0.1 Decay estimates for systems with constant coefficients

Lemma 1.
Let $A = A_{ij}^{\alpha\beta}$ be a constant matrix satisfying the Legendre-Hadamard condition (**??**) for some $\lambda > 0$, let $\Lambda = |A|$ and let $u \in H^1_{loc}(\Omega;\mathbb{R}^m)$ satisfying the system

$$-\sum_{\alpha\beta j} \partial_{x_\alpha}\left(A_{ij}^{\alpha\beta}\partial_{x_\beta}u^j\right) = 0, \qquad \forall i \in \{1,2,\dots,m\} \tag{1}$$

Then for $B_r(x_0) \subset B_R(x_0) \subset\subset \Omega$ it holds

- $$\int_{B_r(x_0)} |u|^2 \,\mathrm{d}x \le c_D \left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |u|^2 \,\mathrm{d}x \tag{2}$$
- $$\int_{B_r(x_0)} \left|u - u_{x_0,r}\right|^2 \mathrm{d}x \le c_E \left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} \left|u - u_{x_0,R}\right|^2 \mathrm{d}x \tag{3}$$

with $c_D = c_D(n,\lambda,\Lambda)$ and $c_E = c_E(n,\lambda,\Lambda)$, having used the notation

$$u_{x_0,s} := \frac{1}{\left|B_s(x_0)\right|}\int_{B_s(x_0)} u(x)\,\mathrm{d}x \tag{4}$$

0.2 Regularity up to the boundary

Let $u \in H^1_0(\Omega;\mathbb{R}^m)$ be a weak solution of

$$-\sum_{\alpha\beta j} \partial_{x_\alpha}\left(A_{ij}^{\alpha\beta}\partial_{x_\beta}u^j\right) = f_i - \sum_\alpha \partial_{x_\alpha} F_i^\alpha, \qquad i \in \{1,2,\dots,m\} \tag{5}$$

We make the following hypothesis:
$f \in L^2(\Omega;\mathbb{R}^m)$, $F \in H^1(\Omega;\mathbb{R}^{m\times n})$, $A \in C^{0,1}(\Omega;\mathbb{R}^{m^2\times n^2})$,
$A(X)$ satisfies the Legendre-Hadamard condition (**??**) uniformly with respect to $x \in \Omega$, Ω has C^2 boundary (we say $\partial\Omega \in C^2$), i.e. the domain Ω is locally the epigraph of a C^2 function up to a rigid motion.

Theorem 1 (Regularity up to the boundary)**.**
Under the assumptions above, the function u belongs to $H^2(\Omega;\mathbb{R}^m)$ and moreover $\exists c = c(\Omega, A, n) > 0$ such that

$$\|u\|_{H^2(\Omega;\mathbb{R}^m)} \le c\left(\|f\|_{L^2(\Omega;R^m)} + \|F\|_{H^1(\Omega;\mathbb{R}^{m\times n})}\right). \tag{6}$$

If both the boundary of the domain and the data are sufficiently regular the method can be iterated to obtain higher Regularity of u.

Theorem 2.
Assume in addition to the hypothesis above that $f \in H^k(\Omega;\mathbb{R}^m)$, $F \in H^{k+1}(\Omega;\mathbb{R}^{m\times n})$, $A \in C^{k,1}(\Omega;\mathbb{R}^{m^2\times n^2})$ with Ω such that $\partial\Omega \in C^{k+2}$. Then $u \in H^{k+2}(\Omega;\mathbb{R}^m)$

0.3 Interior Regularity for Nonlinear Equations

We see here how the Nirenberg's method is appropriate in dealing with nonlinear PDEs as those arising from Euler-Lagrange equations of non-quadratic functionals.
Consider $L \in C^2(\mathbb{R}^{m\times n};\mathbb{R})$ and assume that

(i) there exists a constant $c > 0$ such that $\left|\nabla^2 L(\xi)\right| \leq c$, $\forall \xi \in \mathbb{R}^{m\times n}$

(ii) L satisfies a uniform Legendre condition, i.e.

$$\sum_{\alpha,\beta,i,j} \partial_{p_j^\alpha}\partial_{p_j^\beta} L(p)\xi_i^\alpha \xi_j^\beta \geq \lambda|\xi|^2 \qquad \xi \in \mathbb{R}^{m\times n} \tag{7}$$

for some $\lambda > 0$ independent of p.

To simplify notation we set $B_i^\alpha := \frac{\partial L}{\partial p_i^\alpha}$ and $A_{ij}^{\alpha\beta} := \frac{\partial^2 L}{\partial p_i^\alpha {p_j^\beta}^2}$