Script for the lecture “An introduction to the Regularity Theory of Elliptic Partial Differential Equations” held by Prof. Marco Cicalese in SoSe23

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Chapter 1

TBD

1.1 Elliptic Systems

Some info about existence of weak solutions

We consider functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. We use Greek letters to indicate components of vectors in the starting domain (so that $\alpha, \beta \in \{1, 2, \ldots, n\}$) and we use latin letters to indicate components of vectors in the target domain (so that $i, j \in \{1, 2, \ldots, m\}$). Furthermore, we work with matrices with 4 indices (rank-four tensors). As usually done for elliptic equations we will define ellipticity as the positive semi-definiteness of the tensor, namely the Legendre condition (E) below:

$$\exists c > 0 : \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \geq c|\xi|^2, \qquad \forall \xi \in \mathbb{R}^{m\times n} \tag{E}$$

We can employ the condition (E) to prove existence and uniqueness results for

$$\begin{cases} -\sum_{\alpha,\beta,j} \partial_{x_\alpha}(A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j) = f_i - \sum_\alpha \partial_{x_\alpha} F_i^\alpha & i = 1, \ldots, m \\ u \in H_0^1(\Omega; \mathbb{R}^m) \end{cases} \tag{LS}$$

with data $f_i, F_i^\alpha \in L^2(\Omega; \mathbb{R})$.
The weak formulation of the problem is readily obtained as

$$\int_\Omega \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \partial_{x_\beta} u^i \partial_{x_\alpha} \varphi^i \, \mathrm{d}x = \int_\Omega \left[f_i \varphi^i + \sum_{\alpha,i} F_i^\alpha \partial_{x_\alpha} \varphi^i \right] \mathrm{d}x \qquad \forall \varphi \in C_c^\infty(\Omega; \mathbb{R}^m) \tag{1.1}$$

The matrix $A_{ij}^{\alpha\beta}$ defines a bilinear continuous form on $H_0^1(\Omega; \mathbb{R}^m)$ by means of the formula

$$(\varphi, \psi)_A := \int_\Omega \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \partial_{x_\alpha} \varphi^i \partial_{x_\beta} \psi^j \, \mathrm{d}x \tag{1.2}$$

If moreover $A_{ij}^{\alpha\beta}$ satisfies the Legendre condition (E), then the bilinear form is coercive, and we can use the Lax-Milgram theorem to prove existence and uniqueness of weak solutions. Actually one can prove existence and uniqueness under a weaker assumption known as "Legendre-Hademard" (LH) condition:

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \geq \lambda |\xi|^2 \qquad \forall \xi = a \otimes b \tag{LH}$$

that is the Legendre condition (E) for rank-one matrices $\xi = a \otimes b$.
The (LH) condition is strictly weaker than (E), as the following example shows.

Example 1 ((LH) is weaker than (E))**.**
Let $m = n = 2$ and let $A_{ij}^{\alpha\beta}$ be such that

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j = \det \xi + \varepsilon|\xi|^2 \tag{1.3}$$

for some $\varepsilon \geq 0$ to be chosen later.
Since any rank-one matrix $\xi = a \otimes b$ has $\det \xi = 0$, the (LH) condition is fulfilled with $\lambda = \varepsilon$.
On the other hand for $\overline{\xi} = \mathrm{diag}(t, -t), t \neq 0$, we get

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta}\overline{\xi}_\alpha^i\overline{\xi}_\beta^j = \det \overline{\xi} + \varepsilon|\overline{\xi}|^2 = -t^2 + 2\varepsilon t^2 = t^2(2\varepsilon - 1) \tag{1.4}$$

and the Legendre condition (E) fails for $2\varepsilon - 1 < 0$.

Nevertheless, the following Theorem by Gårding holds true:

Theorem 1.
Assume that the constant matrix $A_{ij}^{\alpha\beta}$ satisfies the Legendre-Hademard (LH) condition for some positive constant λ. Then there exists a unique solution of the linear system (LS).

1.2 Classical regularity theory for the linear problems

We want to study the local behavior of the weak solutions $u \in H^1_{loc}(\Omega;\mathbb{R}^m)$ of a system of equations given by:

$$-\sum_{\alpha,\beta,j} \partial_{x_\alpha}(A_{ij}^{\alpha\beta}\partial_{x_\beta}u^j) = f_i - \sum_\alpha \partial_{x_\alpha}F_i^\alpha \qquad i = 1,\dots,m \tag{1.5}$$

with $A_{ij}^{\alpha\beta} \in L^\infty(\Omega;\mathbb{R}), f_i \in L^2_{loc}(\Omega;\mathbb{R}), F_i^\alpha \in L^2_{loc}(\Omega;\mathbb{R})$.
In what follows we will always assume $\Omega \subset \mathbb{R}^m$ to be an open, bounded and regular domain (here Ω regular means that Ω is locally the epigraph of a C^1 function of $(n-1)$ variables, written in a suitable system of coordinates, near any boundary point).
We will see how to use a Caccioppoli-Leray inequality to prove existence of higher-order weak derivatives of u and suitable integrability results thereof. We will moreover turn such estimates into actual regularity results thanks to Sobolev embeddings. The idea above is due L. Nirenberg.
We use the symbol $|\cdot|$ to denote the Hilbert-Schmidt norm of matrices and tensors, even though some estimates would be true also for the smaller operator norm. We set

$$\left|A_{ij}^{\alpha\beta}\right|^2 = \sum_{\alpha,\beta,j}(A_{ij}^{\alpha\beta})^2 \tag{1.6}$$

Theorem 2 (Caccioppoli-Leray inequality)**.**
If the Borel coefficients $A_{ij}^{\alpha\beta}$ satisfy the Legendre condition with $\lambda > 0$, namely

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \geq \lambda|\xi|^2, \qquad \forall\xi \in \mathbb{R}^{m\times n} \tag{1.7}$$

and

$$\sup_{x\in\Omega}\left|A_{ij}^{\alpha\beta}(x)\right| \leq \Lambda < +\infty \tag{1.8}$$

then there exists a positive constant $C_{CL} = C_{CL}(\lambda, \Lambda)$ such that, for any ball $B_R(x_0) \subset\subset \Omega$ and any $k \in \mathbb{R}^m$ it holds

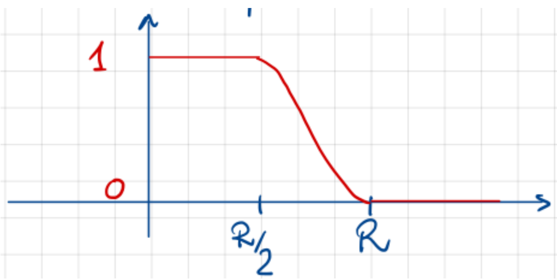
$$C_{CL} \int\limits_{B_{\frac{R}{2}}(x_0)} |\nabla u|^2 \,\mathrm{d}x \leq R^{-2} \int\limits_{B_R(x_0)} \left|u(x) - k\right|^2 \mathrm{d}x + R^2 \int\limits_{B_R(x_0)} \left|f(x)\right|^2 \mathrm{d}x + \int\limits_{B_R(x_0)} \left|F(x)\right|^2 \mathrm{d}x \tag{CLI}$$

