A STRICT MAXIMUM PRINCIPLE FOR NONLOCAL MINIMAL SURFACES

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ABSTRACT. In the setting of fractional minimal surfaces, we prove that if two nonlocal minimal sets are one included in the other and share a common boundary point, then they must necessarily coincide.

This strict maximum principle is not obvious, since the surfaces may touch at an irregular point, therefore a suitable blow-up analysis must be combined with a bespoke regularity theory to obtain this result.

For the classical case, an analogous result was proved by Leon Simon. Our proof also relies on a Harnack Inequality for nonlocal minimal surfaces that has been recently introduced by Xavier Cabré and Matteo Cozzi and which can be seen as a fractional counterpart of a classical result by Enrico Bombieri and Enrico Giusti.

In our setting, an additional difficulty comes from the analysis of the corresponding nonlocal integral equation on a hypersurface, which presents a remainder whose sign and fine properties need to be carefully addressed.

1. Introduction

The maximum principle is a fundamental tool to study geometric properties of minimal hypersurfaces. Roughly speaking, it says that if two sets with smooth boundaries are included one into the other and touch at some point, then the mean curvature of the inner set at the contact point is larger than that of the outer set.

The strict maximum principle aims at imposing a separation between surfaces with the same mean curvature. While this property is rather straightforward for smooth hypersurfaces, it becomes much more delicate in the presence of singularities. For classical minimal surfaces, such a strong maximum principle has been established by Leon Simon in the celebrated article [Sim87], also relying on a Harnack Inequality for minimal surfaces which was put forth by Enrico Bombieri and Enrico Giusti in [BG72].

Related results were put forth in [Mos77]. See also [SW89] for a strict maximum principle for stationary integer rectifiable varifolds in which one of them is smooth and [Ilm96] for

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stationary hypersurfaces under suitable assumptions and relying on an "extrinsic" Harnack inequality.

The goal of this article is to establish a strict maximum principle in the framework of nonlocal minimal surfaces. This result can be seen as a fractional counterpart of the main result in [Sim87] and also leverages a recent Harnack Inequality for nonlocal minimal surfaces introduced by Xavier Cabré and Matteo Cozzi in [CC19].

The setting that we consider is that of nonlocal minimal surfaces, as introduced in [CRS10]. Namely, given a (bounded and Lipschitz) domain $\Omega \subset \mathbb{R}^n$ and a (measurable) set $E \subseteq \mathbb{R}^n$, we say that E is s-minimal in Ω if, whenever a (measurable) set F coincides with E outside Ω , it holds that

$$\operatorname{Per}_s(E,\Omega) \leqslant \operatorname{Per}_s(F,\Omega),$$

where

$$\operatorname{Per}_{s}(E,\Omega) := \iint_{(E\cap\Omega)\times(E^{c}\cap\Omega)} \frac{dx\,dy}{|x-y|^{n+s}} + \iint_{(E\cap\Omega)\times(E^{c}\cap\Omega^{c})} \frac{dx\,dy}{|x-y|^{n+s}} + \iint_{(E\cap\Omega^{c})\times(E^{c}\cap\Omega)} \frac{dx\,dy}{|x-y|^{n+s}}.$$

Here above and in what follows, $s \in (0,1)$ is a fractional parameter and the superscript "c" denotes the complement set in \mathbb{R}^n .

Nonlocal minimal surfaces are a fascinating, and very difficult topic of investigation. While their regularity is known in the plane (see [SV13]) and up to dimension 7 when s is close to 1 (see [CV13]), the full regularity theory of these objects is not well-understood. Though no examples of singular sets are presently known, it is commonly believed that nonlocal minimal surfaces do develop singularities, therefore, for the validity of a strict maximum principle, in general it does not suffice to take into consideration only regular points.

In this setting, our main result is the following:

Theorem 1.1. Let r > 0 and $p \in \mathbb{R}^n$. Let E_1 , E_2 be s-minimal sets in $B_r(p)$, with $E_1 \subseteq E_2$. Assume that $p \in (\partial E_1) \cap (\partial E_2)$.

Then, $E_1 = E_2$.

We recall that a better understanding of nonlocal minimal surfaces is important also in connection with hybrid heat equations (see [CS10]), geometric motions (see [Imb09, CMP15, CNR17, CSV18, SV19, JLM20, CSV20]), nonlocal soap bubble problems (see [CFMN18, CF-SMW18]), capillarity problems (see [MV17, DMV22]), long-range phase coexistence models (see [SV12, DV23]), etc.

Interestingly, the nonlocal perimeter functional interpolate the classical perimeter as $s \nearrow 1$ (see [BBM02, Dáv02, CV11, ADPM11]) with a suitably weighted Lebesgue measure as $s \searrow 0$ (see [MS02, DFPV13]). In this spirit, on the long run, it is expected that nonlocal minimal surfaces can serve as auxiliary tools to better understand classical minimal surfaces as well (see [CCS20, CDSV, CFS]) and can provide new perspectives on classical geometric objects by bringing forth tools alternative to, and different than, differential geometry (see [DTV23]).

For other types of nonlocal minimal surfaces in which the interaction kernel is of integrable type, see [MRT19].

The rest of this paper focuses on the proof of Theorem 1.1. Roughly speaking is that, by a blow-up and dimensional reduction argument, one reduces to the case in which the two s-minimal sets under consideration touch at a point which exhibit the same limit cone for both surfaces.

One then writes a geometric equation coming from the vanishing of the nonlocal mean curvature of one of these surfaces and a suitable translation of the other one and then scales to have a normalized picture in which the separation between these two sheets is of order one (assuming, for a contradiction, that they do not coincide to start with).

Some caveat is in place, since one has to switch from the notion of solution in the smooth or viscosity sense to the one in the weak sense: for this, some energy and capacity estimates will be required.

In this scenario, after a limit procedure, one aims at applying a suitable Harnack Inequality to obtain a contradiction between a linear separation close to the origin and the assumption that the two surfaces share the same¹ tangent cone.

However, a number of difficulties emerge, since the equations under consideration present some complicated kernels, are not defined everywhere, and produce a remainder which needs to be specifically analyzed (in particular, we will need to check that this remainder is bounded and has a sign, to be able to infer uniform Hölder estimates on the normalized oscillation, pass them to the limit, obtain a limit inequality, and apply to it an appropriate version of the Harnack Inequality).

Interestingly, the regularity theory utilized in this process is twofold: first we use Hölder estimates for a nonlocal equation which is not symmetric to perform a compactness argument on a sequence of normalized solutions which encode the distance between the original surfaces and thus obtain a positive supersolution on the limit cone for a fractional Jacobi operator, then we apply to this setting the geometric Harnack Inequality to obtain the desired conclusion.

A slightly more detailed and technical sketch of the proof will be presented in the forth-coming Section 2: after this, we focus on the rigorous arguments needed to make the actual proof work. More specifically, we will present in Section 3 a suitable geometric equation which will play a major role in our analysis. The interplay between viscosity and weak solutions of this equation will be discussed in Section 4, through some capacity and cut-off arguments (some technical estimates being deferred to Appendix A).

The geometric Hölder estimates needed to pass to the limit the rescaled configuration in which the sheets are separated by an order one will be presented in Section 5 (actually, these estimates are of general flavor and can find application in other geometric problems as well).

To apply these Hölder estimates one will also need a uniform bound on solutions of fractional equations in a geometric setting and this argument is contained in Section 6. In turn, in our case, this uniform bound will be a consequence of an integral bound, which is presented in Section 7.

The proof of Theorem 1.1 is thus completed in Section 8 (the details about the good set of regular boundary points utilized in the proof being contained in Appendix B).

¹As usual in geometric measure theory (see e.g. [Sim87]), here and in what follows by the "same tangent cone" we mean that any blow-up sequence possesses a subsequence along which the two dilated surfaces share the same tangent cone in the limit.

That is, the two surfaces have the same tangent cone in the "strong sense", namely along each convergent subsequence and not only along one particular convergent subsequence.

2. The plan of the proof

The proof will combine the blow-up methods introduced in [Sim87] with the fine analysis of the nonlocal setting needed in our context. The idea of the proof is to consider blow-up sequences of the minimal surfaces which approach the same limit cone at the origin and take a normal parameterization of these surfaces away from the singular set. We then look at the difference between these parameterizations, say w (or, more precisely, w_k , since it depends on the step). The inclusion of the two sets guarantees that w has a sign, say $w \ge 0$ (and, by contradiction, we can suppose that w > 0 at some point).

The function w may behave very badly in the vicinity of the singular set, thus we perform our analysis in the complement of a neighborhood of the singular set (we will shrink this neighborhood at the end of the proof, relying on the regularity theory of the nonlocal minimal surfaces). Moreover, since the convergence of $w = w_k$ is only local, we restrict our attention to a given ball B_R (we will send $R \to +\infty$ in the end).

Roughly speaking, the gist is to choose a point x_{\star} (in B_R and away from a given small neighborhood of the singular set) and a sequence of sets so that

$$(2.1) 2w_k(x_\star) \geqslant t^{-1}w_k(tx_\star)$$

for all $t \in (0,1]$: it is indeed possible to fulfill this inequality (or a slight modification of it) since the two minimal surfaces share the same tangent cone (hence their difference is "sublinear" at the origin, see Appendix B for full details about this technical construction). We can also normalize w_k at x_{\star} to be 1, i.e. look at the normalized function $v_k := \frac{w_k}{w_k(x_{\star})}$.

In this way, we see that $w = w_k$ (and therefore $v = v_k$) satisfies a suitable nonlocal equation of geometric type (in view of Theorem 3.4) and we show that v remains locally bounded (owing to Lemma 6.1) and therefore Hölder continuous in compact sets that avoid the singular set (thanks to Lemma 5.1).

We can thereby pick a convergent subsequence to a limit function v_{∞} which is a positive supersolution to a suitable Jacobi operator on the cone (first in the viscosity sense and then in the distributional sense, due to Lemma 4.1). From this, we will apply the weak Harnack Inequality of [CC19] and deduce that v_{∞} is locally bounded below by a universal constant and we contradict in this way the inequality in (2.1) (normalized and passed to the limit).

The details to carry over this plan are very technical and several complications arise from the nonstandard form of the integro-differential equations involved (such as the domain of integration not being flat and the kernels involved not being symmetric) and by the fact of dealing with non-standard domains of definitions (such as large balls with neighbourhoods of singular sets removed). These important technicalities occupy the rest of this paper.

3. A GEOMETRIC EQUATION

Now we consider the case in which two s-minimal sets, one included into the other, can be written locally as a graph in terms of another smooth hypersurface (concretely, in our case, the regular part of their common limit cone, though the arguments presented are of general flavor).

In this scenario, we obtain a geometric inequality for the difference between the parameterizations of the nonlocal minimal surfaces, which is reminiscent of the Jacobi field for nonlocal minimal surfaces (see e.g. [DdPW18, SV19]). The advantage of this approach in general is that these Jacobi fields leverage additional cancellations with respect to the two individual equations for the nonlocal mean curvature of the two surfaces. The specialty of

our setting is however that the two nonlocal minimal surfaces are nice graphs with respect to the limit cone only away from the possible singularities of the cone (hence, in our application, we will have to consider graphs converging to the limit cone away from its singularities and carefully estimate the error terms).

Actually, the methods that we develop work in a greater generality, which is also useful to appreciate the geometric structures arising in the limit construction. Thus, we split our calculations in different regions of the space (namely, near a regular point in Lemma 3.1, in an intermediate ring in Lemma 3.2, and far away in Lemma 3.3) and we collect these calculations into the general form given by Theorem 3.4.

This type of results are influenced by the classical theory of minimal surfaces but present significant differences with respect to the classical case. For instance, in the local setting, one can apply directly the theory of Jacobi fields and obtain a linear equation describing the normal displacement of a surface approaching a limit cone (see equation (7) in [Sim87]). Instead, in the nonlocal setting, remote interactions highly complicate the structure of the equations and a fine theory of cancellations is needed to obtain the convergence of the desired approximation and effective estimates on the remainder terms.

In the forthcoming calculations, we use the notation

$$\widetilde{\chi}_E(x) := \chi_{\mathbb{R}^n \setminus E}(x) - \chi_E(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}^n \setminus E, \\ -1 & \text{if } x \in E. \end{cases}$$

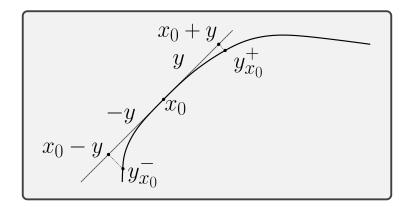


FIGURE 1. Projections along the tangent hyperplane.

Also, here below we will take into consideration suitable kernels defined on a hypersurface, specifically, a certain regular subset of ∂E_1 , with $E_1 \subseteq \mathbb{R}^n$: in this situation, given x_0 on this hypersurface and a vector y (with small norm) in the linear tangent hyperplane at x_0 we will denote by $y_{x_0}^+$ the point on the hypersurface whose projection onto the affine tangent hyperplane at x_0 coincides with $x_0 + y$ and by $y_{x_0}^-$ the point on the hypersurface whose projection onto the affine tangent hyperplane at x_0 coincides with $x_0 - y$, see Figure 1.

