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Calculus of Variations

Graph properties for nonlocal minimal surfaces

Serena Dipierro[1,2,3] · **Ovidiu Savin**[4] ·
Enrico Valdinoci[1,2,3,5]

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Abstract In this paper we show that a nonlocal minimal surface which is a graph outside a cylinder is in fact a graph in the whole of the space. As a consequence, in dimension 3, we show that the graph is smooth. The proofs rely on convolution techniques and appropriate integral estimates which show the pointwise validity of an Euler–Lagrange equation related to the nonlocal mean curvature.

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1 Introduction

This paper deals with the geometric properties of the minimizers of a nonlocal perimeter functional.

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✉ Enrico Valdinoci
enrico@math.utexas.edu

Serena Dipierro
serena.dipierro@ed.ac.uk

Ovidiu Savin
savin@math.columbia.edu

1 School of Mathematics and Statistics, University of Melbourne, Richard Berry Building, Parkville, VIC 3010, Australia

2 School of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley, Perth, WA 6009, Australia

3 Weierstraß Institut für Angewandte Analysis und Stochastik, Hausvogteiplatz 5/7, 10117 Berlin, Germany

4 Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027, USA

5 Dipartimento di Matematica, Università degli studi di Milano, Via Saldini 50, 20133 Milan, Italy

More precisely, given $s \in (0, 1/2)$, and an open set $\Omega \subseteq \mathbb{R}^n$, the s-perimeter of a set $E \subseteq \mathbb{R}^n$ in Ω was defined in [7] as

$$\mathrm{Per}_s(E, \Omega) := L(E \cap \Omega, E^c) + L(\Omega \setminus E, E \setminus \Omega),$$

where $E^c := \mathbb{R}^n \setminus E$ and, for any disjoint sets F and G,

$$L(F, G) := \iint_{F \times G} \frac{dx\, dy}{|x - y|^{n+2s}}.$$

This nonlocal perimeter captures the global contributions between the set E and its complement and it is related to some models in geometry and physics, such as the motion by nonlocal mean curvature (see [8]) and the phase transitions in presence of long-range interactions (see [20]).

As customary in the calculus of variation literature, one says that E is s-minimal in a bounded open set Ω if $\mathrm{Per}_s(E, \Omega) < +\infty$ and $\mathrm{Per}_s(E, \Omega) \leqslant \mathrm{Per}_s(F, \Omega)$ among all the sets F which coincide with E outside Ω (with a slight abuse of language, when Ω is unbounded, we say that E is s-minimal in Ω if it is s-minimal in any bounded open subsets of Ω, see also [17] for further details).

Clearly, the above definition does not distinguish, in principle, between sets whose symmetric difference has vanishing Lebesgue measure. As in the case of minimal surfaces, we will implicitly identify sets up to sets of zero measure. More precisely, whenever necessary, we will take a representation of the set which coincides with the "measure theoretic interior" of the original set, i.e. those points for which there exists a ball of positive radius around them which is included in the set, up to sets of measure zero (we refer to Appendix A at the end of this paper for a detailed discussion).

Several analytic and geometric properties of s-minimal sets have been recently investigated, in terms, for instance, of asymptotics [1,3,9,13,18], regularity [10,15,21] and classification [4,11]. Some examples of s-minimal sets (or, more generally, of sets which possess vanishing nonlocal mean curvatures) have been given in [12,14].

The main result of this paper establishes that an s-minimal set is a subgraph, if so are its exterior data:

Theorem 1.1 *Let Ω_o be an open and bounded subset of $\mathbb{R}^{n-1}$ with boundary of class $C^{1,1}$, and let $\Omega := \Omega_o \times \mathbb{R}$. Let E be an s-minimal set in Ω. Assume that*

$$E \setminus \Omega = \{x_n < u(x'),\ x' \in \mathbb{R}^{n-1} \setminus \Omega_o\}, \tag{1.1}$$

for some continuous function $u : \mathbb{R}^{n-1} \to \mathbb{R}$. Then

$$E \cap \Omega = \{x_n < v(x'),\ x' \in \Omega_o\}, \tag{1.2}$$

for some $v : \mathbb{R}^{n-1} \to \mathbb{R}$, where v is a uniformly continuous function in Ω_o, with $v = u$ outside Ω_o.

We mention that, in general, s-minimal surfaces are not continuous up to the boundary of the domain (even if the datum outside is smooth), and indeed boundary stickiness phenomena occur (see [14] for concrete examples). The possible discontinuity at the boundary makes the proof of Theorem 1.1 quite delicate, since the graph property "almost fails" in a cylinder (see Theorem 1.2 in [14]), and, in general, the graph property cannot be deduced only from the outside data but it may also depend on the regularity of the domain.

More precisely, we stress that, given the examples in [14], even in simple cases, boundary stickiness of nonlocal minimal surfaces has to be expected, therefore the Dirichlet problem for nonlocal minimal surfaces cannot be solved in the class of functions that are continuous up to the boundary (even for smooth Dirichlet data). Nevertheless, Theorem 1.1 says that one can solve the Dirichlet problem in the class of (not necessarily continuous) graphs.

Due to the boundary stickiness (i.e., due to the possibly discontinuous behavior of the solution of the Dirichlet problem), the proof of Theorem 1.1 cannot be just a simple generalization of the proof of similar results for classical minimal surfaces. As a matter of fact, in principle, once the surface reaches a vertical tangent, it might be possible that it bends in the "wrong" direction, which would make the graph property false (our result in Theorem 1.1 excludes exactly this possibility).

The key point for this is that, due to the study of the obstacle problem for nonlocal minimal surfaces treated in [6], we more or less understand how the minimal surfaces separate from the boundary of the cylinder if a vertical piece is present.

In addition, since we are not assuming any smoothness of the surface to start with, some care is needed to compute the fractional mean curvature in a pointwise sense.

The proof of Theorem 1.1 is based on a sliding method, but some (both technical and conceptual) modifications are needed to make the classical argument work, due to the contributions "coming from far". We stress that, since the s-minimal set is not assumed to be smooth, some supconvolutions techniques are needed to take care of interior contact points. Moreover, a fine analysis of the possible contact points which lie on the boundary (and at infinity) is needed to complete the arguments.

As a matter of fact, we think that it is an interesting open problem to determine whether or not Theorem 1.1 holds true without the assumption that $\partial\Omega_o$ is of class $C^{1,1}$ (for instance, whether or not a similar statement holds by assuming only that $\partial\Omega_o$ is Lipschitz). We remark that, under weak regularity assumptions, one cannot make use of the results in [6].

The results in Theorem 1.1 may be strengthen in the case of dimension 3, by proving that two-dimensional minimal graphs are smooth. Indeed, we have:

Theorem 1.2 *Let Ω_o be an open and bounded subset of $\mathbb{R}^2$ with boundary of class $C^{1,1}$, and let $\Omega := \Omega_o \times \mathbb{R}$. Let E be an s-minimal set in Ω. Assume that*

$$E\setminus\Omega = \{x_n < u(x'),\ x' \in \mathbb{R}^{n-1}\setminus\Omega_o\},$$

for some continuous function $u : \mathbb{R}^{n-1} \to \mathbb{R}$. Then

$$E \cap \Omega = \{x_3 < v(x'),\ x' \in \Omega_o\}, \tag{1.3}$$

for some $v \in C^\infty(\Omega_o)$.

The proof of Theorem 1.2 relies on Theorem 1.1 and on a Bernstein-type result of [15].

The rest of the paper is organized as follows. In Sect. 2 we discuss the notion of supconvolutions and subconvolutions for a nonlocal minimal surface, presenting the geometric and analytic properties that we need for the proof of Theorem 1.1.

In Sect. 3 we collect a series of auxiliary results needed to compute suitable integral contributions and obtain an appropriate fractional mean curvature equation in a pointwise sense (i.e., not only in the sense of viscosity, as done in the previous literature).

The proof of Theorem 1.1 is given in Sect. 4 and the proof of Theorem 1.2 is given in Sect. 5.

Finally, in Appendix A, we observe that we can modify the s-minimal set E by a set of measure zero in order to identify E with the set of its interior points in the measure theoretic sense.

2 Supconvolution and subconvolution of a set

In this section, we introduce the notion of supconvolution and discuss its basic properties. This is the nonlocal modification of a technique developed in [5] for the local case.

Given $\delta > 0$, we define the supconvolution of the set $E \subseteq \mathbb{R}^n$ by

$$E_\delta^\sharp := \bigcup_{x \in E} \overline{B_\delta(x)}.$$

Lemma 2.1 *We have that*

$$E_\delta^\sharp = \bigcup_{\substack{v \in \mathbb{R}^n \\ |v| \leqslant \delta}} (E + v).$$

Proof Let $y \in \overline{B_\delta(x)}$, with $x \in E$. Let $v := y - x$. Then $|v| \leqslant \delta$ and $y = x + v \in E + v$, and one inclusion is proved.

Viceversa, let now $y \in E + v$, with $|v| \leqslant \delta$. We set in this case $x := y - v$. Hence $|y - x| = |v| \leqslant \delta$, thus $y \in \overline{B_\delta(x)}$. In addition, $x \in (E + v) - v = E$, so the other inclusion is proved. □

Corollary 2.2 *If $p \in \partial E_\delta^\sharp$, then there exist $v \in \mathbb{R}^n$, with $|v| = \delta$, and $x_o \in \partial E$ such that $p = x_o + v$ and $B_\delta(x_o) \subseteq E_\delta^\sharp$.*

Also, if $E_\delta^\sharp$ is touched from the outside at p by a ball B, then E is touched from the outside at x_o by $B - v$.

Proof Since $p \in \overline{E_\delta^\sharp}$, we have that there exists a sequence $p_j \in E_\delta^\sharp$ such that $p_j \to p$ as $j \to +\infty$. By Lemma 2.1, we have that $p_j \in E + v_j$, for some $v_j \in \mathbb{R}^n$ with $|v_j| \leqslant \delta$. That is, there exists $x_j \in E$ such that $p_j = x_j + v_j$. By compactness, up to a subsequence we may assume that $v_j \to v$ as $j \to +\infty$, for some $v \in \mathbb{R}^n$ with

$$|v| \leqslant \delta. \tag{2.1}$$

Therefore

$$x_j = p_j - v_j \to p - v =: x_o \tag{2.2}$$

as $j \to +\infty$. By construction,

$$x_o \in \overline{E} \tag{2.3}$$

and

$$p = x_o + v. \tag{2.4}$$

Now we show that

$$x_o \in \overline{E^c}. \tag{2.5}$$

For this, since $p \in \overline{\mathbb{R}^n \setminus E_\delta^\sharp}$, we have that there exists a sequence $q_j \in \mathbb{R}^n \setminus E_\delta^\sharp$ such that $q_j \to p$ as $j \to +\infty$.

