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I hereby declare that this thesis is entirely the result of my own work except where otherwise indicated. I have only used the resources given in the list of references.

Munich, Date of submission Mohamed Noah Abdel Wahab

Abstract

Abstract

Nonlocal minimal surfaces confined within a cylinder exhibit unique behaviors dependent on external data. This thesis delves into these surfaces, which incorporate long-range spatial interactions compared to classical minimal surfaces. We consider two variations of the model discussed in [4], a minimal surfaces confined within a cylinder.

We investigate two scenarios: varying the height and width of data outside a separating slab. The results show that when the slab is wide, the minimal surface becomes disconnected from the data, while a narrow slab allows connection. This allows us to predict the behavior of similar models with symmetrically placed data. Additionally, the research reveals that for sufficiently narrow slabs, the surface "sticks" to the cylinder.

Finally, we present an example where the minimizer is completely disconnected from the external data, a phenomenon unique to nonlocal minimal surfaces. This work provides valuable insights into the behavior of these emerging mathematical objects and their interaction with external data.

Zusammenfassung

In Zylindern eingeschlossene nichtlokale Minimalflächen zeigen ein einzigartiges Verhalten, das von externen Daten abhängt. Diese Arbeit befasst sich mit diesen Flächen, die im Vergleich zu klassischen Minimalflächen weitreichende räumliche Wechselwirkungen berücksichtigen. Wir betrachten zwei Varianten des in [4] diskutierten Modells, einer in einem Zylinder eingeschlossenen Minimalfläche.

Dabei untersuchen wir zwei Szenarien: die Variation der Höhe und der Breite von Daten außerhalb einer trennenden Platte. Die Ergebnisse zeigen, dass die Minimalfläche bei breiter Platte von den Daten getrennt wird, während eine schmale Platte eine Verbindung ermöglicht. Dies erlaubt uns, das Verhalten ähnlicher Modelle mit symmetrisch angeordneten Daten vorherzusagen. Darüber hinaus zeigt die Forschung, dass die Fläche bei ausreichend schmalen Platten am Zylinder "haftet".

Schließlich präsentieren wir ein Beispiel, bei dem der Minimierer vollständig von den externen Daten getrennt ist, ein Phänomen, das für nichtlokale Minimalflächen einzigartig ist. Diese Arbeit liefert wertvolle Erkenntnisse über das Verhalten dieser neuen mathematischen Objekte und ihre Wechselwirkung mit externen Daten.

Contents

1 Introduction

Idea: Start with short historical background
18th century: Lagrange, Euler
20th Century: DeGiorgi Perimeter and localized entity
2009 Cafarelli, Roquejoffre, Savin: Nonlocal minimal surfaces
Perimeter and nonlocal perimeter as the (semi)norm of an indicator function
Define the usual problem considered
Better regularity than classical minimal surfaces

Chapter 01
Model 01

Chapter 02
Model 02 and further discussion on the similar problems

Chapter 03
Fully disconnected minimizer

In 2009 Cafarelli, Roquejoffre, and Savin [1] introduced a new concept of minimal surfaces. By incooperating long-range corelations into the classical perimeter, they defined the nonlocal perimeter as the (semi)norm of an indicator function. Minimizing over suitable set with given external data gives us *nonlocal minimal surfaces*. Instead…

Use the introduction in [8] as inspiration.

2 Model 01

For $n \geq 2$ consider the model as follows:

$$E_0 := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t.} |x'| \leq R,\ |x_n| \geq M\} \tag{2.1}$$

$$\Omega := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t.} |x'| \leq 1,\ |x_n| \leq M\} \tag{2.2}$$

for $R \geq 1$ and $M > 0$. The figure fig. 2.1 illustrates the setting.

