1 Figure of a graph

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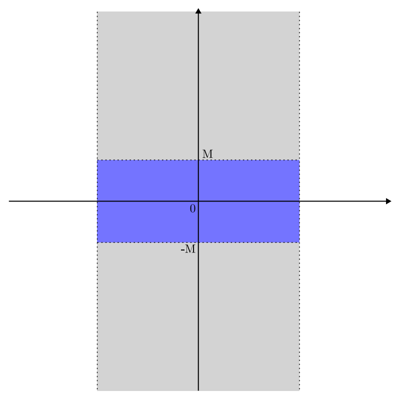

Figure 1: Test

huhfiusdbf iuhfui sdhfuihsduf sdiuhsd fiusdhf dsfiusdhf suisdfh dshiusdhf sdiuhsdf uidiu fdsf sdfsdknf fds oihfiwuehf udshfuidshf uidhf usdhf dshfisdufh hfds fiusdhf uihfu hsuifh iusdhf uisdhf sdhuifhsdiuhfusdhf uhiufhuisdhf uihsduifh suihfusdhfuh iushdfuihsd uifhsduifsd fhsdiuf hsduifh uisdhuihsuidhfiu shfuihsdiu fhsdiufh sdifhsdiu fuisdhf hiu sduif sduifh dsfuidshf sdiufh iusdfhiusd fisudfh dsufihsdiuf sduifhsdui fhdsuifhsdui fsduifh sdiufh sdiufhsdiufh

2 Result 01

Theorem 2.1 (Model 01: connected). *Let* $\Omega \coloneqq \{(x', x_n)\ s.t.\ |x'| \leq 1, |x_n| \leq M\}$ *and* $E_0 \coloneqq \{(x', x_n)\ s.t.\ |x'| \leq R, |x_n| \geq M\}$ *for some* $R, M > 0$*. Then there exists an* M_0 *s.t. for all* $M \leq M_0$ *the minimizer of the fractional perimeter is connected and given by* $E_M = \Omega \cup E_0$*.*

Proof. Proof by contradiction. Assume E_M is not $E_0 \cup \Omega$, then we can slide a ball of radius r down and at some point it will touch E_M. We consider the

ball $B_r(te_n)$. Since E_M not cylindrical, there exists $r_0 \in (0,1)$ and $t_0 > 0$ s.t. $\partial B_{r_0}(t_0 e_n) \cap \partial E_M \neq \emptyset$ and $B_{r_0}(te_n) \subset E_M$ for all $t > t_0$.
Since E_M is a minimizer it is also a variational solution and the inequality holds

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{n+s}} \, \mathrm{d}y \geq 0$$

whereas $q \in \partial B_{r_0}(t_0 e_n) \cap \partial E_M$.
We show that the left hand side is negative. Split the domain into four parts, as seen in the Figure:

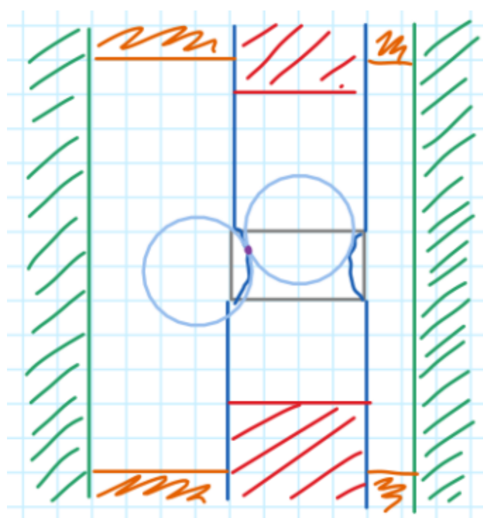

Figure 2:

We define

$$\begin{aligned} A &\coloneqq \{(x', x_n) \text{ s.t. } |x' - q'| \geq R+1\} \text{ Green Area} \\ B &\coloneqq \{(x', x_n) \text{ s.t. } |x'| < R, |x_n - q_n| > 2M\} \\ C &\coloneqq \{(x', x_n) \text{ s.t. } |x'| \geq R, |x' - q'| \leq R+1, |x_n - q_n| > \Lambda M\} \end{aligned}$$

Everything else $\subset S \coloneqq \{(x', x_n) \text{ s.t. } |x' - q'| \leq R+1, |x_n - q_n| \leq \Lambda M\}$