Lemma 3.1. Let $E_1 \subseteq E_2 \subseteq \mathbb{R}^n$. Suppose that there exist $\overline{X} \in \mathbb{R}^n$ and $\delta > 0$ such that $(\partial E_1) \cap B_{\delta}(\overline{X})$ is a hypersurface of class C^2 . Let ν be the unit external normal to E_1 in $B_{\delta}(\overline{X})$ Suppose that

$$(E_2 \setminus E_1) \cap B_{\delta}(\overline{X}) = \left\{ x + t\nu(x), \text{ with } x \in \partial E_1, t \in [0, w(x)) \right\} \cap B_{\delta}(\overline{X}),$$

for some function w of class C^2 with values in $[0, \delta/4]$.

Given $x_0 \in (\partial E_1) \cap B_{\delta/8}(\overline{X})$, let

$$\widetilde{E}_1 := E_1 + w(x_0)\nu(x_0)$$
 and $X_0 := x_0 + w(x_0)\nu(x_0)$.

Then, if δ and $||w||_{C^2(B_{\delta}(\overline{X}))}$ are small enough,

$$\frac{1}{2} \int_{B_{\delta/4}(\overline{X})} \frac{\widetilde{\chi}_{E_2}(X) - \widetilde{\chi}_{\widetilde{E}_1}(X)}{|X - X_0|^{n+s}} dX$$
(3.1)
$$= \int_{(\partial E_1) \cap B_{\delta/4}(\overline{X})} \left(w(x_0) - w(x) \right) K_1(x, x_0) d\mathcal{H}_x^{n-1}$$

$$- w(x_0) \int_{(\partial E_1) \cap B_{\delta/4}(\overline{X})} \left(1 - \nu(x_0) \cdot \nu(x) \right) K_2(x, x_0) d\mathcal{H}_x^{n-1} + O(w^2(x_0)),$$

where $K_j: ((\partial E_1) \cap B_{\delta/4}(\overline{X}))^2 \to [0, +\infty]$, with $j \in \{1, 2\}$, satisfies, for a suitable $C \geqslant 1$, that

(3.2) for all
$$x \neq x_0$$
, we have that $\frac{1}{C|x-x_0|^{n+s}} \leqslant K_j(x,x_0) \leqslant \frac{C}{|x-x_0|^{n+s}}$, and, for all $y \in B_\delta \setminus \{0\}$, $K_j(y_{x_0}^+, x_0) - K_j(y_{x_0}^-, x_0) = O(|y|^{1-n-s})$.

The "big O" terms here above depend only on the curvatures of ∂E_1 in $B_{9\delta/10}(\overline{X})$, on $\|w\|_{C^2(B_{\delta}(\overline{X}))}$, on n and s.

Also, the kernel K_j approaches, up to a normalizing constant, the kernel $|x-x_0|^{-n-s}$ when $||w||_{C^2(B_\delta(\overline{X}))}$ tends to zero.

Proof. Given $X \in B_{\delta}(\overline{X})$, we write $X = x + t\nu(x)$, with $x \in \partial E_1$ and $t \in \mathbb{R}$. If $\delta > 0$ is small enough, the map linking X to (x, t) is a diffeomorphism and the intersection with ∂E_1 occurs only for t = 0.

Thus, we denote the corresponding geometric Jacobian determinant by J(x,t) (set to zero when $x + t\nu(x) \notin B_{\delta/4}(\overline{X})$) and we have that

(3.3)
$$\int_{B_{\delta/4}(\overline{X})} \frac{\widetilde{\chi}_{E_2}(X) - \widetilde{\chi}_{\widetilde{E}_1}(X)}{|X - X_0|^{n+s}} dX$$

$$= \iint_{((\partial E_1) \cap B_{\delta/4}(\overline{X})) \times \mathbb{R}} \frac{\widetilde{\chi}_{E_2}(x + t\nu(x)) - \widetilde{\chi}_{\widetilde{E}_1}(x + t\nu(x))}{|x - x_0 + t\nu(x) - w(x_0)\nu(x_0)|^{n+s}} J(x, t) d\mathcal{H}_x^{n-1} dt.$$

Now, given $x \in (\partial E_1) \cap B_{\delta/4}(\overline{X})$, we have that

$$\widetilde{\chi}_{E_2}(x + t\nu(x)) = \begin{cases} 1 & \text{if } t \geqslant w(x), \\ -1 & \text{if } t < w(x). \end{cases}$$

Also, $\widetilde{\chi}_{\widetilde{E}_1}(x+t\nu(x))=-1$ if and only if $x+t\nu(x)\in\widetilde{E}_1$, and so if and only if $x(t):=x+t\nu(x)-w(x_0)\nu(x_0)\in E_1$.

Now, the signed distance of x(t) to the tangent hyperplane of E_1 at x is equal to

$$d(x) := (x(t) - x) \cdot \nu(x) = t - w(x_0)\nu(x_0) \cdot \nu(x).$$

Moreover, the projection of x(t) - x onto the tangent plane is

$$(x(t) - x) - ((x(t) - x) \cdot \nu(x))\nu(x) = t\nu(x) - w(x_0)\nu(x_0) - (t - w(x_0)\nu(x_0) \cdot \nu(x))\nu(x)$$

$$= -w(x_0)\nu(x_0) + w(x_0)\nu(x_0) \cdot \nu(x)\nu(x) = w(x_0)\Big(\nu(x_0) \cdot \nu(x)\nu(x) - \nu(x_0)\Big),$$

which has length equal to

$$w(x_0) | \nu(x_0) \cdot \nu(x) \nu(x) - \nu(x_0) | = w(x_0) \sqrt{1 - (\nu(x_0) \cdot \nu(x))^2}.$$

Since ∂E_1 detaches at most quadratically from its tangent hyperplane, we have that

$$\widetilde{\chi}_{\widetilde{E}_1}(x+t\nu(x)) = \begin{cases} 1 & \text{if } t \geqslant \widetilde{w}(x), \\ -1 & \text{if } t < \widetilde{w}(x), \end{cases}$$

where

$$\widetilde{w}(x) := w(x_0)\nu(x_0) \cdot \nu(x) + O(w^2(x_0)|\nu(x) - \nu(x_0)|^2).$$

From these observations, we infer that, for all $x \in (\partial E_1) \cap B_{\delta/4}(\overline{X})$,

$$\int_{\mathbb{R}} \frac{\widetilde{\chi}_{E_2}(x+t\nu(x))}{|x-x_0+t\nu(x)-w(x_0)\nu(x_0)|^{n+s}} J(x,t) dt = h_1(w(x)) - h_2(w(x))$$
and
$$\int_{\mathbb{R}} \frac{\widetilde{\chi}_{\widetilde{E}_1}(x+t\nu(x))}{|x-x_0+t\nu(x)-w(x_0)\nu(x_0)|^{n+s}} J(x,t) dt = h_1(\widetilde{w}(x)) - h_2(\widetilde{w}(x)),$$

where

(3.4)
$$h_1(\xi) := \int_{\xi}^{+\infty} \frac{J(x,t) dt}{|x - x_0 + t\nu(x) - w(x_0)\nu(x_0)|^{n+s}}$$
 and
$$h_2(\xi) := \int_{-\infty}^{\xi} \frac{J(x,t) dt}{|x - x_0 + t\nu(x) - w(x_0)\nu(x_0)|^{n+s}}.$$

We insert this information into (3.3) and we find that

$$\int_{B_{\delta/4}(\overline{X})} \frac{\widetilde{\chi}_{E_2}(X) - \widetilde{\chi}_{\widetilde{E}_1}(X)}{|X - X_0|^{n+s}} dX$$

$$= \int_{(\partial E_1) \cap B_{\delta/4}(\overline{X})} \left(h_1(w(x)) - h_1(\widetilde{w}(x)) - h_2(w(x)) + h_2(\widetilde{w}(x)) \right) d\mathcal{H}_x^{n-1}$$

$$= \int_{(\partial E_1) \cap B_{\delta/4}(\overline{X})} \left(h(w(x)) - h(\widetilde{w}(x)) \right) d\mathcal{H}_x^{n-1},$$

where

$$(3.6) h := h_1 - h_2.$$

We also observe that

(3.7)
$$-h'(\xi) = -h'_1(\xi) + h'_2(\xi) = \frac{2J(x,\xi)}{|x - x_0 + \xi \nu(x) - w(x_0)\nu(x_0)|^{n+s}}.$$

Therefore, letting

$$A(x_0) := \text{Id} + w(x_0)II(x_0)$$

and
$$\sigma(x) := A(x_0)(x - x_0) = (\text{Id} + w(x_0)II(x_0))(x - x_0),$$

where II denotes the second fundamental form along ∂E_1 , and using the change of variable $\theta := \frac{\xi - w(x_0)}{|\sigma(x)|}$, we have that

$$\frac{h(a+w(x_0)) - h(b+w(x_0))}{2} \\
= -\frac{1}{2} \int_{a+w(x_0)}^{b+w(x_0)} h'(\xi) d\xi \\
= \int_{a+w(x_0)}^{b+w(x_0)} \frac{J(x,\xi) d\xi}{|x-x_0+\xi\nu(x)-w(x_0)\nu(x_0)|^{n+s}} \\
= \int_{a/|\sigma(x)|}^{b/|\sigma(x)|} \frac{|\sigma(x)| J(x,w(x_0)+\theta|\sigma(x)|) d\theta}{|x-x_0+(w(x_0)+\theta|\sigma(x)|)\nu(x)-w(x_0)\nu(x_0)|^{n+s}}.$$

We also remark that, as $|x-x_0| \to 0$, the vector $x-x_0$ becomes tangent to ∂E_1 , therefore $(x-x_0)\cdot\nu(x)=O(|x-x_0|^2)$.

Moreover,

$$\nu(x) - \nu(x_0) = II(x_0)(x - x_0) + O(|x - x_0|^2).$$

For this reason,

$$(w(x_0) + \theta |\sigma(x)|)\nu(x) - w(x_0)\nu(x_0)$$

$$= (w(x_0) + \theta |\sigma(x)|)(\nu(x_0) + II(x_0)(x - x_0) + O(|x - x_0|^2)) - w(x_0)\nu(x_0)$$

$$= w(x_0)II(x_0)(x - x_0) + \theta |\sigma(x)|\nu(x_0) + O(|x - x_0|^2).$$

As a result,

$$\begin{aligned} & \left| x - x_0 + \left(w(x_0) + \theta |\sigma(x)| \right) \nu(x) - w(x_0) \nu(x_0) \right|^2 \\ &= \left| \left(\text{Id} + w(x_0) H(x_0) \right) (x - x_0) + \theta |\sigma(x)| \nu(x_0) + O(|x - x_0|^2) \right|^2 \\ &= \left| \sigma(x) + \theta |\sigma(x)| \nu(x_0) + O(|x - x_0|^2) \right|^2 \\ &= (1 + \theta^2) |\sigma(x)|^2 + 2\theta |\sigma(x)| \sigma(x) \cdot \nu(x_0) + O(|x - x_0|^3). \end{aligned}$$

Actually, since

$$\sigma(x) \cdot \nu(x_0) = (w(x_0)H(x_0)(x - x_0)) \cdot \nu(x_0) + O(|x - x_0|^2)$$

$$= w(x_0)(\nu(x) - \nu(x_0)) \cdot \nu(x_0) + O(|x - x_0|^2)$$

$$= w(x_0)(\nu(x) \cdot \nu(x_0) - 1) + O(|x - x_0|^2)$$

$$= -\frac{w(x_0)}{2} |\nu(x) - \nu(x_0)|^2 + O(|x - x_0|^2)$$

$$= O(|x - x_0|^2),$$

we find that

$$\left| x - x_0 + \left(w(x_0) + \theta |\sigma(x)| \right) \nu(x) - w(x_0) \nu(x_0) \right|^2 = (1 + \theta^2) |\sigma(x)|^2 + O(|x - x_0|^3).$$

This gives that

$$\begin{aligned} & \left| x - x_0 + \left(w(x_0) + \theta |\sigma(x)| \right) \nu(x) - w(x_0) \nu(x_0) \right|^{-(n+s)} \\ &= \left((1 + \theta^2) |\sigma(x)|^2 + O(|x - x_0|^3) \right)^{-\frac{n+s}{2}} \\ &= (1 + \theta^2)^{-\frac{n+s}{2}} |\sigma(x)|^{-(n+s)} \left(1 + O(|x - x_0|) \right)^{-\frac{n+s}{2}} \\ &= (1 + \theta^2)^{-\frac{n+s}{2}} |\sigma(x)|^{-(n+s)} \left(1 + O(|x - x_0|) \right) \end{aligned}$$

and consequently, in view of (3.8),

(3.9)
$$\frac{h(a+w(x_0)) - h(b+w(x_0))}{2}$$

$$= \int_{a/|\sigma(x)|}^{b/|\sigma(x)|} \frac{|\sigma(x)| J(x, w(x_0) + \theta|\sigma(x)|) (1 + O(|x-x_0|))}{(1+\theta^2)^{\frac{n+s}{2}} |\sigma(x)|^{n+s}} d\theta.$$

Now we take

$$a := w(x) - w(x_0)$$
 and $b := \widetilde{w}(x) - w(x_0)$.