Remark.
- the validity of (CLI) on all $k \in \mathbb{R}^m$ depends on the fact that the PDE is invariant under translation of u, meaning that if u is a solution, then also $u + k$ $\forall k$ is a solution
- note also that the inequality is scale invariant: think about u being adimensional, then all the terms in (CLI) have dimension $(\text{length})^{n-2}$
- the inequality is surely non-trivial (the gradient of a function cannot be controlled by the variation of the function!). (CLI) can already be regarded as a first regularity result, meaning that we are gaining specific information on the behavior of a function from the fact that it is a solution of a PDE

Proof. W.l.o.g we take $x_0 = 0$ and $k = 0$. We choose as test function φ in the weak formulation

$$\int\limits_{B_R} \langle A\nabla u, \nabla\varphi\rangle \,\mathrm{d}x - \int\limits_{B_R} \langle f, \varphi\rangle \,\mathrm{d}x - \int\limits_{B_R} \langle F, \nabla\varphi\rangle \,\mathrm{d}x = 0 \tag{1.9}$$

the function $\varphi := u\eta^2$, where $\eta \in C_c^\infty(B_R; \mathbb{R})$ is a cut-off function with $\eta \equiv 1$ in $B_{R/2}$, $0 \leq \eta \leq 1$ and $\|\nabla\eta\|_\infty \leq \frac{4}{R}$



$$\frac{\partial\varphi_i}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha} = \eta^2 \frac{\partial u^i}{\partial x_\alpha} + 2\eta \frac{\partial\eta}{\partial x_\alpha} u^i, \tag{1.10}$$

that is

$$\nabla\varphi = \eta^2 \nabla u + 2\eta u \otimes \nabla\eta. \tag{1.11}$$

Plugging in the last equality in the weak formulation

$$0 = \int\limits_{B_R} \eta^2 \langle A\nabla u, \nabla u\rangle + 2\int\limits_{B_R} \eta \langle A\nabla u, u \otimes \nabla\eta\rangle \tag{1.12}$$

$$- \int\limits_{B_R} \eta^2 \langle f, u\rangle - \int\limits_{B_R} \eta^2 \langle F, \nabla u\rangle - 2\int\limits_{B_R} \eta \langle F, u \otimes \nabla\eta\rangle \tag{1.13}$$

$$=: I_1 + I_2 - I_3 - I_4 - I_5 \tag{1.14}$$

$$I_1 := \int_{B_R} \eta^2 \langle A\nabla u, \nabla u\rangle \,\mathrm{d}x = \int_{B_R} \sum_{\alpha,\beta,i,j} \eta^2 A_{ij}^{\alpha\beta} \partial_{x_\alpha} u^i \partial_{x_\beta} u^j \,\mathrm{d}x \geq \lambda \int_{B_R} \eta^2 |\nabla u|^2 \,\mathrm{d}x \tag{1.15}$$

$$I_2 := 2\int_{B_R} \eta \langle A\nabla u, u\otimes\nabla\eta\rangle \,\mathrm{d}x = 2\int_{B_R} \eta \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \partial_{x_\alpha} u^i u^j \partial_{x_\beta}\eta \,\mathrm{d}x \tag{1.16}$$

$$\overset{\text{Cauchy-Schwarz}}{\leq} 2\int_{B_R} \eta |A||u||\nabla u||\nabla\eta| \,\mathrm{d}x \tag{1.17}$$

$$\overset{\text{boundedness of } |A| \text{ and } \|\nabla\eta\|_\infty}{\leq} \frac{(\Lambda)}{R} \int_{B_R} (\eta|\nabla u|)|u| \,\mathrm{d}x \tag{1.18}$$

$$\overset{\text{Young } ab \leq \frac{a^2}{2}+\frac{b^2}{2}}{\leq} \frac{4\Lambda}{R}\varepsilon \int_{B_R} \eta^2|\nabla u|^2 \,\mathrm{d}x + \frac{4\Lambda}{R\varepsilon}\int_{B_R} |u|^2 \,\mathrm{d}x \tag{1.19}$$

$$I_3 := \int_{B_R} \left\langle f, \eta^2 u\right\rangle \mathrm{d}x = \int_{B_R} \eta^2 \sum_i f_i u^i \,\mathrm{d}x \overset{Young}{\leq} \frac{1}{2R^2}\int_{B_R} |u|^2 \,\mathrm{d}x + \frac{R^2}{2}\int_{B_R} |f|^2 \,\mathrm{d}x \tag{1.20}$$

$$I_4 := \int_{B_R} \eta^2 \langle F, \nabla u\rangle \,\mathrm{d}x = \int_{B_R} \eta^2 \sum_{\alpha,i} F_i^\alpha \partial_{x_\alpha} u^i \,\mathrm{d}x \leq \frac{\lambda}{4}\int_{B_R} \eta^2|\nabla u|^2 \,\mathrm{d}x + \frac{1}{\lambda}\int_{B_R} |F|^2 \,\mathrm{d}x \tag{1.21}$$

By Cauchy-Schwarz inequality, $\|\nabla\eta\|_\infty \leq \frac{4}{R}$ and Young inequality we have

$$I_5 := 2\int_{B_R} \eta \langle F, u\otimes\nabla\eta\rangle \,\mathrm{d}x = 2\int_{B_R} \sum_{\alpha,i} F_i^\alpha u^i \partial_{x_\alpha}\eta \,\mathrm{d}x \leq 4\int_{B_R} |F|^2 \,\mathrm{d}x + \frac{4}{R^2}\int_{B_R} |u|^2 \,\mathrm{d}x \tag{1.22}$$

Therefore from the weak formulation with $\varphi = \eta^2 u$ we obtain

$$\lambda \int_{B_R} \eta^2|\nabla u|^2 \,\mathrm{d}x \leq \int_{B_R} \eta^2 \langle A\nabla u, \nabla u\rangle \,\mathrm{d}x \tag{1.23}$$

$$\leq \underbrace{\left(\frac{4\Lambda\varepsilon}{R} + \frac{\lambda}{4}\right)\int_{B_R} \eta^2|\nabla u|^2 \,\mathrm{d}x}_{\text{Dirichlet term}} + \left(\frac{4\Lambda}{R\varepsilon} + \frac{1}{2R^2} + \frac{4}{R^2}\right)\int_{B_R} |u|^2 \,\mathrm{d}x + \tag{1.24}$$

$$+ \frac{R^2}{2}\int_{B_R} |f|^2 \,\mathrm{d}x + \left(\frac{1}{\lambda} + 4\right)\int_{B_R} |F|^2 \,\mathrm{d}x \tag{1.25}$$

We can choose ε so small that $\frac{4\Lambda\varepsilon}{R} = \frac{\lambda}{4}$ and absorb the Dirichlet term on the r.h.s. of the equation (meaning that it can be subtracted from both sides and still give a positive term on the left).
Finally, one concludes the proof observing that

$$\int_{B_R} \eta^2|\nabla u|^2 \,\mathrm{d}x \geq \int_{B_{\frac{R}{2}}} |\nabla u|^2 \,\mathrm{d}x \tag{1.26}$$

$\square$

Observe that the proof shows that in I_2 one can have a better estimate noting that $|\nabla u| = 0$ on $B_{R/2}$.

$$I_2 := 2 \int\limits_{B_R} \eta |A||u||\nabla u||\nabla \eta| \,\mathrm{d}x = 2 \int\limits_{B_R \setminus B_{\frac{R}{2}}} \eta |A||u||\nabla u||\nabla \eta| \,\mathrm{d}x\,. \tag{1.27}$$

This observation is indeed the starting point of the Widman's technique below.

