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Nonlocal minimal surfaces: recent developments, applications, and future directions

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Abstract
Recently, we have found that nonlocal minimal surfaces enjoy notably stronger regularity estimates than classical minimal surfaces. This survey paper discusses these findings and their exciting applications in phase transitions and minimal surfaces.

1 Introduction

This survey serves as an invitation to the study of nonlocal minimal surfaces, delving into their foundational concepts, recent advancements, and potential research avenues.

It begins by exploring the motivation behind the pursuit of nonlocal alternatives to classical minimal surfaces, which share similar qualitative features. The survey then spotlights the distinctive aspects of nonlocal minimal surfaces, particularly their strong a priori estimates and the compactness found in finite Morse index critical points. This is in sharp contrast to the non-compactness commonly associated with classic minimal surfaces of finite index.

Moreover, the discussion includes an examination of s-minimal surfaces in Riemannian manifolds, assessing their role in offering fresh insights into established mathematical problems like Yau's conjecture or Plateau's problem.

The survey concludes by mapping out potential future research directions in the field: It suggests a framework for defining higher codimensional s-minimal surfaces and compiles a set of open questions.

1.1 From surface tension to perimeter-like energies

Many physical phenomena feature interfaces that, influenced by forces similar to surface tension, minimize their surface area on a large scale. This behavior stems from the system's natural tendency to reduce its energy, which is heightened in interfacial regions.

While surface area is scale-invariant, real-world energies display different characteristics on microscopic and macroscopic scales.

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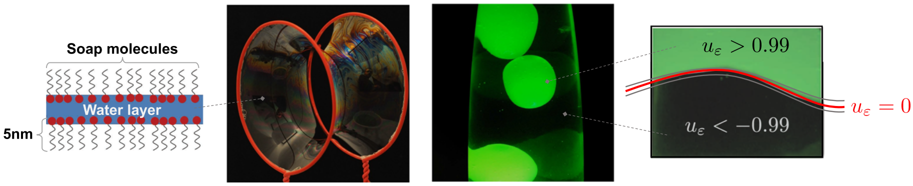

Fig. 1 Surface tension effects in a lava lamp and a soap film

Consider soap films as the quintessential real-world example of area-minimizing surfaces (see Fig. 1, left). As we zoom into the molecular scale, roughly 5 nanometers, the situation changes drastically. The film is no longer thin in comparison to the size of the molecules, which now appear quite large and engage in intricate interactions. This means that the usual method of modeling the soap film as a two-dimensional surface aiming to minimize its area becomes inadequate.

To accurately capture what happens at this scale, we would need to take into account the three-dimensional nature of the molecules and their complex interactions. This would introduce a new level of complexity to the model, potentially making it impractical for certain applications or analyses. In practice, for addressing scale-dependent phenomena, the scientific community often opts for pragmatic and relatively straightforward phenomenological models.

The Allen–Cahn and Cahn–Hilliard equations serve as prominent examples of scale-dependent models for phase separation. They describe the evolution of a concentration function $u(x, t)$, bounded between $[-1, 1]$. Here, $x \in \mathbb{R}^n$ represents the spatial coordinates, and $t \in \mathbb{R}$ denotes time."

Consider, for instance, a binary fluid separating into two pure domains (see Fig. 1, right). Domains $\{u < -0.99\}$ and $\{u > 0.99\}$ represent these pure components, and the region $\{-0.99 < u < 0.99\}$ is the thin *interface* between them. These models incorporate a small parameter $\varepsilon > 0$ (with lenght units) that quantifies the typical width of the *interface*.

For both models, the dynamics aim to minimize the Ginzburg-Landau energy, given by:

$$E_\varepsilon^{\mathrm{GL}}(u) = \int_{\mathbb{R}^n} \left(\varepsilon \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon} W(u) \right) dx,$$

with $W : [-1, 1] \to \mathbb{R}_+$ representing a double-well potential that has minima at ± 1. Typically, $W(u) = \frac{1}{4}(1 - u^2)^2$. For the Cahn-Hiliard equation, the evolution is constrained by $\int_{\mathbb{R}^n} u\, dx = \text{constant in time}$.

Other related phenomenological models have also been considered in the literature. Among these, is the Peierls-Nabarro equation, first proposed in the early 1940s to model crystal dislocations [86, 88]. It also models phase transitions with line-tension effects [1] and boundary vortices in thin magnetic films [70]. Its energy functional is given by:

$$E_\varepsilon^{\mathrm{PN}}(u) := \frac{1}{|\log \varepsilon|} \left(\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\left|u(x) - u(y)\right|^2}{|x - \bar{x}|^{n+1}}\, dx\, dy + \frac{1}{\varepsilon} \int_{\mathbb{R}^n} W(u)\, dx \right),$$

which combines interactions over long ranges with a typical double-well potential $W(u)$. A common choice in this context is $W(u) = \cos(\pi u/2)$.

It's rigorously established that the interfaces of 'absolute' minimizers of $E_\varepsilon^{\mathrm{GL}}$ and $E_\varepsilon^{\mathrm{PN}}$ behave like area minimizing surfaces at macroscopic scales ($\gg \varepsilon$). For a detailed mathematical treatment, see [1, 18, 85, 95, 96].

Yet, some foundational questions about the linkage between phase transitions and minimal surfaces remain unanswered. For instance: Is it true that interfaces of stable equilibria for $E_\varepsilon^{\mathrm{GL}}$ in $\mathbb{R}^3$ behave like stable minimal surfaces on macroscopic scales?

1.2 Understanding stable configurations: a mathematical challenge

Drawing from evolutionary dynamical systems, $E_\varepsilon^{\mathrm{GL}}(u)$ serves as a Lyapunov functional for both the Allen-Cahn and Cahn-Hilliard equations.

The equilibria of the dynamical system are represented by constant-in-time solutions $u = u(x)$ for the associated Euler-Lagrange PDE

$$-\varepsilon\Delta u + \tfrac{1}{\varepsilon}W'(u) = k \quad \text{in (a domain of) } \mathbb{R}^n. \tag{1.1}$$

In (1.1), k is a Lagrange multiplier: $k = 0$ for Allen-Cahn (unconstrained) while $k \in \mathbb{R}$ for Cahn-Hilliard ($\int_{\mathbb{R}^n} u\,dx = \text{constant}$).

But in practical terms, only a handful of these equilibria reflect observable states in nature; most are unstable, and minor disruptions make them decay to more stable states.

Primarily, our focus lies on the stable equilibria or the system's attractors. From a calculus of variations perspective, these equilibria correspond to stable critical points of the functional (i.e., points with zero first variation and nonnegative second variation).

The central challenge we previously highlighted revolves around discerning whether these solutions genuinely mirror stable minimal surfaces (or surfaces with constant mean curvature when $k \neq 0$) as ε tends to zero.

This intricate issue can be encapsulated (see [105]) by the following:

Conjecture 1.1 Every stable solution u of (1.1) in the whole $\mathbb{R}^3$ (that is, a solution of the PDE satisfying $\int_{\mathbb{R}^3}\left(\varepsilon|\nabla\xi|^2 + \frac{1}{\varepsilon}W''(u)\xi^2\right)dx \geq 0$ for all $\xi \in C_c^\infty(\mathbb{R}^3)$) must be of the form:

$$u(x) = \tanh\Big(\frac{e\cdot x - b}{\sqrt{2}\varepsilon}\Big),$$

for some $e \in \mathbb{S}^2$ and $b \in \mathbb{R}$.

An important special case of Conjecture 1.1, the case of monotone solutions known as De Giorgi conjecture, was solved by Ambrosio and Cabré in [6]. However, despite recent breakthroughs in the analysis of stable solutions of Allen Cahn [34, 105], the question remains largely open. The analog of this conjecture with $E_\varepsilon^{\mathrm{PN}}(u)$ replacing $E_\varepsilon^{\mathrm{GL}}(u)$ was solved by Figalli and the author in [53].

From a modeling viewpoint, the essential takeaway is that Allen-Cahn and minimal surfaces, despite their extensive applications across diverse research domains, serve primarily as pragmatic phenomenological models. Yet their mathematical underpinnings can be intricate and challenging to unravel. Given this context, it becomes sensible to explore alternative mathematical structures that manifest comparable macroscopic characteristics.

Guided by this rationale, our survey delves into nonlocal minimal surfaces and their theoretical implications, offering insights that could potentially inspire novel modeling approaches.

2 Classical minimal surfaces vs nonlocal ones

2.1 The classical perimeter and minimal surfaces

Building upon the foundations laid by Cacciopoli and De Giorgi, the classical perimeter of a measurable set $E \subset \mathbb{R}^n$ is defined through duality:

$$\mathrm{Per}(E) := \int_{\mathbb{R}^n} |\nabla \chi_E| := \sup \left\{ \int_E \mathrm{div}\phi \, dx \mid \phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n) \text{ with } \max_{\mathbb{R}^n} |\phi| \le 1 \right\}.$$

Using the divergence theorem, for any bounded set E with a smooth boundary ∂E:

$$\int_E \mathrm{div}\phi \, dx = \int_{\partial E} \phi \cdot \nu \, d\mathcal{H}^{n-1} \le \int_{\partial E} 1 d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial E),$$

where ν is the unit normal to the boundary and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure. Furthermore, we can show that equality holds if we choose ϕ to be an extension of ν in a neighborhood of ∂E (multiplied by a cutoff function).

Before introducing minimal surfaces, we require a notion of relative perimeter $\mathrm{Per}(E, \Omega)$ for a set E within a bounded open set $\Omega \subset \mathbb{R}^n$. This relative perimeter should satisfy:

(I) $\mathrm{Per}(E, \Omega) - \mathrm{Per}(F, \Omega) = \mathrm{Per}(E) - \mathrm{Per}(F)$ for all (measurable) sets E and F that coincide outside Ω and $\mathrm{Per}(F) < \infty$.

(II) $\mathrm{Per}(E, \Omega) < \infty$ whenever ∂E is a smooth submanifold in an open neighborhood of $\overline{\Omega}$.

Thus, we can define:

$$\mathrm{Per}(E, \Omega) := \int_{\Omega} |\nabla \chi_E| := \sup \left\{ \int_E \mathrm{div}\phi \, dx \mid \phi \in C_c^1(\Omega; \mathbb{R}^n) \text{ with } \max_{\mathbb{R}^n} |\phi| \le 1 \right\}.$$

Let us recall next the definition of classical minimal surface, and also the stable and minimizer classes, from a variational viewpoint.

