Chapter 1

TBD

1.1 Elliptic Systems

Some info abput existence of weak solutions

We consider functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. We use Greek letters to indicate components of vectors in the starting domain (so that $\alpha, \beta \in \{1, 2, \ldots, n\}$) and we use latin letters to indicate components of vectors in teh target domain (so that $i.j \in \{1, 2, \ldots, n\}$). Furthermore, we work with matrices with 4 indices (rank-four tensors). As usually done for elliptic equations we will define ellipticity as the positive semi-definiteness of the tensor, namely the Legendre condition (E) below:

$$\exists c > 0 : \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \geq c|\xi|^2, \forall \xi \in \mathbb{R}^{m\times n} \tag{E}$$

We can employ the condition (E) to prove existence and uniqueness results for

$$\begin{cases} -\sum_{\alpha,\beta,j} \partial_{x_\alpha}(A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j) = f_i - \sum_\alpha \partial_{x_\alpha} F_i^\alpha & i = 1, \ldots, m \\ u \in H_0^1(\Omega; \mathbb{R}^m) \end{cases} \tag{LS}$$

with data $f_i, F_i^\alpha \in L^2(\Omega; \mathbb{R})$.
The weak formulation of the problem is readily obtained as

$$\int_\Omega \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \partial_{x_\beta} u^i \partial_{x_\alpha} \phi^i \, \mathrm{d}x = \int_\Omega \left[f_i \phi^i + \sum_{\alpha,i} F_i^\alpha \partial_{x_\alpha} \phi^i \right] \mathrm{d}x \qquad \forall \phi \in C_c^\infty(\Omega; \mathbb{R}^m) \tag{1.1}$$

The matrix $A_{ij}^{\alpha\beta}$ defines a bilinear continuous form on $H_0^1(\Omega; \mathbb{R}^m)$ by means of the formula

$$(\phi, \psi)_A := \int_\Omega \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \partial_{x_\alpha} \phi^i \partial_{x_\beta} \psi^j \, \mathrm{d}x \tag{1.2}$$

If moreover $A_{ij}^{\alpha\beta}$ satisfies the Legendre condition (E), then the bilinear form is coercive, and we can use the Lax-Milgram theorem to prove existence and uniqueness of weak solutions. Actually one can prove existence and uniqueness under a weaker assumption known as "Legendre-Hademard" (LH) condition:

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \geq \lambda |\xi|^2 \qquad \forall \xi = a \otimes b \tag{LH}$$

that is the Legendre condition (E) for rank-one matrices $\xi = a \otimes b$.
The (LH) condition is strictly weaker than (E), as the following example shows.

Example 1 ((LH) is weaker than (E))**.**
Let $m = n = 2$ and let $A^{\alpha\beta}_{ij}$ be such that

$$\sum_{\alpha,\beta,i,j} A^{\alpha\beta}_{ij}\xi^i_\alpha\xi^j_\beta = \det\{\xi\} + \epsilon|\xi|^2 \tag{1.3}$$

for some $\epsilon \geq 0$ to be chosen later.
Since any rank-one matrix $\xi = a \otimes b$ has $\det\{\xi\} = 0$, the (LH) condition is fulfilled with $\lambda = \epsilon$.
On the other hand for $\overline{\xi} = \mathrm{diag}(t, -t), t \neq 0$, we get

$$\sum_{\alpha,\beta,i,j} A^{\alpha\beta}_{ij}\overline{\xi}^i_\alpha\overline{\xi}^j_\beta = \det\left\{\overline{\xi}\right\} + \epsilon\left|\overline{\xi}\right|^2 = -t^2 + 2\epsilon t^2 = t^2(2\epsilon - 1) \tag{1.4}$$

and the Legendre condition (E) fails for $2\epsilon - 1 < 0$.

Nevertheless, the following Theorem by Gårding holds true:

Theorem 1.
Assume that the constant matrix $A^{\alpha\beta}_{ij}$ satisfies the Legendre-Hademard (LH) condition for some positive constant λ. Then there exists a unique solution of the linear system (LS).

