CALCULUS REVIEW QUESTIONS

(1) (a) Suppose α, β, γ are real numbers, prove that if $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$. **[3 marks]**

(b) Express the half open intervals $(-m, n]$ and $[-\alpha, \alpha)$ in set-builder notation. **[4 marks]**

(2) Solve for x. Express your answer in *interval notation.*

(a) $5 < 3x + 2 \leq 10$. **[2 marks]**

(b) $|2x - 7| > 3$. **[4 marks]**

(3) (a) Find the slope of a line passing through the points $(1, 2)$ and $(2, 3)$. **[2 marks]**

(b) Write the equation of this line. **[2 marks]**

(4) (a) By use of definitions, explain how the application of the Intermediate Value Theorem differs from that of the Fixed point Theorem? **[4 marks]**

(b) (i) Use the Intermediate Value Theorem show that the function $f(x) = x^3 - x - 1$ has a root between 1 and $3/2$. **[4 marks]**

(ii) Use the Fixed point Theorem to determine the points for the continuous function $f(x) = x^2 - 6$ in the closed interval $[-4, 4]$. **[2 marks]**

(5) Let $f(x) = x^2$ and $g(x) = \dfrac{x}{x-1}$. Find;

(a) $f \circ g$, **[2 marks]**

(b) $g \circ f$, **[2 marks]**

(c) $g \circ g$. **[2 marks]**

(6) Compute the following limits:

(i) $\displaystyle\lim_{x\to 2} \frac{x^2 + 3x - 10}{x - 2}$. **[3 marks]**

(ii) $\displaystyle\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}$. **[4 marks]**

(7) (a) Define the derivative of a function $f(x)$ at $x = a$. **[2 marks]**

(b) For the function $\dfrac{1}{x+2}$, determine the $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$. **[6 marks]**

(c) Use the definition of a derivative to find $f'(x)$ of the following functions.

(i) $f(x) = \dfrac{1}{2}x - \dfrac{3}{5}$. **[4 marks]**

(ii) $f(x) = 4 - \sqrt{x+3}$. **[4 marks]**

(iii) $f(x) = x^2$. **[4 marks]**

(8) (a) State the definition of a limit of a function $f(x)$ as $x \to a$. **[2 marks]**

(b) Use the definition in (a) above to:

(i) check whether the $\displaystyle\lim_{x\to 0} \frac{|x|}{4x}$ exists. **[3 marks]**

(ii) show that the $\lim_{x \to -6} \left(\frac{x+4}{2-x} \right) = -\frac{1}{4}$. **[5 marks]**

(c) (i) Use the ($\epsilon - \delta$ definition) to prove that the $\lim_{x \to 3} \left(\frac{x}{4x-9} \right) = 1$. **[5 marks]**

(ii) Use the informal definition of a limit of a function to verify that the $\lim_{x \to 1} h(x)$ does not exist if $h(x) = \begin{cases} x; & x < 1 \\ 3-x; & x \geq 1 \end{cases}$. **[5 marks]**

(9) Find the limit if it exists.

(a) $\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - x - 2}$. **[4 marks]**

(b) $\lim_{x \to 3} \frac{x-3}{\sqrt{x+6} - 3}$. **[4 marks]**

(c) $\lim_{x \to -3^-} \frac{x+2}{x+3}$. **[4 marks]**

(d) $\lim_{x \to 0} x^2 \sin \left(\frac{2\pi}{x} \right)$. **[4 marks]**

(e) $\lim_{x \to -1} \left(\frac{2}{x^2 - 1} + \frac{1}{x+1} \right)$. **[4 marks]**

(10) (a) Consider a piece-wise function defined as

$$f(x) = \begin{cases} 2 + \sqrt{x} & \text{if } x \geq 1, \\ \frac{1}{2}x + \frac{5}{2} & \text{if } x < 1. \end{cases}$$

Use the limit definition of a derivative to show whether or not $f(x)$ is differentiable
at $x = 1$. **[5 marks]**

(b) Determine if the function $f(x) = \begin{cases} x^2 + 2x; & x \leq -2 \\ x^3 - 6x; & x > -2 \end{cases}$ is continuous at $x = -2$. **[5 marks]**

(c) Determine the value of α such that the function

$$f(x) = \begin{cases} x^2 + \alpha; & x \leq \alpha \\ -x - 1; & x > \alpha \end{cases}$$

is continuous on $\mathbb{R}$. **[5 marks]**

(d) Re-define the function

$$h(x) = \frac{x^2 + x - 12}{x - 3}$$

so that it is continuous at $x = 3$. **[5 marks]**

(11) Consider the function $f(x) = \begin{cases} 6 & \text{if } x < -3, \\ 10 - x^2 & \text{if } -3 \leq x < 2, \\ x + 4 & \text{if } x \geq 2. \end{cases}$

(a) Sketch the graph of $y = f(x)$. **[4 marks]**

(b) Evaluate the following (write "does not exist" where apropriate): **[8 marks]**