Definition 2.1 Consider a set $E \subset \mathbb{R}^n$ with finite $\mathrm{Per}(E, \Omega)$. We say that E has *minimal boundary* in Ω if for every smooth variation ψ supported in Ω[1] the function

$$t \mapsto \mathrm{Per}(\psi(E, t), \Omega), \tag{2.1}$$

has a critical point (i.e. has zero derivative) at $t = 0$. We then say that ∂E is a minimal hypersurface in Ω.

If in addition (2.1) has a nonnegative second derivative at $t = 0$ (for every ψ) we say that ∂E is a *stable minimal hypersurface*.

A set is E is termed *minimizer* of $\mathrm{Per}(\,\cdot, \Omega)$ if $\mathrm{Per}(E, \Omega) < \infty$ and $\mathrm{Per}(F, \Omega) \ge \mathrm{Per}(E, \Omega)$ for all sets F mathching E outside of Ω.

(*Technical note:* the following "change of variables formula"

$$\begin{aligned}
\mathrm{Per}(\psi^t(E), \Omega) &= \int_{\Omega} \left|\nabla(\chi_E \circ \psi^{-t})\right| = \int_{\Omega} \left|(\nabla \chi_E) \circ \psi^{-t} \cdot D\psi^{-t}\right| \\
&= \int_{\Omega} \left|(\nabla \chi_E) \cdot (D\psi^{-t}) \circ \psi^t\right| \det\big(D\psi^t\big) \\
&= \int_{\partial^* E \cap \Omega} \left|\nu(y) \cdot \big((D\psi^{-t}) \circ \psi^t\big)(y)\right| \det\big(D\psi^t(y)\big) d\mathcal{H}^{n-1}(y),
\end{aligned}$$

[1] Namely, $\psi : \mathbb{R}^n \times (-1, 1) \to \mathbb{R}^n$ smooth, satisfying $\psi(x, 0) = x$ for all $x \in \mathbb{R}^n$, and with $\partial_t \psi(\cdot, \bar{t})$ compactly supported in Ω for all $\bar{t}$.

where ∂^* denotes the reduced boundary and ν the inwards unit vector to $\partial^* E$, holds true. Using it, one can see that the function (2.1) is infinitely differentiable in a neighborhood of $t = 0$. In particular, the first derivative and second derivatives considered above are always well-defined.)

2.2 Fractional perimeter and nonlocal minimal surfaces

The term *fractional perimeter* seems to have emerged primarily in articles published after 2010—as evidenced in works such as [7, 58, 76]—following the introduction of nonlocal minimal surfaces in the influential 2009 paper by Caffarelli, Roquejoffre, and Savin [19]. However, even if not precisely labeled as such, the concept of fractional perimeter has been in use for decades, particularly in the context of fractional Sobolev spaces, as seen for example in [15, 41].

The earliest instance we are aware of where the idea is clearly presented is in the 1990 SIAM J. Math. Anal. paper by Visintin [107]. In this paper, a family of nonlocal perimeters was introduced as an alternative to the perimeter, aiming to model surface tension effects in multi-phase systems.

The fractional perimeter in the Euclidean space $\mathbb{R}^n$ is defined as follows. For a given parameter $s \in (0, 1)$ and a measurable set $E \subset \mathbb{R}^n$, one defines:

$$\mathrm{Per}_s(E) := 2(1-s) \iint_{E \times E^c} \frac{dx\, dy}{|x-y|^{n+s}},$$

where

$$X^c := \mathbb{R}^n \setminus X \quad \text{denotes the complement of a subset} X \subset \mathbb{R}^n.$$

Similar to the classical perimeter, the fractional perimeter may assume values of zero or infinity, but it is always well-defined for any measurable set.

Why is Per_s considered a generalization of the classical perimeter? A highly convincing answer can be found in the works [15, 41], which established that for every set E with finite classical perimeter ($\mathrm{Per}(E) < \infty$), we have $\mathrm{Per}_s(E) \to c_* \mathrm{Per}(E)$ as $s \uparrow 1$, where c_* is a suitable dimensional constant. If one seeks a more profound result beyond this convergence on a set-by-set basis, the deeper Gamma-convergence result in [7] is also available.

To reinforce the analogy between Per and Per_s it is useful to notice that

$$\mathrm{Per}_s(E) = \frac{(1-s)}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}} dx dy.$$

In other words, up to adjusting the multiplicative constant, $\mathrm{Per}_s(E) = [\chi_E]_{W^{s,1}(\mathbb{R}^n)}$, the $W^{s,1}$ Gagliardo seminorm of the characteristic function of E (s derivatives in L^1). It seems then very natural that as $s \to 1$ it converges to the perimeter, which is, formally, the $W^{1,1}$ seminorm of χ_E.

And why might it be advantageous to employ the fractional perimeter as opposed to the conventional perimeter in certain applications? We will delve into some profound reasons later, but let us begin with a naive yet insightful example from [36]. Consider the framework of images with square pixels of small size $\varrho > 0$. For simplicity, let's examine the image of an ideal square with side length 1, oriented at 45° with respect to the pixel grid (see Fig. 2).

In this configuration, the classical perimeter proves to be a rather imprecise tool for analyzing this image, regardless of how high the image resolution is. The perimeter of the

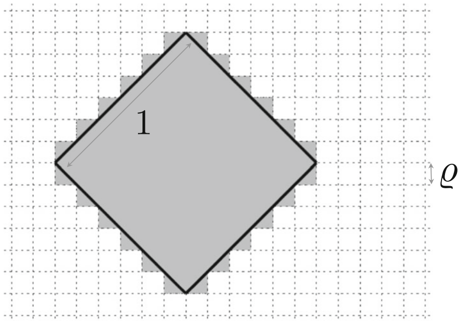

Fig. 2 Pixelled square

ideal square is 4, while the perimeter of the image displayed on the monitor is always greater than $4\sqrt{2}$, regardless of the smallness of the pixel size ϱ.

On the contrary, the fractional perimeter is not that sensitive to the relative orientation of the square and the grid. Indeed, the discrepancy $D_s(\varrho)$ between the s-perimeter of the ideal square and the s-perimeter of the pixelated square is bounded above by the sum of interactions between the "boundary pixel" and their complements. These pixels are the ones that intersect the boundary of the original square, and their count is C/ϱ. Through scaling, we find that the interaction of one pixel with its complement is on the order of ϱ^{2-s}. Consequently, we obtain $D_s(\varrho) \le C\varrho^{1-s}$, which is infinitesimal as $\varrho \downarrow 0$.

Hence, while from a modeling perspective, the classical and local perimeter capture essentially the same information on ideal objects, in some situations—as in this example—using the fractional perimeter may result in much more robust and well-behaved models.

As in the case of Per, we need localized versions of Per_s —satisfying analogs of properties (I) and (II) above. The standard way to define a relative fractional perimeter is:

$$\mathrm{Per}_s(E, \Omega) := 2(1-s) \iint_{(E\times E^c)\setminus(\Omega^c\times\Omega^c)} \frac{dx\,dy}{|x-y|^{n+s}}.$$

To grasp this definition, it can be useful to notice that

$$(E \times E^c)\setminus(\Omega^c \times \Omega^c) = (E\cap\Omega)\times(E^c\cap\Omega) \,\cup\, (E\cap\Omega)\times(E^c\cap\Omega^c) \,\cup\, (E\cap\Omega^c)\times(E^c\cap\Omega).$$

(Notice that the only missing piece to have the full $\mathrm{Per}_s(E)$ is the "interaction" of $E\cap\Omega^c$ and $E^c\cap\Omega^c$.)

The definitions of *s-minimal boundary* (or *s-minimal hypersurface*), *stable s-minimal hypersurface*, and *minimizer of* $\mathrm{Per}_s(\,\cdot\,, \Omega)$ are identical to their classical counterparts in Definition 2.1, with the only difference being the replacement of Per by Per_s throughout. (*Technical note:* the map (2.1) with Per replaced by Per_s is also infinitely differentiable near $t = 0$ as proven in [26].)

2.3 Monotonicity formulae and the fundamental role of hypercones

The classical monotonicity formula for minimal surfaces, due to Fleming [57], states that if ∂E is minimal in Ω and $x_\circ \in \partial E$ is such that $B_\varrho(x_\circ) \subset \Omega$ then

$$r \mapsto \Phi_E(r, x_\circ) := r^{1-n}\mathrm{Per}(E, B_r(x_\circ)) = r^{1-n}\mathcal{H}^{n-1}(\partial E \cap B_r(x_\circ))$$

is monotone nondecreasing for all $r \in (0, \varrho)$. Moreover, it is strictly increasing unless the set E is conical with respect to $x_\circ$ (i.e., $t(E - x_\circ) = E - x_\circ$ for all $t > 0$).

Nonlocal minimal hypersurfaces also enjoy a monotonicity formula. This is a remarkable and highly nontrivial property, established by Caffarelli, Roquejoffre, and Savin in [19], and is formulated in terms of the *(Caffarelli-Silvestre) extension* U_E of E. This *extension* U_E is defined as the (unique) bounded, weak solution U_E of the problem:

$$\begin{cases} \widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}U) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ U = \chi_E - \chi_{E^c} & \text{on } \mathbb{R}^n \times \{0\}, \end{cases}$$

where $\widetilde{\operatorname{div}}$ and $\widetilde{\nabla}$ are the divergence and gradient taken with respect to all $n+1$ coordinates $(x_1, \dots, x_n, z)$ of $\mathbb{R}^n \times (0,\infty)$.

As proven in [19] (see also [25, 84][2]), if ∂E is an s-minimal hypersurface in Ω, $x_\circ \in \partial E$ is such that $B_\varrho(x_\circ) \subset \Omega$, the function

$$r \mapsto \phi_E(r, x_\circ) := r^{s-n} \int_{\widetilde{B}_r^+(x_\circ,0)} z^{1-s}\, |\widetilde{\nabla} U_E(x,z)|^2 \, dx\, dz$$

is monotone nondecreasing in the interval $r \in (0,\varrho)$. Here above, $\widetilde{B}_r^+(x_\circ, 0)$ denotes the upper half-ball of radius r in $\mathbb{R}^{n+1}_+$ centered at $(x_\circ, 0)$. Moreover, $\Phi_E(r, x_\circ)$ is strictly increasing in r unless E is conical with respect to $x_\circ$.