Notice that

$$\overline{B_\delta(q_j)} \cap E = \varnothing. \tag{2.6}$$

Indeed, if not, we would have that there exists $z_j \in \overline{B_\delta(q_j)} \cap E$. So we can define $w_j := q_j - z_j$. We see that $|w_j| \leqslant \delta$ and therefore $q_j = z_j + w_j \in E + w_j \subseteq E^\sharp_\delta$, which is a contradiction.

Having established (2.6), we use it to deduce that $q_j - v_j \in E^c$. Thus passing to the limit

$$x_o = p - v = \lim_{j\to+\infty} q_j - v_j \in \overline{E^c}.$$

This proves (2.5).

From (2.3) and (2.5), we conclude that

$$x_o \in \partial E. \tag{2.7}$$

Now we show that

$$|v| = \delta. \tag{2.8}$$

To prove it, suppose not. Then, by (2.1), we have that $|v| < \delta$. That is, there exists $a \in (0, \delta)$ such that $|v| < \delta - a$. Then, by (2.2),

$$|x_j - p| \leqslant |x_j - x_o| + |x_o - p| = |x_j - x_o| + |v| < \delta - \frac{a}{2},$$

if j is large enough. Hence $B_{a/2}(p) \subseteq B_\delta(x_j) \subseteq E^\sharp_\delta$, that says that p lies in the interior of $E^\sharp_\delta$. This is in contradiction with the assumptions of Corollary 2.2, and so (2.8) is proved.

Now we claim that

$$B_\delta(x_o) \subseteq E^\sharp_\delta. \tag{2.9}$$

To prove this, let $z \in B_\delta(x_o)$. Then, $|z - x_o| \leqslant \delta - b$, for some $b \in (0, \delta)$. Accordingly, by (2.2), we have that $|z - x_j| \leqslant \delta - \frac{b}{2}$ if j is large enough. Hence $z \in B_\delta(x_j) \subseteq E^\sharp_\delta$. This proves (2.9).

Thanks to (2.4), (2.7), (2.8) and (2.9), we have completed the proof of the first claim in the statement of Corollary 2.2.

Now, to prove the second claim in the statement of Corollary 2.2, let us consider a ball B such that $B \subseteq \mathbb{R}^n \setminus E^\sharp_\delta$ and $p \in \partial B$. Then $x_o = p - v \in (\partial B) - v = \partial(B - v)$. Moreover,

$$B - v \subseteq (\mathbb{R}^n \setminus E^\sharp_\delta) - v = \mathbb{R}^n \setminus (E^\sharp_\delta - v).$$

Since $E \subseteq E^\sharp_\delta$, we have that

$$\mathbb{R}^n \setminus (E^\sharp_\delta - v) \subseteq \mathbb{R}^n \setminus (E - v).$$

Consequently, we obtain that $B - v \subseteq \mathbb{R}^n \setminus (E - v)$, which completes the proof of the second claim of Corollary 2.2. □

The supconvolution has an important property with respect to the fractional mean curvature, as stated in the next result:

Lemma 2.3 *Let $p \in \partial E^\sharp_\delta$, $v \in \mathbb{R}^n$ with $|v| \leqslant \delta$ and $x_o \in \partial E$ such that $p = x_o + v$. Then*

$$\int_{\mathbb{R}^n} \frac{\chi_{E^\sharp_\delta}(y) - \chi_{\mathbb{R}^n \setminus E^\sharp_\delta}(y)}{|p - y|^{n+2s}}\, dy \geqslant \int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\mathbb{R}^n \setminus E}(y)}{|x_o - y|^{n+2s}}\, dy.$$

Proof The claim follows simply by the fact that $E^\sharp_\delta \supseteq E + v$ and the translation invariance of the fractional mean curvature. □

Corollary 2.4 *Let E be an s-minimal set in Ω. Let $p \in \partial E^\sharp_\delta$. Assume that $\overline{B_\delta(p)} \subseteq \Omega$ and that $E^\sharp_\delta$ is touched from the outside at p by a ball. Then*

$$\int_{\mathbb{R}^n} \frac{\chi_{E^\sharp_\delta}(y) - \chi_{\mathbb{R}^n \setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\, dy \geqslant 0.$$

Proof By Corollary 2.2, we know that there exist $v \in \mathbb{R}^n$ with $|v| \leqslant \delta$ and $x_o \in \partial E$ such that $p = x_o + v$, and that E is touched by a ball from the outside at x_o.

We remark that $x_o \in \overline{B_\delta(p)} \subseteq \Omega$. So, we can use the Euler–Lagrange equation in the viscosity sense (see Theorem 5.1 in [7]) and obtain that

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\mathbb{R}^n \setminus E}(y)}{|x_o-y|^{n+2s}}\, dy \geqslant 0.$$

This and Lemma 2.3 give the desired result. □

The counterpart of the notion of supconvolution is given by the notion of subconvolution. That is, we define

$$E^\flat_\delta := \mathbb{R}^n \setminus \big((\mathbb{R}^n \setminus E)^\sharp_\delta \big).$$

In this setting, we have:

Proposition 2.5 *Let E be an s-minimal set in Ω. Let $p \in \partial E^\sharp_\delta$. Assume that $\overline{B_\delta(p)} \subseteq \Omega$.*

Assume also that $E^\sharp_\delta$ is touched from above at p by a translation of $E^\flat_\delta$, i.e. there exists $\omega \in \mathbb{R}^n$ such that $E^\sharp_\delta \subseteq E^\flat_\delta + \omega$ and $p \in (\partial E^\sharp_\delta) \cap \big(\partial(E^\flat_\delta + \omega)\big)$.

Then $E^\sharp_\delta = E^\flat_\delta + \omega$.

Proof Notice that

$$p \in \partial(E^\flat_\delta + \omega) = \partial E^\flat_\delta + \omega = \partial\big((\mathbb{R}^n \setminus E)^\sharp_\delta \big) + \omega.$$

Accordingly, by the first claim in Corollary 2.2 (applied to the set $\mathbb{R}^n \setminus E$ and to the point $p - \omega$), we see that there exist $\tilde{v} \in \mathbb{R}^n$, with $|\tilde{v}| = \delta$, and $\tilde{x}_o \in \partial(\mathbb{R}^n \setminus E) = \partial E$ such that $p - \omega = \tilde{x}_o + \tilde{v}$ and $B_\delta(\tilde{x}_o) \subseteq (\mathbb{R}^n \setminus E)^\sharp_\delta$. That is, the set $(\mathbb{R}^n \setminus E)^\sharp_\delta$ is touched from the inside at $p - \omega$ by a ball of radius δ. Taking the complementary set and translating by ω, we obtain that $E^\flat_\delta + \omega$ is touched from the outside at p by a ball of radius δ.

Then, since $E^\flat_\delta + \omega \supseteq E^\sharp_\delta$, we obtain that also $E^\sharp_\delta$ is touched from the outside at p by a ball of radius δ. Thus, making use of Corollary 2.4, we deduce that

$$\int_{\mathbb{R}^n} \frac{\chi_{E^\sharp_\delta}(y) - \chi_{\mathbb{R}^n \setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\, dy \geqslant 0. \tag{2.10}$$

Moreover, by Corollary 2.2, we know that $E^\sharp_\delta$ is touched from the inside at p by a ball of radius δ. By inclusion of sets, this gives that $E^\flat_\delta + \omega$ is touched from the inside at p by a ball of radius δ. Taking complementary sets, we obtain that $(\mathbb{R}^n \setminus E)^\sharp_\delta$ is touched from the outside at $p - \omega$ by a ball of radius δ. Therefore, we can use Corollary 2.4 (applied here to the set $(\mathbb{R}^n \setminus E)^\sharp_\delta$), and get that

$$0 \leqslant \int_{\mathbb{R}^n} \frac{\chi_{(\mathbb{R}^n \setminus E)^\sharp_\delta}(y) - \chi_{\mathbb{R}^n \setminus ((\mathbb{R}^n \setminus E)^\sharp_\delta)}(y)}{|p-\omega-y|^{n+2s}}\, dy$$

$$= \int_{\mathbb{R}^n} \frac{\chi_{\mathbb{R}^n\setminus E_\delta^\flat}(y) - \chi_{E_\delta^\flat}(y)}{|p-\omega-y|^{n+2s}}\,dy = -\int_{\mathbb{R}^n} \frac{\chi_{E_\delta^\flat+\omega}(y) - \chi_{\mathbb{R}^n\setminus(E_\delta^\flat+\omega)}(y)}{|p-y|^{n+2s}}\,dy.$$

By comparing this estimate with the one in (2.10), we obtain that

$$\int_{\mathbb{R}^n} \frac{\chi_{E_\delta^\sharp}(y) - \chi_{\mathbb{R}^n\setminus E_\delta^\sharp}(y)}{|p-y|^{n+2s}}\,dy \geqslant 0 \geqslant \int_{\mathbb{R}^n} \frac{\chi_{E_\delta^\flat+\omega}(y) - \chi_{\mathbb{R}^n\setminus(E_\delta^\flat+\omega)}(y)}{|p-y|^{n+2s}}\,dy.$$

Since $E_\delta^\sharp$ lies in $E_\delta^\flat+\omega$, the inequality above implies that the two sets must coincide. □

A useful variation of Proposition 2.5 consists in taking into account the possibility that the inclusion of the sets only occurs inside a suitable domain. For this, we define the cylinder

$$\mathscr{C}_R := \{x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x'| < R\}. \tag{2.11}$$

Also, given $\eta > 0$, we consider the set Ω_η of points which lie inside the domain at distance greater than η from the boundary, namely we set

$$\Omega_\eta := \{x \in \Omega \text{ s.t. } B_\eta(x) \subseteq \Omega\}. \tag{2.12}$$

Since Ω_o is open and bounded, we may suppose that $\Omega_o \subset B_{R_o}$, for some $R_o > 0$, hence $\Omega \subset \mathscr{C}_{R_o}$. So, for any $R > R_o$, we define

$$\mathscr{D}_{R,\eta} := \mathscr{C}_R\setminus(\Omega\setminus\Omega_{2\eta}) = \Omega_{2\eta} \cup (\mathscr{C}_R\setminus\Omega). \tag{2.13}$$

With this notation, we have:

Proposition 2.6 *Let $R > 4(R_o+1)$ and $\delta, \eta \in (0,1)$. Let E be an s-minimal set in Ω. Let $p \in \partial E_\delta^\sharp$. Assume that*

$$\overline{B_{4(\delta+\eta)}(p)} \subseteq \Omega_{4\eta}. \tag{2.14}$$

Assume also that $E_\delta^\sharp$ is touched in $\mathscr{D}_{R,\eta}$ from above at p by a vertical translation of $E_\delta^\flat$, i.e. there exists $\omega = (\omega', 0) \in \mathbb{R}^n$ such that $E_\delta^\sharp \cap \mathscr{D}_{R,\eta} \subseteq (E_\delta^\flat+\omega) \cap \mathscr{D}_{R,\eta}$ and $p \in (\partial E_\delta^\sharp) \cap (\partial(E_\delta^\flat+\omega))$.