We observe that

$$|w(x) - w(x_0)| = O(\|\nabla w\|_{L^{\infty}(B_{\delta}(\overline{X}))}|x - x_0|)$$

and

$$|\widetilde{w}(x) - w(x_0)| \le w(x_0) |\nu(x_0) \cdot \nu(x) - 1| + O(w^2(x_0)|x - x_0|^2)$$

 $\le O(w(x_0)|x - x_0|^2).$

This says that, in our setting,

(3.10)
$$\frac{a}{|\sigma(x)|} = O(\|\nabla w\|_{L^{\infty}(B_{\delta}(\overline{X}))}) \quad \text{and} \quad \frac{b}{|\sigma(x)|} = O(w(x_0)|x - x_0|).$$

Therefore, with reference to the integral in (3.9), in this setting we have that $\theta = O(\|w\|_{C^1(B_{\delta}(\overline{X}))})$ and therefore

$$J(x, w(x_0) + \theta |\sigma(x)|) = J(x, w(x_0)) + O(|x - x_0|) = J(x_0, w(x_0)) + O(|x - x_0|),$$

with the latter quantity depending on the curvatures of ∂E_1 in $B_{9\delta/10}(\overline{X})$, on $||w||_{C^2(B_\delta(\overline{X}))}$, on n and s.

Gathering this and (3.9) we conclude that

$$\begin{split} \frac{h(a+w(x_0)) - h(b+w(x_0))}{2} \\ &= \frac{|\sigma(x)| \left(J(x_0, w(x_0)) + O(|x-x_0|)\right)}{|\sigma(x)|^{n+s}} \int_{a/|\sigma(x)|}^{b/|\sigma(x)|} \frac{d\theta}{(1+\theta^2)^{\frac{n+s}{2}}} \\ &= \frac{J(x_0, w(x_0)) + O(|x-x_0|)}{|\sigma(x)|^{n+s}} \left(b\Phi\left(\frac{b}{|\sigma(x)|}\right) - a\Phi\left(\frac{a}{|\sigma(x)|}\right)\right), \end{split}$$

where

$$\Phi(\alpha) := \frac{1}{\alpha} \int_0^\alpha \frac{d\theta}{(1+\theta^2)^{\frac{n+s}{2}}}.$$

Now we let

$$(3.11) \qquad \mathcal{A}_1(x, x_0) := \Phi\left(\frac{w(x) - w(x_0)}{|\sigma(x)|}\right) \qquad \text{and} \qquad \mathcal{A}_2(x, x_0) := \Phi\left(\frac{\widetilde{w}(x) - w(x_0)}{|\sigma(x)|}\right)$$

and we see that

$$\frac{h(w(x)) - h(\widetilde{w}(x))}{2} \\
= \frac{J(x_0, w(x_0)) + O(|x - x_0|)}{|\sigma(x)|^{n+s}} \left((\widetilde{w}(x) - w(x_0)) \mathcal{A}_2(x, x_0) - (w(x) - w(x_0)) \mathcal{A}_1(x, x_0) \right) \\
= \frac{J(x_0, w(x_0)) + O(|x - x_0|)}{|\sigma(x)|^{n+s}} \\
\times \left[\left(w(x_0)(\nu(x_0) \cdot \nu(x) - 1) + O(w^2(x_0)|x - x_0|^2) \right) \mathcal{A}_2(x, x_0) - (w(x) - w(x_0)) \mathcal{A}_1(x, x_0) \right] \\
= \left(w(x_0)(\nu(x_0) \cdot \nu(x) - 1) + O(w^2(x_0)|x - x_0|^2) \right) K_2(x, x_0) - (w(x) - w(x_0)) K_1(x, x_0),$$

where

$$K_j(x,x_0) := \frac{\left(J(x_0,w(x_0)) + O(|x-x_0|)\right) \mathcal{A}_j(x,x_0)}{|\sigma(x)|^{n+s}}.$$

By inserting this into (3.5), we have thereby obtained the desired result in (3.1), but we need to check (3.2) in order to ensure the necessary cancellations to make sense of the integrals involved.

To this end, we observe that the Jacobian $J(x_0,t)$ approaches 1 as $t\to 0$. Moreover,

$$\Phi(\alpha) \leqslant \frac{1}{\alpha} \int_0^{\alpha} d\theta = 1$$

and, if $\alpha \in (-1,1)$,

$$\Phi(\alpha) \geqslant \frac{1}{\alpha} \int_0^\alpha \frac{d\theta}{2^{\frac{n+s}{2}}} = \frac{1}{2^{\frac{n+s}{2}}}.$$

This, (3.10) and (3.11) entail that, in the range of interest, also $\mathcal{A}_j(x,x_0) \in \left[\frac{1}{2^{\frac{n+s}{2}}},1\right]$.

Besides, for small $w(x_0)$, we have that $|\sigma(x)| \in \left[\frac{|x-x_0|}{2}, 2|x-x_0|\right]$ and these considerations establish the first claim in (3.2).

Additionally,

$$|\sigma(x_0 \pm y)| = |\pm A(x_0)y| = |A(x_0)y|.$$

Furthermore,

$$\Phi(\alpha) = \frac{1}{\alpha} \int_0^{\alpha} \left(1 - \frac{n+s}{2} \,\theta^2 + O(\theta^4) \right) \, d\theta = 1 - \frac{n+s}{6} \,\alpha^2 + O(\alpha^4),$$

giving that, if $\alpha \in (-1,1)$, then $|\Phi'(\alpha)| \leq C$ and accordingly

$$|\Phi(\alpha) - \Phi(\beta)| \leqslant C|\alpha - \beta|.$$

We also have that

$$w(y_{x_0}^+) - w(y_{x_0}^-) = O(|y|^2)$$

and

$$\widetilde{w}(y_{x_0}^{\pm}) - w(x_0) = w(x_0) \left(\nu(x_0) \cdot \nu(y_{x_0}^{\pm}) - 1 \right) + O(|y|^2) = O(|y|^2).$$

As a consequence,

$$\begin{aligned} \left| \mathcal{A}_{1}(y_{x_{0}}^{+}, x_{0}) - \mathcal{A}_{1}(y_{x_{0}}^{-}, x_{0}) \right| \\ &= \left| \Phi\left(\frac{w(y_{x_{0}}^{+}) - w(x_{0})}{|A(x_{0})y|} \right) - \Phi\left(\frac{w(y_{x_{0}}^{+}) - w(x_{0})}{|A(x_{0})y|} \right) \right| \\ &\leq C \left| \frac{w(y_{x_{0}}^{+}) - w(y_{x_{0}}^{-})}{|A(x_{0})y|} \right| \leq C|y| \end{aligned}$$

and

$$\left| \mathcal{A}_{2}(y_{x_{0}}^{+}, x_{0}) - \mathcal{A}_{2}(y_{x_{0}}^{-}, x_{0}) \right| \\
= \left| \Phi\left(\frac{\widetilde{w}(y_{x_{0}}^{+}) - w(x_{0})}{|A(x_{0})y|} \right) - \Phi\left(\frac{\widetilde{w}(y_{x_{0}}^{-}) - w(x_{0})}{|A(x_{0})y|} \right) \right| \leqslant C|y|.$$

The proof of (3.2) is thereby complete.

The fact that K_1 and K_2 can be seen as perturbations of the kernel $|x - x_0|^{-n-s}$ when $||w||_{C^2(B_{\delta}(\overline{X}))}$ tends to zero follows since in this asymptotics $\sigma(x)$ approaches $|x - x_0|$, \mathcal{A}_1 and \mathcal{A}_2 approach $\Phi(0) = 1$ and the Jacobian term approaches J(x, 0) = 1.

Lemma 3.2. Let $\delta \in (0,1)$, $R > 1 + 2\delta$, $\overline{X} \in \mathbb{R}^n$ and $E_1 \subseteq E_2 \subseteq \mathbb{R}^n$.

Assume that there exists $\mathcal{G} \subseteq B_{2R}(\overline{X})$ such that $(\partial E_1) \cap \mathcal{G}$ is a hypersurface of class C^2 with unit external normal ν .

Suppose that

$$(E_2 \setminus E_1) \cap \mathcal{G} = \left\{ x + t\nu(x), \text{ with } x \in \partial E_1, t \in [0, w(x)) \right\} \cap \mathcal{G},$$

for some function w of class C^2 with values in $[0, \delta/4]$.

Assume also that

(3.12) the Hausdorff distance between ∂E_1 and ∂E_2 in $B_{2R}(\overline{X})$ is less than $\frac{\delta}{20}$.

Let
$$x_0 \in (\partial E_1) \cap B_{\delta/8}(\overline{X}), \ \nu_0 \in \partial B_1,$$

$$\widetilde{E}_1 := E_1 + w(x_0)\nu_0$$
 and $X_0 := x_0 + w(x_0)\nu_0$.

Let also $\mathcal{G}' \subseteq \mathcal{G}$ and assume that

$$(3.13) B_{2R}(\overline{X}) \setminus \mathcal{G}' \subseteq \bigcup_{j \in \mathbb{N}} B_{r_j}(p_j),$$

for some balls $\{B_{r_j}(p_j)\}_{j\in\mathbb{N}}$, with

(3.14)
$$\eta := \sum_{j \in \mathbb{N}} r_j^{n-1} < 1.$$

Then, if δ is small enough, and η and $\|w\|_{L^{\infty}(B_R(\overline{X})\cap\mathcal{G})}$ are small compared to δ and 1/R,

$$\frac{1}{2} \int_{B_R(\overline{X}) \setminus B_{\delta/4}(\overline{X})} \frac{\widetilde{\chi}_{E_2}(X) - \widetilde{\chi}_{\widetilde{E}_1}(X)}{|X - X_0|^{n+s}} dX$$

$$= \int_{((\partial E_1) \cap \mathcal{G}') \setminus B_{\delta/4}(\overline{X})} \left(w(x_0) - w(x) \right) K_1(x, x_0) d\mathcal{H}_x^{n-1}$$

$$-w(x_0) \int_{((\partial E_1) \cap \mathcal{G}') \setminus B_{\delta/4}(\overline{X})} \left(1 - \nu_0 \cdot \nu(x)\right) K_2(x, x_0) d\mathcal{H}_x^{n-1}$$

$$+ \frac{1}{2} \int_{(B_R(\overline{X}) \setminus B_{\delta/4}(\overline{X})) \setminus \mathcal{G}'} \frac{\widetilde{\chi}_{E_2}(X) - \widetilde{\chi}_{E_1}(X)}{|X - X_0|^{n+s}} dX + O\left(w(x_0)\eta\right),$$

where the "big O" term here above depends only on δ , R, n and s.

In addition, $K_j: (((\partial E_1) \cap \mathcal{G}') \setminus B_{\delta/4}(\overline{X}))^2 \to [0, +\infty]$, with $j \in \{1, 2\}$, satisfies (3.2) and approaches, up to a normalizing constant, the kernel $|x - x_0|^{-n-s}$ when $||w||_{C^2(B_R(\overline{X}) \cap \mathcal{G})}$ tends to zero.

Proof. As in the proof of Lemma 3.1, given $X \in (B_R(\overline{X}) \setminus B_{\delta/4}(\overline{X})) \cap \mathcal{G}'$, we write $X = x + t\nu(x)$, with $x \in \partial E_1$ and $t \in \mathbb{R}$.

By (3.12), we see that

if
$$|t| \geqslant \frac{\delta}{10}$$
, then $\widetilde{\chi}_{E_2}(x + t\nu(x)) - \widetilde{\chi}_{\widetilde{E}_1}(x + t\nu(x)) = 0$.

In this setting, if $\delta > 0$ is small enough,

$$(3.15) \int_{(B_{R}(\overline{X})\backslash B_{\delta/4}(\overline{X}))\cap \mathcal{G}'} \frac{\widetilde{\chi}_{E_{2}}(X) - \widetilde{\chi}_{\widetilde{E}_{1}}(X)}{|X - X_{0}|^{n+s}} dX$$

$$= \iint_{((\partial E_{1})\cap (B_{R}(\overline{X})\backslash B_{\delta/4}(\overline{X}))\cap \mathcal{G}')\times \mathbb{R}} \frac{\widetilde{\chi}_{E_{2}}(x + t\nu(x)) - \widetilde{\chi}_{\widetilde{E}_{1}}(x + t\nu(x))}{|x - x_{0} + t\nu(x) - w(x_{0})\nu_{0}|^{n+s}} J(x, t) d\mathcal{H}_{x}^{n-1} dt,$$

where J(x,t) denotes the geometric Jacobian determinant (set to zero when $x + t\nu(x) \notin B_R(\overline{X})$).