(A version of this monotonicity formula on Riemannian manifolds is found in [25].)

The monotonicity formulae confer a central role to minimal and s-minimal hypercones: Every converging *blow-up* sequence ($R_k(E - x_\circ)$, with $R_k \uparrow \infty$) or *blow-down* sequence ($r_k(E - x_\circ)$, with $r_k \downarrow 0$) must have a conical limit. Thus, classifying minimal hypercones is a crucial step in both classical and fractional regularity theories.

Concerning the classification of classical stable minimal hypercones, we have:

Theorem 2.2 (Simons [99]) *Assume that $\partial E \subset \mathbb{R}^n$ is a stable minimal hypersurface that is conical about the origin (i.e. $tE = E$ for all $t > 0$) and such that $\partial E \setminus \{0\}$ is smooth and stable. Then, for dimensions $3 \le n \le 7$, ∂E must be a hyperplane.*

Moreover, in dimensions $n \ge 8$, the boundary of Simons cone

$$\{x_1^2 + x_2^2 + x_3^2 + x_4^2 < x_5^2 + x_6^2 + x_7^2 + x_8^2\} \subset \mathbb{R}^n \tag{2.2}$$

(translation invariant in the last $n-8$ directions) is a stable minimal hypercone (and is not a hyperplane).

The analog of Theorem 2.2, when we assume that $E \subset \mathbb{R}^n$ is a hypercone that minimizes $\mathrm{Per}_s(\,\cdot\,, B_1)$, is known to be true for s sufficiently close to 1—see [22].

However, the question of whether the analog of Theorem 2.2 holds for s-minimal surfaces (i.e. only assuming stability) for s sufficiently close to 1 is much more difficult and remains largely open. It is then convenient to introduce the following:

Definition 2.3 Given $s \in (0,1)$, we define $n_s^\star$ as the minimum dimension $n \ge 3$ such that there exists an s-minimal cone, smooth and stable in $\mathbb{R}^n \setminus \{0\}$, and that is not a hyperplane.

It is not difficult to see[3] that the boundary of the Simons cone (2.2) is a stable s-minimal hypersurface for $n = 8$, provided s is sufficiently close to 1. (Recall that in the limit $s \uparrow 1$ the

[2] The proof given in [19] is for minimizers and [25, 84] confirm that the same monotonicity formula also works for general s-minimal surfaces and in Riemannian manifolds.

[3] Through an appropriate separation of variables argument and using the continuity of a certain spherical eigenvalue when we vary the operator.

nonlocal perimeter Per_s converges to the local perimeter Per.) Hence we know that $n_s^\star \le 8$ for $s \in (0,1)$ sufficiently close to 1.

In view of Theorem 2.2, we believe it is natural to expect the following:

Conjecture 2.4 $n_s^\star = 8$ for all $s \in (0,1)$ sufficiently close to 1.

As we will see, this conjecture seems still out of reach, but nevertheless, we have obtained very substantial progress establishing that $n_s^\star \ge 5$; see [17, 31] and Remark 3.10 below.

2.4 Minimal surface equations and linearization around hyperplanes

The minimal surface equation was found by Lagrange [74] in 1761. Indeed, he proved that if a piece of an area-minimizing surface is smooth, then it must satisfy a PDE: Employing modern terminology, its mean curvature (sum of the principal curvatures) must be identically zero. This PDE can be used to prove, among other things, the analyticity of the surface.

So a crucial question is: when can we ensure that an area-minimizing surface will be smooth in a neighborhood of one of its points?

The standard criterion was found by De Giorgi [43]:

Theorem 2.5 [43] *There exists a dimensional constant $\varepsilon_\circ > 0$ such that the following holds. Assume that $E \subset \mathbb{R}^n$ is a minimizer of* $\mathrm{Per}(\,\cdot\,, B_1)$ *with $\partial E \cap B_1 \subset \{|x_n| \le \varepsilon_\circ\}$. Then, ∂E is a $C^{1,\alpha}$ graph, with estimates, in $B_{1/2}$.*[4]

De Giorgi's theorem and its generalizations and variants have become fundamental pillars in the modern regularity theories for area-minimizing surfaces and several other variational problems (e.g., in many free boundary problems).

We now turn our attention to the nonlocal minimal surface equation. Given $\Omega \subset \mathbb{R}^n$ open and bounded; suppose that ∂E intersected with an open neighborhood of $\overline{\Omega}$ is an $(n-1)$-submanifold of $\mathbb{R}^n$ of class C^2. Then, ∂E is s-minimal in Ω if, and only if,

$$(1-s)\,\mathrm{p.v.}\int_{\mathbb{R}^n} \frac{(\chi_{E^c} - \chi_E)(y)}{|x-y|^{n+s}}\,dy = 0 \quad \text{for all } x \in \partial E \cap \Omega. \tag{2.3}$$

Here a "p.v." stands for *in the principal value sense*[5]. Equation (2.3) is called the *s-minimal surface equation*; see [19, 52].

There is another way useful way to write the equation (2.3) as a (non-singular) integral on ∂E. Integrating by parts[6], we obtain

$$(1-s)\,\mathrm{p.v.}\int_{\mathbb{R}^n} \frac{(\chi_{E^c} - \chi_E)(y)}{|x-y|^{n+s}}\,dy = \frac{2(1-s)}{s}\int_{\partial E\cap\Omega} \frac{\nu(y)\cdot(x-y)}{|x-y|^{n+s}}\,d\mathcal{H}^{n-1}(y) + \begin{bmatrix}\text{boundary and}\\ \text{exterior terms}\end{bmatrix}.$$

The extra terms above, which are carefully computed in [31], are smooth inside Ω. Thus, they can be neglected from the viewpoint of regularity theory.

[4] More precisely, there exists $g : B_1' \to \mathbb{R}$ with $\|g\|_{C^{1,\alpha}} \le C(n,\alpha)$ such that ∂E coincides with $\{x_n = g(x')\}$ in $B_{1/2}$. For example, all principal curvatures of the surface at one of its points in $B_{1/2}$ are absolutely bounded by dimensional constants.

[5] That is $\mathrm{p.v.}\int_{\mathbb{R}^n} \frac{(\chi_{E^c}-\chi_E)(y)}{|x-y|^{n+s}}\,dy := \lim_{r\downarrow 0}\int_{\mathbb{R}^n\setminus B_r(x)} \frac{(\chi_{E^c}-\chi_E)(y)}{|x-y|^{n+s}}\,dy$.

[6] In both domains $E^c \cap \Omega$ and $E \cap \Omega$, using $\mathrm{div}_y\left(\frac{x-y}{|x-y|^{n+s}}\right) = \frac{s}{|x-y|^{n+s}}$.

De Giorgi's theorem remains valid in the nonlocal setup: simply substituting $\mathrm{Per}(\,\cdot\,, B_1)$ by $\mathrm{Per}_s(\,\cdot\,, B_1)$. Certainly, $\varepsilon_\circ$ must also depend on s (in addition to the dimension), but for the rest, the statement is identical; see [19]. That $\varepsilon_\circ$ does not degenerate as $s \uparrow 1$ was proven in [22, 23].

This "flat implies smooth" principle and the monotonicity formula are the reasons why the classification of stable minimal hypercones from Theorem 2.2 is so fundamental to the regularity theory. Indeed, by the monotonicity formula, given a point $x_\circ$ in an area-minimizing hypersurface $\partial E \subset \Omega \subset \mathbb{R}^n$ any blow-up sequence $\partial E_k := \partial(r_k^{-1}(E - x_\circ))$ with $r_k \downarrow 0$ converges, up to subsequence, to an area-minimizing (and hence stable) hypercone. Then, in dimensions $n \le 7$, it must be a hyperplane. Therefore, for $k_\circ$ sufficiently large, one can apply Theorem 2.5 to a $\partial E_{k_\circ}$ and deduce that it is smooth inside $B_{1/2}$. In other words, we find that ∂E is smooth inside the ball of radius $r_{k_\circ}/2$ centered at $x_\circ$. The same reasoning holds valid in the nonlocal case.

It's crucial to note that Theorem 2.5 would not hold if we replaces"area-minimizin" with simply "minimal" (i.e., when dealing with general critical points). Take, for instance, a sequence of catenoids with neck size going to zero —as in (3.1) below. While these catenoids can be contained between any two closely spaced hyperplanes in B_1, they are neither graphs nor have bounded curvatures in $B_{1/2}$.

However, the narrative shifts when nonlocal minimal surfaces enter the picture. As demonstrated in [19], the nonlocal version of Theorem 2.5 is applicable to all embedded s-minimal surfaces, not just those that minimize the fractional perimeter. This unique characteristic is a significant advantage of nonlocal minimal surfaces compared to their local counterparts.

3 Properties of stable and finite Morse index minimal surfaces

3.1 Finite index minimal surfaces: definition and motivation

In variational problems, one often seeks critical points of a functional, typically in function spaces. The Mountain Pass Theorem and the Min-Max Principle are archetypal variational methods to construct critical points. They guarantee the existence of critical points with specific properties, which are valuable in many applications

Although the constructed critical points are typically not stable, they locally minimize the energy with respect to all perturbations orthogonal to a certain finite dimensional subspace. In other words, they are so-called finite Morse index critical points.

Let us give below the precise definitions of finite index critical points within the setups of classical and nonlocal minimal surfaces.

Definition 3.1 (*Morse index* $\le m$) Let ∂E be a minimal surface [respectively, an s-minimal surface] in $\Omega \subset \mathbb{R}^n$ open and bounded (we refer to Sects. 2.1 and 2.2 for the definitions). Given $m \ge 0$ integer, we say that ∂E has *(Morse) index* $\le m$ in Ω if for every $(m+1)$ vector fields $X_0, \ldots, X_m \in C_c^\infty(\Omega; \mathbb{R}^n)$, there exists some linear combination $X = a_0 X_0 + \ldots + a_m X_m$ with $a_0^2 + a_1^2 + \ldots a_m^2 = 1$ such that

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Per}(\psi^X(E,t), \Omega) \ge 0. \quad \text{[respectively, Per replaced by } \mathrm{Per}_s]$$

Here, $\psi^X = \psi^X(x,t)$ denotes the vector flow of X.