1.3 Widman "holes filling technique"

A sharp version of the Caccioppoli-Leray inequality (CLI) has been proven by Widman. We can illustrate that in the simple case of $f = 0, F = 0$.
Observe that, with the notation of the (CLI) proof, since $|\nabla u| \le \frac{4}{R}\chi_{B_R \setminus B_{R/2}}$ one obtains

$$\int\limits_{B_{\frac{R}{2}}} \left|\nabla u(x)\right|^2 \mathrm{d}x \, leq \frac{c}{R^2} \int\limits_{B_R \setminus B_{\frac{R}{2}}} \left|u(x) - k\right|^2 \mathrm{d}x \tag{1.28}$$

for some positive constant c independent of R.
Now the idea is to choose $\kappa := ⨍_{B_R \setminus B_{R/2}} u(x)\,\mathrm{d}x$ so that we can estimate the r.h.s of (1.28) using the Poincarè inequality with explicit scaling, i.e.

$$\int\limits_{B_R \setminus B_{\frac{R}{2}}} \left| u(x) - ⨍\limits_{B_R \setminus B_{\frac{R}{2}}} u \,\mathrm{d}x \right|^2 \mathrm{d}x \le cR^2 \int\limits_{B_R \setminus B_{\frac{R}{2}}} \left|\nabla u(x)\right|^2 \mathrm{d}x \tag{1.29}$$

to get

$$\int\limits_{B_{\frac{R}{2}}} \left|\nabla u(x)\right|^2 \mathrm{d}x \le c \int\limits_{B_R \setminus B_{\frac{R}{2}}} \left|\nabla u(x)\right|^2 \mathrm{d}x \tag{1.30}$$

$$\Leftrightarrow (c+1) \int\limits_{B_{\frac{R}{2}}} \left|\nabla u(x)\right|^2 \mathrm{d}x \le c \int\limits_{B_R} \left|\nabla u(x)\right|^2 \mathrm{d}x \tag{1.31}$$

Setting $\vartheta := \frac{c}{c+1} < 1$ we get

$$\int\limits_{B_{\frac{R}{2}}} \left|\nabla u(x)\right|^2 \mathrm{d}x \le \vartheta \int\limits_{B_R} \left|\nabla u(x)\right|^2 \mathrm{d}x \tag{1.32}$$

Iterating the previous estimates d times for radii

$$2^1 r \to 2^2 r \to 2^3 r \to \cdots \to 2^d r \tag{1.33}$$

and choosing r such that

$$2^d r < R < 2^{d+1} r \tag{1.34}$$

we get

$$\int_{B_R} |\nabla u|^2 \,\mathrm{d}x \leq \vartheta^d \int_{B_R} |\nabla u|^2 \,\mathrm{d}x \tag{1.35}$$

Setting $\alpha \log_2(1/\vartheta)$, i.e. $\vartheta = 1/2^\alpha$ we have

$$\vartheta^d = \frac{1}{2^{\alpha d}} = \left(\frac{1}{2^d}\right)^\alpha \overset{(1.34)}{\leq} 2^\alpha \left(\frac{r}{R}\right)^\alpha . \tag{1.36}$$

Hence,

$$\int_{B_r} |\nabla u|^2 \,\mathrm{d}x \leq 2^\alpha \left(\frac{r}{R}\right)^\alpha \int_{B_R} |\nabla u|^2 \,\mathrm{d}x \,. \tag{1.37}$$

For $n = 2$ the estimate above implies $u \in C^{0,\alpha/2}(\Omega; \mathbb{R}^m)$.
In fact the idea that the decay of the L^p-norm of the gradient is related to its Hölder continuity will play a crucial role in the rest of the course, and we will discuss in detail in the next lectures.

1.4 Continuity via embedding

The Sobolev embedding theorem for $W^{1,p}(\Omega; \mathbb{R})$ says that

$$\begin{cases} p < n & W^{1,p}(\Omega;\mathbb{R}) \hookrightarrow L^{p^\star}(\Omega;\mathbb{R}) \quad \text{continuously} \quad p^\star = \frac{np}{n-p} \\ p = n & W^{1,n}(\Omega;\mathbb{R}) \hookrightarrow L^{q^\star}(\Omega;\mathbb{R}) \quad \text{compactly} \quad \forall 1 \leq q < \infty \\ p > n & W^{1,p}(\Omega;\mathbb{R}) \hookrightarrow C^{0,1-n/p}(\Omega;\mathbb{R}) \quad \text{continuously} \end{cases} \tag{1.38}$$

Hence a way to prove continuity of a Sobolev function is to prove that it belongs to $W^{1,p}$ for $p > n$.

1.5 Embedding for higher order Sobolev spaces

We recall that higher order Sobolev spaces $W^{k,p}(\Omega; \mathbb{R})$ with $k \geq 1$ integer and $1 \leq p \leq \infty$ are recursively defined as

$$W^{k,p}(\Omega, \mathbb{R}) := \{u \in W^{1,p}(\Omega; \mathbb{R}) : \nabla u \in W^{k-1,p}(\Omega; \mathbb{R}^n)\}. \tag{1.39}$$

Another way to prove continuity, applicable if $p < n$, is to use $W^{k,p}$ for k large enough. In fact, it holds true that:

(1) if $kp < n$ then $W^{k,p}(\Omega;\mathbb{R}) \hookrightarrow L^p(\Omega;\mathbb{R})$ for all $1 \leq q \leq p_k^\star$, when $p_k^\star = \frac{np}{n-kp}$

(2) if $kp = n$ then $W^{k,p}(\Omega;\mathbb{R}) \hookrightarrow L^q(\Omega;\mathbb{R})$ for all $1 \leq q \leq \infty$

(3) if $kp > n$ and $k - \frac{n}{p} \notin \mathbb{N}, W^{k,p}(\Omega;\mathbb{R}) \hookrightarrow C^{l,\alpha}(\overline{\Omega;\mathbb{R}})$ for $l = \left\lfloor k - \frac{n}{p} \right\rfloor$ and $0 \leq \alpha \leq k - \frac{n}{p} - l$

(4) if $kp > n$ and $k - \frac{n}{p} = l + 1 \in \mathbb{N}, W^{k,p}(\Omega;\mathbb{R}) \hookrightarrow C^{l,\alpha}(\overline{\Omega};\mathbb{R})$ for all $0 \leq \alpha < 1$

1.6 A priori estimates and the Nirenberg method

If $u \in H^1_{loc}(\Omega;\mathbb{R})$ (for the moment we are not interested at the behavior of u at $\partial\Omega$) is a weak solution of a system of elliptic PDEs we cannot apply previous remark to prove classical regularity, i.e. differentiability of u without assuming existence and some integrability of higher order weak derivatives of u. In fact the previous remark is not really exploiting the equation.

How to gain better integrability?
What follows goes under the name of Nirenberg's method.
Let us consider the simplest setting and consider a solution $u \in H^1_{loc}(\Omega)$ of the Poisson equation

$$-\Delta u = f, \qquad f \in L^2_{loc}(\Omega;\mathbb{R}) \tag{1.40}$$

We want to prove that $u \in H^2_{loc}(\Omega;\mathbb{R}) = W^{2,2}_{loc}(\Omega;\mathbb{R})$, as this will be the first step to transfer regularity information from the data f to the solution u.