1.2 Classical regularity theory for the linear problems

We want to study the local behavior of the weak solutions $u \in H^1_{loc}(\Omega;\mathbb{R}^m)$ of a system of equations given by:

$$-\sum_{\alpha,\beta,j}\partial_{x_\alpha}(A^{\alpha\beta}_{ij}\partial_{x_\beta}u^j) = f_i - \sum_\alpha \partial_{x_\alpha}F^\alpha_i \qquad i = 1,\ldots,m \tag{1.5}$$

with $A^{\alpha\beta}_{ij} \in L^\infty(\Omega;\mathbb{R}), f_i \in L^2_{loc}(\Omega;\mathbb{R}), F^\alpha_i \in L^2_{loc}(\Omega;\mathbb{R})$.
In what follows we will always assume $\Omega \subset \mathbb{R}^m$ to be an open, bounded and regular domain (here Ω regular means that Ω is locally the epigraph of a C^1 function of $(n-1)$-variables, written in a suitable system of coordinates, near any boundary point).
We will see how to use a Caccioppoli-Leray inequality to prove existence of higher-order weak derivatives of u and suitable integrability results thereof. We will moreover turn such estimates into actual regularity results thanks to Sobolev embeddings. The idea above is due L. Nirenberg.
We use the symbol $|\cdot|$ to denote the Hilbert-Schmidt norm of matrices and tensors, even though some estimates would be true also for the smaller operator norm. We set

$$\left|A^{\alpha\beta}_{ij}\right|^2 = \sum_{\alpha,\beta,j}(A^{\alpha\beta}_{ij})^2 \tag{1.6}$$

Theorem 2 (Caccioppoli-Leray inequality)**.**
If the Borel coefficients $A^{\alpha\beta}_{ij}$ satisfy the Legendre condition with $\lambda > 0$, namely

$$\sum_{\alpha,\beta,i,j} A^{\alpha\beta}_{ij}\xi^i_\alpha\xi^j_\beta \geq \lambda|\xi|^2, \forall\xi \in \mathbb{R}^{m\times n} \tag{1.7}$$

and

$$\sup_{x\in\Omega}\left|A_{ij}^{\alpha\beta}(x)\right| \le \Lambda < +\infty \tag{1.8}$$

then there exists a positive constant $C_{CL} = C_{CL}(\lambda,\Lambda)$ such that, for any ball $B_R(x_0) \subset\subset \Omega$ and any $k \in \mathbb{R}^m$ it holds

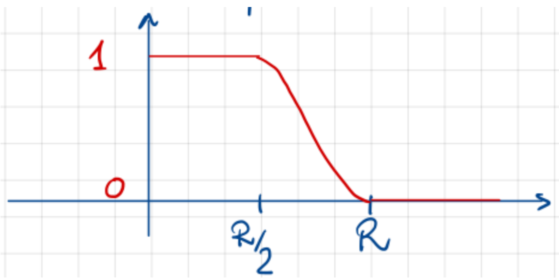
$$C_{CL}\int_{B_{R/2}(x_0)}|\nabla u|^2\,\mathrm{d}x \le R^{-2}\int_{B_R(x_0)}|u(x)-k|^2\,\mathrm{d}x + R^2\int_{B_R(x_0)}|f(x)|^2\,\mathrm{d}x + \int_{B_R(x_0)}|F(x)|^2\,\mathrm{d}x \tag{CLI}$$

Remark.
- the validity of (CLI) on all $k \in \mathbb{R}^m$ depends on the fact that the PDE is invariant under translation of u, meaning that if u is a solution, then also $u + k$ $\forall k$ is a solution
- note also that the inequality is scale invariant: think about u being adimensional, then all the terms in (CLI) have dimension $(\text{length})^{n-2}$
- the inequality is surely non-trivial (the gradient of a function cannot be controlled by the variation of the function!). (CLI) can already be regarded as a first regularity result, meaning that we are gaining specific information on the behavior of a function from the fact that it is a solution of a PDE