Moreover, we say that ∂E has index $\le m$ in $\mathbb{R}^n$ if it has index $\le m$ in every bounded open subset of $\Omega \subset \mathbb{R}^n$.

Notice that, by definition, an s-minimal surface with Morse index 0 (i.e, $m = 0$) is the same as a stable s-minimal surface.

A very useful and insightful property that is an immediate consequence of Definition 3.1, is the following: If $\partial E \subset \mathbb{R}^n$ is a minimal surface with Morse index $\le m$, then given any $m + 1$ disjoint open sets $\Omega_0, \Omega_1, \dots, \Omega_m$, then ∂E must be stable in at least one of the Ω_i.

The same property is not exactly true in the nonlocal case. But we have an interesting replacement proven in [25]: If $\partial E \subset \mathbb{R}^n$ is a minimal surface with Morse index $\le m$, then given any $m+1$ open sets $\Omega_0, \Omega_1, \dots, \Omega_m$ such that $d(\Omega_i, \Omega_j) := \inf\{|x-y| \mid x \in \Omega_i,\ y \in \Omega_j\} \ge 1$ for $i \neq j$, then ∂E must be almost stable in at least one of the Ω_i. More precisely, for any vector field $X \in C_c^\infty(\Omega_i; \mathbb{R}^n)$ we have

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Per}_s(\psi^X(E,t), \Omega_i) \ge -C\left(\int_{\Omega_i} |X \cdot \nabla \chi_E|\right)^2 = -C\left(\int_{\partial^* E \cap \Omega_i} |X \cdot \nu| d\mathcal{H}^{n-1}\right)^2,$$

where C is a constant depending only on n, s, and m (converging to 0 as $s \uparrow 1$).

3.2 Classical finite index surfaces are intrinsically noncompact

Consider the sequence of diminishing catenoids:

$$\left\{x_1^2 + x_2^2 = \varepsilon_k^2 \cosh^2(x_3/\varepsilon_k)\right\} \subset \mathbb{R}^3 \qquad \text{with } \varepsilon_k \downarrow 0. \tag{3.1}$$

One might suggest they converge to the plane $\{x_3 = 0\}$. Actually, they approach a "double plane" due to each catenoid having two sheets nearing the plane from above and below.

We can "artificially" guarantee that sequences of minimal surfaces with uniformly bounded index and area will have converging subsequences, provided we choose a weak enough notion of convergence. However, as in the example of the catenoid, such weak convergence will unavoidably have undesired side effects. By enforcing sequences of minimal surfaces with bounded index and area to converge under a weak definition, we experience a loss of valuable structural information. (Sequences originally exhibiting rich, intricate configurations can diminish in complexity upon convergence, and result in trivial objects like a hyperplane.)

3.3 Enhanced regularity properties of finite index *s*-minimal surfaces

Finite index s-minimal surfaces enjoy much stronger a priori estimates and compactness properties than their classical counterparts. Indeed, in [25] we establish the following surprising estimate

Theorem 3.2 [25] *Suppose that ∂E is C^1 embedded s-minimal hypersurface with index $\le m$ in the unit ball $B_1 \subset \mathbb{R}^n$.*

Then,

$$\mathcal{H}^{n-1}(\partial^* E \cap B_{1/2}) = \mathrm{Per}(E, B_{1/2}) \le C = C(n, s, m)$$

(i.e., a constant depending only on n, s, and m).

In other words from a mere control on the Morse index of the s-minimal hypersurface ∂E, we obtain an a priori bound for the (classical!) perimeter of the set E inside $B_{1/2}$.

To correctly appreciate the significance of Theorem 3.2, let us recall that, by definition, $\mathrm{Per}_s(E)$ is the $W^{s,1}$ norm of the characteristic function of E. It is then natural to use the

$W^{s,1}$ norm to define a notion of strong converge of hypersurfaces: We say that $\partial E_k \to \partial E$ strongly as $k \to \infty$ if $[\chi_{E_k} - \chi_E]_{W^{s,1}} \to 0$. The convergence is strong in the sense that the continuity property $\mathrm{Per}_s(E) = \lim_{k\to\infty} \mathrm{Per}_s(E_k)$ is straightforward.

From this viewpoint, Theorem 3.2 gives the following: for any sequence ∂E_k of s-minimal surfaces all with index $\le m$ in B_1, then

$$[\chi_{E_k}]_{W^{1,1}(B_{1/2})} := \int_{B_{1/2}} |\nabla \chi_{E_k}| := \mathrm{Per}(E_k, B_{1/2}) \le C.$$

But since by classical Sobolev-type embeddings $W^{1,1}$ is compactly embedded in $W^{s,1}$ for $s < 1$ the sequence ∂E_k has a subsequence converging strongly inside $B_{1/2}$.

In simpler terms, s-minimal hypersurfaces with an index of m or less in $B_1 \subseteq \mathbb{R}^n$ demonstrate strong pre-compactness within $B_{1/2}$. (For a different perspective on this property, see Millot, Sire, and Wang's work [84]. They use Marstrand's theorem for compactness in sequences of s-minimal surfaces with uniform index and energy bounds. In our approach, these energy bounds are deduced from index bounds, removing the need for separate assumptions.)

In [25], we also establish strong curvature and separation estimates in low dimensions $3 \le n < n_s^\star$. (Although it is conjectured that $n_s^\star = 8$ for s close to 1, the best currently known estimate is that $n_s^\star$ must belong to $\{5, 6, 7, 8\}$; see Remark 3.10.)

Theorem 3.3 ([25]) *Suppose that ∂E is a C^1 embedded s-minimal surface with index $\le m$ in the unit ball $B_1 \subset \mathbb{R}^n$ and $0 \in \partial E$. Assume that $3 \le n < n_s^\star$ (recall Definition 2.3).*

Then, there is $r_\circ = r_\circ(n, s, m) > 0$ such that

- *$\Gamma := \partial E \cap B_{r_\circ}(0)$ is a hypersurface with exactly one connected component*
- *All principal curvatures of Γ are absolutely bounded by $1/r_\circ$.*

Remark 3.4 Theorem 3.3 (combined with the Schauder-type higher regularity estimates for s-minimal surfaces in [9]) implies that in low dimensions $n < n_s^\star$ any sequence ∂E_k of s-minimal surfaces, all with index $\le m$ in B_1 is pre-compact (inside $B_{1/2}$) in the strongest possible sense: that of C^∞ convergence of smooth submanifolds (without multiplicity).[7]

A good example of the striking consequences that Theorem 3.3 (combined with Theorem 3.9, presented later on) entails, is the following Bernstein-type result:

Corollary 3.5 *There exist $s_\circ \in (0, 1)$ such that if $\partial E \subset \mathbb{R}^4$, a submanifold of class C^1, is an s-minimal surface with finite index and $s \in (s_\circ, 1)$, then ∂E must be a hyperplane.*

Notice that this result is "too strong to be true" in the classical case $s = 1$: the catenoid or Costa's surface would be counterexamples. (There exist so-called "nonlocal catenoids", constructed in [42], but Corollary 3.5 shows they must have infinite index.)

In particular, we obtain that for $s \in (s_\circ, 1)$ stable s-minimal submanifolds of $\mathbb{R}^4$ must be hyperplanes. We underscore that in the classical case $s = 1$, this has only been established very recently in a groundbreaking paper of Chodosh and Li [32] (see also [24, 33]).

[7] Indeed thanks to such an estimate the surfaces can be locally written (in balls of radius $r_\circ/C$) as a single graphs with bounded $C^{1,1}$ norms. Then, by a simple application of the Arzelà-Ascoli theorem on these graphs we obtain their uniform convergence (up to subsequence). This convergence is automatically upgraded to C^k convergence for all $k \ge 1$ (up to subsequences) thanks to the estimates in [9].

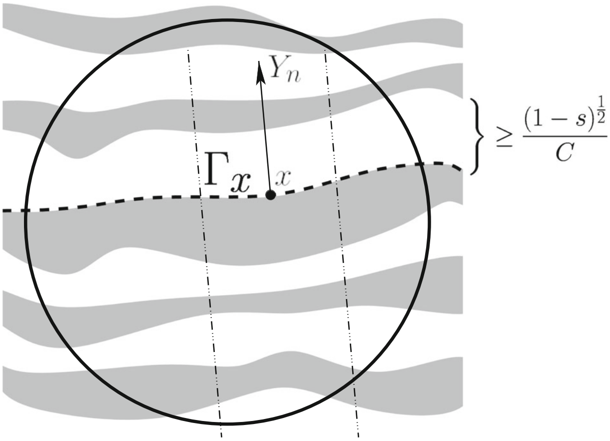

Fig. 3 Sketch of the results in Theorem 3.6

3.4 Optimal curvature and sheet separation estimates for stable surfaces as $s \uparrow 1$

In the previous section, we described remarkably strong estimates for finite index s-minimal surfaces. The constants in these estimates depend on s and degenerate as $s \uparrow 1$.

Therefore, a natural question arises: Is it possible to obtain some robust estimates as $s \uparrow 1$? Surprisingly the answer is affirmative. And, as we will explain later on, such robust estimates play a central role in some of the most striking applications of our results.

In the recent paper [31], Chan, Dipierro, the author, and Valdinoci obtain robust (as $s \uparrow 1$) $C^{2,\alpha}$ estimates and *optimal* sheet separation estimates for stable s-minimal surfaces in $\mathbb{R}^3$ (generalizations to 3-manifolds hold with similar proofs, but we will focus on $\mathbb{R}^3$ for simplicity). Our result reads as follows:

Theorem 3.6 ([31]) *There exist dimensional constants $s_* \in (0,1)$, $\alpha \in (0,1)$, $r_\circ \in (0,1)$, and $C > 0$ such that the following holds true when $s \in [s_*, 1)$.*

Let $E \subset \mathbb{R}^3$ and suppose that ∂E is a stable s-minimal set in B_1. Assume that $\partial E \cap B_1$ is a submanifold of class C^1 (a qualitative assumption) and let ν denote a unit normal vector field.

For any given $x \in \partial E \cap \overline{B_{1/2}}$, let Y_1, Y_2, Y_3 be an Euclidean coordinate system with origin at x and Y_3-axis pointing in the $\nu(x)$ direction.

Let also Γ_x be the connected component $\partial E \cap \{Y_1^2 + Y_2^2 + Y_3^2 \leq r_\circ^2\}$ containing x.