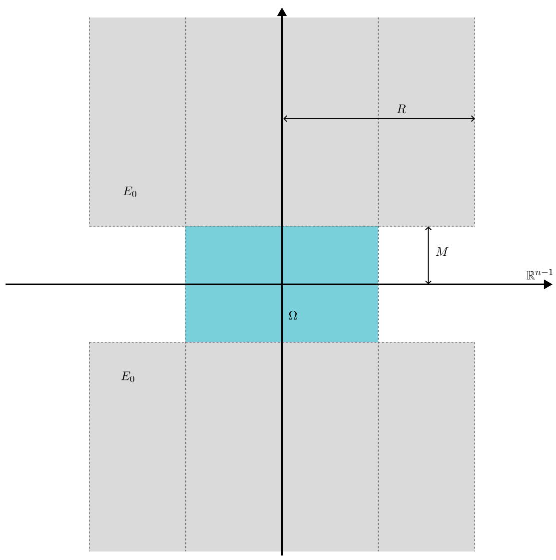

Figure 2.1

We state the following two results, which we will prove afterwards.

Theorem 2.1. *For Ω and E_0 as given above and for all $R \geq 1$, then there exists $M_0 \in (0, 1)$ depending only on the dimension and s, such that for any $M \in (0, M_0)$, the minimizer is $E_M = E_0 \cup \Omega$.*

Theorem 2.2. *For Ω and E_0 as given above and for all $R \geq 1$, then there exists $M_0 > 1$ depending only on the dimension and s, such that for any $M \geq M_0$, the minimizer E_M is disconnected.*

Connect to classical minimal surfaces by observing disconnectedness of the minimizer, but when connected, the minimizer may "stick" to the boundary. Whereas classical minimal surfaces cannot stick to the boundary. (source?)

For the first proof, we will follow a similar construction as in [4].
In [1] the authors have shown that nonlocal minimizer satisfy the Euler-Lagrange equation in the viscosity sense, i.e. if E is a minimizer, there exists some such that $q \in \partial E$ and $B_r(q + r\nu) \subset E$ for some $r > 0$ and unit vector $\nu \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y - q|^{n+s}} \, \mathrm{d}y \geq 0. \tag{2.3}$$

In the proof we will assume that there exist a minimizer which is not $E_0 \cup \Omega$. To bring this assumption to a contradiction, we want to show that the left hand side of eq. (2.3) is negative for M small enough. Thus, we have to construct some suitable ball such that we can apply the Euler-Lagrange equation. Constructing the ball by sliding it down from te_n. If the minimizer is not $E_0 \cup \Omega$, then at some point the ball will touch the minimizer for any $0 < r < 1$ and a point q, then exists. Then we will split the domain into four parts and estimate each part to get the contradiction.

Proof of definition 2.1. Proof by contradiction. Assume E_M is not $E_0 \cup \Omega$, then we can slide a ball of radius r down and at some point it will touch E_M. We consider the ball $B_r(te_n)$. Since E_M not cylindrical, there exists $r_0 \in (0,1)$ and $t_0 > 0$ s.t. $\partial B_{r_0}(t_0 e_n) \cap \partial E_M \neq \emptyset$ and $B_{r_0}(te_n) \subset E_M$ for all $t > t_0$. See figure fig. 2.2.

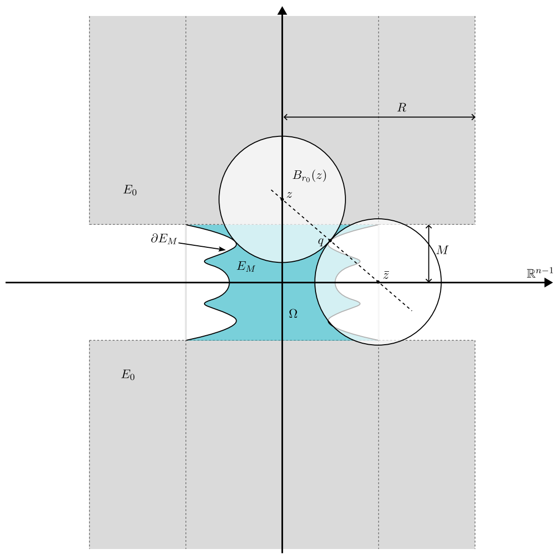

Figure 2.2

Since E_M is a minimizer it is also a variational solution and the inequality holds