(i) $\lim_{x \to -3^-} f(x) =$

(iv) $f(-3) =$

(vi) $\lim_{x \to 2^+} f(x) =$

(ii) $\lim_{x \to -3^+} f(x) =$

(v) $\lim_{x \to 2^-} f(x) =$

(vii) $\lim_{x \to 2} f(x) =$

(iii) $\lim_{x \to -3} f(x) =$

(viii) $f(2) =$

(c) Is $f(x)$ continuous at $x = -3$? Explain your answer using the definition of continuity. **[4 marks]**

(d) Is $f(x)$ continuous at $x = 2$? Explain. **[4 marks]**

(12) (a) Using illustrations, distinguish between an "open-infinite interval" and an "infinite-infinite interval". [2 marks]

(b) Solve the following inequalities:

(i) $\dfrac{(x-1)(x+2)}{(x-3)} < 0.$ [3 marks]

(ii) $(x-1)(x-2)(x-5) \leq 0.$ [3 marks]

(13) (a) By use of definitions, explain how the application of the Intermediate Value Theorem differs from that of the Fixed point Theorem? [4 marks]

(b) Choose an appropriate theorem from 2 (a) above and use it to:

(i) show that the function $f(x) = 4x^3 - 6x^2 + 3x - 2$ has a root between 1 and 2. [4 marks]

(ii) determine the points for the continuous function $f(x) = x^2 - 6$ in the closed interval $[-4, 4]$. [2 marks]

(14) (a) Suppose that $S(x) = \sqrt{x}$ for $x \geq 0$, and let f and g be differentiable functions about which the following is known: $f(3) = 2$, $f'(3) = 7$, $g(3) = 4$, $g'(3) = 5$. Compute the following;

(i) $(f + g)'(3)$. [2 marks]

(ii) $(f \cdot g)'(3)$. [2 marks]

(iii) $\left(\dfrac{f}{g}\right)'(3)$. [2 marks]

(iv) $(S \circ g)'(3)$. [2 marks]

(v) $\left(\dfrac{f \cdot g}{f - g}\right)'(3)$. [4 marks]

(b) State the Sandwich Theorem. [2 marks]

(c) Using a combination of the properties in (a) above and the theorem in (b) above, compute the following limits:

(i) $\lim\limits_{x \to -\infty} \dfrac{x^2(\sin x + \cos^3 x)}{(x^2+1)(x-1)}$. [4 marks]

(ii) $\lim\limits_{x \to \infty} \dfrac{\cos^2(2x)}{3-2x}$. [4 marks]

(15) (a) Define the derivative of a function $f(x)$ at $x = a$. [2 marks]

(b) Using your definition in (a) above, prove that $\dfrac{d}{dx}(x^n) = nx^{n-1}$. [6 marks]

(c) Find $\dfrac{dy}{dx}$ if:

(i) $y - xy^3 + 3x^2y - 2 = 0$. [4 marks]

(ii) $y = x^{\sin x} + e^{x^2} - x^x$. [4 marks]

(iii) $x = 2\cos t - \sin 2t, \quad y = 2\sin t - \cos 2t$. [4 marks]

(16) (a) State the definition of a limit of a function $f(x)$ as $x \to a$. [2 marks]

(b) Use the definition in (a) above to:

(i) check whether the $\lim\limits_{x \to 0} \dfrac{|x|}{4x}$ exists. [3 marks]

(ii) show that the $\lim\limits_{x \to -6} \left(\dfrac{x+4}{2-x}\right) = -\dfrac{1}{4}$. [6 marks]

(c) (i) Given that $X \subset \mathbb{R}$, $a \in \mathbb{R}$, $f : X \to \mathbb{R}$, $\lim\limits_{x \to a} h(x) = L$, and $\lim\limits_{x \to a} h(x) = L'$,
show that the limits are unique if and only if $L = L'$. [5 marks]

(ii) Use the formal definition in c(i) above to prove that the $\lim\limits_{x \to 1} h(x)$ does not exist if

$$h(x) = \begin{cases} x; & x < 1 \\ 3 - x; & x \geq 1 \end{cases}$$

[4 marks]

(17) Given the function $f(x) = \dfrac{2x^2}{x^2 - 1}$,

(a) find:

(i) the x and y intercepts. [3 marks]

(ii) where the curve is increasing and decreasing and classify the turning points. [5 marks]

(iii) where the function is concave up and concave down and establish the inflection points. [6 marks]

(iv) the horizontal and vertical asymptotes. [3 marks]

(b) Use all the results in (a) above to sketch the curve $f(x) = \dfrac{2x^2}{x^2-1}$. [3 marks]

(18) (a) Determine if the function $f(x) = \begin{cases} x^2+2x; & x \leq -2 \\ x^3-6x; & x > -2 \end{cases}$ is continuous at $x = -2$. [7 marks]

