(x) = 4x$*, and from the right of* 0*, we use* $f(x) = x^2 + 1$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0}(4x) = 0 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0}(x^2 + 1) = 1$$

Since

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

The limit DNE

Example 2.1.8 Find the limit of the mapping below at $x = 0$

$$f(x) = \frac{|x|}{x}$$

$$f(x) = \begin{cases} \frac{x}{x}\ , & x \geq 0 \\ -\frac{x}{x}\ , & x < 0 \end{cases}$$

Which implies that

$$f(x) = \begin{cases} 1\ , & x \geq 0 \\ -1\ , & x < 0 \end{cases}$$

whose limit we want to get as x tends to zero.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} -1 = -1 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0} 1 = 1$$

Since

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

The limit DNE

Example 2.1.9 Compute the limit of the function as $x \to -2$

$$f(x) = \frac{x^2 - 1}{x + 2}$$

Since the function is not defined at $x = -2$, the limit does not exist.

Example 2.1.10 Find

$$\lim_{x \to 0} \frac{|x|}{x}$$

We define

$$|x| = \begin{cases} x\ , & x \geq 0 \\ -x\ , & x < 0 \end{cases}$$

Thus

$$\lim_{x \to 0} \frac{|x|}{x} = \begin{cases} \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0} 1 = 1 & x \geq 0 \\ \\ \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0} -1 = -1 & x < 0 \end{cases}$$

Though this limit is finite, it is not unique (one and only one) We note that the $\lim_{x \to 0} \frac{|x|}{x}$ does not exist, since the left hand limit (-1) is not equal to the right hand limit (1)

2.2 Computation of limits

2.2.1 Substitution Method

If a function remain analytic ('defined' or "denominator $\neq 0$") at $x = a$, we just substitute in x as a.

Example 2.2.1 Compute the limits of the following functions

(i)

$$\lim_{x\to 2}(4x^2 - 1) = 4.2^2 - 1 = 15$$

we have simply substituted because $4x^2 - 1$ is a polynomial function and polynomials are analytic over the entire number line.

(ii)

$$\lim_{x\to \frac{\pi}{4}} \sin x = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$\sin x$ is also analytic on $\mathbb{R}$

(iii)

$$\lim_{x\to 1}\left(\frac{x^2+2x+1}{x+1}\right) = \left(\frac{1^2+2(1)+1}{1+1}\right) = \frac{4}{2} = 2$$

(In fact the rational function $\frac{x^2+2x+1}{x+1}$ is defined-analytic at all points except at the poles but $x = 1$ is not a pole of this rational function,thus we merely substitute to get the limit 2).

(iv)

$$\lim_{x\to 4}\left(\frac{x^2+2}{x-3}\right) = \left(\frac{4^2+2}{4-3}\right) = 18$$

Example 2.2.2 Compute the following limits

(i) $\lim_{x\to 0}(4x+6) = 6$

(ii) $\lim_{x\to 1}\left(\frac{2x}{x+1}\right) = 1$

(iii) $\lim_{x\to \pi}\cos 2x = 1$

(iv) $\lim_{x\to 3}\left(\frac{x-2}{x-3}\right)$ DNE

2.2.2 Numerator Factorisation [Non-analytic Technique]

When the functional value $f(a)$ is not defined, not analytic,
or "denominator $= 0$".
we first factorise the function, cancel out terms, so that it becomes analytic and we just substitute [(A)].

Example 2.2.3 Find the

$$\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)$$

Point $x = 1$ is a pole of this rational function. Hence function not defined at $x = 1$

$$\begin{aligned} \texttt{therefore}\ \lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right) &= \lim_{x\to 1}\frac{(x+1)(x-1)}{x-1} \\ &= \lim_{x\to 1}(x+1) \\ &= 1+1=2 \end{aligned}$$

Example 2.2.4 Find the

$$\lim_{x\to 3}\left(\frac{x^2+2x-15}{x-3}\right)$$

The point $x = 3$ is a pole of this rational function. Hence function not defined - non analytic, a need to first factorise the numerator.

$$\begin{aligned} \texttt{therefore}\ \lim_{x\to 3}\left(\frac{x^2+2x-15}{x-3}\right) &= \lim_{x\to 3}\frac{(x-3)(x+5)}{x-3} \\ &= \lim_{x\to 3}(x+5) \\ &= 3+5=8 \end{aligned}$$

Example 2.2.5 Find the

$$\lim_{x\to 2}\left(\frac{x^2-2x}{x-2}\right)$$

The point $x = 2$ is a pole of this rational function.

$$\begin{aligned} \texttt{therefore}\ \lim_{x\to 2}\left(\frac{x^2-2x}{x-2}\right) &= \lim_{x\to 2}\frac{x(x-2)}{x-2} \\ &= \lim_{x\to 2}(x) \\ &= 2 \end{aligned}$$

Example 2.2.6 Find the

$$\lim_{x\to 1}\frac{1}{x-1}$$

$\lim\limits_{x\to 1}\frac{1}{x-1}$ does not exist Since $\frac{1}{1-1}=\frac{1}{0}$ is not defined.

Example 2.2.7

$$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}=\lim_{x\to 2}(x+2)=4$$

Example 2.2.8

$$\lim_{x\to 3}\left(\frac{x^2-5x+6}{x-3}\right)=\lim_{x\to 3}\frac{(x-2)(x-3)}{(x-3)}=\lim_{x\to 3}(x-2)=1$$

Example 2.2.9

$$\lim_{x\to 0}\left(\frac{x^2+5x}{x}\right) = \lim_{x\to 0}\frac{x(x+5)}{x} = \lim_{x\to 0}(x+5) = 5$$

Example 2.2.10

$$\lim_{x\to 1}\frac{x^2-1}{(x+2)(x-1)} = \lim_{x\to 1}\frac{(x-1)(x+1)}{(x+2)(x-1)} = \lim_{x\to 1}\frac{(x+1)}{(x+2)} = \frac{2}{3}$$

Example 2.2.11 Find the limit below

$$\lim_{x\to 4}\frac{(x^2-5x+6)}{(x-4)}$$

The limit DNE by methods of substitution and non-analytic functions.

2.2.3 Infinity

Limits at infinity and infinite limits

Limits at infinity

When asked the limits at infinity, we first divide through the function by the highest power of x, then find the limit of the function. [Recall $\lim_{x\to\infty} \frac{a}{x^n} = 0$ where a,n are constants] A function $f(x)$ may approach a finite number L as x goes to infinity i.e as x becomes very big but positive or negative.

Definition 2.2.1 we say that $\lim_{x\to\infty} f(x) = L_1$ (finite number) if the value of $f(x)$ approaches L_1 as x in creases beyond bound (becomes very big)
We also say that $\lim_{x\to-\infty} f(x) = L_2$ (a finite number) means that the value of $f(x)$ approaches L_2 as x decreases beyond bound (i.e becomes big in absolute value but negative in sign.

Example 2.2.12 Find

(i) $\lim\limits_{x\to+\infty} \frac{x+1}{x}$

(ii) $\lim\limits_{x\to-\infty} \frac{x+1}{x}$

By dividing through by the highest power of x

(i)

$$\begin{aligned}\lim_{x\to+\infty} \frac{x+1}{x} &= \lim_{x\to+\infty} \left(1+\frac{1}{x}\right) \\ &= 1+0=1\end{aligned}$$

The idea is that we divide by the variable with the highest power to get the dominant terms.

(ii)

$$\begin{aligned}\lim_{x\to-\infty} \frac{x+1}{x} &= \lim_{x\to-\infty} \left(1+\frac{1}{x}\right) \\ &= 1+0=1\end{aligned}$$

Example 2.2.13 Find

$$\lim_{x\to\infty} \frac{x^2+3x+1}{3x^2+x+2}$$

The idea is to divide all through by x^2 (variable with highest power)

$$\begin{aligned}\texttt{therefore } \lim_{x\to\infty} \frac{x^2+3x+1}{3x^2+x+2} &= \lim_{x\to\infty} \frac{1+\frac{3}{x}+\frac{1}{x^2}}{3+\frac{1}{x}+\frac{2}{x^2}} \\ &= \frac{1+0+0}{3+0+0} \\ &= \frac{1}{3}\end{aligned}$$

The interpretation is that as x blows beyond bound (becomes very big) the function

$$f(x) = \frac{x^2+3x+1}{3x^2+x+2}$$

approaches a finite number $\frac{1}{3}$ or behaves like the straight line $f(x) = \frac{1}{3}$

Example 2.2.14 Compute the

$$\lim_{x\to\infty}\left(\frac{x^2+4x+1}{-4x^2-6}\right)$$

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x^2+4x+1}{-4x^2-6}\right) &= \lim_{x\to\infty}\left(\frac{\frac{x^2}{x^2}+\frac{4x}{x^2}+\frac{1}{x^2}}{\frac{-4x^2}{x^2}-\frac{6}{x^2}}\right)\\
&= \lim_{x\to\infty}\left(\frac{1+\frac{4}{x}+\frac{1}{x^2}}{-4-\frac{6}{x^2}}\right)\\
&= \left(\frac{1+0+0}{-4-0}\right)\\
&= -\frac{1}{4}
\end{aligned}$$

Since

$$\begin{aligned}
\lim_{x\to\infty}\frac{1}{x} &= \frac{1}{10},\frac{1}{1000},\frac{1}{10000},\frac{1}{100000000},\frac{1}{1000000000000000}\approx 0\\
\lim_{x\to\infty}\frac{1}{x^2} &= \frac{1}{100},\frac{1}{1000000},\frac{1}{100000000},\frac{1}{100000000000000000000}\approx 0\\
\lim_{x\to-\infty}\frac{1}{x} &= \frac{-1}{10},\frac{-1}{1000},\frac{-1}{10000},\frac{-1}{100000000},\frac{-1}{1000000000000000}\approx 0\\
\lim_{x\to-\infty}\frac{1}{x^3} &= \frac{-1}{1000},\frac{-1}{10000000},\frac{-1}{10000000000000},\frac{-1}{1000000000000000000}\approx 0
\end{aligned}$$

Example 2.2.15 Compute the

$$\lim_{x\to-\infty}\left(\frac{4x^2-2x}{7x^3+8x^2}\right)$$

$$\begin{aligned}
\lim_{x\to-\infty}\left(\frac{4x^2-2x}{7x^3+8x^2}\right) &= \lim_{x\to-\infty}\left(\frac{\frac{4x^2}{x^3}-\frac{2x}{x^3}}{\frac{7x^3}{x^3}+\frac{8x^2}{x^3}}\right)\\
&= \lim_{x\to-\infty}\left(\frac{\frac{4}{x}-\frac{2}{x^2}}{7+\frac{8}{x}}\right)\\
&= \left(\frac{0-0}{7+0}\right)\\
&= 0
\end{aligned}$$

Example 2.2.16 Compute the

$$\lim_{x\to\infty}\left(\frac{5-9x^4}{2x+x^3}\right)$$

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{5-9x^4}{2x+x^3}\right) &= \lim_{x\to\infty}\left(\frac{\frac{5}{x^4}-\frac{9x^4}{x^4}}{\frac{2x}{x^4}+\frac{x^3}{x^4}}\right)\\
&= \lim_{x\to\infty}\left(\frac{\frac{5}{x^4}-9}{\frac{2}{x^3}+\frac{1}{x}}\right)=\left(\frac{0-9}{0+0}\right)\\
&= \text{DNE}
\end{aligned}$$

Example 2.2.17 Compute the

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x}\right)$$

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x^2+1}{x}\right) &= \lim_{x\to\infty}\left(x+\frac{1}{x}\right)\\ &= (\infty+0)\\ &= \infty\\ &\quad \text{DNE}\end{aligned}$$

Infinite limits

A function $f(x)$ can blow beyond bound as x approaches a finite number.

Definition 2.2.2 We define

$$\lim_{x\to a} f(x) = +\infty$$

If $f(x)$ increases without bound as x approaches the number a We say that

$$\lim_{x\to a} f(x) = -\infty$$

If $f(x)$ decreases without bound as x approaches the number a. We say that

$$\lim_{x\to +\infty} f(x) = +\infty$$

If $f(x)$ increases without bound as x increases without bound,
and the

$$\lim_{x\to -\infty} f(x) = -\infty$$

If $f(x)$ decreases without bound as x decreases without bound.
In such cases we say that the limit does not exist.

Example 2.2.18 Given the function

$$f(x) = \tan x$$

Use the knowledge of curve sketching (asymptotes)

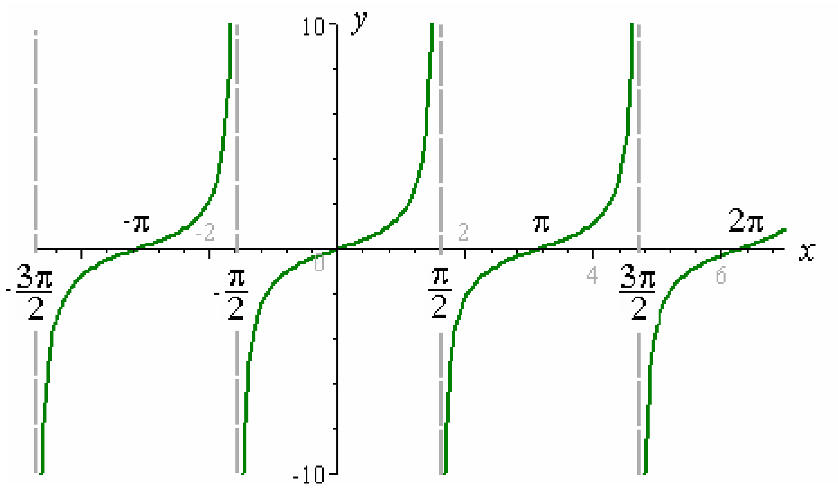


Figure 2.1: Graph of $y = \tan x$

to find that

(i) $\lim_{x \to \frac{\pi}{2}^-} f(x) = +\infty$

(ii) $\lim_{x \to -\frac{\pi}{2}^+} f(x) = -\infty$

Note 2.2.1 The curve above indicates that, some limits can actually be infinity, do not exist.

Exercise 2.1 Evaluate the limit

$$\lim_{x \to \infty} \frac{\sqrt{x^2 + 9}}{x + 1}$$

Example 2.2.19 Given

$$f(x) = \frac{x^2}{1-x^2},$$

graph the function and hence find

(i) $\lim_{x\to 1^-} f(x)$

(ii) $\lim_{x\to 1^+} f(x)$

(iii) $\lim_{x\to -1^+} f(x)$

(iv) $\lim_{x\to -1^-} f(x)$

The knowledge of Curve sketching is assumed. However, later in the lectures you will learn how to sketch such curves of rational functions. Here we only present the sketch without detail.

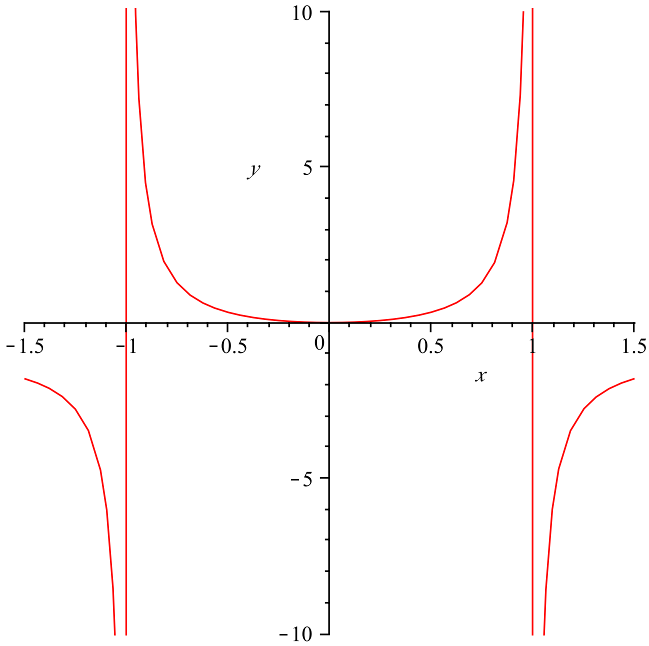


Figure 2.2: Graph of $y = \frac{x^2}{1-x^2}$

(i) $\lim_{x\to 1^-} \frac{x^2}{1-x^2} = +\infty$

(ii) $\lim_{x\to 1^+} \frac{x^2}{1-x^2} = -\infty$

(iii) $\lim_{x\to -1^+} \frac{x^2}{1-x^2} = +\infty$

(iv) $\lim_{x\to -1^-} \frac{x^2}{1-x^2} = -\infty$

2.2.4 Using La'Hopital rule

The La'Hopital rule states that, if

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}\quad\text{or}\quad\lim_{x\to a}\frac{f(x)}{g(x)}=\pm\frac{\infty}{\infty}$$

then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example 2.2.20 Compute

$$\lim_{x\to 0}\frac{\sin x}{x}=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}]=\lim_{x\to 0}\frac{\cos x}{1}=\frac{1}{1}=1$$

Example 2.2.21 Compute

$$\lim_{x\to 0}\frac{x^2-x}{x^3+x}=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}]=\lim_{x\to 0}\frac{2x-1}{3x^2+1}=\frac{-1}{1}=-1$$

Example 2.2.22 Compute the

$$\lim_{x\to 3}\frac{(x^2-9)}{(x-3)}=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}]=\lim_{x\to 3}\frac{2x}{1}=6$$

Example 2.2.23 Use the La'Hopital technique to compute

$$\lim_{x\to 1}\frac{(2x^2-2)}{(x-1)}=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}]=\lim_{x\to 1}\frac{4x}{1}=4$$

Example 2.2.24 Find the limit

$$\lim_{x\to 0}\frac{x^3}{4x^2}=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}]=\lim_{x\to 0}\frac{3x^2}{8x}=\lim_{x\to 0}\frac{6x}{8}=0$$

You differentiate again and again, whenever La'Hopital applies, till cannot apply the theorem anymore.

Example 2.2.25 Find the limit

$$\lim_{x\to 0}\left[\frac{\sin x}{2x}\right]=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}]=\lim_{x\to 0}\left[\frac{\cos x}{2}\right]=\frac{1}{2}$$

Exercise 2.2 By La'Hopital rule, show the following

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 x}{x}&=0\\ \lim_{x\to 0}\frac{1-\cos x}{x}&=0\end{aligned}$$

Example 2.2.26

$$\begin{aligned}\lim_{x\to 0}\frac{\sin \pi x}{\pi x}&=\lim_{y\to 0}\frac{\sin y}{y}\\ &=\lim_{y\to 0}\frac{\cos y}{1}\\ &=1\end{aligned}$$

Example 2.2.27 Applying the rule three times

$$\begin{aligned}
\lim_{x\to 0}\left[\frac{2\sin x-\sin 2x}{x-\sin x}\right] &= \frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}] = \lim_{x\to 0}\left[\frac{2\cos x-2\cos 2x}{1-\cos x}\right]\\
&= \frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}] = \lim_{x\to 0}\left[\frac{-2\sin x+4\sin 2x}{\sin x}\right]\\
&= \frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}] = \lim_{x\to 0}\left[\frac{-2\cos x+8\cos 2x}{\cos x}\right]\\
&= \frac{-2+8}{1}\\
&= 6
\end{aligned}$$

Example 2.2.28

$$\lim_{x\to 1}\frac{2\ln x}{x-1}=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}] = \lim_{x\to 1}\frac{\frac{d}{dx}(2\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1}\frac{\frac{2}{x}}{1} = 2$$

Example 2.2.29

$$\lim_{x\to 0}\left[\frac{e^x-1}{x^2}\right]=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}] = \lim_{x\to 0}\frac{\frac{d}{dx}(e^x-1)}{\frac{d}{dx}(x^2)} = \lim_{x\to 0}\frac{e^x}{2x} = \infty$$

Example 2.2.30

$$\lim_{x\to\infty}\frac{e^x}{x}=\frac{\infty}{\infty}\ [\Rightarrow\ \text{La'Hopital}] = \lim_{x\to\infty}\frac{e^x}{1} = \infty$$

Sometimes it is necessary to use La'Hopital's Rule several times in the same problem:

Example 2.2.31

$$\lim_{x\to 0}\left[\frac{1-\cos x}{x^2}\right]=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}] = \lim_{x\to 0}\frac{\sin x}{2x} = \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$$

Occasionally, a limit can be re-written in order to apply La'Hopital's Rule:

Example 2.2.32 La'Hopital when it is ∞/∞

$$\begin{aligned}
\lim_{x\to\infty}\left[\frac{x^3+3}{x^2+e^x}\right] &= \frac{\infty}{\infty}\ [\Rightarrow\ \text{La'Hopital}] = \lim_{x\to\infty}\left[\frac{3x^2}{2x+e^x}\right]\\
&= \frac{\infty}{\infty}\ [\Rightarrow\ \text{La'Hopital}] = \lim_{x\to\infty}\frac{6x}{2+e^x}\\
&= \frac{\infty}{\infty}\ [\Rightarrow\ \text{La'Hopital}] = \lim_{x\to\infty}\frac{6}{e^x} = 0
\end{aligned}$$

Example 2.2.33 Solve for the limit

$$\lim_{x\to 2}\frac{\sqrt{x+2}\ -2}{x-2}$$

This is usual La'Hopital rule 0/0.

$$\begin{aligned}
\lim_{x\to 2}\frac{\sqrt{x+2}\ -2}{x-2}=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}] &= \lim_{x\to 2}\frac{1}{2\sqrt{x+2}}\\
&= \frac{1}{4}
\end{aligned}$$

Example 2.2.34

$$\lim_{x\to-\infty}\frac{-x}{e^{-x}}=\frac{\infty}{\infty}\ [\Rightarrow\ \text{La'Hopital}]\quad=\quad\lim_{x\to-\infty}\frac{-1}{-e^{-x}}=0$$

Example 2.2.35

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2+2e^{-x}}{3x^2} &= \frac{\infty}{\infty}\ [\Rightarrow\ \text{La'Hopital}]=\lim_{x\to\infty}\frac{2x-2e^{-x}}{6x}\\ &= \frac{\infty}{\infty}\ [\Rightarrow\ \text{La'Hopital}]=\lim_{x\to\infty}\frac{2+2e^{-x}}{6}\\ &= \frac{2}{6}\end{aligned}$$

Example 2.2.36 Here is another example involving 0/0:

$$\lim_{x\to0}\frac{e^x-1-x}{x^2}=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}]=\lim_{x\to0}\frac{e^x-1}{2x}=\frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}]=\lim_{x\to0}\frac{e^x}{2}=\frac{1}{2}$$

Example 2.2.37 This example involves ∞/∞. Assume n is a positive integer. Then

$$\lim_{x\to\infty}x^ne^{-x}=\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}=n\lim_{x\to\infty}\frac{x^{n-1}}{e^x}$$

Repeatedly apply La'Hopital's rule until the exponent is zero to conclude that the limit is zero.

The most important thing to learn about La'Hopital's rule is when it **should not be used**:

(1). When the limits of the two parts are not both 0, or both infinite. In this case the rule is likely to give a wrong answer! Example:

$$\lim_{x\to0^+}\frac{(\cos x)}{x}$$

is positive infinity, because the numerator approaches 1 while the denominator approaches 0. If we incorrectly apply La'Hopital's rule, we get

$$\lim_{x\to0^+}\frac{(-\sin x)}{1}=0$$

(2). When there is a better way to get the answer.

2.2.5 Indeterminate Forms and La'Hopital's Rule

With indeterminate forms, for limit evaluation cases (by substitution) there are competing interests or rules and it's not clear which will win out (thus answer not known).

Other than the

$$\frac{0}{0}\ ,\ \frac{\infty}{\infty}$$

some other types of indeterminate forms are,

$$0^0\ ,\ 1^\infty\ ,\ \infty^0\ ,\ \infty-\infty\ ,\ (0)(\infty)$$

These all have competing interests or rules that tell us what should happen and it's just not clear which, if any, of the interests or rules will win out. The topic of this section is how to deal with these kinds of limits.

We deal with this kind of limits by introducing in a La'Hopital's rule conditions, a quotient function by

(i) creating in a denominator

(ii) having one denominator by LCM

(iii) taking logarithms (natural log) if function has powers in x (exponents).

Example 2.2.38 Evaluate the following limit.

$$\lim_{x\to0^+} x\ln x$$

Note that we really do need to do the right-hand limit here. We know that the natural logarithm is only defined for positive x and so this is the only limit that makes any sense.

Now, in the limit, we get the indeterminate form $(0)(\infty)$. La'Hopital's Rule won't work on products, it only works on quotients. However, we can turn this into a fraction if we rewrite things a little.

$$\lim_{x\to0^+} x\ln x = \lim_{x\to0^+}\frac{\ln x}{1/x}$$

The function is the same, just rewritten, and the limit is now in the form $\frac{\infty}{\infty}$ and we can now use La'Hopital's Rule.

$$\lim_{x\to0^+} x\ln x = \lim_{x\to0^+}\frac{\ln x}{1/x} = \lim_{x\to0^+}\frac{1/x}{-1/x^2} = \lim_{x\to0^+}(-x) = 0$$

Example 2.2.39 Evaluate the following limit

$$\lim_{x\to-\infty} xe^x$$

By using method of analytic at $x=a$ (substitution), we get the indeterminate form $(-\infty)(0)$. Thus we force in a La'Hopital's rule. This means that we'll need to write it as a quotient. Moving the x to the denominator worked in the previous example so let's try that with this problem as well.

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty}\frac{e^x}{1/x} = \lim_{x\to-\infty}\frac{e^x}{-1/x^2} = \lim_{x\to-\infty}\frac{e^x}{2/x^3} = \lim_{x\to-\infty}\frac{e^x}{-6/x^4} = \cdots$$

Hummmm ..... This doesn't seem to be getting us anywhere. With each application of La'Hopital's Rule we just end up with another 0/0 indeterminate form and in fact the derivatives seem to be getting worse and worse. Also note that if we simplified the quotient back into a product we would just end up with either $(-\infty)(0)$ or $(0)(\infty)$ and so that won't do us any good.

This does not mean however that the limit can't be done. It just means that we moved the wrong function to the denominator. Let's move the exponential function instead.

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{1/e^x} = \lim_{x\to-\infty} \frac{x}{e^{-x}}$$

The quotient is now an indeterminate form of $-\infty/\infty$ and use La'Hopital's Rule gives,

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{1/e^x} = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0$$

Example 2.2.40 Evaluate the following limit

$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

In the limit this is the indeterminate form . We're actually going to spend most of this problem on a different limit. Let's first define the following.

$$y = x^{\frac{1}{x}}$$

Now, if we take the natural log of both sides we get,

$$\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

Let's now take a look at the following limit.

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

This limit was just a La'Hopital's Rule problem and we know how to do those. So, what did this have to do with our limit? Well first notice that,

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y}$$

We can now use the limit above to finish this problem.

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim\limits_{x\to\infty} \ln y} = e^0 = 1$$

Example 2.2.41 To evaluate a limit involving $\infty - \infty$, convert the difference of two functions to a quotient:

$$\begin{aligned}
\lim_{x\to1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) &= \lim_{x\to1} \frac{x\ln x - x + 1}{(x-1)\ln x} = \frac{0}{0}\ [\Rightarrow \text{ La'Hopital}] && (2.2)\\
&= \lim_{x\to1} \frac{\ln x + \frac{x}{x} - 1}{\frac{(x-1)}{x} + \ln x} = \lim_{x\to1} \frac{\ln x}{\frac{(x-1)}{x} + \ln x} && (2.3)\\
&= \lim_{x\to1} \frac{x\ln x}{(x-1) + x\ln x} = \frac{0}{0}\ [\Rightarrow \text{ La'Hopital}] && (2.4)\\
&= \lim_{x\to1} \frac{\ln x + \frac{x}{x}}{1 + \ln x + \frac{x}{x}} = \lim_{x\to1} \frac{1+\ln x}{1+1+\ln x} && (2.5)\\
&= \lim_{x\to1} \frac{1+\ln x}{2+\ln x} = \frac{1+0}{2+0}\\
&= \frac{1}{2}
\end{aligned}$$

where La'Hopital's rule was applied in going from (2.2) to (2.3) and then again in going from (2.4) to (2.5).

Note 2.2.2 La'Hopital's rule can be used on indeterminate forms involving *exponents* by using *logarithms* to "move the exponent down".

Example 2.2.42 Here is an example involving the indeterminate form 0^0:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim\limits_{x\to 0^+}(x\ln x)}$$

It is valid to move the limit inside the exponential function because the exponential function is continuous. Now the exponent x has been "moved down" But (using La'Hopital rule)

$$\lim_{x\to 0^+} x\ln x = 0$$

Thus

$$\lim_{x\to 0^+} x^x = e^0 = 1$$

Example 2.2.43 Although La'Hopital's rule is a powerful way of evaluating otherwise hard-to-evaluate limits, it is not always the easiest way. Consider

$$\lim_{|x|\to\infty} x\sin\frac{1}{x}$$

This limit may be evaluated using La'Hopital's rule:

$$\begin{aligned}
\lim_{|x|\to\infty} x\sin\frac{1}{x} &= \lim_{|x|\to\infty} \frac{\sin\frac{1}{x}}{1/x} \\
&= \lim_{|x|\to\infty} \frac{-x^{-2}\cos\frac{1}{x}}{-x^{-2}} \\
&= \lim_{|x|\to\infty} \cos\frac{1}{x} \\
&= \cos\left(\lim_{|x|\to\infty}\frac{1}{x}\right) \\
&= 1.
\end{aligned}$$

It is valid to move the limit inside the cosine function because the cosine function is continuous.

Another way to evaluate this limit is to use a substitution. $y = 1/x$. As $|x|$ approaches infinity, y approaches zero. So,

$$\lim_{|x|\to\infty} x\sin\frac{1}{x} = \lim_{y\to 0}\frac{\sin y}{y} = 1$$

The example may be evaluated using La'Hopital's rule or by noting that it is the definition of the derivative of the sine function at zero.

Still another way to evaluate this limit is to use a Taylor series expansion:

$$\begin{aligned}
\lim_{|x|\to\infty} x\sin\frac{1}{x} &= \lim_{|x|\to\infty} x\left(\frac{1}{x} - \frac{1}{3!\,x^3} + \frac{1}{5!\,x^5} - \cdots\right) \\
&= \lim_{|x|\to\infty} 1 - \frac{1}{3!\,x^2} + \frac{1}{5!\,x^4} - \cdots \\
&= 1 + \lim_{|x|\to\infty}\frac{1}{x}\left(-\frac{1}{3!\,x} + \frac{1}{5!\,x^3} - \cdots\right)
\end{aligned}$$

For $|x| = 1$, the expression in parentheses is bounded, so the limit in the last line is zero.

The other better and easier way, is to apply Sandwich theorem (to be seen in the next section).

Example 2.2.44

$$\lim_{x\to\infty} e^{\left(\frac{4x^3+2x}{x^3-1}\right)} = e^{\lim_{x\to\infty}\left(\frac{4x^3+2x}{x^3-1}\right)} = e^4$$

Note 2.2.3 The *lim* and the exponential e, can always change positions.

Example 2.2.45 Compute the limit

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{3x}$$

This is an indeterminate form 1^∞. Since it is of the the exponent form (with powers in x), we wil take the natural log

$$\begin{aligned}
\texttt{let } y &= \left(1+\frac{1}{2x}\right)^{3x} \\
\texttt{take logs } \ln y &= 3x\ln\left(1+\frac{1}{2x}\right) \\
\texttt{take lim } \lim_{x\to\infty}\ln y &= \lim_{x\to\infty} 3x\ln\left(1+\frac{1}{2x}\right) \\
&= (\infty)(0)
\end{aligned}$$

Still an indeterminate form, but now without exponents, meaning we can force in a La'Hopital by creating in a denominator

$$\begin{aligned}
\lim_{x\to\infty}\ln y &= \lim_{x\to\infty} 3x\ln\left(1+\frac{1}{2x}\right) \\
&= 3\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{x}} = \frac{0}{0}\ [\Rightarrow\ \text{La'Hopital}] \\
&= 3\lim_{x\to\infty}\frac{-\frac{1}{2x^2}/\left(1+\frac{1}{2x}\right)}{-\frac{1}{x^2}} \\
&= 3\lim_{x\to\infty}\frac{1}{2}\left(1+\frac{1}{2x}\right) \\
&= \frac{3}{2}
\end{aligned}$$

We were not looking for $\lim_{x\to\infty}\ln y$, but for $\lim_{x\to\infty} y$, we get it by taking exponentials to remove the natural log.

$$\begin{aligned}
\lim_{x\to\infty} y &= \lim_{x\to\infty} e^{\ln y} \\
&= e^{\lim_{x\to\infty}\ln y} \\
&= e^{\frac{3}{2}}
\end{aligned}$$

Thus

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{3x} = e^{\frac{3}{2}}$$

Example 2.2.46 Evaluate

$$\lim_{x\to\infty}(x+e^x)^{\frac{1}{x}}$$

$$\begin{aligned}\lim_{x\to\infty}(x+e^x)^{\frac{1}{x}} &= \infty^0, \quad \text{an indeterminate form}\\ \text{let } y &= (x+e^x)^{\frac{1}{x}}\\ \ln y &= \frac{\ln(x+e^x)}{x} \quad \text{to have quotient in order to go La'Hopital}\end{aligned}$$

Taking limits on both sides

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(x+e^x)}{x} = \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{(1+e^x)}{(x+e^x)} = \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{e^x}{(1+e^x)} = \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{e^x}{e^x} = \lim_{x\to\infty}1 = 1$$

Thus

$$\lim_{x\to\infty}\ln y = 1$$

Never wanted limit of $\left(\lim_{x\to\infty}\ln y\right)$ but $\left(\lim_{x\to\infty}y\right)$ so we take exponential on both sides to remove the natural logarithm ln

$$\begin{aligned}\lim_{x\to\infty}\ln y &= 1\\ e^{\left(\lim_{x\to\infty}\ln y\right)} &= e^1\\ \lim_{x\to\infty}e^{\ln y} &= e^1\\ \lim_{x\to\infty}y &= e^1\\ \Rightarrow \quad \lim_{x\to\infty}(x+e^x)^{\frac{1}{x}} &= e^1 = e\end{aligned}$$

Exercise 2.3 Use La'Hopital's Rule (for indeterminate forms) to evaluate the limit

(a)

$$\lim_{x\to\infty}(\ln x)^{\frac{2}{x}}$$

(b)

$$\lim_{x\to 0}(\sin x)^{\frac{1}{x}}$$

2.2.6 Pinching/Sandwich/Squeeze theorem

Theorem 2.2.1 The Sandwich or Pinching Theorem states that if $h(x)$ is such that

$$f(x) \leq h(x) \leq g(x) \; \forall x \in [\alpha, \beta]$$

and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} g(x)$$

then

$$\lim_{x \to a} h(x) = L$$

with $a \in [\alpha, \beta]$

Example 2.2.47 Use the Squeeze law (Sandwich theorem) to find $\lim_{x \to \infty} h(x)$ if

$$\frac{x^2+1}{x^2-1} \leq h(x) \leq \frac{x+1}{x} \;, \; \forall x \in \mathbb{R}, x > 1$$

Since

$$\begin{aligned} \lim_{x \to \infty} \frac{x^2+1}{x^2-1} &= 1 \\ \lim_{x \to \infty} \frac{x+1}{x} &= 1 \end{aligned}$$

Then

$$\lim_{x \to \infty} h(x) = 1$$

by Squeeze law.

Example 2.2.48 Compute the limit

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$$

it seems the function will be undefined, we can apply the Squeeze law Since

$$\begin{aligned} -1 \leq & \quad \sin \tfrac{1}{x} & \leq 1 \\ -x \leq & \quad x \sin \tfrac{1}{x} & \leq x \end{aligned}$$

But

$$\begin{aligned} \lim_{x \to 0}(-x) &= 0 \\ \lim_{x \to 0}(x) &= 0 \end{aligned}$$

$$\Rightarrow \quad \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

by Squeeze law.

Example 2.2.49 Compute the limit

$$\lim_{x \to 0} x^2 \cos^3\left(\frac{2}{x^3}\right)$$

cannot be ascertained through the limit law

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$$

because

$$\lim_{x\to 0}\cos^3\left(\frac{2}{x^3}\right)$$

does not exist.

However, by the definition of the cosine function,

$$-1 \le \cos^3\left(\frac{2}{x^3}\right) \le 1$$

It follows that

$$-x^2 \le x^2\cos^3\left(\frac{2}{x^3}\right) \le x^2$$

Since

$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$$

by the squeeze theorem, $\lim\limits_{x\to 0} x^2\cos^3\left(\frac{2}{x^3}\right)$ must also be 0.

Example 2.2.50 By the Sandwich theorem, compute the limit

$$\lim_{x\to\infty}\frac{2-\cos x}{x+3}$$

Note that

$$-1 \le \cos x \le +1$$

because of the well-known properties of the cosine function. Now multiply by -1, reversing the inequalities and getting

$$+1 \ge -\cos x \ge -1$$

or

$$-1 \le -\cos x \le +1$$

Next, add 2 to each component to get

$$1 \le 2-\cos x \le 3$$

Since we are computing the limit as x goes to infinity, it is reasonable to assume that $x+3>0$. Thus,

$$\frac{1}{x+3} \le \frac{2-\cos x}{x+3} \le \frac{3}{x+3}$$

Since

$$\lim_{x\to\infty}\frac{1}{x+3} = 0 = \lim_{x\to\infty}\frac{3}{x+3}$$

it follows from the Squeeze Principle that

$$\lim_{x\to\infty}\frac{2-\cos x}{x+3} = 0$$

Example 2.2.51 Compute

$$\lim_{x\to\infty} \frac{x^2(2+\sin^2 x)}{x+100}$$

First note that

$$-1 \le \sin x \le +1$$

so that

$$0 \le \sin^2 x \le 1$$

and

$$2 \le 2+\sin^2 x \le 3$$

Since we are computing the limit as x goes to infinity, it is reasonable to assume that $x+100 > 0$. Thus, dividing by $x+100$ and multiplying by x^2, we get

$$\frac{2}{x+100} \le \frac{2+\sin^2 x}{x+100} \le \frac{3}{x+100}$$

and

$$\frac{2x^2}{x+100} \le \frac{x^2(2+\sin^2 x)}{x+100} \le \frac{3x^2}{x+100}$$

Then

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^2}{x+100} &= \lim_{x\to\infty} \frac{2\frac{x^2}{x^2}}{\frac{x}{x^2}+\frac{100}{x^2}} \\
&= \lim_{x\to\infty} \frac{2}{\frac{1}{x}+\frac{100}{x^2}} \\
&= \frac{2}{0+0} \\
& \quad DNE
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^2}{x+100} &= \lim_{x\to\infty} \frac{3\frac{x^2}{x^2}}{\frac{x}{x^2}+\frac{100}{x^2}} \\
&= \lim_{x\to\infty} \frac{3}{\frac{1}{x}+\frac{100}{x^2}} \\
& \quad DNE
\end{aligned}$$

Thus, it follows from the Squeeze Principle that

$$\lim_{x\to\infty} \frac{x^2(2+\sin^2 x)}{x+100} \quad DNE$$

Example 2.2.52 Use the Sandwich theorem to compute

$$\lim_{x\to-\infty} \frac{5x^2-\sin 3x}{x^2+10}$$

First note that

$$-1 \le \sin 3x \le +1$$

so that

$$-1 \le -\sin 3x \le +1$$

$$5x^2 - 1 \leq 5x^2 - \sin 3x \leq 5x^2 + 1$$

and

$$\frac{5x^2 - 1}{x^2 + 10} \leq \frac{5x^2 - \sin 3x}{x^2 + 10} \leq \frac{5x^2 + 1}{x^2 + 10}$$

Then

$$\begin{aligned}
\lim_{x \to -\infty} \frac{5x^2 - 1}{x^2 + 10} &= \lim_{x \to -\infty} \frac{5x^2 - 1}{x^2 + 10} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x \to -\infty} \frac{5 - \frac{1}{x^2}}{1 + \frac{10}{x^2}} \\
&= \frac{5 - 0}{1 + 0} \\
&= 5
\end{aligned}$$

Similarly,

$$\lim_{x \to -\infty} \frac{5x^2 + 1}{x^2 + 10} = 5$$

Thus, it follows from the Squeeze Principle that

$$\lim_{x \to -\infty} \frac{5x^2 - \sin 3x}{x^2 + 10} = 5$$

Example 2.2.53 Compute

$$\lim_{x \to -\infty} \frac{x^2(\sin x + \cos^3 x)}{(x^2 + 1)(x - 3)}$$

Realise that

$$-1 \leq \sin x \leq +1$$

and

$$-1 \leq \cos x \leq +1$$

so that

$$-1 \leq \cos^3 x \leq +1$$

and

$$-2 \leq \sin x + \cos^3 x \leq +2$$

Since we are computing the limit as x goes to negative infinity, it is reasonable to assume that $x - 3 < 0$. Thus, dividing by $x - 3$, we get

$$\frac{-2}{x - 3} \geq \frac{\sin x + \cos^3 x}{x - 3} \geq \frac{2}{x - 3}$$

or

$$\frac{2}{x - 3} \leq \frac{\sin x + \cos^3 x}{x - 3} \leq \frac{-2}{x - 3}$$

Now divide by $x^2 + 1$ and multiply by x^2, getting

$$\frac{2x^2}{(x^2 + 1)(x - 3)} \leq \frac{x^2(\sin x + \cos^3 x)}{(x^2 + 1)(x - 3)} \leq \frac{-2x^2}{(x^2 + 1)(x - 3)}$$

Then

$$\begin{aligned}
\lim_{x\to-\infty} \frac{2x^2}{(x^2+1)(x-3)} &= \lim_{x\to-\infty} \frac{2x^2}{x^3-3x^2+x-3} \\
&= \lim_{x\to-\infty} \frac{2x^2}{x^3-3x^2+x-3} \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\
&= \lim_{x\to-\infty} \frac{\frac{2}{x}}{1-\frac{3}{x}+\frac{1}{x^2}-\frac{3}{x^3}} \\
&= \frac{0}{1-0+0-0} \\
&= 0
\end{aligned}$$

Similarly,

$$\lim_{x\to-\infty} \frac{-2x^2}{(x^2+1)(x-3)} = 0$$

It follows from the Squeeze Principle that

$$\lim_{x\to-\infty} \frac{x^2(\sin x + \cos^3 x)}{(x^2+1)(x-3)} = 0$$

Example 2.2.54 Assume that $\lim\limits_{\theta\to-1^+} f(\theta)$ exists and

$$\frac{\theta^2+\theta-2}{\theta+3} \le \frac{f(\theta)}{\theta^2} \le \frac{\theta^2+2\theta-1}{\theta+3}$$

Find $\lim\limits_{\theta\to-1^+} f(\theta)$?

Since

$$\lim_{\theta\to-1^+} \frac{\theta^2+\theta-2}{\theta+3} = \frac{(-1)^2+(-1)-2}{(-1)+3} = -1$$

and

$$\lim_{\theta\to-1^+} \frac{\theta^2+2\theta-1}{\theta+3} = \frac{(-1)^2+2(-1)-1}{(-1)+3} = -1$$

it follows from the Squeeze Principle that

$$\lim_{\theta\to-1^+} \frac{f(\theta)}{\theta^2} = -1$$

that is,

$$-1 = \lim_{\theta\to-1^+} \frac{f(\theta)}{\theta^2} = \frac{\lim\limits_{\theta\to-1^+} f(\theta)}{\lim\limits_{\theta\to-1^+} \theta^2} = \frac{\lim\limits_{\theta\to-1^+} f(\theta)}{(-1)^2} = \lim_{\theta\to-1^+} f(\theta)$$

Thus,

$$\lim_{\theta\to-1^+} f(\theta) = -1$$

Example 2.2.55 Probably the most well-known examples of finding a limit by squeezing are the proofs of the equality

$$\lim_{x\to 0}\frac{\sin x}{x} \quad = \quad 1 \tag{2.6}$$

The first follows by means of the squeeze theorem from the fact that

$$-\frac{1}{x} < \frac{\sin x}{x} < \frac{1}{x}$$

for x close enough, but not equal to 0.

These two limits, Equation (2.6) and (2.7)

$$\lim_{x\to 0}\frac{1-\cos x}{x} \quad = \quad 0 \tag{2.7}$$

are used in proofs of the fact that the derivative of the sine function is the cosine function. That fact is relied on in other proofs of derivatives of trigonometric functions.

2.3 Properties of limits

If

$$\begin{aligned} \lim_{x\to a} f(x) &= L_1 \quad \text{(a finite number)} \\ \text{and } \lim_{x\to a} g(x) &= L_2 \quad \text{(a finite number)}, \end{aligned}$$

then

(i) $\lim_{x\to a} [f(x) + g(x)] = L_1 + L_2$

(ii) $\lim_{x\to a} [f(x) - g(x)] = L_1 - L_2$

(iii) $\lim_{x\to a} \alpha f(x) = \alpha L_1$ (α is a constant)

(iv) $\lim_{x\to a} f(x)g(x) = L_1 L_2$

(v) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2} \quad L_2 \neq 0$

Example 2.3.1 Illustrate the given properties above. Let

$$\begin{aligned} f(x) &= x^2, \\ g(x) &= x^2 - 1 \\ \texttt{therefore } \lim_{x\to 2} f(x) &= 2^2 = 4 = L_1 \\ \text{and } \lim_{x\to 2} g(x) &= 2^2 - 1 = 3 = L_2 \end{aligned}$$

Therefore

part(i)

$$\begin{aligned} \lim_{x\to 2} [f(x) + g(x)] &= \lim_{x\to 2}(x^2 + x^2 - 1) \\ &= \lim_{x\to 2}(2x^2 - 1) \\ &= 8 - 1 = 7 \\ &= 4 + 3 = L_1 + L_2 \end{aligned}$$

part(ii)

$$\lim_{x\to 2} [f(x) - g(x)] = \lim_{x\to 2}(x^2 - x^2 + 1)$$

$$= \quad 4 - 3 = L_1 - L_2$$

part(iii) $\lim_{x\to 2} \alpha f(x) = \lim_{x\to 2} \alpha x^2 = 4\alpha = \alpha L_1$

part(iv)

$$\lim_{x\to 2} f(x)g(x) = \lim_{x\to 2} x^2(x^2 - 1)$$

$$\begin{aligned} &= \lim_{x\to 2}(x^4 - x^2) \\ &= 16 - 4 = 12 = 4.3 = L_1 L_2 \end{aligned}$$

part(v) $\lim_{x\to 2}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{x^2}{x^2-1} = \frac{4}{3} = \frac{L_1}{L_2}$

Example 2.3.2 Illustrate the properties of limits (even for limits at infinity) if

$$f(x) = \frac{2x-1}{x},\ g(x) = \frac{3x}{x+1}$$

$$\begin{aligned}
\texttt{therefore } \lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\frac{2x-1}{x} \\
&= \lim_{x\to\infty}\left(2-\frac{1}{x}\right) = 2 = L_1 \\
\text{and } \lim_{x\to\infty} g(x) &= \lim_{x\to\infty}\frac{3x}{x+1} \\
&= \lim_{x\to\infty}\left(\frac{3}{1+\frac{1}{x}}\right) = 3 = L_2
\end{aligned}$$

part(i)

$$\begin{aligned}
\lim_{x\to\infty}[f(x)+g(x)] &= \lim_{x\to\infty}\left(\left(\frac{2x-1}{x}\right)+\frac{3x}{x+1}\right) \\
&= \lim_{x\to\infty}\left(\frac{(x+1)(2x-1)+3x^2}{x(x+1)}\right) \\
&= \lim_{x\to\infty}\left(\frac{2x^2-x+2x-1+3x^2}{x^2+x}\right) \\
&= \lim_{x\to\infty}\left(\frac{5x^2+x-1}{x^2+x}\right) \\
&= \lim_{x\to\infty}\left(\frac{5+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}}\right) \\
&= 5 \\
&= 2+3 = L_1+L_2
\end{aligned}$$

part(ii)

$$\begin{aligned}
\lim_{x\to\infty}[f(x)-g(x)] &= \lim_{x\to\infty}\left(\frac{2x-1}{x}-\frac{3x}{x+1}\right) \\
&= \lim_{x\to\infty}\left(\frac{(2x-1)(x+1)-3x^2}{x(x+1)}\right) \\
&= \lim_{x\to\infty}\left(\frac{2x^2+x-1-3x^2}{x^2+x}\right) \\
&= \lim_{x\to\infty}\left(\frac{-x^2+x-1}{x^2+x}\right) \\
&= \lim_{x\to\infty}\left(\frac{-1+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}}\right) \\
&= -1 = 2-3 = L_2-L_3
\end{aligned}$$

part(iii)

$$\begin{aligned}
\lim_{x\to\infty} \alpha f(x) &= \lim_{x\to\infty} \alpha\left(\frac{2x-1}{x}\right)\\
&= \alpha \lim_{x\to\infty}\left(2-\frac{1}{x}\right)\\
&= 2\alpha = \alpha L_1
\end{aligned}$$

part(iv)

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{2x-1}{x}.\frac{3x}{x+1}\right) &= \lim_{x\to\infty}\left(\frac{6x-3}{x+1}\right)\\
&= \lim_{x\to\infty}\frac{6-\frac{3}{x}}{1+\frac{1}{x}}\\
&= 6 = 2.3 = L_1 L_2
\end{aligned}$$

part(v)

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right) &= \lim_{x\to\infty}\left(\frac{\frac{2x-1}{x}}{\frac{3x}{x+1}}\right)\\
&= \lim_{x\to\infty}\left(\frac{\frac{2x-1}{x}}{\frac{x+1}{3x}}\right)\\
&= \lim_{x\to\infty}\left(\frac{2x^2+x-1}{3x^2}\right)\\
&= \lim_{x\to\infty}\frac{2+\frac{1}{x}-\frac{1}{x^2}}{3}\\
&= \frac{2}{3} = \frac{L_1}{L_2}
\end{aligned}$$

Example 2.3.3 Given that

$$\lim_{x\to 2} g(x) = 4 \quad , \quad \lim_{x\to 2} f(x) = 5$$

Compute

(i)

$$\lim_{x\to 2} 4f(x)g(x) = 4\lim_{x\to 2} f(x)g(x) = \quad 4\left[\lim_{x\to 2} f(x)\cdot \lim_{x\to 2} g(x)\right] = 4(5)(4) = 80$$

(ii)

$$\lim_{x\to 2}[f(x)+g(x)] = \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x) = 5+4 = 9$$

(iii)

$$\lim_{x\to 2}\left[\frac{f(x)}{3+g(x)}\right] = \frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2}[3+g(x)]} = \frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2} 3 + \lim_{x\to 2} g(x)} = \frac{5}{3+4} = \frac{5}{7}$$

Example 2.3.4 By the product property of limits,

$$\lim_{x\to -4}\frac{x^2-16}{x+4}\ln|x| = \left(\lim_{x\to -4}\frac{x^2-16}{x+4}\right)\cdot\left(\lim_{x\to -4}\ln|x|\right) = -8\ln(4)$$

Example 2.3.5 Use the techniques of limits to compute

(i)

$$\lim_{u\to\infty}\frac{u}{\sqrt{u^2+1}}$$

Solution: Limit at infinity (highest power is u) or La'hopital, $L = 1$

(ii)

$$\lim_{x\to 8}\frac{(x-8)(x+2)}{|x-8|}$$

Solution: Limit from left and right, L *DNE*

(iii)

$$\lim_{x\to 0}\frac{\cos mx - \cos nx}{x^2}$$

Solution: Double La'hopital, $L = (-m^2+n^2)/2$

2.4 Formal definition of a limit of a function

Definition 2.4.1 We say that L is the limit of $f(x)$ as x approaches a if for every $\epsilon > 0$ (however small but positive) there exists a corresponding $\delta > 0$ also dependent on ϵ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$

The ϵ is an arbitrary small positive number called epsilon and $\delta = \delta(\epsilon)$ is a small positive constant.

2.4.1 Limits of linear functions

Example 2.4.1 Prove that $\lim_{x\to 1} 2x = 2$ (use $\epsilon - \delta$ definition)

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x - 1| < \delta$, then $|f(x) - 2| < \epsilon$.
Begin with $|f(x) - 2|$ and "solve for (have in a term)" $|x - 1|$. Then

$$\begin{aligned}
|f(x) - 2| &= |(2x) - 2| \\
&= |2x - 2| \\
&= |2(x - 1)| \\
&= |2||(x - 1)| \\
&= 2|x - 1| \\
&< 2\delta \\
&< \epsilon \text{ if we choose } \delta = \frac{\epsilon}{2}
\end{aligned}$$

Thus if $|x - 1| < \frac{\epsilon}{2}$, that is if $\delta < \frac{\epsilon}{2}$, it follows that $|f(x) - 2| < \epsilon$. This completes the proof.

Example 2.4.2 Prove that $\lim_{x\to 2}(3x - 1) = 5$ (use $\epsilon - \delta$ definition)
Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x - 2| < \delta$, then $|f(x) - 5| < \epsilon$.

Begin with $|f(x) - 5|$ and "solve for (have in a term)" $|x - 2|$. Then

$$\begin{aligned}
|f(x) - 5| = |(3x - 1) - 5| = |3x - 6| = |3(x - 2)| &= |3||(x - 2)| \\
&= 3|x - 2| \\
&< 3\delta \\
&< \epsilon \text{ if we choose } \delta = \frac{\epsilon}{3}
\end{aligned}$$

Thus if $|x - 2| < \frac{\epsilon}{3}$, that is if $\delta < \frac{\epsilon}{3}$, it follows that $|f(x) - 5| < \epsilon$. This completes the proof.

Example 2.4.3 Show that

$$\lim_{x\to 5} 7 = 7$$

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ so that if $0 < |x - 5| < \delta$, then $|f(x) - 7| < \epsilon$, i.e., $|7 - 7| < \epsilon$, i.e., $|0| < \epsilon$. But this trivial inequality is always true, no matter what value is chosen for δ. For example, $\delta = \frac{1}{2}$ will work. Thus if $0 < |x - 5| < \delta$, then it follows that $|f(x) - 7| < \epsilon$. This completes the proof.

Example 2.4.4 Prove that

$$\lim_{x\to 10}(3x + 5) = 35$$

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x - 10| < \delta$, then $|f(x) - 35| < \epsilon$.

Begin with $|f(x) - 35|$ and "solve for (have in a term)" $|x - 10|$. Then

$$\begin{aligned} |f(x) - 35| = |(3x + 5) - 35| = |3x - 30| &= |3(x - 10)| \\ &= |3||(x - 10)| \\ &= 3|x - 10| \\ &< 3\delta \\ &< \epsilon \text{ if we choose } \delta = \frac{\epsilon}{3} \end{aligned}$$

Thus if $|x - 10| < \frac{\epsilon}{3}$, that is if $\delta < \frac{\epsilon}{3}$, it follows that $|f(x) - 35| < \epsilon$. This completes the proof.

Example 2.4.5 Prove that

$$\lim_{x \to -\frac{3}{2}} (1 - 4x) = 7$$

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x - \left(-\frac{3}{2}\right)| < \delta$, that is $\left|x + \frac{3}{2}\right| < \delta$ then $|f(x) - 7| < \epsilon$.