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{n+s}} \, \mathrm{d}y \geq 0$$

whereas $q \in \partial B_{r_0}(t_0 e_n) \cap \partial E_M$.
We show that the left hand side is negative. Split the domain into four parts, as seen in the Figure fig. 2.3. We define

$$\begin{aligned} A &:= \{(x', x_n) \text{ s.t.} |x' - q'| \geq R + 1\} \text{ Green Area} \\ B &:= \{(x', x_n) \text{ s.t.} |x'| < R, |x_n - q_n| > 2M\} \\ C &:= \{(x', x_n) \text{ s.t.} |x'| \geq R, |x' - q'| \leq R + 1, |x_n - q_n| > \Lambda M\} \\ \text{Everything else} \subset S &:= \{(x', x_n) \text{ s.t.} |x' - q'| \leq R + 1, |x_n - q_n| \leq \Lambda M\} \end{aligned}$$

Integration over the first part:

$$\int_A \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \, \mathrm{d}y \overset{A \subseteq E^c}{=} \int_{|y'| > R+1} \frac{1}{|y|^{n+s}} \, \mathrm{d}y \leq c(n) \int_{R+1}^{\infty} r^{-s-2} \, \mathrm{d}y \leq c(n,s) R^{-(1+s)}$$

Integration over the second part:

$$\int_B \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \, \mathrm{d}y \overset{B \subseteq E}{=} - \int_B \frac{1}{|y-q|^{n+s}} \, \mathrm{d}y \leq -c(n,s) M^{-s} \qquad \text{Idea: Consider ball with factor } 2^{-n}$$

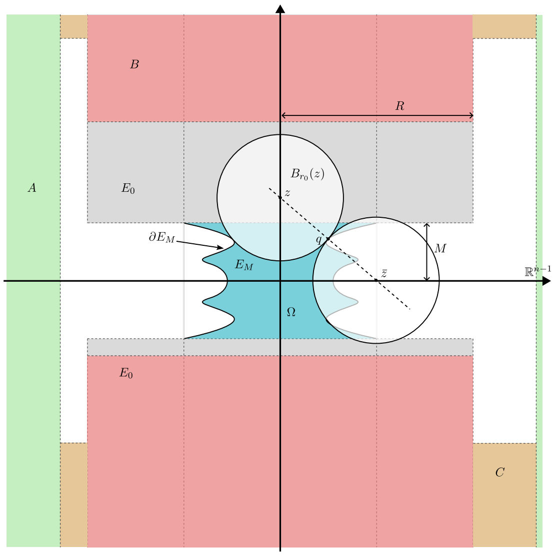

Figure 2.3

Integration over the third part:

$$\int_C \frac{\chi_{E^c}-\chi_E}{|y-q|^{n+s}}\,\mathrm{d}y \overset{C\subseteq E^C}{=} \int_C \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y \le c(n)\int_{R-1}^{R+1}\int_{\Lambda M}^{\infty} \frac{r^{n-2}}{(r^2+y_n^2)^{\frac{n+s}{2}}}\,\mathrm{d}y_n\,\mathrm{d}r$$

$$\overset{r^2\le r^2+y_n^2}{\le} c(n)\int_{R-1}^{R+1}\int_{2\Lambda M}^{\infty}\frac{1}{(r^2+y_n^2)^{\frac{s+2}{2}}}\,\mathrm{d}y_n\,\mathrm{d}r \overset{\text{convexity}}{\le} \int_{R-1}^{R+1}\int_{\Lambda M}^{\infty}\frac{1}{(r+y_n)^{s+2}}\,\mathrm{d}y_n\,\mathrm{d}r$$

$$\le c(n,s)\int_{R-1}^{R+1}\frac{1}{(r+\Lambda M)^{s+1}} \le c(n,s)(R-1+\Lambda M)^{-s} \le c(n,s)(\Lambda M)^{-s}$$