(b) Find the value of β such that the function

$$g(x) = \begin{cases} 3x^3 - x^2 - \beta x; & x > 1 \\ \beta x - 2; & x \leq 1 \end{cases}$$

is continuous at $x = 1$. [6 marks]

(c) Re-define the function $h(x) = \dfrac{x^2-16}{x-4}$ so that it is continuous at $x = 4$. [7 marks]

(19) (a) (i) A population of bacteria undergoes an exponential growth. If at noon, there are 1000 bacteria, and there are 2000 by 2 pm, when does the number of bacteria reach 8000? [5 marks]

(ii) Suppose a 5-meter ladder is sliding down a wall at a rate of 1.2 meters per second, how fast is the distance from the wall to the ladder's bottom changing when the top of the ladder is 3 meters above the ground? [5 marks]

(b) Let a function $y = \dfrac{x^2+1}{x^3(x-1)^2}$. Use logarithmic differentiation to find $\dfrac{dy}{dx}$. [5 marks]

(c) (i) What does the Intermediate Value Theorem say about whether the function

$$f(x) = x^3 - 2x + 1$$

has a root on the interval $[0, 2]$ of x-values? [2 marks]

(ii) The conclusion of the Mean Value Theorem (MVT) includes the equality

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Draw and label a graph to show what the MVT is referring to with this equality. [3 marks]

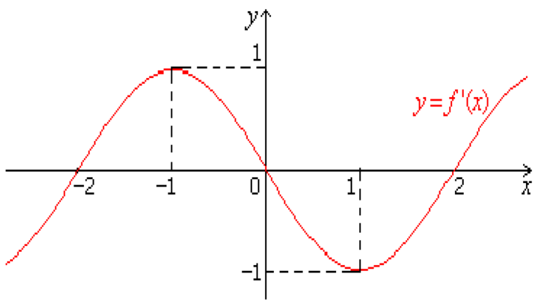
(20) (a) Differentiate between an even and an odd function. [2 marks]

(b) Let $f(x) = \frac{x^2+3x}{x^4-3x^2+4}$. Is $f(x)$ even or odd? Justify your answer. [3 marks]

(c) Give an example of a function that is neither even nor odd. [1 mark]

(21) Solve the inequality: $1 \leq |3x - 5|$ and show its solution set on the real line. [3 marks]

(22) (a) If $f(x) = x + 2\cos x$, find an equation of the tangent line to the curve $f(x)$ at the point where the curve crosses the $y-$axis. [3 marks]



(b) Find $\frac{dy}{dx}$ if $y = (\sin x)^{\sin x}$. [3 marks]

(23) Given that $f\left(\dfrac{3x-2}{2x+1}\right) = x$, find $f'(1)$. [2 marks]

(24) Give the $\epsilon - \delta$ definition of: $\lim_{x\to x_0} f(x) = l$. Use the definition to prove that

$$\lim_{x\to 3}(3x-3) = 6.$$

[3 marks]

(25) Use L'Hôpitals rule to evaluate:

$$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right).$$

[4 marks]

(26) (a) Sketch

$$g(x) = \begin{cases} \frac{x-2}{|x-2|}; & \text{if } \ x \neq 2, \\ \\ 0; & \text{if } \ x = 2. \end{cases}$$

[2 marks]

(b) Find the left and right hand limit of $g(x)$ at $x = 0$ if they exist. [1 mark]

(c) Is $g(x)$ continuous at $x = 0$? Justify your answer. [2 marks]

(27) Explain why it would not be appropriate to use the Intermediate Value Theorem to conclude that $f(x) = \frac{2}{x}$ takes on a value of 0 on $[-1, 1]$. [1 mark]

(28) Give an example of two different functions $f(x)$ and $g(x)$ such that

$$f'(0) = f''(0) = g'(0) = g''(0)$$

[2 Marks]

(29) The graph of a derivative $f'(x)$ of a function $f(x)$ is shown below:

(a) State and classify the local extrema of $f(x)$. [3 Marks]

(b) Determine where the graph of f is concave down in $[-2, 2]$. [1 Mark]

(c) Find the points of inflection of f. [2 Marks]

(30) Sketch the graph of a function that satisfies the following conditions:

$f(-3) = f(3) = f'(0) = 0$

$f'(x) > 0$ if $-3 \leq x \leq 0$,

$f'(x) < 0$, if $0 \leq x \leq 3$

$f''(x) < 0$, if $-3 \leq x \leq 3$

[2 Marks]

(31) (a) (i) Define an injective function.

(ii) Define a strictly monotonic increasing function in interval (a, b).

(iii) Let $f(x)$ be defined on domain D. Prove that $f^{-1}(x)$ exists if $f(x)$ is strictly monotonic increasing on D.