Begin with $|f(x) - 7|$ and "solve for (have in a term)" $\left|x + \frac{3}{2}\right|$. Then

$$\begin{aligned} |f(x) - 7| &= |(1 - 4x) - 7| \\ &= |-6 - 4x| = \left|(-4)\left(\frac{3}{2} + x\right)\right| = \left|(-4)\left(x + \frac{3}{2}\right)\right| = |-4|\left|x + \frac{3}{2}\right| = 4\left|x + \frac{3}{2}\right| \\ &< 4\delta \\ &< \epsilon \text{ if we choose } \delta = \frac{\epsilon}{4} \end{aligned}$$

Thus if $\left|x + \frac{3}{2}\right| < \frac{\epsilon}{4}$, that is if $\delta < \frac{\epsilon}{4}$, it follows that $|f(x) - 7| < \epsilon$. This completes the proof.

Example 2.4.6 Given that

$$\lim_{x \to 8} \frac{x}{4} = 2$$

What is δ when $\varepsilon = 0.01$ for this limit?
Solution: $\delta = 4\varepsilon = 0.04$, *but any smaller number will also work.*

Example 2.4.7 Use the rigorous definition of the limit in order to show that

$$\lim_{x \to 1} (5x - 3) = 2$$

Given $\epsilon > 0$ we find a $\delta(\epsilon) > 0$ such that if $|x - a| < \delta$, then $|f(x) - L| < \epsilon$. In otherwords, we need to show that if $|x - 1| < \delta$ then $|f(x) - 2| < \epsilon$

Begin with $|f(x) - 2|$ and "solve for (have in a term)" $|x - 1|$. Then

$$\begin{aligned} |f(x) - 2| &= |(5x - 3) - 2| \\ &= |5x - 5| = |5(x - 1)| \\ &= |5||(x - 1)| \\ &= 5|x - 1| \\ &< 5\delta \\ &< \epsilon \text{ if we choose } \delta = \frac{\epsilon}{5} \end{aligned}$$

Thus if $|x - 1| < \frac{\epsilon}{5}$, that is if $\delta < \frac{\epsilon}{5}$, it follows that $|f(x) - 2| < \epsilon$.

2.4.2 Uniqueness theorem of limits

Theorem 2.4.1 (Uniqueness of limits) *Let $X \subset \mathbb{R}$, $a \in \mathbb{R}$, and $f : X \to \mathbb{R}$. If*

$$\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} f(x) = L'$$

then

$$L = L'$$

Proof by Contradiction: If $L \neq L'$ then

$$|L - L'| > \epsilon \tag{2.8}$$

From the definition of limit. For all $\epsilon > 0$ there is $\delta > 0$ such that $0 < |x - a| < \delta$ whenever

$$|f(x) - L| < \frac{\epsilon}{2}$$
$$|f(x) - L'| < \frac{\epsilon}{2}$$

By the Triangle Inequality

$$\begin{aligned} |L - L'| &= |L - f(x) + f(x) - L'| \\ &< |L - f(x)| + |f(x) - L'| \\ &< |f(x) - L| + |f(x) - L'| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon \\ \Rightarrow \quad |L - L'| &< \epsilon \end{aligned} \tag{2.9}$$

a contradiction for (2.8) and (2.9). Thus a limit is unique.

Note 2.4.1 We can use $\leq$ and $<$ interchangeably when considering the $\epsilon - \delta$ definition, since looking at a very small neighborhood.

2.4.3 Limits of non linear functions

Example 2.4.8 Prove that $\lim_{x\to 1} x^2 = 1$ (use $\epsilon - \delta$ definition)

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x-1| < \delta$, then $|f(x) - 1| < \epsilon$.

Begin with $|f(x) - 1|$ and "solve for (have in a term)" $|x-1|$. Then

$$\begin{aligned} |f(x) - 1| &= |(x^2) - 1| \\ &= |x^2 - 1| \\ &= |(x-1)(x+1)| \\ &= |x-1||x+1| \end{aligned}$$

We will now "replace" the term $|x+1|$ with an appropriate constant and keep the term $|x-1|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta < 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work).
Then implies that $-1 < x - 1 < 1$ and $0 < x < 2$ so that $1 < |x+1| < 3$ (Make sure that you understand this step before proceeding). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof).

$$\begin{aligned} |f(x) - 1| &= |(x^2) - 1| \\ &= |x^2 - 1| \\ &= |(x-1)(x+1)| \\ &= |x-1||x+1| \\ &= 3|x-1| \\ &< 3\delta \end{aligned}$$

Now choose $\delta = \min\left\{1, \frac{\epsilon}{3}\right\}$. (This guarantees that both assumptions made about in the course of this proof are taken into account simultaneously).

Thus if $0 < |x-1| < \delta$, it follows that $|f(x) - 1| < \epsilon$. This completes the proof.

Example 2.4.9 Prove that $\lim_{x\to 3}(x^2 - 3x + 2) = 2$ (use $\epsilon - \delta$ definition)

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x-3| < \delta$, then $|f(x) - 2| < \epsilon$.

Begin with $|f(x) - 2|$ and "solve for (have in a term)" $|x-3|$. Then

$$\begin{aligned} |f(x) - 2| &= |(x^2 - 3x + 2) - 2| \\ &= |x^2 - 3x| \\ &= |x(x-3)| \\ &= |x||x-3| \end{aligned}$$

We will now "replace" the term $|x|$ with an appropriate constant and keep the term $|x-3|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta < 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work).

Then implies that $-1 < x - 3 < 1$ and $2 < x < 4$, so that $2 < |x| < 4$ (Make sure that you understand this step before proceeding). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof).

$$\begin{aligned} |f(x) - 2| &= |(x^2 - 3x + 2) - 2| \\ &= |x^2 - 3x| \\ &= |x(x-3)| \\ &= |x||x-3| \\ &= 4|x-3| \\ &< 4\delta \end{aligned}$$

Now choose $\delta = \min\left\{1, \frac{\epsilon}{4}\right\}$. (This guarantees that both assumptions made about in the course of this proof are taken into account simultaneously).

Thus if $0 < |x - 3| < \delta$, it follows that $|f(x) - 2| < \epsilon$. This completes the proof.

Example 2.4.10 Prove that $\lim\limits_{x\to 3}(2x^2 - 10) = 8$ (use $\epsilon - \delta$ definition)

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x - 3| < \delta$, then $|f(x) - 8| < \epsilon$.

Begin with $|f(x) - 8|$ and "solve for (have in a term)" $|x - 3|$. Then

$$\begin{aligned} |f(x) - 8| &= |(2x^2 - 10) - 8| \\ &= |2x^2 - 18| \\ &= |2(x^2 - 9)| \\ &= |2(x-3)(x+3)| \\ &= |2||x+3||x-3| \end{aligned}$$

We will now "replace" the term $|x+3|$ with an appropriate constant and keep the term $|x-3|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta < 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work).
Then implies that $-1 < x - 3 < 1$ and $2 < x < 4$, so that $5 < |x + 3| < 7$ (Make sure that you understand this step before proceeding). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof).

$$\begin{aligned} |f(x) - 8| &= |(2x^2 - 10) - 8| \\ &= |2x^2 - 18| \\ &= |2(x^2 - 9)| \\ &= |2(x-3)(x+3)| \\ &= |2||x+3||x-3| \\ &= 14|x-3| \\ &< 14\delta \end{aligned}$$

Now choose $\delta = \min\left\{1, \frac{\epsilon}{14}\right\}$. (This guarantees that both assumptions made about in the course of this proof are taken into account simultaneously).

Thus if $0 < |x - 3| < \delta$, it follows that $|f(x) - 8| < \epsilon$. This completes the proof.

Example 2.4.11 Prove that

$$\lim_{x \to 1}(x^2 + 3) = 4$$

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x - 1| < \delta$, then $|f(x) - 4| < \epsilon$.

Begin with $|f(x) - 4|$ and "solve for (leave in a term)" $|x - 1|$. Then

$$\begin{aligned} |f(x) - 4| &= |(x^2 + 3) - 4| \\ &= |x^2 - 1| \\ &= |(x - 1)(x + 1)| \\ &= |x - 1||x + 1| \end{aligned}$$

We will now "replace" the term $|x+1|$ with an appropriate constant and keep the term $|x-1|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta < 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work).
Then implies that $-1 < x - 1 < 1$ and $0 < x < 2$ so that $1 < |x + 1| < 3$ (Make sure that you understand this step before proceeding). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof).

$$\begin{aligned} |f(x) - 4| &= |(x^2 + 3) - 4| \\ &= |x^2 - 1| \\ &= |(x - 1)(x + 1)| \\ &= |x - 1||x + 1| \\ &= 3|x - 1| \\ &< 3\delta \end{aligned}$$

Now choose $\delta = \min\left\{1, \frac{\epsilon}{3}\right\}$. (This guarantees that both assumptions made about in the course of this proof are taken into account simultaneously).

Thus if $0 < |x - 1| < \delta$, it follows that $|f(x) - 4| < \epsilon$. This completes the proof.

Example 2.4.12 Prove that

$$\lim_{x \to 2}(3x^2 - x) = 10$$

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x - 2| < \delta$, then $|f(x) - 10| < \epsilon$.

Begin with $|f(x) - 10|$ and "solve for (have in a term)" $|x - 2|$. Then

$$|f(x) - 10| = |(3x^2 - x) - 10| = |3x^2 - x - 10| = |(3x + 5)(x - 2)| = |3x + 5||x - 2|$$

We will now "replace" the term $|3x+5|$ with an appropriate constant and keep the term $|x-2|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta < 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work).
Then implies that $-1 < x - 2 < 1$ and $1 < x < 3$ so that $8 < |3x + 5| < 14$ (Make sure that you understand this step before proceeding). It follows that (Always make this "replacement"

between your last expression on the left and ϵ. This guarantees the logic of the proof).

$$\begin{aligned} |f(x) - 10| &= |(3x^2 - x) - 10| \\ &= |3x^2 - x - 10| \\ &= |(3x+5)(x-2)| \\ &= |3x+5||x-2| \\ &= 14|x-2| \\ &< 14\delta \end{aligned}$$

Now choose $\delta = \min\left\{1, \frac{\epsilon}{14}\right\}$. (This guarantees that both assumptions made about in the course of this proof are taken into account simultaneously).

Thus if $0 < |x-2| < \delta$, it follows that $|f(x) - 10| < \epsilon$. This completes the proof.

Example 2.4.13 Prove that

$$\lim_{x \to 3} \frac{2}{x+3} = \frac{1}{3}$$

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x-3| < \delta$, then $\left|f(x) - \frac{1}{3}\right| < \epsilon$.

Begin with $\left|f(x) - \frac{1}{3}\right|$ and "solve for (have in a term)" $|x-3|$. Then

$$\begin{aligned} \left|f(x) - \frac{1}{3}\right| &= \left|\frac{2}{x+3} - \frac{1}{3}\right| = \left|\frac{6-(x+3)}{3(x+3)}\right| = \left|\frac{3-x}{3(x+3)}\right| \\ &= \frac{|3-x|}{|3||x+3|} = \frac{|(-1)(x-3)|}{|3||x+3|} = \frac{|-1||x-3|}{|3||x+3|} = \frac{1}{3}\frac{|x-3|}{|x+3|} \\ &= \frac{1}{3}|x-3|\frac{1}{|x+3|} \end{aligned}$$

We will now "replace" the term $\frac{1}{|x+3|}$ with an appropriate constant (by first thinking of $|x+3|$ and later take the reciprocal) and keep the term $|x-3|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta < 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work).
Then implies that $-1 < x-3 < 1$ and $2 < x < 4$ so that $5 < |x+3| < 7$ and thus

$$\frac{1}{7} < \frac{1}{|x+3|} < \frac{1}{5}$$

Make sure that you understand this step before proceeding. It follows that - Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof.

$$\left|f(x) - \frac{1}{3}\right| = \frac{1}{3}|x-3|\frac{1}{|x+3|} = \frac{1}{3}|x-3|\frac{1}{5} = \frac{1}{15}|x-3| < \frac{1}{15}\delta$$

Now choose $\delta = \min\{1, 15\epsilon\}$. (This guarantees that both assumptions made about in the course of this proof are taken into account simultaneously).

Thus if $0 < |x-3| < \delta$, it follows that $\left|f(x) - \frac{1}{3}\right| < \epsilon$. This completes the proof.

Example 2.4.14 Prove that
$$\lim_{x\to -6}\frac{x+4}{2-x}=-\frac{1}{4}$$
Begin by letting $\epsilon>0$ be given. Find $\delta>0$ (which depends on ϵ) so that if $0<|x+6|<\delta$, then $\left|f(x)+\frac{1}{4}\right|<\epsilon$.

Begin with $\left|f(x)+\frac{1}{4}\right|$ and "solve for" $|x+6|$. Then
$$\begin{aligned}\left|f(x)+\frac{1}{4}\right| &= \left|\frac{x+4}{2-x}+\frac{1}{4}\right|=\left|\frac{4(x+4)+(2-x)}{4(2-x)}\right|\\ &= \left|\frac{3x+18}{4(2-x)}\right|=\left|\frac{3(x+6)}{4(2-x)}\right|=\frac{|3||x+6|}{|4||2-x|}=\frac{3}{4}\frac{|x+6|}{|2-x|}\\ &= \frac{3}{4}|x+6|\frac{1}{|2-x|}\end{aligned}$$

We will now "replace" the term $\frac{1}{|2-x|}$ with an appropriate constant (by first thinking of $|2-x|$ and later take the reciprocal) and keep the term $|x+6|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta<1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work).
Then implies that $-1<x+6<1$ and $-7<x<-5$ so that $7<|2-x|<9$ (The number 9 is bigger than 7, so take care when writting)and thus
$$\frac{1}{9}<\frac{1}{|2-x|}<\frac{1}{7}$$
(Make sure that you understand this step before proceeding). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof).
$$\left|f(x)+\frac{1}{4}\right| = \frac{3}{4}|x+6|\frac{1}{|2-x|}=\frac{3}{4}|x+6|\frac{1}{7}=\frac{3}{28}|x+6|<\frac{3}{28}\delta$$
Now choose $\delta=\min\left\{1,\frac{28}{3}\epsilon\right\}$. (This guarantees that both assumptions made about in the course of this proof are taken into account simultaneously).

Thus if $0<|x+6|<\delta$, it follows that $\left|f(x)+\frac{1}{4}\right|<\epsilon$. This completes the proof.

Example 2.4.15 Prove that
$$\lim_{x\to 3}\frac{x}{4x-9}=1$$
Begin by letting $\epsilon>0$ be given. Find $\delta>0$ (which depends on ϵ) so that if $0<|x-3|<\delta$, then $|f(x)-1|<\epsilon$.

Begin with $|f(x)-1|$ and "solve for" $|x-3|$. Then
$$|f(x)-1| = \left|\frac{x}{4x-9}-1\right|=\left|\frac{9-3x}{4x-9}\right|=\left|\frac{(-3)(x-3)}{4x-9}\right|=|-3|\frac{|x-3|}{|4x-9|}=3|x-3|\frac{1}{|4x-9|}$$
We will now "replace" the term $\frac{1}{|4x-9|}$ with an appropriate constant (by first thinking of $|4x-9|$ and later take the reciprocal) and keep the term $|x-3|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta<1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work).

Then $|x-3|<1$ implies that $-1<x-3<1$ and $2<x<4$.

However, this range of x values is not appropriate since the function $f(x)=\frac{x}{4x-9}$ is not defined at $x=\frac{9}{4}$! Fortunately, this problem can be easily resolved. We simply pick small enough to avoid $x=\frac{9}{4}$. For example, assume that $\delta<\frac{1}{4}$

Then $|x-3|<\frac{1}{4}$ implies that $-\frac{1}{4}<x-3<\frac{1}{4}$ and $\frac{11}{4}<x<\frac{13}{4}$ so that $2<|4x-9|<4$ and thus

$$\frac{1}{4}<\frac{1}{|4x-9|}<\frac{1}{2}$$

(Make sure that you understand this step before proceeding). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof).

$$|f(x)-1| \;=\; 3|x-3|\frac{1}{|4x-9|}=3|x-3|\frac{1}{2}=\frac{3}{2}|x-3|<\frac{3}{2}\delta$$

Now choose $\delta=\min\left\{\frac{1}{4},\frac{2}{3}\epsilon\right\}$. (This guarantees that both assumptions made about in the course of this proof are taken into account simultaneously).

Thus if $0<|x-3|<\delta$, it follows that $|f(x)-1|<\epsilon$. This completes the proof.

Example 2.4.16 Prove that

$$\lim_{x\to 9}\left(2+\sqrt{x}\,\right)=5$$

Begin by letting $\epsilon>0$ be given. Find $\delta>0$ (which depends on ϵ) so that if $0<|x-9|<\delta$, then $|f(x)-5|<\epsilon$.

Begin with $|f(x)-5|$ and "solve for (have the term)" $|x-9|$. Then

$$|f(x)-5| \;=\; \left|\left(2+\sqrt{x}\,\right)-5\right|=\left|\sqrt{x}-3\right|$$

(At this point, we need to figure out a way to make $|x-9|$ "appear" in our computations. Appropriate use of the conjugate will suffice).

$$|f(x)-5| \;=\; \left|(\sqrt{x}-3)\frac{(\sqrt{x}+3)}{(\sqrt{x}+3)}\right|=\left|\frac{(x-9)}{(\sqrt{x}+3)}\right|=\frac{|x-9|}{|\sqrt{x}+3|}=|x-9|\frac{1}{|\sqrt{x}+3|}$$

We will now "replace" the term $\frac{1}{|\sqrt{x}+3|}$ with an appropriate constant (by first thinking of $|\sqrt{x}+3|$ and later take the reciprocal) and keep the term $|x-9|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta<1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work).
Then $0<|x-9|<1$ implies that $-1<x-9<1$ and $8<x<10$ so that $\sqrt{8}+3<|\sqrt{x}+3|<\sqrt{10}+3$ and thus

$$\frac{1}{\sqrt{10}+3}<\frac{1}{|\sqrt{x}+3|}<\frac{1}{\sqrt{8}+3}$$

(Make sure that you understand this step before proceeding). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof).

$$|f(x)-5| \;=\; |x-9|\frac{1}{|\sqrt{x}+3|}=|x-9|\frac{1}{\sqrt{8}+3}<\frac{1}{\sqrt{8}+3}\delta$$

Now choose $\delta=\min\left\{1,(\sqrt{8}+3)\epsilon\right\}$. (This guarantees that both assumptions made about in the course of this proof are taken into account simultaneously).

Thus if $0<|x-9|<\delta$, it follows that $|f(x)-5|<\epsilon$. This completes the proof.

Exercise 2.4 Prove that

$$\lim_{x\to 4}\sqrt{x+5} = 3 \qquad\qquad Ans: \quad \delta = \min\left\{1, (\sqrt{8}+3)\epsilon\right\}$$

Example 2.4.17 Prove that

$$\lim_{x\to 1}\frac{x+3}{1+\sqrt{x}} = 2$$

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x-1| < \delta$, then $|f(x) - 2| < \epsilon$.

Begin with $|f(x) - 2|$ and "solve for" $|x-1|$. Then

$$\begin{aligned} |f(x)-2| &= \left|\frac{x+3}{1+\sqrt{x}} - 2\right| \\ &= \left|\frac{x+1-2\sqrt{x}}{1+\sqrt{x}}\right| \end{aligned}$$

(At this point, we need to figure out a way to make $|x-1|$ "appear" in our computations. A simple use of constants will get us started).

$$\begin{aligned} |f(x)-2| &= \left|\frac{x-1+1+1-2\sqrt{x}}{1+\sqrt{x}}\right| \\ &= \left|\frac{x-1+2-2\sqrt{x}}{1+\sqrt{x}}\right| \\ &= \left|\frac{x-1+2(1-\sqrt{x}\ )}{1+\sqrt{x}}\right| \end{aligned}$$

We need to be able to factor $(x-1)$ from the numerator. Apply the conjugate to the term $(1-\sqrt{x}\ )$

$$\begin{aligned} |f(x)-2| &= \left|\frac{x-1}{1+\sqrt{x}} + \frac{2(1-\sqrt{x}\ )}{1+\sqrt{x}}\right| \\ &= \left|\frac{x-1}{1+\sqrt{x}} + 2\frac{1-\sqrt{x}}{1+\sqrt{x}}\frac{1+\sqrt{x}}{1+\sqrt{x}}\right| \\ &= \left|\frac{x-1}{1+\sqrt{x}} + 2\frac{1-x}{(1+\sqrt{x})^2}\right| \\ &= \left|\frac{x-1}{1+\sqrt{x}} - 2\frac{x-1}{(1+\sqrt{x})^2}\right| \end{aligned}$$

(Now get a common denominator)

$$\begin{aligned} |f(x)-2| &= \left|\frac{(x-1)(1+\sqrt{x}) - 2(x-1)}{(1+\sqrt{x})^2}\right| \\ &= \left|\frac{(x-1)(\sqrt{x}-1)}{(1+\sqrt{x})^2}\right| \\ &= |x-1|\ |\sqrt{x}-1|\ \frac{1}{(1+\sqrt{x})^2} \end{aligned}$$

Since the last term has a square, no need to take its modulus

We will now "replace" the term $\sqrt{x}-1$ with an appropriate constant and $(1+\sqrt{x})^2$ and keep

the term $|x-1|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta < 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work).
Then $|x-1| < 1$ implies that $-1 < x-1 < 1$ and $0 < x < 2$ so that

$$\begin{aligned} 0 &< \quad |\sqrt{x}-1| \quad < 1 && (2.10)\\ 1 &< \quad 1+\sqrt{x} \quad < 1+\sqrt{2} \\ 1^2 &< \quad (1+\sqrt{x}\,)^2 \quad < (1+\sqrt{2}\,)^2 \\ \frac{1}{(1+\sqrt{2}\,)^2} &< \quad \tfrac{1}{(1+\sqrt{x}\,)^2} \quad < 1 && (2.11)\end{aligned}$$

(Make sure that you understand these steps before proceeding). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof).

$$|f(x)-2| \quad = \quad |x-1|\;|\sqrt{x}-1|\;\frac{1}{(1+\sqrt{x})^2} = |x-1|(1)(1) = |x-1|(1)(1) < \delta$$

Now choose $\delta = \min\{1, \epsilon\}$. (This guarantees that both assumptions made about in the course of this proof are taken into account simultaneously).

Thus if $0 < |x-1| < \delta$, it follows that $|f(x)-2| < \epsilon$. This completes the proof.

Exercise 2.5 Prove that

$$\lim_{x\to a} \sin x = \sin a$$

Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x-a| < \delta$, then $|f(x) - \sin a| < \epsilon$.

Begin with $|f(x) - \sin a|$ and "solve for" $|x-a|$. Then

$$|f(x) - \sin a| < \epsilon \quad \text{iff} \quad |\sin x - \sin a| < \epsilon$$

At this point, we need to figure out a way to introduce the term $|x-a|$ into our computations. Begin with $|f(x) - \sin a|$ and "solve for" $|x-a|$. Then

$$\begin{aligned}
|\sin x - \sin a| &= \left|2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x+a}{2}\right)\right| \\
&= 2\left|\sin\left(\frac{x-a}{2}\right)\right|\left|\cos\left(\frac{x+a}{2}\right)\right| \\
&= 2\left|\sin\left(\frac{x-a}{2}\right)\right| \\
&= 2\left|\frac{x-a}{2}\right| \\
&= |x-a| \\
&< \delta
\end{aligned}$$

Now choose $\delta = \min\{1, \epsilon\}$. (This guarantees that both assumptions made about in the course of this proof are taken into account simultaneously).

Thus if $0 < |x-a| < \delta$, it follows that $|f(x) - \sin a| < \epsilon$. This completes the proof.

Exercise 2.6 Compute the following limits

(i) $\lim\limits_{x\to 2}(x^2 + 2x + 1)$

(ii) $\lim\limits_{x\to -2} \frac{2x^3+2x^2+1}{x+3}$

(iii) $\lim\limits_{x\to 3} \frac{x^2-9}{x-3}$

(iv) $\lim\limits_{x\to 0} \sin x$

Exercise 2.7 Prove that

$$\lim_{x\to a} \ln x = \ln a \qquad \left[\delta = \frac{a\epsilon}{2}\right]$$

Exercise 2.8

(1. Compute the following functions

(i) $\lim\limits_{x\to 1} \frac{x}{1-x^2}$

(ii) $\lim\limits_{x\to -1} \frac{x}{1-x^2}$

(iii) $\lim\limits_{x\to -\infty} \frac{2x^2+3}{3x^2-3x+1}$

(iv) $\lim\limits_{x\to -\infty} \frac{x^5-4}{x^4+6x^3}$

(2. Find $\lim\limits_{x\to 0}(x + \frac{1}{x})$ and sketch a graph to support your answer.

(3. Find $\lim\limits_{x\to 1}(\frac{1}{x-1})$ and support your answer with a relevant sketch.

(4. Compute $\lim\limits_{x\to \frac{\pi}{2}} |\tan x|$

Exercise 2.9

(1. Use the techniques of computing limits to find'

(i) $\lim_{x\to 1} 3x$ (iii) $\lim_{x\to -2} 4x^2+3$ (v) $\lim_{\theta\to 0} \frac{1-\cos\alpha\theta}{\theta}$

(ii) $\lim_{x\to 3} x^2-1$ (iv) $\lim_{\theta\to 0} \frac{\sin\alpha\theta}{\theta}$ (vi) $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$

(2. Let

$$f(x) = \begin{cases} x+k & x \leq -1 \\ x^2, & x > -1 \end{cases}$$

find the value of the constant k so that $\lim_{x\to -1} f(x)$ exists and find it . Then sketch the graph of $f(x)$

(3. Given

$$f(x) = \begin{cases} 2, & x \leq -1 \\ \alpha x, & -1 < x \leq 1 \\ \beta x^2, & x > 1 \end{cases}$$

find α and β so that the following limits exist (find the stated limits)

(i) $\lim_{x\to -1} f(x)$ (ii) $\lim_{x\to 1} f(x)$

(4. For

$$g(x) = \begin{cases} \frac{1}{x} & x > 0 \\ -x^2, & x \leq 0 \end{cases}$$

Check whether the function has a limit as x approaches zero.

Exercise 2.10 Suppose f is defined on $[-1, 3]$ and satisfies:

$$f(x) = \begin{cases} x, & -1 \leq x < 0 \\ -\frac{1}{2}x^2, & 0 \leq x < 1 \\ \sqrt{1-(x-2)^2}, & 1 \leq x \leq 3, x \neq 2 \\ 0, & x = 2 \end{cases}$$

(i) Sketch the graph of the function given above.

(ii) Does $\lim_{x\to 2} f(x)$ exist? Justify your answer.

(iii) Does $\lim_{x\to 1} f(x)$ exist? Justify your answer.

(iv) Does $\lim_{x\to 4} f(x)$ exist? Justify your answer.

Exercise 2.11

(1). Suppose that a function $f(x)$ is defined for all $x \in [-2, 2]$. Can anything be said about the existence of $\lim_{x\to 0} f(x)$? Give reasons for your answer.

(2). Suppose that g is a function defined for all x. If $g(1) = 5$, must $\lim_{x\to 1} g(x)$ exist? If it does, then must $\lim_{x\to 1} g(x) = 5$? Can we conclude anything about $\lim_{x\to 1} g(x)$? Explain!

Exercise 2.12 Find the following limits.

(a) $\lim\limits_{h\to 0} \frac{\sqrt{5h+4}-2}{h}$ (c) $\lim\limits_{t\to 0} \frac{1+t+\sin(t)}{3\cos(t)}$ (e) $\lim\limits_{x\to 0} \frac{\sin x}{\sqrt{x}}$

(b) $\lim\limits_{x\to -2} \frac{x+2}{\sqrt{x^2+5}-3}$ (d) $\lim\limits_{u\to 1} \frac{u^6-1}{u^4-1}$

Exercise 2.13 Use Sandwich Theorem and limit laws to show that

(1). $\lim\limits_{t\to 0} t^2\cos(20\pi t) = 0$ (3). $\lim\limits_{h\to 0} \left(h^2\cos(\frac{2}{h})+1\right)\left(h^2\cos(\frac{2}{h})-1\right) = -1$

(2). $\lim\limits_{x\to 0} \sqrt{x^3+x^2}\sin(\frac{\pi}{x}) = 0$ (4). $\lim\limits_{u\to 0} u^2 4^{\sin(\frac{\pi}{u})} = 0$

Example 2.4.18 Compute

$$\lim_{x\to 0} \frac{\sin x}{x}$$

by first expanding $\sin x$ (an alternative to applying the La'Hopital's rule, but come to the same answer)

Using the Taylor series expansion at $x = 0$

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x}{x} &\approx \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x} \\ &\approx \lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ &= 1 - 0 + 0 - 0 \\ &= 1 \end{aligned}$$

Example 2.4.19 Compute

$$\lim_{x\to 0} \frac{2x+1}{x^2}$$

Applying all the 5 methods, the limit DNE

Example 2.4.20 Compute

$$\lim_{x\to 0} \frac{(2x+3)^2-9}{x}$$

At first sight, might think the limit DNE, but if expand the Numerator

$$\lim_{x\to 0} \frac{(2x+3)^2-9}{x} = \lim_{x\to 0} \frac{(4x^2+12x+9)-9}{x} = \lim_{x\to 0} \frac{4x^2+12x}{x} = \lim_{x\to 0}(4x+12) = 12$$

Note that, we could also have used the La'Hopital's rule.

Exercise 2.14 Prove that

(a)

$$\lim_{x\to 4}(x-4) = 0 \quad \texttt{and} \quad \lim_{x\to 1}(4x+2) = 6$$

(b)

$$\lim_{x\to x_0}(ax+b) = ax_0 + b$$

with a, b, x_0 constants. Comment on the value of δ if $a = 0$.

(c)

$$\lim_{x \to 0} \alpha = \alpha$$

where α is a scalar.

Exercise 2.15 Use the $\epsilon - \delta$ definition of limits to prove that for

$$f(x) = \begin{cases} x + 2\,, & x \geq 1 \\ x - 3\,, & x < 1 \end{cases}$$

(i)

$$\lim_{x \to 1^+} f(x) = 3$$

(ii)

$$\lim_{x \to 1^-} f(x) = -2$$

Exercise 2.16 Show that

(a)

$$\lim_{x \to 2} 5x = 10$$

(b)

$$\lim_{x \to 4} (3 - x) = -1$$

2.5 Chapter Examples

1. Use the definition of a limit to prove that

$$\lim_{x\to a}(mx+c) = ma + c$$

Given $\epsilon > 0$, we find $\delta(\epsilon) > 0$ such that for $0 < |x-a| < \delta$ then $|f(x) - L| < \epsilon$.

Given

$$0 < \ |x-a| \ < \delta$$

$$\begin{aligned} |f(x) - L| &= |(mx+c) - (ma+c)| \\ &= |mx - ma| = |m(x-a)| \\ &= |m||x-a| \\ &= m|x-a| \\ &< m\delta \\ &< \epsilon \ , \ \text{ if choose } \ \delta = \frac{\epsilon}{m} \end{aligned}$$

Therefore, for $\delta = \frac{\epsilon}{m}$, $\Rightarrow$ $|f(x) - L| < \epsilon$

2. Use L'hôpital's rule to find

   (a) $\lim\limits_{x\to\infty} \frac{\ln x}{x}$

   `La'Hopital is not by quotient rule`

   (b) $\lim\limits_{x\to\infty} \frac{x^n}{e^x}$; n is a positive integer.

   $$\texttt{La'Hopital many times, n times} : \ \lim_{x\to\infty} \frac{n!x^0}{e^x} = 0$$

   (c) $\lim\limits_{x\to\infty} xe^{\frac{1}{x}} - x$

   `Indeterminate form` $\Rightarrow$ `La'Hopital`

3. *True* or *False*? If $f(a)$ does not exist then neither does $\lim\limits_{x\to a} f(x)$.

4. *True* or *False*? Given

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty} \ , \ \frac{0}{0}$$

Then the La'Hopital's rule states that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

5. *True* or *False*?

   (a) Let$f(x) = |2x - 3|$ then f is a one-to-one (injective) function.

   (b) If

   $$f(x) = \frac{x+10}{x-10}$$

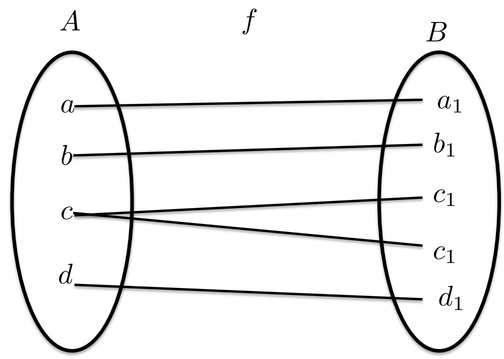
   then the Intermediate Value Theorem guarantees a number $c \in [20, 30]$ such that $f(c) = 2.5$

(c) Let $f(x)$ be defined on $\mathbb{R}$ and $x_0 \in \mathbb{R}$. If $\lim\limits_{x\to x_0} f(x)$ exists, then

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \quad \text{exists.}$$

`False as definition of limit not of a derivative`

(d) Consider $f : A \to B$ represented by the mapping below:



then f is a one-to-one (injective) function.

6. What is $\lim\limits_{x\to 0}\frac{e^{2x}-1}{\tan x}$

(A) -1 (B) 0 (C) 1 (D) 2 (E) The limit does not exist.

7. Of the following choices of δ, which is the largest that could be used successfully with an arbitrary ϵ in an epsilon-delta proof of $\lim\limits_{x\to 2}(1-3x)=-5$?

(A) $\delta = 3\epsilon$ (B) $\delta = \epsilon$ (C) $\delta = \frac{\epsilon}{2}$ (D) $\delta = \frac{\epsilon}{4}$ (E) $\delta = \frac{\epsilon}{5}$

We use $\delta < \frac{\epsilon}{3}$. Of the five choices, the largest satisfying this condition is $\delta = \frac{\epsilon}{4}$

8. let $f(x) = 3x+1$ for all real x and let $\epsilon > 0$. For which of the following choices of δ is $|f(x)-7| < \epsilon$ whenever $|x-2| < \delta$?

(A) $\frac{\epsilon}{4}$ (B) $\frac{\epsilon}{2}$ (C) $\frac{\epsilon}{\epsilon+1}$ (D) $\frac{\epsilon+1}{\epsilon}$ (E) 3ϵ

`Always need a smaller` δ`, your choice should be one less than computed one, if missing.`

9. (a) State the Sandwich/Squeeze Law/Pinching Theorem. Hence, find $\lim\limits_{x\to 0}\sqrt[3]{x}\sin\frac{1}{x}$.

(b) Compute the following limits

(i) $\lim\limits_{x\to 16}\frac{16-x}{4-\sqrt{x}}$ (ii) $\lim\limits_{x\to\infty}(1+e^x)^{e^{-x}}$ (iii) $\lim\limits_{x\to 0}\left(\frac{2}{sinx}-\frac{2}{x}\right)$

10. Evaluate the limits

(a) $\lim\limits_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x^3+8}$

(b) $\lim\limits_{x\to -\infty}\frac{5x^3-\sin 3x}{x^2+10}$

(c) $\lim\limits_{x\to\infty}\left(1+e^{x^2}\right)^{\frac{1}{x^2}}$

(d) $\lim\limits_{x\to -\infty}\frac{4x^3+4x}{|x|(2x+1)(x+5)}$

(e) $\lim\limits_{x\to -2}\frac{x^2+2x}{\sqrt{2-x}-\sqrt{-2x}}$

(f) $\lim\limits_{x\to\infty}\frac{\cos^2 2x}{3-2x}$

11. Explain why it is not completely correct to say that
$$\frac{x^2-64}{x-8} = (x+8)$$
but it is correct to say that
$$\lim_{x\to 8}\frac{x^2-64}{x-8} = \lim_{x\to 8}(x+8)$$

12. State whether the following statement is true or false? Thereafter explain your answer. "If $f(x)$ is not defined at $x=a$ then $\lim_{x\to a} f(x)$ does not exist"

13. State whether the following expression is true or false? Thereafter explain your answer.
$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \left.\frac{x^2-9}{x-3}\right|_{x=3}$$

14. Write an appropriate ordinary (English) statement that best fits the following theorem
$$\lim_{x\to x_0} f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = f(x_0)$$

15. Use La'Hopital's rule or otherwise to evaluate the following limit
$$\lim_{x\to 0}\frac{\tan 3x}{\tan 53}$$

For each of the equations in exercises $16-21$, y is determined as a quadratic function of x. Graph each by first completing square.

16. $3x^2 - 2y - 8 = 0$
17. $y - 3x^2 + x - \frac{1}{2} = 0$
18. $4x^2 + y - 24x + 34 = 0$
19. $3x^2 - y - 7 = 0$
20. $y - 2x^2 + 4x - 5 = 0$
21. $y - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{11}{24} = 0$

In each of the exercises $22-29$, graph the function.

22. $f(x) = \frac{1}{x-1}$
23. $f(x) = \sqrt{2x+1}$
24. $f(x) = \begin{cases} 7-x & , \quad x \le 2 \\ 2x+1 & , \quad x > 2 \end{cases}$
25. $f(x) = x(4-2x)$
26. $f(x) = \frac{x^2-1}{x-1}$
27. $f(x) = \begin{cases} \sqrt{x+2} & , \quad -2 \le x \le 2 \\ 4-x & , \quad x > 2 \end{cases}$
28. $f(x) = (x-1)^{\frac{3}{2}}$
29. $f(x) = (x+2)^{\frac{2}{3}}$

In exercises $30-35$, sketch the graph by first noting where the expression inside the absolute value signs changes sign.

30. $f(x) = x + |x|$
31. $y = |1-x^2|$
32. $y = \frac{x+1}{|x-1|}$
33. $f(x) = |x^2 - x + 2|$
34. $f(t) = \frac{1}{|t|}$
35. $g(s) = |s^2 - 1|$

In exercises 36 – 63, find the indicated limit if exists.

36. $\lim\limits_{x\to 2}(3+7x)$

37. $\lim\limits_{x\to 0}(x^3-7x+5)$

38. $\lim\limits_{x\to 0}\frac{(3+x)^2-9}{x}$

39. $\lim\limits_{x\to 3}\frac{x^2-9}{x-3}$

40. $\lim\limits_{h\to 0}\frac{h^2-1}{h-1}$

41. $\lim\limits_{x\to 0}\frac{(x+3)^3-27}{x}$

42. $\lim\limits_{x\to -2}\frac{x^2-x-6}{x+2}$

43. $\lim\limits_{h\to 0}\frac{(4+h)^2-16}{h}$

44. $\lim\limits_{x\to 0}\frac{1-\cos^2 x}{\sin x\cos x}$

45. $\lim\limits_{x\to 0}\frac{\tan x}{\cos x}$

46. $\lim\limits_{\theta\to 2}\frac{\theta^4-2^4}{\theta-2}$

47. $\lim\limits_{x\to \frac{\pi}{2}}\sin 2x\csc x$

48. $\lim\limits_{x\to \frac{\pi}{2}}\frac{\sin 2x}{\cos x}$

49. $\lim\limits_{x\to -1}\frac{x^2-2x-3}{x+1}$

50. $\lim\limits_{x\to 4}\frac{x^2-2x-8}{x-4}$

51. $\lim\limits_{x\to \frac{\pi}{2}}\frac{\sec x\cos x}{x}$

52. $\lim\limits_{x\to 1}\frac{x^3+3x^2-x-3}{x^2-1}$

53. $\lim\limits_{x\to -1}\frac{x^3+3x^2-x-3}{x^2-1}$

54. $\lim\limits_{x\to 1}\frac{x^3-7x+6}{x^2+2x-3}$

55. $\lim\limits_{x\to -3}\frac{x^3-7x+6}{x^2+2x-3}$

56. $\lim\limits_{x\to 1}\frac{x^{\frac{5}{2}}-x^{\frac{1}{2}}}{x^{\frac{3}{2}}-x^{\frac{1}{2}}}$

57. $\lim\limits_{x\to 2}\frac{x^{\frac{13}{4}}-2x^{\frac{9}{4}}}{x^{\frac{5}{4}}-2x^{\frac{1}{4}}}$

58. $\lim\limits_{x\to -2}\frac{x^{\frac{7}{3}}+x^{\frac{4}{3}}-2x^{\frac{1}{3}}}{x^{\frac{4}{3}}+2x^{\frac{1}{3}}}$

59. $\lim\limits_{h\to 0}\frac{3-\sqrt{9+h}}{h}$

60. $\lim\limits_{x\to 4}\frac{\sqrt{x}-2}{x-4}$

61. $\lim\limits_{x\to 1}\frac{(1/x)-1}{x-1}$

62. $\lim\limits_{x\to -1}\frac{x^{-2}-1}{x+1}$

63. $\lim\limits_{h\to 0}\frac{\sin 2h}{\sin h}$

Hint: $\sin 2h = 2\sin h\cos h$

In exercises 64 – 71, sketch the graph $y=f(x)$ and determine the limit of $f(x)$ as $x\to 0$, if it exists. If the limit does not exist, explain why.

64. $f(x)=\begin{cases} x+2 & , \ x<0 \\ 2x+2 & , \ x>0 \end{cases}$

65. $f(x)=\begin{cases} x+1 & , \ x<0 \\ x^2+2 & , \ x>0 \end{cases}$

66. $f(x)=\begin{cases} (x+2)^2 & , \ x<0 \\ (x+2)^3 & , \ x>0 \end{cases}$

67. $f(x)=\begin{cases} (x+1)^2 & , \ x<0 \\ (x+1)^3 & , \ x>0 \end{cases}$

68. $f(x)=\begin{cases} x^2-3x+3 & , \ x<0 \\ \frac{(x-1)^3+1}{x} & , \ x>0 \end{cases}$

69. $f(x)=\begin{cases} 1 & , \ x<0 \\ \frac{\sin x}{x} & , \ x>0 \end{cases}$

70. $f(x)=\begin{cases} x^2+1 & , \ x<0 \\ \frac{\sin x}{x} & , \ x>0 \end{cases}$

71. $f(x)=\begin{cases} \frac{|x|}{x} & , \ x<0 \\ (x-1)^2 & , \ x>0 \end{cases}$

72. Let f be the function defined by

$$f(x)=\begin{cases} \frac{\sin x}{2x} & , \ x<0 \\ (x+c)^2 & , \ x>0 \end{cases}$$

Find the number(s) c such that the limit of $f(x)$ as $x\to 0$ exists. What is the limit of $f(x)$ as $x\to 0$ in this case?

73. Give an example of a function f for which

$$\lim_{x\to a} f(x)=L \quad \text{and} \quad \lim_{x\to b} f(x)=L$$

with $a\neq b$. This shows that a function can have the same limit at more than one number x.

In exercise 74 – 79, complete the table below using the function specified. Given this numerical evidence, predict the limit of $f(x)$ as $x \to 0$.

x	f(x)
1.000	
0.500	
0.100	
0.050	
0.010	
0.005	
−0.005	
−0.010	
−0.050	
−0.100	
−0.500	
−1.000	

74. $f(x) = \frac{x^2}{1-\cos x}$
75. $f(x) = \frac{(\cos 2x)-1}{x^2}$
76. $f(x) = \frac{x-\sin x}{x^3}$
77. $f(x) = \frac{x^2-\sin x^2}{x^6}$
78. $f(x) = \frac{1-\cos x^2}{x^4}$
79. $f(x) = \frac{x^6}{x^2-\tan x^2}$

80. (a) Let f be defined by

$$f(x) = \frac{(x+1)^2-2}{x} \qquad x \neq 0$$

Explain why the limit does not exist.

(b) Let

$$g(x) = \frac{(x+1)^2-c}{x} \qquad x \neq 0$$

Find the unique number c such that the limit of $g(x)$ as $x \to 0$ exists. What is the limit when it does exist?

In exercise 81 – 96 use theorems

(a) Assume that $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$ both exist. Let c be any scalar. Then the following limits exists, with the values indicated:

(i) $\lim\limits_{x\to a} [f(x) + g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) = L + M$

(ii) $\lim\limits_{x\to a}[cf(x)] = c\left[\lim\limits_{x\to a} f(x)\right] = cL$

(iii) $\lim\limits_{x\to a}[f(x)g(x)] = \left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} f(x)\right] = LM$

(iv) $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{L}{M}$

(b) Assume that $\lim\limits_{x\to a} f(x) = L$ exists. For any positive integer $n = 1, 2, \ldots$

(i) $\lim\limits_{x\to a} x^n = a^n$

(ii) $\lim\limits_{x\to a}[f(x)]^n = [\lim\limits_{x\to a} f(x)]^n = L^n$

(c) Assume that the limit of $g(x)$ and that of $h(x)$ both exist as $x \to a$ and that

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$$

If the function f satisfies the inequality

$$g(x) \leq f(x) \leq h(x)$$

for all x in the open interval containing a (except possibly at $x = a$) then also

$$\lim_{x \to a} f(x) = L$$

81. $\lim_{x \to 4} \sqrt{x}(1 - x^2)$
82. $\lim_{x \to 2}(4\sqrt{x} - 5x^2)$
83. $\lim_{x \to 1}(3x^9 - \sqrt[3]{x})$
84. $\lim_{x \to 2} \frac{x^3-6x+5}{x^2+2x+2}$
85. $\lim_{x \to 3} \frac{x^2-2x-3}{x-3}$
86. $\lim_{x \to 3} \frac{x^2-10x+21}{x-3}$
87. $\lim_{x \to 4} \frac{x^{\frac{3}{2}}+2\sqrt{x}}{x^{\frac{5}{2}}+\sqrt{x}}$
88. $\lim_{x \to 0} \frac{\sin x}{\sin 2x}$
89. $\lim_{x \to 0} \frac{\tan x}{\sin 2x}$
90. $\lim_{x \to 1} \frac{x^3+2x^2+2x-5}{x^2-1}$
91. $\lim_{x \to -8} \frac{x^{\frac{2}{3}}-x}{x^{\frac{5}{3}}}$
92. $\lim_{x \to 4} \frac{x^{\frac{3}{2}}-x^{\frac{5}{2}}}{4+\sqrt{x}}$
93. $\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$
94. $\lim_{x \to -3} \frac{x^2-6x-27}{x^2+3x}$
95. $\lim_{x \to -1} \left(x^{\frac{7}{3}} - 2x^{\frac{2}{3}}\right)^2$
96. $\lim_{x \to -1} \frac{x^3+x^2-x-1}{x^3+2x^2+2x+1}$

In exercises $97 - 104$, use the fact that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

to find the limit.

97. $\lim_{x \to 0} \frac{\sin x}{2x}$
98. $\lim_{x \to 0} \frac{3\sin x}{x}$
99. $\lim_{x \to 0} \frac{\tan x}{4x}$
100. $\lim_{x \to 0} \frac{\sec x \tan x}{x}$
101. $\lim_{x \to 0} x^2 \cot x$
102. $\lim_{x \to 0} \frac{\sin^2 x}{x}$
103. $\lim_{x \to 0} \frac{\sin x}{\sqrt[5]{x}}$
104. $\lim_{x \to 0} \frac{\sin^2 x}{x^{\frac{4}{3}}}$

105. Suppose that $1 - x^2 \leq f(x) \leq 1 + x^2$ for all x. Find $\lim_{x \to 0} f(x)$

106. Suppose that $6x - x^2 \leq f(x) \leq x^2 - 6x + 18$ for all x. Find $\lim_{x \to 3} f(x)$

107. Suppose that $1 - x^4 \leq f(x) \leq \sec x$ for all $x \in (\frac{\pi}{2}, -\frac{\pi}{2})$. Find $\lim_{x \to 0} f(x)$

108. Let $f(x) = \frac{|x|}{x}$ and $g(x) = -\frac{|x|}{x}$, $x \neq 0$
    (a) Find the function $h = f + g$
    (b) Show that

$$\lim_{x \to 0}[h(x)] = \lim_{x \to 0}[f(x) + g(x)]$$

    exists but that neither $\lim_{x \to 0} f(x)$ nor $\lim_{x \to 0} g(x)$ exists.

109. Find two functions f and g so that $\lim_{x \to 0}[f(x)g(x)]$ exists but that either $\lim_{x \to 0} f(x)$ or $\lim_{x \to 0} g(x)$ fails to exist.

110. Is it possible to find the function f and a constant c so that $\lim_{x \to 0}[cf(x)]$ exists but $\lim_{x \to 0} f(x)$ does not exist? What if we require $c \neq 0$?

111. Find the number(s) a so that $\lim_{x\to a}(x^2 - 2x - 5) = 10$

112. Evaluate
$$\lim_{x\to\infty}\left[\sqrt{x+1} - \sqrt{x}\right]$$

113. Consider the function f defined by
$$f(x) = \frac{x^2+5}{x^2-5}$$

Evaluate

(a) $\lim_{x\to\pm\infty} f(x)$

(b) $\lim_{x\to\sqrt{5}^-} f(x)$ and $\lim_{x\to\sqrt{5}^+} f(x)$

(c) $\lim_{x\to-\sqrt{5}^-} f(x)$ and $\lim_{x\to-\sqrt{5}^+} f(x)$

Hence sketch the graph of f.

114. Find the following limits

(a)
$$\lim_{x\to 0}\left[\frac{1}{x} - \frac{1}{e^x - 1}\right]$$

(b)
$$\lim_{x\to 0} x^{\sin x}$$

115. Consider a function $f(x)$ with a property that $\lim_{x\to a} f(x) = 0$. Now consider another function $g(x)$ with a property that $\lim_{x\to a} g(x) = 8$. Then $\lim_{x\to a}[f(x)\cdot g(x)] = 0$

`True, properties of limits`

116. Consider a function $f(x)$ with a property that $\lim_{x\to a} f(x) = 0$. Now consider another function $g(x)$ also defined near a. Then $\lim_{x\to a}[f(x)\cdot g(x)] = 0$

`False, Properties of limits. Can have possibility of` $0(\infty)$ `say`
$$fg = \frac{\sin x}{x} = \frac{1}{x}\sin x$$
`where eventually limit is not zero, but one.`

117. Evaluate
$$\lim_{x\to 0^+}\frac{1-\cos\sqrt{x}}{x}$$

`Squeeze theorem fails, but why?. La'Hopital twice to have` $\frac{1}{2}$`. Have you realised why` $x \to 0^+$ `and not` $x \to 0$`? Square root, only positive values.`

118. The reason
$$\lim_{x\to\infty}\sin x$$
does not exist is because?

`The function is oscillating between` -1 `and` 1 `even as` $x \to \infty$

119.

120.

121.

Chapter 3

Continuity of Functions

3.1 Informal definition of Continuity of a function

Definition 3.1.1 We start by stating what we call an informal definition of continuity of a function. This clearly presents what exactly is meant by a function $f(x)$ being Continuous at the point $x = a$.
We say that a function $f(x)$ is continuous at $x = a$ if

(a) the $\lim_{x\to a} f(x)$ exists ie

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

(b) the functional value $f(a)$ exists.

(c) $\lim_{x\to a} f(x) = f(a)$

From the three conditions, it is sufficient to say that a function $f(x)$ is continuous at $x = a$ if

$$\lim_{x\to a} f(x) \quad = \quad f(a)$$

However, if $f(x)$ is not Continuous at $x = a$ we say that it is discontinuous at $x = a$.

Note 3.1.1

1. Polynomial functions are continuous at all points on the real axis $\mathbb{R}$

2. Rational functions are Continuous on the entire axis $\mathbb{R}$ except at the poles.

Note 3.1.2 If one of the conditions above fails, then the function is not continuous at that point.

Example 3.1.1 Show that the polynomial function $f(x) = x^2 + 2x + 1$ is continuous at $x = 2$

Since

(a) $\lim_{x\to 2}(x^2 + 2x + 1) = 2^2 + 2(2) + 1 = 9$ i.e exists

(b) $f(2) = 2^2 + 2(2) + 1 = 9$ i.e exists

(c) $\lim_{x\to 2} f(x) = f(2) = 9$

Hence $f(x) = x^2 + 2x + 1$ is continuous at $x = 2$. In fact being a polynomial function, it is continuous at all points on $\mathbb{R}$

Example 3.1.2 Show that the rational function,

$$f(x) = \frac{x+2}{x-1}$$

is continuous at $x = 3$ but discontinuous at $x = 1$

Since $x = 3$ is not a pole of the rational function for all substitution of 3 in the function, the denominator does not go to zero. Checking through the conditions of continuity,

(a)

$$\lim_{x\to 3} \frac{x+2}{x-1} = \frac{3+2}{3-1} = \frac{5}{2}$$

(b)

$$f(3) = \frac{3+2}{3-1} = \frac{5}{2}$$

(c)

$$\lim_{x\to 3} f(x) = f(3) = \frac{5}{2}$$

Therefore $f(x) = \frac{x+2}{x-1}$ is continuous at $x = 3$, indeed $f(x)$ is continuous at all points $\mathbb{R}$ except $x = 1$. The function is not continuous at $x = 1$ since the limit does not exist. In fact, also $f(1)$ does not exist.

Example 3.1.3 Check whether the function

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2x + 2, & x \geq 3 \end{cases}$$

is continuous at $x = 3$

(a) $\lim_{x \to 3} f(x)$??

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3}(x^2 - 1) = 8 \\ \lim_{x \to 3^+} f(x) &= \lim_{x \to 3}(2x + 2) = 8 \end{aligned}$$

Thus the limit exists and equal to 8, that is

$$\lim_{x \to 3} f(x) = 8$$

(b) $f(3) = 2(3) + 2 = 8$

(c)

$$\lim_{x \to 3} f(x) = f(3) = 8$$

Therefore, the function is continuous.

Example 3.1.4 Check whether the function $f(x)$ below is continuous at $x = 1$.

$$f(x) = \begin{cases} 2x + 1, & x \geq 1 \\ 4x, & x < 1 \end{cases}$$

(a) $\lim_{x \to 1} f(x)$??

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1}(4x) = 4 \quad \& \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1}(2x + 1) = 3$$

Thus the limit does not exists.

(b) No need to check for $f(1) = 2(1) + 1 = 3$

Therefore, the function is not continuous, since one of the properties fails.

Example 3.1.5 Given the function,

$$f(x) = \begin{cases} x^3, & x \leq 2 \\ \\ \alpha - x, & x > 2 \end{cases}$$

Find the scalar α for which $f(x)$ is continuous at $x = 2$.

$f(x)$ is continuous at $x = 2$ if $\lim\limits_{x\to 2} f(x)$ exists and $\lim_{x\to 2} f(x) = f(2)$. But for the $\lim\limits_{x\to 2} f(x)$ to exist,

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^+} f(x) \\ \lim_{x\to 2} x^3 &= \lim_{x\to 2}(\alpha - x) \\ \Rightarrow 8 &= \alpha - 2 \\ \Rightarrow \alpha &= 10 \end{aligned}$$

Therefore when $\alpha = 10$,

$$\lim_{x\to 2} f(x) = f(2)$$

Implying that $f(x)$ is continuous at $x = 2$

Example 3.1.6 Determine the value of k such that,

$$f(x) = \begin{cases} x^2 - k^2, & x \leq 2 \\ \\ kx + 5, & x > 2 \end{cases}$$

$f(x)$ is continuous at $x = 2$.

$f(x)$ is continuous at $x = 2$ if the $\lim\limits_{x\to 2} f(x)$ do exist,

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^+} f(x) \\ \lim_{x\to 2} x^2 - k^2 &= \lim_{x\to 2} kx + 5 \\ 4 - k^2 &= 2k + 5 \\ \Rightarrow\ k &= -1 \end{aligned}$$

Example 3.1.7 Modify the definition of $f(x)$ such that it is continuous at the point $x = a$ if

$$f(x) = \frac{x^2 - 1}{x - 1} \qquad (a \neq 1)$$

To modify is to redefine a function, so that it is defined everywhere, this is done by defining the function, as its limits where was initially undefined.