Let us start with supposing that we already knew that $\partial_{x_\alpha} u \in H^1_{loc}(\Omega;\mathbb{R})$, then we know that

$$-\Delta(\partial_{x_\alpha} u) = \partial_{x_\alpha} f \qquad \text{in a weak sense} \tag{1.41}$$

To check it, test $-\Delta u = f$ with $\partial_{x_\alpha}\varphi$ and integrate by parts to get

$$\int \nabla u \nabla(\partial_{x_\alpha}\varphi)\,\mathrm{d}x = \int f\partial_{x_\alpha}\varphi\,\mathrm{d}x \tag{1.42}$$

$$=$$

$$\int \nabla\partial_{x_\alpha}(\nabla\varphi)\,\mathrm{d}x \overset{I.P.}{=} -\int \underbrace{\partial_{x_\alpha}\Delta u}_{\Delta(\partial_{x_\alpha}u)} \cdot \Delta\varphi\,\mathrm{d}x = -\int \Delta(\partial_{x_\alpha}u)\Delta\varphi\,\mathrm{d}x \tag{1.43}$$

Weak derivatives commute:

$$\int \partial_1\partial_2 u\varphi\,\mathrm{d}x \overset{I.P.}{=} \int u\partial_1\partial_2\varphi\,\mathrm{d}x \overset{\varphi \text{ is regular}}{=} \int u\partial_2\partial_1\varphi\,\mathrm{d}x \overset{I.P.}{=} \int \partial_2\partial_1 u\varphi\,\mathrm{d}x \tag{1.44}$$

Hence $\int \nabla(\partial_{x_\alpha}u)\nabla\varphi\,\mathrm{d}x = -\int f\partial_{x_\alpha}\varphi\,\mathrm{d}x$ or $-\nabla(\partial_{x_\alpha}u) = \partial_{x_\alpha} f$.
Hence, for every ball $B_R(x_0) \subset\subset \Omega$ ($\overline{B_R(x_0)} \subset \Omega$) we use the Caccioppoli-Leray inequality (CLI) to get:

$$C_{CL} \int\limits_{B_{\frac{R}{2}}(x_0)} \left|\nabla(\partial_{x_\alpha}u)\right|^2 \mathrm{d}x \le \frac{1}{R^2} \int\limits_{B_R(x_0)} \left|\partial_{x_\alpha}U(x)\right|^2 \mathrm{d}x + R^2 \int\limits_{B_R(x_0)} \left|f(x)\right|^2 \mathrm{d}x \qquad (\text{CLI}) \tag{1.45}$$

that provides an explicit bound on the H^2_{loc} norm of u in terms of its H^1 norm.

In the previous discussion we have considered $u \in H_{loc^1}(\Omega;\mathbb{R})$ to be a solution of the Poisson equation $-\nabla u = f$ in Ω with $f \in L^2(\Omega)$. Assuming $u \in H^2_{loc}$ we have obtained the following Caccioppoli-Leray estimate:

$$C_{CL} \int\limits_{B_{\frac{R}{2}}(x_0)} \left|\nabla(\partial_{x_\alpha}u)\right|^2 \mathrm{d}x \le \frac{1}{R^2} \int\limits_{B_R(x_0)} \left|\partial_{x_\alpha}U(x)\right|^2 \mathrm{d}x + R^2 \int\limits_{B_R(x_0)} \left|f(x)\right|^2 \mathrm{d}x \qquad (\text{CLI}) \tag{1.46}$$

Can we remove the "a priori" regularity assumption?
Here, for the Poisson equation, it is simple.
Consider the convolution $u * \rho_\varepsilon$. Since $-\Delta u = f$ we have $-\Delta(u * \rho_\varepsilon) = f * \rho_\varepsilon$.
Now observe that

$$C_{CL} \int\limits_{B_{\frac{R}{2}}} \left|\nabla(\partial_{x_\alpha} u * \rho_\varepsilon)\right| * 2 \,\mathrm{d}x \leq \frac{1}{R^2} \int\limits_{B_R} \left|\partial_{x_\alpha} u * \rho_\varepsilon\right|^2 \mathrm{d}x + R^2 \int\limits_{B_R} \left|f * \rho_\varepsilon\right|^2 \mathrm{d}x \tag{1.47}$$

$$\leq \frac{1}{R^2} \int\limits_{B_R} \left|\partial_{x_\alpha} u\right|^2 \mathrm{d}x + R^2 \int\limits_{B_R} |f|^2 \,\mathrm{d}x \tag{1.48}$$

As a result $\forall \alpha, \beta \left\|\partial_{x_\alpha}\partial_{x_\beta}(u * \rho_\varepsilon)\right\|_{L^2_{loc}} \leq C$. This means that, up to subsequences $\partial_{x_\alpha}\partial_{x_\beta}(u * \rho_\varepsilon) \xrightharpoonup[\varepsilon\to 0]{L^2} g$. Since $\partial_{x_\alpha}(u * \rho_\varepsilon) \xrightarrow{L^2} \partial_{x_\alpha} u$, we have that $g = \partial_{x_\alpha}\partial_{x_\beta} u$ and that the whole sequence $\partial_{x_\alpha}\partial_{x_\beta}(u * \rho_\varepsilon)$ converges to $\partial_{x_\alpha}\partial_{x_\beta} u$.
As a result $\left\|\partial_{x_\alpha}\partial_{x_\beta} u\right\|_{L^2_{loc}} \leq C$ and $u \in H^2_{loc}(\Omega)$.

We have used the following result:

Lemma 1 (Stability of weak derivatives)**.**
$u_k \in W^{1,p}(\Omega)$ for some $1 < p < \infty, \forall k \in \mathbb{N}$. Assume $u_k \to u$ in L^p and $\sup_k \|\nabla u_k\|_p \leq C$, then $u \in W^{1,p}(\Omega)$ and $\nabla u_k \rightharpoonup \nabla u$ in L^p.

The same idea does not work so easily when the coefficients $A^{\alpha\beta}_{ij}$ are not constant. In fact in this case differentiating the equation produces "extra terms".
Nirenberg's idea is to use difference quotients instead of derivatives. We introduce the notation

$$\Delta_{h,\alpha} u(x) = \frac{u(x + he_\alpha) - u(x)}{h} =: \frac{\tau_{h,\alpha} U(x) - u(x)}{h} \tag{1.49}$$

The following properties can be checked to hold true:

- Discrete Leibniz rule

$$\Delta_{h,\alpha}(uv) = (\tau_{h,\alpha} u)\Delta_{h,\alpha} v + (\Delta_{h,\alpha} u)v \tag{1.50}$$
$$= (\tau_{h,\alpha} v)\Delta_{h,\alpha} u + (\Delta_{h,\alpha} v)u \tag{1.51}$$

- Integration by parts rule

$$\int\limits_\Omega \varphi(x)\Delta_{h,\alpha} u(x) \,\mathrm{d}x = -\int\limits_\Omega u(x)\Delta_{-h,\alpha}\varphi(x) \,\mathrm{d}x \tag{1.52}$$

  for all $\varphi \in C^1_c(\Omega; \mathbb{R}), |h| < \operatorname{dist}(\operatorname{spt}\varphi, \partial\Omega)$

The following lemma provides a characterization of $W^{1,p}$ functions with $p > 1$, in terms of uniform L^p bounds of the corresponding discrete partial derivatives.