Then, for η sufficiently small,

$$\int_{\mathscr{D}_{R,\eta}} \frac{\chi_{(E_\delta^\flat+\omega)\setminus E_\delta^\sharp}(y) - \chi_{E_\delta^\sharp\setminus(E_\delta^\flat+\omega)}(y)}{|p-y|^{n+2s}}\,dy \leqslant C\left(R^{-2s} + \frac{\eta}{\big(\operatorname{dist}(p,\partial\Omega)\big)^{n+2s}}\right),$$

for some $C > 0$, independent of δ, η and R.

Proof The proof is a measure theoretic version of the one in Proposition 2.5. We give the full details for the convenience of the reader.

Notice that

$$p \in \partial(E_\delta^\flat+\omega) = \partial E_\delta^\flat + \omega = \partial\big((\mathbb{R}^n\setminus E)_\delta^\sharp\big) + \omega.$$

Accordingly, by the first claim in Corollary 2.2 (applied to the set $\mathbb{R}^n\setminus E$ and to the point $p-\omega$), we see that there exist $\tilde v \in \mathbb{R}^n$, with $|\tilde v| = \delta$, and $\tilde x_o \in \partial(\mathbb{R}^n\setminus E) = \partial E$ such that $p-\omega = \tilde x_o + \tilde v$ and $B_\delta(\tilde x_o) \subseteq (\mathbb{R}^n\setminus E)_\delta^\sharp$. That is, the set $(\mathbb{R}^n\setminus E)_\delta^\sharp$ is touched from the inside at $p-\omega$ by a ball of radius δ. Taking the complementary set and translating by ω, we obtain that $E_\delta^\flat+\omega$ is touched from the outside at p by a ball of radius δ. Notice also that, in view of (2.14), such ball lies in $\Omega_{4\eta}$, which in turn lies in $\mathscr{D}_{R,\eta}$.

Then, since $(E^\flat_\delta+\omega)\cap\mathscr{D}_{R,\eta}\supseteq E^\sharp_\delta\cap\mathscr{D}_{R,\eta}$, we obtain that also $E^\sharp_\delta$ is touched from the outside at p by a ball of radius δ. Thus, making use of Corollary 2.4, we deduce that

$$\int_{\mathbb{R}^n}\frac{\chi_{E^\sharp_\delta}(y)-\chi_{\mathbb{R}^n\setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\,dy\geqslant 0. \tag{2.15}$$

Moreover, by Corollary 2.2, we know that $E^\sharp_\delta$ is touched from the inside at p by a ball of radius δ. By inclusion of sets, this gives that $E^\flat_\delta+\omega$ is touched from the inside at p by a ball of radius δ. Taking complementary sets, we obtain that $(\mathbb{R}^n\setminus E)^\sharp_\delta$ is touched from the outside at $p-\omega$ by a ball of radius δ. Therefore, we can use Corollary 2.4 (applied here to the set $(\mathbb{R}^n\setminus E)^\sharp_\delta$), and get that

$$\begin{aligned}0&\leqslant\int_{\mathbb{R}^n}\frac{\chi_{(\mathbb{R}^n\setminus E)^\sharp_\delta}(y)-\chi_{\mathbb{R}^n\setminus\left((\mathbb{R}^n\setminus E)^\sharp_\delta\right)}(y)}{|p-\omega-y|^{n+2s}}\,dy\\&=\int_{\mathbb{R}^n}\frac{\chi_{\mathbb{R}^n\setminus E^\flat_\delta}(y)-\chi_{E^\flat_\delta}(y)}{|p-\omega-y|^{n+2s}}\,dy=-\int_{\mathbb{R}^n}\frac{\chi_{E^\flat_\delta+\omega}(y)-\chi_{\mathbb{R}^n\setminus(E^\flat_\delta+\omega)}(y)}{|p-y|^{n+2s}}\,dy.\end{aligned}$$

By comparing this estimate with the one in (2.15), we obtain that

$$\int_{\mathbb{R}^n}\frac{\chi_{E^\sharp_\delta}(y)-\chi_{\mathbb{R}^n\setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\,dy\geqslant 0\geqslant\int_{\mathbb{R}^n}\frac{\chi_{E^\flat_\delta+\omega}(y)-\chi_{\mathbb{R}^n\setminus(E^\flat_\delta+\omega)}(y)}{|p-y|^{n+2s}}\,dy.$$

Since $E^\sharp_\delta\cap\mathscr{D}_{R,\eta}$ lies in $(E^\flat_\delta+\omega)\cap\mathscr{D}_{R,\eta}$, the inequality above implies that

$$\begin{aligned}\int_{\mathscr{D}_{R,\eta}}\frac{\chi_{(E^\flat_\delta+\omega)\setminus E^\sharp_\delta}(y)-\chi_{E^\sharp_\delta\setminus(E^\flat_\delta+\omega)}(y)}{|p-y|^{n+2s}}\,dy&\leqslant 2\int_{\mathbb{R}^n\setminus\mathscr{D}_{R,\eta}}\frac{dy}{|p-y|^{n+2s}}\\&=2\int_{\mathbb{R}^n\setminus\mathscr{C}_R}\frac{dy}{|p-y|^{n+2s}}+2\int_{\Omega\setminus\Omega_{2\eta}}\frac{dy}{|p-y|^{n+2s}}.\end{aligned}\tag{2.16}$$

Notice now that, if $y\in\mathbb{R}^n\setminus\mathscr{C}_R$, then $|p-y|\geqslant|p'-y'|\geqslant|y'|-|p'|\geqslant R-R_o\geqslant R/2$. Hence changing variable $\zeta:=p-y$, we have

$$\int_{\mathbb{R}^n\setminus\mathscr{C}_R}\frac{dy}{|p-y|^{n+2s}}\leqslant\int_{\mathbb{R}^n\setminus B_{R/2}}\frac{d\zeta}{|\zeta|^{n+2s}}\leqslant CR^{-2s}, \tag{2.17}$$

for some $C>0$. Moreover, using again (2.14), we see that $\operatorname{dist}(p,\partial\Omega)\geqslant 4\eta$. Hence, if $y\in\Omega\setminus\Omega_{2\eta}$, we have that

$$|p-y|\geqslant\operatorname{dist}(p,\partial\Omega)-2\eta\geqslant\frac{\operatorname{dist}(p,\partial\Omega)}{2}.$$

As a consequence,

$$\int_{(\Omega\setminus\Omega_{2\eta})\cap\{|p_n-y_n|\leqslant 1\}}\frac{dy}{|p-y|^{n+2s}}\leqslant\frac{C\eta}{(\operatorname{dist}(p,\partial\Omega))^{n+2s}}. \tag{2.18}$$

On the other hand,

$$\begin{aligned}\int_{(\Omega\setminus\Omega_{2\eta})\cap\{|p_n-y_n|>1\}}\frac{dy}{|p-y|^{n+2s}}&\leqslant\int_{(\Omega\setminus\Omega_{2\eta})\cap\{|p_n-y_n|>1\}}\frac{dy}{|p_n-y_n|^{n+2s}}\\&\leqslant C\eta\int_{\{|p_n-y_n|>1\}}\frac{dy_n}{|p_n-y_n|^{n+2s}}\leqslant C\eta,\end{aligned}$$

for some $C > 0$ (possibly different from step to step). The latter estimate and (2.18) imply that

$$\int_{\Omega\setminus\Omega_{2\eta}} \frac{dy}{|p-y|^{n+2s}} \leqslant \frac{C\eta}{\big(\mathrm{dist}\,(p,\partial\Omega)\big)^{n+2s}},$$

up to renaming C. By inserting this and (2.17) into (2.16), we obtain the desired result. □

3 Auxiliary integral computations and a pointwise version of the Euler–Lagrange equation

We collect here some technical results, which are used during the proofs of the main results. First, we recall an explicit estimate on the weighted measure of a set trapped between two tangent balls.

Lemma 3.1 *For any $R > 0$ and $\lambda \in (0, 1]$, let*

$$P_{R,\lambda} := \Big\{x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x'| \leqslant \lambda R \text{ and } |x_n| \leqslant R - \sqrt{R^2 - |x'|^2}\Big\}.$$

Then

$$\int_{P_{R,\lambda}} \frac{dx}{|x|^{n+2s}} \leqslant \frac{CR^{-2s}\lambda^{1-2s}}{1-2s},$$

for some $C > 0$ only depending on n.

Proof By scaling $y := x/R$, we see that

$$\int_{P_{R,\lambda}} \frac{dx}{|x|^{n+2s}} = R^{-2s}\int_{P_{1,\lambda}} \frac{dy}{|y|^{n+2s}},$$

so it is enough to prove the desired claim for $R = 1$.

To this goal, we observe that, if $\rho \in [0, 1]$ then

$$1 - \sqrt{1-\rho^2} \leqslant C\rho^2,$$

for some $C > 0$ (independent of n and s). Therefore

$$\int_0^\lambda \frac{1-\sqrt{1-\rho^2}}{\rho^{2+2s}}\,d\rho \leqslant \frac{C\lambda^{1-2s}}{1-2s}, \tag{3.1}$$

up to renaming $C > 0$.

In addition, using polar coordinates in $\mathbb{R}^{n-1}$ (and possibly renaming constants which only depend on n), we have

$$\begin{aligned}\int_{P_{1,\lambda}} \frac{dx}{|x|^{n+2s}} &\leqslant \int_{P_{1,\lambda}} \frac{dx}{|x'|^{n+2s}} = C\int_{\{|x'|\leqslant\lambda\}} \left(\int_0^{1-\sqrt{1-|x'|^2}} \frac{dx_n}{|x'|^{n+2s}}\right) dx' \\ &= C\int_{\{|x'|\leqslant\lambda\}} \frac{1-\sqrt{1-|x'|^2}}{|x'|^{n+2s}}\,dx' = C\int_0^\lambda \frac{1-\sqrt{1-\rho^2}}{\rho^{2+2s}}\,d\rho.\end{aligned}$$

This and (3.1) yield the desired result. □

A variation of Lemma 3.1 deals with the case of trapping between two hypersurfaces, as stated in the following result:

Lemma 3.2 *Let $C_o > 0$ and $\alpha > 2s$. For any $L > 0$, let*

$$P_L := \left\{ x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x'| \leqslant L \text{ and } |x_n| \leqslant C_o\,|x'|^{1+\alpha} \right\}.$$

Then

$$\int_{P_L} \frac{dx}{|x|^{n+2s}} \leqslant \frac{C\,C_o\,L^{\alpha-2s}}{\alpha - 2s},$$

for some $C > 0$ only depending on n.