Integration over the fourth part:
Justification that we can estimate with S: Only negative part of the integration is fully in the set we want to estimate and the rest in S is positive.
We split S into four parts:

i) $S\cap B_{\Lambda M}(q)\cap B_{r_0}(z)$

ii) $S\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})$

iii) $S\cap (B_{\Lambda M}(q)\setminus (B_{r_0}(z)\cup B_{r_0}(\overline{z})))$

iv) $S\setminus B_{\Lambda M}(q)$

where $\overline{z} := z+2(q-z)$ and $\Lambda > 4$ chosen big enough and M chosen small enough s.t. $\Lambda M \le 1$.
We estimate the first and second part:

$$\int_{S\cap B_{\Lambda M}(q)\cap B_{r_0}(z)\cup S\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{\chi_{E^c}-\chi_E}{|y-q|^{n+s}}\,\mathrm{d}y$$

$$\le \int_{S\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y - \int_{S\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y \le 0$$

These two integrals cancel because of symmetry.
We estimate the third part:

$$\int_{S\cap(B_{\Lambda M}(q)\setminus(B_{r_0}(z)\cup B_{r_0}(z)))} \frac{\chi_{E^c}-\chi_E}{|y-q|^{n+s}}\,\mathrm{d}y \le \int_{P_{1,\Lambda M}} \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y \le C\Lambda^{1-s}M^{1-s}$$

where we used lemma 3.1 in [5] with $R = r_0 = 1 and \lambda = \Lambda M$ (we can choose $r_0 = 1$, since if we can show the bound for $r_0 = 1$ then it holds for all smaller balls as well).
We estimate the fourth part:

$$\int_{S \setminus B_{\Lambda M}(q)} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y \le \int_{B_{R+2} \setminus B_{\Lambda M}} \frac{1}{|y|^{n+s}} \,\mathrm{d}y = c(n,s)((\Lambda M)^{-s} - (R+2)^{-s})$$

since $S \subset B_{R+2}$ for $R \ge 1$ since $((\Lambda M)^2 + (R+1)^2)^{\frac{1}{2}} \le (R^2 + 4R + 4)^{\frac{1}{2}} = R + 2$.

Thus in total we get:

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y &\le -c_1 M^{-s} + c_0(R^{-(1+s)} + (\Lambda M)^{-s} + (\Lambda M)^{-s} - (R+2)^{-s} + \Lambda^{1-s} M^{1-s}) \\ &\le -c_1 M^{-s}(1 - + \frac{c_0}{c_1}(R^{-(1+s)} M^s + 2\Lambda^{-s} - (R+2)^{-s} M^s + \Lambda^{1-s} M\end{aligned}$$

Choose Λ large and M small enoguh

$$\le -c_2 M^{-s} < 0$$

□

Interesting to see, that the contribution of the cylinder of radius 1 is enough to get connectedness of the minimizer and even stickiness to the boundary. Also see, that the model seems (maybe prove that) to converge to the problem, considered in [4].

Proof of definition 2.2. In theorems 1.2 in [4] the authors have shown that that $\exists M_0 > 1$, such that..

$$E_M \subset F_M \quad E_M^c \subset F_M^c \tag{2.4}$$

□

3 Model 02

For $n \geq 2$ consider the model as follows:

$$E_0 \coloneqq \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } M|x_n| \geq R + M\} \tag{3.1}$$

$$\Omega \coloneqq \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x'| \leq 1,\ |x_n| \leq M\} \tag{3.2}$$

for $R > 0$ and $M > 0$. The figure.. illustrates the setting.

We state the following two results, which we will prove afterwards.

Theorem 3.1. *Let Ω and E_0 as given above and for all $R > ..$, then there exists $M_0 \in (0, 1)$ depending only on the dimension and s, such that for any $M \in (0, M_0)$, the minimizer is $E_M = E_0 \cup \Omega$. For $R \leq ..$, the cylinder $A \coloneqq \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x'| \leq .., |x_n| \leq M\}$ is in the minimizer, i.e. $E_M \supset E_0 \cup A$.*

Note. *Bound on R depends on the construction of the proof.*

Theorem 3.2. *For Ω and E_0 as given above and for all $R > 0$, then there exists $M_0 > ..$ depending only on the dimension and s, such that for any $M \geq M_0$, the minimizer E_M is disconnected.*

Again, similar proofs as in chapter 2.
Add some more discussion.