Lemma 2.
Consider $u \in L^p_{loc}(\Omega; \mathbb{R})$, with $1 < p \leq \infty$ and fix $\alpha \in \{1, 2, \ldots, n\}$. The partial derivative $\partial_{x_\alpha} u$

belongs to $L^p_{loc}(\Omega;\mathbb{R})$ if and only if the family $\Delta_{h,\alpha}u$ is uniformly bounded in L^p_{loc} as $h \to 0$. More precisely, if $\forall \Omega' \subset\subset \Omega \, \exists C = C(\Omega')$ such that

$$\left| \int_{\Omega'} (\Delta_{h,\alpha}u)\varphi \, \mathrm{d}x \right| \leq c\|\varphi\|_{L^{p'}(\Omega';\mathbb{R})} \qquad \varphi \in C^1_c(\Omega';\mathbb{R}) \tag{1.53}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$ and $|h| < \frac{1}{2}\,\mathrm{dist}(\Omega'; \partial\Omega)$.

We now see how the previous lemma allows us to obtain regularity. We stick to the Poisson equation for the moment.

Suppose $f \in H^1_{loc}(\Omega;\mathbb{R})$ and $-\Delta u = f$ for some $u \in H^1_{loc}(\Omega;\mathbb{R})$.
Being the equation translation invariant, we can write $-\Delta(\tau_{h,\alpha}u) = \tau_{h,\alpha}f$, hence $-\Delta(\Delta_{h,\alpha}u) = \Delta_{h,\alpha}f$ for any $\Omega' \subset\subset \Omega$ and $|h| < \mathrm{dist}(\Omega', \partial\Omega)$.
By Lemma 2 it holds $\Delta_{h,\alpha}f$ is bounded in L^2_{loc} uniformly in h. By (CLI) $\left|\nabla\Delta_{h,\alpha}U\right|$ is bounded in $L^2_{loc}(\Omega;\mathbb{R})$, thanks to the Lemma 2 (applied componentwise) we have that

$$\partial_{x_\alpha}(\nabla u) \in L^2_{loc}(\Omega;\mathbb{R}^n) \tag{1.54}$$

That is, by the arbitrariness of $\alpha \in \{1, 2, \ldots, n\}$, $u \in H^2_{loc}(\Omega;\mathbb{R})$. We are left to prove Lemma 2.

We now state and prove the first interior regularity theorem.

Theorem 3 (H^2-regularity)**.**
Let Ω be an open domain in $\mathbb{R}$. Consider a map $A \in C^{0,1}_{loc}(\Omega;\mathbb{R}^{m^2\times n^2})$ such that $A(x) := A^{\alpha\beta}_{ij}(x)$ satisfies the Legendre-Hademard condition (LH) for some continuous and positive ellipticity function $\lambda : \Omega \to \mathbb{R}$, as well as the uniform bound

$$\sup_{x\in\Omega} \left| A^{\alpha\beta}_{ij}(x) \right| \leq \Lambda < \infty.$$

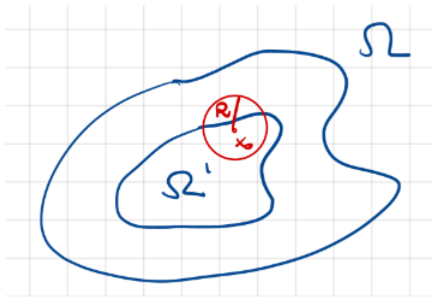
Then, for every $u \in H^1_{loc}(\Omega;\mathbb{R}^m)$ weak solution of the equation

$$-\sum_{\alpha,\beta,j} \partial_{x_\alpha}(A^{\alpha\beta}_{ij}\partial_{x_\beta}u^j) = f_i - \sum_{\alpha} \partial_{x_\alpha}F^\alpha_i \qquad i = 1, 2, \ldots, m$$

with data $f \in L^2_{loc}(\Omega;\mathbb{R}^m)$ and $F \in H^1_{loc}(\Omega;\mathbb{R}^{m\times n})$, one has that $u \in H^2_{loc}(\Omega;\mathbb{R}^m)$ and for every $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exists $c := c(\Omega', \Omega'', A)$ such that

$$\int_{\Omega'} \left|\nabla^2 u\right|^2 \mathrm{d}x \leq c \left(\int_{\Omega''} |u|^2 \, \mathrm{d}x + \int_{\Omega''} |f|^2 \, \mathrm{d}x + \int_{\Omega''} |\nabla F|^2 \, \mathrm{d}x \right)$$

Remark.
Even if we have stated the theorem for a generic $\Omega' \subset\subset \Omega$, it is enough to prove it for balls inside Ω. More precisely. It is enough to prove it for balls $B_R(x_0)$ where $x_0 \in \Omega'$ and $R < \frac{1}{2} \overset{dist}{(} \Omega', \partial\Omega)$.



The general result can then be obtained by a compactness and covering argument (Exercise). For the case of a ball we need to prove that

$$\int_{B_{\frac{R}{2}(x_0)}} \left|\nabla^2 u\right|^2 \mathrm{d}x \leq c\left(\int_{B_{2R}(x_0)} |u|^2 \,\mathrm{d}x + \int_{B_{2R}(x_0)} |f|^2 \,\mathrm{d}x + \int_{B_{2R}(x_0)} |\nabla F|^2 \,\mathrm{d}x\right)$$

for every $x_0 \in \Omega'$.

Proof. • Assume w.l.o.g that $x_0 = 0$ and $F = 0$. (note that the term $\sum_\alpha \partial_{x_\alpha} F_i^\alpha$ can always be absorbed into f. In fact $\left\|f + \operatorname{div} F^i\right\|_2 \leq \|f\|_2 + \|\nabla F\|_2$)
• Moreover we assume that λ is constant (it is possible to reduce to the general case, see next lectures)
We start observing that the equation in its weak formulation reads as

$$\int_\Omega \langle A\nabla u, \nabla\varphi\rangle \,\mathrm{d}x = \int_\Omega \langle f, \varphi\rangle \,\mathrm{d}x\,, \qquad \forall \varphi \in C_c^\infty(\Omega;\mathbb{R})$$

In order to simplify the notation in the proof we let e_γ be a fixed vector and set $\tau_h := \tau_{h,\gamma}$ and $\Delta_{h:=\Delta_{h,\gamma}}$.
We take as test function $\tau_{-h}\varphi$, for h small enough and change variables to get

$$\int_\Omega \big\langle \tau_h(A\nabla u), \nabla\varphi\big\rangle \,\mathrm{d}x = \int_\Omega \langle \tau_h f, \varphi\rangle \,\mathrm{d}x$$

subtracting the two previous equations anf dividing by h we have that (using Leibniz)

$$\int_\Omega \frac{1}{h}\left[\langle A\nabla u, \nabla\varphi\rangle\right] - \big\langle \tau_h(A\nabla u), \nabla\varphi\big\rangle \,\mathrm{d}x \tag{1.55}$$

$$= \int_\Omega \left\langle \underbrace{A\nabla u - \tau_h(A\nabla u)}_{h}, \nabla\varphi\right\rangle \mathrm{d}x \tag{1.56}$$

$$= \int_\Omega \langle \Delta_h(A\nabla u), \nabla\varphi\rangle \,\mathrm{d}x \tag{1.57}$$

$$= \int_\Omega \big\langle \tau_h A\nabla(\Delta_h u), \nabla\varphi\big\rangle + \big\langle (\Delta_h A)\nabla u, \nabla\varphi\big\rangle \,\mathrm{d}x \tag{1.58}$$

$$= \int_\Omega \langle \Delta_h f, \varphi\rangle \,\mathrm{d}x\,, \tag{1.59}$$

i.e,

$$\int_\Omega \big\langle(\tau_h A)\nabla(\Delta_h u)\big\rangle \,\mathrm{d}x = \int_\Omega \langle \Delta_h f, \varphi\rangle \,\mathrm{d}x - \int_\Omega \big\langle(\Delta_h A)\nabla u, \nabla\varphi\big\rangle \,\mathrm{d}x$$