(iv) Let $f(x) = \frac{6x}{x^2-9}$ and $g(x) = \sqrt{3x}$, find the domain of $(f \circ g)(x)$. [8 marks]

(b) An estate investor handles 120 houses. When the rent of each house is \$80 per month, all the houses are full. However, for each \$2 increase in rent, one of the houses will become vacant. Each occupied house requires an average of \$4 per month for repairs and service. What rent should be charged to make the most profit? [7 marks]

(32) Given

$$f(x) = \frac{2x^2}{x^2 - 1},$$

find

(i) the x and $y-$ intercepts. [2 Marks]

(ii) intervals where $f(x)$ is increasing and/or decreasing [2 Marks]

(iii) the critical numbers and classify them. [2 Marks]

(iv) where the function is concave up or concave down. [2 Marks]

(v) the inflection points. [2 Marks]

(vi) the horizontal and vertical asymptotes. [2 Marks]

Hence, sketch the graph of $f(x)$. [3 Marks]

(33) (a) Give the $\epsilon - \delta$ definition of continuity of a function $f(x)$ at a point x_0. Hence, prove that $f(x) = 3x + 5$ is continuous at $x = 1$. [5 marks]

(b) Let f be a continuous function defined on $[0, 1]$ with range $[0, 1]$. Show that there is an x in $[0, 1]$ such that $f(x) = 1 - x$. [5 marks]

(c) State the Sandwich Theorem. Hence, find $\lim_{x \to 0} x f(x)$ where $f(x)$ is a function such that $|f(x)| \leq B$, for any $x \neq 0$. [5 marks]

(34) (a) Find a linear approximation of the function $f(x) = \sqrt[3]{1 + x}$ at $x_0 = 0$ and use it to approximate $\sqrt[3]{1.1}$. How good is your approximation? [5 marks]

(b) A ladder 10 metres long rests along a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 metres per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 metres from the wall. [5 marks]

(c) Determine the scalars α and β for the function

$$f(x) = \begin{cases} \alpha x - 3, & x < 4 \\ -x^2 + 10x - \beta, & x \geq 4 \end{cases}$$

so that it satisfies the hypotheses of Mean Value Theorem on the interval $[2, 6]$. [5 Marks]

(35) State the limit definition of a derivative of a function at a point x_0. Use this definition to find the derivative of the function $f(x) = x^2 + 2\sqrt{x}$ at $x_0 = 1$. [6 marks]

(36) Sketch the graph of a continuous function on the interval $[-3, 10]$ that has;

(a) An absolute maximum, an absolute minimum, a local maximum, a local minimum,
all different. [3 marks]

(b) An absolute maximum and an absolute minimum but no local extrema. [3 marks]

(37) At what points do the vertical tangents touch the circle $(x-1)^2+(y-2)^2 = 9$? [6 marks]

(38) Derive a linear approximation function for $f(x) = \frac{e^{2x}}{(x+1)}$ near the point $x = 1$. Hence evaluate $f(1.01)$ using linear approximation. What is the magnitude of the error?
[6 marks]

(39) Use L'hôpital's rule to find

(a) $\lim_{x\to\infty} \left(1 + \frac{1}{2x}\right)^{3x}$ [3 marks]

(b) $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$ [3 marks]

In Problems 40-44, sketch the graphs of the given functions. Determine the domain and the range of each function.

(40) $f(x) = |x|$.

(41) $f(x) = \dfrac{|x-1|}{x-1}$.

(42) $f(x) = \sqrt{4 - x^2}$.

(43) $f(x) = |x-1| + |x-2|$.

(44) $f(x) = \begin{cases} 5, & 0 < x \leq 1 \\ 10, & 1 < x \leq 2 \\ 15, & 2 < x \leq 3 \\ 20, & 3 < x \leq 4 \end{cases}$

Let $A, B \subseteq \mathbb{R}$. A function $f : A \longrightarrow B$ is even if $f(x) = f(-x)$ and odd if $f(-x) = -f(x)$. In Problems 45-48, determine whether the given function is odd or even.

(45) $f(x) = x^2$.

(46) $f(x) = x^3$.

(47) $f(x) = \cos x$.

(48) $f(x) = \sin x$.

(49) If $f(x) = \dfrac{x-1}{x+1}$ show that $f\left(\dfrac{1}{x}\right) = -f(x)$ and $f\left(-\dfrac{1}{x}\right) = -\dfrac{1}{f(x)}$.

(50) If $f(x) = x^2 - x$, show that $f(x+1) = f(-x)$.

(51) If $f(x) = \dfrac{1}{x}$, show that $f(a) - f(b) = f\left(\dfrac{ab}{b-a}\right)$.

(52) If $y = f(x) = \dfrac{5x+3}{4x-5}$, show that $x = f(y)$.

(53) If $f(x) = 2^x$, show that $f(x+3) - f(x-1) = \dfrac{15}{2} f(x)$ and $\dfrac{f(x+3)}{f(x-1)} = f(4)$.

(54) Use the Squeeze Law to calculate the limits in Problems 54a-54e.