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & a \neq 1 \\ \\ 2, & a = 1 \end{cases}$$

Note that the limit 2 has been got by finding the limit of $f(x)$ by La'Hopital rule.

Example 3.1.8 Show that the function $f(x)$ below is discontinuous at $x = -2$.

$$f(x) = \frac{x^3 + x - 2}{x^3 - x^2 - 6x}$$

We realise that, $f(x)$ is not defined at $x = -2$, thus function *is not* continuous.

Example 3.1.9 Show that the function $f(x) = x^2 + 2x + 1$ is continuous at $x = 2$

It is enough to show that it satisfies the three conditions of continuity.

Example 3.1.10 Determine if the following function is continuous at $x = 1$.

$$f(x) = \begin{cases} 3x - 5, & \text{if } \; x \neq 1 \\ \\ 7, & \text{if } \; x = 1 \end{cases}$$

(a) The limit

$$\lim_{x \to 1} f(x) = \lim_{x \to 1}(3x - 5) = -2$$

i.e.,

$$\lim_{x \to 1} f(x) = -2$$

(b) But $f(1) = 7$, the function $f(x)$ is defined at $x = 1$.

(c) Since $\lim\limits_{x \to 1} f(x) \neq f(1)$, then condition (c) is not satisfied and function f *is not* continuous at $x = 1$.

Example 3.1.11 Determine if the following function is continuous at $x = -2$.

$$f(x) = \begin{cases} x^2 + 2x, & \text{if } x \leq -2 \\ \\ x^3 - 6x, & \text{if } x > -2 \end{cases}$$

(a) The left-hand limit

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2} (x^2 + 2x) = 0$$

The right-hand limit

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2} (x^3 - 6x) = 4$$

Since the left-hand and right-hand limits are not equal, $\lim\limits_{x \to -2} f(x)$ does not exist.

(b) Although the function f is defined at $x = -2$ since

$$f(-2) = (-2)^2 + 2(-2) = 4 - 4 = 0$$

Thus, function f *is not* continuous at $x = -2$ as the first condition failed.

Example 3.1.12 Determine if the following function is continuous at $x = 0$.

$$f(x) = \begin{cases} \frac{x-6}{x-3}, & \text{if } x < 0 \\ 2, & \text{if } x = 0 \\ \sqrt{4 + x^2}, & \text{if } x > 0 \end{cases}$$

(a) The left-hand limit

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{x - 6}{x - 3} = \frac{-6}{-3} = 2$$

The right-hand limit

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \sqrt{4 + x^2} = \sqrt{4 + (0)^2} = 2$$

Thus, $\lim\limits_{x \to 0} f(x)$ exists with $\lim\limits_{x \to 0} f(x) = 2$.

(b) The function f is defined at $x = 0$ since$f(0) = 2$,

(c) Since $\lim\limits_{x \to 0} f(x) = 2 = f(0)$, all *three* conditions satisfied $\Rightarrow f$ is continuous at $x = 0$.

Example 3.1.13 Check the following function for continuity at $x = 3$ and $x = -3$.

$$f(x) = \begin{cases} \frac{x^3-27}{x^2-9}, & \text{if } x \neq 3 \\ \dfrac{9}{2}, & \text{if } x = 3 \end{cases}$$

<u>`Continuity at` $x = 3$</u>

(a) The limit (since function not a piecewise, we compute to test existence of a limit)

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^3 - 27}{x^2 - 9} = \frac{\text{"}0\text{"}}{\text{"}0\text{"}}$$

(Circumvent this indeterminate form by factoring the numerator and the denominator).

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^3 - 27}{x^2 - 9} = \lim_{x\to 3} \frac{x^3 - 3^3}{x^2 - 3^2}$$

Recall that $A^2 - B^2 = (A - B)(A + B)$ and $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^3 - 3^3}{x^2 - 3^2} = \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{x^2+3x+9}{x+3} = \frac{(3)^2+3(3)+9}{(3)+3} = \frac{9}{2}$$

Could even have applied the La'Hopital rule to compute the limit above. i.e.,

$$\lim_{x\to 3} f(x) = \frac{9}{2}$$

(b) The function f is defined at $x = 3$ since$f(3) = \dfrac{9}{2}$

(c) Since,

$$\lim_{x\to 3} f(x) = \frac{9}{2} = f(3)$$

all three conditions are satisfied, and f is continuous at $x = 3$. Now, check for continuity at $x = -3$.

<u>`Continuity at` $x = -3$</u>

Function f is not defined at $x = -3$ because of division by zero. Thus, $f(-3)$ does not exist, condition (b) is violated, and thus f *is not* continuous at $x = -3$.

Example 3.1.14 Show that the function $f(x) = \sin x$ is continuous at all numbers x.

First, f is defined for all $x \in \mathbb{R}$. Let a be an arbitrary real number. Then

$$\begin{aligned}
\lim_{h\to 0} f(a+h) &= \lim_{h\to 0} \sin(a+h) \\
&= \lim_{h\to 0}[\sin a \cos h + \cos a \sin h] \\
&= (\sin a)\left(\lim_{h\to 0} cosh\right) + \cos a \left(\lim_{h\to 0} \sin h\right) \\
&= (\sin a)(1) + \cos a(0) \\
&= \sin a \\
&= f(a)
\end{aligned}$$

Since a was arbitrary f is continuous on $\mathbb{R}$.

Example 3.1.15 For what values of x is the function $f(x) = \dfrac{x^2+3x+5}{x^2+3x-4}$ continuous?

Functions $y = x^2 + 3x + 5$ and $y = x^2 + 3x - 4$ are continuous for all values of x since both are polynomials. Thus, the quotient of these two functions, $f(x) = \dfrac{x^2+3x+5}{x^2+3x-4}$ is continuous for all values of x where the denominator, $y = x^2 + 3x - 4 = (x-1)(x+4)$ does NOT equal zero. Since $(x-1)(x+4) = 0$ for $x = 1$ and $x = -4$, function f is continuous for all values of x except $x = 1$ and $x = -4$.

Example 3.1.16 For what values of x is the function $g(x) = (\sin(x^{20}+5))^{1/3}$ continuous?

First describe function g using functional composition. Let $f(x) = x^{\frac{1}{3}}$, $h(x) = \sin x$, and $k(x) = x^{20} + 5$. Function k is continuous for all values of x since it is a polynomial, and functions f and h are well-known to be continuous for all values of x. Thus, the functional compositions

$$h(k(x)) = \sin(k(x)) = \sin(x^{20}+5)$$

and

$$f(h(k(x))) = (h(k(x)))^{1/3} = (\sin(x^{20}+5))^{1/3}$$

are continuous for all values of x. Since

$$g(x) = (\sin(x^{20}+5))^{1/3} = f(h(k(x)))$$

function g is continuous for all values of x.

Example 3.1.17 For what values of x is the function $f(x) = \sqrt{x^2 - 2x}$ continuous ?

First describe function f using functional composition. Let $g(x) = x^2 - 2x$ and $h(x) = \sqrt{x}$ Function g is continuous for all values of x since it is a polynomial, and function h is well-known to be continuous for $x \geq 0$. Since $g(x) = x^2 - 2x = x(x-2)$, it follows easily that $g(x) \geq 0$ for $x \leq 0$ and $x \geq 2$. Thus, the functional composition

$$h(g(x)) = \sqrt{g(x)} = \sqrt{x^2 - 2x}$$

is continuous for $x \leq 0$ and $x \geq 2$. Since

$$f(x) = \sqrt{x^2 - 2x} = h(g(x))$$

function f is continuous for $x \leq 0$ and $x \geq 2$.

Example 3.1.18 For what values of x is the function $f(x) = \ln\left(\frac{x-1}{x+2}\right)$ continuous? First describe function f using functional composition. Let $g(x) = \frac{x-1}{x+2}$ and $h(x) = \ln x$ Since g is the quotient of polynomials $y = x - 1$ and $y = x + 2$, function g is continuous for all values of x except where $x + 2 = 0$, i.e., except for $x = -2$. Function h is well-known to be continuous for $x > 0$. Since $g(x) = \frac{x-1}{x+2}$, it follows easily that $g(x) > 0$ for $x < -2$ and $x > 1$. Thus, the functional composition

$$h(g(x)) = \ln(g(x)) = \ln\left(\frac{x-1}{x+2}\right)$$

is continuous for $x < -2$ and $x > 1$. Since

$$f(x) = \ln\left(\frac{x-1}{x+2}\right) = h(g(x))$$

function f is continuous for $x < -2$ and $x > 1$

[The ln is not defined at negative values. But also the quotient is always taken seriously with the denominator]

Example 3.1.19 For what values of x is the function $f(x) = \dfrac{e^{\sin x}}{4 - \sqrt{x^2 - 9}}$ continuous?

First describe function f using functional composition. Let $g(x) = \sin x$ and $h(x) = e^x$, both of which are well-known to be continuous for all values of x. Thus, the numerator $y = e^{\sin x} = h(g(x))$ is continuous (the functional composition of continuous functions) for all values of x. Now consider the denominator $y = 4 - \sqrt{x^2 - 9}$. Let

$$g(x) = 4, h(x) = x^2 - 9, \text{ and } k(x) = \sqrt{x}$$

Functions g and h are continuous for all values of x since both are polynomials, and it is well-known that function k is continuous for $x \geq 0$. Since $h(x) = x^2 - 9 = (x - 3)(x + 3) = 0$ when $x = 3$ or $x = -3$, it follows easily that $h(x) \geq 0$ for $x \geq 3$ and $x \leq -3$, so that $y = \sqrt{x^2 - 9} = k(h(x))$ is continuous (the functional composition of continuous functions) for $x \geq 3$ and $x \leq -3$. Thus, the denominator $y = 4 - \sqrt{x^2 - 9}$ is continuous (the difference of continuous functions) for $x \geq 3$ and $x \leq -3$.

There is one other important consideration. We must insure that the denominator is never zero. If

$$y = 4 - \sqrt{x^2 - 9} = 0$$

then

$$4 = \sqrt{x^2 - 9}$$

Squaring both sides, we get

$$16 = x^2 - 9$$

so that

$$x^2 = 25$$

when

$$x = 5 \text{ or } x = -5$$

Thus, the denominator is zero if $x = 5$ or $x = -5$. Summarizing, the quotient of these continuous functions, $f(x) = \dfrac{e^{\sin x}}{4 - \sqrt{x^2 - 9}}$, is continuous for $x \geq 3$ and $x \leq -3$, but not for $x = 5$ and $x = -5$.

Example 3.1.20 For what values of x is the following function continuous ?

$$f(x) = \begin{cases} \dfrac{x-1}{\sqrt{x}-1} & \text{if } \ x > 1 \\ 5-3x, & \text{if } \ -2 \leq x \leq 1 \\ \dfrac{6}{x-4} & \text{if } \ x < -2 \end{cases}$$

Consider separately the three component functions which determine f. Function

$$y = \frac{x-1}{\sqrt{x}-1}$$

is continuous for $x > 1$ since it is the quotient of continuous functions and the denominator is never zero.

Function $y = 5 - 3x$ is continuous for $-2 \leq x \leq 1$ since it is a polynomial.

Function

$$y = \frac{6}{x-4}$$

is continuous for $x < -2$ since it is the quotient of continuous functions and the denominator is never zero.

Now check for continuity of f where the three components are joined together, i.e., check for continuity at $x = 1$ and $x = -2$.

For $x = 1$:

(a) The right-hand limit

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x-1}{\sqrt{x}-1} = \frac{\text{"}0\text{"}}{\text{"}0\text{"}}$$

Circumvent this indeterminate form one of two ways. Either factor the numerator as the difference of squares, or multiply by the conjugate of the denominator over itself.

$$= \lim_{x\to 1^+} \frac{(\sqrt{x})^2 - (1)^2}{\sqrt{x}-1} = \lim_{x\to 1^+} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \lim_{x\to 1^+} (\sqrt{x}+1) = (\sqrt{1}+1) = 2$$

or applying the La'Hopital rule.

The left-hand limit

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (5-3x) = 5 - 3(1) = 2$$

Thus,

$$\lim_{x\to 1} f(x) = 2$$

(b) function f is defined since $f(1) = 5 - 3(1) = 2$.

(c) Since

$$\lim_{x\to 1} f(x) = 2 = f(1)$$

all three conditions are satisfied, and function f is continuous at $x = 1$.

For $x = -2$:

(a) The right-hand limit

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2} (5 - 3x) = 5 - 3(-2) = 11$$

The left-hand limit

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2} \frac{6}{x - 4} = \frac{6}{(-2) - 4} = \frac{6}{-6} = -1$$

Since the left- and right-hand limits are different,

$$\lim_{x \to -2} f(x)$$

does not exist,

(b) Although the function f is defined at $x = -2$ since $f(-2) = 5 - 3(-2) = 11$

condition (a) is violated, and function f is not continuous at $x = -2$.

Summarizing, function f is continuous for all values of x except $x = -2$.

Example 3.1.21 Determine all values of the constant A so that the following function is continuous for all values of x.

$$f(x) = \begin{cases} A^2x - A, & \text{if } x \geq 3 \\ 4, & \text{if } x < 3 \end{cases}$$

First, consider separately the two components which determine function f.

Function $y = A^2x - A$ is continuous for $x \geq 3$ for any value of A since it is a polynomial.

Function $y = 4$ is continuous for $x < 3$ since it is a polynomial.

Now determine A so that function f is continuous at $x = 3$.

(a) The right-hand limit

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3}(A^2x - A) = 3A^2 - A$$

The left-hand limit

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} 4 = 4$$

For the limit to exist, the right- and left-hand limits must exist and be equal. Thus,

$$\lim_{x\to 3} f(x) = 3A^2 - A = 4 \quad \Rightarrow$$

$$3A^2 - A - 4 = 0 \quad \Rightarrow \quad (3A - 4)(A + 1) = 0 \quad \Rightarrow \quad A = \frac{4}{3} \text{ or } A = -1$$

(b) The function is defined at $x = 3$

$$f(3) = A^2(3) - A = 3A^2 - A$$

For either choice of A,

(c)

$$\lim_{x\to 3} f(x) = 4 = f(3)$$

all three conditions are satisfied, and f is continuous at $x = 3$. Therefore, function f is continuous for all values of x if $A = \dfrac{4}{3}$ or $A = -1$

Example 3.1.22 Determine all values of the constants A and B so that the following function is continuous for all values of x.

$$f(x) = \begin{cases} Ax - B, & \text{if } \; x \leq -1 \\ \\ 2x^2 + 3Ax + B, & \text{if } \; -1 < x \leq 1 \\ \\ 4, & \text{if } \; x > 1 \end{cases}$$

First, consider separately the three components which determine function f.

Function $y = Ax - B$ is continuous for $x \leq -1$ for any values of A and B since it is a polynomial.

Function $y = 2x^2 + 3Ax + B$ is continuous for $-1 < x \leq 1$ for any values of A and B since it is a polynomial.

Function $y = 4$ is continuous for $x > 1$ since it is a polynomial.

Now determine A and B so that function f is continuous at $x = -1$ and $x = 1$.

`Continuity at` $x = -1$`:`

(a) The left-hand limit

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1} (Ax - B) = A(-1) - B = -A - B$$

The right-hand limit

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1} (2x^2 + 3Ax + B) = 2(-1)^2 + 3A(-1) + B = 2 - 3A + B$$

For the limit to exist, the right- and left-hand limits must exist and be equal. Thus,

$$\lim_{x \to -1} f(x) = -A - B = 2 - 3A + B$$

so that

$$2A - 2B = 2$$

or

$$A - B \;\; = \;\; 1 \tag{3.1}$$

(b) The function will be defined at $x = -1$ as $f(-1) = A(-1) - B = -A - B$

Now consider `Continuity at` $x = 1$`:`

(a) The left-hand limit

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (2x^2 + 3Ax + B) = 2(1)^2 + 3A(1) + B = 2 + 3A + B$$

The right-hand limit

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} 4 = 4$$

For the limit to exist, the right- and left-hand limits must exist and be equal. Thus,

$$\lim_{x \to 1} f(x) = 2 + 3A + B = 4$$

or

$$3A + B \;\; = \;\; 2 \tag{3.2}$$

(b) The function will be defined at $x = 1$ since $f(1) = 2(1)^2 + 3A(1) + B = 2 + 3A + B$

For continuity at both $\underline{x = -1 \texttt{ and } x = 1}$, we solve Equations (3.1) and (3.2) simultaneously. Thus,

$$\begin{aligned} A &= \frac{3}{4} \\ B &= -\frac{1}{4} \end{aligned}$$

For this choice of A and B it can easily be shown that

$$\begin{aligned} \lim_{x \to 1} f(x) &= 4 = f(1) \texttt{ and} \\ \lim_{x \to -1} f(x) &= -\frac{1}{2} = f(-1) \end{aligned}$$

so that all three conditions are satisfied at both $x = 1$ and $x = -1$, and function f is continuous at both $x = 1$ and $x = -1$. Therefore, function f is continuous for all values of x if

$$A = \frac{3}{4} \text{ and } B = -\frac{1}{4}$$

Example 3.1.23 Show that the following function is continuous for all values of x.

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

First describe f using functional composition. Let $g(x) = -\frac{1}{x^2}$ and $h(x) = e^x$. Function h is well-known to be continuous for all values of x.

Function g is the quotient of functions continuous for all values of x, and is therefore continuous for all values of x except $x = 0$, that x which makes the denominator zero. Thus, for all values of x except $x = 0$,

$$f(x) = h(g(x)) = e^{g(x)} = e^{-\frac{1}{x^2}}$$

is a continuous function (the functional composition of continuous functions).

Now check for continuity of f at $x = 0$. Function f is defined at $x = 0$ since

(a) The limit

$$\lim_{x\to 0} \frac{-1}{x^2} = \frac{-1}{0^+} = -\infty$$

The numerator approaches -1 and the denominator is a positive number approaching zero.

so that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{-1/x^2} = e^{-\infty} = \frac{"1"}{e^{\infty}} = \frac{"1"}{\infty} = 0$$

i.e.,

$$\lim_{x\to 0} f(x) = 0$$

(b)

$$f(0) = 0$$

(c)

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

all three conditions are satisfied, and f is continuous at $x = 0$. Thus, f is continuous for all values of x.

Example 3.1.24 Assume that

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) , & x \neq 0 \\ 0 , & x = 0 \end{cases}$$

Show that f is continuous at $x = 0$.

Recall that function f is continuous at $x = 0$ if

(a) $\lim_{x \to 0} f(x)$ exists,

(b) $f(0)$ is defined (exists), and

(c) $\lim_{x \to 0} f(x) = f(0)$.

First note that it is given that

(a) Use the Squeeze Principle to compute

$$\lim_{x \to 0} f(x)$$

For $x \neq 0$ we know that

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq +1$$

so that

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Since

$$\lim_{x \to 0}(-x^2) = 0 = \lim_{x \to 0} x^2$$

it follows from the Squeeze Principle that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \; x^2 \sin\left(\frac{1}{x}\right) = 0$$

(b) $f(0) = 0$.

(c) Finally, $\lim_{x \to 0} f(x) = 0 = f(0)$,

confirming that function f is continuous at $x = 0$.

Example 3.1.25 Determine whether the following function is continuous

$$f(x) = \begin{cases} 3x-5 & ; \ x \neq 1 \\ \\ 2 & ; \ x = 1 \end{cases}$$

at $x = 1$

$$\begin{aligned} \lim_{x \to 1} &= \lim_{x \to 1}(3x-5) = -2 \\ f(1) &= 2 \end{aligned}$$

Since are not the same, the function is not continuous.

Example 3.1.26 Determine the values of constants a, b so that the function $f(x)$

$$f(x) = \begin{cases} 2ax+b & ; \ x < 3 \\ \\ ax+3b & ; \ x > 3 \\ \\ 10 & ; \ x = 3 \end{cases}$$

is continuous at $x = 3$

The limits at a point should be equal and equal to the value of the function at that point.

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3}(2ax+b) = 6a+b \\ \lim_{x \to 3^+} f(x) &= \lim_{x \to 3}(ax+3b) = 3a+3b \\ f(3) &= 10 \end{aligned}$$

All the three equations above should be equal, i.e

$$\begin{aligned} 6a+b &= 10 \\ 3a+3b &= 10 \end{aligned}$$

Solving simultaneously gives

$$\begin{aligned} a &= \frac{4}{3} \\ b &= 2 \end{aligned}$$

Example 3.1.27 Find the values of a and b for which the function

$$f(x) = \begin{cases} 3x-6a\ , & x < 1 \\ \\ 2ax-b\ , & 1 \leq x \leq 3 \\ \\ x-2b\ , & 3 < x \end{cases}$$

is continuous at 1 and 3

<u>`To be continuous at` $x = 1$</u>

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^+} f(x) \\ \lim_{x \to 1}(3x-6a) &= \lim_{x \to 1}(2ax-b) \\ 3-6a &= 2a-b \end{aligned} \tag{3.3}$$

`To be continuous at` $x = 3$

$$\begin{aligned} \lim_{x\to 3^-} f(x) &= \lim_{x\to 3^+} f(x) \\ \lim_{x\to 3}(2ax - b) &= \lim_{x\to 3}(x - 2b) \\ 6a - b &= 3 - 2b \end{aligned} \tag{3.4}$$

Solving the two equations (3.3) and (3.4) simultaneously

$$\begin{aligned} 8a - b &= 3 \\ 6a + b &= 3 \end{aligned}$$

to have

$$\begin{aligned} a &= \frac{3}{7} \\ b &= \frac{3}{7} \end{aligned}$$

Example 3.1.28 Determine the values of constants a, b so that the function $f(x)$

$$f(x) = \begin{cases} a + bx & ; \ x > 2 \\ 3 & ; \ x = 2 \\ b - ax^2 & ; \ x < 2 \end{cases}$$

is continuous at $x = 2$

The limits at a point should be equal and equal to the value of the function at that point.

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2} (b - ax^2) = b - 4a \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2} (a + bx) = a + 2b \\ f(2) &= 3 \end{aligned}$$

All the three equations above should be equal, i.e

$$\begin{aligned} b - 4a &= 3 \\ a + 2b &= 3 \end{aligned}$$

Solving simultaneously gives

$$a = -\frac{1}{3}, \ b = \frac{5}{3}$$

Example 3.1.29 Determine if the function $h(x) = \dfrac{x^2 + 1}{x^3 + 1}$ is continuous at $x = -1$

Function h is not defined at $x = -1$ since it leads to division by zero. Thus, $h(-1)$ does not exist, condition (b) is violated, and function h is not continuous at $x = -1$.

Theorem 3.1.1 *Let f be continuous at a point $x = a$ in the domain of f and let g be continuous at a point $f(a)$ in its domain. Then the composite function gof is continuous at $x = a$.*

Theorem 3.1.2 *Let f be defined on an open interval containing the number a. f is continuous at a if and only if*

$$\lim_{x \to a} f(a + h) = f(a)$$

Because of the if and only if this statement can be used as an alternative definition of continuity.

3.1.1 Removable discontinuity

There are discontinuities which can be removed by redefining the function.

Definition 3.1.2 If L^- and L^+ at x_0 exist, are finite, and are equal to $L = L^- = L^+$. Then, if $f(x_0)$ is not equal to L, x_0 is called a *removable discontinuity.* This discontinuity can be "removed to make f continuous at x_0".

Removable discontinuity $\equiv$ A hole in a graph. That is, a discontinuity that can be "repaired" by filling in a single point. In other words, a removable discontinuity is a point at which a graph is not connected but can be made connected by filling in a single point.

Example 3.1.30 Show that the function

$$f(x) = \frac{x^2 - 9}{x - 3}$$

Is discontinuous at $x = 3$

Checking through conditions of continuity, we have for $f(x) = \frac{x^2-9}{x-3} \quad \Rightarrow \quad f(3) = \frac{0}{0}$ is not defined . Hence $f(x)$ must be discontinuous at $x = 3$.
But since the

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right) = \lim_{x\to 3}\frac{(x+3)(x-3)}{x-3} = 6 \text{ exists}$$

and for continuity of $f(x)$ at $x = 3$

$$\lim_{x\to 3} f(x) = f(3)$$

Thus if we redefine $f(x)$ at $x = 3$ we can remove the discontinuity at $x = 3$. In fact if $f(x) = 6$ at $x = 3$, then the function becomes continuous at $x = 3$ i.e

$$f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$$

is continuous at $x = 3$

Such discontinuity which can be removed by redefining the function at the discontinuity are called removable discontinuities

Example 3.1.31 Redefine the function

$$f(x) = \frac{1 - \cos^2 x}{\sin x}$$

so that it is continuous at $x = 0$

Since

$$\lim_{x\to 0}\left(\frac{1-\cos^2 x}{\sin x}\right) = \lim_{x\to 0}\frac{\sin^2 x}{\sin x} = \lim_{x\to 0}(\sin x) = 0$$

$f(x)$ to be continuous at $x = 0$,

$$\lim_{x\to x_0} f(x) = f(x_0) \quad \Rightarrow \quad f(0) = \lim_{x\to 0} f(x)$$

Therefore if we redefine $f(0) = 0$ i.e

$$f(x) = \begin{cases} (\frac{1-\cos^2 x}{\sin x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

then $f(x)$ is continuous at $x = 0$.

Example 3.1.32 Redefine the function

$$f(x) = \frac{x^2 - 5x + 6}{(x-2)}$$

so that it is continuous at $x = 2$.

The function is not defined at $x = 2$. The function is continuous if

$$f(2) = \lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - 5x + 6}{(x-2)} = \lim_{x\to 2} \frac{(x-2)(x-3)}{(x-2)} = \lim_{x\to 2}(x-3) = -1$$

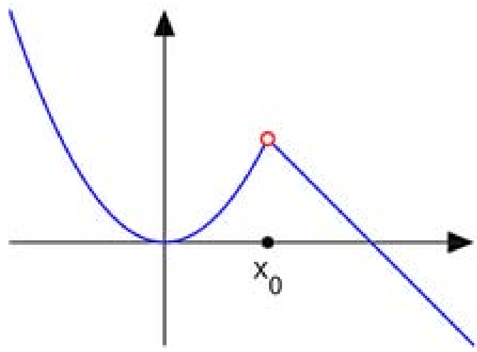
Thus the redefined continuous function is

$$f(x) = \begin{cases} \frac{x^2-5x+6}{(x-2)}, & \text{if } \; x \neq 2 \\ -1, & \text{if } \; x = 2 \end{cases}$$

Such discontinuity is termed as removable discontinuity.

Example 3.1.33 Consider the function

$$f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ 2 - x & \text{for } x > 1 \end{cases}$$



Then, the point $x_0 = 1$ is a removable discontinuity.

3.1.2 Jump discontinuity

Definition 3.1.3 The limits L^- and L^+ exist and are finite, but not equal. Then, x_0 is called a *jump discontinuity or step discontinuity.* For this type of discontinuity, the function f may have any value in x_0.

Example 3.1.34 Consider the function

$$f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ 2 - (x-1)^2 & \text{for } x > 1 \end{cases}$$

Then, the point $x_0 = 1$ is a jump discontinuity.

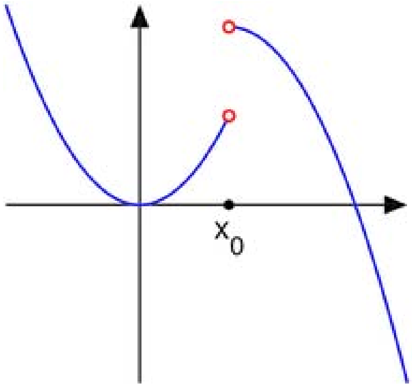


Figure 3.1: The function in Example 3.1.34, with a jump discontinuity at $x_0 = 1$

3.1.3 Essential discontinuity

Definition 3.1.4 One or both of the limits L^- and L^+ does not exist or is infinite. Then, x_0 is called an *essential discontinuity*, or *infinite discontinuity*.

Example 3.1.35 Consider the function

$$f(x) = \begin{cases} \sin \frac{5}{x-1} & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ \frac{0.1}{x-1} & \text{for } x > 1 \end{cases}$$

Then, the point $x_0 = 1$ is an essential discontinuity (sometimes called infinite discontinuity).

For it to be an essential discontinuity, it would have sufficed that only one of the two one-sided limits did not exist or were infinite.

However, given this example the discontinuity is also an essential discontinuity for the extension of the function into complex variables.

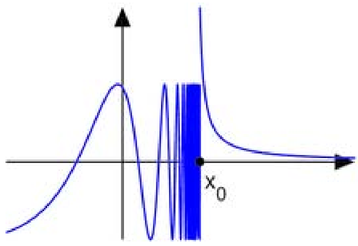


Figure 3.2: The function in example 3.1.35, is an essential discontinuity

3.1.4 Formal definition of continuity of function $f(x)$ at $x = a$

Definition 3.1.5 A function $f(x)$ is continuous at $x = a$ if and only if given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$

Example 3.1.36 Prove that

$$f(x) = (3x + 5)$$

is continuous at $x = 10$. Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $|x - 10| \leq \delta$, then $|f(x) - 35| \leq \epsilon$.

Begin with $|f(x) - 35|$ and "solve for" $|x - 10|$. Then

$$\begin{aligned}
|f(x) - 35| &= |(3x + 5) - 35| \\
&= |3x - 30| \\
&= |3(x - 10)| \\
&\leq |3||(x - 10)| \\
&\leq 3|x - 10| \\
&\leq \epsilon \\
&\text{iff} \quad |x - 10| \leq \frac{\epsilon}{3}
\end{aligned}$$

Now choose $\delta = \frac{\epsilon}{3}$.

Thus if $|x - 10| \leq \frac{\epsilon}{3}$, that is $\delta \leq \frac{\epsilon}{3}$, it follows that $|f(x) - 35| \leq \epsilon$. This completes the proof.

3.1.5 Continuity at end points of domain

We say that a function is continuous at a left endpoint α of its domain if,

$$\lim_{x \to \alpha^+} f(x) = f(\alpha)$$

Likewise we say that a function $f(x)$ is continuous at a right endpoint β of its domain if

$$\lim_{x \to \beta^-} f(x) = f(\beta)$$

Exercise 3.1 Why is the function

$$f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 8, & x = 3 \end{cases}$$

not continuous at $x = 3$? Redefine the function f to make it continuous at $x = 3$.

Example 3.1.37 Given that

$$f(x) = \begin{cases} x^2+1; & \text{if } x < 2, \\ 3x-1; & \text{if } x > 2, \end{cases}$$

(a) Sketch the graph of $f(x)$.

(b) Does $\lim\limits_{x \to 2} f(x)$ exist? Justify your answer.

(c) Why is $f(x)$ NOT continuous at $x = 2$? Re-define $f(x)$ to make it continuous at $x = 2$

The redefined continuous function is

$$f(x) = \begin{cases} x^2+1; & \text{if } x < 2, \\ 5; & \text{if } x = 2, \\ 3x-1; & \text{if } x > 2 \end{cases}$$

Example 3.1.38 Redefine the function

$$f(x) = \frac{x^2-3x+2}{x-1}$$

such that it is continuous at $x = 1$.

The continuous redefined function is

$$f(x) = \begin{cases} \frac{x^2-3x+2}{x-1}; & \text{if } x \neq 1, \\ -1; & \text{if } x = 1 \end{cases}$$

3.2 Intermediate Value Theorem, IVT

Theorem 3.2.1 Let f be continuous on the closed interval $[a, b]$ with $f(a) \neq f(b)$. Let d be any number between $f(a)$ and $f(b)$. Then there exists at least one number $c \in (a, b)$ with $f(c) = d$.

All that this theorem says is that a continuous function cannot skip any number in pasing from any of its values to another. The IVT is an existence theorem. It guarantees the existence of the number c, though it does not say how to find it.
It also guarantees that a continuous function cannot change sign without becoming zero at some point, that is, if a continuous function g has positive and negative values for some two points on an interval, then there exists a point c in the interval at which $g(c) = 0$.

Example 3.2.1 Show that the function $f(x) = x^2 - 4$ has a roots between -3 and -1 and also between 1 and 3.
A root of f is a number x for which $f(x) = 0$. Now $f(-3) = 5$ and $f(-1) = -3$. Since $0 \in [-3, 5]$, by the IVT there exists a number $c \in (-3, -1)$ such that $f(c) = 0$.
Similarry A root of f is a number x for which $f(x) = 0$. Now $f(1) = -3$ and $f(3) = 5$. Since $0 \in [-3, 5]$, by the IVT (We can use the theorem since $f(x)$ is a continuous function everywhere) there exists a number $c \in (1, 3)$ such that $f(c) = 0$.

Example 3.2.2 Show that the function $f(x) = 4x^3 - 6x^2 + 3x - 2$ has a root between 1 and 2.
A root of f is a number x for which $f(x) = 0$. Now $f(1) = -1$ and $f(2) = 12$. Since $0 \in [-1, 12]$, by the IVT there exists a number $c \in (1, 2)$ such that $f(c) = 0$.

Example 3.2.3 If a child grows from 1m to 1.5m between the ages of 2 years and 6 years, then, at some time between 2 years and 6 years of age, the child's height must have been 1.25m.

Example 3.2.4 Show that the function $f(x) = \ln(x) - 1$ has a solution in $[2, 3]$.

Example 3.2.5 Show that the function $f(x) = x^5 + 2x^3 + x - 5$ has only one real solution. Hint: Use $x = 1$ and $x = 2$

Example 3.2.6 Use the Intermediate Value Theorem to show that there is a positive number c such that $c^2 = 2$.
Let $f(x) = x^2$. Then f is continuous and $f(0) = 0 < 2 < 4 = f(2)$. By the IVT there is $c \in (0, 2)$ such that $c^2 = f(c) = 2$.

Example 3.2.7 If $f(x) = x^3 - x^2 + x$, show that there is $c \in \mathbb{R}$ such that $f(c) = 10$. But $f(1) = 1$ and $f(3) = 3^3 - 3^2 + 3 = 27 - 9 + 3 = 21$, so $f(1) < 10 < f(3)$. Since f is continuous everywhere, there must be $c \in \mathbb{R}$ such that $f(c) = 10$.

Example 3.2.8 Let f be a continuous function on $[0, 1]$. Show that if $-1 \leq f(x) \leq 1$ for all $x \in [0, 1]$ then there is $c \in [0, 1]$ such that $[f(c)]^2 = c$.
If $f(x)$ is continuous on $[0, 1]$ then so is $[f(x)]^2$. Set $g(x) = [f(x)]^2 - x$. Then g is also continuous on $[0, 1]$. Now $g(0) = [f(0)]^2 - 0 = [f(0)]^2 \geq 0$ and $g(1) = [f(1)]^2 - 1 \leq 0$, so by IVT there is $c \in [0, 1]$ such that $g(c) = 0$. Then $[f(c)]^2 - c = 0$ or $[f(c)]^2 = c$.

3.3 Fixed Point Theorem

Theorem 3.3.1 Let $f : [0,1] \to [0,1]$ be a continuous function. Then there is at least one number c that is fixed by f, that is, for which

$$f(c) = c$$

Definition 3.3.1 In mathematics, a fixed point (sometimes shortened to fix point, also known as an invariant point) of a function is a point that is mapped to itself by the function. A set of fixed points is sometimes called a fixed set.

Example 3.3.1 For example, if f is defined on the real numbers by

$$f(x) = x^2 - 3x + 4,$$

then 2 is a fixed point of f, because $f(2) = 2$.

$$\begin{aligned} f(c) &= c \\ c^2 - 3c + 4 &= c \\ c^2 - 4c + 4 &= 0 \\ c &= \frac{4 \pm \sqrt{4^2 - 4(1)(4)}}{2(1)} = 2 \end{aligned}$$

The point $c = 2$ is the fixed point for the function $f(x) = x^2 - 3x + 4$

Example 3.3.2 Find the fixed points for the continuous function

$$f(x) = \frac{(x+1)}{2}$$

in the interval $[-1, 1]$

For a fixed point

$$\begin{aligned} f(c) &= c \\ \frac{(c+1)}{2} &= c \\ c + 1 &= 2c \\ c &= 1 \end{aligned}$$

Since $c = 1 \in [-1, 1]$, $c = 1$ is a fixed point.

Example 3.3.3 Find all the fixed points for the function

$$f(x) = x^2 - 6$$

in the closed interval $[-4, 4]$

For a fixed point

$$\begin{aligned} f(c) &= c \\ c^2 - 6 &= c \\ c^2 - c - 6 &= 0 \\ c &= \frac{1 \pm \sqrt{1 + 4(1)(6)}}{2(1)} = -2, 3 \end{aligned}$$

Thus the points $c = -2$ and $c = 3$ are the fixed points of $f(x) = x^2 - 6$ since $-2, 3 \in [-4, 4]$

Example 3.3.4 Find all the fixed points for the function

$$f(x) = x^2 - 2x + 2$$

in the closed interval $[-6, 1]$

For a fixed point

$$\begin{aligned} f(c) &= c \\ c^2 - 2c + 2 &= c \\ c^2 - 3c + 2 &= 0 \\ c &= 1, 2 \end{aligned}$$

Thus the point $c = 1$ is a fixed point but $c = 2$ is not a fixed point of $f(x) = x^2 - 2x + 2$ since $1 \in [-6, 1]$ but $2 \notin [-6, 1]$

Example 3.3.5 A continuous function that maps $[0, 1]$ into itself has a fixed point.

Example 3.3.6 A continuous function that maps a disk into itself has a fixed point.

Example 3.3.7 A continuous function that maps a spherical ball into itself necessarily has a fixed point.

Exercise 3.2

1. Find values of x for which the following functions are discontinuous.

   (i) $\frac{x^2+2}{x-1}$

   (ii) $\frac{\cos x}{x^2}$

   (iii) $\frac{x+1}{x^2-1}$

   (iv) $\frac{x}{x^3-1}$

   (v) $f(x) = \begin{cases} 2-x, & x \leq 2 \\ x-2, & x > 2 \end{cases}$

   (vi) $\frac{x^4-9x^2}{x^4-3x^3}$

2. Redefine $f(x)$ so that it is continuous at the given points

   (i) $\frac{x^2-2}{x-2}$ at $x = 2$

   (ii) $\frac{x^2-1}{x-1}$ at $x = 1$

   (iii) $\frac{1-\cos^2 x}{\sin^2 x}$ at $x = 0$

   (iv) $\frac{1-\cos^2 x}{\sin x \cos x}$ at $x = 0$

   (v) $\frac{\sin x}{x}$ at $x = 0$

3. Given

$$f(x) = \begin{cases} 4 - x^2, & x \leq -1 \\ x+1, & x > -1 \end{cases}$$

   Discuss the continuity of $f(x)$ at $x = -1$

4. Find the constant k that will make the function f continuous at $x = 1$ if

$$f(x) = \begin{cases} \frac{x^3-3x^2+2}{x^2-1}, & for\ x \neq 1 \\ k, & for\ x = 1 \end{cases}$$

5. Given

$$f(x) = \begin{cases} \sin x, & \text{if } 2n\pi < x < 2(n+1)\pi \text{ for n even} \\ \cos x, & \text{if } 2n\pi < x < 2(n+1)\pi \text{ for n odd} \end{cases}$$

   (a) Sketch $f(x)$

   (b) Find $f(\pi), f(2\pi)$ and $f(3\pi)$

   (c) Find $\lim_{x\to 2\pi} f(x)$ if it exists

   (d) Find $\lim_{x\to 3\pi} f(x)$ if it exists

3.4 Questions with Solutions

3.4.1 Questions

[Limits & Continuity]

(a) State the definition of a limit of a function $f(x)$ as $x \to a$.

(b) Compute the following limits.

(i) $\lim_{x\to -3} \frac{2x^2+5x-3}{x+3}$

(ii) $\lim_{x\to 1} \sqrt{2x^2+1}$

(iii) $\lim_{x\to 1} \frac{x^3-1}{x-1}$

(iv) $\lim_{x\to\infty} \frac{7x^2-9x^3+x}{-18x^3-5x^2-x}$

(c) Find λ such that $\lim_{x\to -2}$ exists where

$$f(x) = \begin{cases} 6-x, & x \le -2 \\ \lambda x^2, & x > -2 \end{cases}$$

(d) Does the $\lim_{x\to 0} \frac{|x|}{4x}$ exist ? Give reasons for your answer.

e(i) Define what is meant by the function $f(x)$ being continuous at $x = a$.

(ii) Let

$$f(x) = \frac{x^2-9}{x-3}$$

(ii1) Show that $f(x)$ is discontinuous at $x = 3$

(ii2) Reduce $f(x)$ so that it is continuous at $x = 3$

(f) Check whether the function,

$$f(x) = \begin{cases} x^3, & x \le 2 \\ 10x, & x > 2 \end{cases}$$

is continuous at $x = 2$.

3.4.2 Solutions

[Limits & Continuity]

(a) We say that L is the limit of $f(x)$ as x approaches a if for every $\epsilon > 0$ (however small but positive) there exists a corresponding $\delta > 0$ also dependent on ϵ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$.

(b) (i)

$$\begin{aligned} \lim_{x\to -3} \frac{2x^2+5x-3}{x+3} &= \frac{0}{0} \text{ thus by La'Hopitals' rule} \\ &= \lim_{x\to -3} \frac{4x+5}{1} = -7 \end{aligned}$$

(ii)

$$\lim_{x\to 1}\sqrt{2x^2+1} = \sqrt{2(1)^2+1}=\sqrt{3}$$

(iii)

$$\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}=\lim_{x\to 1}(x^2+x+1)=3$$

(iv)

$$\lim_{x\to\infty}\frac{7x^2-9x^3+x}{-18x^3-5x^2-x}=\lim_{x\to\infty}\frac{\frac{7}{x}-9+\frac{1}{x^2}}{-18-\frac{5}{x}-\frac{1}{x^2}}=\frac{-9}{-18}=\frac{1}{2}$$

(c) For the limit to exist

$$\begin{aligned}\lim_{x\to -2^-} f(x) &= \lim_{x\to -2^+} f(x)\\ 8 &= 4\lambda\\ 2 &= \lambda\end{aligned}$$

(d) The limit does not exist since for

$$f(x)=\begin{cases}-\frac{x}{4x}, & x<0\\ \frac{x}{4x}, & x\geq 0\end{cases} = \begin{cases}-\frac{1}{4}, & x<0\\ \frac{1}{4}, & x\geq 0\end{cases}$$

$$\text{since} \quad \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x) \text{ that is } \left[-\frac{1}{4}\neq\frac{1}{4}\right]$$

e(i) We say that a function $f(x)$ is continuous at $x=a$ if

(i) the $\lim\limits_{x\to a} f(x)$ exists ie $\lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a^+} f(x)$

(ii) the functional value $f(a)$ exists.

(iii) $\lim_{x\to a} f(x)=f(a)$

OR
A function $f(x)$ is continuous at $x=a$ if and only if given $\epsilon>0$ there exists $\delta>0$ such that $|f(x)-f(a)|<\epsilon$ whenever $0\leq|x-a|<\delta$

(ii) Let $f(x)=\frac{x^2-9}{x-3}$

(ii1) Since $f(x)$ is not defined at $x=3$, then is discontinuous at that point.

(ii2) Using limits

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3}(x+3)=6$$

$$f(x)=\begin{cases}\frac{x^2-9}{x-3}, & x\neq 3\\ 6, & x=3\end{cases}$$

(f)

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= 2^3 = 8 \\ \text{and} \quad \lim_{x\to 2^+} &= 10(2) = 20 \\ \text{since} \quad \lim_{x\to 2^-} f(x) &\neq \lim_{x\to 2^+} \end{aligned}$$

The limit does not exist, thus the function is not continuous.

Exercise 3.3 Suppose that $6x - x^2 \leq f(x) \leq x^2 - 6x + 18$ for all x. Find $\lim_{x\to 3} f(x)$

3.5 Chapter Examples

1. If f is a continuous function on $[a,b]$, which of the following is necessarily *true*

   (A) f' exists on (a,b)

   (B) If $f(x_0)$ is a maximum of f, then $f'(x_0) = 0$

   (C) $\lim_{x\to x_0} f(x) = f\left(\lim_{x\to x_0} x\right)$ for $x_0 \in (a,b)$

   (D) $f'(x) = 0$ for some $x \in [a,b]$

   (E) The graph of f' is a straight line

   `Why do you think (1B) is False, but (1C) is True, How?`

2. *True* or *False*? When a function is continuous at a point, then the left hand limit exists at that point.

3. *True* or *False*? The existence of both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ always guarantees the existence of $f(a)$

   `False, they need also to be equal`

4. Identify a function that is not continuous at a given point

   (A) $y = |x|$ at $x = 4$

   (B) $y = \sin x$ at $x = 0$

   (C) $f(x) = \begin{cases} 2x, & x < 5 \\ 10, & x > 5 \end{cases}$ at $x = 5$

   (D) $y = 10^x$ at $x = 3$

   `If a function not defined at that point, it is enough not to be continuous`

5. $f(x) = x^2$ is not

   (A) a function

   (B) even

   (C) One-to-One

   (D) continuous at $x = 0$

6. State the interval on which $f(x) = \frac{1}{\sqrt{x+1}}$ is continuous on $\mathbb{R}$

   `There would be a slight error if I say` $x \geq -1$, `what is that mistake?`

7. What are a jump and essential discontinuities? Identify and classify the discontinuities of the following functions:

   (a)
   $$f(x) = \frac{x^2 - 3x - 28}{x - 7}$$

   (b)
   $$g(x) = \begin{cases} x + 1, & x \geq 3 \\ -2, & x < 3 \end{cases}$$

8. Give two possible reasons (characteristics) of $f(x)$ that might lead to a case where $\lim_{x\to a^+} f(x)$ exists but $\lim_{x\to a} f(x)$ does not exist

9. Does the existence of both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ always guarantee the existence of $\lim_{x \to a^+} f(x)$? Explain your answer.

10. Clearly describe how a function $f(x)$ with an essential discontinuity at $x = a$ can be redefined to make it continuous at $x = a$.

11. Using both a graph and actual functions, give one example of a function that has a limit at $x = 1$ but is not continuous at that point.

12. Using both a graph and an actual function, give one example of a function that is continuous at $x = 0$ but does not have a limit at that point.

13. Let f be a continuous function defined on $[0, 1]$ with range $[0, 1]$. Use the Intermediate Value Theorem to show that there is an $x \in [0, 1]$ such that $f(x) = 1 - x$.

14. Determine whether the following functions are continuous

(a)

$$f(x) = \begin{cases} 3x - 5 & ; x \neq 1 \\ 2 & ; x = 1 \end{cases}$$

at $x = 1$

(b)

$$f(x) = \begin{cases} \frac{x-6}{x-3}, & x < 0 \\ 2, & x = 0 \\ (4 + x^2)^{\frac{1}{2}}, & x > 0 \end{cases}$$

at $x = 0$

In Exercise 15 – 21, state the intervals on which the function is continuous.

15. $f(x) = \frac{x}{\cos x}$

16. $f(x) = \frac{x+2}{x^2-x-2}$

17. $y = \frac{x^2+x+1}{x^3+2x^2-3x}$

18. $y = x^{\frac{2}{3}} - x^{\frac{-2}{3}}$

19. $f(x) = \frac{1}{1+x^{\frac{2}{3}}}$

20. $f(x) = \begin{cases} 1 - x & , x \leq 2 \\ x - 1 & , x > 2 \end{cases}$

21. $y = \begin{cases} x^2 & , x < 0 \\ 3x & , x \geq 0 \end{cases}$

In exercise 22 – 25 the given function has a removable discontinuity at $x = a$. Determine how to define $f(a)$ so that the function is continuous at a.

22. $f(x) = \frac{x^2-1}{x-1}$, $a = 1$

23. $f(x) = \begin{cases} x^2 + 1 & , x < 1 \\ \sqrt{3 + x} & , x > 1 \end{cases}$, $a = 1$

24. $f(x) = \frac{\cos^2 x - 1}{\sin x}$, $a = 0$

25. $f(x) = \frac{x^2+x-2}{x^3-x^2-6x}$, $a = -2$

In the exercises 26 – 28, find the constant k that makes the function continuous at $x = a$.

26. $y = \begin{cases} x^k & , x \leq 2 \\ 10 - x & , x > 2 \end{cases}$, $a = 2$

27. $y = \begin{cases} k & , x \geq 1 \\ \frac{1}{\sqrt{kx^2+k}} & , x < 1 \end{cases}$, $a = 1$

28. $h = \begin{cases} (x-k)(x+k) & , \quad x \leq 2 \\ kx+5 & , \quad x > 2 \end{cases} \quad , \quad a = 2$

In Exercises $29 - 36$, solve the inequality $f(x) > 0$ or $f(x) < 0$ using the Intermediate Value Theorem.

29. $(x-3)(x+1) < 0$
30. $x(x+6) < -8$
31. $x^2 + x < 0$
32. $x^3 + x^2 - 2x > 0$
33. $x^2 + x + 7 > 19$
34. $x^4 - 9x^2 > 0$
35. $x \sec x > 0, \quad -2\pi \leq x \leq 2\pi$
36. $\sin x \cos x > 0, \quad -2\pi \leq x \leq 2\pi$
37. Let $f(x) = \frac{1}{(x-2)}$. Note that $f(0) = -\frac{1}{2}$ and $f(3) = 1$. Is there a number x between 0 and 3 such that $f(x) = 0$? Does this contradict the Intermediate Value Theorem?
38. Verify the fact that the function $f(x) = x^2$ satisfys the hypothesis of the Intermediate Value Theorem on the interval $[1, 2]$. Use the Intermediate Value theorem to explain why $\sqrt{2}$ is between 1 and $\frac{3}{2}$ and why $\sqrt{3}$ is between $\frac{3}{2}$ and 2.
39. The function $f(x) = \begin{cases} |x|, & x < 10 \\ 10, & x > 10 \end{cases}$ is continuous on $\mathbb{R}$.

    `False, but why?`

40. If f is continuous on $[a, b]$, then
    A. There must be a local extreme values but there may or may not be an absolute max or absolute min value of the function.
    B. There must be numbers m and M such that $m \leq f(x) \leq M$ for $x \in [a, b]$.
    C. Any absolute maximum or minimum would either be at the end point of the interval or at the places in the domain where $f'(x) = 0$.
    D. There must be a point c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$
    E. f is differentiable on (a, b).

    `Give reasons why other parts are false other than 40B`

41.
42.

Chapter 4

Differentiation

4.1 Derivative of a function

Definition 4.1.1 A derivative of a function $f(x)$ denoted as $f'(x)$ is said to exists if

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \text{exists}$$

ie if

$$\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$$

The derivative of a function $f(x)$ denoted by $f'(x)$ if exists, is given by

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}} \tag{4.1}$$

Remark 4.1.1 A derivative of a function $f(x)$ `at a point` $x=a$ denoted as $f'(a)$ is said to exists if [substituting $x=a$ in Equation (4.1)]

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \tag{4.2}$$

Definition 4.1.2 A derivative of a function $f(x)$ `at a point` $x=a$ denoted as $f'(a)$ is said to exists if

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \qquad \text{exists} \quad \Rightarrow$$

$$\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

The derivative of a function $f(x)$ at a point $x=a$ denoted as $f'(a)$ if exists, is given by

$$\boxed{f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}} \tag{4.3}$$

This definition of the derivative is illustrated by the following figure.

The slope of the chord AB is

$$\frac{\Delta y}{h} = \frac{f(x_0+h)-f(x_0)}{h}$$

and indeed as $h\to 0$, this quotient tends to the slope of the tangent to the curve at $x=x_0$ which is $f'(x_0)$.

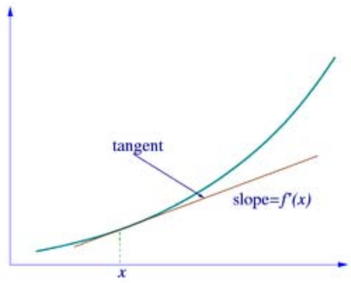


Figure 4.1: Illustration of the derivative of $f(x_0)$

Example 4.1.1 Suppose $f(x) = \alpha$ (constant). Use the definition of a derivative to compute $f'(x)$

By definition,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\alpha - \alpha}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

Note 4.1.1 This result shows that the derivative of a constant is 0 . Thus

$$\frac{d}{dx}(10) = \frac{d}{dx}(a) = \frac{d}{dx}(90) = 0$$

Example 4.1.2 Suppose $f(x) = x^2$, use the definition of a derivative to find $f'(x)$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
\text{but } f(x+h) &= (x+h)^2 \\
\text{therefore } \ f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} (2x + h) = 2x \\
\Rightarrow \ \frac{d}{dx}(x^2) &= 2x
\end{aligned}$$

Example 4.1.3 Let us start with the function $f(x) = x^2 + 1$. We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^2 + 1] - [x^2 + 1]}{h} \quad \text{bracketing } f(x+h) \text{ and } f(x) \\
&= \lim_{h\to 0} \frac{[x^2 + 2xh + h^2 + 1] - [x^2 + 1]}{h} \quad \text{the } f(x) \text{ terms have to cancel} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \quad \text{which of the six techniques of finding limits to use?} \\
&= \lim_{h\to 0} \frac{(2x + h)(h)}{h} \\
&= \lim_{h\to 0} (2x + h) = 2x \quad \text{can use method I now, of substitution}
\end{aligned}$$

Example 4.1.4 For $f(x) = x^n$ where $n \geq 1$ integer. Use the definition of a derivative to compute $\frac{d}{dx}(x^n)$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ \text{since } f(x+h) &= (x+h)^n \\ \text{and } f(x) &= x^n \\ \Rightarrow f'(x) &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \end{aligned}$$

We use Binomial theorem to expand $(x+h)^n$

$$\begin{aligned} \Rightarrow f'(x) &= \lim_{h\to 0}\left(\frac{x^n + nx^{n-1}h + \frac{n(n-1)x^{n-2}(h)^2}{2!} + \cdots + h^n - x^n}{h}\right) \\ &= \lim_{h\to 0}\left[nx^{n-1} + \frac{hn(n-1)x^{n-2}}{2!} + ... + h^{n-1}\right] \quad \Rightarrow \\ \frac{d}{dx}(x^n) &= nx^{n-1} \end{aligned}$$

This result is an important differentiation formula . The formula is valid for all $n \in \mathbb{R}$ (real numbers)

Example 4.1.5 Suppose $f(x) = x$, using the definition of a derivative find $f'(x)$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{x+h-x}{h} \\ &= \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1 \end{aligned}$$

The result shows that

$$\frac{d}{dx}(x) = 1$$

Example 4.1.6 Consider the function

$$f(x) = 1/x$$

for $x \neq 0$. We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
f'(x) &= \lim_{h\to 0} \frac{x - (x+h)}{hx(x+h)} \\
f'(x) &= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}
\end{aligned}$$

Exercise 4.1 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = (x+1)^{\frac{1}{3}}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{[(x+1)+h]^{\frac{1}{3}} - (x+1)^{\frac{1}{3}}}{h} \quad \text{Binomial expansion, fractional powers} \\
f'(x) &= \lim_{h\to 0} \frac{\left[(x+1)^{\frac{1}{3}} + \frac{1}{3}(x+1)^{\frac{-2}{3}}h + \frac{\frac{1}{3}\frac{-2}{3}(x+1)^{\frac{-5}{3}}h^2}{2!} + \cdots\right] - \left[(x+1)^{\frac{1}{3}}\right]}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\frac{1}{3}(x+1)^{\frac{-2}{3}}h + \frac{\frac{1}{3}\frac{-2}{3}(x+1)^{\frac{-5}{3}}h^2}{2!} + \cdots}{h} \\
f'(x) &= \lim_{h\to 0} \left[\frac{1}{3}(x+1)^{\frac{-2}{3}} + \frac{\frac{1}{3}\frac{-2}{3}(x+1)^{\frac{-5}{3}}h}{2!}\right] \\
&= \frac{1}{3}(x+1)^{\frac{-2}{3}}
\end{aligned}$$

Note 4.1.2 For any value of n, whether positive, negative, integer or non-integer, the value of the nth power of a binomial is given by:

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \cdots + b^n$$

Example 4.1.7 Using the definition of a derivative, find $\frac{d}{dx}(\sin x)$

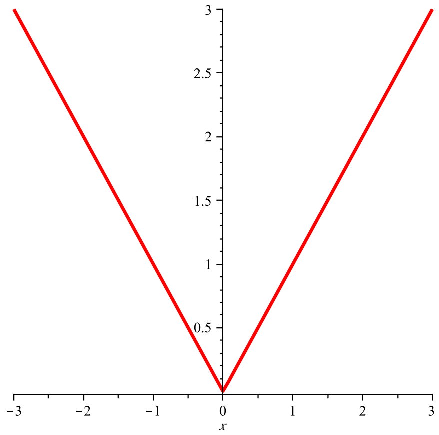
$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
\sin'(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}\\
&= \sin x\lim_{h\to 0}\frac{(\cos h-1)}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin x\lim_{h\to 0}\frac{(-\sin h)}{1}+\cos x\lim_{h\to 0}\frac{\cos h}{1}\\
&= \sin x(0)+\cos x(1)\\
&= \cos x
\end{aligned}
$$

Example 4.1.8 Let a function f at a point $x=2$ be defined by $f(x)=(x+3)^{10}x$. Find

$$\left.\frac{df}{dx}\right|_{x=2}$$

The function $f(x)$ given to you is at only $x=2$, but we do not have the function as $x\to 2^+$ or the $f(x)$ as $x\to 2^-$, so we cannot compute the derivative since the function is not known.