Proof of definition 3.1. We show that for every $R > 0$ at least the tube $\{|x_n| < r_0\}$ is in the minimizer for some $r_0 > 0$.
We do that analogously to theorem **??** by contradiction. We assume that E_M is disconnected, thus we can slide a ball of radius r down and for all $r_0 \in (0, 1)$ there exists a $t_0 > 0$ s.t. $\partial B_{r_0}(t_0 e_n) \cap \partial E_M \neq \emptyset$. If we can show that there exists a r_0 s.t. this conntradicts then the tube is in the minimizer. It is enough to show that for one r_0 since if we can contradict this for one r_0 then for all smaller there is no touching as well.
For that we split into four parts as seen in the figure: We define

$$\begin{aligned}
A &\coloneqq \{(x', x_n) \text{ s.t. } |x_n| \geq M + R\} \\
B &\coloneqq \{(x', x_n) \text{ s.t. } |x_n| \leq M, |x' - q'| > 2\} \\
C &\coloneqq E_0 \setminus S \\
S &\coloneqq \{(x', x_n) \text{ s.t. } |x_n - q_n| \leq M + R, |x' - q'| \leq 2\}
\end{aligned}$$

Integration over the first part:

$$\begin{aligned}
&\int_A \frac{\chi_{E^c} - \chi_E}{|y - q|^{n+s}} \,\mathrm{d}y \overset{A \subset E^c}{\leq} \int_{|y_n| \geq R} \frac{1}{|y|^{n+s}} \,\mathrm{d}y \leq c(n) \int_0^\infty \int_R^\infty \frac{r^{n-2}}{(r^2 + y_n^2)^{\frac{n+s}{2}}} \,\mathrm{d}y_n \,\mathrm{d}r \\
&\leq c(n) \int_0^\infty \int_R^\infty \frac{1}{(r^2 + y_n^2)^{\frac{s+2}{2}}} \,\mathrm{d}y_n \,\mathrm{d}r \leq c(n) \int_0^\infty \int_R^\infty \frac{1}{(r + y_n)^{s+2}} \,\mathrm{d}y_n \,\mathrm{d}r \\
&= c(n, s) \int_0^\infty \frac{1}{(r + R)^{s+1}} \,\mathrm{d}r = c(n, s) R^{-s}
\end{aligned}$$

Integration over the second part:

$$\begin{aligned}
&\int_B \frac{\chi_{E^c} - \chi_E}{|y - q|^{n+s}} \,\mathrm{d}y \overset{B \subset E^c}{\leq} c(n) \int_0^M \int_2^\infty \frac{r^{n-2}}{(r^2 + y_n^2)^{\frac{n+s}{2}}} \,\mathrm{d}r \,\mathrm{d}y_n \\
&\leq c(n) \int_0^M \int_2^\infty \frac{1}{(r + y_n)^{s+2}} \,\mathrm{d}r \,\mathrm{d}y_n = c(n, s) \int_0^M \frac{1}{(2 + y_n)^{s+1}} \,\mathrm{d}r \\
&= c(n, s)(2^{-s} - (2 + M)^{-s}) \leq c(n, s) 2^{-s}
\end{aligned}$$

Integration over the third part (here we need $R > M$):

$$\int_C \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}}\,\mathrm{d}y = -\int_C \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y \le -c(n)\int_{B_M(\dots)} \frac{1}{|y|^{n+s}}\,\mathrm{d}y \le -c(n,s)M^{-s}$$

Idea: Move part of the stripe outside, restrict to ball with radius M and multiply with $\frac{1}{2}$ since not whole ball may be in the set.