(a) $\lim_{x \to 1} (x-1) \cos\left(\dfrac{1}{x-1}\right)$.

(b) $\lim_{x \to 0} \sin x \sin \dfrac{1}{x}$

(c) $\lim_{x \to \infty} \dfrac{\sin x}{x}$

(d) $\lim_{x \to 0} \dfrac{x \sin x}{x^2+1}$.

(e) $\lim_{x \to 2} g(x)$ if $|g(x) + 3| < (2-x)^{10}$ for all $x \in \mathbb{R}$.

(55) Find the limits in Problems 55a-55f.

(a) $\lim_{x \to 2} \dfrac{|x-2|}{x-2}$.

(b) $\lim_{x \to -\frac{1}{3}} \dfrac{|3x+1|}{3x+1}$.

(c) $\lim_{x \to 5} \dfrac{\sqrt{x+20}-5}{x-5}$.

(d) $\lim_{x \to 5} \dfrac{\sqrt{x+9}-\sqrt{14}}{x-5}$.

(e) $\lim_{x \to 0} \dfrac{\sqrt{x+a}-\sqrt{a}}{x}$, $a > 0$.

(f) $\lim_{x \to 1} \dfrac{1-x^3}{2-\sqrt{x^2+3}}$.

(56) Given that $\lim_{x \to 0} (\sin x / x) = 1$, evaluate the limits in Problems 56a-56h.

(a) $\lim_{x \to 0} \dfrac{\tan 2x}{\tan 3x}$.

(b) $\lim_{x \to 0} \dfrac{\cos 2x - 1}{x}$.

(c) $\lim_{x \to 0} \dfrac{x^3}{3 \sin^2 2x}$.

(d) $\lim_{x \to 0} \dfrac{(\sin x/2)}{(\sin x/5)}$.

(e) $\lim_{x \to 0} \dfrac{\sin(\sin x)}{x}$.

(f) $\lim_{x \to 0+} \dfrac{\sqrt{x}}{\sqrt{x} + \sin\sqrt{x}}$.

(g) $\lim_{x \to \infty} x \sin \dfrac{1}{x}$.

(h) $\lim_{x \to 0} \dfrac{\cos x^2 - 1}{x^4}$.

(57) Find the values of the infinite limits in Problems 57a-57d.

(a) $\lim_{x\to-\infty} \dfrac{6x^5-3x^2+7x+2}{2x^5+3x^2+5}$.

(b) $\lim_{x\to\infty} \dfrac{x^3+2x+4}{x^5+3x^2+7}$.

(c) $\lim_{x\to\infty} (\sqrt{x+1}-\sqrt{x})$.

(d) $\lim_{x\to\infty} f(x)$ if $f(x+1)=\sqrt{1+f(x)}$ and $\lim_{x\to\infty} f(x)$ exists.

(58) Provide solutions to the following problems.

(a) Show that the function $f(x)=3+x-x^2, x\in[0,3]$ satisfies the Intermediate Value Theorem if $1\in f([0,3])$.

(b) Show that the function $f(x)=x^2-(a+b-1)x+ab$ takes on the value $(a+b)/2$.

(c) Prove that the functions $f(x)=\sin x$ and $g(x)=\cos x$ are continuous at every $x_0\in\mathbb{R}$.

(d) Show that the function $g(x)=x^3-15x+1$ has at least has at least three zeros in $[-4,4]$.

(e) Show that the equation $xe^{\sin x}=\cos x$ has a solution in $(0,\pi/2)$.

(f) Find the values of a and b for which the function

$$f(x)=\begin{cases} x^2+2x-a, & x<2 \\ 6, & x=2 \\ 2ax+b, & x>2 \end{cases}$$

is continuous at $x=2$.

(g) Show that the equation $2\sin x=x^2-1$ has a solution between 1 and 2.

(h) Let a function $f:[0,1]\longrightarrow\mathbb{R}$ be defined by

$$f(x)=\begin{cases} x, & x \textit{ is rational} \\ 1-x, & x \textit{ is irrational} \end{cases}$$

Discuss the continuity of f.

(i) Let $f:\mathbb{R}\longrightarrow\mathbb{R}$

$$f(x)=\begin{cases} x, & x \textit{ is rational} \\ x^2, & x \textit{ is irrational} \end{cases}$$

Discuss the continuity of f.

(j) Find all fixed points of the function $f(x)=2x^3-3x^2-x+3$ defined on the closed interval $[-2,2]$.

(59) Use the definition of a derivative in Problems 59a-59e to find

(a) $g'(2)$ if $g(x)=\sqrt{4x-6}$.

(b) $f'(x)$ if $f(x)=\cos x$.

(c) $f'(3)$ if $f(x)=\dfrac{1}{\sqrt{x}}$.

(d) $f'(1)$ if $f(x)=(x+1)^3$.

(e) $f'(2)$ if $f(x)=\dfrac{3+x}{3-x}, x\neq 3$.