Proof. W.l.o.g we take $x_0 = 0$ and $k = 0$. We choose as test function ϕ in the weak formulation

$$\int_{B_R}\langle A\nabla u,\nabla\phi\rangle\,\mathrm{d}x - \int_{B_R}\langle f,\phi\rangle\,\mathrm{d}x - \int_{B_R}\langle F,\nabla\phi\rangle\,\mathrm{d}x = 0 \tag{1.9}$$

the function $\phi := u\eta^2$, where $\eta \in C_c^\infty(B_R;\mathbb{R})$ is a cut-off function with $\eta \equiv 1$ in $B_{R/2}$, $0 \le \eta \le 1$ and $\|\nabla\eta\|_\infty \le \frac{4}{R}$



$$\frac{\partial\phi_i}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha} = \eta^2\frac{\partial u^i}{\partial x_\alpha} + 2\eta\frac{\partial\eta}{\partial x_\alpha}u^i, \tag{1.10}$$

that is

$$\nabla\phi = \eta^2\nabla u + 2\eta u\otimes\nabla\eta. \tag{1.11}$$

Plugging in the last equality in the weak formulation

$$0 = \int_{B_R}\eta^2\langle A\nabla u,\nabla u\rangle + 2\int_{B_R}\eta\langle A\nabla u, u\otimes\nabla\eta\rangle \tag{1.12}$$

$$-\int_{B_R}\eta^2\langle f,u\rangle - \int_{B_R}\eta^2\langle F,\nabla u\rangle - 2\int_{B_R}\eta\langle F,u\otimes\nabla\eta\rangle \tag{1.13}$$

$$=: I_1 + I_2 - I_3 - I_4 - I_5 \tag{1.14}$$

$$I_1 := \int\limits_{B_R} \eta^2 \langle A\nabla u, \nabla u\rangle \,\mathrm{d}x = \int\limits_{B_R} \sum_{\alpha,\beta,i,j} \eta^2 A_{ij}^{\alpha\beta} \partial_{x_\alpha} u^i \partial_{x_\beta} u^j \,\mathrm{d}x \geq \lambda \int\limits_{B_R} \eta^2 |\nabla u|^2 \,\mathrm{d}x \tag{1.15}$$

$$I_2 := 2\int\limits_{B_R} \eta \langle A\nabla u, u \otimes \nabla\eta\rangle \,\mathrm{d}x = 2\int\limits_{B_R} \eta \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \partial_{x_\alpha} u^i u^j \partial_{x_\beta}\eta \,\mathrm{d}x \tag{1.16}$$

$$\overset{\text{Cauchy-Schwarz}}{\leq} 2\int\limits_{B_R} \eta |A||u||\nabla u||\nabla\eta| \,\mathrm{d}x \tag{1.17}$$

$$\overset{\text{boundedness of } |A| \text{ and } \|\nabla\eta\|_\infty}{\leq} \frac{(\Lambda)}{R} \int\limits_{B_R} (\eta|\nabla u|)|u| \,\mathrm{d}x \tag{1.18}$$

$$\overset{\text{Young } ab \leq \frac{a^2}{2} + \frac{b^2}{2}}{\leq} \frac{4\Lambda}{R}\epsilon \int\limits_{B_R} \eta^2 |\nabla u|^2 \,\mathrm{d}x + \frac{4\Lambda}{R\epsilon} \int\limits_{B_R} |u|^2 \,\mathrm{d}x \tag{1.19}$$

$$I_3 := \int\limits_{B_R} \left\langle f, \eta^2 u \right\rangle \mathrm{d}x = \int\limits_{B_R} \eta^2 \sum_i f_i u^i \,\mathrm{d}x \overset{Young}{\leq} \frac{1}{2R^2} \int\limits_{B_R} |u|^2 \,\mathrm{d}x + \frac{R^2}{2} \int\limits_{B_R} |f|^2 \,\mathrm{d}x \tag{1.20}$$

$$I_4 := \int\limits_{B_R} \eta^2 \langle F, \nabla u\rangle \,\mathrm{d}x = \int\limits_{B_R} \eta^2 \sum_{\alpha,i} F_i^\alpha \partial_{x_\alpha} u^i \,\mathrm{d}x \leq \frac{\lambda}{4} \int\limits_{B_R} \eta^2 |\nabla u|^2 \,\mathrm{d}x + \frac{1}{\lambda} \int\limits_{B_R} |F|^2 \,\mathrm{d}x \tag{1.21}$$