Integration over the fourth part:
We split S into four parts:

i) $S \cap B_{\Lambda M}(q) \cap B_{r_0}(z)$

ii) $S \cap B_{\Lambda M}(q) \cap B_{r_0}(\overline{z})$

iii) $S \cap (B_{\Lambda M}(q) \setminus (B_{r_0}(z) \cup B_{r_0}(z)))$

iv) $S \setminus B_{\Lambda M}(q)$

where $\overline{z} := z + 2(q-z)$ and $\Lambda > 4$ chosen big enough and M chosen small enough s.t. $\Lambda M \le 1$.
Again the first and second part are in sum smaller than zero.
We estimate the third part:

$$\begin{aligned}&\int_{S\cap(B_{\Lambda M}(q)\setminus(B_{r_0}(z)\cup B_{r_0}(\overline{z})))} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}}\,\mathrm{d}y \\ &\le \int_{P_{r_0,1}} \frac{1}{|y|^{n+s}}\,\mathrm{d}y + \int_{B_{\Lambda M}\setminus B_{r_0}} \frac{1}{|y|^{n+s}}\,\mathrm{d}y \le c(n,s)(r_0^{-s} - (\Lambda M)^{-s})\end{aligned}$$

We estimate the fourth part:

$$\begin{aligned}&\int_{S\setminus B_{\Lambda M}(q)} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}}\,\mathrm{d}y \\ &\le c(n)\int_{\Lambda M}^{R+3} \frac{1}{r^{s+1}}\,\mathrm{d}r \le c(n,s)((\Lambda M)^{-s} - (R+3)^{-s})\end{aligned}$$

Thus we estimate the domain S with

$$\int_S \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}}\,\mathrm{d}y \le c(n,s)(r_0^{-s} - (R+3)^{-s}) \le c(n,s)r_0^{-s}$$

Thus in total we get:

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}}\,\mathrm{d}y &\le -c_0M^{-s} + c_1(R^{-s} + 2^{-s} + r_0^{-s}) \\ &\le -c_0M^{-s}(1 - \frac{c_1}{c_0}(R^{-s}M^s + 2^{-s}M^s + r_0^{-s}M^s))\end{aligned}$$

Now choose $r_0 = \frac{R}{2}$ and at most 2

$$\le -c_0M^{-s}(1 - \frac{c_1}{c_0}(R^{-s}M^s + 2^{-s}M^s + \left(\frac{2M}{R}\right)^s))$$

Choose Λ large and M small enoguh

$$\le -c_2M^{-s} < 0$$

□

Disscussion about connectedness in case of small R and refer to next chapter. Behavior unique to nonlocal minimal surfaces.
Talk about the contribution of the complement.

Proof of definition 3.2. In theorems 1.2 in [4] the authors have shown that that $\exists M_0 > 1$, such that ..

$$E_M \subset F_M \quad E_M^c \subset F_M^c \tag{3.3}$$

□

Discussion about extending the model to arbitrary models with symmetric external data. Enough to consider discs of radius.. and heigth.. to have connectedness and even stickiness at some point.
New idea: If there is a minimizer E_M, can it ever be non sticky to the boundary?
Maybe able to give own interpretation of nonlocal minimal surfaces. Idea about Volume or Gravity?

4 Disconnected Minimizer

Example of a minimizer that has a non-empty set in Ω, while $d(E_0, \Omega) =: d > 0$.
Compare to classical case, where this cannot happen. Refer to .. and .. where discussion about the behavior of the perimeter for $s \to 1^-$ and $s \to 0^+$ was done.
Connect to the discussion in chapter 3..
Add discussion why $n = 1$ doesn't make sense or has a special standing.

Idea: If $d(E_0, \Omega) = 0$, does there exist a connected component $F \subset E$ s.t. $d(E_0, F) > 0$?

Conclusion

dicussion of the results, comparison to classical case, open problems, future work,..

1. Change of Topology in the models (barrier construction)
2. Cubic construction for arbitrary external data
3. Existence of s_0 for all external data and prescribed sets
4. Minimizer touching the boundary of the prescribed set (Calculations with of 3. with arbitrary parameter shows, no)
5. Can we give an estimate of the amount of connected components?

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