Proof Using polar coordinates in $\mathbb{R}^{n-1}$, we have

$$\begin{aligned} \int_{P_L} \frac{dx}{|x|^{n+2s}} &\leqslant \int_{\{|x'|\leqslant L\}} \left(\int_{\{|x_n|\leqslant C_o\,|x'|^{1+\alpha}\}} \frac{dx_n}{|x'|^{n+2s}} \right) dx' \\ &= \int_{\{|x'|\leqslant L\}} \frac{2C_o\,|x'|^{1+\alpha}}{|x'|^{n+2s}}\, dx' = \frac{C\,C_o\,L^{\alpha-2s}}{\alpha - 2s}, \end{aligned}$$

for some $C > 0$. □

Now we show that an s-minimal set does not have spikes going to infinity:

Lemma 3.3 *Let Ω_o be an open and bounded subset of $\mathbb{R}^{n-1}$ and let $\Omega := \Omega_o \times \mathbb{R}$. Let E be an s-minimal set in Ω.*

Assume that

$$E \setminus \Omega \subseteq \{x_n \leqslant v(x')\}, \tag{3.2}$$

for some $v : \mathbb{R}^{n-1} \to \mathbb{R}$, and that, for any $R > 0$,

$$M_R := \sup_{|x'|\leqslant R} v(x') < +\infty.$$

Then

$$E \cap \Omega \subseteq \{x_n \leqslant M\}$$

for some $M \in \mathbb{R}$ (which may depend on s, n, Ω_o and v).

Proof Assume that $\Omega_o \subseteq \{|x'| < R_o\}$, for some R_o and let $R > R_o + 1$, to be chosen suitably large. We show that

$$E \subseteq \left\{ x_n \leqslant 2M_{5R} + \frac{3}{2}R \right\}. \tag{3.3}$$

For any $t \geqslant 2M_{5R} + 2R$ we slide a ball centered at $\{x_n = t\}$ of radius $R/2$ "from left to right". For this, we observe that

$$B_{R/2}(-2R, 0, \ldots, 0, t) \subseteq E^c. \tag{3.4}$$

Indeed, if $x \in B_{R/2}(-2R, 0, \ldots, 0, t)$, then

$$\begin{aligned} \big| |x'| - 2R \big| &= \big| |x'| - |(-2R, 0, \ldots, 0)| \big| \leqslant \big| x' - (-2R, 0, \ldots, 0) \big| \\ &\leqslant \big| x - (-2R, 0, \ldots, 0, t) \big| \leqslant \frac{R}{2}. \end{aligned}$$

In particular,

$$|x'| \in (R,\ 3R).$$

In addition,

$$x_n \geqslant t - \frac{R}{2} \geqslant 2M_{5R} + 2R - \frac{R}{2} > 2M_{5R} \geqslant v(x').$$

These considerations and (3.2) imply that $x \in E^c$, thus establishing (3.4).

As a consequence of (3.4), we can slide the ball $B_{R/2}(-2R, 0, \dots, 0, t)$ in direction e_1 till it touches ∂E. Notice that if no touching occurs for any t, then (3.3) holds true and we are done. So we assume, by contradiction, that there exists $t \geqslant 2M_{5R} + 2R$ for which a touching occurs, namely there exists a ball $B := B_{R/2}(\rho, 0, \dots, 0, t)$ for some $\rho \in [-2R, 2R]$ such that

$$B \subset E^c \tag{3.5}$$

and there exists $p \in (\partial B) \cap (\partial E) \cap \overline{\Omega}$.

Let now B' be the ball symmetric to B with respect to p, and let K be the convex envelope of $B \cup B'$.

Notice that if $x \in B'$ then $x_n \geqslant t - \frac{3}{2}R \geqslant 2M_{5R} + \frac{R}{2} > 2M_{5R}$. That is, $B \cup B' \subseteq \{x_n > 2M_{5R}\}$ and so, by convexity

$$K \subseteq \{x_n > 2M_{5R}\}. \tag{3.6}$$

Now we claim that

$$K \subseteq \{x_n > v(x')\}. \tag{3.7}$$

Indeed, if $x \in K$ then $|x'| \leqslant \rho + 2R \leqslant 4R$, hence (3.7) follows from (3.6).

From (3.2) and (3.7) we conclude that

$$K \backslash \Omega \subseteq E^c. \tag{3.8}$$

Now define $B_\star := B_1(p + (2R_o + 2)e_1)$ and we observe that

$$B_\star \subseteq \Omega^c. \tag{3.9}$$

Indeed, if $x \in B_\star$, then

$$\begin{aligned} |x'| &\geqslant \big|\big(p' + (2R_o + 2)e_1\big)\big| - \big|x' - \big(p' + (2R_o + 2)e_1\big)\big| \\ &\geqslant 2R_o + 2 - |p'| - \big|x - \big(p + (2R_o + 2)e_1\big)\big| \geqslant 2R_o + 2 - R_o - 1 > R_o, \end{aligned}$$

which proves (3.9).

Now we check that

$$B_\star \subseteq K. \tag{3.10}$$

Indeed,

$$\text{if } x \in B_\star, \quad \text{then } |x - p| \leqslant 2R_o + 3, \tag{3.11}$$

and so in particular $|x - p| < \frac{R}{4}$ if R is large enough, and this proves (3.10).

In light of (3.8), (3.9) and (3.10), we have that

$$B_\star \subseteq K \cap \Omega^c = K \backslash \Omega \subseteq E^c. \tag{3.12}$$

Also, since we have slided the balls from left to right, we have that $B_\star$ is on the right of B and hence it lies outside B. Hence, (3.10) can be precised by saying that $B_\star \subseteq K \backslash B$.

Thus, as a consequence of (3.5) and (3.12),

$$\begin{aligned}\int_K \frac{\chi_{E^c}(y)-\chi_E(y)}{|p-y|^{n+2s}}\,dy &= \int_{K\setminus B_\star} \frac{\chi_{E^c}(y)-\chi_E(y)}{|p-y|^{n+2s}}\,dy + \int_{B_\star} \frac{dy}{|p-y|^{n+2s}}\\ &\geqslant \int_B \frac{dy}{|p-y|^{n+2s}} - \int_{K\setminus(B\cup B_\star)} \frac{dy}{|p-y|^{n+2s}} + \int_{B_\star} \frac{dy}{|p-y|^{n+2s}}\\ &\geqslant \int_B \frac{dy}{|p-y|^{n+2s}} - \int_{K\setminus B} \frac{dy}{|p-y|^{n+2s}} + \int_{B_\star} \frac{dy}{|p-y|^{n+2s}},\end{aligned}$$

in the principal value sense. Hence, the contributions in B and B' cancel out by symmetry and, in virtue of Lemma 3.1 (used here with $\lambda := 1$), we obtain that

$$\int_K \frac{\chi_{E^c}(y)-\chi_E(y)}{|p-y|^{n+2s}}\,dy \geqslant -CR^{-2s} + \int_{B_\star} \frac{dy}{|p-y|^{n+2s}},$$

up to renaming $C>0$. Now if $y\in B_\star$ we have that $|p-y|\leqslant 2R_o+3\leqslant C$, for some $C>0$, thanks to (3.11). Also, if $y\in\mathbb{R}^n\setminus K$ then $|p-y|\geqslant R/4$. As a consequence, up to renaming $C>c>0$ step by step,

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y)-\chi_E(y)}{|p-y|^{n+2s}}\,dy &\geqslant \int_K \frac{\chi_{E^c}(y)-\chi_E(y)}{|p-y|^{n+2s}}\,dy - CR^{-2s}\\ &\geqslant -CR^{-2s} + \int_{B_\star} \frac{dy}{|p-y|^{n+2s}} \geqslant -CR^{-2s} + c\,|B_\star| \geqslant -CR^{-2s}+c,\end{aligned}$$

which is strictly positive if R is large enough. This is in contradiction with the Euler–Lagrange equation in the viscosity sense (see Theorem 5.1 in [7]) and so it proves (3.3). □

Next result gives the continuity of the fractional mean curvature at the smooth points of the boundary:

Lemma 3.4 *Let*

$$\alpha\in(2s,1]. \tag{3.13}$$

Let $E\subseteq\mathbb{R}^n$ *and* $x_o\in\partial E$. *Assume that* $(\partial E)\cap B_R(x_o)$ *is of class* $C^{1,\alpha}$, *for some* $R>0$. *Then*

$$\lim_{\substack{x\to x_o\\ x\in\partial E}} \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y)-\chi_E(y)}{|x-y|^{n+2s}}\,dy = \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y)-\chi_E(y)}{|x_o-y|^{n+2s}}\,dy.$$

Proof Up to a rigid motion, we suppose that $x_o=0$ and that, in the vicinity of the origin, the set E is the subgraph of a function $u\in C^{1,\alpha}(\mathbb{R}^{n-1})$ with $u(0)=0$ and $\nabla u(0)=0$. By formulas (49) and (50) in [2], we can write the fractional mean curvature in terms of u, as long as $|x'|$ is small enough. More precisely, there exist an odd and smooth functions F, with $F(0)=0$, $|F|+|F'|\leqslant C$, for some $C>0$, a function $\Psi\in C^{1,\alpha}(\mathbb{R}^{n-1})$, and a smooth, radial and compactly supported function ζ such that, if $|x'|$ is small and $x_n=u(x')$,

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y)-\chi_E(y)}{|x-y|^{n+2s}}\,dy = \int_{\mathbb{R}^{n-1}} F\left(\frac{u(x'+y')-u(x')}{|y'|}\right)\frac{\zeta(y')}{|y'|^{n-1+2s}}\,dy' + \Psi(x'),$$

in the principal value sense. Since also, by symmetry,

$$\int_{\mathbb{R}^{n-1}} F\left(\frac{\nabla u(x')\cdot y'}{|y'|}\right)\frac{\zeta(y')}{|y'|^{n-1+2s}}\,dy' = 0$$

in the principal value sense, we write

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+2s}}\,dy = \int_{\mathbb{R}^{n-1}} \left[F\left(\frac{u(x'+y')-u(x')}{|y'|}\right) - F\left(\frac{\nabla u(x')\cdot y'}{|y'|}\right)\right] \times \frac{\zeta(y')}{|y'|^{n-1+2s}}\,dy' + \Psi(x'). \tag{3.14}$$

So we define

$$G(x', y') := \left[F\left(\frac{u(x'+y')-u(x')}{|y'|}\right) - F\left(\frac{\nabla u(x')\cdot y'}{|y'|}\right)\right] \frac{\zeta(y')}{|y'|^{n-1+2s}}.$$