Example 4.1.9 Show that the function $y = |x|$ is not differentiable at $x = 0$.



Recall that

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

with the graph of $y = |x|$ on the left

Now check for differentiability at $x = 0$, i.e., compute $f'(0)$. Then

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \end{aligned}$$

Finding a limit for a piecewise function, we check from left and from right, if equal, that is the limit, otherwise, the limit does not exist.

From the right of point $x = 0$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(x) - (0)}{x} = \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = 1$$

From the left of point $x = 0$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(-x) - (0)}{x} = \lim_{x \to 0} \frac{-x}{x} = \lim_{x \to 0} -1 = -1$$

Since the one-sided limits exist but are *not equal though finite*, $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist, and f is not differentiable at $x = 0$. This implies that the derivative of $f(x) = |x|$ does not exist at $x = 0$

Example 4.1.10 Show that the derivative of the function $f(x) = x|x|$ is given by.
The function is also given by

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

Convince your self that

$$f'(x) = \begin{cases} 2x, & x > 0 \\ 0, & x = 0 \\ -2x, & x < 0 \end{cases} = 2|x|$$

Example 4.1.11 Use the definition of derivatives to compute $f'(x)$ given $f(x) = mx + c$

Example 4.1.12 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = \frac{1}{2}x - \frac{3}{5}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\left(\frac{1}{2}(x+h) - \frac{3}{5}\right) - \left(\frac{1}{2}x - \frac{3}{5}\right)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\frac{1}{2}x + \frac{1}{2}h - \frac{3}{5} - \frac{1}{2}x + \frac{3}{5}}{h}
\end{aligned}$$

Algebraically and arithmetically simplify the expression in the numerator.

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{2}h}{h}$$

The term h now divides out and the limit can be calculated.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Example 4.1.13 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = 5x^2 - 3x + 7$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{(5(x+h)^2 - 3(x+h) + 7) - (5x^2 - 3x + 7)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 - 3x - 3h + 7 - 5x^2 + 3x - 7}{h}
\end{aligned}$$

Algebraically and arithmetically simplify the expression in the numerator.

$$f'(x) = \lim_{h\to 0} \frac{10xh + 5h^2 - 3h}{h}$$

The term h now divides out and the limit can be calculated.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}(10x + 5h - 3) \\
&= 10x - 3
\end{aligned}$$

Example 4.1.14 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = 4 - \sqrt{x+3}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\left(4 - \sqrt{(x+h)+3}\,\right) - \left(4 - \sqrt{x+3}\,\right)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+3} - \sqrt{(x+h)+3}}{h}
\end{aligned}$$

Eliminate the square root terms in the numerator of the expression by multiplying by the conjugate of the numerator divided by itself

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+3}-\sqrt{x+h+3}}{h} \frac{\sqrt{x+3}+\sqrt{x+h+3}}{\sqrt{x+3}+\sqrt{x+h+3}} \\
f'(x) &= \lim_{h\to 0} \frac{(x+3)-(x+h+3)}{h\left(\sqrt{x+3}+\sqrt{x+h+3}\right)} \\
f'(x) &= \lim_{h\to 0} \frac{-h}{h\left(\sqrt{x+3}+\sqrt{x+h+3}\right)}
\end{aligned}$$

The term h now divides out and the limit can be calculated.

$$f'(x) = \lim_{h\to 0} \frac{-1}{\left(\sqrt{x+3}+\sqrt{x+h+3}\right)} = \frac{-1}{\left(\sqrt{x+3}+\sqrt{x+3}\right)} = \frac{-1}{2\sqrt{x+3}}$$

Example 4.1.15 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = \frac{x+1}{2-x}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{(x+h)+1}{2-(x+h)} - \frac{x+1}{2-x}}{h} \\
f'(x) &= \lim_{h\to 0} \frac{[(x+h+1)(2-x)-(x+1)(2-x-h)]}{(2-x-h)(2-x)} \frac{1}{h}
\end{aligned}$$

Algebraically and arithmetically simplify the expression in the numerator. It is important to note that the denominator of this expression should be left in factored form so that the term h can be easily eliminated later.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{2x+2h+2-x^2-xh-x-\{2x-x^2-xh+2-x-h\}}{(2-x-h)(2-x)h} \\
f'(x) &= \lim_{h\to 0} \frac{3h}{(2-x-h)(2-x)h}
\end{aligned}$$

The term h now divides out and the limit can be calculated

$$f'(x) = \lim_{h\to 0} \frac{3}{(2-x-h)(2-x)} = \frac{3}{(2-x)(2-x)} = \frac{3}{(2-x)^2}$$

Example 4.1.16 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = x^{\frac{2}{3}}$$

This problem may be more difficult than it first appears.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^{\frac{2}{3}} - x^{\frac{2}{3}}}{h}$$

Note that $A-B$ can be written as the difference of cubes,

$$x^3 - y^3 = (x-y)(x^2+xy+y^2)$$

so that

$$\begin{aligned} A-B=\left(A^{\frac{1}{3}}\right)^3-\left(B^{\frac{1}{3}}\right)^3 &= \left(A^{\frac{1}{3}}-B^{\frac{1}{3}}\right)\left(A^{\frac{2}{3}}+A^{\frac{1}{3}}B^{\frac{1}{3}}+B^{\frac{2}{3}}\right) \\ \Rightarrow \left(A^{\frac{1}{3}}-B^{\frac{1}{3}}\right) &= \frac{A-B}{\left(A^{\frac{2}{3}}+A^{\frac{1}{3}}B^{\frac{1}{3}}+B^{\frac{2}{3}}\right)} \end{aligned}$$

This will help explain the next step.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{(x+h)^{\frac{2}{3}}-x^{\frac{2}{3}}}{h} = \lim_{h\to 0}\frac{\{(x+h)^2\}^{\frac{1}{3}}-\{x^2\}^{\frac{1}{3}}}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h\left\{(x+h)^{\frac{4}{3}}+(x+h)^{\frac{2}{3}}x^{\frac{2}{3}}+x^{\frac{4}{3}}\right\}} \\ &= \lim_{h\to 0}\frac{(x^2+2xh+h^2)-x^2}{h\left\{(x+h)^{\frac{4}{3}}+(x+h)^{\frac{2}{3}}x^{\frac{2}{3}}+x^{\frac{4}{3}}\right\}} \\ &= \lim_{h\to 0}\frac{h(2x+h)}{h\left\{(x+h)^{\frac{4}{3}}+(x+h)^{\frac{2}{3}}x^{\frac{2}{3}}+x^{\frac{4}{3}}\right\}} \\ &= \lim_{h\to 0}\frac{2x+h}{\left\{(x+h)^{\frac{4}{3}}+(x+h)^{\frac{2}{3}}x^{\frac{2}{3}}+x^{\frac{4}{3}}\right\}} \\ &= \frac{2x}{\left\{x^{\frac{4}{3}}+x^{\frac{2}{3}}x^{\frac{2}{3}}+x^{\frac{4}{3}}\right\}} = \frac{2x}{3x^{\frac{4}{3}}} = \frac{2}{3x^{\frac{1}{3}}} \end{aligned}$$

Example 4.1.17 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = \cos 3x$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\cos 3(x+h)-\cos 3x}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\cos(3x+3h)-\cos 3x}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\{\cos 3x\cos 3h-\sin 3x\sin 3h\}-\cos 3x}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\cos 3x(\cos 3h-1)-\sin 3x\sin 3h}{h}
\end{aligned}$$

Since

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

Recall the following two well-known trigonometry limits (La'Hopital rule):

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0\ ,\ \lim_{h\to 0} \frac{\sin h}{h} = 1 \ \Rightarrow\ \lim_{h\to 0} \frac{\sin 3h}{h} = 3$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\cos 3x(\cos 3h-1)-\sin 3x\sin 3h}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\cos 3x(\cos 3h-1)}{h} - \lim_{h\to 0} \frac{\sin 3x\sin 3h}{h} \\
f'(x) &= \cos 3x \lim_{h\to 0} \frac{(\cos 3h-1)}{h} - \sin 3x \lim_{h\to 0} \frac{\sin 3h}{h}
\end{aligned}$$

Terms without h were factored out, as

$$\lim_{x\to a} \alpha f(x) = \alpha \lim_{x\to a} f(x)$$

$$\begin{aligned}
f'(x) &= \cos 3x \lim_{h\to 0} \frac{(\cos 3h-1)}{h} - \sin 3x \lim_{h\to 0} \frac{\sin 3h}{h} \\
f'(x) &= \cos 3x\ (0) - \sin 3x\ (3) \\
f'(x) &= -3\sin 3x
\end{aligned}$$

Example 4.1.18 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = \frac{x-1}{x^2+3x}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{(x+h)-1}{(x+h)^2+3(x+h)} - \dfrac{x-1}{x^2+3x}}{h}
\end{aligned}$$

Get a common denominator for the expression in the numerator.

$$= \lim_{h\to 0} \frac{(x^3+3x^2+x^2h+3xh-x^2-3x-x^3-2x^2h-xh^2-3x^2-3xh+x^2+2xh+h^2+3x+3h)}{(x^2+2xh+h^2+3x+3h)(x^2+3x)} \frac{1}{h}$$

Algebraically and arithmetically simplify the expression in the numerator. The terms $x^3, 2x^2, -3x$, and $3xh$ will subtract out. It is important to note that the denominator of this expression should be left in factored form so that the term h can be easily eliminated later.

$$= \lim_{h\to 0} \frac{-x^2h+2xh+h^2+3h}{(x^2+2xh+h^2+3x+3h)(x^2+3x)h}$$

Factor h from the numerator.

$$= \lim_{h\to 0} \frac{h(-x^2+2x+h+3)}{(x^2+2xh+h^2+3x+3h)(x^2+3x)h}$$

The term h now divides out and the limit can be calculated.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{-x^2+2x+h+3}{(x^2+2xh+h^2+3x+3h)(x^2+3x)} \\ f'(x) &= \frac{-x^2+2x+3}{(x^2+3x)(x^2+3x)} \\ f'(x) &= \frac{2x+3-x^2}{(x^2+3x)^2} \end{aligned}$$

Example 4.1.19 Use the limit definition to compute the derivative, $f'(x)$, for

$$f(x) = \sqrt{x^3-x}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^3-(x+h)}-\sqrt{x^3-x}}{h}$$

Eliminate the square root terms in the numerator of the expression by multiplying by the conjugate of the numerator divided by itself.

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^3-(x+h)}-\sqrt{x^3-x}}{h} \frac{\sqrt{(x+h)^3-(x+h)}+\sqrt{x^3-x}}{\sqrt{(x+h)^3-(x+h)}+\sqrt{x^3-x}}$$

Algebraically and arithmetically simplify the expression in the numerator. It is important to note that the denominator of this expression should be left in factored form so that the term h can be easily eliminated later.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[(x+h)^3-(x+h)]-(x^3-x)}{h\sqrt{(x+h)^3-(x+h)}+\sqrt{x^3-x}} \\ f'(x) &= \lim_{h\to 0} \frac{[x^3+3xh^2+3x^2h+h^3-x-h]-(x^3-x)}{h\sqrt{(x+h)^3-(x+h)}+\sqrt{x^3-x}} \\ f'(x) &= \lim_{h\to 0} \frac{3xh^2+3x^2h+h^3-h}{h\sqrt{(x+h)^3-(x+h)}+\sqrt{x^3-x}} \end{aligned}$$

Factor h from the numerator.

$$f'(x) = \lim_{h\to 0} \frac{h\,[3xh+3x^2+h^2-1]}{h\sqrt{(x+h)^3-(x+h)}+\sqrt{x^3-x}}$$

The term h now divides out and the limit can be calculated.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{3xh+3x^2+h^2-1}{\sqrt{(x+h)^3-(x+h)}+\sqrt{x^3-x}} \\
f'(x) &= \frac{3x^2-1}{\sqrt{x^3-x}+\sqrt{x^3-x}} \\
f'(x) &= \frac{3x^2-1}{2\sqrt{x^3-x}}
\end{aligned}$$

Example 4.1.20 Assume a piecewise function $f(x)$ defined as

$$f(x)=\begin{cases} 2+\sqrt{x}, & \text{if } x\geq 1 \\ \frac{1}{2}x+\frac{5}{2}, & \text{if } x<1 \end{cases}$$

Show whether or not $f(x)$ is differentiable at $x=1$, i.e., use the limit definition of the derivative to compute $f'(1)$.

To compute $f'(1)$: Lets first compute $f(1)=2+\sqrt{1}=3$, then

$$\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\
f'(1) &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1}
\end{aligned}$$

Finding a limit for a piecewise function, we check from left and from right, if equal, that is the limit, otherwise, the limit does not exist.

From the right of point $x=1$

$$\begin{aligned}
\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} &= \lim_{x\to 1}\frac{(2+\sqrt{x}\ )-(3)}{x-1} \\
&= \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{(\sqrt{x}-1)}{(\sqrt{x}-1)(\sqrt{x}+1)} \\
&= \lim_{x\to 1}\frac{1}{(\sqrt{x}+1)}=\frac{1}{2}
\end{aligned}$$

From the left of point $x=1$

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{\left(\frac{1}{2}x+\frac{5}{2}\right)-(3)}{x-1}=\lim_{x\to 1}\frac{\frac{x-1}{2}}{x-1}=\lim_{x\to 1}\frac{1}{2}=\frac{1}{2}$$

Since the one-sided limits exists and are *equal*, $f'(1)=\lim\limits_{x\to 1}\dfrac{f(x)-f(1)}{x-1}=\dfrac{1}{2}$ does exist, and thus f is differentiable at $x=1$.

Example 4.1.21 Assume that

$$f(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right), & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

Show that f is differentiable at $x = 0$, i.e., use the limit definition of the derivative to compute $f'(0)$.

To have from right and from left, we use the Squeeze law to create functions from left and right.

$$x^2 \sin\left(\frac{1}{x}\right) = \begin{cases} x^2, & \text{if } \ x > 0 \\ \\ -x^2, & \text{if } \ x < 0 \end{cases}$$

The derivative at $x = 0$ is

$$f'(0) = 0$$

Show the solution above.

Remark 4.1.2 What follows is a common *incorrect* attempt to solve this problem using another method. Since $f(x) = x^2 \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$, it follows, using the product rule and chain rule, that

$$f'(x) \ = \ x^2 \cos\left(\frac{1}{x}\right)\left\{\frac{-1}{x^2}\right\} + 2x \sin\left(\frac{1}{x}\right) = -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)$$

for $x \neq 0$. Then

$$\begin{aligned} f'(0) &= \lim_{x\to 0} f'(x) \\ &= \lim_{x\to 0} \left\{ -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)\right\} \end{aligned}$$

Because the term $-\cos\left(\dfrac{1}{x}\right)$ oscillates between 1 and -1 as h approaches zero, this limit does not exist.

An *incorrect* conclusion would be that $f'(0)$ does not exist, i.e., f is not differentiable at $x = 0$. If f' were continuous at $x = 0$, this would be a valid method to compute $f'(0)$.

Example 4.1.22 Use the limit definition to compute the derivative, $f'(x)$, for a piecewise function

$$f(x) = |x^2 - 3x|$$

First rewrite $f(x)$. That is,

$$f(x) = |x^2 - 3x| = \begin{cases} (x^2 - 3x), & x \in (-\infty, 0] \cup [3, \infty) \\ \\ -(x^2 - 3x), & 0 < x < 3 \end{cases}$$

In other words, the region for positive, $+(x^2 - 3x)$

$$\begin{aligned} (x^2 - 3x) \geq 0 \ &\Rightarrow \ x(x-3) \geq 0 \ \Rightarrow \\ \text{either (both positives) } x \geq 0 \ \& \ (x-3) \geq 0 \ &\text{ or } \ \text{(both negatives) } x \leq 0 \ \& \ (x-3) \leq 0 \\ \text{either } \ x \geq 0 \ \& \ x \geq 3 \ &\text{ or } \ x \leq 0 \ \& \ x \leq 3 \\ \text{either } \ x \geq 3 \ &\text{ or } \ x \leq 0 \\ \Rightarrow \ & \ \ x \in (-\infty, 0] \cup [3, \infty) \end{aligned}$$

and the region for negative, $-(x^2 - 3x)$

$$\begin{array}{rcl} (x^2-3x)<0 & \Rightarrow & x(x-3)<0 \Rightarrow \\ \text{either (one negative) } x>0 \ \& \ (x-3)<0 & \text{or} & \text{(the other negative) } x<0 \ \& \ (x-3)>0 \\ \text{either } \ x>0 \ \& \ x<3 & \text{or} & x<0 \ \& \ x>3 \\ \text{either } \ (0,3) & \text{or} & \text{no solution} \\ \Rightarrow & & x \in (0,3) \end{array}$$

This can be summarized as

$$f(x) = \begin{cases} (x^2-3x), & \text{if } \ x \le 0 \\ -(x^2-3x), & \text{if } \ 0<x<3 \\ (x^2-3x), & \text{if } \ x \ge 3 \end{cases}$$

1. Check for differentiability at $x = 0$, i.e., compute $f'(0)$. Then

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ f'(0) &= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \end{aligned}$$

   Finding a limit for a piecewise function, we check from left and from right, if equal, that is the limit, otherwise, the limit does not exist.

   From the right of point $x = 0$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{(3x-x^2)-(0)}{x} = \lim_{x\to 0} \frac{3x-x^2}{x} = \lim_{x\to 0} 3-x = 3$$

   From the left of point $x = 0$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{(x^2-3x)-(0)}{x} = \lim_{x\to 0} \frac{x^2-3x}{x} = \lim_{x\to 0} x-3 = -3$$

   Since the one-sided limits exist but are *not equal*, $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist, and f is not differentiable at $x = 0$.

2. Now check for differentiability at $x = 3$, i.e., compute $f'(3)$: $f(3) = (3)^2 - 3(3) = 0$ Then

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ f'(3) &= \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} \end{aligned}$$

   Finding a limit for a piecewise function, we check from left and from right, if equal, that is the limit, otherwise, the limit does not exist.

   From the right of point $x = 3$

$$\lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3} \frac{(x^2-3x)-(0)}{x-3} = \lim_{x\to 3} \frac{x^2-3x}{x-3} = \lim_{x\to 3} x = 3$$

From the left of point $x = 3$

$$\lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3} \frac{(3x-x^2)-(0)}{x-3} = \lim_{x\to 3} \frac{3x-x^2}{x-3} = \lim_{x\to 3} -x = -3$$

Since the one-sided limits exist but are *not equal*, $f'(3) = \lim_{x\to 3} \dfrac{f(x)-f(3)}{x-3}$ does not exist, and f is not differentiable at $x = 3$. Since the one-sided limits exist but are *not equal*, $f'(3) = \lim_{x\to 3} \dfrac{f(x)-f(3)}{x-3}$ does not exist, and f is not differentiable at $x = 3$.

3. Assume that $x < 0$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-3(x+h)-(x^2-3x)}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2-3h}{h} = \lim_{h\to 0} \frac{h(2x+h-3)}{h} = \lim_{h\to 0} (2x+h-3) = 2x-3. \end{aligned}$$

4. Assume that $x > 3$. Then it is also true (the same function of $f(x) = (x^2-3x)$) that

$$f'(x) = 2x-3$$

5. Assume that $0 < x < 3$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)-(x+h)^2-(3x-x^2)}{h} \\ &= \lim_{h\to 0} \frac{3x+3h-(x^2+2xh+h^2)-3x+x^2}{h} = \lim_{h\to 0} \frac{3x+3h-x^2-2xh-h^2-3x+x^2}{h} \\ &= \lim_{h\to 0} \frac{3h-2xh-h^2}{h} = \lim_{h\to 0} \frac{h(3-2x-h)}{h} = \lim_{h\to 0} (3-2x-h) = 3-2x. \end{aligned}$$

6. Summarizing, the derivative of f (the function f is not differentiable at $x = 0$ or $x = 3$) is

$$f'(x) = \begin{cases} 2x-3, & \text{if } x < 0 \\ \text{DNE}, & \text{if } x = 0 \\ 3-2x, & \text{if } 0 < x < 3 \\ \text{DNE}, & \text{if } x = 3 \\ 2x-3, & \text{if } x > 3 \end{cases}$$

Example 4.1.23 Assume that

$$f(x) = \begin{cases} \dfrac{1}{4}x^3 - \dfrac{1}{2}x^2, & \text{if } \ x \geq 2 \\ \\ \dfrac{-3x+6}{x^2+2}, & \text{if } \ x < 2 \end{cases}$$

Determine if f is differentiable at $x = 2$, i.e., determine if

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} \quad \text{exists}$$

From right, for region $x > 2$

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\left\{\frac{1}{4}x^3 - \frac{1}{2}x^2 - 0\right\}}{x - 2} = \lim_{x \to 2} \frac{\frac{1}{4}x^2(x-2)}{(x-2)} = \lim_{x \to 2} \frac{1}{4}x^2 = 1$$

From left, for region $x < 2$

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\left\{\frac{-3x+6}{x^2+2} - 0\right\}}{x - 2} = \lim_{x \to 2} \frac{-3x+6}{(x-2)(x^2+2)} = \lim_{x \to 2} \frac{-3(x-2)}{(x-2)(x^2+2)} = -\frac{1}{2}$$

Since the derivatives are not equal, the derivative does not exist.

Remark 4.1.3 : Use of the limit definition of the derivative of f at $x = 2$ also leads to a correct solution to this problem.

Remark 4.1.4 : What follows is a common incorrect attempt to solve this problem using another method.

For $x > 2$

$$f'(x) = \frac{3}{4}x^2 - x$$

For $x < 2$

$$f'(x) = \frac{(x^2+2)(-6) - (-6x-6)(2x)}{(x^2+2)^2} = \frac{6x^2 + 12x - 12}{(x^2+2)^2}$$

Then

$$\lim_{x \to 2^+} f'(x) = \lim_{x \to 2^+} \left\{\frac{3}{4}x^2 - x\right\} = \frac{3}{4}(2)^2 - 2 = 1$$

and

$$\lim_{x \to 2^-} f'(x) = \lim_{x \to 2^-} \frac{6x^2 + 12x - 12}{(x^2+2)^2} = \frac{6(2)^2 + 12(2) - 12}{((2)^2+2)^2} = 1$$

An *incorrect* conclusion would be that $f'(2) = 1$. If f' were continuous at $x = 2$, this would be a valid method to compute $f'(2)$.

Example 4.1.24 Given a piecewise function

$$f(x) = \begin{cases} x^2, & \text{if } \ x < 0 \\ \\ 2, & \text{if } \ 0 \leq x \leq 3 \\ \\ 4 - x, & \text{if } \ x > 3 \end{cases}$$

Show that (using the definitions of differentiability)

$$f'(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x < 3 \\ -1, & \text{if } x > 3 \end{cases}$$

Exercise 4.2 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} \frac{1+x}{2}, & \text{if } x < 1 \\ 1, & \text{if } x = 1 \\ \sqrt{x}, & \text{if } x > 1 \end{cases}$$

(a) Find $f'(1)$

(b) Find $f''(1)$

Example 4.1.25 Show that the piecewise function

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is not differentiable at $x = 0$. $[0 \neq -1]$
It can also be seen from the graph, having a sharp turn at $x = 0$.

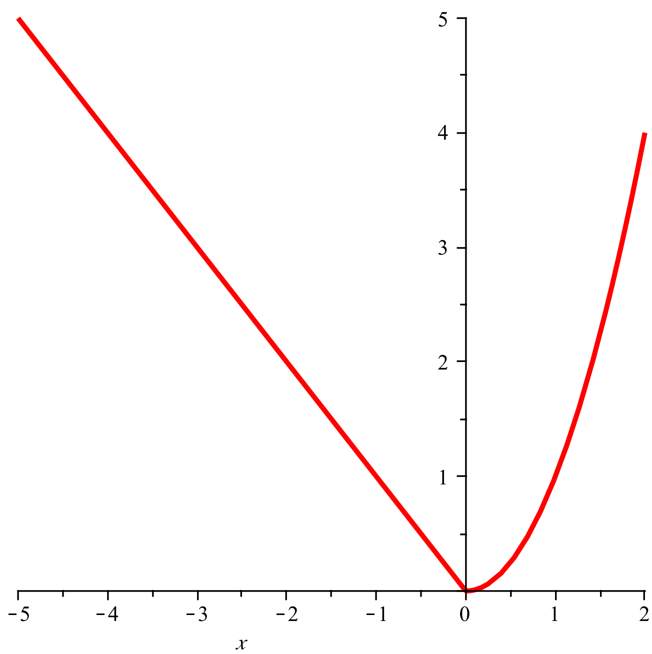


Figure 4.2: Sharp turn at $x = 0$, not differentiable

Example 4.1.26 Find the derivative of

$$y = |5x - 2|$$

The function is given by

$$y = \begin{cases} (5x-2), & x \geq \frac{2}{5} \\ \\ -(5x-2), & x < \frac{2}{5} \end{cases}$$

Note that, we have not been asked $f'\left(\frac{2}{5}\right)$ [at a point], but $f'(x)$ [everywhere], but since we have a special point $x = \frac{2}{5}$, to differentiate everywhere, it is to differentiate at $x = \frac{2}{5}$, $x > \frac{2}{5}$ and $x < \frac{2}{5}$

(1). `Differentiable at` $x = \frac{2}{5}$:

Derivative is given by

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ f'\left(\frac{2}{5}\right) &= \lim_{x\to \frac{2}{5}} \frac{f(x)-f\left(\frac{2}{5}\right)}{x-\frac{2}{5}} \end{aligned}$$

Since finding limit of a piecewise function, we use the informal definition of limits:

From the right of point $x = \frac{2}{5}$

$$\begin{aligned} \lim_{x\to \frac{2}{5}^+} \frac{f(x)-f\left(\frac{2}{5}\right)}{x-\frac{2}{5}} &= \lim_{x\to \frac{2}{5}} \frac{(5x-2)-0}{x-\frac{2}{5}} \\ &= \lim_{x\to \frac{2}{5}} \frac{5\left(x-\frac{2}{5}\right)-0}{\left(x-\frac{2}{5}\right)} \\ &= \lim_{x\to \frac{2}{5}} 5 = 5 \end{aligned}$$

From the left of point $x = \frac{2}{5}$

$$\begin{aligned} \lim_{x\to \frac{2}{5}^-} \frac{f(x)-f\left(\frac{2}{5}\right)}{x-\frac{2}{5}} &= \lim_{x\to \frac{2}{5}} \frac{-(5x-2)-0}{x-\frac{2}{5}} \\ &= \lim_{x\to \frac{2}{5}} \frac{-5\left(x-\frac{2}{5}\right)-0}{\left(x-\frac{2}{5}\right)} \\ &= \lim_{x\to \frac{2}{5}} -5 = -5 \end{aligned}$$

since

$$\lim_{x\to \frac{2}{5}^+} \frac{f(x)-f\left(\frac{2}{5}\right)}{x-\frac{2}{5}} \neq \lim_{x\to \frac{2}{5}^-} \frac{f(x)-f\left(\frac{2}{5}\right)}{x-\frac{2}{5}} \Rightarrow \lim_{x\to \frac{2}{5}} \frac{f(x)-f\left(\frac{2}{5}\right)}{x-\frac{2}{5}} = f'\left(\frac{2}{5}\right) \quad \text{DNE}$$

the function is not differentiable at $x = \frac{2}{5}$.

(2). `Differentiability for` $x > \frac{2}{5}$:

For the region $x > \frac{2}{5}$, $f(x) = y = (5x-2)$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[5(x+h)-2]-[(5x-2)]}{h} \\ &= \lim_{h\to 0} \frac{5h}{h} = \lim_{h\to 0} 5 = 5 \end{aligned}$$

(3). **Differentiability for $x < \frac{2}{5}$:**

For the region $x < \frac{2}{5}$, $f(x) = y = -(5x - 2)$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[-\{5(x+h) - 2\}] - [-(5x-2)]}{h} \\ &= \lim_{h \to 0} \frac{-5h}{h} = \lim_{h \to 0} -5 = -5 \end{aligned}$$

In summary, the derivative of the function y denoted as y' is given by

$$y' = \begin{cases} 5, & x > \frac{2}{5} \\ \text{DNE}, & x = \frac{2}{5} \\ -5, & x < \frac{2}{5} \end{cases}$$

Example 4.1.27 Given a function

$$y = \frac{|4x-3|}{4x-3}$$

Find the derivative of y

$$y = \begin{cases} \frac{(4x-3)}{4x-3}, & x \geq \frac{3}{4} \\ \\ \frac{-(4x-3)}{4x-3}, & x < \frac{3}{4} \end{cases} \quad = \quad \begin{cases} 1, & x \geq \frac{3}{4} \\ \\ -1, & x < \frac{3}{4} \end{cases}$$

(1). `Differentiability at` $x = \frac{3}{4}$:

$$\lim_{x\to\frac{3}{4}^-} \frac{f(x) - f(\frac{3}{4})}{x - \frac{3}{4}} \neq \lim_{x\to\frac{3}{4}^+} \frac{f(x) - f(\frac{3}{4})}{x - \frac{3}{4}} \quad \Leftrightarrow \quad \lim_{x\to\frac{3}{4}} \frac{f(x) - f(\frac{3}{4})}{x - \frac{3}{4}} \text{ DNE}$$

$y'\left(\frac{3}{4}\right)$ does not exist.

(2). `Differentiability in region` $x > \frac{3}{4}$: consider $f(x) = 1$

$$y' = 0$$

(3). `Differentiability in region` $x < \frac{3}{4}$: consider $f(x) = -1$

$$y' = 0$$

$$y' = \begin{cases} 0, & x > \frac{3}{4} \\ \\ \text{DNE}, & x = \frac{3}{4} \\ \\ 0, & x < \frac{3}{4} \end{cases}$$

Exercise 4.3 Using the definition of derivative, calculate the derivative of

$$f(x) = (3x+2)^{\frac{2}{3}}$$

Apply Binomial expansion at a certain point.

Exercise 4.4 Using the definition of derivative, calculate the derivative of

$$f(x) = \frac{5}{3x+4}$$

Exercise 4.5 Find the derivative of the function $\sqrt{x}$ at the point $x = 1$. $f'(1) = \frac{1}{2}$

Example 4.1.28 Given

$$f(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ \\ \cos x, & x < 0 \end{cases}$$

Show that $f(x)$ is differentiable at $x = 0$, and $f'(0) = 0$

Remark 4.1.5 Even if f does have a derivative, it may not have a second derivative. For example, let

$$f(x) = \begin{cases} +x^2, & \text{if } x \geq 0 \\ \\ -x^2, & \text{if } x < 0 \end{cases}$$

Calculation shows that f is a differentiable function whose derivative is

$$f'(x) = \begin{cases} +2x, & \text{if } x \geq 0 \\ \\ -2x, & \text{if } x < 0 \end{cases} \quad = \quad 2|x|$$

an absolute function which does not have a derivative, thus $f''(x)$ does not exist.

4.2 Continuity Versus Differentiability

Theorem 4.2.1 *Let f be differentiable at x_0. Then f is continuous at x_0.*

Proof: What is known is that f is differentiable at x_0. That is

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \qquad \text{exists say } \xi \tag{4.4}$$

Now that known, we need to prove that the function $f(x)$ is continuous. A function is said to be continuous if the limit exists and equal to function at that point.

Lets compute the limit at x_0 but using known information Equation (4.4)

$$\begin{aligned}
\lim_{x\to x_0} f(x) &= \lim_{x\to x_0} [f(x)-f(x_0)+f(x_0)] \\
&= \lim_{x\to x_0} [f(x)-f(x_0)] + \lim_{x\to x_0} f(x_0) \\
&= \lim_{x\to x_0} \frac{[f(x)-f(x_0)]\,(x-x_0)}{x-x_0} + \lim_{x\to x_0} f(x_0) \\
&= \lim_{x\to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0}\right] \lim_{x\to x_0} (x-x_0) + \lim_{x\to x_0} f(x_0) \\
&= \xi\cdot[0] + \lim_{x\to x_0} f(x_0) \\
&= 0 + \lim_{x\to x_0} f(x_0) \\
&= \lim_{x\to x_0} f(x_0) \\
&= f(x_0)
\end{aligned}$$

Hence

$$\lim_{x\to x_0} f(x) = f(x_0)$$

so f is continuous at $x = x_0$.

Remark 4.2.1 *A differentiable function is a continuous function but the reverse is not true*

Example 4.2.1 Show that the function $f(x) = |x|$ is continuous but not differentiable at $x = 0$. Check whether the function

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is continuous at $x = 0$

i). $f(0) = 0$

ii). $\lim_{x\to 0} f(x)$?? Since a piecewise function, to compute the limit, we use the informal definition of limits.

$$\begin{aligned} \text{From the left}: \quad \lim_{x\to 0^-} f(x) &= \lim_{x\to 0}(-x) = 0 \\ \text{From the right}: \quad \lim_{x\to 0^+} f(x) &= \lim_{x\to 0}(x) = 0 \end{aligned}$$

Thus the limit exists and equal to 0, that is $\lim_{x\to 0} f(x) = 0$

iii).

$$\lim_{x\to 0} f(x) = f(0) = 0$$

Therefore, the function is continuous at $x = 0$.

Show that the function $y = |x|$ is not differentiable at $x = 0$.

Recall that

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Now check for differentiability at $x = 0$, i.e., compute $f'(0)$. Then

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\ f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \end{aligned}$$

Finding a limit for a piecewise function, we check from left and from right, if equal, that is the limit, otherwise, the limit does not exist.

From the right of point $x = 0$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{(x) - (0)}{x} = \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$$

From the left of point $x = 0$

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{(-x) - (0)}{x} = \lim_{x\to 0} \frac{-x}{x} = \lim_{x\to 0} -1 = -1$$

Since the one-sided limits exist but are *not equal though finite*, $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist, and f is not differentiable at $x = 0$. This implies that the derivative of $f(x) = |x|$ does not exist at $x = 0$ as seen by the sharp curve at the point.

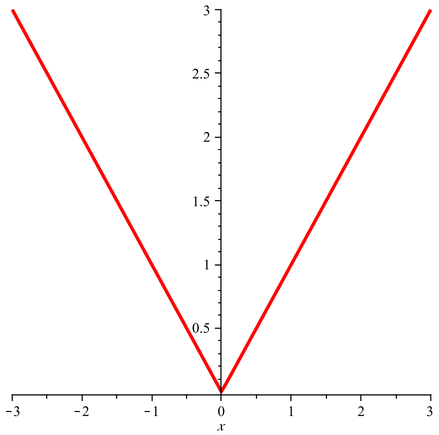


Figure 4.3: Graph of $y = |x|$

Remark 4.2.2 Sometimes a derivative may fail to exist at a point. In general, there are three reasons why a derivative at a point may not exist.

1. The graph of the function has a sharp turn or a cusp, e.g. $f(x) = |x|$ at $x = 0$.
2. The graph is not continuous at the point, one of the three properties of continuity fails

$$g(x) = \frac{x^2 + x}{x} \text{ at } x = 0$$

3. The graph has a vertical tangent line at the point, e.g. $h(x) = x^{\frac{1}{3}}$ at $x = 0$.

Note 4.2.1 So what is the derivative, after all? The derivative measures the steepness of the graph of a function at some particular point on the graph

Example 4.2.2 A function is defined by the following formula:

$$f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ a\left[x - \left(\frac{1}{x}\right)\right] + b, & x > 1 \end{cases}$$

Find a and b such that f is continuous and differentiable. Plot the function, if possible.

(a) To be continuous at $x = 1$

i) $f(1) = 1^2 + 2 = 3$

ii) $\lim\limits_{x \to 1} f(x)$??

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1}(x^2 + 2) = 3 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1}\left(a\left[x - \left(\frac{1}{x}\right)\right] + b\right) = b \end{aligned}$$

iii) For continuity, $\lim\limits_{x \to 1} f(x) = f(1) \quad \Rightarrow \quad b = 3$

(b) To be differentiable at $x = 1$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists} \quad \Rightarrow \quad f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \text{ exists}$$

Finding a limit for a piecewise function, we check from left and from right, and to be equal since a derivative exists.

i) From the left of point $x = 1$

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{(x^2 + 2) - (3)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} x + 1 = 2$$

ii) From the right of point $x = 1$

$$\begin{aligned} \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \to 1} \frac{a\left[x - \left(\frac{1}{x}\right)\right] + b - 3}{x - 1} = \lim_{x \to 1} \frac{a\left[x - \left(\frac{1}{x}\right)\right] + 3 - 3}{x - 1} \\ &= \lim_{x \to 1} \frac{a\left[x - \left(\frac{1}{x}\right)\right]}{x - 1} = \lim_{x \to 1} \frac{\frac{a(x^2-1)}{x}}{x - 1} = \lim_{x \to 1} \frac{a\,(x + 1)}{x} = 2a \end{aligned}$$

limits to equal, $2 = 2a \quad \Rightarrow \quad a = 1$

Example 4.2.3 We wish to determine the values of the parameters k and m for which the function below is differentiable at $x = 3$:

$$f(x) = \begin{cases} k\sqrt{x + 1}, & 0 \leq x \leq 3 \\ 5 - mx, & 3 < x \leq 5 \end{cases}$$

For a function to be differentiable at a domain value,

(i) the function must be *continuous* there.

(i) the derivative exists (the pieces must match with the same slope).

(a) To be continuous at $x = 3$

i) $f(3) = k\sqrt{3+1} = 2k$

ii) $\lim_{x\to 3} f(x)$ exists, and since a piecewise function,

$$\begin{aligned}
\lim_{x\to 3^-} f(x) &= \lim_{x\to 3}\left(k\sqrt{x+1}\right) = 2k \\
\lim_{x\to 3^+} f(x) &= \lim_{x\to 3}(5-mx) = 5-3m
\end{aligned}$$

iii) The limits should be equal and equal to $f(3)$ to be continuous.

$$5-3m = 2k \tag{4.5}$$

(b) To be differentiable at $x = 3$

$$f'(3) = \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} \quad \text{exists}$$

Finding a limit for a piecewise function, we check from left and from right, and to be equal since a derivative exists.

i) From the left of point $x = 3$

$$\begin{aligned}
\lim_{x\to 3^-}\frac{f(x)-f(3)}{x-3} &= \lim_{x\to 3}\frac{(k\sqrt{x+1})-(2k)}{x-3} = \frac{0}{0} \quad \Rightarrow \quad \text{La'Hopital} \\
&= \lim_{x\to 3}\frac{\frac{1}{2}k(x+1)^{-\frac{1}{2}}}{1} = \lim_{x\to 3}\frac{\frac{1}{2}k}{\sqrt{x+1}} = \frac{1}{4}k
\end{aligned}$$

ii) From the right of point $x = 3$

$$\begin{aligned}
\lim_{x\to 3^+}\frac{f(x)-f(3)}{x-3} &= \lim_{x\to 3}\frac{(5-mx)-(2k)}{x-3} = \frac{(5-3m)-2k}{0} \\
&= \frac{0: \ \text{Eqn (4.5)}}{0} \quad \Rightarrow \quad \text{La'Hopital} \\
&= \lim_{x\to 3}\frac{-m}{1} = -m
\end{aligned}$$

iii) For derivative to exist, the limits must be equal,

$$\frac{1}{4}k = -m \tag{4.6}$$

To be differentiable, it has to be continuous and derivative exists. Thus solving the simultaneous equations (4.5) and (4.6)

$$\begin{aligned}
5-3m &= 2k \\
\frac{k}{4} &= -m
\end{aligned}$$

$$\Rightarrow \quad k = 4, \quad m = -1$$

Example 4.2.4 Let

$$f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Show that f is continuous for all values of x. Show that f is differentiable for all values of x, but that the derivative, f', is not continuous at $x = 0$.

First show that f is continuous for all values of x. Describe f using functional composition. Let

$$g(x) = \frac{1}{x}, h(x) = \cos x, \text{ and } k(x) = x^2$$

Function h is well-known to be continuous for all values of x.

Function k is a polynomial and is therefore continuous for all values of x.

Function g is the quotient of functions continuous for all values of x, and is therefore continuous for all values of x except $x = 0$, that x which makes the denominator zero. Thus, for all values of x except $x = 0$

$$f(x) = k(x)h(g(x)) = x^2 \cos(g(x)) = x^2 \cos\left(\frac{1}{x}\right)$$

is a continuous function (the product and functional composition of continuous functions). `Continuity of` f `at` $x = 0$. Function f is defined at $x = 0$ since

(a) The limit $\lim\limits_{x\to 0} \cos\left(\dfrac{1}{x}\right)$ does not exist since the values of $\cos\left(\dfrac{1}{x}\right)$ oscillate between -1 and $+1$ as x approaches zero. However, for $x \neq 0$

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq +1$$

so that

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2.$$

Since

$$\lim_{x\to 0}(-x^2) = 0 = \lim_{x\to 0} x^2,$$

it follows from the Squeeze Principle that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

(b) The function is defined at $x = 0$

$$f(0) = 0.$$

(c)

$$\lim_{x\to 0} f(x) = 0 = f(0),$$

all three conditions are satisfied, and f is continuous at $x = 0$. Thus, f is continuous for all values of x.

Now show that f is differentiable for all values of x. For $x \neq 0$ we can differentiate f using the product rule and the chain rule. That is, for $x \neq 0$ the derivative of f is

$$\begin{aligned} f'(x) &= x^2\ D\Big\{\cos\Big(\frac{1}{x}\Big)\Big\} + D\{x^2\}\ \cos\Big(\frac{1}{x}\Big) \\ &= x^2\Big\{-\sin\Big(\frac{1}{x}\Big)\ D\Big\{\frac{1}{x}\Big\}\Big\} + \{2x\}\cos\Big(\frac{1}{x}\Big) \\ &= -x^2\sin\Big(\frac{1}{x}\Big)\Big\{\frac{-1}{x^2}\Big\} + 2x\cos\Big(\frac{1}{x}\Big) \\ &= \sin\Big(\frac{1}{x}\Big) + 2x\cos\Big(\frac{1}{x}\Big) \end{aligned}$$

Use the limit definition of the derivative to differentiate f at $x = 0$. Then

$$\begin{aligned} f'(0) &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0}\frac{f(h)-0}{h} \\ &= \lim_{h\to 0}\frac{(h)^2\cos\Big(\frac{1}{h}\Big)}{h} \\ &= \lim_{h\to 0}\ h\cos\Big(\frac{1}{h}\Big) \end{aligned}$$

Use the Squeeze Principle to evaluate this limit. For $h \neq 0$

$$-1 \leq \cos\Big(\frac{1}{h}\Big) \leq +1.$$

If $h > 0$, then

$$-h \leq h\cos\Big(\frac{1}{h}\Big) \leq h.$$

If $h < 0$, then

$$-h \geq h\cos\Big(\frac{1}{h}\Big) \geq h.$$

In either case,

$$\lim_{h\to 0}\ (-h) = 0 = \lim_{h\to 0}\ h,$$

and it follows from the Squeeze Principle that

$$f'(0) = \lim_{h\to 0}\ h\cos\Big(\frac{1}{h}\Big) = 0.$$

Thus, f is differentiable for all values of x. Check to see if f' is continuous at $x = 0$. The function f' is defined at $x = 0$ since

(a) However,

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\Big[\sin\Big(\frac{1}{x}\Big) + 2x\cos\Big(\frac{1}{x}\Big)\Big]$$

does not exist since the values of $\sin\Big(\frac{1}{x}\Big)$ oscillate between -1 and $+1$ as x approaches zero.

(b)

$$f'(0) = 0$$

Thus, condition (a) is violated, and the derivative, f', is not continuous at $x = 0$.

Note 4.2.2 The continuity of function f for all values of x also follows from the fact that f is differentiable for all values of x.

4.3 Differentiation Theorems

Let $f(x)$ and $g(x)$ be differentiable and α a scalar, then $\alpha f(x), (f+g)(x), (fg)(x), (\frac{f}{g})(x)$ are all differentiable functions such that

(a) *Constant rule*: if $f(x)$ is constant, then

$$f'(x) = 0$$

(b) The *scalar multiplication*

$$(\alpha f)'(x) = \alpha f'(x)$$

(c) The *sum rule*

$$(f+g)'(x) = f'(x) + g'(x)$$

(d)

$$(fg)'(x) = f'(x).g(x) + g'(x)f(x)$$

This is popularly known as the *product rule*

(e)

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

This is popularly known as the *quotient rule* of differentiation

Example 4.3.1 Prove that

$$(f+g)'(x) \quad = \quad f'(x) + g'(x)$$

$$\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x)
\end{aligned}$$

Exercise 4.6 Prove that $(f-g)'(x) = f'(x) - g'(x)$

Example 4.3.2 Let $f(x)$ and $g(x)$ be differentiable and α a scaler, prove that,

(i) $(\alpha f)'(x) = \alpha f'(x)$

(ii) $(fg)'(x) = f'(x)g(x) + g'(x)f(x)$

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{g(x+h)\left[f(x+h)-f(x)\right]+f(x)\left[g(x+h)-g(x)\right]}{h}\\
&= \lim_{h\to 0}\frac{g(x+h)\left[f(x+h)-f(x)\right]}{h} + \lim_{h\to 0}\frac{f(x)\left[g(x+h)-g(x)\right]}{h}\\
&= \lim_{h\to 0}g(x+h)\lim_{h\to 0}\frac{\left[f(x+h)-f(x)\right]}{h} + \lim_{h\to 0}f(x)\lim_{h\to 0}\frac{\left[g(x+h)-g(x)\right]}{h}\\
&= g(x)f'(x)+f(x)g'(x)
\end{aligned}$$

Exercise 4.7 Show that

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

Example 4.3.3 Use the product rule -part(c) of the theorem to find $\frac{d}{dx}(e^{x^2}\sin x)$

$$\begin{aligned}\text{Let } f(x) &= e^{x^2} \quad \Rightarrow \; f'(x) = 2xe^{x^2} \\ \text{and } g(x) &= \sin x \quad \Rightarrow \; g'(x) = \cos x\end{aligned}$$

$\Rightarrow$ By the theorem, we have that,

$$\begin{aligned}(fg)'(x) &= f'(x)g(x) + g'(x)f(x) \\ &= 2xe^{x^2}\sin x + \cos x e^{x^2} \\ &= e^{x^2}(2x\sin x + \cos x)\end{aligned}$$

Example 4.3.4 Use the quotient rule of the theorem to compute

$$\frac{d}{dx}\left(\frac{\sin^2 x}{1-e^{-x}}\right)$$

$$\begin{aligned}f(x) = \sin^2 x \quad &\Rightarrow \quad f'(x) = 2\sin x\cos x \\ g(x) = 1 - e^{-x} \quad &\Rightarrow \quad g'(x) = e^{-x}\end{aligned}$$

By the theorem, we have that,

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2} \\ &= \frac{2\sin x\cos x(1-e^{-x}) - e^{-x}\sin^2 x}{(1-e^{-x})^2}\end{aligned}$$

Example 4.3.5

$$\frac{d}{dx}\left(\frac{1}{x-2}\right) = \frac{\frac{d}{dx}1\cdot(x-2) - 1\cdot\frac{d}{dx}(x-2)}{(x-2)^2} = \frac{0\cdot(x-2) - 1\cdot 1}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

Example 4.3.6

$$\frac{d}{dx}\left(\frac{x-1}{x-2}\right) = \frac{(x-1)'(x-2) - (x-1)(x-2)'}{(x-2)^2} = \frac{1\cdot(x-2) - (x-1)\cdot 1}{(x-2)^2}$$

$$= \frac{(x-2)-(x-1)}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

Example 4.3.7

$$\frac{d}{dx}\left(\frac{5x^3+x}{2-x^7}\right) = \frac{(5x^3+x)'\cdot(2-x^7) - (5x^3+x)\cdot(2-x^7)}{(2-x^7)^2}$$

$$= \frac{(15x^2+1)\cdot(2-x^7) - (5x^3+x)\cdot(-7x^6)}{(2-x^7)^2}$$

and there's hardly any point in simplifying the last expression, unless someone gives you a good reason. In general, it's not so easy to see how much may or may not be gained in 'simplifying', and we won't make ourselves crazy over it.

Note 4.3.1 One way that the product rule can be useful is in postponing or eliminating a lot of algebra. For example, to evaluate

$$\frac{d}{dx}\left((x^3+x^2+x+1)(x^4+x^3+2x+1)\right)$$

we *could* multiply out and then take the derivative term-by-term as we did with several polynomials above. This would be at least mildly irritating because we'd have to do a bit of algebra. Rather, just apply the product rule *without* feeling compelled first to do any algebra:

$$\begin{aligned}
&\frac{d}{dx}\left((x^3+x^2+x+1)(x^4+x^3+2x+1)\right)\\
&= (x^3+x^2+x+1)'(x^4+x^3+2x+1)+(x^3+x^2+x+1)(x^4+x^3+2x+1)'\\
&= (3x^2+2x+1)(x^4+x^3+2x+1)+(x^3+x^2+x+1)(4x^3+3x^2+2)
\end{aligned}$$

Now if we were somehow still obliged to multiply out, then we'd still have to do some algebra. But *we can take the derivative without multiplying out*, if we want to, by using the product rule.

For that matter, once we see that there is a *choice* about doing algebra either *before* or *after* we take the derivative, it might be possible to make a choice which minimizes our computational labor. This could matter.

Example 4.3.8 Suppose we want to differentiate $y = x^2\cos 3x$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(-3x\sin 3x + 2\cos 3x)$$

Example 4.3.9 Suppose we want to differentiate

$$y = x^3(4-x)^{1/2}$$

$$\begin{aligned}
f(x) = x^3 &\quad\Rightarrow\quad f'(x) = 3x^2\\
g(x) = (4-x)^{1/2} &\quad\Rightarrow\quad g'(x) = -\frac{1}{2}(4-x)^{-1/2}
\end{aligned}$$

$$\begin{aligned}
(fg)'(x) &= f'(x).g(x) + g'(x)f(x)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= (3x^2)(4-x)^{1/2} - \frac{x^3}{2(4-x)^{1/2}}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(4-x)^{1/2}\cdot 3x^2}{1}\cdot\frac{2(4-x)^{1/2}}{2(4-x)^{1/2}} - \frac{x^3}{2(4-x)^{1/2}} = \frac{6x^2(4-x)-x^3}{2(4-x)^{1/2}} = \frac{x^2(24-7x)}{2(4-x)^{1/2}}
\end{aligned}$$

Example 4.3.10 Suppose we want to differentiate $y = (1-x^3)\mathrm{e}^{2x}$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= (1-x^3)\times 2\mathrm{e}^{2x} + \mathrm{e}^{2x}\times(-3x^2)\\
&= \mathrm{e}^{2x}(2-3x^2-2x^3)
\end{aligned}$$

Example 4.3.11 Find the derivative of each of the following:

a) $x \tan x$

$$x \sec^2 x + \tan x$$

b) $x^2 e^{-x}$

$$x(2-x)e^{-x}$$

c) $5e^{-2x} \sin 3x$

$$5e^{-2x}(3\cos 3x - 2\sin 3x)$$

d) $3x^{1/2} \cos 2x$

$$\frac{3}{2}x^{-1/2}(\cos 2x - 4x \sin 2x)$$

e) $2x^6(1+x)^5$

$$2x^5(1+x)^4(6+11x)$$

f) $x^{-2}(1+x^2)^{1/2}$

$$-x^{-3}((1+x^2)^{-1/2}(2+x^2)$$

g) $xe^x \sin x$

$$e^x[(1+x)\sin x + x\cos x]$$

h) $7x^{3/2}e^{-4x} \cos 2x$

$$\frac{7}{2}x^{1/2}e^{-4x}(3\cos 2x - 8x\cos 2x - 4x \sin x)$$

Example 4.3.12 Compute the derivative of $y = (2x^2 + 6x)(2x^3 + 5x^2)$

$$y' = (2x^2 + 6x)(6x^2 + 10x) + (2x^3 + 5x^2)(4x + 6) = 20x^4 + 88x^3 + 90x^2$$

Exercise 4.8 Find $\frac{d}{dx}(f(x))$ given that

(i) $f(x) = 2x^{\frac{1}{2}} - x^3 + 2$

(ii) $f(x) = \alpha x^3 + \beta x^2 + \lambda x + \theta$ where α, β, λ and θ are constants.

4.4 Other techniques of differentiation

Other than the sum, difference, product, quotient or constant differentiation, other forms of differentiation include

(i) Chain rule

(ii) Implicit differentiation

(iii) Parametric differentiation

(iv) Logarithmic differentiation

4.4.1 Chain Rule - Composite differentiation

Theorem 4.4.1 Let $g(x)$ be differentiable at x and $h(g)$ be differentiable at $g(x)$, then $h \circ g(x) = h(g(x))$ is differentiable at x and if $f(x) = h(g(x))$, then

$$(h \circ g)'(x) = \frac{d}{dx}[h(g(x))] = \frac{dh}{dg}\cdot\frac{dg}{dx} \tag{4.7}$$

$$(h \circ g)'(x) = h'(g(x)) \cdot g'(x) \tag{4.8}$$

Proof

$$\begin{aligned}
\frac{d}{dx}[h(g(x))] &= \lim_{h\to 0}\frac{h(g(x+h)) - h(g(x))}{h} \\
&= \lim_{h\to 0}\frac{h(g(x+h)) - h(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h} \\
\text{Let } \Delta g &= g(x+h) - g(x) \\
\lim_{h\to 0}\Delta g &= 0 = \lim_{h\to 0}\frac{h(g+h)-h(g)}{\Delta g}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}
\end{aligned}$$

Note 4.4.1 The chain rule can only be used when you can express a function f given as a composite of two functions h and g.