Then, there is $g : B'_{r_\circ} \to \mathbb{R}$ with

$$\|g\|_{C^{2,\alpha}(B'_{r_\circ})} \leq C, \tag{3.2}$$

where $B'_{r_\circ} \subset \mathbb{R}^2$ is the ball of radius $r_\circ$, such that $Y_3 = g(Y_1, Y_2)$ on Γ_x.

Moreover, the following "sheet separation estimate" holds:

$$\partial E \cap \left\{ Y_1^2 + Y_2^2 + Y_3^2 \leq r_\circ^2 \ \text{ and } \ \left|Y_3 - g(Y_1, Y_2)\right| \leq \frac{(1-s)^{\frac{1}{2}}}{C} \right\} = \Gamma_x. \tag{3.3}$$

A sketch of the result obtained in Theorem 3.6 is provided in Fig. 3. Roughly speaking, Theorem 3.6 provides two precious pieces of information:

1. The graph describing Γ_x enjoys a $C^{2,\alpha}$ estimate that is *robust* as $s \uparrow 1$.

2. While ∂E can consist of many sheets, other sheets must be *separated from* Γ_x *by, at least, a distance* $\frac{(1-s)^{\frac{1}{2}}}{C}$, for a *uniform* $C > 0$ as $s \uparrow 1$.

Remark 3.7 The exponent $\frac{1}{2}$ in the "sheet separation estimate" (3.3) is *sharp*. Indeed, if, for example, one defines

$$E^s := \bigcup_{k \in \mathbb{Z}} \left\{ 2k \leq \frac{x_n}{C_* \sqrt{1-s}} \leq 2k+1 \right\},$$

then, by symmetry, ∂E^s is an s-minimal set for all $s \in (0, 1)$. Moreover, ∂E^s is stable provided that s is sufficiently close to 1 and C_* is chosen large enough (independently of s); see [16, Remark 2.3] for details.

Remark 3.8 Theorem 3.6 can be regarded as the nonlocal minimal surface counterpart to the regularity theory for level sets of stable solutions of Allen Cahn, developed by Chodosh-Mantoulidis [34] and Wang-Wei [105].

3.5 Dávila-Del Pino-Wei system

As $s_k \uparrow 1$, multiple (even *infinitely many*) sheets of a sequence of s_k-minimal surfaces may converge towards *the same smooth minimal surface* (as explained in Remark 3.7, it is easy to construct examples).

When multiple sheets—at a critical distance of order $(1-s)^{\frac{1}{2}}$—are collapsing onto the same surface, crucial information on their nonlocal interactions "survives" in the limit $s \uparrow 1$. This information turns out to be encoded as a nontrivial solution of a certain (local!) PDE system of the following type:

$$\Delta_{\mathbb{R}^{n-1}} \widetilde{g}_i = 2 \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \frac{(-1)^{i-j}}{\widetilde{g}_j(x') - \widetilde{g}_i(x')}, \tag{3.4}$$

where $\widetilde{g}_1 < \widetilde{g}_2 < \cdots < \widetilde{g}_N$ are functions from $\mathbb{R}^{n-1}$ to $\mathbb{R}$.

The system (3.4) is analogous in many aspects to the Toda system (for Allen-Cahn), which plays a crucial role in [34, 105].

To the best of our knowledge, the system (3.4) was first considered in our context by Dávila, del Pino and Wei in [42], where embedded s-minimal surfaces with $N \geq 2$ layers (as well as s-catenoids, with $N = 2$) are constructed. This construction is done by perturbing a particular solution of (3.4), as $s \uparrow 1$. More precisely, the "Ansatz"

$$\widetilde{g}_i = a_i \widetilde{g}, \qquad i = 1, \ldots, N,$$

is made, where $\widetilde{g} : \mathbb{R}^{n-1} \to \mathbb{R}$ solves the Lane-Emden equation with negative exponent

$$\Delta_{\mathbb{R}^{n-1}} \widetilde{g} = \frac{1}{\widetilde{g}}, \tag{3.5}$$

and $a_i \in \mathbb{R}$ satisfy the balancing condition[8]

$$a_i = 2 \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \frac{(-1)^{i-j}}{a_j - a_i}, \qquad i = 1, \ldots, N. \tag{3.6}$$

[8] When $N = 2$ one may simply replace (3.6) with $a_1 = 1$ and $a_2 = -1$.

Coming back to (3.4), the formal computations from [42] strongly suggest that a system of the type (3.4) should be the "right model" for interactions between sheets of rather general s-minimal surfaces as $s \uparrow 1$. However, bounding the "errors" in the formal computations seems extremely delicate (in particular, it does not seem possible for general s-minimal surfaces).

In [31] we establish the following: Put $\Omega := B_1' \times (-1, 1) \subset \mathbb{R}^3$, where B_1' is the unit ball of $\mathbb{R}^2$. Suppose that ∂E_k is a sequence of stable s_k-minimal surfaces (in Ω) with $s_k \uparrow 1$. Assume that $\partial E_k \cap \Omega = \bigcup_{i=1}^{N_k} \Gamma_{k,i}$ where $\Gamma_{k,i} := \{x_3 = g_{k,i}(x')\}$, and $g_{k,1} < g_{k,2} < \cdots < g_{k,N_k}$. Suppose in addition that $\sup_{1\le i\le N_k} \|g_{k,i} - g\|_{L^\infty(B_1')} \to 0$ for some $g : B_1' \to (-1, 1)$. In other words, suppose that all the sheets of the surface $\Gamma_{k,i}$ are converging towards the same graphical surface $\Gamma := \{x_3 = g(x')\}$.[9] Then, under the above hypotheses, the following system of PDEs holds:

$$H[\Gamma_{k,i}](x) = 2\sqrt{\sigma_k}\left(\sum_{j>i}(-1)^{i-j}\frac{\sqrt{\sigma_k}}{d(x,\Gamma_{k,j})} - \sum_{j<i}(-1)^{i-j}\frac{\sqrt{\sigma_k}}{d(x,\Gamma_{k,j})}\right) + O\big((\sqrt{\sigma_k})^{1+\beta}\big), \tag{3.7}$$

where $x \in \Gamma_{k,i}$, $\sigma_k := (1 - s_k) \downarrow 0$ and $\beta > 0$. Here above, we have denoted by $H[\Gamma_{k,i}](x)$ the mean curvature of the surface $\Gamma_{k,i}$ at its point x, and by $d(x, \Gamma_{k,j})$ the distance in $\mathbb{R}^3$ between the point x and the surface $\Gamma_{k,j}$.

We stress that, since we know from the optimal sheet separation estimate of Theorem 3.6 that $d(x, \Gamma_{k,j}) \ge c\sqrt{\sigma_k}$, for some $c > 0$, the error term $O\big((\sqrt{\sigma_k})^{1+\beta}\big)$ in (3.7) is a genuine higher-order term.

We also remark that in the particular case in which N_k remains bounded, $g \equiv 0$ and

$$\widetilde{g}_{k,i} := \frac{g_{k,i}}{\sqrt{\sigma_k}} \to \widetilde{g}_i \in L^\infty(B_1') \quad \text{as } k \to +\infty,$$

the limit of (3.7) as $k \to \infty$ is (3.4). Similarly, when multiple sheets converge towards some non-planar minimal surface, then a system like (3.4) is obtained, with the only difference that $\Delta_{\mathbb{R}^{n-1}}$ is replaced by the Jacobi operator of the minimal surface.

Finally, let us also point out that the stability of ∂E can be rewritten (see [31]) in terms of solutions to the system (3.4). This stability condition is very important in applications, such as in the classification of stable s-minimal cones which we describe in the next section; see (3.9) below.

3.6 Classification of stable s-minimal cones in $\mathbb{R}^4$ for $s \sim 1$

In [31] we show that estimates as in Theorem 3.6 hold in dimension $n = 4$ if we assume that E has a conical structure, with singularity away from the region of interest. (E.g. under the assumption that $t(E - x_\circ) = E - x_\circ$ for all $t > 0$, for some $x_\circ$ with $|x_\circ| \ge 100$.)

This estimate plays a key role in establishing the following classification result for stable s-minimal hypercones in $\mathbb{R}^4$:

Theorem 3.9 [31] *There exists $s_\circ \in (0, 1)$ such that for every $s \in (s_\circ, 1)$ the following holds true. Let $E \subset \mathbb{R}^4$ be an s-minimal hypercone that is stable in $\mathbb{R}^4 \setminus \{0\}$. Suppose that ∂E is nonempty and has a smooth trace on $\mathbb{S}^3$. Then, ∂E must be a hyperplane.*

[9] We stress that, in this assumption, the multiplicity (number of sheets converging to Γ) could be infinite as well. We also observe that, by the estimates in Theorem 3.6, we may assume that $\|g_{k,i}\|_{C^{2,\alpha}(B_1')}$ is uniformly bounded for all k and i (and hence g is $C^{2,\alpha}$).

Remark 3.10 Notice that Theorem 3.9 implies $n_s^\star \geq 5$, if $s \in (0, 1)$ is sufficiently close to 1.

Why is the classification of stable s-minimal hypercones hard? Well, taking a naive approach, one would expect that sequences of stable s_k-minimal cones ∂E_k should converge, as $s_k \uparrow 1$, towards some stable minimal hypercone (hence a hyperplane if $n \leq 7$ using Almgren and Simons' result). However, turning this intuition into an actual mathematical proof poses several serious challenges; for instance:

- Nothing prevents, *a priori*, the sets $\partial E_k \cap \mathbb{S}^{n-1}$ from *converging towards some subset of* $\mathbb{S}^{n-1}$ *with infinite* $(n-2)$*-perimeter!*
- Even if we artificially added the assumption that $\sup_k \mathcal{H}^{n-2}(\partial E_k \cap \mathbb{S}^{n-1}) < \infty$ (and even if we managed to use this extra information to prove that ∂E_k should then converge towards a conical stable minimal integral varifold), the classification problem would remain far from trivial. Indeed, one would still need to rule out *multiplicity* of the limit surface: for instance $\partial E_k \cap \mathbb{S}^{n-1}$ could consist of *several connected components* all converging smoothly towards the "equator" of $\mathbb{S}^{n-1}$.

How can we establish the result? As said above, a version of Theorem 3.6 for conical hypersurfaces in $\mathbb{R}^4$ is a key ingredient. But we also crucially leverage the information provided by the Dávila-Del Pino-Wei system.