Notice that

$$\lim_{x'\to 0} G(x', y') = G(0, y').$$

Also, for any small $|x'|$ and bounded $|y'|$,

$$\left| F\left(\frac{u(x'+y')-u(x')}{|y'|}\right) - F\left(\frac{\nabla u(x')\cdot y'}{|y'|}\right)\right| \leqslant C\,\frac{|u(x'+y')-u(x')-\nabla u(x')\cdot y'|}{|y'|} \leqslant C\,|y'|^{\alpha}.$$

Therefore

$$|G(x', y')| \leqslant \frac{C}{|y'|^{n-1-\alpha+2s}} \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}),$$

thanks to (3.13). Accordingly, by the Dominated Convergence Theorem,

$$\lim_{x'\to 0} \int_{\mathbb{R}^{n-1}} G(x', y')\,dy' = \int_{\mathbb{R}^{n-1}} G(0, y')\,dy'.$$

Consequently,

$$\begin{aligned} &\lim_{x'\to 0} \int_{\mathbb{R}^{n-1}} \left[F\left(\frac{u(x'+y')-u(x')}{|y'|}\right) - F\left(\frac{\nabla u(x')\cdot y'}{|y'|}\right)\right] \frac{\zeta(y')}{|y'|^{n-1+2s}}\,dy' + \Psi(x') \\ &= \int_{\mathbb{R}^{n-1}} \left[F\left(\frac{u(y')-u(0)}{|y'|}\right) - F\left(\frac{\nabla u(0)\cdot y'}{|y'|}\right)\right] \frac{\zeta(y')}{|y'|^{n-1+2s}}\,dy' + \Psi(x'), \end{aligned}$$

which, combined with (3.14), establishes the desired result. □

The result in Lemma 3.4 can be modified to take into account sets with lower regularity properties.

Lemma 3.5 *Let $R > 0$, $E \subseteq \mathbb{R}^n$ and $x_o \in \partial E$. For any $k \in \mathbb{N}$, let $x_k \in \partial E$, with $x_k \to x_o$ as $k \to +\infty$, be such that E is touched from the inside at x_k by a ball of radius R, i.e. there exists $p_k \in \mathbb{R}^n$ such that*

$$B_R(p_k) \subseteq E \tag{3.15}$$

and $x_k \in \partial B_R(p_k)$.

Suppose that

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x_k - y|^{n+2s}}\,dy \leqslant 0.$$

Then

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x_o - y|^{n+2s}}\, dy \leqslant 0. \tag{3.16}$$

Proof Fix $\lambda > 0$, to be taken arbitrarily small in the sequel. Let $q_k := p_k + 2(x_k - p_k)$. We observe that the ball $B_R(q_k)$ is tangent to $B_R(p_k)$ at x_k. Therefore, by Lemma 3.1,

$$\int_{B_\lambda(x_k)\setminus(B_R(p_k)\cup B_R(q_k))} \frac{dy}{|x_k - y|^{n+2s}} \leqslant C R^{-2s} \lambda^{1-2s}, \tag{3.17}$$

for some $C > 0$. Also, using (3.15),

$$\int_{B_\lambda(x_k)} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x_k - y|^{n+2s}}\, dy \geqslant \int_{B_\lambda(x_k)} \frac{\chi_{B_R(p_k)}(y) - \chi_{B_R^c(p_k)}(y)}{|x_k - y|^{n+2s}}\, dy. \tag{3.18}$$

Now we define T_k to be the half-space passing through x_k with normal parallel to $x_k - p_k$ and containing $B_R(p_k)$. By symmetry,

$$\int_{B_\lambda(x_k)} \frac{\chi_{T_k}(y) - \chi_{T_k^c}(y)}{|x_k - y|^{n+2s}}\, dy = 0.$$

Using this, (3.18) and (3.17), we obtain that

$$\begin{aligned}
&\int_{B_\lambda(x_k)} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x_k - y|^{n+2s}}\, dy \\
&\geqslant \int_{B_\lambda(x_k)} \frac{\chi_{B_R(p_k)}(y) - \chi_{B_R^c(p_k)}(y)}{|x_k - y|^{n+2s}}\, dy - \int_{B_\lambda(x_k)} \frac{\chi_{T_k}(y) - \chi_{T_k^c}(y)}{|x_k - y|^{n+2s}}\, dy \\
&= -2 \int_{B_\lambda(x_k)\cap\big(T_k\setminus B_R(p_k)\big)} \frac{dy}{|x_k - y|^{n+2s}} \\
&\geqslant -C R^{-2s} \lambda^{1-2s}.
\end{aligned} \tag{3.19}$$

Now we define

$$f_k(y) := \chi_{B_\lambda^c(x_k)} \cdot \frac{\chi_E(y) - \chi_{E^c}(y)}{|x_k - y|^{n+2s}}.$$

We observe that f_k vanishes in $B_\lambda(x_k)$. Also, if $y \in B_{2\lambda}(x_o)\setminus B_\lambda(x_k)$, we have that $|f_k(y)| \leqslant \frac{1}{\lambda^{n+2s}}$. Moreover, if $y \in \mathbb{R}^n \setminus B_{2\lambda}(x_o)$, we have that

$$|y - x_o| \leqslant |y - x_k| + |x_k - x_o| \leqslant |y - x_k| + \lambda \leqslant |y - x_k| + \frac{|y - x_o|}{2},$$

as long as k is large enough, and so $|y - x_k| \geqslant \frac{|y - x_o|}{2}$, which gives that $|f_k(y)| \leqslant \frac{1}{|x - x_o|^{n+2s}}$ for any $y \in \mathbb{R}^n \setminus B_{2\lambda}(x_o)$. As a consequence of these observations, we can use the Dominated Convergence Theorem and obtain that

$$\begin{aligned}
\lim_{k\to+\infty} \int_{B_\lambda^c(x_k)} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x_k - y|^{n+2s}}\, dy &= \lim_{k\to+\infty} \int_{\mathbb{R}^n} f_k(y)\, dy = \int_{\mathbb{R}^n} \lim_{k\to+\infty} f_k(y)\, dy \\
&= \int_{B_\lambda^c(x_o)} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x_o - y|^{n+2s}}\, dy.
\end{aligned}$$

Thus, if k is large enough,

$$\int_{B_\lambda^c(x_k)} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x_k - y|^{n+2s}}\, dy \geqslant \int_{B_\lambda^c(x_o)} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x_o - y|^{n+2s}}\, dy - R^{-2s} \lambda^{1-2s}.$$

Thus, recalling (3.19),

$$\begin{aligned} 0 &\geqslant \int_{\mathbb{R}^n} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy \\ &= \int_{B_\lambda(x_k)} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy + \int_{B^c_\lambda(x_k)} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy \\ &\geqslant \int_{B^c_\lambda(x_o)} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_o-y|^{n+2s}}\,dy - CR^{-2s}\lambda^{1-2s}, \end{aligned}$$

up to renaming $C > 0$ line after line. Then, (3.16), in the principal value sense, follows by sending $\lambda \to 0$. □

A variation of Lemma 3.5 deals with the touching by sufficiently smooth hypersurfaces, instead of balls. In this sense, the result needed for our scope is the following:

Lemma 3.6 *Let $\Lambda > 0$. Let $E \subseteq \mathbb{R}^n$ and $x_o \in \partial E$. For any $k \in \mathbb{N}$, let $x_k \in \partial E$, with $x_k \to x_o$ as $k \to +\infty$, be such that E is touched from the inside in $B_\Lambda(x_k)$ at x_k by a surface of class $C^{1,\alpha}$, with $C^{1,\alpha}$-norm bounded independently of k and $\alpha \in (2s, 1]$. Suppose that*

$$\int_{\mathbb{R}^n} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_k-y|^{n+2s}}\,dy \leqslant 0.$$

Then

$$\int_{\mathbb{R}^n} \frac{\chi_E(y)-\chi_{E^c}(y)}{|x_o-y|^{n+2s}}\,dy \leqslant 0.$$

Proof The proof is similar to the one of Lemma 3.5. The only difference is that (3.17) is replaced here by

$$\int_{B_\lambda(x_k)\setminus(P_k^+\cup P_k^-)} \frac{dy}{|x_k-y|^{n+2s}} \leqslant C\,\lambda^{\alpha-2s}, \tag{3.20}$$

where $\lambda \in (0, \Lambda)$ can be taken arbitrarily small and P_k^+ is a region with $C^{1,\alpha}$-boundary that is contained in E and P_k^- is the even reflection of P_k^+ with respect to the tangent plane of P_k^+ at x_k. In this framework, (3.20) is a consequence of Lemma 3.2.

The rest of the proof follows the arguments given in the proof of Lemma 3.5, substituting $B_R(p_k)$ and $B_R(q_k)$ with P_k^+ and P_k^-. □

4 Graph properties of s-minimal sets and proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1.

Proof of Theorem 1.1. First we show that $(\partial E) \cap \Omega$ is a graph, namely that

$$\text{formula (1.2) holds true, for some } v : \mathbb{R}^{n-1} \to \mathbb{R}. \tag{4.1}$$

The idea is to slide E from above till it touches itself. Namely, for any $t \geqslant 0$, we let $E_t := E + te_n$. We also define

$$\Gamma := \partial((\partial E) \cap \Omega) \quad \text{and} \quad \Gamma_t := \partial((\partial E_t) \cap \Omega).$$

By Lemma 3.3,

$$\Omega_o \times (-\infty, -M) \subseteq E \cap \Omega \subseteq \Omega_o \times (-\infty, M), \tag{4.2}$$

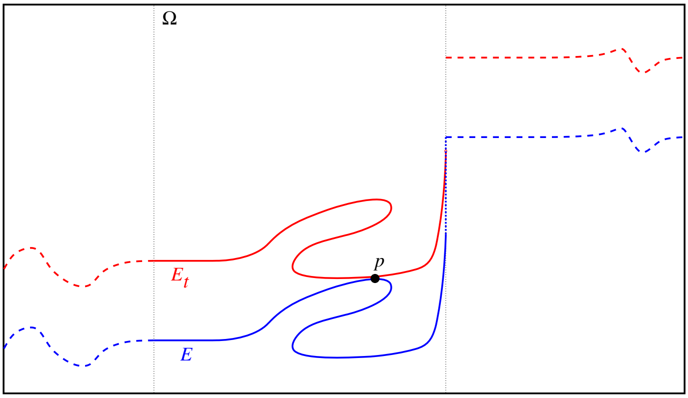

Fig. 1 The case in (4.7)

for some $M \geqslant 0$. Hence, if $t > 2M$, then Γ_t lies above Γ (with respect to the nth coordinate). So we take the smallest t for which such position holds, namely we set

$$t := \inf\{\tau \text{ s.t. } \Gamma_\tau \text{ lies above } \Gamma\}. \tag{4.3}$$

Our goal is to show that

$$t = 0. \tag{4.4}$$

Indeed, if we show that $t = 0$, we could define $v(x') := \inf\{\tau \text{ s.t. } (x', \tau) \in E^c\}$ and obtain that $E \cap \Omega_o$ is the subgraph of v.