(60) Use the definition of a derivative of a function at a point in Problems 60a-60g.

(a) Find $f^{'}(0)$ and $f^{'}(1)$ if $f(x) = \frac{|x|}{(1+|x|)}$.

(b) Let

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Find $f^{'}(0)$.

(c) Let

$$f(x) = \begin{cases} x, & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$$

Find $f^{'}(1+)$ and $f^{'}(1-)$.

(d) Let

$$f(x) = \begin{cases} 1 + x^2, & x \leq 0 \\ x^2, & x > 0 \end{cases}$$

Find $f^{'}(1+)$ and $f^{'}(1-)$ if they exist. Is f differentiable at 0?

(e) Let

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ (x-2)^2, & 1 < x \leq 2 \end{cases}$$

Show that f is not differentiable at $x = 1$.

(f) Prove that the function $f(x) = |x - 3|$ is not differentiable at $x = 3$.

(g) Prove that the function $f(x) = \lfloor x \rfloor, n \leq x < n + 1, n \in \mathbb{Z}$ is not differentiable at $x = 2$.

(61) Find the slope of the tangent to the curve $y = f(x) = \sqrt{x}$ at the point $(4, 2)$ and write down the equation of the tangent line that passes through $(4, 2)$.

(62) Find the equation of the tangent line T to the curve $f(x) = \sqrt{2 - x}$ which is parallel to the straight line $x + 4y = 8$.

(63) Find an equation of the normal line N to the curve $f(x) = x^2$ which is perpendicular to the line $x + 2y = 3$.

(64) Show that $ky = 2a(x + h)$ is a straight line which is tangent to the parabola $y^2 = 4ax$ at the point (h, k) on the parabola.

(65) Find the equations of the tangent lines to the curve $f(x) = x^2 + 4$ which passes through the origin $(0, 0)$.

(66) If $f^{'}(a) = b$, evaluate

$$\lim_{t \to 0} \frac{f(a + 2t) - f(a + t)}{2t}$$

(67) Prove that if f is differentiable at x_0, then

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h} = f^{'}(x_0).$$

(68) Let $f:\mathbb{R} \longrightarrow \mathbb{R}$, $f(x+h) = f(x)f(h)$ and $f(0) \neq 0$. Prove that
  (a) $f(0) = 1$.
  (b) If $f^{'}(0)$ exists, then $f^{'}(x)$ exists for all $x \in \mathbb{R}$ and $f^{'}(x) = f(x)f^{'}(0)$.

(69) Provide solutions to the following problems.
  (a) Prove that the function $f(x) = 4x^3 - 3x^2 - 8x + 6$ has a unique root in the closed interval $[0,1]$.
  (b) Prove that the equation $f(x) = x^3 - 4x^2 + \cos x = 0$ has one and only one solution between 0 and 1.
  (c) Explain why the function $f(x) = 1 - \sqrt[3]{x^2}$ does not satisfy Rolle's Theorem on the closed interval $[-1,1]$.
  (d) Show that the function

$$f(x) = \frac{\sqrt{1-x^2}}{1+x^2}$$

  satisfies the hypotheses of Rolle's Theorem on $[-1,1]$ and find all numbers $c \in (-1,1)$ such that $f^{'}(c) = 0$.
  (e) By using Rolle's Theorem, show that $f(x) = x^{10} + ax - b$ $(a,b \in \mathbb{R})$ has at most two real roots.
  (f) Show that the function

$$f(x) = x^2 + \frac{4}{x+1}$$

  satisfies the hypotheses of the Mean Value Theorem on $[2,5]$. Find all numbers $c \in (2,5)$ satisfying the conclusion of the Mean Value Theorem.
  (g) Use the Mean Value Theorem to show that $1/9 < \sqrt{66} - 8 < 1/8$.
  (h) Use the Mean Value Theorem to estimate the value of $\sqrt{80}$.
  (i) Use the Mean Value Theorem to prove that $\lim_{x\to\infty}\left(\sqrt{x+1} - \sqrt{x}\right) = 0$.
  (j) Prove that if a function f is differentiable and $f^{'}(x) = 0, \forall x \in \mathbb{R}$, then f is a constant function.
  (k) If f and g are continuous functions on $[a,b]$ and differentiable on (a,b), and if $f^{'}(x) = g^{'}(x)$ for all $x \in (a,b)$, then prove that

$$f(x) = g(x) + c \; for \; all \; x \in [a,b],$$

  where c is an arbitrary constant.
  (l) Suppose the $a, b \in \mathbb{R}$ such that $0 < a < b$. Use the Mean Value Theorem to show that

$$1 - \frac{a}{b} < \ln\frac{b}{a} < \frac{b}{a} - 1.$$

(m) Prove that if $a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n = 0$ has a positive root $x = r$, then $a_1 + 2a_2x + \cdots + (n-1)a_{n-2}x^{n-1} + nx^{n-1} = 0$ has a positive root less than r.