Given $x \in (\partial E_1) \cap \mathcal{G}'$, we have that

(3.16)
$$\widetilde{\chi}_{E_2}(x + t\nu(x)) = \begin{cases} 1 & \text{if } t \geqslant w(x), \\ -1 & \text{if } t < w(x). \end{cases}$$

Moreover, $\widetilde{\chi}_{\widetilde{E}_1}(x+t\nu(x))=-1$ if and only if $x+t\nu(x)\in\widetilde{E}_1$, and so if and only if $x(t):=x+t\nu(x)-w(x_0)\nu_0\in E_1$.

Now, the signed distance of x(t) to the tangent hyperplane of E_1 at x is equal to

$$d(x) := (x(t) - x) \cdot \nu(x) = t - w(x_0)\nu_0 \cdot \nu(x).$$

Moreover, the projection of x(t) - x onto the tangent plane is

$$(x(t) - x) - ((x(t) - x) \cdot \nu(x))\nu(x) = t\nu(x) - w(x_0)\nu_0 - (t - w(x_0)\nu_0 \cdot \nu(x))\nu(x)$$

= $w(x_0)\nu_0 \cdot \nu(x)\nu(x) - w(x_0)\nu_0$

which has length equal to

$$\ell(x) := \sqrt{w^2(x_0) - (w(x_0)\nu_0 \cdot \nu(x))^2}.$$

We stress that $\ell(x) \leq w(x_0)$, hence if $w(x_0)$ is small enough then $B_{\ell(x)}(x)$ lies in \mathcal{G} and accordingly ∂E_1 detaches at most quadratically from its tangent hyperplane in this region. This gives that

(3.17)
$$\widetilde{\chi}_{\widetilde{E}_1}(x+t\nu(x)) = \begin{cases} 1 & \text{if } t \geqslant \widetilde{w}(x), \\ -1 & \text{if } t < \widetilde{w}(x), \end{cases}$$

where

$$\widetilde{w}(x) := w(x_0)\nu_0 \cdot \nu(x) + O(w^2(x_0)).$$

By using (3.16) and (3.17), and in the notation of (3.4) and (3.6) (with $\nu(x_0)$ replaced by ν_0 here), we deduce that, if $x \in (\partial E_1) \cap (B_R(\overline{X}) \setminus B_{\delta/4}(\overline{X})) \cap \mathcal{G}'$, then

$$\int_{\mathbb{R}} \frac{\widetilde{\chi}_{E_2}(x + t\nu(x)) - \widetilde{\chi}_{\widetilde{E}_1}(x + t\nu(x))}{|x - x_0 + t\nu(x) - w(x_0)\nu_0|^{n+s}} J(x, t) dt = h(w(x)) - h(\widetilde{w}(x))$$

and therefore

(3.18)
$$\iint_{((\partial E_1) \cap (B_R(\overline{X}) \setminus B_{\delta/4}(\overline{X}) \cap \mathcal{G}') \times \mathbb{R}} \frac{\widetilde{\chi}_{E_2}(x + t\nu(x)) - \widetilde{\chi}_{\widetilde{E}_1}(x + t\nu(x))}{|x - x_0 + t\nu(x) - w(x_0)\nu_0|^{n+s}} J(x, t) d\mathcal{H}_x^{n-1} dt$$
$$= \int_{(\partial E_1) \cap (B_R(\overline{X}) \setminus B_{\delta/4}(\overline{X})) \cap \mathcal{G}'} \left(h(w(x)) - h(\widetilde{w}(x)) \right) d\mathcal{H}_x^{n-1}.$$

Thus, we define

$$K(x,x_0) := -\frac{h(w(x)) - h(\widetilde{w}(x))}{2(w(x) - \widetilde{w}(x))}$$

and we remark that this kernel satisfies (3.2). Indeed, we can take here $K_1 = K_2 = K$ and, since in our setting $|x - x_0| \ge \frac{\delta}{100}$, we only need to check the second claim in (3.2).

And this claim holds true, because, by (3.7),

$$-\frac{h(w(x)) - h(\widetilde{w}(x))}{w(x) - \widetilde{w}(x)} = -\int_0^1 h'(tw(x) + (1 - t)\widetilde{w}(x)) dt$$
$$= \int_0^1 \frac{2J(x, tw(x) + (1 - t)\widetilde{w}(x))}{|x - x_0 + (tw(x) + (1 - t)\widetilde{w}(x))\nu(x) - w(x_0)\nu_0|^{n+s}} dt,$$

which, when $|x-x_0| \ge \frac{\delta}{100}$ is bounded from above and below by $\frac{1}{|x-x_0|^{n+s}}$, up to multiplicative constants.

We also deduce from (3.18) that

(3.19)
$$\frac{1}{2} \int_{(B_{R}(\overline{X})\backslash B_{\delta/4}(\overline{X}))\cap \mathcal{G}'} \frac{\widetilde{\chi}_{E_{2}}(X) - \widetilde{\chi}_{\widetilde{E}_{1}}(X)}{|X - X_{0}|^{n+s}} dX$$

$$= \int_{(\partial E_{1})\cap (B_{R}(\overline{X})\backslash B_{\delta/4}(\overline{X}))\cap \mathcal{G}'} \left(\widetilde{w}(x) - w(x)\right) K(x, x_{0}) d\mathcal{H}_{x}^{n-1}.$$

Now we deal with the term

$$\int_{(B_{R}(\overline{X})\backslash B_{\delta/4}(\overline{X}))\backslash \mathcal{G}'} \frac{\widetilde{\chi}_{E_{2}}(X) - \widetilde{\chi}_{\widetilde{E}_{1}}(X)}{|X - X_{0}|^{n+s}} dX$$

$$= \int_{(B_{R}(\overline{X})\backslash B_{\delta/4}(\overline{X}))\backslash \mathcal{G}'} \frac{\widetilde{\chi}_{E_{2}}(X) - \widetilde{\chi}_{E_{1}}(X)}{|X - X_{0}|^{n+s}} dX + \int_{(B_{R}(\overline{X})\backslash B_{\delta/4}(\overline{X}))\backslash \mathcal{G}'} \frac{\widetilde{\chi}_{E_{1}}(X) - \widetilde{\chi}_{\widetilde{E}_{1}}(X)}{|X - X_{0}|^{n+s}} dX.$$

Specifically, to estimate the latter term, we define F_1 as the symmetric difference between E_1 and \widetilde{E}_1 and we point out that, by (3.13),

$$(3.20) \left| \int_{(B_{R}(\overline{X})\backslash B_{\delta/4}(\overline{X}))\backslash \mathcal{G}'} \frac{\widetilde{\chi}_{E_{1}}(X) - \widetilde{\chi}_{\widetilde{E}_{1}}(X)}{|X - X_{0}|^{n+s}} dX \right| \leq 2 \int_{((B_{R}(\overline{X})\backslash B_{\delta/4}(\overline{X}))\backslash \mathcal{G}')\cap F_{1}} \frac{dX}{|X - X_{0}|^{n+s}} \leq \frac{C}{\delta^{n+s}} \sum_{j \in \mathbb{N}} |B_{r_{j}}(p_{j}) \cap F_{1}|.$$

To complete this estimate, it is useful to observe that, for all r > 0, all $\tau \in \mathbb{R}^n$ and all measurable sets L, we have that

$$\left| B_r \cap \left((L + \tau) \setminus L \right) \right| = \left| \int_{B_r \cap (L + \tau)} dx - \int_{B_r \cap L} dx \right| = \left| \int_{(B_r - \tau) \cap L} dx - \int_{B_r \cap L} dx \right| \\
= \left| L \cap \left((B_r - \tau) \setminus B_r \right) \right| \leqslant \left| (B_r - \tau) \setminus B_r \right| \leqslant \left| B_{r + \tau} \setminus B_r \right| \leqslant Cr^{n - 1} \tau.$$

Plugging this information into (3.20) and exploiting (3.14), we conclude that

$$\left| \int_{(B_R(\overline{X}) \setminus B_{\delta/4}(\overline{X})) \setminus \mathcal{G}'} \frac{\widetilde{\chi}_{E_1}(X) - \widetilde{\chi}_{\widetilde{E}_1}(X)}{|X - X_0|^{n+s}} dX \right| \leqslant \frac{Cw(x_0)}{\delta^{n+s}} \sum_{j \in \mathbb{N}} r_j^{n-1} \leqslant \frac{C\eta w(x_0)}{\delta^{n+s}}.$$

The desired result follows from this inequality and (3.19) (the asymptotics of the kernel being similar to those discussed in Lemma 3.1).

Lemma 3.3. Let $\delta \in (0,1)$, $R > 1 + 2\delta$, $\overline{X} \in \mathbb{R}^n$ and $E_1 \subseteq E_2 \subseteq \mathbb{R}^n$. Let $x_0 \in B_{\delta/8}(\overline{X})$, $\nu_0 \in \partial B_1$, $w_0 \in [0, \delta/4]$

$$\widetilde{E}_1 := E_1 + w_0 \nu_0$$
 and $X_0 := x_0 + w_0 \nu_0$.

Then.

$$\frac{1}{2} \int_{\mathbb{R}^n \backslash B_R(\overline{X})} \frac{\widetilde{\chi}_{E_2}(X) - \widetilde{\chi}_{\widetilde{E}_1}(X)}{|X - X_0|^{n+s}} dX = \frac{1}{2} \int_{\mathbb{R}^n \backslash B_R(\overline{X})} \frac{\widetilde{\chi}_{E_2}(X) - \widetilde{\chi}_{E_1}(X)}{|X - X_0|^{n+s}} dX + O\left(\frac{w_0}{R^{1+s}}\right),$$

where the "big O" term here above depends only on n and s.

Proof. We observe that

$$\begin{split} &\left| \int_{\mathbb{R}^n \backslash B_R(\overline{X})} \frac{\widetilde{\chi}_{E_1}(X) - \widetilde{\chi}_{\widetilde{E}_1}(X)}{|X - X_0|^{n+s}} \, dX \right| \\ &= \left| \int_{\mathbb{R}^n \backslash B_R(\overline{X})} \frac{\widetilde{\chi}_{E_1}(X)}{|X - X_0|^{n+s}} \, dX - \int_{\mathbb{R}^n \backslash B_R(\overline{X} - w_0 \nu_0)} \frac{\widetilde{\chi}_{E_1}(X)}{|X + w_0 \nu_0 - X_0|^{n+s}} \, dX \right| \\ &\leqslant \left| \int_{\mathbb{R}^n \backslash B_R(\overline{X})} \frac{\widetilde{\chi}_{E_1}(X)}{|X - X_0|^{n+s}} \, dX - \int_{\mathbb{R}^n \backslash B_R(\overline{X} - w_0 \nu_0)} \frac{\widetilde{\chi}_{E_1}(X)}{|X - X_0|^{n+s}} \, dX \right| + \frac{Cw_0}{R^{1+s}} \\ &\leqslant \frac{Cw_0}{R^{1+s}}, \end{split}$$

up to renaming C, as usual, from line to line.

Now we recall the notion of nonlocal mean curvature of a set E at a point $X \in \partial E$, namely

$$H_E^s(X) := \int_{\mathbb{R}^n} \frac{\widetilde{\chi}_E(Y)}{|X - Y|^{n+s}} \, dY.$$

Putting together Lemmata 3.1, 3.2 and 3.3, we conclude that:

Theorem 3.4. Let $E_1 \subseteq E_2 \subseteq \mathbb{R}^n$. Let $\delta \in (0,1)$, $R > 1 + 2\delta$ and $\overline{X} \in \mathbb{R}^n$.

Suppose that $(\partial E_1) \cap \mathcal{G}$ is a hypersurface of class C^2 , for some $\mathcal{G} \subseteq \mathbb{R}^n$ such that $\mathcal{G} \supseteq B_{2\delta}(\overline{X})$, and let ν be the unit external normal to E_1 in \mathcal{G} . Suppose that

$$(E_2 \setminus E_1) \cap \mathcal{G} = \left\{ x + t\nu(x), \text{ with } x \in \partial E_1, t \in [0, w(x)) \right\} \cap \mathcal{G},$$

for some function w of class C^2 with values in $[0, \delta/4]$.

Assume also that the Hausdorff distance between ∂E_1 and ∂E_2 in $B_{2R}(\overline{X})$ is less than $\delta/20$. Given $x_0 \in (\partial E_1) \cap B_{\delta/8}(\overline{X})$, let

$$\widetilde{E}_1 := E_1 + w(x_0)\nu(x_0)$$
 and $X_0 := x_0 + w(x_0)\nu(x_0)$.