To prove that $t = 0$, we argue by contradiction, assuming that

$$t > 0, \tag{4.5}$$

and so there is a contact point between Γ and Γ_t.

We remark that, in our framework, the set ∂E may have some vertical portions along $\partial\Omega$ (and indeed, this is the "typical" picture that we deal with, see [14]). Hence, the two sets ∂E and ∂E_t may share some common vertical portions along $\partial\Omega$. Roughly speaking, these vertical portions do not really consist of contact points since they do not prevent the sliding of the sets E and E_t by keeping the inclusion (for instance, in Fig. 1 the only contact point is the black dot named p, while in Fig. 2 we have two contact points given by the black dots p and q).

To formalize this notion, we explicitly define the set of contact points between Γ and Γ_t as

$$\mathscr{K} := \Gamma \cap \Gamma_t = (\partial((\partial E) \cap \Omega)) \cap (\partial((\partial E_t) \cap \Omega)). \tag{4.6}$$

The definition of first contact time given in (4.3) gives that $\mathscr{K} \neq \varnothing$.

We distinguish two cases, according to whether all the contact points are interior, or there are boundary contacts (no other possibilities occur, thanks to (1.1)). Namely, we have that either

$$\text{all the contact points lie in } \Omega_o \times \mathbb{R}, \tag{4.7}$$

i.e. $\mathscr{K} \subseteq \Omega$, or

$$\text{there exists a contact point in } (\partial\Omega_o) \times \mathbb{R}, \tag{4.8}$$

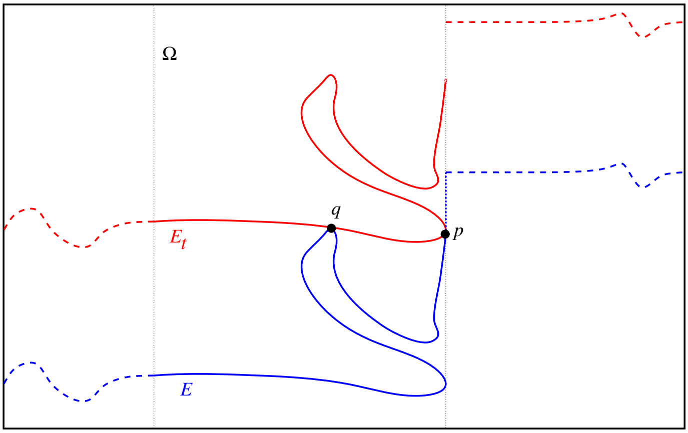

Fig. 2 The case in (4.8)

i.e. $\mathscr{K} \cap (\partial\Omega) \neq \varnothing$.

The case in (4.7) is depicted in Fig. 1 and the case in (4.8) is depicted in Fig. 2. The rest of the proof will take into account these two cases separately. To be precise, when (4.7) holds true, we will get a contradiction by using the supconvolution method (since we did not assume any regularity of the surface that we study), while if (4.8) holds true we will take advantage of the regularity in the obstacle problem for nonlocal minimal surfaces (see [6] and Theorem 5.1 here).

The case in which (4.7) *holds true* Assume first that (4.7) is satisfied. Then we consider the subconvolution of E and we slide it from above till it touches the supconvolution of E (in the notation of Sect. 2). More explicitly, fixed δ and $\eta > 0$, to be taken suitably small in the sequel, and, for any $\tau \in \mathbb{R}$, we consider $E_\delta^\flat + \tau e_n$.

Now we recall the notation in (2.12) and, by taking η and δ sufficiently small, we deduce from (4.7) that $\mathscr{K}$ lies inside $\Omega_{16(\delta+\eta)}$, at some positive distance from $\partial\Omega$ (which is uniform in η and δ).

By (4.2), we have that if τ is large, then

$$(E_\delta^\flat + \tau e_n) \cap \Omega \supseteq E_\delta^\sharp \cap \Omega$$

and so, in particular,

$$(E_\delta^\flat + \tau e_n) \cap \Omega_\eta \supseteq E_\delta^\sharp \cap \Omega_\eta. \tag{4.9}$$

So we take the smallest $\tau = \tau_{\delta,\eta}$ for which such inclusion holds. From (4.5), we have that

$$\tau \geqslant \frac{t}{2} > 0, \tag{4.10}$$

for small δ and η. Also, by (4.7) (recall also the first statement in Corollary 2.2), if δ is small enough, we obtain that $\big(\partial(E_\delta^\flat + \tau e_n)\big) \cap \Omega_\eta$ and $(\partial E_\delta^\sharp) \cap \Omega_\eta$ possess a contact point p in $\Omega_o \times \mathbb{R}$ (namely, p is close to the contact set $\mathscr{K}$ for small δ and η). Now we distinguish two subcases: either this is the first contact point in the whole of the space or not. In the first subcase, we have that (4.9) may be strengthen to $E_\delta^\flat + \tau e_n \supseteq E_\delta^\sharp$, and therefore we can apply Proposition 2.5, and we obtain that $E_\delta^\sharp = E_\delta^\flat + \tau e_n$. By taking δ arbitrarily small and using

(4.10), we obtain that $E = E + \tau_o e_n$, with $\tau_o \geqslant t/2 > 0$, which is in contradiction with (1.1).

The second subcase is when the first contact point p in Ω_η does not prevent the sets to overlap outside Ω_η. In this case, we will show that this overlap only occurs either in $\Omega\setminus\Omega_\eta$ or at infinity, and then we provide a contradiction arising from the contribution in bounded sets. Namely, first of all we recall the notation in (2.11) and (2.13) and we notice that for any $R > 0$ there exist $\delta_R, \eta_R > 0$ such that for any $\delta \in (0, \delta_R]$ and $\eta \in (0, \eta_R]$ we have that

$$(E_\delta^\flat + \tau e_n) \cap \mathscr{D}_{R,\eta} \supseteq E_\delta^\sharp \cap \mathscr{D}_{R,\eta}. \tag{4.11}$$

To prove (4.11), we argue by contradiction. If not, there exist some $R > 0$ and infinitesimal sequences $\delta, \eta \to 0$ such that $\big(E_\delta^\sharp \setminus (E_\delta^\flat + \tau e_n)\big) \cap \mathscr{D}_{R,\eta} \neq \varnothing$. Then, let $q_{\delta,\eta} = (q'_{\delta,\eta}, q_{\delta,\eta,n})$ be a point in such set. By construction $|q_{\delta,\eta,n}| \leqslant 3M + 1$ and $|q'_{\delta,\eta}| \leqslant R$, therefore, up to subsequences, as $\delta, \eta \to 0$, we may suppose that $\tau = \tau_{\delta,\eta} \to \tau_\star$ and $q_{\delta,\eta} \to q_\star = (q'_\star, q_{\star,n}) \in \overline{\big(E\setminus(E + \tau_\star e_n)\big) \cap \mathscr{D}_{R,\eta}}$. Hence, by (4.9), $q_\star \in \mathbb{R}^n\setminus\Omega$ and so, by (1.1), we have that $u(q'_\star) + \tau_\star \leqslant q_{\star,n} \leqslant u(q'_\star)$. This gives that $\tau_\star \leqslant 0$, which is in contradiction with (4.10) and thus completes the proof of (4.11).

Now we fix $R_o > 0$ such that $\Omega \subset \mathscr{C}_{R_o}$, and we suppose that $R > 4(R_o + 1)$. Thanks to (4.11), we can now use Proposition 2.6 and obtain that

$$\int_{\mathscr{D}_{R,\eta}} \frac{\chi_{(E_\delta^\flat+\tau e_n)\setminus E_\delta^\sharp}(y)}{|p-y|^{n+2s}}\,dy = \int_{\mathscr{D}_{R,\eta}} \frac{\chi_{(E_\delta^\flat+\tau e_n)\setminus E_\delta^\sharp}(y) - \chi_{E_\delta^\sharp\setminus(E_\delta^\flat+\tau e_n)}(y)}{|p-y|^{n+2s}}\,dy \leqslant C(R^{-2s}+\eta), \tag{4.12}$$

for some $C > 0$ that does not depend on R, δ and η, provided that δ and η are small enough.

Since u is continuous in $\mathbb{R}^{n-1}$, it is uniformly continuous in compact sets and so we can define

$$\sigma_\delta := \sup_{\substack{|x'|,\ |y'|\leqslant R_o+3\\ |x'-y'|\leqslant 2\delta}} |u(x') - u(y')|,$$

and we have that $\sigma_\delta \to 0$ as $\delta \to 0$.

We claim that, for small $\delta > 0$,

$$\begin{aligned} &\text{if } x = (x', x_n) \in \partial(E_\delta^\flat + \tau e_n), \quad y = (y', y_n) \in \partial E_\delta^\sharp \quad \text{and} \quad x' = y',\\ &\quad \text{with } |x'| \in (R_o + 1, R_o + 2),\\ &\quad \text{then } x_n \geqslant y_n + \frac{t}{4}. \end{aligned} \tag{4.13}$$

To prove it, we use the first statement in Corollary 2.2 to find $x_o \in (\partial E) + \tau e_n$ and $y_o \in \partial E$ such that

$$\max\{|x - x_o|,\ |y - y_o|\} \leqslant \delta.$$

Notice that $x_{o,n} = u(x'_o) + \tau$ and $y_{o,n} = u(y'_o)$. Moreover, $|x' - x'_o| \leqslant \delta$ and $|x' - y'_o| = |y' - y'_o| \leqslant \delta$, hence $|x'_o - y'_o| \leqslant 2\delta$. Therefore

$$\begin{aligned} x_n - y_n &= x_n - x_{o,n} + u(x'_o) + \tau - y_n + y_{o,n} - u(y'_o)\\ &\geqslant \tau - |x - x_o| - |y - y_o| - |u(x'_o) - u(y'_o)| \geqslant \tau - 2\delta - \sigma_\delta. \end{aligned}$$

This and (4.10) imply (4.13), as desired.