Example 4.4.1 Using the chain rule find $f'(x)$ for

$$f(x) = \frac{1}{(4x^2 - x)^5}$$

This can be decomposed as the composite of two functions:

$$\begin{aligned}
f(x) &= h(g(x)) \\
g(x) &= 4x^2 - x, \\
h(g) &= \frac{1}{g^5}
\end{aligned}$$

Their derivatives are:

$$\frac{dg}{dx} = 8x - 1 \quad , \quad \frac{dh}{dg} = -5g^{-6} = \frac{-5}{g^6}$$

The derivative function is therefore:

$$\frac{df}{dx} = \frac{dh}{dg}\cdot\frac{dg}{dx} = \frac{-5}{g^6}\cdot(8x-1) = -\frac{5(8x-1)}{(4x^2-x)^6}$$

Example 4.4.2 Differentiate

$$f(x) = (3x+1)^2$$

In a short form, its

$$\begin{aligned} f'(x) &= 2(3x+1)(3) \\ f'(x) &= 6(3x+1) \end{aligned}$$

To use the chain rule, this can be decomposed as the composite of two functions:

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) &= 3x+1, \\ h(g) &= g^2 \end{aligned}$$

Their derivatives are:

$$\frac{dg}{dx} = 3, \qquad \frac{dh}{dg} = 2g$$

The derivative function is therefore:

$$\frac{df}{dx} = \frac{dh}{dg} \cdot \frac{dg}{dx} = 2g(3) = 6g = 6(3x+1)$$

Example 4.4.3 Differentiate

$$y = \sqrt{13x^2 - 5x + 8}$$

$$\begin{aligned} y' &= \frac{1}{2}(13x^2 - 5x + 8)^{-\frac{1}{2}}(26x - 5) \\ y' &= \frac{26x-5}{2\sqrt{13x^2 - 5x + 8}} \end{aligned}$$

Example 4.4.4 Differentiate

$$y = (1 - 4x + 7x^5)^{30}$$

$$\begin{aligned} y' &= 30(1 - 4x + 7x^5)^{29}(-4 + 35x^4) \\ y' &= 30(35x^4 - 4)(1 - 4x + 7x^5)^{29} \end{aligned}$$

Example 4.4.5 Differentiate

$$y = (4x + x^{-5})^{\frac{1}{3}}$$

$$y' = \frac{1}{3}(4x + x^{-5})^{-\frac{2}{3}}(4 - 5x^{-6})$$

Example 4.4.6 Differentiate

$$y = \left(\frac{8x - x^6}{x^3}\right)^{-\frac{4}{5}}$$

First, begin by simplifying the expression before we differentiate it

$$y = \left(8x^{-2} - x^3\right)^{-\frac{4}{5}}$$

$$y' = -\frac{4}{5}\left(8x^{-2} - x^3\right)^{-\frac{9}{5}}\left(-16x^{-3} - 3x^2\right)$$

Example 4.4.7 The derivative of

$$f(x) = x^4 + \sin(x^2) - \ln(x)e^x + 7$$

$$\begin{aligned} f'(x) &= 4x^{(4-1)} + \frac{d\,(x^2)}{dx}\cos(x^2) - \frac{d\,(\ln x)}{dx}e^x - \ln x\frac{d\,(e^x)}{dx} + 0 \\ &= 4x^3 + 2x\cos(x^2) - \frac{1}{x}e^x - \ln(x)e^x. \end{aligned}$$

Here the second term was computed using the chain rule and third using the product rule. The known derivatives of the elementary functions $x^2, x^4, \sin x, \ln(x)$ and e^x, as well as the constant 7, were also used.

Example 4.4.8 For concreteness, consider the function

$$f(x) = e^{\sin x^2}$$

This can be decomposed as the composite of three functions:

$$\begin{aligned} f(x) &= h(g(p(x))) \\ p(x) &= x^2, \\ g(p) &= \sin p, \\ h(g) &= e^g \end{aligned}$$

Their derivatives are:

$$\frac{dp}{dx} = 2x, \quad \frac{dg}{dp} = \cos p, \quad \frac{dh}{dg} = e^g$$

The derivative function is therefore:

$$\begin{aligned} \frac{df}{dx} &= \frac{dh}{dg}\cdot\frac{dg}{dp}\cdot\frac{dp}{dx} \\ \frac{df}{dx} &= e^{\sin x^2}\cdot\cos x^2\cdot 2x \end{aligned}$$

Exercise 4.9 Differentiate

$$y = \sin(5x)$$

Exercise 4.10 Differentiate

$$y = e^{5x^2+7x-13}$$

Exercise 4.11 Differentiate

$$y = 3\tan\sqrt{x}$$

$$y' = \frac{3\sec^2\sqrt{x}}{2\sqrt{x}}$$

Exercise 4.12 Differentiate

$$y = \cos^2(x^3)$$

Example 4.4.9 Use the chain rule to differentiate

$$f(x) = (x^3 + 5x)^7$$

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) = x^3 + 5x, \qquad & h(g) = g^7 \\ f'(x) &= 7(x^3+5x)^6\cdot(3x^2+5) \end{aligned}$$

Example 4.4.10 Use the chain rule to differentiate

$$f(x) = \sqrt{5\cos x}$$

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) = 5\cos x, \qquad & h(g) = g^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}(5\cos x)^{-\frac{1}{2}} \cdot 5(-\sin x) \end{aligned}$$

Example 4.4.11 Use the chain rule to differentiate

$$f(x) = 7e^{x^2-5}$$

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) = x^2 - 5, \qquad & h(g) = 7e^{g} \\ f'(x) &= 7e^{x^2-5} \cdot (2x) \end{aligned}$$

Example 4.4.12 Use the chain rule to differentiate

$$f(x) = -3\tan(5x^4)$$

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) = 5x^4, \qquad & h(g) = -3\tan g \\ f'(x) &= -3\sec^2(5x^4) \cdot (20x^3) \end{aligned}$$

Example 4.4.13 Use the chain rule to differentiate

$$f(x) = \frac{8}{4 + \sin x}$$

$$\begin{aligned} f(x) &= h(g(x)) \\ g(x) = 4 + \sin x, \qquad & h(g) = \frac{8}{g} \\ f'(x) &= -8(4 + \sin x)^{-2} \cdot \cos x \end{aligned}$$

Example 4.4.14 Find the derivative of $f(x) = \sin(5x)$ using the chain rule.

$$f'(x) = 5 \cdot \Big[\cos(5x)\Big] = 5\cos(5x)$$

Example 4.4.15 Find the derivative of

$$f(t) = \left(t^2 - \frac{2}{t^3}\right)^2$$

We will use the Chain rule. Set $f = y(u(t))$

$$u = t^2 - \frac{2}{t^3} \text{ and } y = u^2$$

The Chain rule implies

$$\frac{df}{dt} = \frac{du}{dt}\frac{dy}{du} = \left(2t + \frac{6}{t^4}\right) 2u = 2\left(2t + \frac{6}{t^4}\right)\left(t^2 - \frac{2}{t^3}\right)$$

Exercise 4.13 Use the chain rule to find $f'(x)$ for

(a) $f(x) = (4 - x)^{\frac{1}{2}}$ $\qquad h(g) = g^{\frac{1}{2}}$

(b) $f(x) = (7x^2 - 5x)^3$ $\qquad h(g) = g^3$

(c) $f(x) = \frac{1}{(3x-2)}$ $\qquad h(g) = g^{-1} = \frac{1}{g}$

(d) $f(x) = 5 + \cos^3 x$

(e) $f(x) = \sqrt[3]{1 + \tan x}$

Exercise 4.14 Given $f = x^2 - 3$ and $g = 4x + 7$, compute

(a) $(f \circ g)'(x)$

(b) $(g \circ f)'(x)$

(c) $(f \circ g)'(5)$

(d) $(g \circ f)'(-4)$

4.4.2 Differentiation of implicit functions

Implicit differentiation is nothing more than a special case of the well-known chain rule for derivatives. The majority of differentiation problems in first-year calculus involve functions y written *explicitly* as functions of x. For example, if

$$y = 3x^2 - \sin(7x + 5)$$

then the derivative of y is

$$y' = 6x - 7\cos(7x + 5)$$

However, some functions y are written *implicitly* as functions of x. A familiar example of this is the equation

$$x^2 + y^2 = 25$$

which represents a circle of radius five centered at the origin. Suppose that we wish to find the slope of the line tangent to the graph of this equation at the point $(3, -4)$

How could we find the derivative of y in this instance ? One way is to first write y explicitly as a function of x. Thus,

$$x^2 + y^2 = 25 \;\Rightarrow\; y^2 = 25 - x^2$$

and

$$y = \pm\sqrt{25 - x^2}$$

where the positive square root represents the top semi-circle and the negative square root represents the bottom semi-circle. Since the point $(3, -4)$ lies on the bottom semi-circle given by

$$y = -\sqrt{25 - x^2}$$

i.e,

$$y' = \frac{x}{\sqrt{25 - x^2}}$$

Thus, the slope of the line tangent to the graph at the point $(3, -4)$ is

$$m = y' = \frac{3}{\sqrt{25 - (3)^2}} = \frac{3}{4}$$

Unfortunately, not every equation involving x and y can be solved explicitly for y

With Implicit differentiation, we differentiate both sides with respect to x, and make y' the subject.

Example 4.4.16 Differentiate

$$x^2 + y^2 = 25$$

$$\begin{aligned} x^2 + y^2 &= 25 \\ D(x^2) + D(y^2) &= D(25) \\ 2x + 2yy' &= 0 \\ 2yy' &= -2x \\ y' &= \frac{-2x}{2y} = \frac{-x}{y} \end{aligned}$$

Thus, the slope of the line tangent to the graph at the point $(3, -4)$ is

$$m = y' = \frac{-x}{y} = \frac{-(3)}{(-4)} = \frac{3}{4}$$

Example 4.4.17 Assume that y is a function of x. Find $y' = dy/dx$ for

$$x^3 + y^3 = 4$$

Begin with $x^3 + y^3 = 4$. Differentiate both sides of the equation, getting

$$\begin{aligned} x^3 + y^3 &= 4 \\ D(x^3 + y^3) &= D(4) \\ D(x^3) + D(y^3) &= D(4) \end{aligned}$$

Remember to use the chain rule on $D(y^3)$

$$3x^2 + 3y^2 y' = 0$$

so that (Now solve for y')

$$3y^2 y' = -3x^2$$

and

$$y' = \frac{-x^2}{y^2}$$

Example 4.4.18 Assume that y is a function of x. Find $y' = dy/dx$ for

$$(x - y)^2 = x + y - 1$$

Begin with $(x - y)^2 = x + y - 1$. Differentiate both sides of the equation, getting

$$\begin{aligned} (x - y)^2 &= x + y - 1 \\ D(x - y)^2 &= D(x + y - 1) \\ D(x - y)^2 &= D(x) + D(y) - D(1) \\ 2(x - y)D(x - y) &= 1 + y' - 0 \\ 2(x - y)(1 - y') &= 1 + y' \end{aligned}$$

Now solve for y'

$$\begin{aligned} y'[-2(x - y) - 1] &= 1 - 2(x - y) \\ y' &= \frac{1 - 2(x - y)}{-2(x - y) - 1} = \frac{2y - 2x + 1}{2y - 2x - 1} \end{aligned}$$

Example 4.4.19 Assume that y is a function of x. Find $y' = dy/dx$ for

$$y = \sin(3x + 4y)$$

Begin with $y = \sin(3x + 4y)$. Differentiate both sides of the equation, getting

$$\begin{aligned} y &= \sin(3x + 4y) \\ D(y) &= D\left(\sin(3x + 4y)\right) \\ y' &= \cos(3x + 4y) D\left(3x + 4y\right) \\ y' &= \cos(3x + 4y)\left(3 + 4y'\right) \end{aligned}$$

Now solve for y'

$$\begin{aligned} y'\left[1 - 4\cos(3x + 4y)\right] &= 3\cos(3x + 4y) \\ y' &= \frac{3\cos(3x + 4y)}{1 - 4\cos(3x + 4y)} \end{aligned}$$

Example 4.4.20 Assume that y is a function of x. Find $y' = dy/dx$ for

$$y = x^2y^3 + x^3y^2$$

Begin with $y = x^2y^3 + x^3y^2$. Differentiate both sides of the equation, getting

$$\begin{aligned} y &= x^2y^3 + x^3y^2 \\ D(y) &= D\left(x^2y^3 + x^3y^2\right) \\ y' &= D(x^2y^3) + D(x^3y^2) \\ y' &= x^2(3y^2y') + (2x)y^3 + x^3(2yy') + (3x^2)y^2 \\ y' &= 3x^2y^2y' + 2xy^3 + 2x^3yy' + 3x^2y^2 \end{aligned}$$

Now solve for y'

$$y' = \frac{2xy^3 + 3x^2y^2}{1 - 3x^2y^2 - 2x^3y}$$

Example 4.4.21 Assume that y is a function of x. Find $y' = dy/dx$ for

$$e^{xy} = e^{4x} - e^{5y}$$

$$y' = \frac{4e^{4x} - ye^{xy}}{xe^{xy} + 5e^{5y}}$$

Example 4.4.22 Assume that y is a function of x. Find $y' = dy/dx$ for

$$\cos^2 x + \cos^2 y = \cos(2x + 2y)$$

$$y' = \frac{[\cos x \sin x - \sin(2x + 2y)]}{[\sin(2x + 2y) - \cos y \sin y]}$$

Example 4.4.23 Assume that y is a function of x. Find $y' = dy/dx$ for

$$x = \sqrt{x^2 + y^2}$$

$$y' = \frac{\sqrt{x^2 + y^2} - x}{y}$$

Example 4.4.24 Assume that y is a function of x. Find $y' = dy/dx$ for

$$\frac{x - y^3}{y + x^2} = x + 2$$

$$y' = \frac{1 - y - 3x^2 - 4x}{3y^2 + x + 2}$$

Example 4.4.25 Assume that y is a function of x. Find $y' = dy/dx$ for

$$(x^2 + y^2)^3 = 8x^2y^2$$

$$y' = \frac{16xy^2 - 6x(x^2 + y^2)^2}{6y(x^2 + y^2)^2 - 16x^2y}$$

Example 4.4.26 For the function, $x^2 + y^3 = 5y$, find $\frac{dy}{dx}$

$$\begin{aligned}
\texttt{Since } x^2 + y^3 &= 5y \\
\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^3) &= \frac{d}{dx}(5y) \\
\Rightarrow 2x + 3y^2\frac{dy}{dx} &= 5\frac{dy}{dx} \\
\Rightarrow \frac{dy}{dx}(3y^2 - 5) &= -2x \\
\Rightarrow \frac{dy}{dx} &= \left(\frac{-2x}{3y^2 - 5}\right)
\end{aligned}$$

Example 4.4.27 Find $\frac{dy}{dx}$ for $x^2y - 2x^3y^2 = 4$

We differentiate term by term with respect to x i.e

$$\begin{aligned}
\Rightarrow \frac{d}{dx}(x^2y) - 2\frac{d}{dx}(x^3y^2) &= \frac{d}{dx}(4) \\
\Rightarrow 2xy + x^2\frac{dy}{dx} - 2(3x^2y^2 + 2x^3y\frac{dy}{dx}) &= 0 \\
\Rightarrow 2xy + x^2\frac{dy}{dx} - 6x^2y^2 - 4x^3y\frac{dy}{dx} &= 0 \\
\Rightarrow \frac{dy}{dx}(x^2 - 4x^3y) &= 6x^2y^2 - 2xy \\
\texttt{therefore } \frac{dy}{dx} &= \frac{6x^2y^2 - 2xy}{x^2 - 4x^3y} \\
&= \frac{6xy^2 - 2y}{x - 4x^2y}
\end{aligned}$$

4.4.3 Parametric equations

To differentiate parametric equations, we must use the *chain rule.*

The equations of a plane curve $f(x, y) = 0$ may be given by equations of the type $x = x(t)$ and $y = y(t)$, where t is the variable called a parameter. These equations are called parametric equations of the curve.

Example 4.4.28 Consider the parametric equations

$$x = \cos t, \ y = \sin t \qquad\qquad \text{for } 0 \le t \le 2\pi$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot t$$

Example 4.4.29 If $x = 2at^2$ and $y = 4at$, find dy/dx and dx/dy

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} = (4a) \cdot \frac{1}{4at} = \frac{1}{t} \\
\frac{dx}{dy} &= \frac{dx}{dt} \cdot \frac{dt}{dy} = (4at) \cdot \frac{1}{4a} = t
\end{aligned}$$

Example 4.4.30 Finding the second derivative is a little trickier.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\cdot\frac{dt}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

or Example (4.4.29)

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx} = \left(\frac{-1}{t^2}\right)\frac{1}{4at}$$

Exercise 4.15 Given

$$x = t^3 - t \qquad\qquad y = 4 - t^2$$

Compute

(a) dx/dy

(b) dy/dx

(c) $(dy/dx)^2$

(d) d^2y/dx^2

Example 4.4.31 Find the equation of a Curve whose parametric equations are

(i) $x = t$ and $y = 5t + 6$

(ii) $x = \alpha\cos\theta$ and $y = \beta\sin\theta$

To solve, just eliminate the parameter

(i) We eliminate the parameter t from the equations i.e $y = 5x + 6$

(ii)

$$\begin{aligned} x^2 &= \alpha^2\cos^2\theta \\ \Rightarrow \frac{x^2}{\alpha^2} &= \cos^2\theta \qquad (4.9) \\ \texttt{and } y^2 &= \beta^2\sin^2\theta \\ \Rightarrow \frac{y^2}{\beta^2} &= \sin^2\theta \qquad (4.10) \end{aligned}$$

Adding equations (4.9) to (4.10) we have

$$\begin{aligned} \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} &= \cos^2\theta + \sin^2\theta \\ \Rightarrow \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} &= 1 \end{aligned}$$

Which is an ellipse

Derivatives of parametrically defined curves

If we let $x = x(t)$ and $y = y(t)$ be parametric equations of $f(x, y) = 0$ this means that,

$$\begin{aligned} \frac{dx}{dt} &= x'(t), \quad \frac{dy}{dt} = y'(t) \\ \texttt{but } \frac{dy}{dx} &= \frac{dy}{dt}\cdot\frac{dt}{dx} \ (\texttt{Chain rule }) \\ &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(x)}{x'(t)} \ \texttt{provided } (x'(t) \neq 0) \end{aligned}$$

Example 4.4.32 Compute dy/dx for the following parametric equations

(i) $x = t^2$ and $y = 4t^2 + 5$

(ii) $x = t$ and $y = \frac{t^3}{3} + \frac{1}{7t^2}$

(iii) $x = e^t \cos t$ and $y = e^t \sin t$ $(0 \leq t \leq \pi)$

(i) Since $x(t) = t^2 \quad \Rightarrow x'(t) = 2t$

$$\begin{aligned} \text{and } y &= 4t^2 + 5 \Rightarrow y'(t) = 8t \\ \text{therefore } \frac{dy}{dx} &= \frac{y'(t)}{x'(t)} = \frac{8t}{2t} = 4 \end{aligned}$$

(ii) $x = t \ \Rightarrow x'(t) = 1$

$$\begin{aligned} \text{and } y &= \frac{t^3}{3} + \frac{1}{7}t^{-2} \Rightarrow y'(t) = t^2 - \frac{2}{7}t^{-3} \\ \text{therefore } \frac{dy}{dx} &= \frac{y'(t)}{x'(t)} = t^2 - \frac{2}{7t^3} \end{aligned}$$

(iii)

Since $x = e^t \cos t, \quad y = e^t \sin t$ $(0 \leq t \leq \pi)$

$$\begin{aligned} x'(t) &= e^t \cos t - e^t \sin t \\ y'(t) &= e^t \sin t + e^t \cos t \\ \Rightarrow \frac{dy}{dx} &= \frac{e^t(\cos t + \sin t)}{e^t(\cos t - \sin t)} \\ &= \frac{(\cos t + \sin t)}{(\cos t - \sin t)} \end{aligned}$$

This holds only when $\cos t \neq \sin t$

Example 4.4.33 Find the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \quad , \quad y = t^2$$

$$\frac{dy}{dx} = \frac{2}{5t^3 - 12t}$$

Example 4.4.34 Given

$$x(t) = t^3 \ , \quad y(t) = t^4$$

$$\frac{dy}{dx} = \frac{4}{3}t \qquad \frac{d^2y}{dx^2} = \frac{4}{9t^2}$$

Example 4.4.35

$$x = t + \cos t, \quad y = \sin t$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos t}{1 - \sin t} \\ \frac{d^2y}{dx^2} &= \frac{-\sin t + 1}{(1 - \sin t)^3} = \frac{1}{(1 - \sin t)^2} \end{aligned}$$

Example 4.4.36 Find the second derivative if $x = t - t^2,\ y = t - t^3$

$$\frac{d^2y}{dx^2} = \frac{1 - 3t^2}{(1 - 2t)^2}$$

Exercise 4.16 Given an equation

$$\begin{aligned} y &= \sin 2p \\ x &= cosp \end{aligned}$$

Compute the ordinary differential equations

(a) dx/dy

(b) $(dy/dx)^2$

4.4.4 Logarithmic differentiation

Logarithmic differentiation is a powerful technique for differentiating functions . However, the method is "uneconomical " for simple functions like polynomial functions.

This is done by

(i). Let y, be the function to differentiate

(ii). Take logs on both sides, take $\log_e = \ln$

(iii). Differentiate both sides

(iv). Make dy/dx the subject

(v). Substitute back the y value.

We here present some common suitable forms for the logarithmic differentiation.

(a)

$$y = u(x)v(x)$$

where $u(x)$ and $v(x)$ are quite big expressions .On differentiating, we take logarithms to base e on both sides i.e

$$\begin{aligned}
\ln y &= \ln u(x) + \ln v(x) \\
\Rightarrow \frac{1}{y}\frac{dy}{dx} &= \frac{1}{u(x)}.u'(x) + \frac{1}{v(x)}v'(x) \\
\Rightarrow \frac{dy}{dx} &= y(\frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)}) \\
&= v(x)u(x)(\frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)}) \\
&= v(x)u'(x) + u(x)v'(x)
\end{aligned}$$

(b)

$$y = \frac{u(x)v(x)}{h(x)g(x)}$$

Taking logs to both sides, we have

$$\begin{aligned}
\ln y &= \ln u(x) + \ln v(x) - \ln h(x) - \ln g(x) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)} - \frac{h'(x)}{h(x)} - \frac{g'(x)}{g(x)} \\
\Rightarrow \frac{dy}{dx} &= y[(\frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)} - \frac{h'(x)}{h(x)} - \frac{g'(x)}{g(x)})].
\end{aligned}$$

(c)

$$y = (u(x))^{v(x)}$$

Taking logs on both sides we have,

$$\begin{aligned}
\ln y &= v(x)\ln u(x) \\
\frac{1}{y}\frac{dy}{dx} &= v'(x)\ln u(x) + v(x)\frac{u'(x)}{u(x)}
\end{aligned}$$

Example 4.4.37 Find the derivative of

$$\frac{(x^2+1)^3(x+1)^4}{x\sqrt{(x-1)}\sqrt{(x+3)}}$$

Using logarithmic differentiation,

$$\begin{aligned}
\text{Let } y &= \frac{(x^2+1)^3(x+1)^4}{x\sqrt{(x-1)}\sqrt{(x+3)}}\\
\Rightarrow \ln y &= 3\ln(x^2+1)+4\ln(x+1)-\ln x-\frac{1}{2}\ln(x-1)-\frac{1}{2}\ln(x+3)\\
\Rightarrow \frac{1}{y}\frac{dy}{dx} &= \left\{\frac{6x}{x^2+1}+\frac{4}{x+1}-\frac{1}{x}-\frac{1}{2(x-1)}-\frac{1}{2(x+3)}\right\}\\
\text{therefore } \frac{dy}{dx} &= \left[\frac{6x}{x^2+1}+\frac{4}{x+1}-\frac{1}{x}-\frac{1}{2(x-1)}-\frac{1}{2(x+3)}\right]y\\
&= \left[\frac{6x}{x^2+1}+\frac{4}{x+1}-\frac{1}{x}-\frac{1}{2(x-1)}-\frac{1}{2(x+3)}\right]\frac{(x^2+1)^3(x+1)^4}{x\sqrt{(x-1)}\sqrt{(x+3)}}
\end{aligned}$$

Example 4.4.38 Given

$$y=(\sin x)^{\cos x}$$

find the derivative dy/dx

$$\begin{aligned}
\text{Since } y &= (\sin x)^{cosx}\\
\Rightarrow \ln y &= \cos x\ln(\sin x)\\
\Rightarrow \frac{1}{y}\frac{dy}{dx} &= -\sin x\ln\sin x+\frac{\cos^2 x}{\sin x}\\
\Rightarrow \frac{dy}{dx} &= \left(\frac{\cos^2 x}{\sin x}-\sin x\ln\sin x\right)(\sin x)^{\cos x}
\end{aligned}$$

Note 4.4.2 In this case it is only logarithmic differentiation which is applicable

Example 4.4.39 Differentiate

$$y=x^x$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply* ! The function must first be revised before a derivative can be taken. Begin with

$$y=x^x$$

Apply the natural logarithm to both sides of this equation getting

$$\begin{aligned}
\ln y &= \ln x^x\\
\ln y &= x\ln x
\end{aligned}$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule on the right-hand side. Thus, differentiating, we get

$$\frac{1}{y}y'=x\frac{1}{x}+(1)\ln x=1+\ln x$$

Multiply both sides of this equation by y (making y' the subject), getting

$$y'=y(1+\ln x)=x^x(1+\ln x)$$

Example 4.4.40 Differentiate the function

$$y = \frac{(x+2)(x-6)^3(x+4)^2}{(x-3)}$$

$$\begin{aligned}
\ln y &= \ln\left[\frac{(x+2)(x-6)^3(x+4)^2}{(x-3)}\right] = \ln(x+2) + \ln(x-6)^3 + \ln(x+4)^2 - \ln(x-3)\\
\frac{\frac{dy}{dx}}{y} &= \frac{1}{(x+2)} + \frac{3(x-6)^2}{(x-6)^3} + \frac{2(x+4)}{(x+4)^2} - \frac{1}{(x-3)}\\
&= \frac{1}{(x+2)} + \frac{3}{(x-6)} + \frac{2}{(x+4)} - \frac{1}{(x-3)}\\
\frac{dy}{dx} &= y\left[\frac{1}{(x+2)} + \frac{3}{(x-6)} + \frac{2}{(x+4)} - \frac{1}{(x-3)}\right]\\
&= \left(\frac{(x+2)(x-6)^3(x+4)^2}{(x-3)}\right)\left[\frac{1}{(x+2)} + \frac{3}{(x-6)} + \frac{2}{(x+4)} - \frac{1}{(x-3)}\right]
\end{aligned}$$

Example 4.4.41 Differentiate

$$y = x^{(e^x)}$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation DO NOT APPLY ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln x^{(e^x)} = e^x \ln x$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule on the right-hand side.

$$\begin{aligned}
\frac{1}{y}y' &= e^x\left\{\frac{1}{x}\right\} + e^x \ln x\\
\frac{1}{y}y' &= \frac{e^x}{x} + \left\{\frac{x}{x}\right\}e^x \ln x = \frac{e^x}{x} + \frac{xe^x \ln x}{x} = \frac{e^x + xe^x \ln x}{x} = \frac{e^x(1 + x\ln x)}{x}
\end{aligned}$$

Multiply both sides of this equation by y, getting (by combining the powers of x)

$$y' = y\frac{e^x(1+x\ln x)}{x} = x^{(e^x)}\frac{e^x(1+x\ln x)}{x^1} = x^{(e^x-1)}e^x(1+x\ln x)$$

Example 4.4.42 Differentiate

$$y = (3x^2+5)^{\frac{1}{x}}$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply* ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln(3x^2+5)^{1/x} = \left(\frac{1}{x}\right)\ln(3x^2+5) = \frac{\ln(3x^2+5)}{x}$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the quotient rule and the chain rule on the right-hand side. Thus,

$$\frac{1}{y}y' = \frac{x\left\{\dfrac{1}{3x^2+5}\right\}(6x) - \ln(3x^2+5)(1)}{x^2}$$

Get a common denominator and combine fractions in the numerator.

$$\frac{1}{y}y' = \frac{\dfrac{6x^2}{3x^2+5} - \ln(3x^2+5)\left\{\dfrac{3x^2+5}{3x^2+5}\right\}}{\dfrac{x^2}{1}}$$

Dividing by a fraction is the same as multiplying by its reciprocal.

$$\begin{aligned} \frac{1}{y}y' &= \frac{6x^2 - (3x^2+5)\ln(3x^2+5)}{3x^2+5}\frac{1}{x^2} \\ \frac{1}{y}y' &= \frac{6x^2 - (3x^2+5)\ln(3x^2+5)}{x^2(3x^2+5)} \end{aligned}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{6x^2 - (3x^2+5)\ln(3x^2+5)}{x^2(3x^2+5)} = (3x^2+5)^{\frac{1}{x}}\,\frac{6x^2 - (3x^2+5)\ln(3x^2+5)}{x^2(3x^2+5)^1}$$

Combine the powers of $(3x^2+5)$

$$y' = \frac{(3x^2+5)^{\left(\frac{1}{x}-1\right)}\left\{6x^2 - (3x^2+5)\ln(3x^2+5)\right\}}{x^2}$$

Example 4.4.43 Differentiate

$$y = (\sin x)^{x^3}$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply* ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln(\sin x)^{x^3} = x^3\ln(\sin x)$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule and the chain rule on the right-hand side.

$$\frac{1}{y}y' = x^3\left\{\frac{1}{\sin x}\right\}\cos x + (3x^2)\ln(\sin x)$$

Get a common denominator and combine fractions on the right-hand side.

$$\frac{1}{y}y' = \frac{x^3\cos x}{\sin x} + 3x^2\ln(\sin x)\left\{\frac{\sin x}{\sin x}\right\} = \frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{\sin x}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{\sin x} = (\sin x)^{x^3}\frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{(\sin x)^1}$$

Combine the powers of $(\sin x)$.

$$y' = (\sin x)^{(x^3-1)}\left\{x^3\cos x + 3x^2\sin x\ln(\sin x)\right\}$$

Example 4.4.44 Differentiate

$$y = 7x(\cos x)^{\frac{x}{2}}$$

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply* ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting

$$\ln y = \ln\left((7x)(\cos x)^{\frac{x}{2}}\right) = \ln(7x) + \ln(\cos x)^{\frac{x}{2}} = \ln(7x) + \left(\frac{x}{2}\right)\ln(\cos x)$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x . Use the product rule and the chain rule on the right-hand side.

$$\frac{1}{y}y' = \left\{\frac{7}{7x}\right\} + \left(\frac{x}{2}\right)\left\{\frac{1}{\cos x}\right\}(-\sin x) + \left(\frac{1}{2}\right)\ln(\cos x) = \frac{1}{x} - \frac{x\sin x}{2\cos x} + \frac{\ln(\cos x)}{2}$$

Get a common denominator and combine fractions on the right-hand side.

$$\frac{1}{y}y' = \frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2x\cos x}$$

Multiply both sides of this equation by y, getting

$$\begin{aligned}
y' &= y\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2x\cos x}\\
y' &= 7x(\cos x)^{\frac{x}{2}}\left[\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2x\cos x}\right]\\
y' &= 7(\cos x)^{x/2}\left[\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2(\cos x)^1}\right]\\
y' &= \left(\frac{7}{2}\right)(\cos x)^{\left(\frac{x}{2}-1\right)}\left\{2\cos x - x^2\sin x + x\cos x\ln(\cos x)\right\}
\end{aligned}$$

Example 4.4.45 Differentiate
$$y = \sqrt{x}^{\sqrt{x}} e^{x^2}$$
Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply* ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting
$$\begin{aligned}
\ln y &= \ln\left(\sqrt{x}^{\sqrt{x}} e^{x^2}\right) \\
&= \ln\left(\sqrt{x}^{\sqrt{x}}\right) + \ln\left(e^{x^2}\right) \\
&= \sqrt{x}\ln(\sqrt{x}) + x^2\ln(e) \\
&= \sqrt{x}\ln(\sqrt{x}) + x^2(1) \\
&= \sqrt{x}\ln(\sqrt{x}) + x^2
\end{aligned}$$
Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule and the chain rule on the right-hand side.
$$\frac{1}{y}y' = \sqrt{x}\left\{\frac{1}{\sqrt{x}}\right\}(1/2)x^{-1/2} + (1/2)x^{-1/2}\ln(\sqrt{x}) + 2x = \frac{1}{2\sqrt{x}} + \frac{\ln(\sqrt{x})}{2\sqrt{x}} + 2x$$
Get a common denominator and combine fractions on the right-hand side.
$$y' = \frac{1}{2\sqrt{x}} + \frac{\ln(\sqrt{x})}{2\sqrt{x}} + 2x\left\{\frac{2\sqrt{x}}{2\sqrt{x}}\right\} = \frac{1+\ln(\sqrt{x})+4x^{1+1/2}}{2\sqrt{x}} = \frac{1+\ln(\sqrt{x})+4x^{3/2}}{2\sqrt{x}}$$
Multiply both sides of this equation by y, getting
$$y' = y\frac{1+\ln(\sqrt{x})+4x^{3/2}}{2\sqrt{x}} = \sqrt{x}^{\sqrt{x}} e^{x^2}\frac{1+\ln(\sqrt{x})+4x^{3/2}}{2\sqrt{x}^1}$$
Combine the powers of $\sqrt{x}$.
$$y' = (1/2)\sqrt{x}^{(\sqrt{x}-1)} e^{x^2}\left\{1+\ln(\sqrt{x})+4x^{3/2}\right\}$$
Example 4.4.46 Differentiate
$$y = x^{\ln x}(\sec x)^{3x}$$
Because a variable is raised to a variable power in this function, the ordinary rules of differentiation *do not apply* ! The function must first be revised before a derivative can be taken.

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting
$$\begin{aligned}
\ln y &= \ln\left(x^{\ln x}(\sec x)^{3x}\right) \\
&= \ln x^{(\ln x)} + \ln(\sec x)^{3x} \\
&= (\ln x)(\ln x) + 3x\ln(\sec x) \\
&= (\ln x)^2 + 3x\ln(\sec x)
\end{aligned}$$
Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule and the chain rule on the right-hand side.
$$\frac{1}{y}y' = 2(\ln x)\left\{\frac{1}{x}\right\} + 3x\left\{\frac{1}{\sec x}\right\}(\sec x\tan x) + (3)\ln(\sec x)$$

Divide out a factor of $\sec x$.

$$\frac{1}{y}y' = \frac{2\ln x}{x} + 3x\tan x + 3\ln(\sec x)$$

Get a common denominator and combine fractions on the right-hand side.

$$\begin{aligned}\frac{1}{y}y' &= \frac{2\ln x}{x} + 3x\tan x\left\{\frac{x}{x}\right\} + 3\ln(\sec x)\left\{\frac{x}{x}\right\}\\ &= \frac{2\ln x + 3x^2\tan x + 3x\ln(\sec x)}{x}\end{aligned}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{2\ln x + 3x^2\tan x + 3x\ln(\sec x)}{x} = x^{\ln x}(\sec x)^{3x}\frac{2\ln x + 3x^2\tan x + 3x\ln(\sec x)}{x^1}$$

Combine the powers of x.

$$y' = x^{(\ln x - 1)}(\sec x)^{3x}\left\{2\ln x + 3x^2\tan x + 3x\ln(\sec x)\right\}$$

Example 4.4.47 Consider the function

$$f(x) = \frac{x^5e^x(4x+3)}{5^{\ln x}(3-x)^2}$$

Find an equation of the line tangent to the graph of f at $x = 1$.

First note that

$$f(1) = \frac{(1)^5e^1\,(4(1)+3)}{5^{\ln 1}(3-1)^2} = \frac{7}{4}e^1$$

so that the tangent line passes through the point

$$(x, y) = \left(1, \frac{7}{4}e^1\right)$$

Then getting the tangent, by differentiating

$$\begin{aligned}\ln f(x) &= 5\ln x + x + \ln(4x+3) - (\ln 5)\ln x - 2\ln(3-x)\\ \frac{1}{f(x)}f'(x) &= \frac{5}{x} + 1 + \frac{4}{4x+3} - \frac{\ln 5}{x} + \frac{2}{3-x}\\ f'(x) &= f(x)\left\{\frac{5}{x} + 1 + \frac{4}{4x+3} - \frac{\ln 5}{x} + \frac{2}{3-x}\right\}\end{aligned}$$

The slope of the line tangent to the graph of f at $x = 1$ is

$$\begin{aligned}f'(1) &= f(1)\left\{\frac{5}{1} + 1 + \frac{4}{4+3} - \frac{\ln 5}{1} + \frac{2}{3-1}\right\}\\ &= \frac{7}{4}e^1\left(7 + \frac{4}{7} - \ln 5\right)\\ &= \frac{7}{4}e^1\left(\frac{53}{7} - \ln 5\right)\end{aligned}$$

Thus, the equation of the line tangent to the graph of f at $x = 1$ is

$$\begin{aligned}\frac{y - \frac{7}{4}e^1}{x-1} &= \frac{7}{4}e^1\left(\frac{53}{7} - \ln 5\right)\\ y &= \frac{7}{4} + \frac{7}{4}e^1\left(\frac{53}{7} - \ln 5\right)(x-1)\end{aligned}$$

Example 4.4.48 Consider the function

$$f(x) = \pi^2 + 2^x + x^2 + x^{\frac{1}{x}}$$

Determine the slope of the line perpendicular to the graph of f at $x = 1$.

In this function the only terms that requires logarithmic differentiation is $x^{\frac{1}{x}}$ and 2^x

$$\begin{aligned}
\ln y &= \frac{\ln x}{x} \\
\frac{1}{y}y' &= \frac{1 - \ln x}{x^2} \\
y' &= \frac{x^{\frac{1}{x}}(1 - \ln x)}{x^2}
\end{aligned}$$

Now return to the original function $f(x) = \pi^2 + 2^x + x^2 + x^{\frac{1}{x}}$. Differentiating, we get

$$\begin{aligned}
f'(x) &= (0) + 2^x \ln 2 + 2x + \frac{x^{\frac{1}{x}}(1 - \ln x)}{x^2} \\
&= 2^x \ln 2 + 2x + \frac{x^{\frac{1}{x}}(1 - \ln x)}{x^2}
\end{aligned}$$

The slope of the line tangent to the graph of f at $x = 1$ is

$$\begin{aligned}
f'(1) &= 2^{(1)} \ln 2 + 2(1) + \frac{(1)^{\frac{1}{1}}(1 - \ln 1)}{1^2} \\
&= 3 + \ln 4
\end{aligned}$$

Thus, the slope of the line perpendicular to the graph of f at $x = 1$ is

$$m = \frac{-1}{3 + \ln 4}$$

Example 4.4.49 Differentiate $y = x^{(x^{(x^4)})}$

$$\begin{aligned}
y &= x^{(x^{(x^4)})} \\
\ln y &= x^{(x^4)} \ln x \\
\ln(\ln y) &= \ln\left(x^{(x^4)} \ln x\right) \\
&= \ln x^{(x^4)} + \ln(\ln x) = x^4 \ln x + \ln(\ln x)
\end{aligned}$$

Differentiate both sides of this equation.

$$\begin{aligned}
\ln(\ln y) &= x^4 \ln x + \ln(\ln x) \\
\left\{\frac{1}{\ln y}\right\}\left\{\frac{1}{y}\right\} y' &= x^4 \left\{\frac{1}{x}\right\} + \left(4x^3\right) \ln x + \left\{\frac{1}{\ln x}\right\}\left\{\frac{1}{x}\right\} \\
\left\{\frac{1}{\ln y}\right\}\left\{\frac{1}{y}\right\} y' &= \left(x^3 + 4x^3 \ln x\right) + \frac{1}{x \ln x} \\
\left\{\frac{1}{\ln y}\right\}\left\{\frac{1}{y}\right\} y' &= \frac{x^4(1 + 4\ln x)\ln x + 1}{x \ln x}
\end{aligned}$$

$$y' = x^{(x^{(x^4)} + x^4 - 1)} \left\{x^4(1 + 4\ln x)\ln x + 1\right\}$$

Example 4.4.50 Differentiate

$$y = \frac{(\ln x)^x}{2^{3x+1}}$$

$$\begin{aligned}
\ln y &= x\ln(\ln x) - (3x+1)\ln 2 \\
\frac{1}{y}y' &= \frac{1}{\ln x} + \ln(\ln x) - \ln 2^3 \\
y' &= \frac{(\ln x)^{(x-1)}\left\{1 + (\ln x)\ln(\ln x) - (\ln 8)\ln x\right\}}{2^{3x+1}}
\end{aligned}$$

Example 4.4.51 Differentiate

$$y = \frac{x^{2x}(x-1)^3}{(3+5x)^4}$$

$$\begin{aligned}
\ln y &= (2x)\ln x + 3\ln(x-1) - 4\ln(3+5x) \\
\frac{1}{y}y' &= 2 + 2\ln x + \frac{3}{x-1} - \frac{20}{3+5x} = \frac{2(\ln x)(x-1)(3+5x) + 10x^2 - 9x + 23}{(x-1)(3+5x)} \\
y' &= \frac{x^{2x}(x-1)^2\left\{2(\ln x)(x-1)(3+5x) + 10x^2 - 9x + 23\right\}}{(3+5x)^5}
\end{aligned}$$

Exercise 4.17 Differentiate

(i) $y = x^{x^x}$ (ii) $y = 2^x$ (iii) $y = 2^{\cot x}$

Exercise 4.18 Let $y = x^x$, find $\frac{dy}{dx}$.

Exercise 4.19 Differentiate the following functions with respect to x

(i) x^{x-1} (iii) $(x-1)^{\ln x}$ (v) $\frac{(x^2-1)^3(x-1)^{\frac{1}{2}}}{(x-1)^3}$

(ii) $x^{\sin x}$ (iv) $\frac{xe^x}{\sin x}$ (vi) $\frac{1}{(x+1)^6}$

Exercise 4.20 Find $\frac{dy}{dx}$ given that,

(i) $x = t^3 - 2,\ y = t^2 + 2$ (iii) $x = \frac{1}{t^2}\, y = 4t^3 + 8$

(ii) $x = \cos t,\ y = 6\sin t$ (iv) $x = 2 + \sqrt{t},\ y = 2 - \sqrt{t}$

Exercise 4.21 Find $y'(x)$ given that

(i) $xy^2 - 3x^2y = 10$ (iii) $\cot y = 2x^3 + \cot(x+y)$

(ii) $(yx)^{\frac{1}{2}} + y^{\frac{1}{2}} = 0$ (iv) $(x+y^2)^3 + x^2y = \alpha^2$

4.5 Applications of differentiation

4.5.1 Maxima and Minima

When using mathematics to model the physical world in which we live, we frequently express physical quantities in terms of **variables**. Then, **functions** are used to describe the ways in which these variables change. A scientist or engineer will be interested in the ups and downs of a function, its maximum and minimum values (optimization problems), its turning points. Drawing a graph of a function using a graphical calculator or computer graph plotting package will reveal this behavior, but if we want to know the precise location of such points we need to turn to algebra and differential calculus.

Definition 4.5.1 *A function f is said to have a* **local maximum** *at x_0 (or a* **relative maximum** *at x_0) if for all x in some interval containing x_0 we have*

$$f(x_0) \geq f(x)$$

Similarly, a function f is said to have a **local minimum** *or* **relative minimum** *at x_0 if $f(x_0) \leq f(x)$*

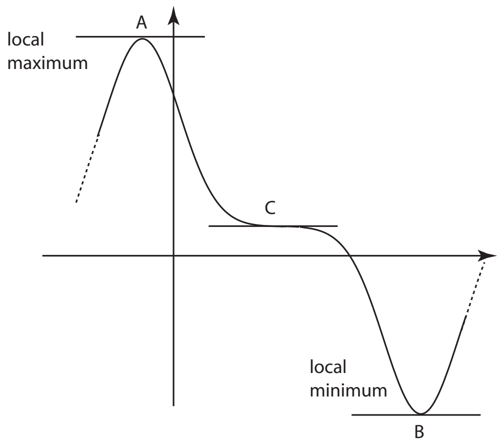
Note 4.5.1 A local maximum at one point can have a higher value of f than a local maximum at another point.

Definition 4.5.2 *A function f defined on a domain D is said to have a* **maximum** *(or an* **absolute maximum** *or a* **global maximum***at x_0) at $x_0 \in D$* if

$$f(x_0) \geq f(x) \; \forall \; x \in D$$

Similarly, a function f defined on a domain D is said to have a **minimum** *(or an* **absolute minimum** *or a* **global minimum***at x_0) at $x_0 \in D$* if $f(x_0) \leq f(x) \; \forall \; x \in D$

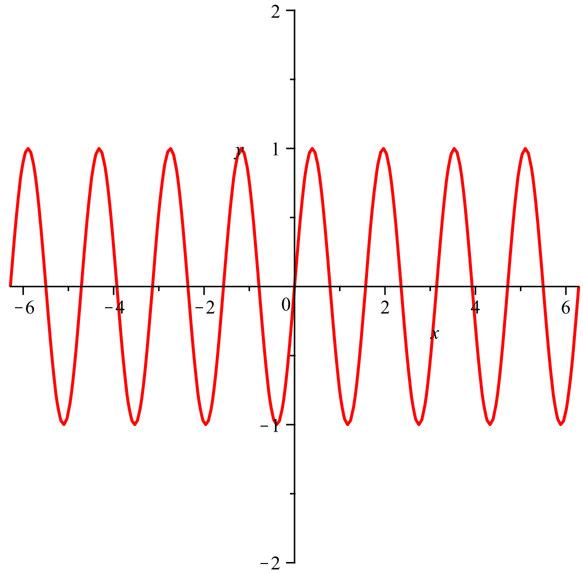
Note 4.5.2 The number $f(x_0)$ is called the maximum value of f on the domain D. The maximum and minimum values of f are called extreme values of f.



Example 4.5.1 For the function

$$f(x) = \sin 4x, \text{ on } -2\pi \leq x \leq 2\pi$$

with the curve



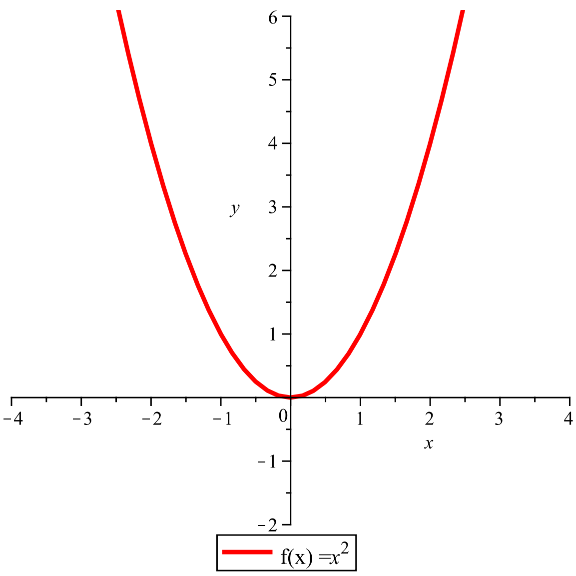
(i) $f(x) = -1$ is a local and an absolute (global) minimum

(ii) $f(x) = 1$ is a local and a global maximum

Example 4.5.2 For the function

$$f(x) = x^2, \text{ on } -4 \leq x \leq 4$$

which is represented by the graph



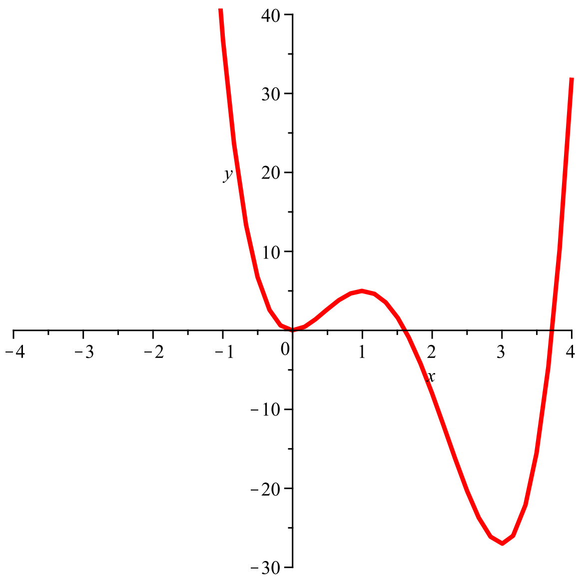
(i) $f(0) = 0$ is an absolute (global) minimum

(ii) $f(x)$ has no global maximum

Example 4.5.3 For the function

$$f(x) = 3x^4 - 16x^3 + 18x^2, \text{ on } -1 \leq x \leq 4$$

which can be graphed as



(i) $f(-1) = 37 > f(4) = 32$ is an absolute (global) maximum for the interval $-1 \leq x \leq 4$

(ii) $f(3) = -27$ is a local and a global minimum

(iii) $f(0) = 0$ is a local minimum on interval $[-1, 1]$

(iv) $f(1) = 5$ is a local maximum on the interval $[0, 2]$

Example 4.5.4 Sketch the graph on $[-2, 4]$ where

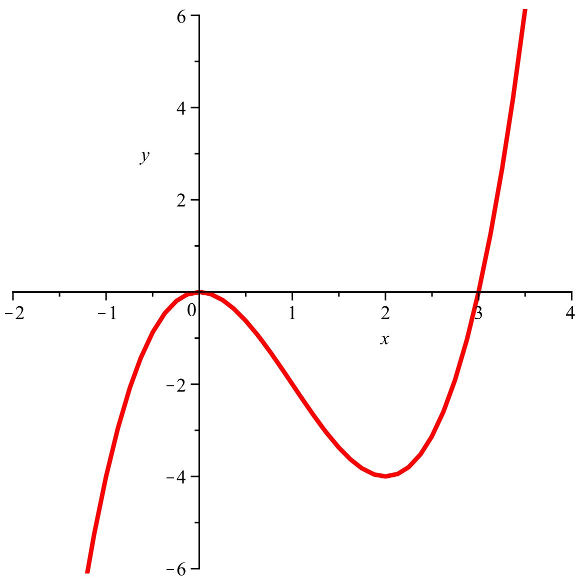
(i) $f(-2)$ is an absolute minimum

(ii) $f(0) = 0$, and $f(0)$ is a relative maximum for interval $[-1, 2)$

(iii) $f(2)$ is a relative minimum on interval $[0, 3]$

(iv) $f(3) = 0 \Rightarrow$ cutting the x-axis

(v) $f(4)$ is an absolute maximum



Theorem 4.5.1 *Let f be a continuous function on a closed finite interval $[a, b]$. Then f has both a maximum and minimum.*

Definition 4.5.3 Let f be a function defined on interval I and c be an interior point of I. A *critical number* for f is a number c for which $f'(c) = 0$ `or f'(c) fails to exist`.

Note 4.5.3 A critical number is a Stationary point or Turning point.

Theorem 4.5.2 *Suppose f is differentiable in (a, b) and $x_0 \in (a, b)$. If f has a local extremum (local maxima or minima) at x_0, then $f'(x_0) = 0$.*

Note 4.5.4 A point of inflection is not an extrema point.

Note 4.5.5 If point c is a critical point or an extrema, then $f'(c) = 0$, but if $f'(c) = 0$, it does not necessarily mean that point c is an extrema, it could be an inflection point.

Example 4.5.5 Find all the critical numbers and the maximum and minimum values for $f = \frac{1}{4}x^4 - 2x^2$ on the given interval $-2 \leq x \leq 2$

$$\begin{aligned} f(x) &= \frac{1}{4}x^4 - 2x^2 \\ \Rightarrow \quad f'(x) &= x^3 - 4x = x(x^2 - 4) \end{aligned}$$

For the critical values

$$\begin{aligned} f'(x) = 0 \quad \Rightarrow & \quad x(x^2 - 4) = 0 \\ x(x-2)(x+2) &= 0 \end{aligned}$$

The critical values are $x = 0, -2, 2$.

The extremas are

$$\begin{aligned} f(-2) &= -4 \quad \text{an absolute minimum} \\ f(0) &= 0 \quad \text{a relative maximum on } -2 \leq x \leq 2 \\ f(2) &= -4 \quad \text{an absolute minimum} \end{aligned}$$

Theorem 4.5.3 First Derivative Test: *Let c be a critical number of f and f continuous at c. If there exists a $\delta > 0$ such that*

(a) $f'(x) < 0$ for all $x \in (c-\delta, c)$ and $f'(x) > 0$ for all $x \in (c, c+\delta)$ the $f(c)$ is a local minimum.

(b) $f'(x) > 0$ for all $x \in (c-\delta, c)$ and $f'(x) < 0$ for all $x \in (c, c+\delta)$ the $f(c)$ is a local maximum.

(c) $f'(x)$ has the same sign on $(c-\delta, c) \cup (c, c+\delta)$ then $f(c)$ is neither a local maximum nor minimum.

Example 4.5.6 Think about what happens to the gradient of the graph as we travel through the minimum turning point, from *left to right*, that is as x increases. Study Figure (4.4) to help you do this.

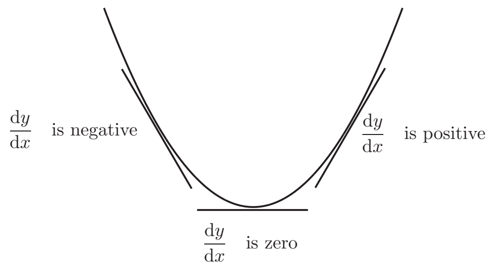


Figure 4.4: Minima

Notice that to the left of the minimum point, dy/dx is negative because the tangent has negative gradient. At the minimum point, $dy/dx = 0$. To the right of the minimum point dy/dx is positive, because here the tangent has a positive gradient.

Example 4.5.7 Now think about what happens to the gradient of the graph as we travel through the maximum turning point, from left to right, that is as x increases. Study Figure (4.5) to help you do this.

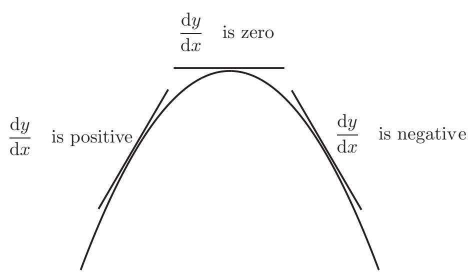


Figure 4.5: Maxima

Notice that to the left of the maximum point, dy/dx is positive because the tangent has positive gradient. At the maximum point, $dy/dx = 0$. To the right of the maximum point dy/dx is negative, because here the tangent has a negative gradient. So, dy/dx goes from positive, to zero, to negative as x increases.