By Cauchy-Schwarz inequality, $\|\nabla\eta\|_\infty \leq \frac{4}{R}$ and Young inequality we have

$$I_5 := 2\int\limits_{B_R} \eta \langle F, u \otimes \nabla\eta\rangle \,\mathrm{d}x = 2\int\limits_{B_R} \sum_{\alpha,i} F_i^\alpha u^i \partial_{x_\alpha}\eta \,\mathrm{d}x \leq 4\int\limits_{B_R} |F|^2 \,\mathrm{d}x + \frac{4}{R^2} \int\limits_{B_R} |u|^2 \,\mathrm{d}x \tag{1.22}$$

Therefore from the weak formulation with $\phi = \eta^2 u$ we obtain

$$\lambda \int\limits_{B_R} \eta^2 |\nabla u|^2 \,\mathrm{d}x \leq \int\limits_{B_R} \eta^2 \langle A\nabla u, \nabla u\rangle \,\mathrm{d}x \tag{1.23}$$

$$\leq \underbrace{(\frac{4\Lambda\epsilon}{R} + \frac{\lambda}{4}) \int\limits_{B_R} \eta^2 |\nabla u|^2 \,\mathrm{d}x}_{\text{Dirichlet term}} + (\frac{4\Lambda}{R\epsilon} + \frac{1}{2R^2} + \frac{4}{R^2}) \int\limits_{B_R} |u|^2 \,\mathrm{d}x + \tag{1.24}$$

$$+ \frac{R^2}{2} \int\limits_{B_R} |f|^2 \,\mathrm{d}x + (\frac{1}{\lambda} + 4) \int\limits_{B_R} |F|^2 \,\mathrm{d}x \tag{1.25}$$

We can choose ϵ so small that $\frac{4\Lambda\epsilon}{R} = \frac{\lambda}{4}$ and absorb the Dirichlet term on the r.h.s. of the equation (meaning that it can be subtracted from both sides and still give a positive term on the left).
Finally, one concludes the proof observing that

$$\int\limits_{B_R} \eta^2 |\nabla u|^2 \,\mathrm{d}x \geq \int\limits_{B_{R/2}} |\nabla u|^2 \,\mathrm{d}x \tag{1.26}$$

$\square$

Observe that the proof shows that in I_2 one can have a better estimate noting that $|\nabla u| = 0$ on $B_{R/2}$.

$$I_2 := 2 \int_{B_R} \eta |A||u||\nabla u||\nabla \eta| \,\mathrm{d}x = 2 \int_{B_R \setminus B_{R/2}} \eta |A||u||\nabla u||\nabla \eta| \,\mathrm{d}x\,. \tag{1.27}$$

This observation is indeed the starting point of the Widman's technique below.

1.3 Widman "holes filling technique"

A sharp version of the Caccioppoli-Leray inequality (CLI) has been proven by Widman. We can illustrate that in the simple case of $f = 0, F = 0$.
Observe that, with the notation of the (CLI) proof, since $|\nabla u| \le \frac{4}{R}\chi_{B_R \setminus B_{R/2}}$ one obtains

$$\int_{B_{R/2}} \left|\nabla u(x)\right|^2 \mathrm{d}x\, leq \frac{c}{R^2} \int_{B_R \setminus B_{R/2}} \left|u(x) - k\right|^2 \mathrm{d}x \tag{1.28}$$

for some positive constant c independent of R.
Now the idea is to choose $\kappa := ⨍_{B_R \setminus B_{R/2}} u(x)\,\mathrm{d}x$ so that we can estimate the r.h.s of (1.28) using the Poincarè inequality with explicit scaling, i.e.