Let also $\mathcal{G}' \subseteq \mathcal{G}$ with $\mathcal{G}' \supseteq B_{2\delta}(\overline{X})$ and assume that

$$B_{2R}(\overline{X}) \setminus \mathcal{G}' \subseteq \bigcup_{j \in \mathbb{N}} B_{r_j}(p_j),$$

for some balls $\{B_{r_i}(p_j)\}_{j\in\mathbb{N}}$, with

(3.21)
$$\eta := \sum_{j \in \mathbb{N}} r_j^{n-1} < 1.$$

Then, if δ is small enough, and η and $\|w\|_{C^2(B_R(\overline{X})\cap\mathcal{G})}$ are small compared to δ and 1/R,

$$\frac{H_{E_2}^s(X_0) - H_{\widetilde{E}_1}^s(X_0)}{2} = \int_{(\partial E_1) \cap \mathcal{G}'} \left(w(x_0) - w(x) \right) K_1(x, x_0) d\mathcal{H}_x^{n-1}
- w(x_0) \int_{(\partial E_1) \cap \mathcal{G}'} \left(1 - \nu(x_0) \cdot \nu(x) \right) K_2(x, x_0) d\mathcal{H}_x^{n-1}
+ \frac{1}{2} \int_{\mathbb{R}^n \setminus \mathcal{G}'} \frac{\widetilde{\chi}_{E_2}(X) - \widetilde{\chi}_{E_1}(X)}{|X - X_0|^{n+s}} dX + O\left(w(x_0) \eta \right) + O\left(\frac{w(x_0)}{R^{1+s}} \right),$$

where $K_j: ((\partial E_1) \cap \mathcal{G}')^2 \to [0, +\infty]$, with $j \in \{1, 2\}$, satisfies, for a suitable $C \geqslant 1$, that

(3.23) for all
$$x \neq x_0$$
, we have that $\frac{1}{C|x-x_0|^{n+s}} \leqslant K_j(x,x_0) \leqslant \frac{C}{|x-x_0|^{n+s}}$, and, for all $y \in B_\delta \setminus \{0\}$, $K_j(y_{x_0}^+, x_0) - K_j(y_{x_0}^-, x_0) = O(|y|^{1-n-s})$.

All the "big O" terms here above depend only on the curvatures of ∂E_1 in $B_{9\delta/10}(\overline{X})$, on $\|w\|_{C^2(B_R(\overline{X})\cap\mathcal{G})}$, on δ , R, n and s, except for the last one in (3.22), which depends only on n and s.

Also, the kernel K_j approaches, up to a normalizing constant, the kernel $|x - x_0|^{-n-s}$ when $||w||_{C^2(B_R(\overline{X})\cap\mathcal{G})}$ tends to zero.

4. Cut-off arguments

Having obtained a geometric equation for a nonnegative function in Theorem 3.4, our objective would be to apply a Harnack Inequality to it (actually, to a suitable limit of it, and this will indeed be implemented in the forthcoming Section 8). This is however rather delicate, since in principle, in our setting, the equation will only be valid in a pointwise sense

along the regular part of a limit s-minimal cone, but we do not have any information about the equation along the singular points of this cone.

Also, the equation obtained needs to be set into a distributional framework, which, again, is not for free since the quantities involved may explode at singular points, not allowing for a formulation in a suitable energy space.

These difficulties can be overcome by an appropriate cut-off argument, based on a fine covering of the singular points (if any) of the limit cone. Namely, the regularity theory of s-minimal cones (see [SV13, BFV14]) will allow us to confine these singular points within a small region, which can be covered by suitable balls that produce a negligible energy term.

Similar difficulties arose in [CC19, Section 6] and this issue was solved there in Lemma 6.3 via a bespoke capacity argument. We provide here a self-contained approach. The technical result that we use goes as follows:

Lemma 4.1. Let $\mathcal{R} \subset \mathbb{R}^n$ and $S := \overline{\mathcal{R}} \setminus \mathcal{R}$. Assume that \mathcal{R} is locally a C^2 -smooth hypersurface and that S is closed and has Hausdorff dimension $d \leq n-3$.

Assume also that, for every r > 0 and $p \in \mathbb{R}^n$,

$$\mathcal{H}^{n-1}(\mathcal{R} \cap B_r(p)) \leqslant Cr^{n-1},$$

for some C > 0.

Let $u: \mathcal{R} \to [0, +\infty)$ be such that, for every $x \in \mathcal{R}$,

(4.2)
$$\int_{\mathcal{R}} \frac{u(x) - u(y)}{|x - y|^{n+s}} d\mathcal{H}_y^{n-1} \geqslant 0.$$

Given Q > 0, let $u_Q := \min\{u, Q\}$.

Then, for every $\zeta \in W^{1,\infty}(\mathbb{R}^n, [0, +\infty))$ with support of $\zeta|_{\mathcal{R}}$ contained in \mathcal{R} , we have that

(4.3)
$$\iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x) - u_Q(y))(\zeta(x) - \zeta(y))}{|x - y|^{n+s}} d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \geqslant 0.$$

Moreover, for every $R_0 > 0$,

(4.4)
$$\iint_{(\mathcal{R} \cap B_{R_0}) \times \mathcal{R}} \frac{(u_Q(x) - u_Q(y))^2}{|x - y|^{n+s}} d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} < +\infty$$

and, for every $\phi \in C_0^{\infty}(\mathbb{R}^n, [0, +\infty))$,

(4.5)
$$\iint_{\mathcal{R} \times \mathcal{R}} \frac{(u_Q(x) - u_Q(y))(\phi(x) - \phi(y))}{|x - y|^{n+s}} d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \geqslant 0.$$

For the reader's facility, the rather technical proof of Lemma 4.1 will be given in Appendix A.

5. Hölder estimates for fractional operators in a geometric setting

Here we present a Hölder regularity result in a geometric setting. Namely, differently from the cases already treated in the literature, the equation considered here is defined by an integral on a portion of a hypersurface and the kernel is not necessarily symmetric, but only symmetric up to a suitable remainder.

Lemma 5.1. Let \mathcal{R} be a portion of a C^2 hypersurface in B_2 that is C^2 -diffeomorphic to $B_2 \cap \{x_n = 0\}$. Let ν be the unit normal vector field of \mathcal{R} and $f \in L^{\infty}(B_2)$.

Let $K: \mathbb{R} \times \mathbb{R} \to [0, +\infty]$ be such that

(5.1) for all
$$x \neq x_0$$
, we have that $\frac{1}{C|x - x_0|^{n+s}} \leqslant K(x, x_0) \leqslant \frac{C}{|x - x_0|^{n+s}}$, and, for all $y \in B_\delta \setminus \{0\}$, $|K(y_{x_0}^+, x_0) - K(y_{x_0}^-, x_0)| \leqslant C|y|^{1-n-s}$,

for some $C \geqslant 1$.

Assume that, for all $x_0 \in \mathcal{R} \cap B_1$, the function $v \in L^{\infty}(\mathcal{R})$ is a solution of

$$\int_{\mathcal{R}} \left(v(x_0) - v(x) \right) K(x, x_0) d\mathcal{H}_x^{n-1} = f(x_0).$$

Then, v is Hölder continuous in $\mathcal{R} \cap B_{1/2}$. More precisely,

(5.2)
$$||v||_{C^{\alpha}(\mathcal{R}\cap B_{1/2})} \leq C_0 \Big(||f||_{L^{\infty}(B_2)} + ||v||_{L^{\infty}(\mathcal{R})} \Big),$$

where $\alpha \in (0,1)$ and $C_0 > 0$ depend only on n, s, the regularity parameters of \mathcal{R} and the structural constant C of the kernel K in (5.1).

Proof. We let $u := v\chi_{B_{3/4}}$ and we see that, for all $x_0 \in \mathcal{R} \cap B_{2/3}$,

$$\int_{\mathcal{R}} (u(x_0) - u(x)) K(x, x_0) d\mathcal{H}_x^{n-1} = \int_{\mathcal{R}} (v(x_0) - v(x)\chi_{B_{3/4}}(x)) K(x, x_0) d\mathcal{H}_x^{n-1}$$
$$= f(x_0) + \int_{\mathcal{R}\setminus B_{3/4}} v(x) K(x, x_0) d\mathcal{H}_x^{n-1} =: g(x_0)$$

and

$$|g(x_0)| \leq ||f||_{L^{\infty}(B_2)} + C||v||_{L^{\infty}(\mathcal{R})} \int_{\mathcal{R}\setminus B_{3/4}} \frac{d\mathcal{H}_x^{n-1}}{|x - x_0|^{n+s}}$$
$$\leq C\Big(||f||_{L^{\infty}(B_2)} + \mathcal{H}^{n-1}(\mathcal{R})||v||_{L^{\infty}(\mathcal{R})}\Big) =: C_{\star},$$

up to renaming C from line to line.

We now take $B_2' := B_2 \cap \{x_n = 0\}$ and a C^2 -diffeomorphism $\phi : \mathcal{R} \to B_2'$. We can also adapt the diffeomorphism so that $\phi(\mathcal{R} \cap B_{2/3})$ contains $B_{3/4}'$. We define $U(y) := u(\phi^{-1}(y))$ and we observe that, for all $y_0 = \phi(x_0) \in B_{3/4}'$,

(5.3)
$$G(y_0) := g(\phi^{-1}(y_0)) = \int_{\mathcal{R}} \left(u(\phi^{-1}(y_0)) - u(x) \right) K(x, \phi^{-1}(y_0)) d\mathcal{H}_x^{n-1}$$
$$= \int_{B_2'} \left(U(y_0) - U(y) \right) K_*(y, y_0) d\mathcal{H}_y^{n-1},$$

where

(5.4)
$$K_*(y, y_0) := K(\phi^{-1}(y), \phi^{-1}(y_0)) J(y),$$

for a suitable Jacobian function J, which is bounded and bounded away from zero.

Thus, by (5.1), we have that $K_*(y, y_0)$ is bounded from above and below, up to constants, by $|\phi^{-1}(y) - \phi^{-1}(y_0)|^{-(n+s)}$, which in turn is comparable to $|y - y_0|^{-(n+s)}$, since ϕ is a diffeomorphism.

Additionally, if

$$L(y_0, z) := K_*(y_0 + z, y_0) - K_*(y_0 - z, y_0),$$

we have that

$$|L(y_{0},z)| \leq |K(\phi^{-1}(y_{0}+z),\phi^{-1}(y_{0})) - K(\phi^{-1}(y_{0}-z),\phi^{-1}(y_{0}))| J(y_{0}+z) + |K(\phi^{-1}(y_{0}-z),\phi^{-1}(y_{0}))| |J(y_{0}+z) - J(y_{0}-z)|$$

$$\leq C|K(\phi^{-1}(y_{0}+z),\phi^{-1}(y_{0})) - K(\phi^{-1}(y_{0}-z),\phi^{-1}(y_{0}))| + C|z|^{-n-s} |J(y_{0}+z) - J(y_{0}-z)|$$

$$\leq C|z|^{1-n-s},$$

up to renaming C line after line.

Actually, up to renaming G by an additional bounded function, if we extend U by zero outside B'_2 we can rewrite (5.3) in the form

(5.6)
$$\int_{\mathbb{R}^{n-1}} (U(y_0) - U(y)) K_*(y, y_0) d\mathcal{H}_y^{n-1} = G(y_0),$$

for all $y_0 \in B'_{3/4}$.

We can thereby apply the regularity theory for integro-differential equations (see e.g. [Sil06, Theorem 5.4 and Remark 4.4], or [CLD12, Theorem 3.1], or [KW, Theorem 1.1(ii)]) and obtain the desired Hölder estimate for U, and therefore for u, and then for v.

6. An upper bound in a geometric setting

In the proof of our main result, as it will be presented in Section 8, a somewhat delicate issue comes from the possibility of extracting a convergent sequence from the renormalized distance between minimal sheets. Roughly speaking, the plan would be to remove an arbitrarily small neighborhood of the singular set and focus on an arbitrarily large ball, obtain estimates in this domain which are uniform with respect to the sequences under consideration (with constants possibly depending on the neighborhood of the singular set and on the radius of the large ball), pass the sequence to the limit, and only at the end of the argument shrink the neighborhood of the singular set and invade the space by larger and larger balls.

In this setting, Lemma 5.1 is instrumental to provide the necessary compactness. However, to use this result, one needs a bound in L^{∞} , as dictated by the right-hand side of (5.2).

Such a bound does not come completely for free, not even in the situation in which the distance between minimal sheets is normalized to be 1 at some point, due to the possible divergence of this function at the singular set and the correspondingly large values that this function could, in principle, attain in the domain under consideration. To get around such a difficulty, we present here an L^{∞} bound in terms of an integral estimate (as made precise by (6.3) here below). This integral control will be then checked in our specific case via a sliding method (as made precise by the forthcoming Lemma 7.1).

This strategy will thus allow us to pass to the limit locally away from the singular set: summarizing, one needs to just establish an integral bound (that will come from Lemma 7.1), to deduce from it a bound in L^{∞} (coming from the next result), and thus obtain a bound in C^{α} (coming from Lemma 5.1), which in turns provides the desired compactness property (by the Arzelà-Ascoli Theorem).

Lemma 6.1. Let \mathcal{R} be a portion of a C^2 hypersurface in B_2 that is C^2 -diffeomorphic to $B_2 \cap \{x_n = 0\}$. Let ν the unit normal vector field of \mathcal{R} .