Integration over the first part:

$$\int_A \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \, \mathrm{d}y \overset{A \subseteq E^c}{=} \int_{|y'| > R+1} \frac{1}{|y|^{n+s}} \, \mathrm{d}y \leq c(n) \int_{R+1}^{\infty} r^{-s-2} \, \mathrm{d}y \leq c(n,s) R^{-(1+s)}$$

Integration over the second part:

$$\int_B \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \, \mathrm{d}y \overset{B \subseteq E}{=} -\int_B \frac{1}{|y-q|^{n+s}} \, \mathrm{d}y \leq -c(n,s) M^{-s} \qquad \text{Idea: Consider ball with factor } 2^{-n}$$

Integration over the third part:

$$\int_C \frac{\chi_{E^c}-\chi_E}{|y-q|^{n+s}}\,\mathrm{d}y \overset{C\subseteq E^C}{=} \int_C \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y \le c(n)\int_{R-1}^{R+1}\int_{\Lambda M}^{\infty} \frac{r^{n-2}}{(r^2+y_n^2)^{\frac{n+s}{2}}}\,\mathrm{d}y_n\,\mathrm{d}r$$

$$\overset{r^2\le r^2+y_n^2}{\le} c(n)\int_{R-1}^{R+1}\int_{2\Lambda M}^{\infty} \frac{1}{(r^2+y_n^2)^{\frac{s+2}{2}}}\,\mathrm{d}y_n\,\mathrm{d}r \overset{\text{convexity}}{\le} \int_{R-1}^{R+1}\int_{\Lambda M}^{\infty} \frac{1}{(r+y_n)^{s+2}}\,\mathrm{d}y_n\,\mathrm{d}r$$

$$\le c(n,s)\int_{R-1}^{R+1} \frac{1}{(r+\Lambda M)^{s+1}} \le c(n,s)(R-1+\Lambda M)^{-s} \le c(n,s)(\Lambda M)^{-s}$$

Integration over the fourth part:
Justifiction that we can estimate with S: Only negative part of the integration is fully in the set we want to estimate and the rest in S is positive.
We split S into four parts:

i) $S\cap B_{\Lambda M}(q)\cap B_{r_0}(z)$

ii) $S\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})$

iii) $S\cap (B_{\Lambda M}(q)\setminus (B_{r_0}(z)\cup B_{r_0}(\overline{z})))$

iv) $S\setminus B_{\Lambda M}(q)$

where $\overline{z} \coloneqq z+2(q-z)$ and $\Lambda>4$ chosen big enough and M chosen small enough s.t. $\Lambda M\le 1$.
We estimate the first and second part:

$$\int_{S\cap B_{\Lambda M}(q)\cap B_{r_0}(z)\cup S\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{\chi_{E^c}-\chi_E}{|y-q|^{n+s}}\,\mathrm{d}y$$

$$\le \int_{S\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y - \int_{S\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y \le 0$$

These two intgerals cancel because of symmetry.
We estimate the third part:

$$\int_{S\cap(B_{\Lambda M}(q)\setminus(B_{r_0}(z)\cup B_{r_0}(z)))} \frac{\chi_{E^c}-\chi_E}{|y-q|^{n+s}}\,\mathrm{d}y \le \int_{P_{1,\Lambda M}} \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y \le C\Lambda^{1-s}M^{1-s}$$

where we used lemma 3.1 in 2016 dipierro-savin-valdinoci with $R=r_0=1 and \lambda=\Lambda M$ (we can choose $r_0=1$, since if we can show the bound for $r_0=1$ then it holds for all smaller balls as well).
We estimate the fourth part:

$$\int_{S\setminus B_{\Lambda M}(q)} \frac{\chi_{E^c}-\chi_E}{|y-q|^{n+s}}\,\mathrm{d}y \le \int_{B_{R+2}\setminus B_{\Lambda M}} \frac{1}{|y|^{n+s}}\,\mathrm{d}y = c(n,s)((\Lambda M)^{-s}-(R+2)^{-s})$$

since $S \subset B_{R+2}$ for $R \geq 1$ since $((\Lambda M)^2+(R+1)^2)^{\frac{1}{2}} \leq (R^2+4R+4)^{\frac{1}{2}} = R+2$.