This is the weak formulation of the equation

$$-\sum_{\alpha,\beta,j} \partial_{x_\alpha}\Big((\tau_h A)_{ij}^{\alpha\beta}\partial_{x_\beta} v^j\Big) = f_i' - \sum_\alpha \partial_{x_\alpha} G_i^\alpha, \qquad i = 1, 2, \dots, m \tag{EQ}$$

where $v = \Delta_h u$. $f' = \Delta_h f$ and $G = -(\Delta_h A)\nabla u$. The basic idea now is to use (CLI). A direct application of it would lead to an estimate in terms of the L^2 norm of $f' = \Delta_h f$ which we know can be uniformly bounded in h only if $f \in H^1_{loc}(\Omega)$ (by the characterization of Sobolev spaces in terms of difference quotients). Since we have only assumed $f \in L^2_{loc}(\Omega)$, we need to proceed carefully and "adapt" the proof of (CLI). We consider the cut-off function η compactly supported in B_R, $\eta \in [0,1]$, $\eta \equiv 1$ on $B_{R/2}$ and $|\nabla\eta| \le 4/R$. We need test (EQ) with $\varphi := \eta^2\delta_h u = \eta^2 v$, where $|h| < R/2$.
As in the proof of (CLI) we get

$$\frac{3\lambda}{4}\int_{B_R} \eta^2|\nabla v|^2 \,\mathrm{d}x \le \frac{4\Lambda\varepsilon}{R}\int_{B_R} \eta^2|\nabla v|^2 \,\mathrm{d}x + \Big(\frac{4\Lambda}{R\varepsilon} + \frac{4}{R^2}\Big)\int_{B_R} |v|^2 \,\mathrm{d}x + \int_{B_R} \eta^2 v\Delta_h f \,\mathrm{d}x + \Big(\frac{1}{\lambda} + 4\Big)\int_{B_R} |G|^2 \,\mathrm{d}x\,.$$

Choosing $\varepsilon > 0$ we absorb $\frac{4\Lambda\varepsilon}{R}\int_{B_R} \eta^2|\nabla v|^2 \,\mathrm{d}x$ in the L.H.S and we get that for some constant $c = c(\lambda, \Lambda, R) > 0$

$$c\int_{B_R} \eta^2|\nabla v|^2 \,\mathrm{d}x \le \underbrace{\int_{B_R} |v|^2 \,\mathrm{d}x}_{(1.61)} + \underbrace{\int_{B_R} \eta^2 v\Delta_h f \,\mathrm{d}x}_{(1.62)} + \underbrace{\int_{B_R} |G|^2 \,\mathrm{d}x}_{(1.71)} \tag{1.60}$$

We consider the different terms separately. We notice that (see (1.28) in the proof of Lemma 2)

$$\int_{B_R} |v|^2 \,\mathrm{d}x = \int_{B_R} |\Delta_h u|^2 \,\mathrm{d}x \le \int_{B_{R+h}} |\nabla u|^2 \,\mathrm{d}x \tag{1.61}$$

The R.H.S of the inequality above can be estimated by the (CLI). In fact $\int_{B_{R+h}} |\nabla u|^2 \,\mathrm{d}x \le \int_{B_{3/2R}} |\nabla u|^2 \,\mathrm{d}x$ which can be in turn be estimated by (CLI) for u between the balls $B_{3R/2}$ and B_{2R}, with an upper bound of the type we are looking for. Concerning the term (1.62) we have

$$\begin{aligned}\left|\int_{B_R} \eta^2 v\Delta_h f \,\mathrm{d}x\right| &\overset{\text{discrete by I.P.}}{=} \left|\int_{B_R} -\Delta_{-h}(\eta^2 v) f \,\mathrm{d}x\right| \\ &\overset{\text{Young } p=q=2}{\le} \widetilde{\varepsilon}\int_{B_R} \Big|\Delta_{-h}(\eta^2 v)\Big|^2 \,\mathrm{d}x + \frac{1}{\widetilde{\varepsilon}}\int_{B_R} |f|^2 \,\mathrm{d}x\end{aligned} \tag{1.62}$$

The term $\int_{B_R} |f|^2 \,\mathrm{d}x \leq \int_{B_{2R}} |f|^2 \,\mathrm{d}x$ is already fine for the estimate we want. For the other term we have

$$\widetilde{\varepsilon} \int\limits_{B_R} \left|\Delta_h(\eta^2 v)\right|^2 \mathrm{d}x \overset{\text{Corollary of } 2}{\leq} \widetilde{\varepsilon} \int\limits_{B_{h+r}} \left|\nabla(\eta^2 v)\right|^2 \mathrm{d}x \tag{1.63}$$

$$= c\widetilde{\varepsilon} \int\limits_{B_{R+h}} \left|(\nabla\eta^2)v + \eta^2\nabla v\right|^2 \mathrm{d}x \tag{1.64}$$

$$\overset{(1.69)}{\leq} c\widetilde{\varepsilon}\left[\frac{128}{R^2}\int\limits_{B_{R+h}} |v|^2 \,\mathrm{d}x + 2\int\limits_{B_{R+h}} \eta^4|\nabla v|^2 \,\mathrm{d}x\right] \tag{1.65}$$

$$\leq c\widetilde{\varepsilon}2\left[\int\limits_{B_{R+h}} \left|\nabla\eta^2\right|^2|v|^2 \,\mathrm{d}x + \int\limits_{B_{R+h}} \eta^4|\nabla v|^2 \,\mathrm{d}x\right] \tag{1.66}$$

$$\overset{(1.70)}{\leq} c\widetilde{\varepsilon}\left[\frac{128}{R^2}\int\limits_{B_{R+h}} |v|^2 \,\mathrm{d}x + 2\int\limits_{B_{R+h}} \eta^4|\nabla v|^2 \,\mathrm{d}x\right] \tag{1.67}$$

$$\overset{\eta^4\leq\eta^2}{\leq} c\widetilde{\varepsilon}\frac{128}{R^2}\int\limits_{B_{R+h}} |v|^2 \,\mathrm{d}x + c\widetilde{\varepsilon}2\int\limits_{B_{R+h}} \eta^2|\nabla v|^2 \,\mathrm{d}x \tag{1.68}$$

For (1.69), see that

$$(a+b)^2 = 4(\frac{1}{2}a + \frac{1}{2}b)^2 \overset{\text{conv}}{\leq} 4\frac{(}{1}2a^2 + \frac{1}{2}b) = 2(a^2+b^2) \tag{1.69}$$

and for (1.70), see that

$$|\eta| \leq 1,\ |\nabla\eta| \leq \frac{4}{R} \Rightarrow \left|\nabla\eta^2\right|^2 = |2\eta\nabla\eta|^2 < (\frac{8}{R})^2 = \frac{64}{R^2} \tag{1.70}$$