Let $K: \mathbb{R} \times \mathbb{R} \to [0, +\infty]$ be such that

(6.1) for all
$$x \neq x_0$$
, we have that $\frac{1}{C|x - x_0|^{n+s}} \leqslant K(x, x_0) \leqslant \frac{C}{|x - x_0|^{n+s}}$, and, for all $y \in B_\delta \setminus \{0\}$, $|K(y_{x_0}^+, x_0) - K(y_{x_0}^-, x_0)| \leqslant C|y|^{1-n-s}$,

for some $C \geqslant 1$.

Assume that, for all $x_0 \in \mathcal{R} \cap B_1$, the function v satisfies

(6.2)
$$\int_{\mathcal{R}} (v(x_0) - v(x)) K(x, x_0) d\mathcal{H}_x^{n-1} - a(x_0)v(x_0) \leqslant M,$$

for some $M \geqslant 0$ and $a \in L^{\infty}(\mathbb{R}^n)$, and that

(6.3)
$$\int_{\mathcal{R}} v^{+}(x) d\mathcal{H}_{x}^{n-1} \leqslant M',$$

for some $M' \ge 0$.

Then, v is bounded from above in $\mathcal{R} \cap B_{1/2}$, with

$$\sup_{\mathcal{R} \cap B_{1/2}} v \leqslant C_0 \left(1 + M + M' \right),$$

with $C_0 > 0$ depending only on n, s, $||a||_{L^{\infty}(\mathbb{R}^n)}$, the regularity parameters of \mathcal{R} and the structural constant C of the kernel K in (6.1).

Proof. The argument presented here extends to the geometric framework and for more general kernels a method of proof utilized in [CDSS17, Lemma 5.2]. Specifically, recalling the notation $B'_r := B_r \cap \{x_n = 0\}$, as in (5.4) we consider a C^2 -diffeomorphism $\phi : \mathcal{R} \to B'_2$, define $V(y) := v(\phi^{-1}(y))$ and look at the corresponding integral equation driven by the kernel K_*

We look at the function $B_1' \ni y \mapsto \Psi(y) := \Theta(1-|y|^2)^{-n}$, with $\Theta > 0$ suitably large, and we slide the graph of Ψ by above in B_1' till we touch the graph of V. That is, we find $t \in \mathbb{R}$ such that $V \leqslant \Psi + t$ in B_1' with equality holding at some $q \in B_1'$. We can assume that $t \geqslant 0$ (otherwise, $v \leqslant \Psi + t \leqslant \Psi$ and this would give the desired bound).

Let also d := 1 - |q|. Let $r \in (0, \frac{d}{2})$ to be taken suitably small. We consider the tangent plane of the barrier $\Psi + t$ at q, namely the linear function

$$\ell(y) := \nabla \Psi(q) \cdot (y - q).$$

It would be desirable to freely subtract tangent planes in the operators, but this is not possible in our framework, due to the lack of symmetry of the kernel, therefore we need to take care of an additional remainder. Namely,

$$\left| 2 \int_{B'_{r}(q)} \ell(y) K_{*}(y,q) d\mathcal{H}_{y}^{n-1} \right| = \left| 2 \nabla \Psi(q) \cdot \int_{B'_{r}(q)} (y-q) K_{*}(y,q) d\mathcal{H}_{y}^{n-1} \right|$$
$$= \left| \nabla \Psi(q) \cdot \int_{B'_{r}} z \left(K_{*}(q+z,q) - K_{*}(q-z,q) \right) d\mathcal{H}_{z}^{n-1} \right|.$$

Now, as observed in (5.5), it follows from (6.1) that

$$|K_*(q+z,q) - K_*(q-z,q)| \le C|z|^{1-n-s}$$

As a result,

$$\left| 2 \int_{B'_r(q)} \ell(y) K_*(y,q) d\mathcal{H}_y^{n-1} \right| \leq |\nabla \Psi(q)| \int_{B'_r} |z| \left| K_*(q+z,q) - K_*(q-z,q) \right| d\mathcal{H}_z^{n-1}$$

$$\leq C |\nabla \Psi(q)| \int_{B'_z} |z|^{2-n-s} d\mathcal{H}_z^{n-1} \leq C |\nabla \Psi(q)| r^{1-s}.$$

Consequently, if $q := \phi(p)$ and $\mathcal{R}_r(p) := \phi^{-1}(B'_r(q))$,

$$\int_{\mathcal{R}_{r}(p)} (v(p) - v(x)) K(x, p) d\mathcal{H}_{x}^{n-1}$$

$$= \int_{B'_{r}(q)} (V(q) - V(y) + \ell(y)) K_{*}(y, q) d\mathcal{H}_{y}^{n-1} - \int_{B'_{r}(q)} \ell(y) K_{*}(y, q) d\mathcal{H}_{y}^{n-1}$$

$$\geqslant - \int_{B'_{r}(q)} (V(y) - V(q) - \ell(y)) K_{*}(y, q) d\mathcal{H}_{x}^{n-1} - C|\nabla \Psi(q)| r^{1-s}.$$

Now we use the notation of positive and negative parts of a function, namely $g = g_+ - g_-$, where $g_+ := \max\{g, 0\}$ and $g_- := \max\{-g, 0\}$, to see that, if $y \in B'_r(q)$,

$$|y| \le |q| + |y - q| \le 1 - d + r \le 1 - \frac{d}{2}$$

and, as a consequence,

$$\begin{split} \left(V(y) - V(q) - \ell(y)\right)_+ &\leqslant & \max\{\Psi(y) - \Psi(q) - \ell(y), 0\} \\ &\leqslant & \frac{C|y - q|^2}{d^{n+2}}. \end{split}$$

For this reason,

(6.5)
$$\int_{B'_{r}(q)} \left(V(y) - V(q) - \ell(y) \right)_{+} K_{*}(y,q) d\mathcal{H}_{y}^{n-1} \leqslant \frac{C}{d^{n+2}} \int_{B'_{r}(q)} |y - q|^{2} K_{*}(y,q) d\mathcal{H}_{y}^{n-1}$$

$$\leqslant \frac{C}{d^{n+2}} \int_{0}^{r} \frac{d\rho}{\rho^{s}} = \frac{Cr^{1-s}}{d^{n+2}}.$$

Moreover,

$$(V(y) - V(q) - \ell(y))_{-} \geqslant V(q) - V(y) + \ell(y)$$
$$\geqslant \Psi(q) + t - V^{+}(y) + \ell(y)$$

and therefore

$$\int_{B'_{r}(q)} \left(V(y) - V(q) - \ell(y) \right)_{-} K_{*}(y, q) d\mathcal{H}_{y}^{n-1}
\geqslant \frac{1}{C} \int_{B'_{r}(q)} \left(V(y) - V(q) - \ell(y) \right)_{-} \frac{d\mathcal{H}_{y}^{n-1}}{|y - q|^{n+s}}
\geqslant \frac{1}{Cr^{n+s}} \int_{B'_{r}(q)} \left(V(y) - V(q) - \ell(y) \right)_{-} d\mathcal{H}_{y}^{n-1}
\geqslant \frac{1}{Cr^{n+s}} \int_{B'_{r}(q)} \left(\Psi(q) + t - V^{+}(y) + \ell(y) \right) d\mathcal{H}_{y}^{n-1}
\geqslant \frac{1}{Cr^{n+s}} \left(\left(\Psi(q) + t \right) r^{n-1} - \int_{B'_{r}(q)} V^{+}(y) d\mathcal{H}_{y}^{n-1} \right) - \frac{C|\nabla \Psi(q)|}{r^{s}}
\geqslant \frac{\Psi(q) + t}{Cr^{1+s}} - \frac{C}{r^{n+s}} \int_{\mathcal{R}_{r}(p)} v^{+}(x) d\mathcal{H}_{x}^{n-1} - \frac{C|\nabla \Psi(q)|}{r^{s}}.$$

Note that

$$\Psi(q) = \Theta(1 - |q|^2)^{-n} = \Theta(1 + |q|)^{-n}(1 - |q|)^{-n} \geqslant 2^{-n}\Theta d^{-n}$$

and

$$|\nabla \Psi(q)| = \frac{2n\Theta|q|}{(1-|q|^2)^{n+1}} \leqslant \frac{2n\Theta}{d^{n+1}},$$

giving that

$$\int_{B'_{r}(q)} \left(V(y) - V(q) - \ell(y) \right)_{-} K_{*}(y, q) d\mathcal{H}_{y}^{n-1}$$

$$\geqslant \frac{\Theta}{C d^{n} r^{1+s}} + \frac{t}{C r^{1+s}} - \frac{C}{r^{n+s}} \int_{\mathcal{R}_{-}(p)} v^{+}(x) d\mathcal{H}_{x}^{n-1} - \frac{C\Theta}{d^{n+1} r^{s}}.$$

Notice that

$$\frac{\Theta}{Cd^nr^{1+s}} - \frac{C\Theta}{d^{n+1}r^s} = \frac{\Theta}{Cd^nr^s} \left(\frac{1}{r} - \frac{C^2}{d}\right) > \frac{\Theta}{Cd^nr^{1+s}}$$

up to renaming C, if r is smaller than a small constant times d, and therefore

$$\int_{B'_{r}(q)} (V(y) - V(q) - \ell(y))_{-} K_{*}(y, q) d\mathcal{H}_{y}^{n-1}$$

$$\geqslant \frac{\Theta}{Cd^{n}r^{1+s}} + \frac{t}{Cr^{1+s}} - \frac{C}{r^{n+s}} \int_{\mathcal{R}_{r}(p)} v^{+}(x) d\mathcal{H}_{x}^{n-1}.$$

Combining this estimate and (6.5), we infer that

$$\int_{B'_{r}(q)} \left(V(y) - V(q) - \ell(y) \right) K_{*}(y, q) d\mathcal{H}_{y}^{n-1}$$

$$\leq -\frac{\Theta}{Cd^{n}r^{1+s}} - \frac{t}{Cr^{1+s}} + \frac{C}{r^{n+s}} \int_{\mathcal{R}_{x}(p)} v^{+}(x) d\mathcal{H}_{x}^{n-1} + \frac{Cr^{1-s}}{d^{n+2}}.$$

From this and (6.4) we conclude that

(6.6)
$$\int_{\mathcal{R}_{r}(p)} \left(v(p) - v(x) \right) K(x, p) d\mathcal{H}_{x}^{n-1} \\ \geqslant \frac{\Theta}{Cd^{n}r^{1+s}} + \frac{t}{Cr^{1+s}} - \frac{C}{r^{n+s}} \int_{\mathcal{R}_{r}(p)} v^{+}(x) d\mathcal{H}_{x}^{n-1} - \frac{Cr^{1-s}}{d^{n+2}},$$

where, as customary, we have renamed constants line after line. Also,

$$v(p) = V(q) = \Psi(q) + t \geqslant \Psi(q) \geqslant 0$$

and therefore

$$\int_{\mathcal{R}\backslash\mathcal{R}_{r}(p)} \left(v(p) - v(x)\right) K(x,p) d\mathcal{H}_{x}^{n-1} \geqslant -C \int_{\mathcal{R}\backslash\mathcal{R}_{r}(p)} v^{+}(x) \frac{d\mathcal{H}_{x}^{n-1}}{|x - p|^{n+s}}$$

$$\geqslant -\frac{C}{r^{n+s}} \int_{\mathcal{R}\backslash\mathcal{R}_{r}(p)} v^{+}(x) d\mathcal{H}_{x}^{n-1} \geqslant -\frac{C}{r^{n+s}} \int_{\mathcal{R}} v^{+}(x) d\mathcal{H}_{x}^{n-1}.$$

This, (6.2) and (6.6) yield that, if Θ is large as specified above,

$$M \geqslant \int_{\mathcal{R}} (v(p) - v(x)) K(x, p) d\mathcal{H}_{x}^{n-1} - a(p)v(p)$$

$$\geqslant \frac{\Theta}{Cd^{n}r^{1+s}} + \frac{t}{Cr^{1+s}} - \frac{C}{r^{n+s}} \int_{\mathcal{R}} v^{+}(x) d\mathcal{H}_{x}^{n-1} - \frac{Cr^{1-s}}{d^{n+2}} - a(p)(\Psi(q) + t)$$

$$\geqslant \frac{\Theta}{Cd^{n}r^{1+s}} + \frac{t}{Cr^{1+s}} - \frac{C}{r^{n+s}} \int_{\mathcal{R}} v^{+}(x) d\mathcal{H}_{x}^{n-1} - \frac{Cr^{1-s}}{d^{n+2}} - a(p)\Psi(q)$$

$$\geqslant \frac{\Theta}{Cd^{n}r^{1+s}} + \frac{t}{Cr^{1+s}} - \frac{C}{r^{n+s}} \int_{\mathcal{R}} v^{+}(x) d\mathcal{H}_{x}^{n-1} - \frac{Cr^{1-s}}{d^{n+2}} - \frac{C\Theta}{d^{n}}.$$