Example 4.5.8 Sketch the graph on $[-3, 8]$ such that

(i) $f(4) = 0$

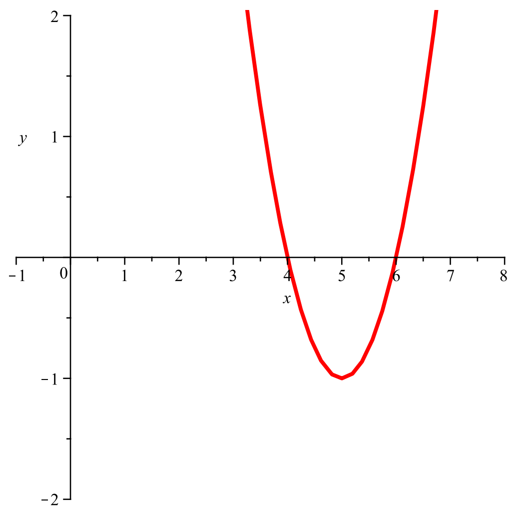
(ii) $f(6) = 0$

(iii) $f'(5) = 0$

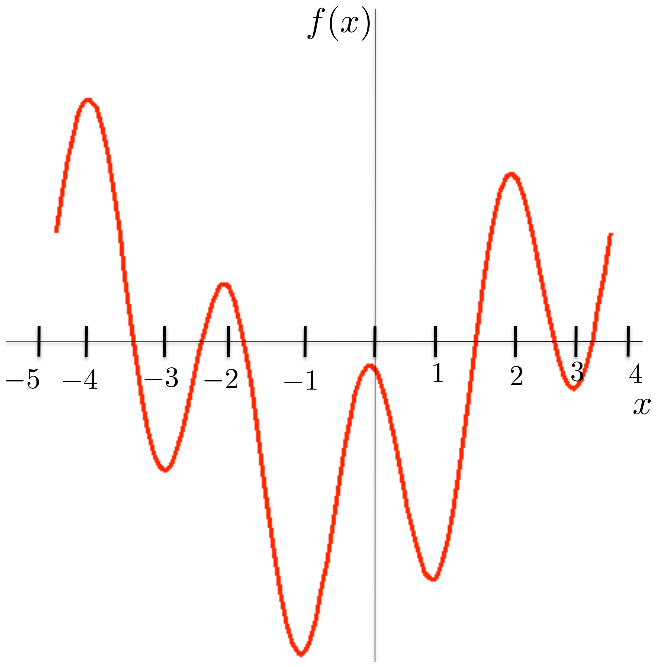
(iv) $f'(4.88) < 0$

(v) $f'(5.02) > 0$

Using the first derivative test, we realise that the critical point, $x = 5$ is a local minima since on left, the derivative is negative, and on right of 5, the derivative is positive. But the curve has to cut the x-axis at 4 and 6.



Example 4.5.9 Describe the type of extremas for a function $f(x)$ by



$$
\begin{aligned}
f(0) &\equiv \texttt{Relative maxima in } (-1,1) \\
f(-4) &\equiv \texttt{Absolute maxima} \\
f(3) &\equiv \texttt{Local minima in } (2,4) \\
f(2) &\equiv \texttt{local maxima in } (-3,4) \\
f(-1) &\equiv \texttt{Minima} \\
f(1) &\equiv \texttt{Relative minima in } (0,4) \\
f(-3) &\equiv \,?? \\
f(-2) &\equiv \,??
\end{aligned}
$$

4.5.2 Mean Value Theorem MVT

Theorem 4.5.4 Mean Value Theorem: *Let*

(a) f be continuous on $[a, b]$

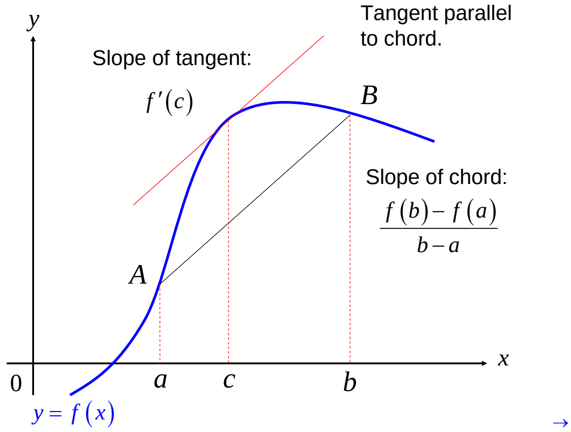
(b) f be differentiable in (a, b) and

then there exists at least one number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{4.11}$$

Mean Value Theorem - Geometrical

Look at the secant line through $(a, f(a))$ and $(b, f(b))$. We expect that somewhere between a and b there is a point c where the tangent is parallel to this secant.



That is, the slopes of these two lines are equal. This is formalized in the MVT.

Proof of Mean Value Theorem

To prove the MVT, we apply the Rolle's theorem

Theorem 4.5.5 Rolle's Theorem: *Let*

(a) f be continuous on $[a, b]$

(b) f be differentiable in (a, b) and

(c) $f(a) = f(b)$

then there exists at least one number $c \in (a, b)$ such that $f'(c) = 0$

Back to proof of MVT: We define a function

$$h(x) = f(x) - \left[f(a) \ + \ \frac{x-a}{b-a}\,[f(b) - f(a)] \right]$$

represents the difference in height between the curve $y = f(x)$ and the line joining its end points.

Clearly, by the nature of our function $h(x)$

(a) $h(x)$ be continuous on $[a, b]$ by sum of continuous functions

(b) $h(x)$ be differentiable in (a, b) by the sum of differentiable functions, and

(c) $h(a) = h(b)$ by substitution

satisfying all the conditions of Rolle's Theorem. Hence there is some $c \in (a, b)$ such that

$$\begin{aligned} h'(c) &= 0 \text{ (Rolle's Theorem)} \\ f'(c) - \frac{f(b) - f(a)}{b - a} &= 0 \end{aligned}$$

Thus, the proof

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Corollary 4.5.1 *Let $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).*

Proof: Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable in (a, b) - the derivative given- it is differentiable in (x_1, x_2) and is continuous on $[x_1, x_2]$ - all differentiable functions, are continuous. Then by MVT, there exists a number $c \in (x_1, x_2)$ such that

$$\begin{aligned} f'(c) &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ 0 &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \end{aligned}$$

Since we were given $f'(x) = 0$ for all $x \in (a, b)$.

$$f(x_1) = f(x_2)$$

for any arbitrarily chosen numbers $x_1, x_2 \in (a, b)$. Hence f is constant on (a, b).

Corollary 4.5.2 *Let $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on (a, b).*

Corollary 4.5.3 *Let $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).*

Example 4.5.10 For a function $f(x) = -x^2 + 6x - 6$, find a c on $[1, 3]$ that satisfies the Mean Value Theorem.

Since $f(x)$ is continuous and differentiable (because a polynomial), it satisfies the hypotheses of MVT, $a = 1, b = 3, f'(x) = -2x + 6$, therefore,

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ -2c + 6 &= \frac{f(3) - f(1)}{3 - 1} \\ -2c + 6 &= 2 \end{aligned}$$

And a $c \in (1, 3)$ is such that, $f'(c) = -2c + 6 = 2$ this implies that $c = 2$.

Example 4.5.11 For a function $f(x) = x^3 - x$ find a c on $[0, 2]$ that satisfies the mean value theorem.

The function $f(x)$ is differentiable in $(0, 2)$ and continuous on $[0, 2]$ and $f'(x) = 3x^2 - 1$, therefore,

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ 3c^2 - 1 &= \frac{6 - 0}{2 - 0} \\ 3c^2 - 1 &= 3 \end{aligned}$$

And a $c \in (0, 2)$ is such that, $f'(c) = 3c^2 - 1$ this implies that $c = \pm 2/\sqrt{3}$. So the c in the interval is $c = 2/\sqrt{3}$

Exercise 4.22 Using Intermediate Value Theorem, show that $x^3 - x + 1$ has only one real root on $[-2, -1]$ and that other two roots are complex.

Example 4.5.12 Determine a and b for the function:

$$f(x) = \begin{cases} ax - 3, & x < 4 \\ -x^2 + 10x - b, & x \geq 4 \end{cases}$$

If it satisfies the hypothesis of Mean Value Theorem on the interval $[2, 6]$. Hint: To satisfy the MVT, it has to be continuous on $[2, 6]$ and differentiable in $(2, 6)$ [a,b] = [2,19]

Example 4.5.13 Determine a c that satisfies the MVT for the function

$$f(x) = \begin{cases} x + 1, & x < 1 \\ x - 1, & x \geq 1 \end{cases}$$

on the interval $[0, 3]$.

The MVT hypotheses not satisfied, since function not continuous at $x = 1$.

Example 4.5.14 Determine a c that satisfies the MVT for the function $f(x) = |x|$ on $[-2, 5]$

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The MVT hypotheses not satisfied, since function not differentiable at $x = 0$.

Exercise 4.23 Let $f(x) = |x^2 - x - 2|$. Determine if the Mean Value Theorem applies to f on the interval $[a, b] = [0, 3]$. If the MVT does not apply, state the hypothesis that is not satisfied. If the MVT does apply, identify all numbers c in the interval where $f'(c) = \frac{f(b)-f(a)}{b-a}$. In either case, include a graph that supports your conclusion.

Example 4.5.15 Use the mean value theorem to prove that for any two real numbers a and b,

$$|\cos a - \cos b| \leq |a - b|$$

The function $\cos x$ is continuous and differentiable for all real numbers. Using the mean value theorem, using 2 real numbers a and b to write

$$\begin{aligned}
(\cos x)' &= \frac{[\cos a - \cos b]}{[a-b]} \\
|(\cos x)'| &= \left|\frac{[\cos a - \cos b]}{[a-b]}\right| \\
\left|\frac{[\cos a - \cos b]}{[a-b]}\right| &\leq 1 \\
\frac{|\cos a - \cos b|}{|a-b|} &\leq 1 \\
|\cos a - \cos b| &\leq |a-b|
\end{aligned}$$

Example 4.5.16 Given the piecewise defined function

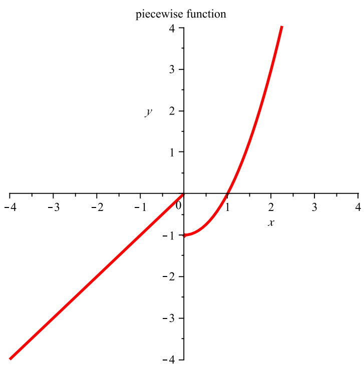
$$f(x) = \begin{cases} x, & x < 0 \\ x^2 - 1, & x \geq 0 \end{cases}$$

(i) determine if the function satisfies the Mean Value Theorem in $(1,3)$ and $(-1,1)$

The function $f(x) = x^2 - 1$ is continuous and differentiable in the interval $[1,3]$ and therefore satisfies the initial conditions (hypotheses) of the MVT in the interval $(1,3)$.

The function $f(x)$ is neither continuous nor differentiable in the interval at a point $x = 0$ and therefore $f(x)$ does not satisfy the initial conditions (hypotheses) of the MVT in the interval $(-1,1)$.

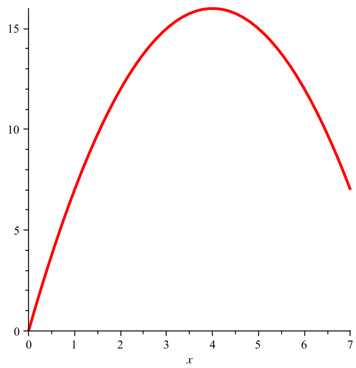
(ii) plot the function



(iii) find all values of c that satisfy the conclusion of the Mean Value Theorem.

$$\begin{aligned}
f'(c) &= \frac{f(b) - f(a)}{b-a} \\
2c &= \frac{f(3) - f(1)}{3-1} \\
2c &= \frac{8-0}{3-1} = 4 \\
c &= 2
\end{aligned}$$

The function, $f(x)$, is not continuous at $x = 0$ and therefore is not continuous in $[-1,1]$. The MVT does not apply in this interval.



Example 4.5.17 Sketch the graph where: $f'(x) > 0, 1 \leq x < 5$ and $f'(x) < 0, 5 < x \leq 7$

Exercise 4.24 For the numbers 1-7, find a number c which satisfies the MVT.

1). $\frac{x-1}{x+1}$ on $[0, 1]$

2). $x^{\frac{2}{5}}$ on $[0, 8]$

3). $|5 - x^2|$ on $[-2, 2]$

4). $\tan x$ on $[0, \frac{\pi}{4}]$

5). x^2 on $[0, 3]$

6). x^3 on $[0, 4]$

7). $x^3 - 2x^2 + 3x + 1$ on $[0, 2]$

8). Is Rolle's theorem applicable to the function $f(x) = |x - 1|$ on the interval $[0, 2]$?

9). Determine if the function $f(x) = x - x^3$ satisfies the conditions of Rolle's theorem on the interval $[-1, 0]$ and $[0, 1]$. In the affirmative case, determine the values of c.

10). Does the function $f(x) = 1 - x$ satisfy the conditions of Rolle's theorem on the interval $[-1, 1]$?

11). Prove that the equation $1 + 2x + 3x^2 + 4x^3 = 0$ has a unique solution.

12). How many roots does the equation $x^3 + 6x^2 + 15x - 25 = 0$ have?

13). Prove that the equation $2x^3 - 6x + 1 = 0$ has only one real solution on the interval $(0, 1)$.

14). Can the mean value theorem be applied to $f(x) = 4x^2 - 5x + 1$ on $[0, 2]$?

15). Can the mean value theorem be applied to $f(x) = 1/x^2$ on $[0, 2]$?

16). In the segment of the parabola between the points $A = (1, 1)$ and $B = (3, 0)$, find a point whose tangent is parallel to the chord.

17). Calculate a point on the interval $[1, 3]$ in which the tangent to the curve $y = x^3 - x2 + 2$ is parallel to the line determined by the points $A = (1, 2)$ and $B = (3, 20)$. What theorem guarantees the existence of this point?

4.5.3 Second Derivative Test and Concavity

Definition 4.5.4 *Let f be differentiable in (a, b), and let $p \in (a, b)$.*

(a) We call point p a stationary (or critical) point for f if $f'(p) = 0$,

(b) we call point p a point of inflection for f if $f''(p) = 0$,

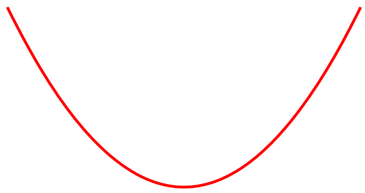
(c) we call f (strictly) concave upwards if f' is (strictly) increasing on (a, b),

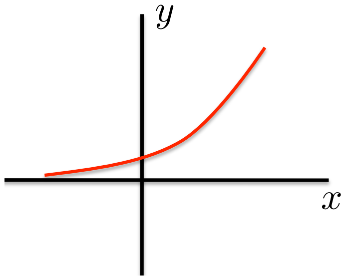
(d) we call f (strictly) concave downwards if f' is (strictly) decreasing on (a, b).

Concave up is also referred to as positive curvature and concave down is referred as negative curvature.

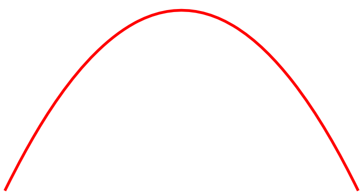
Theorem 4.5.6 On Concavity: Let f be twice differentiable in (a, b).

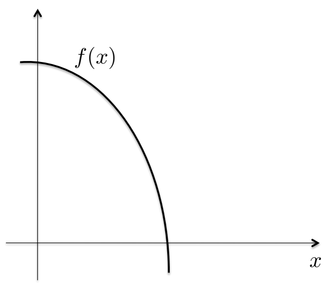
(i) If $f''(x) \geq 0 \;\forall\, x \in (a, b)$ then f is concave upwards,





(ii) If $f''(x) \leq 0 \;\forall\, x \in (a, b)$ then f is concave downwards,





(iii) If $f''(x) > 0 \;\forall\, x \in (a, b)$ then f is strictly concave upwards,

(iv) If $f''(x) < 0 \;\forall\, x \in (a, b)$ then f is strictly concave downwards.

Theorem 4.5.7 Second derivative Test: Let f be defined on an open interval containing point p. Suppose $f'(p) = 0$ and $f''(p)$ exists.

(i) If $f''(p) < 0$ then $f(p)$ is a relative maximum

(ii) If $f''(p) > 0$ then $f(p)$ is relative minimum

(iii) If $f''(p) = 0$ then $f(p)$,the test is not informative, there is no conclusion

Definition 4.5.5 *Let f be differentiable in (a, b), and let $p \in (a, b)$, a point p is a point of inflection for f if*

(i) $f''(p) = 0$, *and*

(ii) at point p, the concavity of a function f switches from up to down or down to up, that is

$$f''(p-\delta)\cdot f''(c+\delta) < 0$$

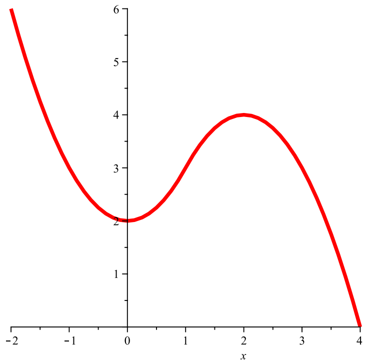
Example 4.5.18 Sketch the graph where

(i) $f(x)$ is concave upwards $[f''(x) > 0]$ where $-1 < x \leq 1$, and $f(x)$ is concave downwards $[f''(x) < 0]$ where $1 < x \leq 3$

(ii) $f''(0) > 0$[`relative minima`], $f''(2) < 0$ [`local maxima`]

(iii) $f'(x) > 0$[`f increasing`], $x \in (0, 2)$ and $f'(x) < 0$[`f decreasing`], $x \in (2, 4)$

The sketch after using the first derivative, second derivative tests and the concavity definitions is as shown [the curve can also be in the negative/lower side]



The graph can even be drawn in the negative/lower side

Example 4.5.19 Given a function

$$f(x) = x^3 - 3x^2$$

(a) The Stationary or critical points

$$f'(x) = 0 \quad\Rightarrow\quad 3x^2 - 6x = 0 \;\Rightarrow\; 3x(x-2) = 0 \;\Rightarrow\; x = 0, x = 2$$

(b) Maxima or Minima.
Since

$$\begin{array}{lcl} f''(x) & = & 6x - 6 \\ x = 0 & & \text{a maxima point } (x, y) = (0, 0) \\ x = 2 & & \text{a minima point } (x, y) = (2, -4) \end{array}$$

Thus $f(x) = f(2) = -4$ is a relative minima since $f''(2) = 6(2) - 6 = 6 > 0$.
Thus $f(x) = f(0) = 0$ is a relative maxima since $f''(0) = 6(0) - 6 = -6 < 0$.

Since $f''(x) = 6x - 6$

$$\begin{aligned} f''(x) &< 0 \text{ if } x < 1. \text{ f is concave down in the interval} \\ f''(x) &> 0 \text{ if } x > 1. \text{ f is concave up in the interval} \end{aligned}$$

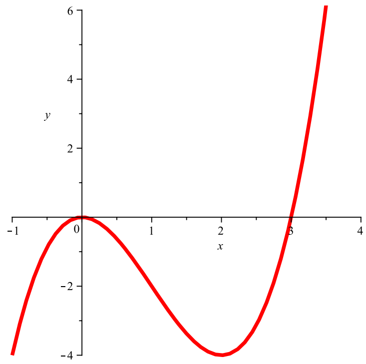


Figure 4.6: A curve $f(x) = x^3 - 3x^2$

Example 4.5.20 Show that for the function $f(x) = x^3$

$$\begin{aligned} x = 0 &: \text{ is a point of inflection} \\ x \in [0, \infty) &: \text{ f is concave up in the interval} \\ x \in (-\infty, 0] &: \text{ f is concave down in the interval} \end{aligned}$$

as could be visualised from the graph

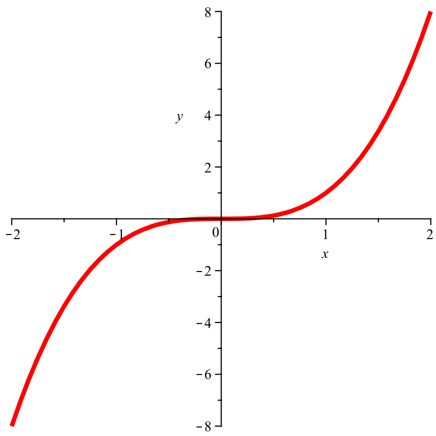


Figure 4.7: A curve $f(x) = x^3$

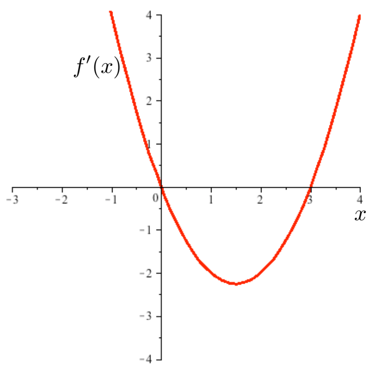
Summary of Derivatives

	First Derivative $f'(x)$	**Second Derivative** $f''(x)$
1.	c *a critical points* $\Rightarrow$ $f'(c) = 0$ or $f'(c)$ DNE *If* $f'(c) = 0$ $\Rightarrow$ c *a critical points*	*Concavity* Concave up: $f''(x) \geq 0,\ x \in (a, b)$ Concave down: $f''(x) \leq 0,\ x \in (a, b)$
2.	*Minima/Maxima* Maxima: $f'(x) > 0\ ;\ x \in (c-\delta, c)$ & $f'(x) < 0\ ;\ x \in (c, c+\delta)$ Minima: $f'(x) < 0\ ;\ x \in (c-\delta, c)$ & $f'(x) > 0\ ;\ x \in (c, c+\delta)$ No conclusion: $f'(x) < 0\ ;\ x \in (c-\delta, c)$ & $f'(x) < 0\ ;\ x \in (c, c+\delta)$ $f'(x) > 0\ ;\ x \in (c-\delta, c)$ & $f'(x) > 0\ ;\ x \in (c, c+\delta)$	*Minima/Maxima* Maxima: $f''(p) < 0$ Minima: $f''(p) > 0$ No conclusion (Points of inflection): $f''(p) = 0$
3.	*Increasing/Decreasing* $f'(x) > 0, x \in I$ $\Rightarrow$ $f(x)$ is increasing $f'(x) < 0, x \in I$ $\Rightarrow$ $f(x)$ is decreasing	

There is always a minima and a maxima in any interval
A point of inflection is not an extremum
If $f(k) = 0$ $\Rightarrow$ k is an x-intercept, where the curve cuts the $x-$axis

Absolute = Global : Domain D (everywhere)
Relative = Local : Interval I

Example 4.5.21 Given a graph of $f'(x)$ below



$f'(c) = 0, \Rightarrow$ $c \equiv$ *Critical points/Turning points/Stationary points*: $c = 0, 3$

$$f'(0-\delta, 0) > 0,\ f'(0, 0+\delta) < 0 \quad \Rightarrow \quad f(0) \texttt{ is a relative maxima}$$
$$f'(3-\delta, 3) < 0,\ f'(3, 3+\delta) > 0 \quad \Rightarrow \quad f(3) \texttt{ is a relative minima}$$

Example 4.5.22 Let g be a function whose derivative g' is continuous and has the graph shown below.

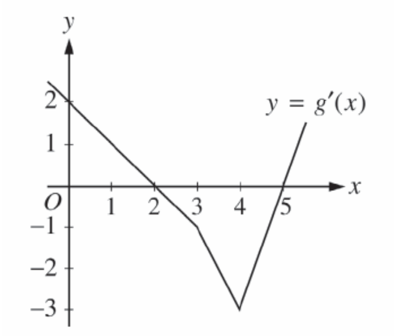


Figure 4.8: Curve of $g'(x)$

(i) State the turning points of $g(x)$? Turning point or critical points are when $g'(x) = 0$, that is at $x = 2$ and $x = 5$

(ii) Which of the following values of g is largest?

(a) $g(1)$ (b) $g(2)$ (c) $g(3)$ (d) $g(4)$ (e) $g(5)$

Its $g(2)$ by first derivative test, a maxima, before $g'(x) > 0$ and after $g'(x) < 0$. At $x = 5$, it is a minima

(iii) Sketch the curve of $g(x)$. Draw any sketch where there is maxima at $x = 2$, and a minima at $x = 5$.

Example 4.5.23 Sketch the graph for the function $f(x)$ on the interval $-4 \leq x \leq 6$ where

(i) $f(-3) = f(-1) = f(5) = 0$

(ii) $f'(-2) = f'(3) = 0$

(iii) $f'(x) < 0, x \in (-4, -2) \cup (3, 6)$ and $f'(x) > 0, x \in (-2, 3)$

(iv) $f''(x) \geq 0, x \in (-4, 2)$ and $f''(x) \leq 0, x \in (2, 6)$

Example 4.5.24 Given

$$f'(x) = \begin{cases} x\ , & x < 0 \\ -x\ , & x \geq 0 \end{cases}$$

Determine the interval on which $f(x)$ is increasing. [Region DNE]

Example 4.5.25 If $f'(x)$ and $g'(x)$ exist and $f'(x) > g'(x)$ for all real x, then the graph of $y = f(x)$ and the graph of $y = g(x)$

(A) intersect exactly once

(B) intersect not more than once

(C) do not intersect

(D) could intersect more than once

Convince your self that the best answer is option (B). You might consider an example of $f = 2x, g = x$

Example 4.5.26 If a function f is continuous for all x and if f has a relative maximum at the point $(-1, 4)$ and a relative minimum at the point $(3, -2)$, which of the following statements must be true

(A) The graph of f has a point of inflection somewhere between $x = -1$ and $x = 3$

(B) $f'(-1) = 0$

(C) The graph of f has a horizontal asymptote

(D) The graph of f has a horizontal line at $x = 3$

(E) The graph of f intersects both axes

The best option is (E) since we have the word "must" in the question. Option (B) is not correct since it might be $f'(-1) = 0$ or $f'(-1)$ DNE as in Defintion (4.5.3)

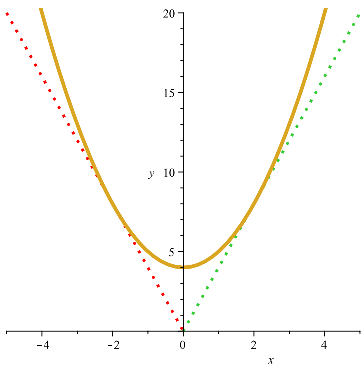
Example 4.5.27 State the definition of a derivative at a point x_0. Hence find $f'(x_0), x_0 > 0$ if $f(x) = \sqrt{x} + 7$.

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{(\sqrt{x} + 7) - (\sqrt{x_0} + 7)}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} \\ &= \frac{1}{2}\sqrt{x_0} \end{aligned}$$

By rationalisation or La'Hopital
To indicate that $x_0 > 0$, because we cannot have a square root of a negative number.

Example 4.5.28 Find the equations of the tangent lines to the curve $f(x) = x^2 + 9$ which pass through the origin $(0, 0)$.

Where (x_0, y_0) is a point on a curve and m is the gradient



$$\begin{aligned} f'(x) &= 2x = 2x_0 \\ \frac{0 - y_0}{0 - x_0} &= 2x_0 \quad \Rightarrow \quad y_0 = 2x_0^2 \\ & \qquad y_0 = x_0^2 + 9 \\ \Rightarrow & \qquad x_0 = 3, y_0 = 18 \end{aligned}$$

$$\frac{y - y_0}{x - x_0} = m$$

$$\frac{y - 0}{x - 0} = 6 \quad \Rightarrow \quad y = 2x \ \& \ y = -2x$$

Exercise 4.25 For the circle $(x - 2)^2 + (y - 2)^2 = 16$, find $\frac{dy}{dx}$. Also find the slope of the horizontal and vertical tangent lines to the circle. At what points do the horizontal tangents touch the circle?

$$\begin{aligned} \texttt{Tangent} &: \frac{dy}{dx} = \frac{2 - x}{y - 2} \\ \texttt{Horizontal} &: \frac{dy}{dx} = 0 \ \Rightarrow \ 2 - x = 0 \ \Rightarrow (x, y) = (2, \pm 4 + 2) \\ \texttt{Vertical} &: \frac{dy}{dx} \ \texttt{undefined} \ \Rightarrow \ y - 2 = 0 \ \Rightarrow (x, y) = (\pm 4 + 2, 2) \end{aligned}$$

Exercise 4.26 Use MVT to prove that:

$$\lim_{x\to\infty}\left(\sqrt{x+2}-\sqrt{x}\right)=0$$

Let $f(t)=\sqrt{t}$ on $[x,x+2]$, then $\exists\ c\in(x,x+2)$ such that

$$\begin{aligned}
\frac{f(x+2)-f(x)}{(x+2)-x}=f'(c) &= \frac{1}{2\sqrt{c}}\\
\Leftrightarrow\ f(x+2)-f(x) &= \frac{1}{2\sqrt{c}}\left[(x+2)-x\right]=\frac{1}{\sqrt{c}}\\
\texttt{but also}\ f(x+2)-f(x) &= \sqrt{x+2}-\sqrt{x}\\
\textit{Thus}\ \lim_{x\to\infty}(\sqrt{x+2}-\sqrt{x}) &= \lim_{c\to\infty}\frac{1}{\sqrt{c}}=0
\end{aligned}$$

Example 4.5.29 Given the function $g(x) = 2x^3 + 12x^2 + 18x + 12$, find stationary points on the curve and determine their nature.

$$\begin{aligned} f'(x) &= 6x^2 + 24x + 18 \\ \Rightarrow \text{ for } f'(x) &= 0 \Rightarrow 6(x^2 + 4x + 3) = 0 \\ \Rightarrow (x+3)(x+1) &= 0 \Rightarrow x = -3 \text{ or } x = -1 \\ f''(x) &= 6(2x+4) = 12(x+2) \\ \text{but } at\ x = -1, f''(x) = 12 > 0. \end{aligned}$$

Hence $x = -1$ is a relative minimum At $x = -3$, $f''(x) = -12 < 0$, hence $x = -3$ is a relative maximum. But when $f''(x) = 0 \ \Rightarrow 12(x+2) = 0 \ \Rightarrow \ x = -2$, therefore $x = -2$ is an inflection point which is a stationary point .

Example 4.5.30 Find the turning points of the function $y = x^3 - 3x + 2$ and distinguish between them.

$$1: \text{ minima }, \quad -1: \text{ maxima}$$

Example 4.5.31 Suppose we wish to find points on the curve $y(x)$ given by

$$y = x^3 - 6x^2 + x + 3$$

where the tangents are parallel to the line $y = x + 5$.

If the tangents have to be parallel to the line then they must have the same gradient. The standard equation for a straight line is $y = mx + c$, where m is the gradient. So what we gain from looking at this standard equation and comparing it with the straight line $y = x+5$ is that the gradient, m, is equal to 1. Thus the gradients of the tangents we are trying to find must also have gradient 1.

We know that if we differentiate $y(x)$ we will obtain an expression for the gradients of the tangents to $y(x)$ and we can set this equal to 1. Differentiating, and setting this equal to 1 we find

$$\frac{dy}{dx} = 3x^2 - 12x + 1 = 1$$

from which

$$3x^2 - 12x = 0$$

This is a quadratic equation which we can solve by factorisation.

$$\begin{aligned} 3x^2 - 12x &= 0 \\ 3x(x-4) &= 0 \end{aligned}$$

$$3x = 0 \qquad \text{or} \qquad x - 4 = 0$$

$$x = 0 \qquad \text{or} \qquad x = 4$$

Now having found these two values of x we can calculate the corresponding y coordinates. We do this from the equation of the curve: $y = x^3 - 6x^2 + x + 3$.

when $x = 0$: $\quad y = 0^3 - 6.0^2 + 0 + 3 = 3$.

when $x = 4$: $\quad y = 4^3 - 6.4^2 + 4 + 3 = 64 - 96 + 4 + 3 = -25$.

So the two points are $(0, 3)$ and $(4, -25)$

These are the two points where the gradients of the tangent are equal to 1, and so where the tangents are parallel to the line that we started out with, i.e. $y = x + 5$.

Exercise 4.27 For each of the functions given below determine the equation of the tangent at the points indicated.

a) $f(x) = 3x^2 - 2x + 4$ at $x = 0$ and 3.

$$y = -2x + 4, \; y = 16x - 23$$

b) $f(x) = 5x^3 + 12x^2 - 7x$ at $x = -1$ and 1.

$$y = -16x - 2, \; y = 32x - 22$$

c) $f(x) = x\mathrm{e}^x$ at $x = 0$.

$$y = x$$

d) $f(x) = \left(x^2 + 1\right)^3$ at $x = -2$ and 1.

$$y = -300x - 475, \; y = 24x - 16$$

e) $f(x) = \sin 2x$ at $x = 0$ and $\frac{\pi}{6}$.

$$y = 2x, \; y = x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

f) $f(x) = 1 - 2x$ at $x = -3$, 0 and 2.

$$y = 1 - 2x, \; y = 1 - 2x, \; y = 1 - 2x$$

Exercise 4.28 Find the equation of each tangent of the function $f(x) = x^3 - 5x^2 + 5x - 4$ which is parallel to the line $y = 2x + 1$.

$$y = 2x - \frac{95}{27}, \; y = 2x - 13$$

Exercise 4.29 Find the equation of each tangent of the function $f(x) = x^3 + x^2 + x + 1$ which is perpendicular to the line $2y + x + 5 = 0$.

$$y = 2x + 2, \; y = 2x + \frac{22}{27}$$

Example 4.5.32 Suppose we wish to find the equation of the tangent and the equation of the normal to the curve

$$y = x + \frac{1}{x}$$

at the point where $x = 2$.

First of all we shall calculate the y coordinate at the point on the curve where $x = 2$:

$$y = 2 + \frac{1}{2} = \frac{5}{2}$$

Next we want the gradient of the curve at the point $x = 2$. We need to find dy/dx.

Noting that we can write y as $y = x + x^{-1}$ then

$$\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

Furthermore, when $x = 2$

$$\frac{dy}{dx} = 1 - \frac{1}{4} = \frac{3}{4}$$

This is the gradient of the tangent to the curve at the point $(2, \frac{5}{2})$. We know that the standard equation for a straight line is

$$\frac{y - y_1}{x - x_1} = m$$

With the given values we have

$$\frac{y - \frac{5}{2}}{x - 2} = \frac{3}{4}$$

Rearranging

$$\begin{aligned} y - \tfrac{5}{2} &= \tfrac{3}{4}(x-2) \\ 4\left(y - \tfrac{5}{2}\right) &= 3(x-2) \\ 4y - 10 &= 3x - 6 \\ 4y &= 3x + 4 \end{aligned}$$

So the equation of the tangent to the curve at the point where $x = 2$ is $4y = 3x + 4$.

Now we need to find the equation of the normal to the curve.

Let the gradient of the normal be m_2. Suppose the gradient of the tangent is m_1. Recall that the normal and the tangent are perpendicular and hence $m_1 m_2 = -1$. We know $m_1 = \frac{3}{4}$. So

$$\frac{3}{4} \times m_2 = -1$$

and so

$$m_2 = -\frac{4}{3}$$

So we know the gradient of the normal and we also know the point on the curve through which it passes, $\left(2, \frac{5}{2}\right)$.

As before,

$$\begin{aligned} \tfrac{y-y_1}{x-x_1} &= m \\ \tfrac{y-\frac{5}{2}}{x-2} &= -\tfrac{4}{3} \end{aligned}$$

Rearranging

$$\begin{aligned} 3\left(y - \tfrac{5}{2}\right) &= -4(x-2) \\ 3y - \tfrac{15}{2} &= -4x + 8 \\ 3y + 4x &= 8 + \tfrac{15}{2} \\ 6y + 8x &= 31 \end{aligned}$$

This is the equation of the normal to the curve at the given point.

Example 4.5.33 Consider the curve $xy = 4$. Suppose we wish to find the equation of the normal at the point $x = 2$. Further, suppose we wish to know where the normal meet the curve again, if it does.

Notice that the equation of the given curve can be written in the alternative form $y = \frac{4}{x}$. A graph of the function $y = \frac{4}{x}$ is shown in Figure (4.9).

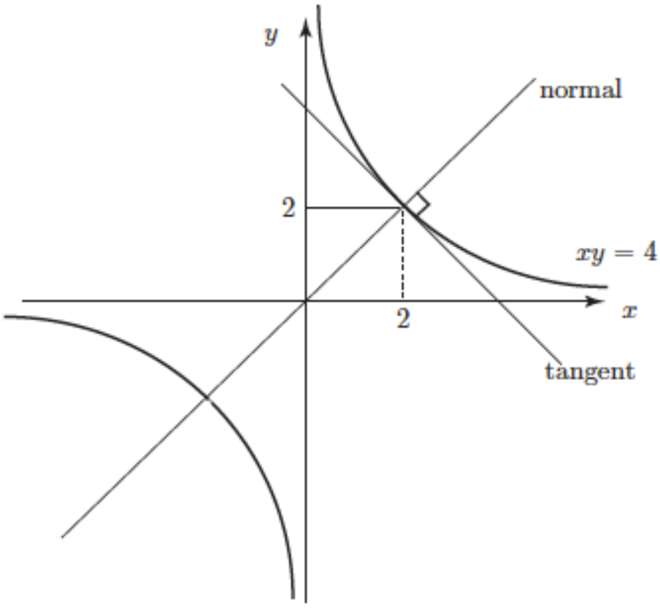


Figure 4.9: A graph of the curve $xy = 4$ showing the tangent and normal at $x = 2$.

From the graph we can see that the normal to the curve when $x = 2$ does indeed meet the curve again (in the third quadrant). We shall determine the point of intersection. Note that when $x = 2$, $y = \frac{4}{2} = 2$.

We first determine the gradient of the tangent at the point $x = 2$. Writing

$$y = \frac{4}{x} = 4x^{-1}$$

and differentiating, we find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -4x^{-2} = -\frac{4}{x^2}$$

Now, when $x = 2$ $\quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{4} = -1$.

So, we have the point $(2, 2)$ and we know the gradient of the tangent there is -1. Remember that the tangent and normal are at right angles and for two lines at right angles the product of their gradients is -1. Therefore we can deduce that the gradient of the normal must be $+1$. So, the normal passes through the point $(2, 2)$ and its gradient is 1.

As before, we use the equation of a straight line in the form:

$$\frac{y - y_1}{x - x_1} = m$$

$$\frac{y - 2}{x - 2} = 1 \quad \Rightarrow \quad y - 2 = x - 2 \quad \Rightarrow \quad y = x$$

So the equation of the normal is $y = x$.

We can now find where the normal intersects the curve $xy = 4$. At any points of intersection both of the equations

$$xy = 4 \qquad \text{and} \qquad y = x$$

are true at the same time, so we solve these equations simultaneously. We can substitute $y = x$ from the equation of the normal into the equation of the curve:

$$\begin{aligned} xy &= 4 \\ x \cdot x &= 4 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

So we have two values of x where the normal intersects the curve. Since $y = x$ the corresponding y values are also 2 and -2. So our two points are $(2, 2)$, $(-2, -2)$. These are the two points where the normal meets the curve. Notice that the first of these is the point we started off with.

Exercise 4.30 For each of the functions given below determine the equations of the tangent and normal at each of the points indicated.

(a) $f(x) = x^2 + 3x + 1$ at $x = 0$ and 4.

At $x = 0$: $y = 3x + 1$, $y = -\frac{1}{3}x + 1$, At $x = 4$: $y = 11x - 15$, $y = -\frac{1}{11}x + \frac{323}{11}$

(b) $f(x) = 2x^3 - 5x + 4$ at $x = -1$ and 1.

At $x = -1$: $y = x + 8$, $y = -x + 6$, At $x = 1$: $y = x$, $y = -x + 2$

Exercise 4.31 Find the equation of each normal of the function $f(x) = \frac{1}{3}x^3 + x^2 + x - \frac{1}{3}$ which is parallel to the line $y = -\frac{1}{4}x + \frac{1}{3}$

Exercise 4.32 Find the x co-ordinate of the point where the normal to $f(x) = x^2 - 3x + 1$ at $x = -1$ intersects the curve again. $\frac{21}{5}$

Exercise 4.33 A total of x feet of fencing is to form three sides of a level rectangular yard. What is the maximum possible area of the yard, in terms of x ?

(a) $\frac{x^2}{9}$ (b) $\frac{x^2}{8}$ (c) $\frac{x^2}{4}$ (d) x^2 (e) $2x^2$

Exercise 4.34 Let f and g be twice-differentiable real-valued functions defined on $\mathbb{R}$. If $f'(x) > g'(x)$ for all $x > 0$, which of the following inequalities must be true for all $x > 0$?

(A) $f(x) > g(x)$

(B) $f''(x) > g''(x)$

(C) $f(x) - f(0) > g(x) - g(0)$

(D) $f'(x) - f'(0) > g'(x) - g'(0)$

(E) $f''(x) - f''(0) > g''(x) - g''(0)$

Exercise 4.35 Find an equation of the line tangent to the graph of $y = x + e^x$ at $x = 0$.

Example 4.5.34 Find two non-negative numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximum.

Let variables x and y represent two non-negative numbers. The sum of the two numbers is given to be $9 = x + y$,so that $y = 9 - x$.

We wish to *maximize* the product

$$P = xy^2 = x(9-x)^2$$

Now differentiate this equation using the product rule and chain rule, getting

$$\begin{aligned} P' &= x(2)(9-x)(-1) + (1)(9-x)^2 \\ &= (9-x)[-2x + (9-x)] \\ &= (9-x)[9-3x] = (9-x)(3)[3-x] = 0 \end{aligned}$$

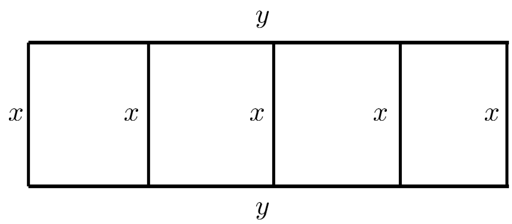
for $x = 9$ or $x = 3$.

Note that since both x and y are non-negative numbers and their sum is 9, it follows that $0 \le x \le 9$.

If $x = 3$ and $y = 6$, then $P = 108$ is the largest possible product.

Example 4.5.35 Jesse is to build a rectangular pen with four parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?

Let variable x be the width of the pen and variable y the length of the pen.



The total amount of fencing is given to be $500 = 5(\text{width}) + 2(\text{length}) = 5x + 2y$, so that $2y = 500 - 5x \;\Rightarrow\; y = 250 - \frac{5}{2}x$

We wish to maximize the total area of the pen

$A = (\text{width})(\text{length}) = xy = x\left(250 - \frac{5}{2}x\right) = 250x - \frac{5}{2}x^2.$

Now differentiate this equation, getting

$$A' = 250 - (5/2)2x = 250 - 5x = 5(50 - x) = 0, \; x = 50$$

Note that since there are 5 lengths of x in this construction and 500 feet of fencing, it follows that $0 \le x \le 100$. For $x = 50$ ft. then $y = 125$ ft.,and $A = 6250\text{ft}^2$ is the largest possible area of the pen.

Exercise 4.36 The daily profit, P, of an oil refinery is given by

$$P = 8x - 0.02x^2,$$

where x is the number of barrels of oil refined. How many barrels will give maximum profit and what is the maximum profit? x = 200, P = $800

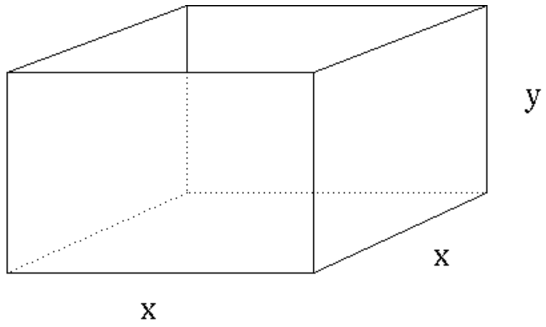
Exercise 4.37 A rectangular storage area is to be constructed along the side of a tall building. A security fence is required along the remaining 3 sides of the area. What is the maximum area that can be enclosed with 800 m of fencing?

$$x = 200, y = 400, \text{ and } A = 80000m^2$$

Exercise 4.38 A rectangular box with a square base and no top is to have a volume of 108 cubic inches. Find the dimensions for the box that require the least amount of material. $(x, y, h) = (6, 6, 3)$

Example 4.5.36 An open rectangular box (no top) with square base is to be made from 48ft^2 of material. What dimensions will result in a box with the largest possible volume?

Let the sizes of the square base be x, and the height h.



The total surface area S of the box is given to be

$$\begin{aligned} S &= x^2 + 4(xh) \\ 48 &= x^2 + 4(xh) \end{aligned}$$

$$\Rightarrow \quad h = \frac{48 - x^2}{4x} = \frac{48}{4x} - \frac{x^2}{4x} = \frac{12}{x} - \frac{1}{4}x$$

The total volume V of the box is given by

$$V = x^2 h = x^2 \left(\frac{12}{x} - \frac{1}{4}x\right) = 12x - \frac{1}{4}x^3$$

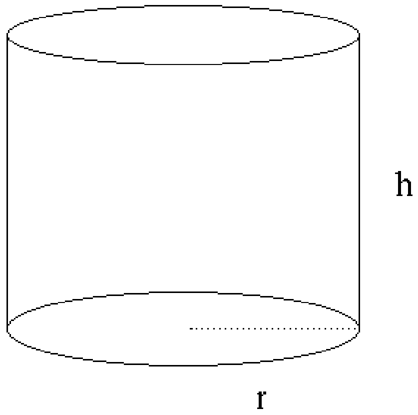
Maximizing the total volume V of the box, getting extremas

$$\begin{aligned} V' &= 0 \\ 12 - \frac{3}{4}x^2 = \frac{3}{4}(16 - x^2) = \frac{3}{4}(4 - x)(4 + x) &= 0 \\ \Rightarrow \qquad & x = 4, -4 \end{aligned}$$

But $x \neq -4$ since variable x measures a distance and $x > 0$.
Thus to maximise the volume, $x = 4$ Or we can use second derivative test, to determine which x value gives the maximum volume.

Example 4.5.37 A container in the shape of a right circular cylinder with no top has surface area $3\pi\text{ft}^2$. What height h and base radius r will maximize the volume of the cylinder ?

Let variable r be the radius of the circular base and variable h the height of the cylinder.



The total surface area of the cylinder is given to be

$$3\pi = (\text{area of base}) + (\text{area of the curved side}) = \pi r^2 + (2\pi r)h$$

so that $h = \dfrac{3\pi - \pi r^2}{2\pi r} = \dfrac{3}{2r} - \dfrac{1}{2}r$

We wish to maximize the total volume of the cylinder

$$V = \pi r^2 h = \pi r^2\left(\frac{3}{2r} - \frac{1}{2}r\right) = \frac{3}{2}\pi r - \frac{1}{2}\pi r^3$$

Now differentiate this equation, getting

$$V' = (3/2)\pi - (1/2)\pi 3r^2 = (3/2)\pi(1 - r^2) = (3/2)\pi(1 - r)(1 + r) = 0$$

$r = 1$ or $r = -1$. But $r \neq -1$ since variable r measures a distance and $r > 0$. Since the base of the box is a circle and there are $3\pi\text{ft}^2$ of material, it follows that $0 < r \leq \sqrt{3}$.

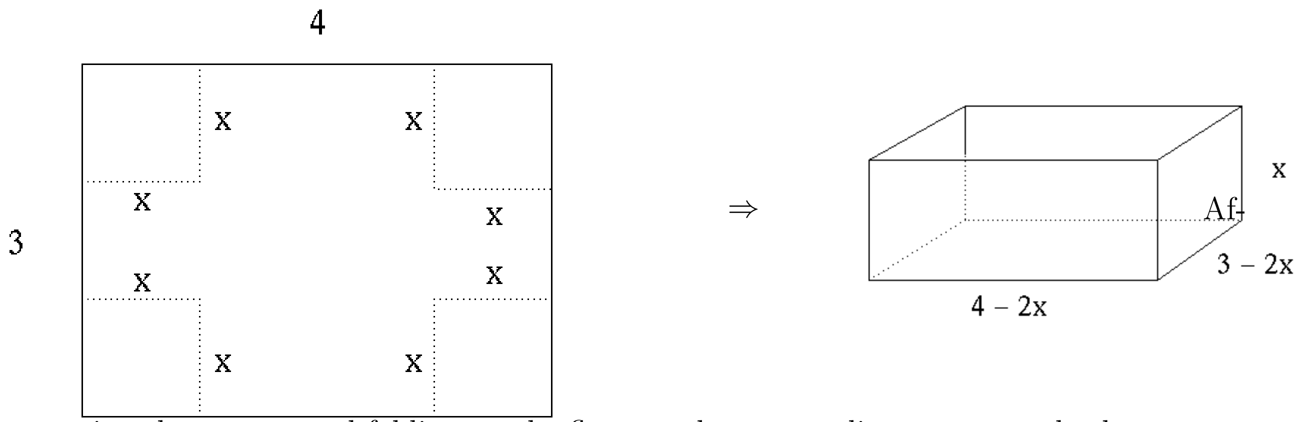
Alternatively,

$$V'' = \frac{3}{2}\pi(-2r) \;\Rightarrow\; V''_{r=-1} = 3\pi > 0 \text{ a minima, } V''_{r=1} = -3\pi < 0 \text{ a maxima}$$

For $r = 1$ft and $h = 1$ft,then $V = \pi\text{ft}^3$ is the largest possible volume of the cylinder.

Example 4.5.38 A sheet of cardboard 3ft by 4ft will be made into a box by cutting equal-sized squares from each corner and folding up the four edges. What will be the dimensions of the box with largest volume?

Let variable x be the length of one edge of the square cut from each corner of the sheet of cardboard.



Af-ter removing the corners and folding up the flaps, we have an ordinary rectangular box.

We wish to maximize the total volume of the box
V = (length) (width) (height) = $(4-2x)(3-2x)(x)$. Now differentiate this equation using the triple product rule, getting

$$\begin{aligned} V' &= (-2)(3-2x)(x)+(4-2x)(-2)(x)+(4-2x)(3-2x)(1) \\ &= 12x^2-28x+12 \\ &= 4(3x2-7x+3) \\ &= 0 \end{aligned}$$

for (Use the quadratic formula).

$$x = \frac{-(-7)\pm\sqrt{(-7)^2-4(3)(3)}}{2(3)} = \frac{7\pm\sqrt{13}}{6} \approx 0.57 \text{ or } x \approx 1.77$$

For $x \approx 0.57$ft (use second derivative to show for maxima), then

$$V \approx 3.03\text{ft}^3$$

is largest possible volume of the box.

Example 4.5.39 Find the maximum and minimum value of $A(x) = |2x|$ on the interval $[-1, 6]$.

The graph of $A(x)$ is

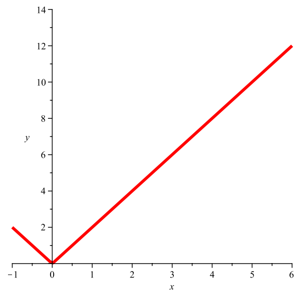


Figure 4.10: A curve of $A(x) = |2x|$ on [-1,6]

$A'(x)$ is undefined at $x = 0$ but $A(0) = 0$. Therefore $x = 0$ is a critical point. the minimum occurs at this point, $x = 0$ and is equal to zero. $A(-1) = 2$ and $A(6) = 12$. Therefore the maximum occurs at the endpoint, $x = 6$ and is equal to 12. Since

$$A(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases}$$

by definition of absolute value.

Example 4.5.40 Suppose $f(x) = x^3 - 3x^2 + x - 2$. Let's determine where the graph of f is concave up and where it is concave down. Since f is twice-differentiable for all x, we use the result given above and first determine that

$$f''(x) = 6(x-1)$$

Thus, $f''(x) > 0$ if $x > 1$ and $f(x) < 0$ if $x < 1$. By the Concavity Theorem, the graph of f is concave up for $x > 1$ and concave down for $x < 1$.

Example 4.5.41 An investor has 100 houses. When the rent of every house is \$80 per month, all houses are occupied. However, for every \$4 increase in rent, one house becomes vacant. Each occupied unit requires a monthly average of \$8 for repairs. If there are no other expenses, what rent should be charged to make the most profit?

Let the number of vacant houses be x. Let the new income (revenue $R(x)$) and cost (Expenditure $E(x)$) and Profits $P(x)$ are

$$\begin{aligned} R(x) &= \text{Price} \times \text{Quantity} = (80+4x)(100-x) \\ E(x) &= \text{cost} \times \text{Quantity} = (8)(100-x) \\ P(x) &= R(x) - E(x) = (80+4x)(100-x) - 8(100-x) \\ &= (100-x)(72+4x) \\ \frac{dP}{dx} &= 4(100-x) - (72+4x) = 0 \quad \Rightarrow \quad x = 41 \\ \frac{d^2P}{dx^2} &= -8 < 0 \end{aligned}$$

at $x = 41$. So $P(x)$ is maximum when $x = 41$. Thus, the maximum rent is $R(41) = 80 + 164 = 244$. That is, the profit is maximum when the monthly rent is \$244 with 59 units occupied.

Example 4.5.42 During Christmas time Jackie makes and sells necklaces on the beach. Last Christmas she sold the necklaces for \$10 each and her sales averaged 20 per day. When she increased the price by \$1, she found that the average decreased by two sales per day. If the material of each necklace costs Jackie \$6, what should the selling price be to maximize her profit?
Let unsold number of necklaces be x,

$$\begin{aligned} R(x) &= \text{Price} \times \text{Quantity} = (10+x)(20-2x) \\ E(x) &= \text{cost} \times \text{Quantity} = (6)(20-2x) \\ P(x) &= R(x) - E(x) \\ &= (10+x)(20-2x) - 6(20-2x) \end{aligned}$$

$$\frac{dP}{dx} = 12 - 4x = 0 \quad \Rightarrow \quad x = 3$$

$$\frac{d^2P}{dx^2} = -4 < 0$$

at $x = 3$. So $P(x)$ is maximum when $x = 3$. That is, the profit is maximum when $x = 3$.

Example 4.5.43 Given an equation $y = f(x) = x^2$, find the shortest distance between the parabola and the point $(6, 3)$.

Let D denote the distance between the parabola and the point $(6, 3)$. If (x, y) is a point on the parabola, then

$$\begin{aligned} D^2 &= (x-6)^2 + (y-3)^2, \ y = x^2 \\ &= (x-6)^2 + (x^2-3)^2 \\ 2D\frac{dD}{dx} &= 2(x-6) + 2(x^2-3)(2x) \\ &= 2(x-6) + 4x(x^2-3) \\ \Leftrightarrow \quad D\frac{dD}{dx} &= 2x^3 - 5x - 6 \end{aligned}$$

So $dD/dx = 0 \ \Rightarrow \ 2x^3 - 5x - 6 = 0 \ \Leftrightarrow \ x = 2$ is a critical point of D. Now,

$$\begin{aligned} D\frac{dD}{dx} = 2x^3 - 5x - 6 \quad &\Rightarrow \quad \frac{dD}{dx} = \frac{2x^3 - 5x - 6}{D} \\ \frac{d^2D}{dx^2} = \frac{D(6x^2-5) - (2x^3 - 5x - 6)\frac{dD}{dx}}{D^2} \quad &\Rightarrow \quad \frac{d^2D}{dx^2} = \frac{19}{\sqrt{17}} > 0 \end{aligned}$$

at $x = 2$.
Thus, D is minimum when $x = 2$ and $y = 4$, $D^2 = (-4)^2 + (1)^2 = 17 \ \Rightarrow \ \min D = \sqrt{17}$

4.5.4 Approximation of functions and Rates of change

From the definition of a derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ f'(x_0) &\approx \frac{f(x + \Delta x) - f(x_0)}{\Delta x} \end{aligned}$$

$$\boxed{f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x} \tag{4.12}$$

Where

$$x = x_0 + \Delta x$$

Example 4.5.44 Use differentials to approximate $\sqrt{65}$

$$\begin{aligned} f(x) &= \sqrt{x} \\ f'(x) &= \frac{1}{2\sqrt{x}} \\ f(x) &= \sqrt{65} = \sqrt{64+1} \end{aligned}$$

$$\begin{aligned} f(x_0 + \Delta x) &\approx f(x_0) + f'(x_0)\Delta x \\ \sqrt{64+1} &\approx \sqrt{64} + f'(64)(1) \\ \sqrt{65} &\approx \sqrt{64} + \frac{1}{2\sqrt{64}}(1) \\ \sqrt{65} &\approx \sqrt{64} + \frac{1}{16}(1) \\ \sqrt{65} &\approx 8 + 0.0625 \\ &\approx 8.0625 \end{aligned}$$

Example 4.5.45 Use differentials to approximate $e^{0.1}$

$$\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f(x) &= e^{0.1} = e^{0+0.1} \end{aligned}$$

$$\begin{aligned} f(x_0 + \Delta x) &\approx f(x_0) + f'(x_0)\Delta x \\ e^{0.1} &\approx e^0 + (0.1)f'(0) \\ e^{0.1} &\approx e^0 + (0.1)e^0 \\ e^{0.1} &\approx 1 + 0.1 \\ &\approx 1.1 \end{aligned}$$

Exercise 4.39 Why is approximating $e^{0.5}$ give a more divergent answer from the exact one?