Notice also that $\mathscr{D}_{R,\eta} \supset \mathscr{C}_{R_o+2} \setminus \mathscr{C}_{R_o+1}$. So we use (4.11) and (4.13) to deduce that, fixed $R > 4(R_o + 1)$ and $\delta > 0$ small enough (possibly in dependence of R),

$$\int_{\mathscr{D}_{R,\eta}} \frac{\chi_{(E^\flat_\delta + \tau e_n)\setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\, dy \geqslant \int_{\mathscr{C}_{R_o+2}\setminus\mathscr{C}_{R_o+1}} \frac{\chi_{(E^\flat_\delta + \tau e_n)\setminus E^\sharp_\delta}(y)}{|p-y|^{n+2s}}\, dy \geqslant c_o t,$$

for some $c_o > 0$ (possibly depending on the fixed R_o and M). From this and (4.12), we obtain that $t \leqslant \tilde{C}(R^{-2s} + \eta)$, for some $\tilde{C} > 0$ and so, by taking η as small as we wish and R as large as we wish, we conclude that $t = 0$. This is in contradiction with (4.5), and so we have completed the proof of Theorem 1.1 under assumption (4.7).

The case in which (4.8) *holds true* Now we deal with the case in which (4.8) is satisfied. Hence, there exists a contact point $p = (p', p_n) \in (\partial E_t) \cap (\partial E)$ with $p' \in \partial\Omega_o$. More explicitly, we notice that, by (4.6),

$$p \in \left(\overline{(\partial E_t) \cap \Omega}\right) \cap \left(\overline{(\partial E) \cap \Omega}\right). \tag{4.14}$$

Now, we observe that E is a variational subsolution in a neighborhood of p (according to Definition 2.3 in [7]): namely, if $A \subseteq E \cap \Omega$ and $p \in \overline{A}$, we have that

$$0 \geqslant \mathrm{Per}_s(E, \Omega) - \mathrm{Per}_s(E\setminus A, \Omega) = L(A, E^c) - L(A, E\setminus A).$$

Therefore (see Theorem 5.1 in [7]) we have that

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\mathbb{R}^n\setminus E}(y)}{|p-y|^{n+2s}}\, dy \geqslant 0. \tag{4.15}$$

in the viscosity sense (i.e. (4.15) holds true provided that E is touched by a ball from outside at p).

Our goal is now to establish fractional mean curvature estimates in the strong sense. For this, we notice that, by (4.5), either

$$p_n \neq u(p') \tag{4.16}$$

or

$$p_n \neq u(p') + t. \tag{4.17}$$

We focus on the case in which (4.16) holds true (the case in (4.17) can be treated similarly, by exchanging the roles of E and E_t).

Then, either $B_r(p)\setminus\Omega \subseteq E$ or $B_r(p)\setminus\Omega \subseteq E^c$, for a small $r > 0$. In any case, by Theorem 5.1, we have that $(\partial E) \cap B_r(p)$ is a $C^{1,\frac{1}{2}+s}$-graph in the direction of the normal of Ω at p, up to renaming r.

Let $\nu(p) = (\nu'(p), \nu_n(p))$ be such normal, say, in the interior direction. Since Ω is a cylinder, we have that $\nu_n(p) = 0$. Also, up to a rotation we can suppose that $\nu'(p) = e_1$. In this framework, we can write ∂E in the vicinity of p as a graph $G := \{x_1 = \Psi(x_2, \dots, x_n)\}$, for a suitable $\Psi \in C^{1,\frac{1}{2}+s}(\mathbb{R}^{n-1})$, with $\Psi(p_2, \dots, p_n) = p_1$.

We observe that

$$\text{there exists a sequence of points } p^{(k)} \in G \text{ such that } p^{(k)} \in \Omega \text{ and } p^{(k)} \to p \text{ as } k \to +\infty. \tag{4.18}$$

Indeed, if not, we would have that ∂E in the vicinity of p lies in Ω^c. This is in contradiction with (4.14) and so it proves (4.18).

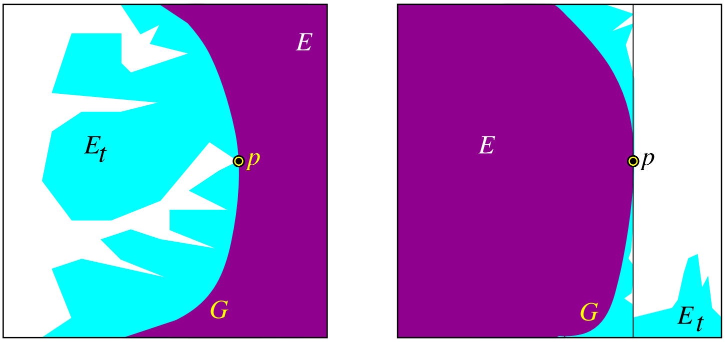

Fig. 3 The alternative in (4.21) and (4.22)

From (4.18), we obtain that there exists a sequence of points $p^{(k)} \to p$, such that

$$\partial E \text{ near } p^{(k)} \text{ is a graph of class } C^{1,\frac{1}{2}+s} \tag{4.19}$$

and

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|p^{(k)} - y|^{n+2s}}\, dy = 0.$$

As a consequence of this, (4.19), and Lemma 3.6 (applied to both E and E^c) we obtain that

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|p - y|^{n+2s}}\, dy = 0.$$

Hence, since $E_t \supseteq E$ (and they are not equal, thanks to (4.5)),

$$\int_{\mathbb{R}^n} \frac{\chi_{E_t}(y) - \chi_{E_t^c}(y)}{|p - y|^{n+2s}}\, dy > 0. \tag{4.20}$$

Also, since $E_t \supseteq E$, we have that $(\partial E_t) \cap B_{\frac{r}{4}}(p)$ can only lie on one side of the graph G, i.e.

$$\text{either } E_t \cap B_{\frac{r}{4}}(p) \supseteq \{x_1 \geqslant \Psi(x_2, \ldots, x_n)\} \tag{4.21}$$

$$\text{or } E_t \cap B_{\frac{r}{4}}(p) \subseteq \{x_1 \leqslant \Psi(x_2, \ldots, x_n)\}, \tag{4.22}$$

see Fig. 3.

In any case (recall (4.14)), we have that there exists a sequence of points $\tilde{p}^{(k)} \in (\partial E_t) \cap \Omega$ that can be touched by a surface of class $C^{1,\frac{1}{2}+s}$ lying in E_t (indeed, for this we can either enlarge balls centered at G, or slide a translation of G, see Fig. 4).

Then

$$\int_{\mathbb{R}^n} \frac{\chi_{E_t}(y) - \chi_{E_t^c}(y)}{|\tilde{p}^{(k)} - y|^{n+2s}}\, dy \le 0.$$

Hence, by Lemma 3.6,

$$\int_{\mathbb{R}^n} \frac{\chi_{E_t}(y) - \chi_{E_t^c}(y)}{|p - y|^{n+2s}}\, dy \le 0.$$

This is in contradiction with (4.20) and so the proof of (4.1) is complete.

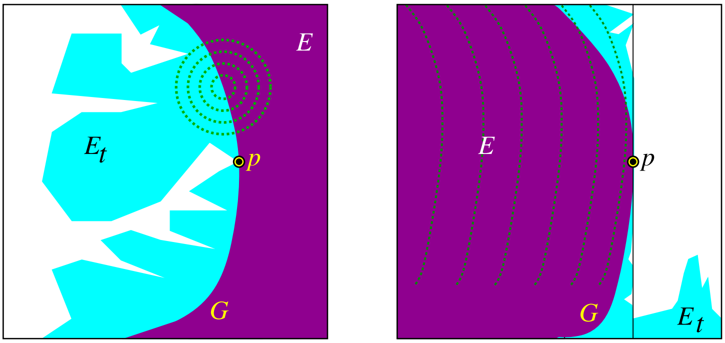

Fig. 4 Touching ∂E_t, according to the alternative in (4.21) and (4.22)

Now, to finish the proof of Theorem 1.1, we need to check the properties on the function v stated in Theorem 1.1. By construction, v and u coincide outside Ω_o, since they both describe the boundary of E. Moreover, v is continuous on $\overline{\Omega_o}$: to prove this, suppose by contradiction that

$$\ell_+ := \limsup_{\substack{x'\to p' \\ x'\in\Omega_o}} u(x') > \liminf_{\substack{x'\to p' \\ x'\in\Omega_o}} u(x') =: \ell_-,$$

for some $p' \in \overline{\Omega_o}$. Then $(x', u(x')) \in (\partial E) \cap \Omega$ and so both (p', ℓ_-) and (p', ℓ_+) belong to $\partial((\partial E) \cap \Omega)$. As a consequence, sliding E_t from above as in (4.3), we would obtain that $t \geqslant \ell_+ - \ell_- > 0$, in contradiction with (4.4). Hence the proof of Theorem 1.1 is complete. $\square$

5 Smoothness in dimension 3 and proof of Theorem 1.2

The goal of this section is to prove Theorem 1.2. For this, we state the following result, concerning the boundary regularity of nonlocal minimal surfaces, which is a direct consequence of [6]:

Theorem 5.1 *Let Ω be an open and bounded subset of $\mathbb{R}^n$ with boundary of class $C^{1,\alpha}$, with $\alpha \in \left(s + \frac{1}{2},\ 1\right)$.*

Let E be an s-minimal set in Ω and suppose that $p \in (\partial\Omega) \cap (\partial E)$.

Assume also that either

$$B_r(p)\setminus\Omega \subseteq E \tag{5.1}$$

or

$$B_r(p)\setminus\Omega \subseteq E^c, \tag{5.2}$$

for some $r > 0$.

Then, there exists $r' \in (0, r)$, depending on n, s, α and the $C^{1,\alpha}$ regularity of Ω, such that $(\partial E) \cap B_{r'}(p)$ is of class $C^{1,\frac{1}{2}+s}$.