Notice that for r small enough, the latter term can be reabsorbed. In particular, taking $r = \alpha d$, for a small $\alpha \in (0, 1)$, we find that

$$M \geqslant \frac{\Theta}{Cd^{n}r^{1+s}} + \frac{t}{Cr^{1+s}} - \frac{C}{r^{n+s}} \int_{\mathcal{R}} v^{+}(x) d\mathcal{H}_{x}^{n-1} - \frac{Cr^{1-s}}{d^{n+2}}$$

$$= \frac{\Theta}{C\alpha^{1+s}d^{n+1+s}} + \frac{t}{C\alpha^{1+s}d^{1+s}} - \frac{C}{\alpha^{n+s}d^{n+s}} \int_{\mathcal{R}} v^{+}(x) d\mathcal{H}_{x}^{n-1} - \frac{C\alpha^{1-s}}{d^{n+1+s}}$$

$$\geqslant \frac{t}{C\alpha^{1+s}d^{1+s}} + \frac{\Theta}{\alpha^{1+s}d^{n+1+s}} \left[\frac{1}{C} - \frac{Cd}{\alpha^{n-1}\Theta} \int_{\mathcal{R}} v^{+}(x) d\mathcal{H}_{x}^{n-1} - \frac{C\alpha^{2}}{\Theta} \right].$$

The smallness of α having played its role, we omit it from the notation from now on, absorbing it into the constants. In particular, if

$$\Theta := C \left(1 + \int_{\mathcal{R}} v^{+}(x) \, d\mathcal{H}_{x}^{n-1} \right),$$

with C large enough, we have that

$$M \geqslant \frac{t}{Cd^{1+s}} + \frac{\Theta}{Cd^{n+1+s}} \geqslant \frac{t}{Cd^{1+s}}.$$

All in all, we have found that, if $\xi \in \mathcal{R} \cap B_{1/2}$,

$$v(\xi) = V(\phi(\xi)) \leqslant \Psi(\phi(\xi)) + t$$

$$\leqslant C \left(1 + M + \int_{\mathcal{R}} v^{+}(x) d\mathcal{H}_{x}^{n-1} \right) (1 - |\phi(\xi)|^{2})^{-n}$$

$$\leqslant C \left(1 + M + \int_{\mathcal{R}} v^{+}(x) d\mathcal{H}_{x}^{n-1} \right),$$

up to renaming C.

7. An integral bound in a geometric setting

In retrospect, the Hölder regularity theory established in Lemma 5.1 relied on a specific hypothesis, namely an L^{∞} bound, which in turn was reduced to an integral bound by means of Lemma 6.1.

However, in our setting, integral bounds do not come completely for free. As already mentioned, the complication arises from the fact that we need to apply this regularity theory to a normalized parameterization between minimal sheets: thus, in view of the possible divergence of normal parameterizations at singular points, L^{∞} bounds, and even L^{1} bounds, may be nontrivial. The next result fills this gap and ensures an L^{1} bound, which, in our application, will lead to an L^{∞} bound (via Lemma 6.1) and thus to a Hölder estimate (owing to Lemma 5.1), from which one will deduce suitable compactness properties of the normalized distance of minimal sheets.

Lemma 7.1. Let \mathcal{R} be a portion of a C^2 hypersurface in B_2 that is C^2 -diffeomorphic to $B_2 \cap \{x_n = 0\}$. Let ν be the unit normal vector field of \mathcal{R} .

Let $K: \mathbb{R} \times \mathbb{R} \to [0, +\infty]$ be such that

(7.1) for all
$$x \neq x_0$$
, we have that $\frac{1}{C|x - x_0|^{n+s}} \leqslant K(x, x_0) \leqslant \frac{C}{|x - x_0|^{n+s}}$, and, for all $y \in B_\delta \setminus \{0\}$, $|K(y_{x_0}^+, x_0) - K(y_{x_0}^-, x_0)| \leqslant C|y|^{1-n-s}$,

for some $C \geqslant 1$.

Assume that, for all $x_0 \in \mathcal{R} \cap B_1$, the function v satisfies

(7.2)
$$\int_{\mathcal{R}} (v(x_0) - v(x)) K(x, x_0) d\mathcal{H}_x^{n-1} - a(x_0) v(x_0) \in [-M_0, M_0],$$

for some $M_0 > 0$ and $a \in L^{\infty}(\mathbb{R}^n, [0, +\infty))$.

Assume also that v is nonnegative and that $v(x_*) = 1$, for some $x_* \in \mathcal{R} \cap B_1$. Then,

$$\int_{\mathcal{R}} v(x) \, d\mathcal{H}_x^{n-1} \leqslant C_0,$$

with $C_0 > 0$ depending only on n, s, M_0 , $||a||_{L^{\infty}(\mathbb{R}^n)}$, the regularity parameters of \mathcal{R} and the structural constant C of the kernel K in (7.1).

Proof. We straighten \mathcal{R} by a diffeomorphism $\phi: \mathcal{R} \to B_2' := B_2 \cap \{x_n = 0\}$ and set $V(y) := v(\phi^{-1}(y))$. We let $y_* := \phi(x_*)$ and slide the parabola $-M|y - y_*|^2$ from below till we touch the graph of V at some point y_{\sharp} . Notice that

$$M|y_{\sharp} - y_{\star}|^{2} \leqslant V(y_{\sharp}) + M|y_{\sharp} - y_{\star}|^{2} \leqslant V(y_{\star}) = v(x_{\star}) = 1.$$

Thus, the parameter M > 0 is fixed (once and for all) sufficiently large such that the touching point y_{\sharp} lies in $B'_{3/2}$.

We remark that

$$P(y) := V(y_{\sharp}) + M|y_{\sharp} - y_{\star}|^{2} - M|y - y_{\star}|^{2} \leqslant V(y)$$

and in particular $P(y_{\sharp}) = V(y_{\sharp}) \geqslant 0$.

Hence, if $\widetilde{V} := V - P$, we have that $\widetilde{V}(y_{\sharp}) = 0$ and thus, recalling the notation in (5.4),

$$\int_{B'_{2}} \widetilde{V}(y) K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1} = \int_{B'_{2}} \left(\widetilde{V}(y) - \widetilde{V}(y_{\sharp}) \right) K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1}
= \int_{B'_{2}} \left(V(y) - V(y_{\sharp}) \right) K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1} - \int_{B'_{2}} \left(P(y) - P(y_{\sharp}) \right) K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1}
\leq \int_{B'_{2}} \left(V(y) - V(y_{\sharp}) \right) K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1} + C_{0}
= \int_{\mathcal{P}} \left(v(x) - v(x_{\sharp}) \right) K(x, x_{\sharp}) d\mathcal{H}_{x}^{n-1} + C_{0},$$

with C_0 bounded in terms of n, s, the regularity parameters of \mathcal{R} and the structural constant C of the kernel K in (7.1).

From this observation and (7.2), we infer that

$$\int_{B_2'} \widetilde{V}(y) K_*(y, y_{\sharp}) d\mathcal{H}_y^{n-1} \leqslant M_0 - a(x_{\sharp}) v(x_{\sharp}) + C_0 \leqslant C_0,$$

with C_0 now depending also on M_0 .

Since $\tilde{V} \geqslant 0$, we thereby conclude that

$$\int_{B'_{1/100}(y_{\sharp})} \widetilde{V}(y) K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1} \leqslant C_{0}.$$

We point out that, if $y \in B'_{1/100}(y_{\sharp})$,

$$K_*(y, y_{\sharp}) \geqslant \frac{1}{C|y - y_{\sharp}|^{n+s}} \geqslant \frac{100^{n+s}}{C},$$

and thus

$$\int_{B'_{1/100}(y_{\sharp})} \widetilde{V}(y) \, d\mathcal{H}_{y}^{n-1} \leqslant \frac{C}{100^{n+s}} \int_{B'_{1/100}(y_{\sharp})} \widetilde{V}(y) \, K_{*}(y, y_{\sharp}) \, d\mathcal{H}_{y}^{n-1} \leqslant C_{0}.$$

As a result, up to keeping renaming C_0 ,

(7.3)
$$\int_{B'_{1/100}(y_{\sharp})} V(y) d\mathcal{H}_{y}^{n-1} \leq \int_{B'_{1/100}(y_{\sharp})} P(y) d\mathcal{H}_{y}^{n-1} + C_{0} \leq C_{0}.$$

Furthermore,

$$\int_{B'_{1/100}(y_{\sharp})} \left(V(y_{\sharp}) - V(y) \right) K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1} \\
\leqslant \int_{B'_{1/100}(y_{\sharp})} \left(P(y_{\sharp}) - P(y) \right) K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1} \leqslant C_{0}$$

and therefore one deduces from (7.2) that

$$C_{0} + \int_{B'_{2} \setminus B'_{1/100}(y_{\sharp})} \left(V(y_{\sharp}) - V(y) \right) K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1}$$

$$\geqslant \int_{B'_{2}} \left(V(y_{\sharp}) - V(y) \right) K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1}$$

$$= \int_{\mathcal{R}} \left(v(x_{\sharp}) - v(x) \right) K(x, x_{\sharp}) d\mathcal{H}_{x}^{n-1}$$

$$\geqslant a(x_{\sharp}) v(x_{\sharp}) - C_{0}$$

$$\geqslant -C_{0}.$$

This gives that

$$\int_{B'_{2} \setminus B'_{1/100}(y_{\sharp})} V(y) K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1}
\leq V(y_{\sharp}) \int_{B'_{2} \setminus B'_{1/100}(y_{\sharp})} K_{*}(y, y_{\sharp}) d\mathcal{H}_{y}^{n-1} + C_{0}
= C_{0} (V(y_{\sharp}) + 1)
\leq C_{0} (V(y_{\star}) - M|y_{\sharp} - y_{\star}|^{2} + 1)
\leq C_{0},$$

up to keeping renaming C_0 , and consequently

$$\int_{B_2' \setminus B_{1/100}'(y_\sharp)} V(y) \, d\mathcal{H}_y^{n-1} \leqslant C_0.$$

This and (7.3) yield the desired result.

We now present a useful variant² of Lemma 7.1. Its utility in our context is that the geometric equation that we deal with will present an integral term coming from the nonlocal mean curvature of two minimal sets that we cannot reabsorb into "smooth" objects (due to the fact that the domain of integration is a "bad set", say \mathcal{B} , containing far away points and points close to the singular set, for which any regularity information is missing). The next result however will provide a uniform control of such additional term.

Lemma 7.2. Let \mathcal{R} be a portion of a C^2 hypersurface in B_2 that is C^2 -diffeomorphic to $B_2 \cap \{x_n = 0\}$. Let ν be the unit normal vector field of \mathcal{R} .

Let
$$K : \mathcal{R} \times \mathcal{R} \to [0, +\infty]$$
 be such that

(7.4) for all
$$x \neq x_0$$
, we have that $\frac{1}{C|x-x_0|^{n+s}} \leqslant K(x,x_0) \leqslant \frac{C}{|x-x_0|^{n+s}}$, and, for all $y \in B_{\delta} \setminus \{0\}$, $|K(y_{x_0}^+, x_0) - K(y_{x_0}^-, x_0)| \leqslant C|y|^{1-n-s}$,

for some $C \geqslant 1$.

²Actually, not only the proofs of Lemmata 7.1 and 7.2 are similar, but one could state just one single, albeit more complicated, result to condensate Lemmata 7.1 and 7.2 into a single statement. For the sake of simplicity, however, we prefer to keep the two statements separate.

Assume that, for all $\tilde{x} \in \mathcal{R} \cap B_{3/2}$, the nonnegative function w satisfies

(7.5)
$$\int_{\mathcal{R}} \left(w(\widetilde{x}) - w(x) \right) K(x, \widetilde{x}) d\mathcal{H}_{x}^{n-1} - \int_{\mathcal{B}} \frac{\Phi(X)}{|X - \widetilde{X}|^{n+s}} dX \geqslant -\mu,$$

where \mathcal{B} is some measurable set of $\mathbb{R}^n \setminus \mathcal{R}$, $\Phi \in L^{\infty}(\mathbb{R}^n, [0, 1])$, $\mu \geqslant 0$, and $\widetilde{X} := \widetilde{x} + w(\widetilde{x})\nu(\widetilde{x})$. Assume that there exists a point x_0 in $\mathcal{R} \cap B_1$ whose distance from \mathcal{B} is bounded from below by some $r_0 > \widetilde{C} \|w\|_{L^{\infty}(\mathcal{R} \cap B_2)}$, with $\widetilde{C} > 1$.

Then, there exists $C_0 > 0$, depending only on n, s, r_0 , the regularity parameters of \mathcal{R} and the structural constant C of the kernel in (7.4), such that, if $\widetilde{C} \geq C_0$,

(7.6)
$$\int_{\mathcal{B}} \frac{\Phi(X)}{|X - X_0|^{n+s}} dX \leqslant C_0 \left(\|w\|_{L^{\infty}(\mathcal{R} \cap B_2)} + \mu \right),$$

where $X_0 := x_0 + w(x_0)\nu(x_0)$.