The term $2c\widetilde{\varepsilon}\int_{B_{R+h}} \eta^2|\nabla v|^2 \,\mathrm{d}x$ can be absorbed in the l.h.s. of (1.60) on choosing $\widetilde{\varepsilon}$ small enough, while the term $\frac{128c}{R^2}\widetilde{\varepsilon}\int_{B_{R+h}} |v|^2 \,\mathrm{d}x$ can again be estimated as in (1.61) using (CLI).
We are left with

$$\int\limits_{B_R} |G|^2 \,\mathrm{d}x = \int\limits_{B_R} |\Delta_h A \cdot \nabla u|^2 \,\mathrm{d}x \overset{\text{A is locally Lip}}{\leq} c\int\limits_{B_R} |\nabla u|^2 \,\mathrm{d}x\,, \tag{1.71}$$

hence we can use (CLI) again.
We eventually collect all the estimates to find out that we control uniformly as $h \to 0$ the term

$$\int\limits_{B_{\frac{R}{2}}} |\nabla v|^2 \,\mathrm{d}x = \int\limits_{B_{\frac{R}{2}}} |\nabla\Delta_h u|^2 \,\mathrm{d}x = \int\limits_{B_{\frac{R}{2}}} |\Delta_h\nabla u|^2 \,\mathrm{d}x \tag{1.72}$$

Thanks to Lemma 2 we obtain the control over $\int_{B_{R/2}} \left|\partial_{x_\gamma}\nabla u\right|^2 \mathrm{d}x$. By the arbitrariness of γ we control $\int_{B_{R/2}} \left|\nabla^2 u\right|^2 \mathrm{d}x$ as claimed. $\square$

1.7 Decay estimates for systems with constant coefficients

Lemma 3.
Let $A = A_{ij}^{\alpha\beta}$ be a constant matrix satisfying the Legendre-Hadamard condition (LH) for some $\lambda > 0$, let $\Lambda = |A|$ and let $u \in H^1_{loc}(\Omega; \mathbb{R}^m)$ satisfying the system

$$-\sum_{\alpha,\beta,j} \partial_{x_\alpha} \left(A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j \right) = 0, \qquad \forall i \in \{1, 2, \ldots, m\} \tag{1.73}$$

Then for $B_r(x_0) \subset B_R(x_0) \subset\subset \Omega$ it holds

- $$\int\limits_{B_r(x_0)} |u|^2 \,\mathrm{d}x \leq c_D \left(\frac{r}{R} \right)^n \int\limits_{B_R(x_0)} |u|^2 \,\mathrm{d}x \tag{1.74}$$
- $$\int\limits_{B_r(x_0)} \left| u - u_{x_0,r} \right|^2 \mathrm{d}x \leq c_E \left(\frac{r}{R} \right)^{n+2} \int\limits_{B_R(x_0)} \left| u - u_{x_0,R} \right|^2 \mathrm{d}x \tag{1.75}$$

with $c_D = c_D(n, \lambda, \Lambda)$ and $c_E = c_E(n, \lambda, \Lambda)$, having used the notation

$$u_{x_0,s} := \frac{1}{\left| B_s(x_0) \right|} \int\limits_{B_s(x_0)} u(x) \,\mathrm{d}x \tag{1.76}$$

1.8 Regularity up to the boundary

Let $u \in H^1_0(\Omega; \mathbb{R}^m)$ be a weak solution of

$$-\sum_{\alpha,\beta,j} \partial_{x_\alpha} \left(A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j \right) = f_i - \sum_{\alpha} \partial_{x_\alpha} F_i^\alpha, \qquad i \in \{1, 2, \ldots, m\} \tag{1.77}$$

We make the following hypothesis:
$f \in L^2(\Omega; \mathbb{R}^m)$, $F \in H^1(\Omega; \mathbb{R}^{m \times n})$, $A \in C^{0,1}(\Omega; \mathbb{R}^{m^2 \times n^2})$,
$A(X)$ satisfies the Legendre-Hadamard condition (LH) uniformly with respect to $x \in \Omega$, Ω has C^2 boundary (we say $\partial\Omega \in C^2$), i.e. the domain Ω is locally the epigraph of a C^2 function up to a rigid motion.

Theorem 4 (Regularity up to the boundary)**.**
Under the assumptions above, the function u belongs to $H^2(\Omega; \mathbb{R}^m)$ and moreover $\exists c = c(\Omega, A, n) > 0$ such that

$$\|u\|_{H^2(\Omega;\mathbb{R}^m)} \leq c \left(\|f\|_{L^2(\Omega;R^m)} + \|F\|_{H^1(\Omega;\mathbb{R}^{m\times n})} \right). \tag{1.78}$$

If both the boundary of the domain and the data are sufficiently regular the method can be iterated to obtain higher Regularity of u.

Theorem 5.
Assume in addition to the hypothesis above that $f \in H^k(\Omega; \mathbb{R}^m)$, $F \in H^{k+1}(\Omega; \mathbb{R}^{m\times n})$, $A \in C^{k,1}(\Omega; \mathbb{R}^{m^2 \times n^2})$ with Ω such that $\partial\Omega \in C^{k+2}$. Then $u \in H^{k+2}(\Omega; \mathbb{R}^m)$

1.9 Interior Regularity for Nonlinear Equations

We see here how the Nirenberg's method is appropriate in dealing with nonlinear PDEs as those arising from Euler-Lagrange equations of non-quadratic functionals.
Consider $L \in C^2(\mathbb{R}^{m\times n};\mathbb{R})$ and assume that

(i) there exists a constant $c > 0$ such that $\left|\nabla^2 L(\xi)\right| \leq c$, $\forall \xi \in \mathbb{R}^{m\times n}$

(ii) L satisfies a uniform Legendre condition, i.e.

$$\sum_{\alpha,\beta,i,j} \partial_{p_j^\alpha}\partial_{p_j^\beta} L(p)\xi_i^\alpha \xi_j^\beta \geq \lambda|\xi|^2 \qquad \xi \in \mathbb{R}^{m\times n} \tag{1.79}$$

for some $\lambda > 0$ independent of p.

To simplify notation we set $B_i^\alpha := \frac{\partial L}{\partial p_i^\alpha}$ and $A_{ij}^{\alpha\beta} := \frac{\partial^2 L}{\partial p_i^\alpha \partial p_j^\beta}$ and notice that $A_{ij}^{\alpha\beta}$ is symmetric w.r.t the transformation $(\alpha, i) \to (\beta, j)$.
Let $\Omega \subset \mathbb{R}^n$ be an open domain and let $u \in H^1_{loc}(\Omega;\mathbb{R}^m)$ be local minimizer (see later for the precise definition) of the functional

$$w \mapsto \mathcal{L}(w) := \int_\Omega L(\nabla w)\,\mathrm{d}x\,. \tag{1.80}$$

We will discuss the implication

$$L \in C^\infty \Rightarrow u \in C^\infty \tag{1.81}$$

which is strictly related ro Hilbert's XIX problem (initially posed for analytic functions of two variables).