Let us explain the roadmap we follow in [31] to establish Theorem 3.9. We need to carefully analyze the behavior as $s \uparrow 1$ of the traces Σ of stable s-minimal cones $\partial E \subset \mathbb{R}^4$ on the sphere $\mathbb{S}^3$, that is:

$$\Sigma := \partial E \cap \mathbb{S}^3.$$

Step 1: Structure of trace on $\mathbb{S}^3$ of s-minimal cones with bounded second fundamental form. A key reduction is to show Σ is the union of graphs over the equator with maximal height $O(\sqrt{1-s})$. To do this, we first use a version applicable to Σ of the $C^{2,\alpha}$ estimate of Theorem 3.6, combined with a blow-up argument in the spirit of B. White, in order to prove that the second fundamental form of Σ must be bounded by a constant independent of s (as $s \uparrow 1$). This argument relies on the classification of complete embedded stable minimal surfaces in $\mathbb{R}^3$.

Then, we prove that any embedded C^2 almost-minimal surface in $\mathbb{S}^3$ with bounded second fundamental form (not necessarily stable) must be a union of graphs over some closed (and smooth) minimal surface in $\mathbb{S}^3$.

Using the stability of Σ, we then deduce that (provided s is sufficiently close to 1) Σ must be the union of graphs over the equator with maximal height $O(\sqrt{1-s})$. Thanks to the optimal separation estimate, we obtain, in addition, that the number of graphs in this union remains bounded as $s \uparrow 1$.

Step 2: Limiting system and stability inequality (this step works for $n \leq 7$). Once we know that, for s sufficiently close to 1, the trace Σ of a s-minimal cone on $\mathbb{S}^{n-1}$ is a union of graphs over its equator, $\bigcup_{i=1}^N \{x_n = g_i(\vartheta)\}$, with $\vartheta \in \mathbb{S}^{n-1} \cap \{x_n = 0\} \cong \mathbb{S}^{n-2}$, $|g_i| \leq C\sqrt{1-s}$, and $\|g_i\|_{C^{2,\alpha}} \leq C$ (by the previous step we know it when $n = 4$, but interestingly the last step actually works for $n \leq 7$), we can consider the re-normalized functions

$$\widetilde{g}_i := \frac{g_i}{\sqrt{1-s}}.$$

We prove that these functions $\tilde{g}_i$, up to small errors,

- solve the Dávila-Del Pino-Wei system;

- satisfy the stability condition.

These results can be stated respectively as:

$$\Delta_{\mathbb{S}^{n-2}}\widetilde{g}_i(\vartheta) + (n-2)\widetilde{g}_i(\vartheta) = 2\sum_{\substack{1\le j\le N\\ j\ne i}} \frac{(-1)^{i-j}}{\widetilde{g}_{i+1}(\vartheta)-\widetilde{g}_i(\vartheta)} + O\big((1-s)^{\gamma'}\big), \tag{3.8}$$

$$\frac{2}{N}\sum_{\substack{1\le i<j\le N\\ j-i \text{ odd}}} \int_{\mathbb{S}^{n-2}} \frac{4}{\big|\widetilde{g}_j(\vartheta)-\widetilde{g}_i(\vartheta)\big|^2}\, d\mathcal{H}^{n-2}_\vartheta \le \frac{(n-3)^2+\varepsilon}{4}\mathcal{H}^{n-2}\big(\mathbb{S}^{n-2}\big) + C_\varepsilon(1-s)^{\gamma'}. \tag{3.9}$$

The (small) parameter $\varepsilon > 0$ in stability condition and the "numerology" $\frac{(n-3)^2}{4}$ comes from the fact that we choose a radial test function similar to $r^{-\frac{(n-1)-2}{2}}$ which almost saturates Hardy's inequality in $\mathbb{R}^{n-1}$.

Step 3: Final contradiction. By testing the difference of two suitably chosen equations in (3.8) against the reciprocal of the sheet separation and integrating by parts, and comparing with (3.9) one obtains

$$(n-2) \le \frac{(n-3)^2}{4},$$

which leads to a contradiction if $3 \le n \le 7$. We point out that this dimensional range, which we believe to be optimal, is the same as in Simons' paper [99].

4 *s*-Minimal surfaces in Riemannian manifolds and applications

4.1 Nonlocal minimal (hyper)surfaces in a closed Riemannian manifold

In the paper [25], Caselli, Florit-Simon, and the author introduce nonlocal minimal (hyper)surfaces on closed[10] Riemannian manifolds. Let us recall the definition here, emphasizing the "canonical nature" of these new geometric objects. We refer to [25, 26] for further deteails.

Let (M^n, g) be an n-dimensional, closed Riemannian manifold, with $n \ge 2$. We will use dV to denote the standard volume measure on M. Let us start by giving a canonical definition of the fractional Sobolev seminorm $H^{s/2}(M)$. This can be done in at least three equivalent ways:

(i) Using the *heat kernel*[11] $H_M(t, p, q)$ of M, we can put

$$K_s(p,q) := \int_0^\infty H_M(p,q,t)\,\frac{dt}{t^{1+s/2}}. \tag{4.1}$$

We then define

$$[u]^2_{H^{s/2}(M)} := \iint_{M\times M} (u(p)-u(q))^2 K_s(p,q)\, dV_p\, dV_q. \tag{4.2}$$

[10] I.e., compact, without boundary.

[11] I.e., the fundamental solution of $\partial_t u = \Delta u$ on M, where Δ denotes the Laplace-Beltrami operator on M.

(ii) Following a *spectral approach*, we can set

$$[u]^2_{H^{s/2}(M)} = \sum_{k\geq 1} \lambda_k^{s/2} \langle u, \varphi_k \rangle^2_{L^2(M)} \tag{4.3}$$

where $\{\varphi_k\}_k$ is an orthonormal basis of eigenfunctions of the Laplace-Beltrami operator $(-\Delta_g)$ and $\{\lambda_k\}_k$ are the corresponding eigenvalues. For $s = 2$ this gives the usual $[u]^2_{H^1(M)}$ seminorm.

(iii) Considering a *Caffarelli-Silvestre type extension (cf.* [8, 20]*)*, namely, a degenerate-harmonic extension problem in one extra dimension, we can set

$$[u]^2_{H^{s/2}(M)} = \inf\left\{ \int_{M\times\mathbb{R}_+} z^{1-s} \left|\widetilde{\nabla} U(p,z)\right|^2 dV_p dz \quad \text{s.t.} \quad U(\cdot, 0^+) = u \right\}.$$

Here $\widetilde{\nabla}$ denotes the Riemannian gradient of the manifold $\widetilde{M} = M \times \mathbb{R}_+$, with respect to the natural product metric $\widetilde{g} = g + dz \otimes dz$, and the infimum is taken over all U belonging to the weighted Hilbert space $H^1(\widetilde{M})$ with weight z^{1-s} (see [25] for details).

It turns out that (i)–(iii) define the same norm (not merely equivalent norms), up to explicit multiplicative constants; see [26].

With this canonical $H^{s/2}$ seminorm at hand, we can define fractional perimeters on M.

Definition 4.1 Given $s \in (0, 1)$ and a (measurable) set $E \subset M$, we define the *s-perimeter of* E as

$$\mathrm{Per}_s(E) := [\chi_E]^2_{H^{s/2}(M)} = \frac{1}{4}[\chi_E - \chi_{E^c}]^2_{H^{s/2}(M)} = 2\int_E \int_{E^c} K_s(p,q) dV_p dV_q, \tag{4.4}$$

where χ_E is the characteristic function of E, $E^c := M \backslash E$ and $[\cdot]^2_{H^{s/2}(M)}$ is defined by (4.2).

Remark 4.2 Notice that in $\mathbb{R}^n$ we defined $\mathrm{Per}_s(E)$ as $[\chi]_{W^{1,s}(\mathbb{R}^n)}$ (up to a multiplicative constant). In a closed manifold, instead, it is more natural and canonic to work on the reflexive space $H^{s/2} = W^{s/2,2}$. Notice that in $\mathbb{R}^n$ however the $W^{s,1}$ norm is the same (up to multiplicative constant) as the $W^{s/p,p}$ norm for all $p \in [1, \infty)$:

From the estimates in [26] for the kernel $K_s(p,q)$, one can see that for every set $E \subset M$ with smooth boundary, one has that $(1-s)\mathrm{Per}_s(E) \to \mathrm{Per}(E)$ as $s \uparrow 1$ (up to a multiplicative dimensional constant, see [15] and also [7, 23, 41] for further details on the computation in the case of $\mathbb{R}^n$).

We have now all the necessary prerequisites to define nonlocal minimal surfaces on a closed manifold.

Definition 4.3 Consider a set $E \subset M$ with finite $\mathrm{Per}_s(E, M)$. We say that E has *s-minimal boundary* in M if for every smooth variation ψ[12] the function

$$t \mapsto \mathrm{Per}_s(\psi(E,t), \Omega), \tag{4.5}$$

has a critical point (i.e. has zero derivative) at $t = 0$. We then say that ∂E is an *s-minimal hypersurface*.

If in addition (2.1) has a nonnegative second derivative at $t = 0$ (for every ψ) we say that ∂E is a *stable s-minimal hypersurface*. (*Technical note:* the map (2.1) is infinitely differentiable near $t = 0$, as proven in [26].)

[12] Namely, $\psi : M \times (-1, 1) \to \mathbb{R}^n$ smooth and satisfying $\psi(p, 0) = p$ for all $p \in M$.

4.2 The fractional Allen-Cahn equation

Let $s \in (0,2)$ and $\varepsilon > 0$. Given $u : M \to \mathbb{R}$, we define the *fractional Allen-Cahn energy* of u on M as

$$\mathcal{E}_s(u) := \frac{1}{4} \iint_{M\times M} (u(p)-u(q))^2 K_s(p,q) dV_p dV_q + \varepsilon^{-s} \int_M W(u)\, dV, \tag{4.6}$$

where $W(u) = \frac{1}{4}(1-u^2)^2$ is the standard quartic double-well potential with wells at ± 1.

The double-well potential penalizes functions that are not identical to ± 1, and that is why one expects to find nonlocal s-minimal surfaces as the limits of critical points of this energy when $\varepsilon \to 0$.