(70) Provide solutions to the following problems.

(a) Differentiate

(i) $f(x) = \dfrac{ax+b}{cx-d}$, where a, b, c, d are constants.

(ii) $f(x) = (x^3 + x^2 - 1)\left(1 + \dfrac{1}{x^2}\right)$

For Problems 70b-70e, find every point where the function is not differentiable

(b) $f(x) = |x+5|$

(c) $f(x) = |x^2 - 4|$

(d) $f(x) = \dfrac{x^2}{x^2-1}$

(e) $f(x) = \begin{cases} x^2 - x, & x > 1 \\ x^3 - x^2 + x, & x \leq 1 \end{cases}$

(f) Find the equation of the tangent line to the curve $f(x) = 1/(x+1)$ at the point $(0, 1)$.

(g) Find all points on the graph of $f(x) = 2x^3 + 3x^2 - 12x + 5$ where the tangent line is horizontal.

(h) Find all points on the graph of $f(x) = x^3 + x - 1$ where the tangent line has slope 2.

(i) If the curve $f(x) = a + bx + cx^2$ passes through the point $(0, -3)$ and is tangent to the line $y = 4x - 12$ at $(3, 0)$, then find the values of a, b, and c.

(j) A bug is crawling from left to right along the top of the curve $f(x) = 9 - x^2$. A spider waits at the point $(5, 0)$. Find the distance between the two insects when they first see each other.

(k) At time t seconds, the height s in feet of a ball above the ground is given by

$$s = -16t^2 + 20t + 40.$$

(i) What is its instantaneous velocity at $t = 3$?

(ii) When is its instantaneous velocity 0?

(l) Suppose that the number of people in your school that hear a rumour about you t hours after it is started is given by the function $N(t) = 6\sqrt{t}$.

(i) What is the average rate at which the rumour is spreading between $t = 0$ and $t = 9$?

(ii) What is the instantaneous rate at which the rumour is spreading after 4 hours?

(iii) Is the rate at which the rumour is spreading increasing or decreasing? Hint: Consider the graph of the function.

(m) The value of a new car t years after it is purchased is given by the function

$$V(t) = \frac{25000}{(1+0.02t)^2}.$$

(i) What it the value of the car when it is new?
(ii) What is the value of the car when it is five years old?
(iii) What is the average yearly drop in price during those five years?
(iv) What is the instantaneous rate of change in price after 4 years (that is when $t = 4$)?

(n) If gasoline costs \$.80/L and you drive 24000 km per year, then your annual cost for gasoline will be given by $C(x) = 19200/x$ where x is the number of km/L that your car can travel.
(a) Find the annual gasoline costs if your car gets 3 km/L.
(b) Find your annual gasoline costs if your car gets 8 km/L.
(c) What is the average rate of change in your annual gasoline cost as x changes from 3 to 8?
(d) What is instantaneous rate of change in your annual fuel costs when $x = 5$?

(o) Find $f^{'}(x)$ if

$$f(x) = \begin{cases} x^2, & x \textit{ is rational} \\ \\ x^3, & x \textit{ is irrational} \end{cases}$$

(p) Let $f(x) = |x-2| + |x-3|, \forall x \in \mathbb{R}$. Indicate those points at which f is not differentiable.

(q) Suppose that $f(x+y) = f(x) + f(y) + 2xy$ for all $x, y \in \mathbb{R}$ and $f^{'}(0) = 0$. Find $f(0), f^{'}(x)$, and $f(x)$.

(71) In Problems 71a-71d, find the differential dy of each function.

(a) $y = 2x^3 - 4.$
(b) $y = \dfrac{x+5}{2x+3}$
(c) $y = \dfrac{1}{\sqrt{x}} + \sqrt{x}.$
(d) $y = \dfrac{\tan x}{1+x^2}.$

(72) A square has an edge of 20 cm, with a possible error of 0.001 cm. What is the possible error in the area of the square?

(73) Oil is leaking from a sunken tanker and is forming a circular ring whose radius is increasing at a rate of 10 m/minute. How are the circumference of the ring and the area of the ring changing when the radius of the ring is 600 m?

(74) The length of a rectangle is increasing at a rate of 6 cm/s. If the area of the rectangle is not changing, at what rate is the width of the rectangle decreasing when the length is 14 cm and the width is 7 cm?

(75) A spherical ball is filled with $4500\pi\ m^3$ of helium. A leak in the balloon causes the helium to escape at a rate of $72\pi\ m^3/min$. At what rate was the radius of the balloon decreasing 49 minutes after the leak began?

(76) Water is being poured into a conical vase at a rate of 18 cm^3/s. The diameter of the cone is 30 cm and the height of the cone is 25 cm. At what rate is the water level rising when the water's depth is 20 cm?