1.10 Local minimality

We say that u is a local minimizer for $\mathcal{L}$, if for all $v \in H^1_{loc}(\Omega;\mathbb{R}^m)$ such that $\mathrm{spt}(u-v) \subset \Omega' \subset\subset \Omega$, we have

$$\int_{\Omega'} L(\nabla v)\,\mathrm{d}x \geq \int_{\Omega'} L(\nabla u)\,\mathrm{d}x \tag{1.82}$$

In this case one can obtain the Euler-Lagrange equation considering perturbations of u of the type $V_t = u + t\varphi$ with $\varphi \in C_c^\infty(\Omega;\mathbb{R}^m)$ and imposing that it holds (note that $v_0 = u$)

$$\int_\Omega L(\nabla v_t)\,\mathrm{d}x \geq \int_\Omega L(\nabla v_0)\,\mathrm{d}x \tag{1.83}$$

or in other words the 1D function $\Phi(t) := \int_\Omega L(\nabla v_t)\,\mathrm{d}x$ has a local minimum at $t = 0$, which by the regularity of L gives $\Phi'(0) = 0$, or

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega L(\nabla u + t\nabla\varphi)\,\mathrm{d}x\right]_{t=0} = \sum_{\alpha,i}\int_\Omega B_i^\alpha(\nabla u)\frac{\partial \varphi^i}{\partial x_\alpha}\,\mathrm{d}x \tag{1.84}$$

Applying the same argument to test functions of the form $\tau_{-h,\gamma}\varphi$ (here γ is a fixed coordinate direction corresponding to the unit vector e_γ and $h>0$) we get (upon changing variables)

$$\sum_{\alpha,i}\int_\Omega \tau_{h,\gamma}\left(B_i^\alpha(\nabla u)\right)\frac{\partial\varphi^i}{\partial x_\alpha}\,\mathrm{d}x=0 \tag{1.85}$$

Subtracting the last two equations and dividing by h

$$\sum_{\alpha,i}\int_\Omega \Delta_{h,\gamma}\left(B_i^\alpha(\nabla u)\right)\frac{\partial\varphi^i}{\partial x_\alpha}\,\mathrm{d}x=0 \tag{EL$_h$}$$

Note that, by the regularity assumptions on L we can write that

$$B_i^\alpha(\nabla u(x+he_\gamma))-B_i^\alpha(\nabla u(x)) \tag{1.86}$$

$$=\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}\left[B_i^\alpha\left(t\nabla u(x+he_\gamma)+(1-t)\nabla u(x)\right)\right]\mathrm{d}t \tag{1.87}$$

$$=\sum_{\beta,j}\int_0^1 A_{ij}^{\alpha\beta}\left(t\nabla u(x+he_\gamma)+(1-t)\nabla u(x)\right)\mathrm{d}t\left(\frac{\partial u^j}{\partial x_\beta}(x+he_\gamma)-\frac{\partial u^j}{\partial x_\beta}(x)\right). \tag{1.88}$$

Setting for convenience

$$\widetilde{A}_{ij,h}^{\alpha\beta}(x):=\int_0^1 A_{ij}^{\alpha\beta}\left(t\nabla u(x+he_\gamma)+(1-t)\nabla u(x)\right)\mathrm{d}t \tag{1.89}$$

We rewrite the (EL$_h$) condition as

$$\sum_{\alpha,\beta,i,j}\int_\Omega \widetilde{A}_{ij,h}^{\alpha\beta}(x)\frac{\partial\Delta_{h,\gamma}u^j}{\partial x_\beta}(x)\frac{\partial\varphi^i}{\partial x_\alpha}(x)\,\mathrm{d}x=0 \tag{1.90}$$

Hence the function $w=\Delta_{h,\gamma}u$ solves the system:

$$-\sum_{\alpha,\beta,j}\partial_{x_\alpha}\left(\widetilde{A}_{ij,h}^{\alpha\beta}\partial_{x_\beta}w^j\right)=0 \qquad i=1,2,\dots,n \tag{1.91}$$

since $\widetilde{A}_{ij,h}^{\alpha\beta}$ satisfies uniformly with respect to h both a Legendre condition and an upper bound on the L^∞ norm, we can apply (CLI) to obtain that $\exists c>0$ independent of h such that

$$\int_{B_R(x_0)}\left|\nabla(\Delta_{h,\gamma}u)\right|^2\mathrm{d}x\le\frac{c}{R^2}\int_{B_{2R}(x_0)}\left|\Delta_{h,\gamma}u\right|^2\mathrm{d}x\le\frac{c}{R^2}\int_{B_{2R+h}(x_0)}|\nabla u|^2\,\mathrm{d}x \tag{1.92}$$

for every $B_R(x_0)\subset B_{2R}(x_0)\subset\subset\Omega$. As a result we obtain by Lemma 2 that $u\in H^2_{loc}(\Omega;\mathbb{R}^m)$. Moreover, we have that

(i) $\Delta_{h,\gamma}u\xrightarrow{h\to 0}\partial_{x_\gamma}u$ in $L^2_{loc}(\Omega;\mathbb{R}^m)$ (as usual this is trivial if u is regular. In our case $u\in H^2_{loc}$ this is obtained by approximation)
Notice also that since $u\in H^1_{loc}$ we have $\left\|\Delta_{h,\gamma}u\right\|_2\le c$, with, together with (1.92), gives $\left\|\Delta_{h,\gamma}\right\|_{H^1_{loc}}\le c$.
This means that, up to subsequences, $\Delta_{h,\gamma}u\xrightarrow{h\to 0}\partial_{x_\gamma}u$ weakly in H^1_{loc}.

(ii) As a result of (i) the function $w = \partial_{x_\gamma} u$ satisfies

$$-\sum_{\alpha,\beta,j} \partial_{x_\alpha} \left(A_{ij}^{\alpha\beta}(\nabla u) \partial_{x_\beta} w^j \right) = 0 \qquad i = 1, 2, \dots, m \tag{1.93}$$

in the weak sense. [It is enough to check that $\widetilde{A}_{ij,h}^{\alpha\beta} \to A_{ij}^{\alpha\beta}$ in $L^p_{loc}(\Omega)$, $\forall 1 \leq p < \infty$]

To solve Hilbert's XIX problem, we would like to apply a classical result by Schauder asserting that if w is a weak solution if a problem in divergence form:

$$-\sum_{\alpha,\beta,j} \partial_{x_\alpha} \left(B_{ij}^{\alpha\beta} \partial_{x_\beta} w^j \right) = 0 \qquad i = 1, 2, \dots, m \tag{1.94}$$

them $B \in C^{0,\alpha}(\Omega; \mathbb{R}^{m^2 \times n^2}) \Rightarrow w \in C^{1,\alpha}(\Omega; R^m)$ which is to say $u \in C^{2,\alpha}(\Omega; R^m)$. If we know in addition that $A_{ij}^{\alpha\beta}$ where C^∞, which is the case if $L \in C^\infty$, then

$$A_{ij}^{\alpha\beta}(\nabla u) = B_{ij}^{\alpha\beta} \in C^{0,\alpha}(\Omega; \mathbb{R}^{m^2 \times n^2}) \tag{1.95}$$

and the Schauder's theory would give $w \in C^{2,\alpha} \Rightarrow u \in C^{3,\alpha} \Rightarrow B \in C^{2,\alpha} \Rightarrow \dots$, i.e./ we can bootstrap regularity!
Roughly speaking, the ?? (hölderianity) result is what we need to bootstrap the argument and prove $u \in C^\infty$ if $L \in C^\infty$.
But to do so we first need to improve the regularity of $B(x) = A(\nabla u(x))$, since at the moment we only know that $A(\nabla u) \in H^1_{loc}(\Omega; \mathbb{R}^{m^2 \times n^2})$, while we would need $A(\nabla u) \in C^{0,\alpha}(\Omega; \mathbb{R}^{m^2 \times n^2})$.
In the case $n = 2, m \in \mathbb{N}$ we will apply Widman's improvement of (CLI) to prove that ∇u is a Hölder function. The problem is much harder for $n > 2$ and required new deep ideas. The celebrated DeGiorgi-Nash-Moser theory solves the problem in the scalar case $m = 1$, while for $m > 1$ new difficulties arise.