A function $u : M \to \mathbb{R}$ is a critical point of $\mathcal{E}_s$ if and only if it solves the fractional Allen-Cahn equation

$$(-\Delta)^{s/2} u + \varepsilon^{-s} W'(u) = 0 \quad \text{in } \Omega. \tag{4.7}$$

Here $(-\Delta)^{s/2}$ is the fractional Laplacian on (M, g), and it can be represented as (see [26])

$$(-\Delta)^{s/2} u(p) = \int_M (u(p)-u(q)) K_s(p,q)\, dV_q. \tag{4.8}$$

We also have a definition of Morse index, related to the second variation of the energy.

Proposition 4.4 (Second variation [25]) *Let $u \in H^{s/2}(M)$ be a critical point of $\mathcal{E}_\Omega$. Then, given $\xi \in C^1(M)$, the second variation of $\mathcal{E}_\Omega$ at u is given by*

$$\mathcal{E}''_\Omega(u)[\xi,\xi] = \frac{1}{4} \iint_{M\times M} |\xi(p)-\xi(q)|^2 K_s(p,q)\, dV_p\, dV_q + \varepsilon^{-s} \int_M W''(u)\xi^2\, dV\,. \tag{4.9}$$

Definition 4.5 (*Morse index*) Let $u \in H^{s/2}(M)$ be a critical point of $\mathcal{E}_s$. The *Morse index* of u is defined as the maximum dimension $m \geq 0$ among all linear subspaces $\mathcal{L} \subset C^1_c(\Omega) \subset H^{s/2}(M)$ such that $\mathcal{E}''_\Omega(u)$ is negative definite on $\mathcal{L}$. Moreover, we say that u is *stable* if $m = 0$.

In the paper [25] (see also [16] for the case of stable solutions in domains of $\mathbb{R}^n$) we precisely establish the convergence, as $\varepsilon \to 0$, of finite index solutions of (4.7) to a limit s-minimal surface. We prove the following

Theorem 4.6 [16, 25] *(**Convergence as** $\varepsilon \to 0^+$). Fix $s \in (0,1)$. Let u_{ε_j} be a sequence of solutions of (4.7) on M with parameters $\varepsilon_j \to 0$ and Morse index $\leq m$. Then, there exist a subsequence, still denoted by u_{ε_j}, and a nonlocal s-minimal surface $\Sigma = \partial E$ with Morse index $\leq m$ such that*

$$u_{\varepsilon_j} \xrightarrow{H^{s/2}} u_0 = \chi_E - \chi_{E^c}.$$

In addition, up to changing E in a set of measure zero, we have

$$\mathrm{int}(E) \supseteq \Big\{ p \in M \ \liminf_{r\downarrow 0} \tfrac{|E\cap B_r(p)|}{|B_r(p)|} = 1 \Big\}, \tag{4.10}$$

$$M \setminus \overline{E} \supseteq \Big\{ p \in M \ \limsup_{r\downarrow 0} \tfrac{|E\cap B_r(p)|}{|B_r(p)|} = 0 \Big\}, \tag{4.11}$$

$$\partial E = \Big\{ p \in M : \tfrac{|E\cap B_r(p)|}{|B_r(p)|} \in [c, 1-c] \quad \forall r \in (0, r_\circ(p)), \quad \text{for some } r_\circ(p) > 0 \Big\}, \tag{4.12}$$

and ∂E is the standard topological boundary of E. Also, for all given $c \in (-1, 1)$

$$d_{\mathcal{H}}(\{u_{\varepsilon_j} \geq c\}, E) \to 0, \quad \text{as } j \to \infty, \tag{4.13}$$

where $d_{\mathcal{H}}(X, Y) = \inf\{\rho > 0 : X \subseteq \bigcup_{y \in Y} B_\rho(y)$ and $Y \subseteq \bigcup_{x \in X} B_\rho(x)\}$ is Hausdorff distance.

We also have the following regularity theorem for the interfaces in low dimensions.

Theorem 4.7 [25] *With the same assumptions as in Theorem 4.6. In dimension $3 \leq n < n_s^*$ the limit s-minimal surface ∂E is a smooth submanifold (with estimates)*

Remark 4.8 A straightforward consequence of Theorems 4.6 and 4.7, the interfaces of stable, or even finite index, nonlocal Allen-Cahn with parameter $\varepsilon \ll 1$ behave like s-minimal surfaces on a macroscopic scale.

Related to the previous remark, in [25] we answer positively the analog to Conjecture 1.1:

Theorem 4.9 *(*Finite index nonlocal De Giorgi-type result *[25]) Let $s \in (0, 1)$ and $3 \leq n < n_s^*$, where n_s^* is the critical dimension (see Definition 2.3).*

Then, every finite Morse index solution u of $(-\Delta)^{s/2}u + W'(u) = 0$ in $\mathbb{R}^n$ is a 1D layer solution, namely, $u(x) = \phi(e \cdot x)$ for some $e \in \mathbb{S}^{n-1}$ and increasing function $\phi : \mathbb{R} \to (-1, 1)$.

Under the assumption of stability (Morse index zero) the previous theorem was established in [16] and for minimizers it followed from [19, 48].

The De Giorgi conjecture is a famous related statement about certain entire solutions to the Allen-Cahn equation being one-dimensional, or equivalently about their level sets being hyperplanes in low dimensions — see [6, 53, 64, 95, 96], to cite a few.

4.3 Application 1: Yau-type result

The existence and regularity of minimal hypersurfaces in closed manifolds is one of the central questions in Riemannian geometry. Yau's conjecture (raised in 1982 by S.-T. Yau [110]) is a particularly famous and influential problem. It states that every closed three-dimensional manifold must contain infinitely many smooth, closed, minimal surfaces. This problem exposed the enormous difficulties in applying variational methods to the area functional, defined on the class of "surfaces".

Building on the work of Almgren [3–5] and Pitts [89], Yau's conjecture was first established by Irie, Marques, and Neves [73] (in the case of generic metrics) and by Song [101] (in full generality) —see also [77–80, 98], to cite a few.

We have the following:

Theorem 4.10 [73, 101] *Let (M^n, g) be a closed Riemannian manifold of dimension $3 \leq n \leq 7$. Then, there exists an infinite number of smooth, closed, minimal hypersurfaces in M.*

To construct an infinite number of minimal surfaces, one must consider non-stable critical points of the area functional (since generic closed 3-manifolds contain only a finite number of stable minimal surfaces). These surfaces are naturally constructed using variational min-max (i.e., mountain-pass type) methods. Several critical difficulties arise when employing a min-max scheme in this setup. Indeed, in an optimistic best-case scenario, min-max sequences will behave as finite index minimal surface critical points; an intrinsically non-compact class of surfaces—as we discussed Subsection 3.2.

In the paper [25] we leverage the better properties of finite index s-minimal surfaces (Theorem 4.6) to establish the following nonlocal analog of Yau's conjecture.

Theorem 4.11 Yau-type result *Let (M^n, g) be an n-dimensional, closed Riemannian manifold, with $n \geq 2$. Then, for every natural number $\mathfrak{p} \geq 1$, there exists an s-minimal surface $\partial E^{\mathfrak{p}}$ with index $\leq \mathfrak{p}$ and energy bounds*

$$C^{-1}\mathfrak{p}^{s/n} \leq (1-s)\mathrm{Per}_s(E^{\mathfrak{p}}) \leq C\mathfrak{p}^{s/n}, \tag{4.14}$$

for some $C = C(M) > 1$. In particular, M contains infinitely many s-minimal surfaces.

Morever, for $n \geq 3$, the surfaces $\Sigma^{\mathfrak{p}}$ are smooth submanifolds outside of a closed set $\mathrm{sing}(\Sigma^{\mathfrak{p}})$ of Hausdorff dimension at most $n - n_s^\star$. In particular, $\mathrm{sing}(\Sigma^{\mathfrak{p}}) = \varnothing$ if $n < n_s^\star$. Moreover, in the case $n = n_s^\star$ the set $\mathrm{sing}(\Sigma^{\mathfrak{p}})$ is discrete.

To establish this result, we employ the min-max method on the fractional Allen-Cahn equation, drawing inspiration from [59–62]. We then proceed to take the limit as ε approaches 0. In the scenario where $s = 1$ (the classical case), this transition to the limit is extremely delicate. As of now, it has only been accomplished in 3 dimensions and with a bumpy metric prerequisite in [34]. However, our refined estimates for finite Morse index critical points with $s < 1$ allow us to confidently take the limit as ε approaches zero to prove our result.

It's pivotal to note that once we've constructed an infinite number of s-minimal surfaces in our designated compact manifold M, we can transition to $s \uparrow 1$ to identify classical minimal surfaces. This strategy is feasible due to our comprehensive estimates outlined in Subsections 3.4 and 3.5. We term this the"nonlocal approximation method". Currently, we are delving deeper into its extensive implications for the analysis of classical minimal surfaces.

4.4 Application 2: nonlocal Plateau problem

Given a closed (smooth, regular) curve in $\Gamma \subset \mathbb{R}^3$, can one find a surface S with the least area "spanning" Γ (e.g. with boundary ∂S equal to Γ)?

Raised by Joseph-Louis Lagrange in the 1760s, this problem is one of the most classical and influential ones in the Calculus of Variations and Geometry. It is named after the Belgian physicist J. Plateau (1801–1883), who conducted experiments with soap films and bubbles postulated their geometric laws. By the effect of surface tension, soap films are natural examples of area-minimizing surfaces.

A key modeling challenge that hampered progress in Plateau's problem for two centuries is the incongruence between observed physical soap films and classical surfaces as defined in differential geometry. While these films typically present as finite combinations of smooth surface segments, they can also exhibit singularities, such as when three distinct surface pieces meet at equal angles along a curve.

It's essential to operate within suitable classes of generalized "surfaces" that encompass the diverse structures observed in soap films. The chosen class should not only include these structures but also permit a solution to the area minimization problem. Furthermore, one needs to formalize what "S spans Γ" means in rigorous mathematical terms.

Between the 1930s and the 1970s, outstanding analysts and geometers, including Almgren, De Giorgi, Douglas, Federer, Fleming, Radó, Reifenberg, and Taylor, among others, addressed Plateau's problem and built its modern theory; see, for instance [2, 3, 43, 44, 49, 57, 91, 92, 94, 103]. Thanks to these intensive efforts, Plateau's problem is today well understood. (For up-to-date information on the problem, see, for instance, the excellent work [46] and references therein.)