Example 4.5.46 Use differentials to find the value of $(0.96)^3$

$$\begin{aligned} f(x) &= x^3 \\ f'(x) &= 3x^2 \\ f(x) &= (0.96)^3 = (1-0.04)^3 \end{aligned}$$

$$\begin{aligned} f(x_0+\Delta x) &\approx f(x_0) + f'(x_0)\Delta x \\ (1-0.04)^3 &\approx (1)^3 + f'(1)(-0.04) \\ (0.96)^3 &\approx (1)^3 + 3(1)^2(-0.04) \\ (0.96)^3 &\approx 1 - 0.12 \\ &\approx 0.88 \end{aligned}$$

Example 4.5.47 Find $\sqrt{36.01}$ using differentials

$$\begin{aligned} f(x) &= \sqrt{x} \\ f'(x) &= \frac{1}{2\sqrt{x}} \\ f(x) &= \sqrt{36.01} = \sqrt{36+0.01} \end{aligned}$$

$$\begin{aligned} f(x_0+\Delta x) &\approx f(x_0) + f'(x_0)\Delta x \\ \sqrt{36+0.01} &\approx \sqrt{36} + f'(36)(0.01) \\ \sqrt{36.01} &\approx \sqrt{36} + \frac{1}{2\sqrt{36}}(0.01) \\ \sqrt{36.01} &\approx \sqrt{36} + \frac{1}{12}(0.01) \\ \sqrt{36.01} &\approx 6 + 0.0008333 \\ &\approx 6.0008333 \end{aligned}$$

Example 4.5.48 Show that

$$\sqrt{82} = 9 + \frac{1}{18}$$

using differentials

Example 4.5.49 Find $\sin 42^\circ$ using differentials.

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f(x) &= \sin 42 = \sin(45^\circ - 3^\circ) \end{aligned}$$

$$\Delta x = -3^\circ = \frac{-3\pi}{180} = \frac{-\pi}{60}$$

into radians, since there will be no trigonometric function on Δx, we change from degrees to radians.

$$\begin{aligned} f(x_0+\Delta x) &\approx f(x_0) + f'(x_0)\Delta x \\ \sin(45-3) &\approx \sin 45 + f'(45^\circ)\left(\frac{-\pi}{60}\right) \\ \sin 42 &\approx \sin 45 - \cos 45\left(\frac{\pi}{60}\right) \\ \sin 42 &\approx \sin 45 - \cos 45\left(\frac{\pi}{60}\right) \\ \sin 42 &\approx \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\frac{\pi}{60}\right) \\ &\approx 0.6701 \end{aligned}$$

Example 4.5.50 Show that

$$\cos 62^\circ = 0.46977$$

using differentials (linear approximations)

Example 4.5.51 Show that

$$\cos 42^\circ = 0.74314$$

using linear approximations

Example 4.5.52 Approximate $\sqrt[3]{124}$ using differentials (without using calculators)

$$\begin{aligned} f(x) &= x^{\frac{1}{3}} \\ f'(x) &= \frac{1}{3x^{\frac{2}{3}}} \\ f(x) &= (124)^{\frac{1}{3}} = (125-1)^{\frac{1}{3}} \end{aligned}$$

$$\begin{aligned} f(x_0 + \Delta x) &\approx f(x_0) + f'(x_0)\Delta x \\ (125-1)^{\frac{1}{3}} &\approx (125)^{\frac{1}{3}} + f'(125)(-1) \\ (124)^{\frac{1}{3}} &\approx (125)^{\frac{1}{3}} + \frac{1}{3(125)^{\frac{2}{3}}}(-1) \\ (124)^{\frac{1}{3}} &\approx (125)^{\frac{1}{3}} - \frac{1}{75}(1) \\ (124)^{\frac{1}{3}} &\approx 5 - \frac{1}{75} \\ &\approx 4.9867 \end{aligned}$$

Example 4.5.53 The position of a particle is given by the equation

$$s(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in metres.

(a) Find the velocity at time t.

(b) What is the velocity after 5 seconds?

(c) When is the particle at rest?

Example 4.5.54 Air is being pumped into a spherical balloon such that its radius increases at a rate of 0.75 in/min. Find the rate of change of its volume when the radius is 5 inches ($V = \frac{4}{3}\pi r^3$).

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dr}\frac{dr}{dt} \\ &= 4\pi r^2 \frac{dr}{dt} \\ &= (4\pi r^2)(0.75) = (4\pi 5^2)(0.75) \end{aligned}$$

Example 4.5.55 Gas is being pumped into a spherical balloon at the rate of 2 cm^3 per second. How fast is the surface area of the balloon increasing when the radius is 12 cm.

$$\begin{aligned}\frac{dS}{dt} &= \frac{dS}{dr}\frac{dr}{dt}\\ \frac{dS}{dt} &= \frac{dS}{dr}\frac{dV}{dt}\frac{dr}{dV}\end{aligned}$$

But a sphere of radius r has volume $V = \frac{4}{3}\pi r^3$ and surface area $S = 4\pi r^2$. Thus

$$\begin{aligned}\frac{dV}{dr} &= 4\pi r^2\\ \frac{dS}{dr} &= 8\pi r\\ \frac{dV}{dt} &= 2\end{aligned}$$

$$\begin{aligned}\frac{dS}{dt} &= \frac{dS}{dr}\frac{dV}{dt}\frac{dr}{dV}\\ \frac{dS}{dt} &= (8\pi r)(2)\frac{1}{4\pi r^2} = \frac{4}{r}\end{aligned}$$

So when $r = 12$,

$$\frac{dS}{dt} = \frac{4}{r} = \frac{4}{12} = \frac{1}{3}\ cm^2$$

Example 4.5.56 Pressure (P) and volume (V) of air at room temperature are related by the equation

$$PV^{1.4} = C$$

Here C is a constant. At some instant t_0 the pressure of the gas is 25 kg/cm^2 and the volume is 200 cm^3. Find the rate of change of P if the volume increases at a rate of 10 cm^3/min.

$$\frac{dP}{dt} = \frac{dP}{dV}\cdot\frac{dV}{dt} = -\frac{1.4P}{V}\cdot\frac{dV}{dt} = -1.75\ Kg/cm^2sec$$

Since

$$\frac{dP}{dV}V^{1.4} + 1.4PV^{0.4} = 0$$

Exercise 4.40 Use differentials to approximate

(i) $\sqrt{64.2}$

(ii) $\cos 91°$

(iii) $(0.83)^4$

(iv) $\sqrt{51}$

Exercise 4.41 Find the differential

(i) $\alpha x + \beta$

(ii) $\frac{\sqrt{x-1}}{\sqrt{x+1}}$

(iii) $\sin\sqrt{x}$

Exercise 4.42 The radius of a sphere in creases from 10cm to 10.5cm. Use differentials to approximate the relative and percentage change in its volume.

Exercise 4.43 Use derivatives to estimate $\sqrt{24}$. [4.9]

Exercise 4.44 Estimate $\sqrt[3]{29}$ without using a calculator.

Exercise 4.45 Suppose the radius of a ball changes at a rate of 2 cm/min. At which rate does its volume change when $r = 20\ cm$?

$$3200\pi\ cm^3/min$$

Exercise 4.46 Suppose that a mountain climber ascends at a rate of 0.5 kilometer per hour. The temperature is lower at higher elevations; suppose the rate by which it decreases is $6°C$ per kilometer. To calculate the decrease in air temperature per hour that the climber experiences, one multiplies $6°C$ per kilometer by 0.5 kilometer per hour, to obtain $3°C$ per hour. This calculation is a typical chain rule application.

Exercise 4.47 State Intermediate Value Theorem and use it to show that the equation

$$2x^3 - 3x^2 = 12x + 6$$

has only one real root in the interval $(-1, 0)$.

Exercise 4.48 State the Mean Value Theorem (MVT) and the Roll's theorem. Hence find the constant c that satisfys the

(i) MVT for $f(x) = x^3 - 6x^2 + 9x + 2\ \ ; 0 \leq x \leq 4$

(ii) Roll's theorem for $f(x) = 4 - x^2$ on $[-2, 2]$

Exercise 4.49 Of the following, which is the best approximation of $\sqrt{1.5}\ (266)^{\frac{3}{2}}$

(a) $1,000$ (b) $2,700$ (c) $3,200$ (d) $4,100$ (e) $5,300$

Exercise 4.50 Find the linear approximate value of $y = \sqrt{4 + \sin x}$ at $x = 0.12$ obtained from the tangent to the graph at $x = 0$.

4.5.5 Curve Sketching

Among the steps involved in curve sketching are;

(i) x and y intercepts

(ii) where curve is increasing or decreasing

(iii) and classify the critical numbers

(iv) where function is concave up and concave down

(v) inflection points

(vi) horizontal, oblique and vertical asymptotes

Note 4.5.6 An oblique asymptote is an asymptote defined by a function.

Note 4.5.7 If you divide the polynomial into quotient and remainder, and then the linear part will be your oblique asymptote.

Note 4.5.8 A horizontal asymptote is a horizontal line and an oblique asymptote is a line that is neither horizontal or vertical. In simple terms, it is slanted.

Example 4.5.57 Find the oblique asymptote of

$$f(x) = \frac{(x^2 + 2x + 2)}{(x + 1)}$$

The $f(x)$ can be rewritten as

$$f(x) = (x + 1) + \left[\frac{1}{(x + 1)}\right]$$

Hence the line $y = x + 1$ is the oblique asymptote.

Example 4.5.58 Find the slant asymptote of the following function:

$$y = \frac{x^2 + 3x + 2}{x - 2}$$

The slant asymptote is $y = x + 5$ since

$$y = \frac{x^2 + 3x + 2}{x - 2} = (x + 5) + \frac{12}{x - 2}$$

Example 4.5.59 Sketch the curve

$$y = x^2 - 4$$

(i) x and y intercepts (where curve cuts the axes). i.e

$$\begin{aligned} f(x) = y = 0 \;&\Rightarrow\; (x^2-4) = 0 \;\Rightarrow\; (x-2)(x+2) = 0 \;\Rightarrow\; x = 2, x = -2 \\ x = 0 \;&\Rightarrow\; y = f(x) = 0^2 - 4 \;\Rightarrow\; y = -4 \end{aligned}$$

(ii) where curve is increasing $f'(x) > 0$ or decreasing $f'(x) < 0$.

$$f'(x) = \frac{dy}{dx} = 2x$$

Increasing:

$$f'(x) = 2x > 0 \;\Leftrightarrow\; x > 0$$

Decreasing:

$$f'(x) = 2x < 0 \;\Leftrightarrow\; x < 0$$

(iii) and classify the critical numbers (turning points)

$$\begin{aligned} f'(x) = \frac{dy}{dx} = 0 \;&\Rightarrow\; 2x = 0 \\ &\Rightarrow\; 2x = 0 \;\Rightarrow\; x = 0 \end{aligned}$$

$$f''(x) = \frac{d^2y}{dx^2} = 2$$

$x = 0$ is a local minima, since; $f''(0) = 2 > 0$

(iv) where the function is concave up $f''(x) > 0$ and concave down $f''(x) < 0$.

$$f''(x) = 2$$

concave up

$$\begin{aligned} f''(x) = 2 \;&>\; 0 \;\forall\; x \\ \Rightarrow &\quad x \in (-\infty, +\infty) \\ \Rightarrow &\quad x \in \Re \end{aligned}$$

concave down

$$\begin{aligned} f''(x) = 2 \;&\not<\; 0 \;\forall\; x \\ \Rightarrow &\quad x \notin (-\infty, +\infty) \\ \Rightarrow &\quad x \notin \Re \end{aligned}$$

The curve is only concave up.

(v) Inflection points
An inflection point p is where the concavity of a function f switches from up to down or down to up. That is where $f''(x) = 0$

$$\begin{aligned} f''(x) = 2 &= 0 \\ \Leftrightarrow 2 &= 0 \\ \Leftrightarrow &\quad \text{a contradiction} \end{aligned}$$

The function $f(x) = x^2 - 4$ has no point of inflection.

(vi) horizontal, oblique and vertical asymptotes:
There is no real horizontal asymptotes since

$$\begin{aligned} \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^2 - 4 = & \quad \infty \\ \& \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^2 - 4 \quad = & \quad -\infty \end{aligned}$$

For vertical asymptotes by computing one-sided limits at the zeroes of the denominator, i.e., No vertical asymptote as no denominator.

Thus the curve is

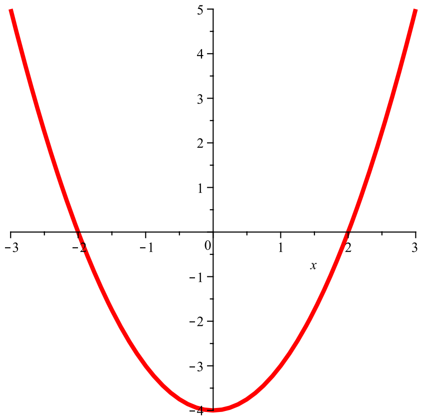


Figure 4.11: Curve $y = x^2 - 4$

Example 4.5.60 Given the function

$$f(x) = \frac{2x^2}{x^2 - 1}$$

find

(i) x and y intercepts

(ii) where curve is increasing or decreasing

(iii) and classify the critical numbers (turning points)

(iv) where function is concave up and concave down

(v) inflection points

(vi) horizontal, oblique and vertical asymptotes

Hence sketch the curve

$$f(x) = \frac{2x^2}{x^2 - 1}$$

(i) x and y intercepts (where curve cuts the axes). i.e

$$f(x) = y = 0 \;\Rightarrow\; \frac{2x^2}{x^2 - 1} = 0 \;\Rightarrow\; 2x^2 = 0 \;\Rightarrow\; x = 0$$
$$x = 0 \;\Rightarrow\; f(x) = \frac{2x^2}{x^2 - 1} \;\Rightarrow\; f(x) = y = 0$$

(ii) where curve is increasing $f'(x) > 0$ or decreasing $f'(x) < 0$.

$$f'(x) = \frac{dy}{dx} = \frac{4x(x^2 - 1) - (2x)(2x^2)}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Increasing:

$$f'(x) = \frac{-4x}{(x^2 - 1)^2} > 0 \;\Leftrightarrow\; x < 0$$

Decreasing:

$$f'(x) = \frac{-4x}{(x^2 - 1)^2} < 0 \;\Leftrightarrow\; x > 0$$

(iii) and classify the critical numbers (turning points)

$$f'(x) = \frac{dy}{dx} = 0 \;\Rightarrow\; \frac{-4x}{(x^2 - 1)^2} = 0$$
$$\Rightarrow\; -4x = 0 \;\Rightarrow\; x = 0$$

$$f''(x) = \frac{d^2y}{dx^2} = \frac{-4(x^2 - 1)^2 + 4x \cdot 2(x^2 - 1) \cdot 2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

$x = 0$ is a local maxima, since; $f''(0) = -4 < 0$

(iv) where function is concave up $f''(x) > 0$ and concave down $f''(x) < 0$.

$$f''(x) = \frac{12x^2+4}{(x^2-1)^3}$$

concave up

$$\begin{aligned} f''(x) = \frac{12x^2+4}{(x^2-1)^3} &> 0 \\ \Leftrightarrow (x^2-1)^3 &> 0 \\ \Leftrightarrow (x^2-1) &> 0 \\ \Leftrightarrow x^2 &> 1 \\ \Rightarrow \quad x &\in (-\infty,-1)\cup(1,+\infty) \end{aligned}$$

concave down

$$\begin{aligned} f''(x) = \frac{12x^2+4}{(x^2-1)^3} &< 0 \\ \Leftrightarrow (x^2-1)^3 &< 0 \\ \Leftrightarrow (x^2-1) &< 0 \\ \Leftrightarrow x^2 &< 1 \\ \Rightarrow \quad x &\in (-1,1) \end{aligned}$$

(v) Inflection points
An inflection point p is where the concavity of a function f switches from up to down or down to up. That is where $f''(x) = 0$

$$\begin{aligned} f''(x) = \frac{12x^2+4}{(x^2-1)^3} = 0 \quad \Leftrightarrow \quad & 12x^2+4 = 0 \\ \Leftrightarrow x^2 &= -\frac{1}{3} \end{aligned}$$

The function $f(x) = \frac{2x^2}{x^2-1}$ has no real point of inflection.

(vi) horizontal, oblique and vertical asymptotes:
There is a horizontal asymptote $y = 2$ since

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{2x^2}{x^2-1} = 2 \quad \& \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x^2}{x^2-1} = 2$$

In this case, the oblique (slant) asymptote is also the horizontal asymptote $y = 2$. For vertical asymptotes by computing one-sided limits at the zeroes of the denominator, i.e.,

$$x^2 - 1 = 0$$

at $x = -1$ and at $x = 1$.

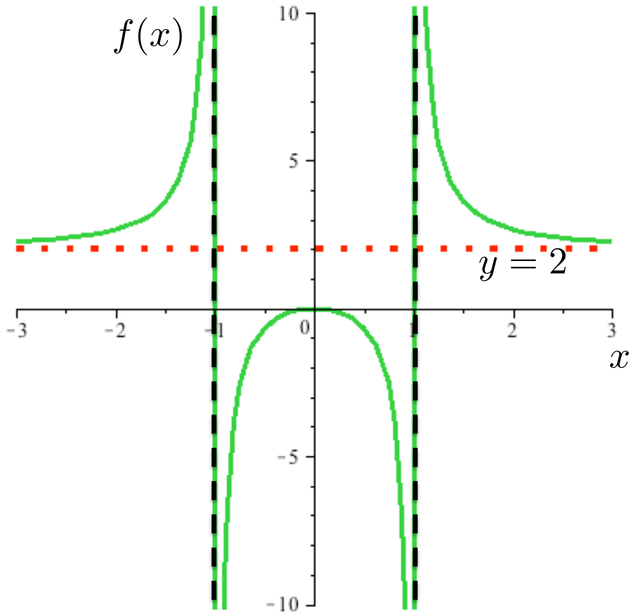


Figure 4.12: A curve $f(x) = \frac{2x^2}{x^2-1}$

Example 4.5.61 Sketch the curve $y = x^3 - 3x^2 - 13x + 15$
Can you realise that

(i) x and y intercepts (where curve cuts the axes). i.e

$$\begin{aligned} f(x) = y = 0 \;&\Rightarrow\; (x-1)(x-5)(x+3) = 0 \;\Rightarrow\; x = 1, x = 5, x = -3 \\ x = 0 \;&\Rightarrow\; f(x) = x^3 - 3x^2 - 13x + 15 \;\Rightarrow\; f(x) = y = 15 \end{aligned}$$

(ii) where curve is increasing $f'(x) > 0$ or decreasing $f'(x) < 0$.

$$f'(x) = \frac{dy}{dx} = 3x^2 - 6x - 13$$

Increasing:

$$\begin{aligned} f'(x) = 3x^2 - 6x - 13 &> 0 \\ \Leftrightarrow \quad & x < -1.31, x > 3.31 \\ \Rightarrow \quad & x \in (-\infty, -1.31) \cup (3.31, +\infty) \end{aligned}$$

Decreasing:

$$\begin{aligned} f'(x) = 3x^2 - 6x - 13 &< 0 \\ \Leftrightarrow \quad & x > -1.31, x < 3.31 \\ \Rightarrow \quad & x \in (-1.31,\ 3.31) \end{aligned}$$

(iii) and classify the critical numbers (turning points)

$$\begin{aligned} f'(x) = \frac{dy}{dx} = 0 \;&\Rightarrow\; 3x^2 - 6x - 13 = 0 \\ &\Rightarrow\; x = \frac{6 \pm \sqrt{6^2 - 4(3)(-13)}}{2(3)} \\ \Rightarrow x &= -1.31,\ 3.31 \end{aligned}$$

$$f''(x) = \frac{d^2y}{dx^2} \quad = \quad 6x - 6$$

$x = -1.31$ is a local maxima, since

$$f''(-1.31) = -13.86 < 0$$

$x = 3.31$ is a local minima, since

$$f''(3.31) = 13.86 > 0$$

(iv) where function is concave up $f''(x) > 0$ and concave down $f''(x) < 0$.

$$f''(x) \quad = \quad 6x - 6$$

concave up

$$\begin{aligned} f''(x) = 6x - 6 \quad &> \quad 0 \\ \Leftrightarrow \; 6(x-1) \quad &> \quad 0 \quad \Leftrightarrow \; x > 1 \; \Leftrightarrow \; x \; \in \; (1, +\infty) \end{aligned}$$

concave down

$$\begin{aligned} f''(x) = 6x - 6 \quad &< \quad 0 \\ \Leftrightarrow \; 6(x-1) \quad &< \quad 0 \quad \Leftrightarrow \; x < 1 \; \Leftrightarrow \; x \; \in \; (-\infty, 1) \end{aligned}$$

(v) Inflection points
An inflection point p is where the concavity of a function f switches from up to down or down to up. That is where $f''(x) = 0$

$$f''(x) = 6x - 6 \quad = \quad 0 \; \Leftrightarrow \; x = 1$$

The function $f(x)$ has a real point of inflection at $x = 1$.

(vi) horizontal, oblique and vertical asymptotes:
There are no horizontal asymptotes since

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (x^3 - 3x^2 - 13x + 15) = \infty \quad \& \quad \lim_{x \to -\infty} f(x) = -\infty$$

No vertical asymptotes since not a rational function

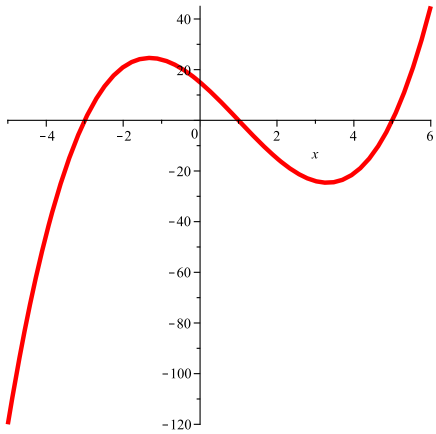


Figure 4.13: Curve $y = x^3 - 3x^2 - 13x + 15$

Example 4.5.62 Graph the curve

$$f(x) = x^3 - 3x^2$$

(i) x and y intercepts (where curve cuts the axes). i.e

$$\begin{array}{lllllll} y = 0 & \Rightarrow & x^3 - 3x^2 = 0 & \Rightarrow & x^2(x-3) = 0 & \Rightarrow & x = 0, x = 3 \\ x = 0 & \Rightarrow & f(x) = x^3 - 3x^2 & \Rightarrow & f(x) = y = 0 & & \end{array}$$

(ii) where the curve is increasing $f'(x) > 0$ or decreasing $f'(x) < 0$.

$$\begin{aligned} f'(x) = \frac{dy}{dx} &= 3x^2 - 6x \\ &= 3x(x-2) \end{aligned}$$

Increasing:

$$f'(x) = 3x(x-2) > 0 \quad \Leftrightarrow \quad \begin{array}{c} x > 0 \;\&\; x > 2 \\ \text{or} \\ x < 0 \;\&\; x < 2 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} x > 2 \\ \text{or} \\ x < 0 \end{array}$$

Decreasing:

$$f'(x) = 3x(x-2) < 0 \quad \Leftrightarrow \quad \begin{array}{c} x > 0 \;\&\; x < 2 \\ \text{or} \\ x < 0 \;\&\; x > 2 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} x \in (0,2) \\ \text{or} \\ x \text{ DNE} \end{array}$$

(iii) and classify the critical numbers (turning points)

$$\begin{aligned} f'(x) = \frac{dy}{dx} = 0 \quad &\Rightarrow \quad 3x(x-2) = 0 \\ &\Rightarrow \quad x = 0, x = 2 \end{aligned}$$

$$f''(x) = \frac{d^2y}{dx^2} \quad = \quad 6x - 6$$

$x = 0$ is a local maxima, since; $f''(0) = -6 < 0$
$x = 2$ is a local minima, since; $f''(2) = 6 > 0$

(iv) where function is concave up $f''(x) > 0$ and concave down $f''(x) < 0$.

$$f''(x) = 6x - 6$$

concave up

$$\begin{aligned} f''(x) = 6x - 6 &> 0 \\ \Leftrightarrow\ 6(x-1) &> 0 \ \Leftrightarrow\ x > 1 \ \Leftrightarrow\ x \in (1, +\infty) \end{aligned}$$

concave down

$$\begin{aligned} f''(x) = 6x - 6 &< 0 \\ \Leftrightarrow\ 6(x-1) &< 0 \ \Leftrightarrow\ x < 1 \ \Leftrightarrow\ x \in (-\infty, 1) \end{aligned}$$

(v) Inflection points
An inflection point p is where the concavity of a function f switches from up to down or down to up. That is where $f''(x) = 0$

$$f''(x) = 6x - 6 = 0 \Leftrightarrow x = 1$$

The function $f(x)$ has a real point of inflection at $x = 1$.

(vi) horizontal, oblique and vertical asymptotes:
There are no horizontal asymptotes since

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} (x^3 - 3x^2) = \infty \quad \& \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x^3 - 3x^2) = -\infty$$

No vertical asymptotes since not a rational function

The graph can be sketched as follows

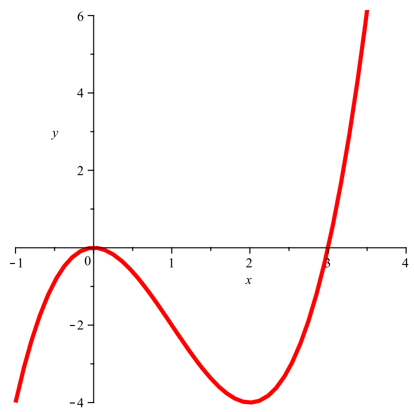


Figure 4.14: A curve $f(x) = x^3 - 3x^2$

Example 4.5.63 Show that the curve

$$f(x) = x^4 - 4x^3$$

is given by

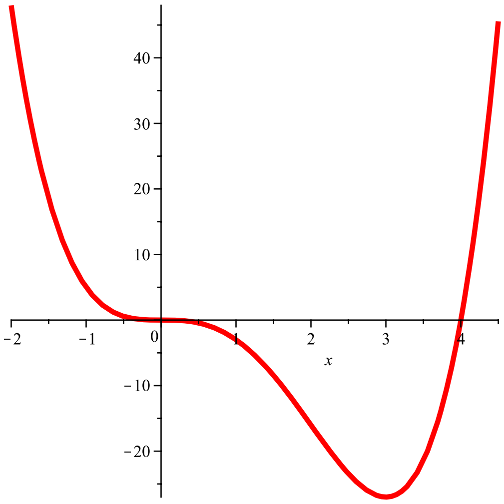


Figure 4.15: A curve $f(x) = x^4 - 4x^3$

Example 4.5.64 Graph the curve

$$f(x) = x^3(x-2)^2$$

(i) x and y intercepts (where curve cuts the axes). i.e

$$\begin{aligned} y = 0 \;&\Rightarrow\; x^3(x-2)^2 = 0 \;\Rightarrow\; x = 0, x = 2 \\ x = 0 \;&\Rightarrow\; y = x^3(x-2)^2 = 0 \;\Rightarrow\; y = 0 \end{aligned}$$

(ii) where curve is increasing $f'(x) > 0$ or decreasing $f'(x) < 0$.

$$f'(x) = \frac{dy}{dx} \;=\; 3x^2(x-2)^2 + 2(x-2)x^3 = x^2(x-2)[2x + 3(x-2)] = x^2(x-2)[5x-6]$$

Increasing:

$$x \;\in\; (-\infty,\; 0) \cup \left(0,\; \frac{6}{5}\right) \cup (2,\; \infty)$$

Decreasing:

$$x \;\in\; \left(\frac{6}{5},\; 2\right)$$

(iii) and classify the critical numbers (turning points)

$$\begin{aligned} \frac{dy}{dx} = 0 \;&\Rightarrow\; 3x^2(x-2)^2 + 2(x-2)x^3 = 0 \\ &\Rightarrow\; x^2(x-2)[2x + 3(x-2)] = x^2(x-2)[5x-6] = 0 \\ &\Rightarrow\; x = 0, x = 2, x = \frac{6}{5} \end{aligned}$$

$$\begin{aligned} f''(x) = \frac{d^2y}{dx^2} &= 4x[5x^2 - 12x + 6] \\ x = \frac{6}{5} & \quad \text{is a local maxima} \\ x = 0, 2 & \quad \text{is a local minima} \end{aligned}$$

(iv) where function is concave up $f''(x) > 0$ and concave down $f''(x) < 0$.

(v) Inflection points

(vi) horizontal, oblique and vertical asymptotes: None

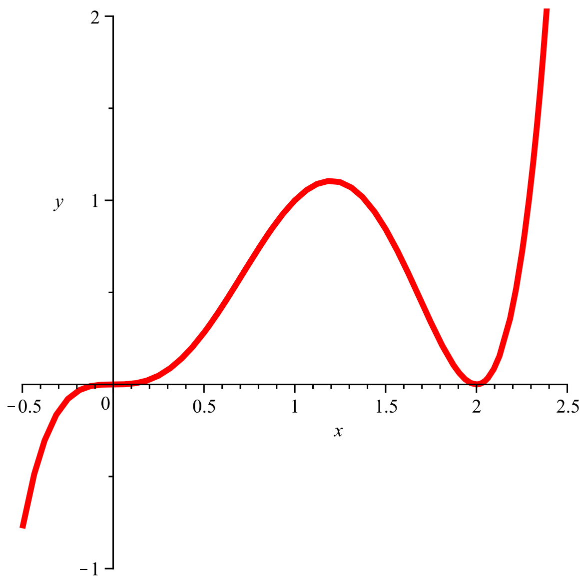


Figure 4.16: A curve $f(x) = x^3(x-2)^2$

Example 4.5.65 Graph the curve

$$f(x) = \frac{2x^2 - 3x}{x - 2}$$

(i) x and y intercepts (where curve cuts the axes). i.e

$$y = 0 \;\Rightarrow\; 2x^2 - 3x = x(2x - 3) = 0 \;\Rightarrow\; x = 0, x = \frac{3}{2}$$

$$x = 0 \;\Rightarrow\; y = \frac{2x^2 - 3x}{x - 2} = 0 \;\Rightarrow\; y = 0$$

(ii) where curve is increasing $f'(x) > 0$ or decreasing $f'(x) < 0$.

$$f'(x) = \frac{dy}{dx} = \frac{2x^2 - 8x + 6}{(x-2)^2} = \frac{2(x-1)(x-3)}{(x-2)^2}$$

Increasing:

$$f'(x) = \frac{2(x-1)(x-3)}{(x-2)^2} > 0 \;\Leftrightarrow\; x \in (-\infty, 1) \cup (3, +\infty)$$

Decreasing:

$$f'(x) = \frac{2(x-1)(x-3)}{(x-2)^2} < 0 \;\Leftrightarrow\; x \in (1, 3)$$

The denominator is always positive.

(iii) and classify the critical numbers (turning points)

$$\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{2x^2 - 8x + 6}{(x-2)^2} = \frac{2(x-1)(x-3)}{(x-2)^2} = 0$$
$$\Rightarrow\; 2(x-1)(x-3) = 0 \;\Rightarrow\; x = 1, x = 3$$

$$f''(x) = \frac{d^2y}{dx^2} = \frac{4}{(x-2)^3}$$

$x = 1$ a maxima

$x = 3$ a minima

(iv) where function is concave up $f''(x) > 0$ and concave down $f''(x) < 0$.

$$f''(x) = \frac{4}{(x-2)^3}$$

concave up

$$f''(x) = \frac{4}{(x-2)^3} > 0 \;\Rightarrow\; x \in (2, \infty)$$

concave down

$$f''(x) = \frac{4}{(x-2)^3} < 0 \;\Rightarrow\; x \in (-\infty, 2)$$

(v) Inflection points
An inflection point p is where the concavity of a function f switches from up to down or down to up. That is where $f''(x) = 0$

$$f''(x) = \frac{4}{(x-2)^3} \quad = \quad 0$$

The function $f(x)$ has no real point of inflection.

(vi) horizontal, oblique and vertical asymptotes:

There is no horizontal asymptote since

$$\lim_{x\to+\infty} f(x) \quad = \quad \lim_{x\to+\infty} \frac{2x^2-3x}{x-2} = DNE$$

and

$$\lim_{x\to-\infty} f(x) \quad = \quad \lim_{x\to-\infty} \frac{2x^2-3x}{x-2} = DNE$$

Remember, a horizontal asymptote exists only if the limit

$$\lim_{x\to\pm\infty} f(x) = L \text{ exists}$$

The oblique (slant) asymptote is $y = 2x$ since

$$\frac{2x^2-3x}{x-2} = 2x + 1 + \frac{1}{x-2}$$

Now check for a vertical asymptote by computing one-sided limits at the zero of the denominator, i.e., at $x = 2$. Thus,

$$\begin{aligned}\lim_{x\to2^+} f(x) \quad &= \quad \lim_{x\to2^+} \frac{2x^2-3x}{x-2} = \frac{2}{0^+} = +\infty \\ \lim_{x\to2^-} f(x) \quad &= \quad \lim_{x\to2^-} \frac{2x^2-3x}{x-2} = \frac{2}{0^-} = -\infty\end{aligned}$$

This shows that the line $x = 2$ is a vertical asymptote for the graph of f. Remember, if either of these one-sided limits is $+\infty$ or $-\infty$, a vertical asymptote exists.

See the adjoining detailed graph of f.

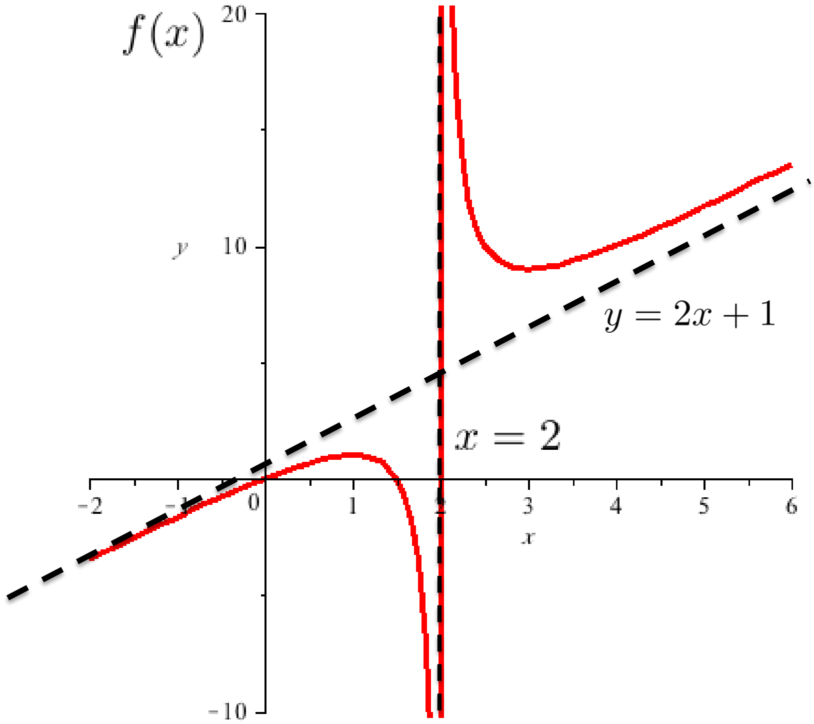


Figure 4.17: A curve $f(x) = \frac{2x^2-3x}{x-2}$

Example 4.5.66 Graph the curve

$$f(x) = \frac{(x-4)^2}{x^2-4}$$

(i) Where curve cuts the axes. i.e x and y intercepts

$$y = 0 \;\Rightarrow\; \frac{(x-4)^2}{x^2-4} = (x-4)^2 = 0 \;\Rightarrow\; x = 4$$
$$x = 0 \;\Rightarrow\; y = \frac{(x-4)^2}{x^2-4} = 0 \;\Rightarrow\; y = -4$$

(ii) Increasing and decreasing function: Exercise

(iii) The turning points (critical or stationary points)

$$\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{8(x-4)(x-1)}{(x^2-4)^2} = 0$$
$$\Rightarrow\; 8(x-4)(x-1) = 0 \;\Rightarrow\; x = 1, x = 4$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{8\left[-2x^3+15x^2-24x+20\right]}{(x^2-4)^3} \\ x = 1 &\quad \text{a maxima; } (x,y) = (1,-3) \\ x = 4 &\quad \text{a minima; } (x,y) = (4,0) \end{aligned}$$

(iv) Concave up and concave down: Exercise

(v) Points of inflection: Exercise

(vi) asymptotes:

There is a horizontal asymptote since

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{(x-4)^2}{x^2-4} = 1$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{(x-4)^2}{x^2-4} = 1$$

Thus, the line $y = 1$ is a horizontal asymptote for the graph of f. The oblique asymptote is also $y = 1$ since

$$f(x) = y = \frac{(x-4)^2}{x^2-4} = 1 + \frac{(-8x+20)}{x^2-4}$$

Now check for vertical asymptotes by computing one-sided limits at the zeroes of the denominator, i.e., at $x = 2$ and at $x = -2$. Thus,

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} \frac{(x-4)^2}{x^2-4} = \frac{4}{0^+} = +\infty \quad \& \quad \lim_{x\to2^-} f(x) = \lim_{x\to2^-} \frac{(x-4)^2}{x^2-4} = \frac{4}{0^-} = -\infty$$
$$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} \frac{(x-4)^2}{x^2-4} = \frac{36}{0^+} = +\infty \quad \& \quad \lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} \frac{(x-4)^2}{x^2-4} = \frac{36}{0^-} = -\infty$$

This shows that the line $x = -2$ and $x = 2$ are vertical asymptote for the graph of f. Remember, if either of these one-sided limits is $+\infty$ or $-\infty$, a vertical asymptote exists.

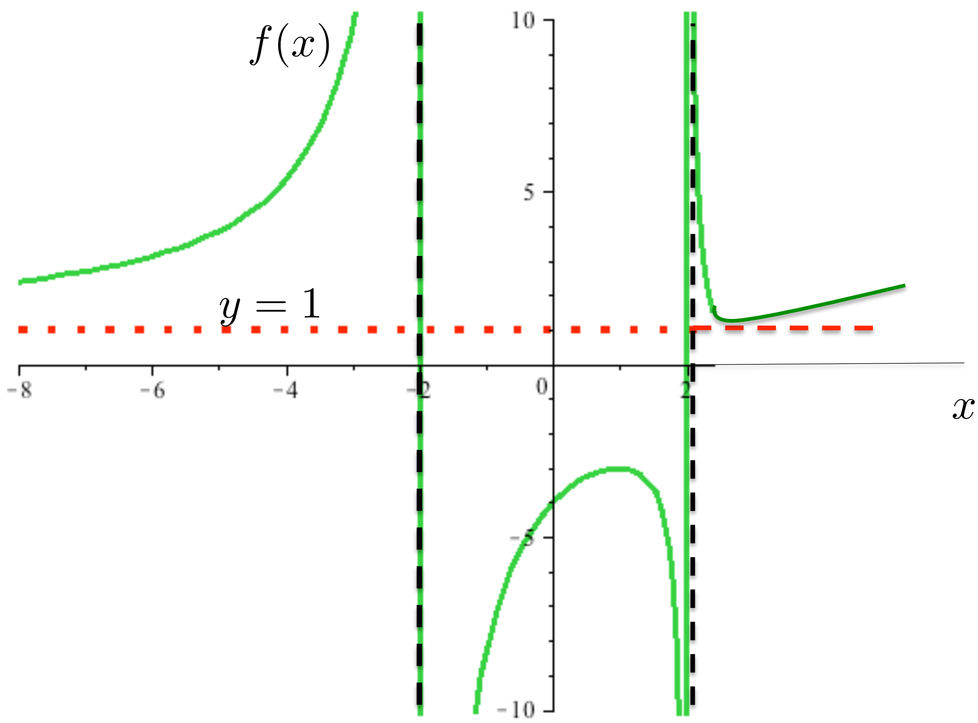


Figure 4.18: A curve $f(x) = \frac{(x-4)^2}{x^2-4}$

Example 4.5.67 Graph the curve

$$f(x) = \frac{x^2|x-3|}{(5+x)}$$

Here the point of trouble is at $x = 3$ above and below, so we are to use two graphs

$$f(x) = \begin{cases} \frac{x^2(3-x)}{(5+x)}, & \text{if } x \leq 3 \\ \\ \frac{x^2(x-3)}{(5+x)}, & \text{if } x > 3 \end{cases}$$

Example 4.5.68 Graph the curve

$$f(x) = x\sqrt{4-x^2}$$

(i) Where curve cuts the axes. i.e

$$\begin{aligned} y = 0 \;&\Rightarrow\; x\sqrt{4-x^2} = x\sqrt{(2-x)(2+x)} = 0 \;\Rightarrow\; x = 0, x = 2, x = -2 \\ x = 0 \;&\Rightarrow\; y = x\sqrt{4-x^2} = 0 \;\Rightarrow\; y = 0 \end{aligned}$$

(ii) The turning points and their nature.

$$\begin{aligned} \frac{dy}{dx} = 0 \;&\Rightarrow\; \frac{2(2-x^2)}{\sqrt{4-x^2}} = 0 \;\Rightarrow\; \frac{2(\sqrt{2}-x)(\sqrt{2}+x)}{\sqrt{4-x^2}} = 0 \\ &\Rightarrow\; 2(\sqrt{2}-x)(\sqrt{2}+x) = 0 \;\Rightarrow\; x = -\sqrt{2}, x = \sqrt{2} \end{aligned}$$

and

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{2x(x^2-6)}{(4-x^2)^{3/2}} \;&\Rightarrow\; x = \sqrt{2} \text{ absolute maxima } (x, y) = (\sqrt{2}, 2\sqrt{2}) \\ &\& \quad x = -\sqrt{2} \text{ absolute minima } (x, y) = (-\sqrt{2}, -2\sqrt{2}) \end{aligned}$$

(iii) Increasing and decreasing function: Exercise

(iv) Concave up and concave down: Exercise

(v) Points of inflection: Exercise

(vi) asymptotes: Exercise

Example 4.5.69 The curve for

$$f(x) = x\sqrt{4-x^2}$$

can be shown to be

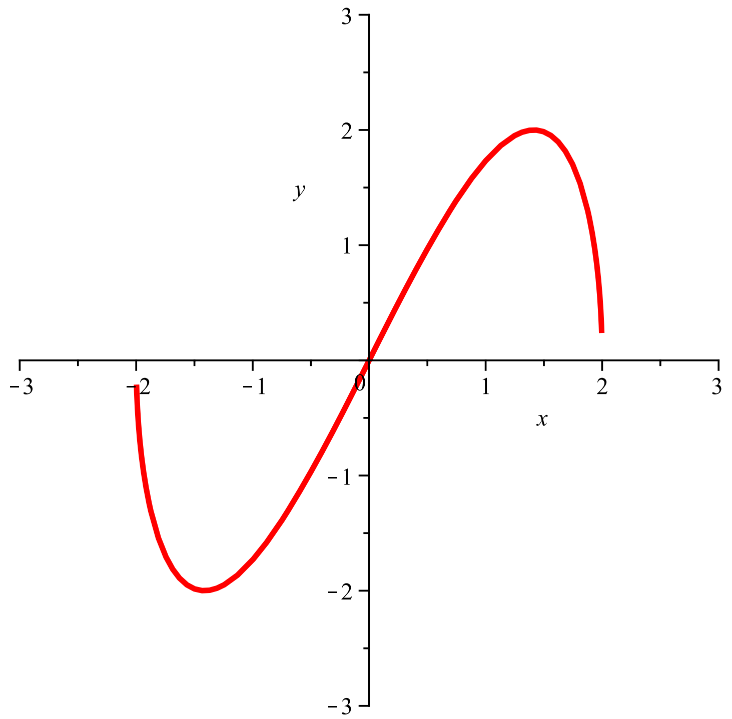


Figure 4.19: A curve $f(x) = x\sqrt{4 - x^2}$

Example 4.5.70 Graph the curve

$$f(x) = x - 3x^{\frac{1}{3}}$$

(i) Where curve cuts the axes. i.e

$$y = 0 \;\Rightarrow\; x - 3x^{\frac{1}{3}} = x^{\frac{1}{3}}(x^{\frac{2}{3}} - 3) = 0 \;\Rightarrow\; x = 0, x = \sqrt{27}, x = -\sqrt{27}$$
$$x = 0 \;\Rightarrow\; y = x - 3x^{\frac{1}{3}} = 0 \;\Rightarrow\; y = 0$$

(ii) The turning points

$$\frac{dy}{dx} = 0 \;\Rightarrow\; 1 - 3(1/3)x^{\frac{1}{3}-1} = 0 \;\Rightarrow\; 1 - x^{-2/3} = \frac{x^{2/3} - 1}{x^{2/3}} = 0 \;\Rightarrow\; x = -1, x = 1$$

(iii) Maxima or Minima: Since $\frac{d^2y}{dx^2} = \frac{2}{3x^{5/3}}$

$$x = -1 \qquad \text{a maxima } (x, y) = (-1, 2)$$
$$x = 1 \qquad \text{a minima } (x, y) = (1, -2)$$

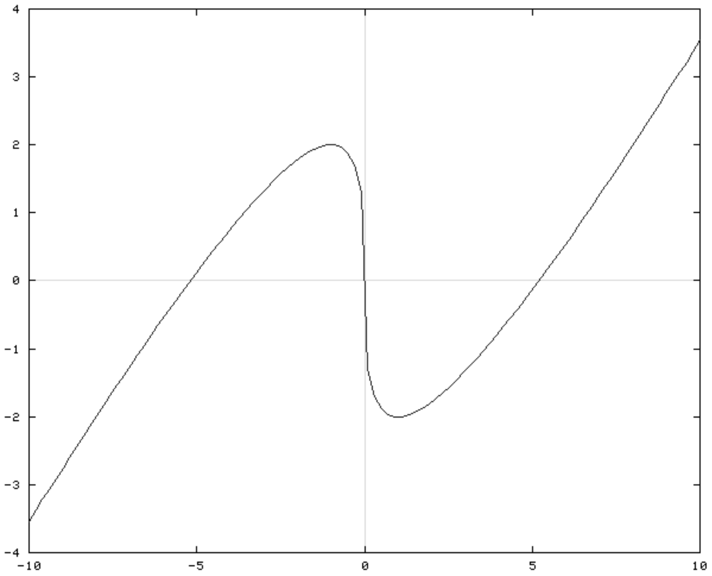


Figure 4.20: A curve $f(x) = x - 3x^{\frac{1}{3}}$

Example 4.5.71 Graph the curve

$$f(x) = \frac{|x+2|x^2}{|x|}$$

Now its not only one point to think of, but now both $x = -2$ and $x = 0$. We need to have different functions for

$$\begin{aligned} x &< -2 \\ -2 \leq x &\leq 0 \\ x &> 0 \end{aligned}$$

$$f(x) = \begin{cases} \frac{-(x+2)x^2}{-(x)}, & \text{if } \ x < -2 \\ \\ \frac{(x+2)x^2}{-(x)}, & \text{if } \ -2 \leq x \leq 0 \\ \\ \frac{(x+2)x^2}{(x)}, & \text{if } \ x > 0 \end{cases}$$

and sketch those function in the different ranges.

Exercise 4.51 Plot the curves;

(i) $y = x^2$ (ii) $y = \frac{x-5}{x^2-9}$ (iii) $y^2 = x$

Example 4.5.72 f is a function given by

$$f(x) = |x - 2|$$

Find the x and y intercepts of the graph of f. Find the domain and range of f. Sketch the graph of f.

(a) The y intercept is given by

$$(0, f(0)) = (0, |-2|) = (0, 2)$$

The x coordinate of the x intercepts is equal to the solution of the equation

$$|x-2| = 0$$

which is $x = 2$

(b) The x intercepts is at the point $(2, 0)$

(c) The domain of f is the set of all real numbers
Since $|x-2|$ is either positive or zero for $x = 2$, the range of f is given by the interval $[0, +\infty)$.

(d) To sketch the graph of $f(x) = |x-2|$, we first sketch the graph of $y = x-2$ and then take the absolute value of y.

The graph of $y = x-2$ is a line with x intercept $(2, 0)$ and y intercept $(0, -2)$.

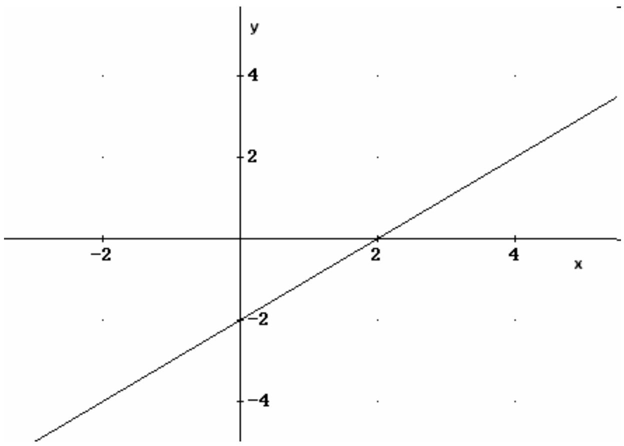
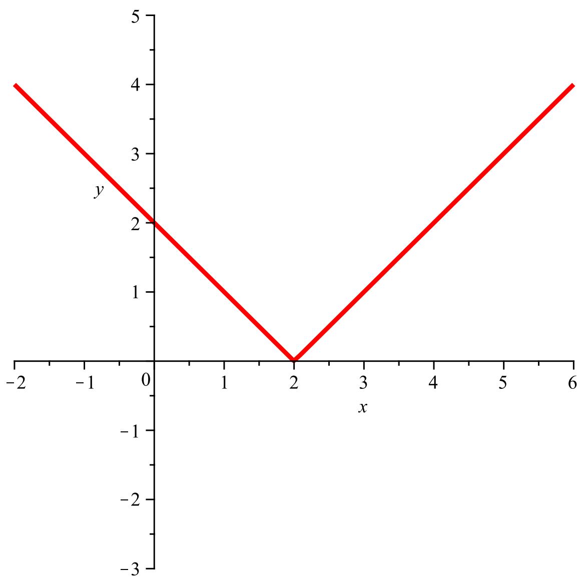


Figure 4.21: A curve $y = x - 2$

(e) But

$$|x-2| = \begin{cases} (x-2)\ , & x \geq 2 \\ -(x-2)\ , & x < 2 \end{cases}$$

whose curve is given on the right hand side



Example 4.5.73 Plot the curve

$$f(x) = |(x-2)^2 - 4|$$

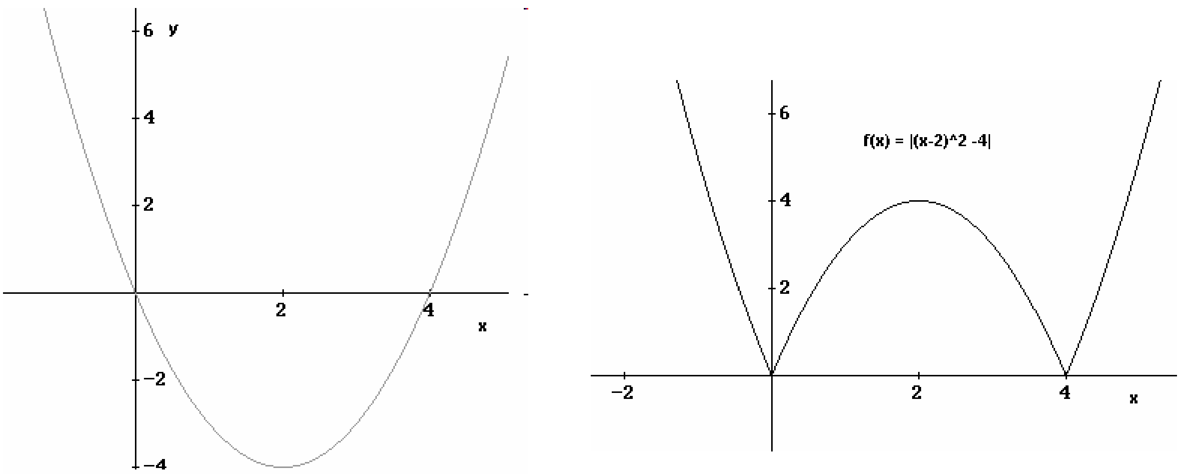


Figure 4.22: The curves $y = (x-2)^2 - 4$ and $y = |(x-2)^2 - 4|$

Example 4.5.74 Graph the curve

$$y = |x-3| + 2$$

Example 4.5.75 Graph the function

$$y = \frac{2x^2 - 11}{x^2 + 9}$$

Take note of the horizontal asymptote

Example 4.5.76 Find the slant (oblique) asymptote of the following function

$$f(x) = \frac{2x^3 + 4x^2 - 9}{3 - x^2}$$

$y = -2x - 4$

Exercise 4.52 Sketch the curve

$$f(x) = \frac{x^2 + 2}{x - 2}$$

4.6 Chapter Examples

1. Prove the Quotient rule

Method I

$$
\begin{aligned}
h'(x) = \left[\frac{f(x)}{g(x)}\right]' &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{1}{h} \left[\frac{f(x+h)g(x) - f(x)g(x)}{g(x+h)g(x)} - \frac{f(x)g(x+h) - f(x)g(x)}{g(x+h)g(x)}\right] \\
&= \lim_{h\to 0} \left[\frac{1}{g(x+h)}\right] \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h}\right] \\
&\quad - \lim_{h\to 0} \left[\frac{f(x)}{g(x+h)g(x)}\right] \lim_{h\to 0} \left[\frac{g(x+h) - g(x)}{h}\right] \\
&= \frac{1}{g(x)} \cdot f'(x) - \frac{f(x)}{[g(x)]^2} \cdot g'(x) \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.
\end{aligned}
$$

We have implicitly assumed here that $f'(x)$ and $g'(x)$ exist and that $g(x) \neq 0$.