Proof Without loss of generality, we can assume that $r = 2$, $p = 0$ and (5.1) holds true. Then we take $\mathscr{O}$ to be a domain with boundary of class $C^{1,\alpha}$ and such that

$$B_1 \cap \Omega^c \subseteq \mathscr{O} \subseteq B_2 \cap \Omega^c.$$

By construction

$$0 \in \partial \mathscr{O} \tag{5.3}$$

and

$$\mathscr{O} \cap B_1 = B_1 \cap \Omega^c \subseteq B_2 \cap \Omega^c \subseteq E. \tag{5.4}$$

Now we observe that if F contains $\mathscr{O} \cap B_1$, then

$$E \cap \Omega^c \cap B_1 = E \cap \mathscr{O} \cap B_1 = \mathscr{O} \cap B_1 = F \cap \mathscr{O} \cap B_1 = F \cap \Omega^c \cap B_1.$$

Also, if $F \setminus B_1 = E \setminus B_1$, then

$$E \cap \Omega^c \cap B_1^c = F \cap \Omega^c \cap B_1^c.$$

Therefore, if F contains $\mathscr{O} \cap B_1$ and $F \setminus B_1 = E \setminus B_1$, then

$$E \cap \Omega^c = (E \cap \Omega^c \cap B_1) \cup (E \cap \Omega^c \cap B_1^c) = (F \cap \Omega^c \cap B_1) \cup (F \cap \Omega^c \cap B_1^c) = F \cap \Omega^c,$$

thus, by the minimality of E,

$$\mathrm{Per}_s(E, \Omega) \leqslant \mathrm{Per}_s(F, \Omega),$$

and therefore

$$\mathrm{Per}_s(E, B_1) - \mathrm{Per}_s(F, B_1) = \mathrm{Per}_s(E, \Omega) - \mathrm{Per}_s(F, \Omega) \leqslant 0. \tag{5.5}$$

Thanks to (5.3), (5.4) and (5.5), we can apply Theorem 1.1 in [6] and conclude that $(\partial E) \cap B_{r'}$ is of class $C^{1, \frac{1}{2}+s}$. □

With this, we are ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. By Theorem 1.1, we know that E is an epigraph, i.e. (1.3) holds true for some $v : \mathbb{R}^2 \to \mathbb{R}$. It remains to show that

$$v \in C^\infty(\Omega_o). \tag{5.6}$$

For this, we take $x_o \in (\partial E) \cap \Omega$ and we show that v is C^∞ in a neighborhood of x_o. Up to a translation, we suppose that x_o is the origin. Now we consider a blow-up E_0 of the set E, i.e., for any $r > 0$, we define $E_r := \frac{E}{r} := \{\frac{x}{r} \text{ s.t. } x \in E\}$ and E_0 to be a cluster point for E_r as $r \to 0$ (see Theorem 9.2 in [7]). In this way, we have that E_0 is an s-minimal set, and it is an epigraph (see e.g. (5.8) in [15]). Thus, by Corollary 1.3 in [15], we deduce that E_0 is a half-space.

Hence, by Theorem 9.4 in [7], we have that ∂E is a graph of class $C^{1,\alpha}$ in the vicinity[1] of the origin – and, as a matter of fact, of class C^∞, thanks to Theorem 1 of [2]. □

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[1] As a technical remark, we point out that with the methods of Theorem 9.4 in [7], one obtains $C^{1,\alpha}$ regularity for any $\alpha < s$, but this exponent can be further improved, since flat minimal surfaces are $C^{1,\beta}$ for any $\beta < 1$, with estimates, as proved in Theorem 2.7 of [6].

Appendix A: Choosing a "good" representative for the s-minimal set

For completeness, in this appendix we give full details about the convenient choice of the representative of an s-minimal set. Indeed, when dealing with an s-minimal set, it is useful to consider a representation of the set which avoids unnecessary pathologies (conversely, a bad choice of the set may lead to the formations of additional boundaries, which come from subsets of measure zero and can therefore be neglected). First of all, one would like to chose the representative of the set with "the smallest possible boundary". We will also show that one can reduce the analysis to an open set, by considering the points of the set which are interior in the sense of measures, according to the following observation:

Lemma A.1 *Let Ω be an open subset of $\mathbb{R}^n$, with $|\partial\Omega| = 0$ and E be an s-minimal set in Ω. Assume that $E\setminus\Omega$ is open in $\mathbb{R}^n\setminus\Omega$. Let*

$$\begin{aligned} E_1 &:= \left\{x \in \mathbb{R}^n \text{ s.t. there exists } r > 0 \text{ s.t. } |E \cap B_r(x)| = |B_r(x)|\right\} \\ &= \left\{x \in \mathbb{R}^n \text{ s.t. there exists } r > 0 \text{ s.t. } |B_r(x)\setminus E| = 0\right\}. \end{aligned} \tag{A.1}$$

Then:

$$E_1 \text{ is an open set}, \tag{A.2}$$

$$E \text{ and } E_1 \text{ coincide, up to a set of measure zero}, \tag{A.3}$$

$$\begin{aligned} &\text{there exists } c \in (0,1) \text{ such that for any } x \in \partial E_1 \text{ and any } r > 0 \text{ for which } B_r(x) \subset \Omega \\ &\text{there exist } y_1, y_2 \in B_r(x) \text{ such that } B_{cr}(y_1) \subset B_r(x) \cap E_1 \text{ and } B_{cr}(y_2) \subset B_r(x)\setminus E_1, \end{aligned} \tag{A.4}$$

$$\text{for any } x \in \partial E_1, \quad 0 < \frac{|E_1 \cap B_r(x)|}{|B_r(x)|} < 1. \tag{A.5}$$

Proof The statement in (A.2) comes directly from (A.1), so we focus on the proof of (A.3), (A.4) and (A.5). The proof of these facts is a consequence of the density estimates of s-minimal sets, see [7]. We provide full details for the convenience of the reader.

First of all, we can reduce from measurable sets to Borel sets, up to sets of measure zero, see e.g. Theorem 2.20(b) in [19]. Hence we may suppose that E is Borel. Also, we observe that, since $E\setminus\Omega$ is relatively open in $\mathbb{R}^n\setminus\Omega$, we have that

$$E_1\setminus\overline{\Omega} \text{ coincides with } E\setminus\overline{\Omega}. \tag{A.6}$$

Also, the symmetric difference of E_1 and E restricted to $\partial\Omega$ has zero measure, since so does $\partial\Omega$, therefore the proof of (A.3) is nontrivial only due to the possible contributions inside Ω.

Now we set

$$\begin{aligned} &E_0 := \left\{x \in \mathbb{R}^n \text{ s.t. there exists } r > 0 \text{ s.t. } |E \cap B_r(x)| = 0\right\}, \\ &\text{and } \tilde{E} := (E \cup E_1)\setminus E_0. \end{aligned}$$

By Proposition 3.1 in [16], one has that $\tilde{E}$ coincides with E up to sets of measure zero, that

$$|E \cap E_0| = |E_1\setminus E| = 0, \tag{A.7}$$

and

$$0 < \frac{|E \cap B_r(x)|}{|B_r(x)|} < 1 \tag{A.8}$$

for any $x \in \partial E$ and any $r > 0$.

Also, by construction, we see that $E_0 \cap E_1 = \varnothing$ and so

$$E_1 \subseteq \tilde{E}. \tag{A.9}$$

We set L to be the set of Lebesgue points of $\tilde{E}$, i.e. the set of all points $x \in \tilde{E}$ such that

$$\lim_{r \to 0} \frac{|\tilde{E} \cap B_r(x)|}{|B_r(x)|} = 1.$$

We recall (see e.g. Theorem 7.7 in [19]) that

$$|\tilde{E} \backslash L| = 0. \tag{A.10}$$

Now we claim that

$$\big(\tilde{E} \backslash (E_0 \cup E_1)\big) \cap \Omega \subseteq \tilde{E} \backslash L. \tag{A.11}$$

To prove this, let $x \in (\tilde{E} \backslash (E_0 \cup E_1)) \cap \Omega$. Then, for any $r > 0$, we have that $|E \cap B_r(x)| > 0$ and $|E^c \cap B_r(x)| > 0$, and so the same inequalities hold for $\tilde{E}$ replacing E.

Accordingly, for any $r > 0$ such that $B_r(x) \subset \Omega$, there exists $p_r \in B_r(x) \cap (\partial \tilde{E})$. By (A.8) and the Clean Ball Condition in Corollary 4.3 of [7], we deduce that there exist points $y_{1,r}$ and $y_{2,r}$ such that $B_{cr}(y_{1,r}) \subset B_r(p_r) \cap \tilde{E}$ and $B_{cr}(y_{2,r}) \subset B_r(p_r) \backslash \tilde{E}$, for a universal $c \in (0, 1)$.

Since $B_r(p_r) \subseteq B_{2r}(x)$, we see that $B_{cr}(y_{1,r}) \subset B_{2r}(x) \cap \tilde{E}$ and $B_{cr}(y_{2,r}) \subset B_{2r}(x) \backslash \tilde{E}$, and therefore

$$\frac{|\tilde{E} \cap B_{2r}(x)|}{|B_{2r}(x)|} \leqslant \frac{|B_{2r}(x) \backslash B_{cr}(y_{2,r})|}{|B_{2r}(x)|} = \frac{2^n - c^n}{2^n}.$$

This implies that

$$\limsup_{r \to 0} \frac{|\tilde{E} \cap B_{2r}(x)|}{|B_{2r}(x)|} \leqslant \frac{2^n - c^n}{2^n} < 1,$$

and so $x \notin L$, which proves (A.11).

By (A.6), (A.10) and (A.11), we obtain that $|\tilde{E} \backslash (E_0 \cup E_1)| = 0$. So, by (A.7),

$$|\tilde{E} \backslash E_1| \leqslant |\tilde{E} \backslash (E_0 \cup E_1)| + |\tilde{E} \cap E_0| = 0 + |E \cap E_0| = 0.$$

Moreover, by (A.7), we also have that

$$|E_1 \backslash \tilde{E}| = |E_1 \backslash E| = 0,$$

therefore E_1 and $\tilde{E}$ agree up to a set of measure zero. This establishes (A.3).

Now we prove (A.4). For this, let $x \in \partial E_1$ and suppose that $B_r(x) \subset \Omega$. By construction, there exist $a_r \in B_{r/4}(x) \cap E_1$ and $b_r \in B_{r/4}(x) \backslash E_1$. So, by (A.9), we have that

$$a_r \in B_{r/4}(x) \cap \tilde{E}. \tag{A.12}$$

Also, by construction,

$$|B_{r/4}(b_r) \backslash \tilde{E}| = |B_{r/4}(b_r) \backslash E| > 0,$$

therefore there exists $c_r \in B_{r/4}(b_r) \backslash \tilde{E}$, and so in particular $c_r \in B_{r/2}(x) \backslash \tilde{E}$.

From this and (A.12), we obtain that there exists $q_r \in B_{r/2}(x) \cap (\partial \tilde{E})$. Then, by the Clean Ball Condition in Corollary 4.3 of [7], we deduce that there exist points y_1 and y_2 such that

$$B_{cr/2}(y_1) \subset B_{r/2}(q_r) \cap \tilde{E} \subset B_r(x) \cap \tilde{E}$$

and

$$B_{cr/2}(y_2) \subset B_r(q_r) \setminus \tilde{E} \subset B_r(x) \setminus \tilde{E},$$

for a universal $c \in (0, 1)$. This says that $B_{cr/2}(y_1)$ lies in E and $B_{cr/2}(y_2)$ lies in E^c, up to sets of measure zero, therefore $B_{cr/2}(y_1) \subseteq E_1$ and $B_{cr/2}(y_2) \subseteq E_0 \subseteq E_1^c$. This completes the proof of (A.4) (up to changing $c/2$ to c).

Now, (A.5) is a direct consequence of (A.4). □

Thanks to Lemma A.1, when dealing with the s-minimal set E in the statement of Theorems 1.1 and 1.2, we implicitly identify E with E_1.

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