Given the intriguing applications of s-minimal surfaces in closed manifolds, one is prompted to question: Does a natural nonlocal variant of Plateau's problem exist? Indeed, there is an affirmative response, and its foundation significantly rests on the established concept of nonlocal minimal surfaces in Riemannian manifolds

We will shortly delineate the foundational concept, which will be elaborated upon and expanded in our joint work with M. Guaraco [68]. The crux of the matter lies in determining if there is a natural notion of nonlocal minimal surface spanning a specified closed curve in $\mathbb{R}^3$.

A compelling solution comes from the covering space method, introduced in [10] for the classical Plateau problem (drawing inspiration from [12]). For clarity, we'll outline this approach in its most straightforward scenario: when the presented contour Γ is a smooth, closed, unknotted, embedded curve.

Consider the Riemannian manifold $M = \mathbb{R}^3 \setminus \Gamma$, which is not simply connected due to its fundamental group being $\mathbb{Z}$. The Riemannian double cover of M can be denoted as $\pi : \widetilde{M} \to M$.

One way to conceptualize $\widetilde{M}$ is through the following "cut and paste" model:

(1) Start by selecting a smooth embedded surface $S_\circ$ that is diffeomorphic to a disk and has Γ as its boundary.
(2) Next, choose a smooth unit normal $\nu : S_\circ \to \mathbb{S}^2$ for $S_\circ$. Given there are two possible unit normals, fix one of them. This establishes a sense of "above" and "below" relative to $S_\circ$.
(3) Make two copies: M_1 and M_2 of $\mathbb{R}^3 \setminus S_\circ$. These will be joined with two copies of $S_\circ$ to construct $\widetilde{M}$.
(4) When piecing together M_1 and M_2, ensure that geodesic paths (i.e., straight lines) in M_1 which approach $S_\circ$ from above and would intersect it transversally, are transitioned into M_2, continuing straight but departing from $S_\circ$ from below. A symmetrical procedure applies for trajectories in M_2 moving towards $S_\circ$ from below.

In essence, this construction ensures that as trajectories approach $S_\circ$ from one side in one copy, they continue seamlessly on the other side in the other copy, making $\widetilde{M}$ a double cover of M. In this construction, it is clear what the projection $\pi : \widetilde{M} \to M$ is. Needless to say, the covering space $(\widetilde{M}, \pi)$ is independent of the choice of $S_\circ$ (up to isomorphism).

There is then a nontrivial deck transformation γ (i.e. an isometry of $\widetilde{M}$ different from the identity such that $\pi = \pi \circ \gamma$) such that $\gamma^2 = \mathrm{id}$. In other words γ is the generator of the group of deck transformations, that is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Then we can formulate the *nonlocal Plateau problem* in $\mathbb{R}^3$ for the contour Γ as follows:

$$\text{Minimize } \mathrm{Per}_s(E) \text{ among sets } E \subset \widetilde{M} \text{ that are } equivariant: \text{ i.e. } \gamma(E) = \widetilde{M} \setminus E.$$

The existence of a minimizer E_* can be easily proven by the direct method, while equivariance guarantees that this minimizer is nontrivial and "spans" Γ. We can then see (using our regularity theory that applies in particular to minimizers of Per_s is a Riemannian manifold) that the boundary $\partial E_* \subset \widetilde{M}$ is smooth ("away from Γ") and that the surface $S_* := \pi(\partial E_*)$ is a smooth submanifold of $\mathbb{R}^3$ that has Γ as boundary. Arguably, S_* is the canonical solution of the nonlocal Plateau problem for the parameter $s \in (0, 1)$.

Of course, the solutions of the nonlocal Plateau problem will converge to solutions to the classical Plateau problem as $s \uparrow 1$.

5 Future directions and open questions

5.1 Higher codimension *s*-minimal surfaces

Let M^n be either a closed and oriented n-dimensional manifold or $\mathbb{R}^n$ (for simplicity). Suppose that $\Gamma \subset M^n$ is a oriented co-dimension $k+1$ submanifold ($k \geq 0$) given by $\Gamma = F^{-1}(y)$, where $F : M \to \mathbb{R}^{k+1}$ is a smooth map and y is a regular value of F. Let $d := n - k - 1$ denote the dimension of Γ. We may assume without loss of generality that $y = 0$.

The map $u_\circ := F/|F|$ is a smooth map from $M \backslash \Gamma \to \mathbb{S}^k$. Moreover, since 0 is a regular value of F around each point $p \in \Gamma$ there exists a chart $\varphi_p : U_p \to V \subset \mathbb{R}^n$, where U_p and V are an open neigborhoods of p and 0 respectively, such that:

$$\varphi_p^i = F^i \quad \text{for } 1 \leq i \leq k+1.$$

Let now $u : M \setminus \Gamma \to \mathbb{S}^k$ be a smooth map (for example $u = u_\circ$). For any embedded $(k+1)$-dimensional oriented disk $D \subset M$ with boundary ∂D (diffeomorphic to $\mathbb{S}^k$ and consistently oriented), we can consider the degree $\deg(u, \partial D) := \deg(u|_{\partial D})$ of the map

$$u|_{\partial D} : \partial D \to \mathbb{S}^k.$$

Recall that this is a homotopy invariant. (If $k = 0$, there are four maps $\{-1, 1\} \to \{-1, 1\}$, and we set the $\deg(1 \mapsto 1, -1 \mapsto -1) = 1$, $\deg(1 \mapsto -1, -1 \mapsto 1) = -1$ and $\deg = 0$ for the other two maps, which are constant.)

Now for $s \in (0, 1)$ let us define the fractional d-dimensional volume of Γ as

$$V_{d,s}(\Gamma) = \inf \left\{ [u]^2_{H^{\frac{k+s}{2}}(M)} \;\middle|\; u : M \setminus \Gamma \to \mathbb{S}^k \text{ smooth and } \textit{linked} \text{ with } \Gamma \right\},$$

where we say that u is *linked* with Γ if the following holds:

(a) Whenever $\varphi : U \to V \subset \mathbb{R}^n$ is a positive submanifold chart of Γ around one of its points q such that $\varphi|_\Gamma$ is also positive.[13] Then for tiny $(k+1)$-disks intersecting Γ transversally of the form

$$D = \varphi^{-1}\big\{(x^1)^2 + \cdots + (x^{k+1})^2 < \delta^2, x^{k+2} = \cdots = x^n = 0\big\}$$

we have $\deg(u, \partial D) = +1$.

(b) For every embedded disk D not intersecting Γ we have $\deg(u, \partial D) = 0$.

We notice that the set of $u : M \setminus \Gamma \to \mathbb{S}^k$ smooth and linked with Γ is nonempty as $u_\circ$, perhaps composed with some reflection, belongs to it.

We also remark that in $\mathbb{R}^n$ the seminorm $H^{\frac{k+s}{2}}$ seminorm has a standard definition, while in a close manifold M we can canonically define it at least in three equivalent ways corresponding to the natural modifications of (i), (ii), (iii) in Sect. 4.1 (see [109] for higher order Caffarelli-Silvestre extensions).

We believe that the previous definition gives a very natural and interesting notion of fractional higher codimension volumes. Notice that for $d = n - 1$, we have $\Gamma = \{F = 0\} = \partial\{F > 0\}$ for some $F : M \to \mathbb{R}$. Also, $V_{d,s}(\Gamma) = 4\mathrm{Per}_s(\{F > 0\})$, so we recover exacly the the fractional perimeter. In the same way that $\mathrm{Per}_s(\{F > 0\})$ appropriately renormalized

[13] More precisely: φ is positively oriented diffeomorphism mapping an open set $U \subset M$ containing q to an open set $W \subset \mathbb{R}^n$. It satisfies $\varphi(\Gamma) = W \cap \{x^1 = \cdots = x^{k+1} = 0\}$. And the map obtained taking the last m components of φ restricted to Γ is also positive.

converges to $\mathrm{Per}(\{F > 0\})$, we expect that $V_{d,s}(\Gamma)$, appropriately renormalized, will give d-dimensional volume of Γ; we refere to [27] for a detailed analysis in the codimension two case ($k = 1$).

It's intriguing to consider d-surfaces Γ which are critical points of $V_{d,s}$, viewing them as "higher codimension s-minimal surfaces". These can be interpreted as the singular sets of fractional harmonic maps into spheres.

Introduced by Da Lio and Rivière [38–40], fractional harmonic maps boast outstanding mathematical features, including a monotonicity formula. For more on this, see [82, 83] and related works. We anticipate that their singular sets will enjoy Allard-type regularity results and showcase properties similar to minimal surfaces. Moreover, compared to their local counterparts, they're likely to exhibit enhanced regularity and compactness-much like the codimension one scenario. This is a topic we're keen on exploring in upcoming research.

Let us highlight the following questions that seem to us particularly interesting and that would have striking applications:

Open problem 1. Suppose that $\Gamma \subset \mathbb{R}^n$ is a local minimizer $V_{d,s}$ inside of Ω (similar definition of localized $V_{d,s}$ to the ones we gave for Per and Per_s). Is Γ analytic outside of a $(d-2)$-dimensional set of points?

Open problem 2. Let M^n be a closed Riemannian manifold. Suppose that $\Gamma \subset M^n$ is a critical point of $V_{d,s}$ with Morse index $\leq m$ (similar definition to the ones we gave for Per and Per_s). Can one bound the (classical) d-dimensional volume of Γ by a constant depending only on M, s, and m ?

Open problem 3. Do there exist infinitely many (finite Morse index) d-dimensional s-minimal surfaces (namely, critical points of $V_{d,s}$) in every closed Riemannian manifold? If $d \geq 2$ are they smooth outside of a $(d-2)$-dimensional set?

5.2 Four *s*-minimal spheres in every $(\mathbb{S}^3, g)$?

A very delicate question in min-max constructions of (classical) minimal surfaces is the control of their topology (e.g. when we want construct minimal spheres in a given manifold); see Simon-Smith [100], Haslhofer-Ketover [72] and Wang-Zhou [106]. In this direction, it would be very interesting to see to what extent the strong a priori estimates for nonlocal minimal surfaces (in low dimension) help in the control of their topology. Let us highlight the following interesting question, which is the nonlocal version of a well-known open problem of S.-T. Yau:

Open problem 4. Let M be a closed Riemannian 3-manifold diffeomorphic to $\mathbb{S}^3$. Can one always find four different s-minimal spheres in M?

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