Thus in total we get:

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y &\leq -c_1 M^{-s} + c_0(R^{-(1+s)} + (\Lambda M)^{-s} + (\Lambda M)^{-s} - (R+2)^{-s} + \Lambda^{1-s}M^{1-s}) \\ &\leq -c_1 M^{-s}(1 - + \frac{c_0}{c_1}(R^{-(1+s)}M^s + 2\Lambda^{-s} - (R+2)^{-s}M^s + \Lambda^{1-s}M\end{aligned}$$

Choose Λ large and M small enoguh

$$\leq -c_2 M^{-s} < 0$$

□

3 Result 02

Theorem 3.1 (Model 01: disconnected)**.** *Let the setting be as in theorem theorem 2.1, then there exists an M_0 s.t. for all $M \geq M_0$ the minimizer of the fractional perimeter is disconnected.*

Proof. Proof analogous to theorem 1.2 in 2016 dipierro-onoue-valdinoci.

□

4 Result 03

Theorem 4.1 (Model 02: connected)**.** *Let $\Omega := \{(x', x_n)$ s.t. $|x'| \leq 1, |x_n| \leq M\}$ and $E_0 := \{(x', x_n)$ s.t. $M \leq |x_n| \leq R + M\}$. For every $R > 0$ there exists a M_0 s.t. for all $M \leq M_0$ the minimizer of the fractional perimeter is connected. (We need $R > M$)*

Note. *To prove this we need to show that minimizers are always connected to Ω^c and i.e. $d(E_0, \Omega) = d(E \setminus E_0, \Omega)$.*

Proof. We show that for every $R > 0$ at least the tube $\{|x_n| < r_0\}$ is in the minimizer for some $r_0 > 0$.
We do that analogously to theorem theorem 2.1 by contradiction. We assume that E_M is disconnected, thus we can slide a ball of radius r down and for all $r_0 \in (0, 1)$ there exists a $t_0 > 0$ s.t. $\partial B_{r_0}(t_0 e_n) \cap \partial E_M \neq \emptyset$. If we can show that there exists a r_0 s.t. this conntradicts then the tube is in the minimizer. It is enough to show that for one r_0 since if we can contradict

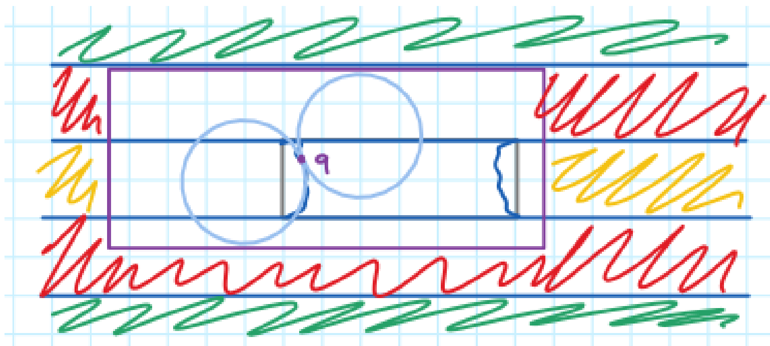

Figure 3:

this for one r_0 then for all smaller there is no touching as well.
For that we split into four parts as seen in the figure:

We define

$$A := \{(x', x_n) \text{ s.t. } |x_n| \geq M + R\}$$
$$B := \{(x', x_n) \text{ s.t. } |x_n| \leq M, |x' - q'| > 2\}$$
$$C := E_0 \setminus S$$
$$S := \{(x', x_n) \text{ s.t. } |x_n - q_n| \leq M + R, |x' - q'| \leq 2\}$$

Integration over the first part:

$$\int_A \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \, \mathrm{d}y \overset{A \subset E^c}{\leq} \int_{|y_n| \geq R} \frac{1}{|y|^{n+s}} \, \mathrm{d}y \leq c(n) \int_0^\infty \int_R^\infty \frac{r^{n-2}}{(r^2 + y_n^2)^{\frac{n+s}{2}}} \, \mathrm{d}y_n \, \mathrm{d}r$$
$$\leq c(n) \int_0^\infty \int_R^\infty \frac{1}{(r^2 + y_n^2)^{\frac{s+2}{2}}} \, \mathrm{d}y_n \, \mathrm{d}r \leq c(n) \int_0^\infty \int_R^\infty \frac{1}{(r + y_n)^{s+2}} \, \mathrm{d}y_n \, \mathrm{d}r$$
$$= c(n,s) \int_0^\infty \frac{1}{(r+R)^{s+1}} \, \mathrm{d}r = c(n,s) R^{-s}$$