Method II

Theorem 4.6.1 *The strong derivative of the reciprocal function $x \mapsto \frac{1}{x}$ is the function*

$$x \mapsto -\frac{1}{x^2}$$

Theorem 4.6.2 *If g is strongly differentiable at x and $g(x) \neq 0$ then*

$$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{(g(x))^2}.$$

Let $a(x) = \dfrac{1}{x}$. Then $\dfrac{1}{g} = a \circ g$. By the Chain Rule and Theorem 4.6.1,

$$\left(\frac{1}{g}\right)'(x) = a'(g(x))g'(x) = -\frac{g'(x)}{(g(x))^2},$$

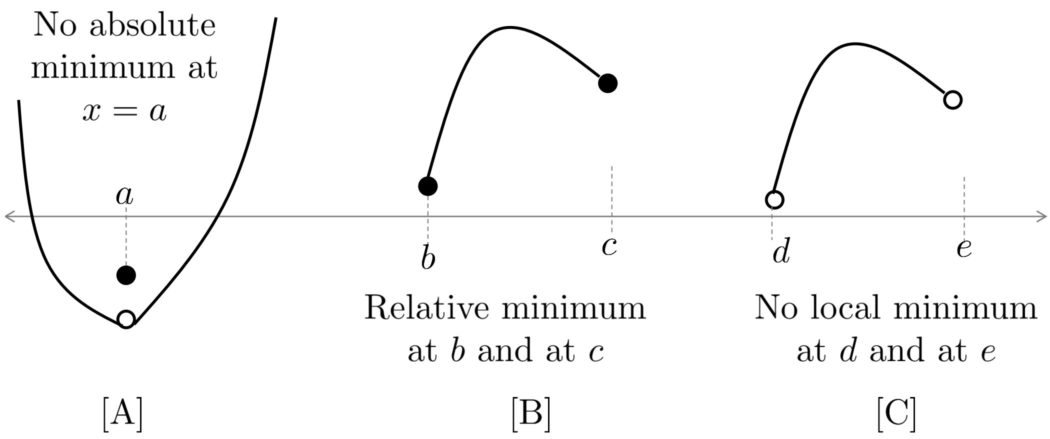
as we needed to show

Using the Product Rule and Theorem 4.6.2,

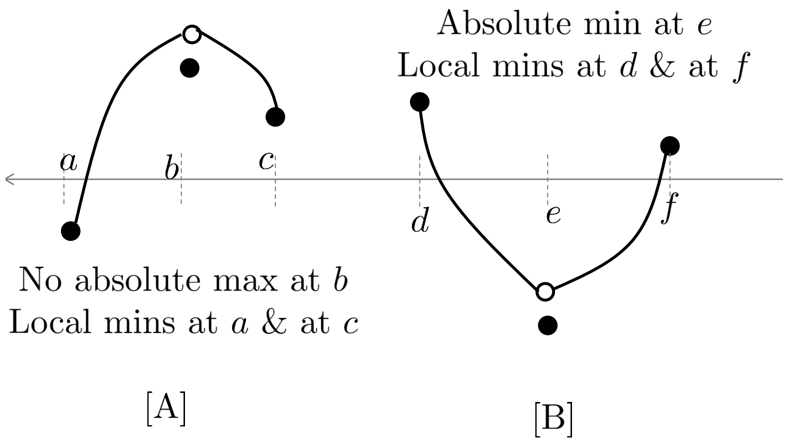
$$
\begin{aligned}
\left(\frac{f}{g}\right)'(x) &= \left(f \cdot \frac{1}{g}\right)'(x) \\
&= f'(x)\frac{1}{g(x)} + f(x)\left(\frac{1}{g}\right)'(x) \\
&= f'(x)\frac{1}{g(x)} + f(x)\left(-\frac{g'(x)}{(g(x))^2}\right) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2},
\end{aligned}
$$

as desired.

2. Examples of relative (local) and global (absolute) minimum and maximum



and



3. (a) The following limit exists and defines the derivative of some function $f(x)$ at the point x_0 :

$$\lim_{h\to 0} \frac{5(1+h)^4 + \frac{1}{1+h} \ - \ 6}{h}$$

Write down $f(x)$ and the point x_0.

`Apply Equation` (4.2) `and not Equation` (4.3)

(b) Given that

$$f(x) = \begin{cases} x^2; & \text{if } \ x < 0, \\ x^2 + 5; & \text{if } \ x > 0, \end{cases}$$

Show that $\lim\limits_{x\to 0^+} f'(x) = \lim\limits_{x\to 0^-} f'(x)$. Does $f'(0)$ exist? Explain fully.

$f(0)$ `not exist` $\Rightarrow$ $f'(0)$ DNE

4. (a) Let $f(x) = |x|$ on $[-2, 2]$. Does the MVT apply to $f(x)$? Justify your answer.

$f(x)$ `not differentiable`

(b) Show that $f(x) = x^2 - 5x + 1$ satisfies the hypothesis of the MVT on $[0, 2]$ and find a $c \in (0, 2)$ which satisfies the conclusion of the theorem.

5. Let $f(x) = x^4 - ax^2 + b;\ a, b \in \mathbb{R}$. Find a and b if $f(\sqrt{2}) = -2$ is the absolute minimum value of $f(x)$ on $[-3, 2]$.

$$\begin{aligned} f(\sqrt{2}) &= -2 \\ f'(\sqrt{2}) &= 0 \end{aligned}$$

6. The number of people in some town m months from today is given by the function

$$f(m) = \frac{520(m + 12)}{m + 60}$$

(a) How many people are in the town now?

`now, means` $m = 0$

(b) Is the population of the town always increasing or decreasing? Justify your answer.

`Increase` $f'(x) > 0$ `Decrease` $f'(x) < 0$

(c) What is the equation of the horizontal asymptote of this function?

(d) Give an interpretation to your answer in (6c) above.

7. Given that $f(x)$ is continuous on the interval $[x_1, x_{10}]$, where $x_1 < x_2 < x_3 \ldots < x_9 < x_{10}$ and further that $x_2 - x_1 = x_3 - x_2 = \cdots = x_{10} - x_9$ and given the following information about $f(x), f'(x)$ and $f''(x)$, sketch the graph of $f(x)$ if;

x	$f(x)$	$f'(x)$	$f''(x)$
x_1	-	$+$	$+$
x_2	0	$+$	-
x_3	$+$	$+$	-
x_4	$+$	0	-
x_5	$+$	-	-
x_6	-	-	$+$
x_7	-	0	$+$
x_8	-	$+$	$+$
x_9	-	$+$	-
x_{10}	$+$	$+$	-

$f(x) \equiv$ [`cutting axis and value of` $f(x)$]

$f'(x) \equiv$ [`Critical points, maxima-minima` 1^{st} `derivative test, increasing-decreasing`]

$f''(x) \equiv$ [`maxima-minima` 2^{nd} `derivative test, Concavity, and Points of inflection`]

8. Give an example of two different functions f and g such that

$$f'(0) = f''(0) = g'(0) = g''(0)$$

`Either a polynomial` $ax^n,\ n \geq 2$

$$\begin{aligned} f(x) &= x^3 \\ g(x) &= x^4 \end{aligned}$$

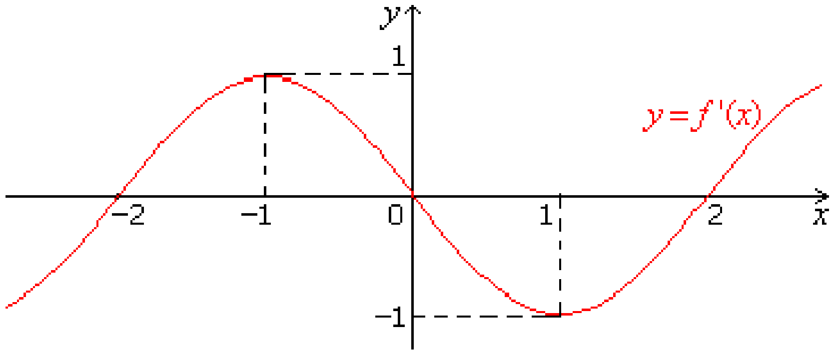
`or an exponential` $ae^{x^n},\ n \geq 2$

$$\begin{aligned} f(x) &= e^{x^3} \\ g(x) &= e^{x^4} \end{aligned}$$

`or a product of` $\sin^n x$ and $\cos x$, $n \geq 2$

$$\begin{aligned} f(x) &= \sin^2 x \\ g(x) &= \cos x \end{aligned}$$

9. The graph of a derivative f' of a function f is shown below.



(a) Where does the curve f cut the x-axis?

`We do not know, since` $f(x) = 0$ `isn't given/known`

(b) State and classify the local extrema of $f(x)$.

p `is a local extrema if` $f'(p) = 0$

`maxima-minima classification by` 1^{st} `derivative test, only` $f'(x)$ `given, not` $f''(x)$ `to go` 2^{nd} `derivative test`

$$\begin{aligned} x &= -2,\ \text{a local minimum since}\ \ f'(-2) = 0 \text{ and }\ f'(-2-\delta, -2) < 0,\ f'(-2, -2+\delta) > 0 \\ x &= 2,\ \text{a local minimum since}\ \ f'(2) = 0 \text{ and }\ f'(2-\delta, 2) < 0,\ f'(2, 2+\delta) > 0 \\ x &= 0,\ \text{a local maximum since}\ \ f'(0) = 0 \text{ and }\ f'(0-\delta, 0) > 0,\ f'(0, 0+\delta) < 0 \end{aligned}$$

for any $\delta > 0$ or can say, by first derivative test.

(c) Determine where the graph of f is concave down in $[-2, 2]$.
`A curve is concave down if` $f''(x) < 0$. To have $f''(x)$, we differentiate $f'(x)$ the curve given, where the derivative/slope of the first derivative (curve given) is negative, that is $x \in (-1, 1)$

(d) Find the points of inflection of f.
An inflection point p is where $f''(x) = 0$ and the concavity of a function f switches from up to down or down to up.

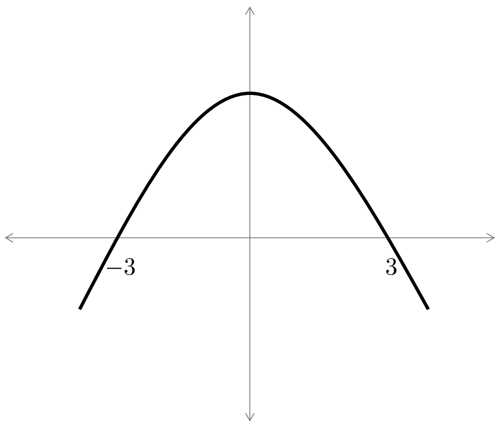
$$\begin{aligned} f''(x) &= 0 \\ &\Rightarrow \quad x = -1, x = 1 \end{aligned}$$

10. Sketch the graph of a function that satisfies all of the given conditions

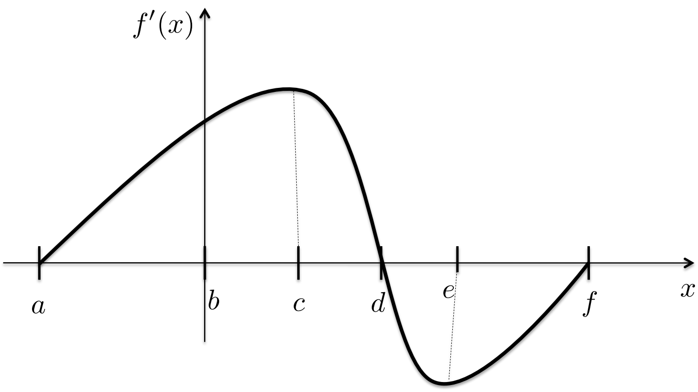
$$f(-3) = f(3) = f'(0) = 0$$
$$f'(x) > 0 \text{ if } -3 \leq x \leq 0,\ f'(x) < 0, \text{ if } 0 \leq x \leq 3$$
$$f''(x) < 0, \text{ if } -3 \leq x \leq 3$$

The curve has a turning point at $x = 0$,
it is increasing in interval $-3 \leq x \leq 0$, but decreasing in the interval $0 \leq x \leq 3$
and concave down in the interval $-3 \leq x \leq 3$

Which can have a sketch as follows



11. The graph of $f'(x)$ is shown below:



State the interval(s) on which f is concave down.

`Defend the interval` (c, e)

12. *True* or *False*? You are given a continuous function for which $f''(x) > 0$ for all values except at $x = a$, then f might have an absolute maximum value at $x = a$.

`"might" makes it True`

13. *True* or *False*? If $f'(x) > 0$ on (a, b) and $f'(x) < 0$ on (b, c) and f is defined at b, then $f'(x)$ may not exist at $x = b$.

14. Sketch one possible graph of a function which is continuous at $x = a$ and yet the derivative does not exist at $x = a$.

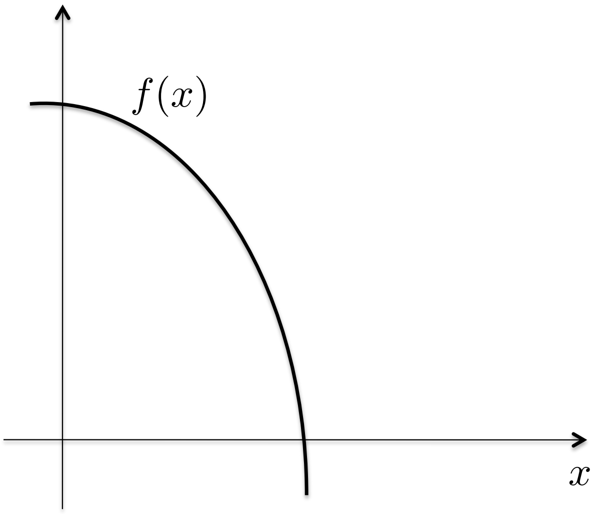
    `The question is sketch. If you have a correct function example` $f(x) = |x|$`, but the sketch is wrong, then the solution is wrong.`

15. Evaluate

$$\frac{d}{dx}\left(e\cos nx + \pi \sin nx\right)$$

    e `is a number, not an exponential`

16. Below is a graph of f, then f is



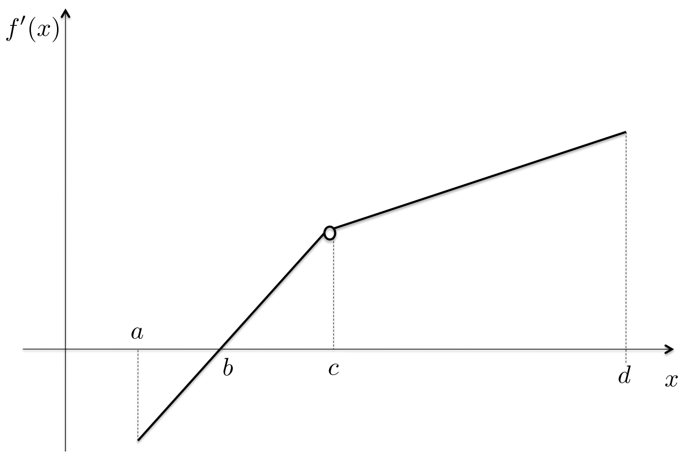
(A) decreasing and concave up
(B) increasing and concave up
(C) decreasing and concave down
(D) increasing and concave down

17. The differentiation rule that helps us understand the derivative of a function of a function is

(A) The chain rule
(B) The product rule
(C) The quotient rule
(D) None of the above

18. A store has been selling 200 Blu-ray disc players a week at \$350 each. A market survey indicates that for each \$10 discount offered to buyers, the number of units sold will increase by 20 a week. How large should the discount be in order for the store to maximize its income?

19. Let $f(x)$ be a continuous function on $[a, d]$ and let the graph of $f'(x)$ be as shown below:



State the interval(s) on which $f(x)$ is increasing.

It is $[b, d]$, `the point` c `is inclusive since increase in never at a point, but on an interval`

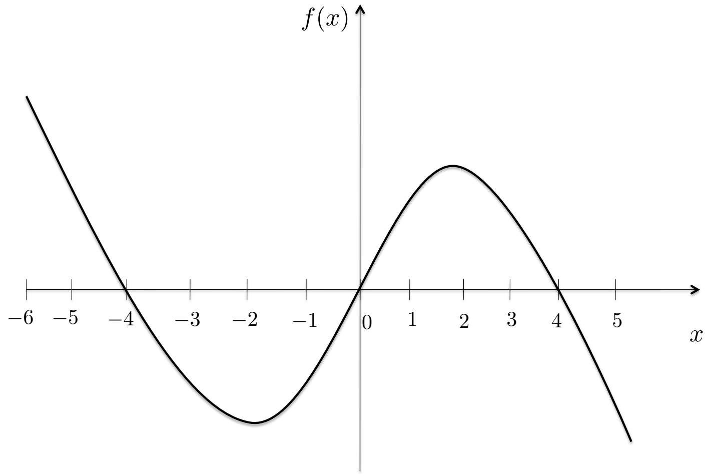
20. *True* or *False*? Differentiability at point $x = x_0$ implies $\lim_{x \to x_0} f(x) = f(x_0)$

    `True, differentiability` $\Rightarrow$ `continuity`

21. Sketch the graph of the function $f(x)$ on the interval $-6 \leq x \leq 5$ where

    (i) $f(-4) = f(0) = f(4) = 0$
    (ii) $f'(-2) = f'(2) = 0$
    (iii) $f'(x) < 0, x \in (-6, -2) \cup (2, 5)$ and $f'(x) > 0, x \in (-2, 2)$
    (iv) $f''(x) \geq 0, x \in (-6, 0)$ and $f''(x) \leq 0, x \in (0, 5)$

    The sketch satisfying all the four properties above is

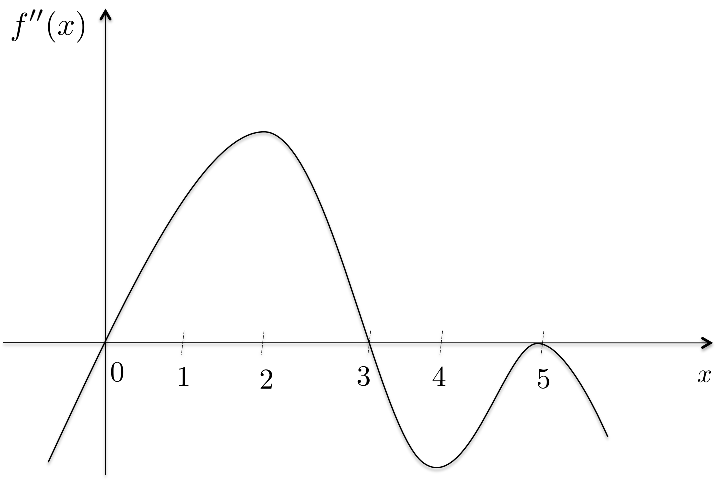


22. *True* or *False*? If $f(-c) = c$, then c is the fixed point of $f(x)$.

    `Do not confuse odd-even functions with fixed points`

23. The length of a rectangle is increasing at a rate of 6 cm/s. If the area of the rectangle is not changing, at what rate is the width of the rectangle decreasing when the length is 14 cm and the width is 7 cm?

24. Given a graph of $f''(x)$ below:



(a) State the points of inflection.

(b) Why is $x = 5$ not a point of inflection.

(c) On the interval $[0, 5]$, state the intervals where the curve is concave up and concave down

(d) Is there any significance/implication on points $x = 2$ and $x = 4$

`Note that, the graph is neither of` $f(x)$ `nor` $f'(x)$

25. Water is being poured into a conical vase at a rate of 18 cm^3/s. The diameter of the cone is 30 cm and the height of the cone is 25 cm. At what rate is the water level rising when the water's level depth is 20 cm?

26. Find the equation of the line tangent to the curve

$$(1 + x^2y)^3 + x\sqrt{y} = 9\cos(x+y)\pi$$

at the point $(1, 1)$.

`Implicit is a better option`

27. A travel agency will plan a group of size 25 or larger. If the group contains exactly 25 people, the cost is \$300 per person. However, each person's cost is reduced by \$10 for each additional person above the 25. What size group will produce the largest revenue for the agency?

28. If

$$f'(4) = \lim_{x \to 4} \frac{2x^{\frac{3}{2}} - 2^4}{x - 4},$$

then

$$f'(5) = ?$$

`An easier option could be to identify` $f(x) = 2x^{\frac{3}{2}}$ `and differentiate it directly, and at` $x = 5$ or go equation (4.3)

29. *True* or *False*? Justify your answer.
If

$$f(x) = x^3$$

then $f(x)$ has an extreme value at $x = 0$.

`The point` $x = 0$ `is a point of inflection not an extremum. If we had said, a critical point, it would be True`

30. If

$$\begin{aligned} y &= \tan u \\ u &= v - \frac{1}{v} \\ v &= \ln x \end{aligned}$$

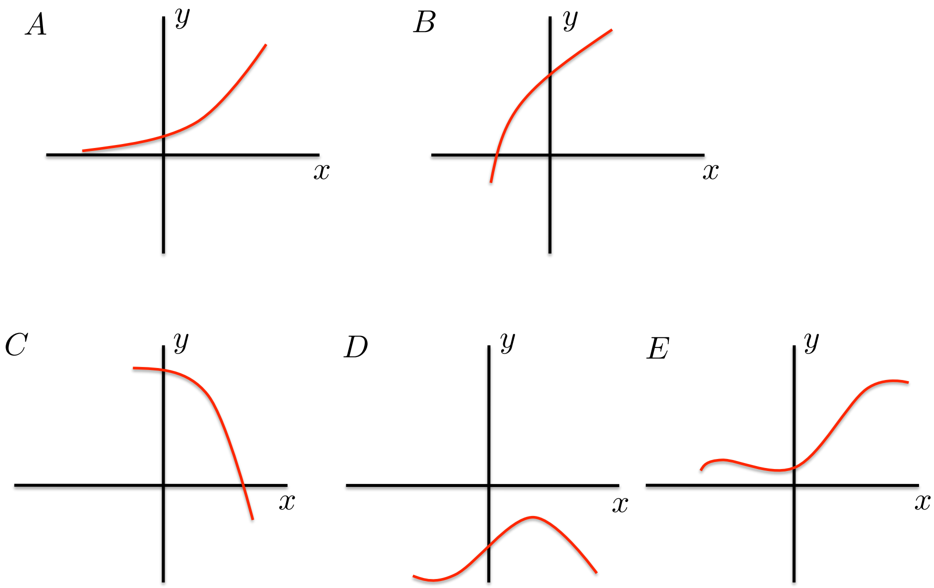
What is the value of $\frac{dy}{dx}$ at $x = e$

`Apply the chain rule`

`At` $x = e$,

$$v = ?\ u = ?\ y = ?,\ \texttt{but solution is } \frac{2}{e}$$

31. If y is a function of x such that $y' > 0$ for all x and $y'' < 0$ for all x, which of the following could be part of the graph of $y = f(x)$?



`In other words, which of the curves is increasing` $(y' > 0)$ `and concave downwards` $(y'' < 0)$`?`

32. Find all the inflection points of the graph of

$$y = 5x^4 - x^5$$

`Why isn't` $(0,0)$ `an inflection point? The only inflection point is` $(3, 162)$

33. If

$$3x^2 + 2xy + y^2 = 2 \ ,$$

then the value of $\frac{dy}{dx}$ at $x = 1$ is?

`What makes` $\frac{dy}{dx}$ `undefined? If` $x = 1$`, what is` y`?`

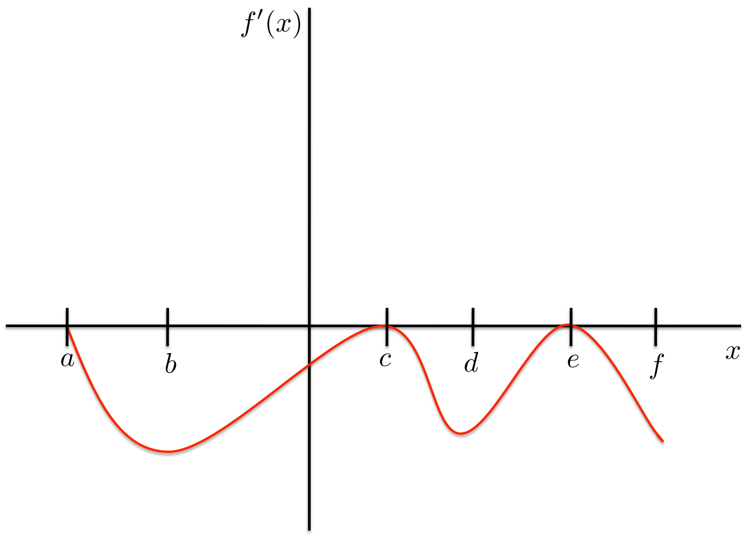
34. Suppose that x, y are both functions of t, and that $x^2 + y^2 = 25$. Express $\frac{dx}{dt}$ in terms of x, y, and $\frac{dy}{dt}$. When $x = 3$ and $y = 4$ and $\frac{dy}{dt} = 6$, what is$\frac{dx}{dt}$?

35. A 2-feet tall dog is walking away from a street light which is on a 10-feet pole. At a certain moment, the tip of the dog's shadow is moving away from the street light at 5 feet per second. How fast is the dog walking at that moment?

`Rates of change`

36. A ladder 13 feet long leans against a house, but is sliding down. How fast is the top of the ladder moving at a moment when the base of the ladder is 12 feet from the house and moving outward at 10 feet per second?

37. Given a graph of $f'(x)$ as



(a) Write down (if any) the interval on which $f(x)$ is increasing.

`The curve is of` $f'(x)$ `not of` $f(x)$`. There is no interval where the curve` $f(x)$ `is increasing. How? - but why?`

`But if needed where the curve` $f'(x)$ `is increasing, we have` $(b, c) \cup (d, e)$

(b) Write down the interval on which $f(x)$ is decreasing.

38. At $x = 0$, which of the following is true of the function f defined by $f(x) = x^2 + e^{-2x}$?

    (A) f is increasing.
    (B) f is decreasing.
    (C) f is discontinuous.
    (D) f has a relative minimum.
    (E) f has a relative maximum.

39. If $f(x) = 2 + |x - 3|$ for all x, then the value of the derivative $f'(x)$ at $x = 3$ is

    (A) -1 (B) 0 (C) 1 (D) 2 (E) nonexistent

40. The function defined by $f(x) = \sqrt{3}\cos x + 3\sin x$ has an amplitude of

    (A) $3 - \sqrt{3}$ (B) $\sqrt{3}$ (C) $2\sqrt{3}$ (D) $3 + \sqrt{3}$ (E) $3\sqrt{3}$

41. If $y = x^2 + 2$ and $u = 2x - 1$, then $\frac{dy}{du} =$

    (A) $\frac{2x^2-2x+4}{(2x-1)^2}$
    (B) $6x^2 - 2x + 4$
    (C) x^2
    (D) x
    (E) $\frac{1}{x}$

42. What is

$$\lim_{h\to 0} \frac{8\left(\frac{1}{2}+h\right)^8 - 8\left(\frac{1}{2}\right)^8}{h}$$

    (A) 0
    (B) $\frac{1}{2}$
    (C) 1
    (D) The limit does not exist
    (E) It cannot be determined from the information given

43. The Mean ValueTheorem guarantees the existence of a special point on the graph of $y = \sqrt{x}$ between $(0, 0)$ and $(4, 2)$. What are the coordinates of this point?

(A) $(2, 1)$ (B) $(1, 1)$ (C) $(2, \sqrt{2})$ (D) $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ (E) None of them

44. For what values of k will $x + \frac{k}{x}$ have a relative maximum at $x = -2$?

(A) 4 (B) -2 (C) 2 (D) 4 (E) None of these

45. If $h(x) = f^2(x) - g^2(x)$, $f'(x) = -g(x)$, and $g'(x) = f(x)$, then $h'(x) =$

(A) 0
(B) 1
(C) $-4f(x)g(x)$
(D) $-(g(x))^2 - (f(x))^2$
(E) $-2(-g(x) + f(x))$

46. If $f(x) = e^x$, which of the following lines is an asymptote to the graph of f?

(A) $y = 0$ (B) $x = 0$ (C) $y = x$ (D) $y = -x$ (E) $y = 1$

`Not specified whether vertical, horizontal (`$x \to \pm\infty$`) or oblique asymptote`

47. The derivative of $f(x) = \frac{x^4}{3} - \frac{x^5}{5}$ attains its maximum value at $x =$

(A) -1 (B) 0 (C) 1 (D) $\frac{4}{3}$ (E) $\frac{5}{3}$

`Maxima of a derivative, not of a function`

48. If $f(x) = x^{\frac{1}{3}}(x-2)^{\frac{2}{3}}$ for all x, then the domain of f' is

(A) $\{x | x \neq 0\}$
(B) $\{x | x > 0\}$
(C) $\{x | 0 \leq x \leq 2\}$
(D) $\{x | x \neq 0 \text{ and } x \neq 2\}$
(E) $\{x | x \text{ is a real number}\}$

`Domain of `f'` not of `f`. What if was domain of `ff'

49. The radius r of a sphere is increasing at the uniform rate of 0.3 meters per second. At the instant when the surface area S becomes 100π square meters, what is the rate of increase, in cubic meters per second, in the volume V? $\left(S = 4\pi r^2 \text{ and } V = \frac{4}{3}\pi r^3\right)$

(A) 10π (B) 12π (C) 22.5π (D) 25π (E) 30π

50. Suppose f is an odd function; i.e, $f(-x) = -f(x)$ for all x. Suppose that $f'(x_0)$ exists. Which of the following must necessarily be equal to $f'(-x_0)$?

(A) $f'(x_0)$ (B) $-f'(x_0)$ (C) $\frac{1}{f'(x_0)}$ (D) $\frac{-1}{f'(x_0)}$ (E) None of them

`Why the statement - "Suppose that `$f'(x_0)$` exists"`

51. If $y = e^{n\pi}$, then $\frac{d^n y}{dx^n} =$

(A) $n^n e^{n\pi}$ (B) $n! e^{n\pi}$ (C) $n e^{n\pi}$ (D) $n^n x^x$ (E) $n! e^x$

52. For small values of h, the function $\sqrt[4]{16 + h}$ is best approximated by which of the following?

(A) $4 + \frac{h}{32}$ (B) $2 + \frac{h}{32}$ (C) $\frac{h}{32}$ (D) $4 - \frac{h}{32}$ (E) $2 - \frac{h}{32}$

53. If $f(x) = x + \frac{1}{x}$, then the set of values for which f increases is

(A) $(-\infty, -1] \cup [1, \infty)$ (C) $(-\infty, \infty)$ (E) $(-\infty, 0) \cup (0, \infty)$
(B) $[-1, 1]$ (D) $(0, \infty)$

54. If $y = \ln(x^2 + y^2)$, then the value of $\frac{dy}{dx}$ at a point $(1, 0)$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) undefined

55. If $x = t^2 - 1$ and $y = 2e^t$, then $\frac{dy}{dx} =$

(A) $\frac{e^t}{t}$ (B) $\frac{2e^t}{t}$ (C) $\frac{e^{|t|}}{t^2}$ (D) $\frac{4e^t}{2t-1}$ (E) e^t

56. Let g be a continuous function on a closed interval $[0, 1]$. Let $g(0) = 1$ and $g(1) = 0$. Which of the following in NOT necessarily true?

(A) There exists a number h in $[0, 1]$ such that $g(h) \geq g(x)$ for all $x \in [0, 1]$.
(B) For all a and b in $[0, 1]$, if $a = b$, then $g(a) = g(b)$.
(C) There exists a number h in $[0, 1]$ such that $g(h) = \frac{1}{2}$.
(D) There exists a number h in $[0, 1]$ such that $g(h) = \frac{3}{2}$.
(E) For all h in the open interval $(0, 1)$, $\lim_{x \to h} g(x) = g(h)$.

`Does it violate continuity definition, MVT, IVT, Rolle's, or functions ? State why the other options are correct. Why say "open interval" in last option?`

57. Which of the following is true about the graph of $y = \ln|x^2 - 1|$ in the interval $(-1, 1)$?

(A) It is increasing.
(B) It attains a relative minimum at $(0, 0)$.
(C) It has a range of all real numbers
(D) It is concave down.
(E) It has an asymptote of $x = 0$.

58. If $f(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 5$ and the domain is the set of all x such that $0 \leq x \leq 9$, then the absolute maximum value of the function f occurs when x is

(A) 0 (B) 2 (C) 4 (D) 6 (E) 9

`Why do you think we have given the domain? Do you have to substitute in all value of` x `given? Maybe not, what theorem do you use? What is the limitation of the theorem you might have thought of?`

59. *True* or *False*?
Concavity classifies maxima or minima.

60. *True* or *False*? $f''(x) = 0 \Leftrightarrow (x, f(x))$ is a point of inflection.

`Answer is False. This is always students' error. Probably` $\Rightarrow$ `is okay, but` $\Leftarrow$ `is incorrect.`

61. If $f(x) = \ln(\ln x)$, then $f'(x) =$

(A) $\frac{1}{x}$ (B) $\frac{1}{\ln x}$ (C) $\frac{\ln x}{x}$ (D) x (E) $\frac{1}{x \ln x}$

62. Let f and g be differentiable functions such that

$$\begin{aligned} f(1) &= 2, & f'(1) &= 3, & f'(2) &= -4, \\ g(1) &= 2, & g'(1) &= -3, & g'(2) &= 5. \end{aligned}$$

If $h(x) = f(g(x))$, then $h'(1) =$

(A) -9 (B) -4 (C) 0 (D) 12 (E) 15

63. *True* or *False*? $\frac{d}{dx} f(y) = f'(y)$

64. *True* or *False*? f is a maximum (or minimum) $\Leftrightarrow$ $f'(x) = 0$.

65. Classifying Critical Points: Complete each statement by choosing one of the four phrases from the box below.

(i) the absolute (global) maximum
(ii) a local (relative) maximum
(iii) the absolute (global) minimum
(iv) a local (relative) minimum

Phrases may be used more than once. Unless otherwise specified, assume each function is defined and continuous for all real numbers.

(a) If $x = 2$ is the only critical point of a function f and $f''(2) < 0$, then $x = 2$ locates _____ value of the function.
(b) If $f'(2) = 0$ and $f'(x)$ changes from negative to positive at $x = 2$, then $x = 2$ locates _____ value of the function f.
(c) If $f'(2) = 0$ and $f''(2) > 0$, then $x = 2$ locates _____ value of the function f.
(d) If $x = 2$ is a critical point of the function f, and $f'(x)$ decreases through $x = 2$, then $x = 2$ locates _____ value of the function.
(e) If a continuous function f increases throughout a closed interval, then the left endpoint of the graph of f on the interval is _____ point of the function.
(f) A student found the critical points of a function f to be $x = 2$ and $x = 4$, and produced the chart below.

Interval	$x < 2$	$2 < x < 4$	$x > 4$
$f'(x)$	positive	negative	positive

The value $x = 2$ locates _____ value of the function.
(g) If $x = 2$ is the only critical point of a function f and $f''(2) = 3$, then $x = 2$ locates a _____ value of the function.
(h) A function f defined and continuous on the interval $-2 \leq x \leq 5$ has critical points (or critical numbers) only at $x = -1$ and $x = 2$. The function f has values as given in the table below.

x	$f(x)$
-2	1
-1	2
2	0
5	2

The value $x = 2$ locates _____ value of the function. The value $f(x) = 2$ is _____ value of the function.

[65*i*]; [65*iv*]; [65*iv*]; [65*ii*]; [65*iii*]; [65*ii*]; [65*iii*]; [65*iii and* 65*i*]

66. Determine if each of the following statements is *True* or *False*. If you decide a statement is false, provide a counterexample to show why it is false and then rewrite the statement in order to make it true. Unless otherwise specified, assume each function is defined and continuous for all real numbers.

a). A critical point (or critical number) of a function f of a variable x is the x-coordinate of a relative maximum or minimum value of the function.

`False, Counterexample:` Note that $x = 0$ is a critical point for the function $f(x) = x^3$, but that $x = 0$ corresponds to neither a relative maximum nor a relative minimum value of f, it is just point of inflection.

b). A continuous function on a closed interval can have only one maximum value.

`True, Maximum = Global maximum, and always is one, unique`

c). If $f''(x)$ is always positive, then the function f must have a relative minimum value.

`False, Counterexample:` For $f(x) = e^x$, $f''(x)$ is always positive, but the function $f(x) = e^x$ has no relative extrema.

d). If a function f has a local minimum value at $x = c$, then $f'(c) = 0$.

`False, Counterexample:` For $f(x) = |x|$ has a local minimum at $x = 0$, but $f'(0)$ is not defined. See definition 4.5.3, or $f'(c)$ fails to exist.

e). If $f'(2) = 0$ and $f''(2) < 0$, then $x = 2$ locates a relative maximum value of f. T

f). If $f''(c) = 0$, then $x = c$ is a point of inflection for the function f and cannot be the x-coordinate of a maximum or minimum point on the graph of f.

`False,` If $f''(c) = 0$ for a function f, then $x = c$ may or may not be an inflection point for f and $x = c$ may or may not correspond to a relative minimum or maximum value of f.
`Counterexample:` For $f(x) = x^4, f''(0) = 0$ is not a point of inflection. Note that $x = 0$ does correspond to a relative and absolute minimum value of f.

g). If a function f is defined on a closed interval and $f'(x) > 0$ for all x in the interval, then the absolute maximum value of the function will occur at the right endpoint of the interval. `True`

h). The absolute minimum value of a continuous function on a closed interval can occur at only one point.

`False, There is exactly one absolute minimum value of a continuous function on a closed interval, but this minimum value can occur at more than one point in the interval.`
`Counterexample:` For $f(x) = \sin x$ takes on the unique (only one) minimum value of -1 at the points $x = \frac{3\pi}{2}, x = \frac{7\pi}{2}$ and $x = \frac{11\pi}{2}$ in the interval $0 < x < 6\pi$

i). If $x = 2$ is the only critical point of a function f and $f''(2) > 0$, then $f(2)$ is the minimum value of the function.

`True, If we had said, relative minima, would it still be true?`

j). To locate the absolute extrema of a continuous function on a closed interval, you need only compare the y-values of all critical points.

`False, To locate the absolute extrema of a continuous function on a closed interval, you must compare the y-values of all critical points AND ENDPOINTS. Counterexample: The function` $f(x) = x^2$ `on the interval` $2 \le x \le 6$ `has its absolute minimum value at` $x = 2$ `and its absolute maximum value at` $x = 6$. `Neither` $x = 2$ `nor` $x = 6$ `is a critical point of the function.`

k). Absolute extrema of a continuous function on a closed interval can occur only at endpoints or critical points. `True`

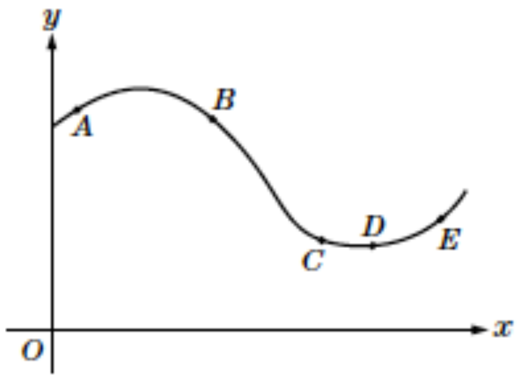
67. Given a differential equation $\frac{dy}{dx} = 2x + y$

(a) How do we know that $(1, -2)$ is a critical point for the function $y = f(x)$?

$f'(x, y) = \frac{dy}{dx} = 0$ `or does not exist. Show it.`

(b) Is the point $(1, -2)$ a relative maximum point, relative minimum point, or neither? Justify your answer.

$f''(x, y) = \frac{d^2y}{dx^2} = 2 + \frac{dy}{dx} = 2 + 0 = 2 > 0$ `is a relative minimum.`

68. 

At which of the five points on the graph in the figure at the left are $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ both negative?

(A) A (B) B (C) C (D) D (E) E

`Why is the solution` B `not` C`? Slope and Concavity both used?`

69. Which of the following statements about the function given by $f(x) = x^4 - 2x^3$ is true?

(A) The function has no relative extremum.

(B) The graph of the function has one point of inflection and the function has two relative extrema.

(C) The graph of the function has two points of inflection and the function has one relative extremum.

(D) The graph of the function has two points of inflection and the function has two relative extrema.

(E) The graph of the function has two points of inflection and the function has three relative extrema.

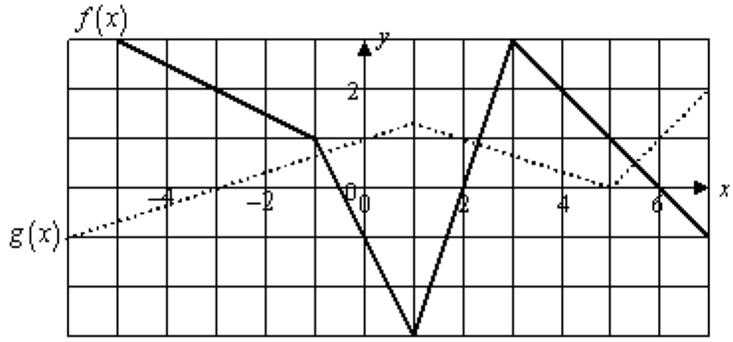
The answer is (69C).

70. Find the 100th derivative of $f(x) = x^5 + e^{2x}$ at $x = 0$.

71. A function is defined for all real numbers and has the following property

$$f(x + h) - f(x) = 4x^2h + 2xh - 6x^3h^2$$

h is a small distance from x, find $f'(-31)$



72. Below are the graphs of $f(x)$ and $g(x)$.

    Find

    (a) $\frac{d}{dx}\left(\frac{f}{g}\right)(0)$ (b) $\frac{d}{dx}g \circ f(3)$

73. The number of live grass hoppers per square meter x hours after spraying with a pesticide is modeled by the function $f(x) = \frac{120}{x+2}$,

    (a) Find the number of live grasshoppers per square meter 8 hours after spraying
    (b) Find the number of live grasshoppers per square meter 58 hours after spraying
    (c) Find $f'(x)$ using the definition of the derivative
    (d) Find $f'(1)$ and interpret the result of your answer giving appropriate units
    (e) Based on your answer in (iii), systematically explain when the grasshoppers are dying at the greatest rate; is it immediately after spraying or a long time after spraying?

74. Let g be a function such that $g(2) = 4$ and whose derivative is known to be

$$g'(x) = \sqrt{x^2 + 2}$$

    use linear approximation to estimate the value of $g(1.95)$.

75. Oil is leaking from a tanker and is forming a circular ring whose radius is increasing at a rate of 10m/minute. How are

    (a) the circumference of the ring; and
    (b) the area of the ring

    changing when the radius of the ring is 600m

76. Suppose g is a differentiable function that satisfies the following three properties:

    1. g is concave up 2. $g(1) = 9$ 3. $g(5) = 3$

    (a) What is the average rate of change of g on the interval $[1, 5]$?
    (b) Which is larger, $g'(2)$ or $g'(4)$? Explain.
    (c) What is the maximum possible value for $g'(3)$? Explain your reasoning.

77. Sketch the graph of a function on the interval $[-3, 10]$ that has:-

(a) An absolute maximum, an absolute minimum, a local maximum, a local minimum, all different

(b) An absolute maximum and an absolute minimum but no local extrema

78. Derive a linear approximating function for $f(x) = \frac{e^{2x}}{(x+1)}$ near the point $x = 1$. Hence evaluate $f(1.01)$ using linear approximation. What is the magnitude of the error?

79. State whether True or False, an explanation is necessary

(a) If $f(x)$ is increasing, then $f'(x)$ is increasing.

(b) Suppose $f'(a) \geq f'(b)$ whenever $a \leq b$. Then f has no points of inflection.

(c) If $f(x)$ is defined for all x, then $f'(x)$ is defined for all x.

(d) If f and g are functions whose second derivatives are defined, then

$$(fg)'' = fg'' + f''g$$

(e) If the radius of a circle is increasing at a constant rate, then so is the area.

(f) If $f(x)$ has an inverse function, then the derivative of the inverse function is $\frac{1}{f'(x)}$.

(g) If $f'(1) = -3.4$ and $g'(1) = 4.1$, then the function $h(x) = f(x) + g(x)$ is increasing at $x = 1$.

(h) The graph of $y = xe^{-0.1x}$ has an inflection point at $x = 20$

(i) Let f be a continuous function on the interval $[1, 10]$ and differentiable on $(1, 10)$. Suppose that $f(5) = 3$ and $f(2) = 1$. Then there is a point c in the interval $(2, 5)$ so that $f(c) = 2$.

(j) If a is a local maximum for the function f on the interval $[2, 50]$, then $f'(a) = 0$.

(k) The 100^{th} derivative of $f(x) = x^5 + e^{2x}$ at $x = 0$ is 2^{100}

(l) If $f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)$, then

$$f'(x) = (x-1) + (x-2) + (x-3) + (x-4) + (x-5)$$

(m) If f is continuous on $[a, b]$, then f has a global maximum and global minimum on that interval

80. Suppose f has a continuous derivative whose values are given in the following table.

x	0	1	2	3	4	5	6	7	8	9	10
$f'(x)$	5	2	1	-2	-5	-3	-1	2	3	1	-1

(a) Using the data in the table, estimate the x-coordinates of indicated critical points of f for $0 < x < 10$.

(b) For each critical point above, indicate if it is a local maximum of f, a local minimum, or neither.

(c) Approximate the interval(s) between $x = 0$ and $x = 10$, if any, for which the data indicates that the graph of f is concave up?

(d) If $f(0) = 4$, approximate the value of $f(0.2)$.

81. Find the number which satisfies the Roll's Theorem for

(a) $f(x) = 4 - x^2 \;\; ; -2 \leq x \leq 2$

(b) $f(x) = x^2 - 9 \;\; ; -4 \leq x \leq 3$

82. Find the critical points and intervals on which $f(x) = x^2+2x+9$ is increasing and decreasing

`f is` *decreasing* `on the interval` $(-\infty, -1)$.
`And f is` *increasing* `on the interval` $(-1, \infty)$

83. Find the critical points and intervals on which $f(x) = x^3 - 12x + 3$ is increasing, decreasing.

`f is` *decreasing* `on` $(-2, +2)$, `and f is` *increasing* `on` $(2, \infty)$.

84. Jonathan has 200 feet of fencing with which he wishes to enclose the largest possible rectangular garden. What is the largest garden you can have? $2500\ ft^2$

85. Approximate $\sin 31^o$ by differentials.

86. Approximate $\ln(x + 2)$ by differentials, in terms of $\ln x$ and x:

This *non-numerical* question is somewhat more sensible. Take $f(x) = \ln\ x$, so that $f'(x) = \frac{1}{x}$. Then

$$\Delta x = (x + 2) - x = 2$$

and by the formulas above

$$\ln(x + 2) = f(x + 2) \approx f(x) + f'(x) \cdot 2 = \ln\ x + \frac{2}{x}$$

87. Approximate $\ln(e + 2)$ in terms of differentials:

Use $f(x) = \ln\ x$ again, so $f'(x) = \frac{1}{x}$. We probably have to imagine that we can 'easily evaluate' both $\ln\ x$ and $\frac{1}{x}$ at $x = e$. (Do we know a numerical approximation to e?). Now

$$\Delta x = (e + 2) - e = 2$$

so we have

$$\ln(e + 2) = f(e + 2) \approx f(e) + f'(e) \cdot 2 = \ln\ e + \frac{2}{e} = 1 + \frac{2}{e}$$

since $\ln\ e = 1$.

In Exercises $88 - 92$, find the rate $f'(t)$ for the given values of g, g', and t.

88. $f(t) = 2(g(t))^3 + 5,\ t = 1,\ g(1) = 3,\ g'(1) = -2$

89. $f(t) = \sqrt{2 + g(t)},\ t = 0,\ g(0) = 3,\ g'(0) = 4$

90. $f(t) = \frac{1}{1+g(t)},\ t = 2,\ g(2) = 3,\ g'(2) = -2$

91. $[f(t)]^2 + [g(t)]^3 = 265,\ t = 1,\ g(1) = 6,\ g'(1) = -2, f(t) = 7$

92. $\sin(f(t)) = [g(t)]^2,\ t = 0,\ g(0) = 1,\ g'(0) = -2,\ f(0) = \frac{\pi}{6}$

93. When a pebble is tossed into a still pond, ripples move out from the point where the stone hits in the form of concentric circles. Find the rate at which the area of the disturbed water is increasing when the radius of the outermost circle equals $10\ m$ if the radius is increasing at a rate of $2\ m/s$.

94. If a ray of monochromatic light traveling in a vacuum makes an angle of incidence α with the normal to the surface of a substance a and the angle of refraction β in the substance, then **Snell's law** states that

$$\frac{\sin\alpha}{\sin\beta} = C_a$$

where C_a is a constant, called the index of refraction for the substance a. For water, the index of refraction is $C_w = 1.33$. If the angle of incidence of the light ray striking water is decreasing at a rate of 0.2 radians per second, find an expression for the rate at which the angle of refraction is decreasing.

95. The velocity of Viscous liquid flowing through a circular tube is not the same at all points of a cross section. Provided the velocity is not too great, the flow has a maximum velocity at the center and decreases to zero at the walls. For a point at a radial distance r from the center of the tube, the velocity of the flow is given by

$$v = \frac{\alpha}{L}(R^2 - r^2)$$

where R is the radius of the tube, L is its length, and α is a constant. Find the acceleration of the fluid moving at the center of the tube if

(a) $L = 25\ cm$ is fixed and R is increasing at a rate of $0.2\ cm/min$ at the instant when $R = 10\ cm$.

(b) $R = 10\ cm$ is fixed and L is increasing at a rate of $0.5\ cm/s$ at the instant when $L = 25\ cm$.

96. True or False, $f(x) = x - |x|$ is differentiable at $x = 0$.

`False, see reasons for differentiability to fail`

97. At $(0, 0)$, the graph of $|x|$

A. has infinity many tangent lines

B. has a tangent line at $y = 0$

C. has the tangent lines $y = -x$ and $y = x$

D. has no tangent line

E. has a vertical tangent line

`A derivative or tangent does not exist, cannot be drawn at sharp turns, why it is 97D`

98. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = f(a)$, then

$$\lim_{h\to 0}\frac{f(a+h) - f(a)}{h} =$$

A. $f(a)$

B. must exist and is $f'(a)$

C. $f'(a+h)$

D. It may not exist

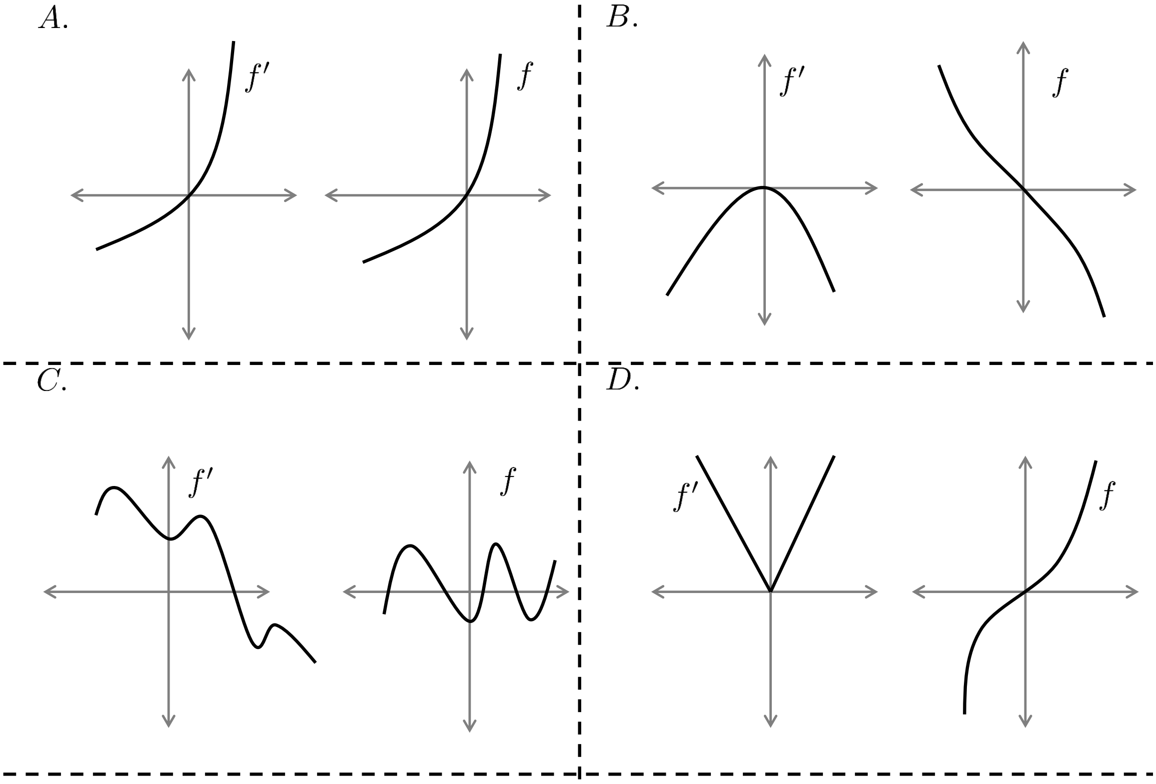
E. it must exist but there is not enough information to determine it exactly.

Continuity does not necessarily imply differentiability, although formula for $f'(a)$, but may not exist, reason for 98D

99. A snowball, in the shape of a sphere, is melting so that the radius is decreasing at a uniform rate of 1 cm/s. How fast is the volume decreasing when the radius equals 6 cm?

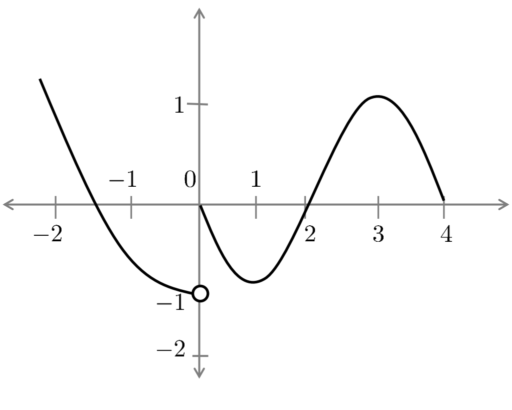
100. A point moves along the graph of $y = x^{\frac{5}{2}}$ so that its x-coordinate increases at the rate of 2 units per second. Find the rate at which the y coordinate is increasing as it passes the point $(4, 32)$.

101. Which of the graphs below properly matches the function and its possible derivative.



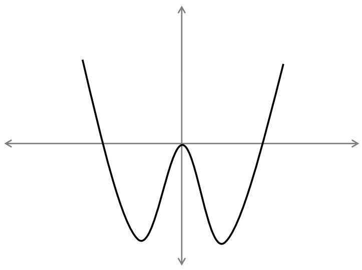
E. None of the above

102. The graph of the function is shown in the figure below. f has a vertical tangent at the point $(2, 0)$ and a horizontal tangent at the points $(1, -1)$ and $(3, 1)$. For what values of $x \in (-2, 4)$ is f not differentiable.

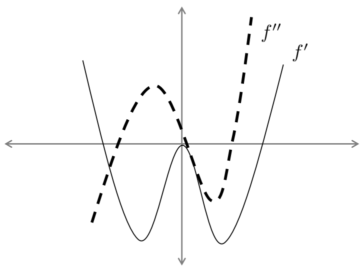


The graph might not be clear at $x = 2$ but there is a vertical tangent from the question. Thus not differentiable at both $x = 2$ and $x = 0$.

103. Given below is the graph of f'. Sketch a possible graph of f''.



`Using tangents on` f' `being positive and negatives, where` f'' `will be above or below the axis`



104. Consider

$$f(x) = |x^2 + x - 2| - 6 \ , \ \ -5 \leq x \leq 5$$

Write down the maximum value of $f(x)$ and the minimum value of $f(x)$ on $-5 \leq x \leq 5$

`Clearly its` 22 `and` -6 `respectively, but how?`

105. Write down the equation of the line tangent to the graph of $y = x \sin x$ at the point $(\pi, 0)$.

$$y = -\pi(x - \pi)$$

106. Find $\frac{dy}{dx}$ if

$$(xy)^{\frac{1}{2}} = xy - x$$

107. Compute y' given

$$x^2y - 4y^3 = 4xy^2 + x - 5 + 2\cos 2y$$

108.

109.

110.

4.7 Bibiliography