(77) A car, travelling north at 48 km/h is approaching an intersection. A truck, travelling east at 60 km/h is moving away from the same intersection. How is the distance between the car and the truck changing when the car is 9 m from the intersection and the truck is 40 m from the intersection?

(78) Use L'hôpital's rule to find the following limits.

(a) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1-x/2}{x^2}$.

(b) $\lim\limits_{x\to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$.

(c) $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{2x}\right)^{5x}$.

(d) $\lim\limits_{x\to 1+} \left(\dfrac{1}{x-1} - \dfrac{1}{\ln x}\right)$.

(e) $\lim\limits_{x\to 0} \dfrac{e^x+e^{-x}-2}{1-\cos 2x}$

(f) $\lim\limits_{x\to\infty} x^{-1/2}\ln x$.

(g) $\lim\limits_{x\to\infty} \dfrac{\ln(\ln x)}{\ln x}$.

(h) $\lim\limits_{x\to\infty} (1+e^x)^{e^{-x}}$.

(i) $\lim\limits_{x\to\infty} \dfrac{x\ln x}{x+\ln x}$.

(j) $\lim\limits_{x\to 0+} (x+\cos x)^{\csc 3x}$.

(k) $\lim\limits_{x\to\infty} \left(\sqrt{x^4+5x^2+3}-x^2\right)$.

(l) $\lim\limits_{x\to\infty} \left(\dfrac{x+a}{x-a}\right)^x$.

(m) $\lim\limits_{x\to\frac{\pi}{2}-} \dfrac{\tan x}{1+\sec x}$.

(n) $\lim\limits_{x\to 0+} x^{x^x}$.

(o) $\lim\limits_{x\to\frac{\pi}{2}-} (\sec x - \tan x)$.

(p) $\lim\limits_{x\to\infty} (x-1)e^{-x^2}$.

(q) $\lim\limits_{x\to 0+} \left(\dfrac{4}{x+3} - \dfrac{x}{x^2+2x-3}\right)$.

(r) $\lim\limits_{x\to\infty} x(e^{1/x}-1)$.

(s) $\lim\limits_{x\to 1-} (1-x)^{\ln x}$.

(t) $\lim\limits_{x\to 0} \dfrac{e^x-\ln(1+x)-1}{x^2}$.

(u) $\lim\limits_{x\to 0} \dfrac{\tan x-\sin x}{x^2\tan x}$.

(v) $\lim\limits_{x\to 0} \dfrac{\sin x - x}{e^{x^2}-1}$.

(w) Let f be a function. If $f''(x)$ exists and f'' is continuous, then find

$$\lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2}.$$

In Problems 79-80, sketch the graph of a function that satisfies all of the given conditions.

(79) (a) $f(-1)=5; f(1)=1; f'(-1)=f'(1)=0$.
(b) $f'(x)<0$ if $|x|<1; f'(x)>0$ if $|x|>1$.
(c) $f''(x)<0$ if $x<0; f''(x)>0$ if $x>0$.

(80) (i) $f(-1)=f(2)=-2; f(-3)=5; f'(-1)=f'(2)=0$.

(ii) $f'(x) = 0$ if $x < -3$; $f'(x) < 0$ if $x \in (-3,-1) \cup (0,2)$; $f'(x) > 0$ if $x \in (-1,0) \cup (2,\infty)$.

(iii) $f''(x) > 0$ if $x \in (-3,0) \cup (0,5)$; $f''(x) < 0$ if $x > 5$.

(81) Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{\sqrt{x^2+9}}{x-3}.$$

(82) Let $f(x) = \dfrac{x(x-5)}{(x-3)(x+2)}$.

(a) Show that $f(x)$ has no turning points.

(b) Sketch the curve $y = f(x)$.

(c) If $g(x) = \dfrac{1}{f(x)}$. Sketch the curve $y = g(x)$ on the same axes. Show asymptotes and where $g(x)$ and $f(x)$ intersect.

(83) Sketch the curve $y = x - \dfrac{8}{x^2}$ for $x > 0$ showing any asymptotes.

(84) A curve is given by $y = \dfrac{(x-1)(x-9)}{(x+1)(x+9)}$.

(a) Determine the turning point of the curve.

(b) Determine the equation of asymptotes to the curve

(c) Sketch the curve.

(85) Let $f(x) = \dfrac{x+1}{x^2+2x}$.

(a) Show that $f(x)$ has no turning points.

(b) Find the equations of asymptotes to the curve.

(c) Sketch the curve.

(86) Let $f(x) = \dfrac{x(x-3)}{(x-1)(x-4)}$.

(a) Show that $f(x)$ has no turning points.

(b) Find the equations of asymptotes to the curve.

(c) Sketch the curve.

(87) Sketch the curve $f(x) = \dfrac{4(x-3)}{x(x+2)}$.

(88) Determine the nature of turning points to the curve $y = \dfrac{x^2-6x+5}{2x-1}$. Sketch the graph of the curve for $x = -2$ to $x = 7$. State any asymptotes.