Integration over the second part:

$$\int_B \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \, \mathrm{d}y \overset{B \subset E^c}{\leq} c(n) \int_0^M \int_2^\infty \frac{r^{n-2}}{(r^2 + y_n^2)^{\frac{n+s}{2}}} \, \mathrm{d}r \, \mathrm{d}y_n$$
$$\leq c(n) \int_0^M \int_2^\infty \frac{1}{(r + y_n)^{s+2}} \, \mathrm{d}r \, \mathrm{d}y_n = c(n,s) \int_0^M \frac{1}{(2 + y_n)^{s+1}} \, \mathrm{d}r$$
$$= c(n,s)(2^{-s} - (2+M)^{-s}) \leq c(n,s) 2^{-s}$$

Integration over the third part (here we need $R > M$):

$$\int_C \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y = -\int_C \frac{1}{|y-q|^{n+s}} \,\mathrm{d}y \leq -c(n) \int_{B_M(\dots)} \frac{1}{|y|^{n+s}} \,\mathrm{d}y \leq -c(n,s) M^{-s}$$

Idea: Move part of the stripe outside, restrict to ball with radius M and multiply with $\frac{1}{2}$ since not whole ball may be in the set.

Integration over the fourth part:
We split S into four parts:

i) $S \cap B_{\Lambda M}(q) \cap B_{r_0}(z)$

ii) $S \cap B_{\Lambda M}(q) \cap B_{r_0}(\overline{z})$

iii) $S \cap (B_{\Lambda M}(q) \setminus (B_{r_0}(z) \cup B_{r_0}(z)))$

iv) $S \setminus B_{\Lambda M}(q)$

where $\overline{z} := z + 2(q - z)$ and $\Lambda > 4$ chosen big enough and M chosen small enough s.t. $\Lambda M \leq 1$.
Again the first and second part are in sum smaller than zero.
We estimate the third part:

$$\begin{aligned} &\int_{S \cap (B_{\Lambda M}(q) \setminus (B_{r_0}(z) \cup B_{r_0}(\overline{z})))} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y \\ &\leq \int_{P_{r_0,1}} \frac{1}{|y|^{n+s}} \,\mathrm{d}y + \int_{B_{\Lambda M} \setminus B_{r_0}} \frac{1}{|y|^{n+s}} \,\mathrm{d}y \leq c(n,s)(r_0^{-s} - (\Lambda M)^{-s}) \end{aligned}$$

We estimate the fourth part:

$$\begin{aligned} &\int_{S \setminus B_{\Lambda M}(q)} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y \\ &\leq c(n) \int_{\Lambda M}^{R+3} \frac{1}{r^{s+1}} \,\mathrm{d}r \leq c(n,s)((\Lambda M)^{-s} - (R+3)^{-s}) \end{aligned}$$

Thus we estimate the domain S with

$$\int_S \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y \leq c(n,s)(r_0^{-s} - (R+3)^{-s}) \leq c(n,s) r_0^{-s}$$

Thus in total we get:

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y &\leq -c_0 M^{-s} + c_1(R^{-s} + 2^{-s} + r_0^{-s}) \\ &\leq -c_0 M^{-s}(1 - \frac{c_1}{c_0}(R^{-s} M^s + 2^{-s} M^s + r_0^{-s} M^s)) \end{aligned}$$

Now choose $r_0 = \frac{R}{2}$ and at most 2

$$\leq -c_0 M^{-s}(1 - \frac{c_1}{c_0}(R^{-s}M^s + 2^{-s}M^s + \left(\frac{2M}{R}\right)^s))$$

Choose Λ large and M small enoguh

$$\leq -c_2 M^{-s} < 0$$

□

5 Result 04

Theorem 5.1 (Model 02: disconnected)**.** *Let the setting be as in theorem theorem 4.1, then there exists an M_0 s.t. for all $M \geq M_0$ the minimizer of the fractional perimeter is disconnected.*

Proof. Proof analogous to theorem 1.2 in 2016 dipierro-onoue-valdinoci. (I think)

□

6 Result 05