$$\int_{B_R \setminus B_{R/2}} \left| u(x) - ⨍_{B_R \setminus B_{R/2}} u\,\mathrm{d}x \right|^2 \mathrm{d}x \le cR^2 \int_{B_R \setminus B_{R/2}} \left|\nabla u(x)\right|^2 \mathrm{d}x \tag{1.29}$$

to get

$$\int_{B_{R/2}} \left|\nabla u(x)\right|^2 \mathrm{d}x \le c \int_{B_R \setminus B_{R/2}} \left|\nabla u(x)\right|^2 \mathrm{d}x \tag{1.30}$$

$$\Leftrightarrow (c+1) \int_{B_{R/2}} \left|\nabla u(x)\right|^2 \mathrm{d}x \le c \int_{B_R} \left|\nabla u(x)\right|^2 \mathrm{d}x \tag{1.31}$$

Setting $\vartheta := \frac{c}{c+1} < 1$ we get

$$\int_{B_{R/2}} \left|\nabla u(x)\right|^2 \mathrm{d}x \le \vartheta \int_{B_R} \left|\nabla u(x)\right|^2 \mathrm{d}x \tag{1.32}$$

Iterating the previous estimates d times for radii

$$2^1 r \to 2^2 r \to 2^3 r \to \cdots \to 2^d r \tag{1.33}$$

and choosing r such that

$$2^d r < R < 2^{d+1} r \tag{1.34}$$

we get

$$\int_{B_R} |\nabla u|^2 \, \mathrm{d}x \leq \vartheta^d \int_{B_R} |\nabla u|^2 \, \mathrm{d}x \tag{1.35}$$

Setting $\alpha \log_2(1/\vartheta)$, i.e. $\vartheta = 1/2^\alpha$ we have

$$\vartheta^d = \frac{1}{2^{\alpha d}} = \left(\frac{1}{2^d}\right)^\alpha \overset{(1.34)}{\leq} 2^\alpha \left(\frac{r}{R}\right)^\alpha . \tag{1.36}$$

Hence,

$$\int_{B_r} |\nabla u|^2 \, \mathrm{d}x \leq 2^\alpha \left(\frac{r}{R}\right)^\alpha \int_{B_R} |\nabla u|^2 \, \mathrm{d}x \, . \tag{1.37}$$

For $n = 2$ the estimate above implies $u \in C^{0,\alpha/2}(\Omega; \mathbb{R}^m)$.
In fact the idea that the decay of the L^p-norm of the gradient is related to its Hölder continuity will play a crucial role in the rest of the course, and we will discuss in detail in the next lectures.

1.4 Continuity via embedding

The Sobolev embedding theorem for $W^{1,p}(\Omega; \mathbb{R})$ says that

$$\begin{cases} p < n & W^{1,p}(\Omega;\mathbb{R}) \hookrightarrow L^{p^\star}(\Omega;\mathbb{R}) \quad \text{continuously} \quad p^\star = \frac{np}{n-p} \\ p = n & W^{1,n}(\Omega;\mathbb{R}) \hookrightarrow L^{q^\star}(\Omega;\mathbb{R}) \quad \text{compactly} \quad \forall 1 \leq q < \infty \\ p > n & W^{1,p}(\Omega;\mathbb{R}) \hookrightarrow C^{0,1-n/p}(\Omega;\mathbb{R}) \quad \text{continuously} \end{cases} \tag{1.38}$$

Hence a way to prove continuity of a Sobolev function is to prove that it belongs to $W^{1,p}$ for $p > n$.

1.5 Embedding for higher order Sobolev spaces

We recall that higher order Sobolev spaces $W^{k,p}(\Omega; \mathbb{R})$ with $k \geq 1$ integer and $1 \leq p \leq \infty$ are recursively defined as

$$W^{k,p}(\Omega, \mathbb{R}) := \{u \in W^{1,p}(\Omega; \mathbb{R}) : \nabla u \in W^{k-1,p}(\Omega; \mathbb{R}^n)\}. \tag{1.39}$$

Another way to prove continuity, applicable if $p < n$, is to use $W^{k,p}$ for k large enough. In fact, it holds true that:

(1) if $kp < n$