Proof. We straighten \mathcal{R} by a diffeomorphism $\phi: \mathcal{R} \to B_2' := B_2 \cap \{x_n = 0\}$ and set $W(y) := w(\phi^{-1}(y))$. We can also suppose for simplicity that $\mathcal{R} \cap B_1$ is mapped by ϕ in B_1' and that $\mathcal{R} \cap B_{3/2}$ is mapped by ϕ in $B_{3/2}'$. Let $x_0 \in \mathcal{R} \cap B_1$ with $B_{r_0}(x_0) \cap \mathcal{B} = \emptyset$. Let also $y_0 := \phi(x_0)$ and notice that $y_0 \in B_1'$.

Given M > 0, we slide the parabola $P(y) := -M|y - y_0|^2$ from below till we touch W at some point \hat{y} . The touching condition between the slid parabola and the graph of W entails that

(7.7)
$$W(\widehat{y}) + M|\widehat{y} - y_0|^2 - M|y - y_0|^2 \leqslant W(y).$$

Here we choose

(7.8)
$$M := C_*^2 \|w\|_{L^{\infty}(\mathcal{R} \cap B_2)} \max \left\{ 1, \frac{1}{r_0^2} \right\},$$

with C_* to be chosen sufficiently large. In this way, by evaluating (7.7) at $y := y_0$,

$$|\widehat{y} - y_0| \leqslant \sqrt{\frac{W(y_0) - W(\widehat{y})}{M}} \leqslant \sqrt{\frac{W(y_0)}{M}} = \sqrt{\frac{w(x_0)}{M}}$$
$$\leqslant \sqrt{\frac{1}{C_*^2 \max\left\{1, \frac{1}{r_0^2}\right\}}} = \frac{1}{C_*} \min\{1, r_0\}.$$

In particular, if C_* is large enough, we have that $\widehat{y} \in B'_{3/2}$ and $|\widehat{y} - y_0| < \frac{r_0}{10}$.

In this way, the point $\widehat{x} := \phi^{-1}(\widehat{y})$ belongs to $\mathcal{R} \cap B_{3/2}$. We can therefore utilize (7.5) at \widehat{x} , finding that

(7.9)
$$\int_{\mathcal{B}} \frac{\Phi(X)}{|X - \widehat{X}|^{n+s}} dX \leqslant \int_{\mathcal{R}} \left(w(\widehat{x}) - w(x) \right) K(x, \widehat{x}) d\mathcal{H}_{x}^{n-1} + \mu$$
$$= \int_{B_{2}^{\prime}} \left(W(\widehat{y}) - W(y) \right) K_{*}(y, \widehat{y}) d\mathcal{H}_{y}^{n-1} + \mu,$$

where $\widehat{X} := \widehat{x} + w(\widehat{x})\nu(\widehat{x})$ and we used the notation in (5.4).

Notice that, on the one hand,

$$|X - \widehat{X}| \le |X - X_0| + |X_0 - \widehat{X}| \le |X - X_0| + |x_0 - \widehat{x}| + 2||w||_{L^{\infty}(\mathcal{R} \cap B_2)}$$

$$\leq |X - X_0| + \frac{r_0}{10} + 2||w||_{L^{\infty}(\mathcal{R} \cap B_2)} \leq |X - X_0| + \frac{3r_0}{10}.$$

On the other hand, if $X \in \mathcal{B}$,

$$|X - X_0| \geqslant |X - x_0| - ||w||_{L^{\infty}(\mathcal{R} \cap B_2)} \geqslant r_0 - ||w||_{L^{\infty}(\mathcal{R} \cap B_2)} \geqslant \frac{9r_0}{10}.$$

By combining these observations, we find that

$$|X - \widehat{X}| \le |X - X_0| + \frac{3}{10} \cdot \frac{10}{9} |X - X_0| \le C_0 |X - X_0|,$$

for some $C_0 > 1$.

This and (7.9), up to renaming C_0 , give that

$$\frac{1}{C_0} \int_{\mathcal{B}} \frac{\Phi(X)}{|X - X_0|^{n+s}} dX \leqslant \int_{B_0'} \left(W(\widehat{y}) - W(y) \right) K_*(y, \widehat{y}) d\mathcal{H}_y^{n-1} + \mu.$$

As a consequence, by (7.7),

$$\frac{1}{C_0} \int_{\mathcal{B}} \frac{\Phi(X)}{|X - X_0|^{n+s}} dX \leqslant M \int_{B_2'} (|y - y_0|^2 - |\widehat{y} - y_0|^2) K_*(y, \widehat{y}) d\mathcal{H}_y^{n-1} + \mu
\leqslant C_0 (M + \mu).$$

The desired result thus follows in view of (7.8).

8. Completion of the proof of Theorem 1.1

We can now complete the proof of Theorem 1.1 by relying on the work carried out so far and on the Harnack Inequality by Cabré and Cozzi in [CC19].

Proof of Theorem 1.1. Suppose, by contradiction, that $E_1 \neq E_2$. Without loss of generality, we may suppose that, in the notation recalled in footnote 1,

(8.1)
$$E_1$$
 and E_2 share the same tangent cone at the origin.

This is a standard procedure in geometric measure theory, based on blow-up, dimensional reduction, and regularity theory. We recall the details for the facility of the reader.

Indeed, if (8.1) does not hold, we have a converging blow-up sequence for E_1 approaching a cone C_1 with the corresponding blow-up sequence for E_2 approaching a cone $C_2 \neq C_1$. We take a rotation \mathcal{R}_1 of C_1 such that $E_1^{(1)} := \mathcal{R}_1 C_1 \subseteq C_2 =: E_2^{(1)}$ and there exists $p^{(1)} \in$ $(\partial E_1^{(1)}) \cap (\partial E_2^{(1)}) \cap (\partial B_1).$

We then consider a blow-up of $E_1^{(1)}$ and $E_2^{(1)}$ at $p^{(1)}$, which produces two new s-minimal cones in \mathbb{R}^n , which will be denoted by $\mathcal{C}_1^{(1)}$ and $\mathcal{C}_2^{(2)}$, with $\mathcal{C}_1^{(1)} \subseteq \mathcal{C}_2^{(2)}$. Now, two cases can occur. If $\mathcal{C}_1^{(1)} = \mathcal{C}_2^{(2)}$, it suffices to replace p, E_1 , E_2 , \mathcal{C}_1 and \mathcal{C}_2 respectively with $p^{(1)}$, $E_1^{(1)}$, $E_2^{(1)}$, $\mathcal{C}_1^{(1)}$ and $\mathcal{C}_2^{(1)}$: in this way we have obtained (8.1) in this new configuration.

If instead $C_1^{(1)} \neq C_2^{(1)}$, we observe that both these cones are cylinders over \mathbb{R}^{n-1} . In this way, by the dimensional reduction (see [CRS10]), we have found two s-minimal cones in \mathbb{R}^{n-1} , which we denote by $\widetilde{E}_1^{(2)}$ and $\widetilde{E}_2^{(2)}$, such that $\widetilde{E}_1^{(2)} \subseteq \widetilde{E}_2^{(2)}$ and $\widetilde{E}_1^{(2)} \neq \widetilde{E}_2^{(2)}$ (and, up to an isometry, $C_j^{(1)} = \widetilde{E}_j^{(2)} \times \mathbb{R}$ for $j \in \{1, 2\}$). We thus repeat the previous algorithm, taking a rotation \mathcal{R}_2 in \mathbb{R}^{n-1} , such that $E_1^{(2)} := \mathcal{R}_2 \widetilde{E}_1^{(2)} \subseteq \widetilde{E}_2^{(1)} =: E_2^{(1)}$ and there exists $p^{(2)} \in (\partial E_1^{(2)}) \cap (\partial E_2^{(2)}) \cap (\partial B_1)$. Then, a blow-up at p_2 produces two s-minimal cones $\mathcal{C}_1^{(2)}$ and $\mathcal{C}_2^{(2)}$, which either coincide (whence we replace p, E_1 , E_2 , \mathcal{C}_1 and \mathcal{C}_2 respectively with $p^{(2)}$, $E_1^{(2)}$, $E_2^{(2)}$, $\mathcal{C}_1^{(2)}$ and $\mathcal{C}_2^{(2)}$) or not, in which case we apply again the dimensional reduction (reducing now to \mathbb{R}^{n-2}) and proceed.

This algorithm will stop at a certain dimension, producing the same s-minimal cones after a blow-up, since, by the regularity of s-minimal cones in dimension 2 (see [SV13]), we know that halfspaces are the only nontrivial minimal cones in \mathbb{R}^2 . The proof of (8.1) is thereby complete.

Hence, from now on, we will denote by \mathcal{C} the common tangent cone of E_1 and E_2 along a blow-up sequence that we are now going to specify. To this end, we write E_2 in normal coordinates with respect to E_1 at its regular points (this is possible up to an initial dilation), that is suppose that, at a set of regular points, $E_2 \setminus E_1$ has the form $x + t\nu(x)$, with $x \in \partial E_1$ and $t \in [0, w_0(x))$, for a suitable function $w_0 > 0$. Then, by Corollary B.2, we find a set \mathcal{M}_0 of regular points for E_1 and an infinitesimal sequence of points $z_{\star,k} \in \mathcal{M}_0$ such that

(8.2)
$$\frac{w_0(x)}{|x|} \leqslant \frac{2w_0(z_{\star,k})}{|z_{\star,k}|}$$

for all $x \in \mathcal{M}_0$ with $|x| \leq |z_{\star,k}|/2$. In addition, by (B.1),

(8.3) the distance of any point $x \in \mathcal{M}_0$ from the singular set is at least $\theta_0|x|$, with $\theta_0 > 0$.

What is more, by (B.2), we also know that, for all $\rho \in (0, \rho_0]$,

$$\mathcal{M}_0 \cap (\partial B_\rho) \neq \varnothing.$$

Hence, we choose

$$(8.5) r_k := |z_{\star,k}|$$

and consider the corresponding blow-up sequences for $j \in \{1, 2\}$ defined by

$$(8.6) E_{j,k} := \frac{E_j - p}{r_k}.$$

We now apply Theorem 3.4. More specifically, the sets E_1 and E_2 in the statement of Theorem 3.4 are here the sets $E_{1,k}$ and $E_{2,k}$, as defined in (8.6), with k large enough.

The gist of the construction is that we cover the singular set of \mathcal{C} by a set of small balls (the fact that the singular set has dimension at most n-3, due to [SV13,BFV14] guarantees that these balls can be chosen to satisfy (3.21) for η as small as we wish).

Since, for k large enough, $E_{1,k}$ and $E_{2,k}$ locally lie in a small neighborhood of \mathcal{C} , we have that both $E_{1,k}$ and $E_{2,k}$ are smooth away from the above covering of the singular set of \mathcal{C} , thanks to the improvement of flatness of nonlocal minimal surfaces put forth in [CRS10, Corollary 4.4 and Theorem 6.1] (in this way, the convergence of $E_{1,k}$ and $E_{2,k}$ to the limit cone \mathcal{C} occurs in the $C^{1,\alpha}$ sense away from any small neighborhood of the singular set of \mathcal{C} , and actually in the C^k sense for any $k \geq 2$, thanks to the bootstrap regularity in [BFV14]).

This allows us to look at a "large" set³ \mathcal{G} containing regular points as in Theorem 3.4 (and at a small shrinkage of it, namely \mathcal{G}') such that all the singular points of $\partial \mathcal{C}$, ∂E_1 and ∂E_2

³Note that this set \mathcal{G} possibly contains \mathcal{M}_0 .

in a large ball B_R lie outside \mathcal{G} (also, \mathcal{G} covers all B_R , up to a negligible covering of balls, as specified in (3.21)).

Accordingly, for $j \in \{1, 2\}$ and k sufficiently large, one can parameterize $E_{j,k}$ as a graph of class C^2 in the normal direction of C away from a small neighborhood of the singular set.

It is however technically simpler to recenter this parameterization on the "middle surface", namely to parameterize $E_2 = E_{2,k}$ in terms of $E_1 = E_{1,k}$, and this corresponds to the normal parameterization $w = w_k$ in Theorem 3.4. We stress that since both $E_{1,k}$ and $E_{2,k}$ converge to \mathcal{C} locally in the C^2 -sense away from the singular points of $\partial \mathcal{C}$, we also know that the C^2 -norm of $w = w_k$ is as small as we like, provided that k is chosen large enough (possibly in dependence also of the covering of the singular set, which is now fixed once and for all).

We can therefore utilize Theorem 3.4 in this context and conclude that the normal parameterization $w = w_k$ of $E_2 = E_{2,k}$ with respect to $E_1 = E_{1,k}$ satisfies the equation in (3.22), with suitable integrable kernels, as defined in (3.23). More specifically, since E_1 and E_2 are s-minimal sets and therefore $H_{E_1}^s = H_{E_2}^s$ at every regular boundary point (see [CRS10, Theorem 5.1]), we obtain from Theorem 3.4 that

$$0 = \int_{(\partial E_1) \cap \mathcal{G}'} \left(w(x_0) - w(x) \right) K_1(x, x_0) d\mathcal{H}_x^{n-1}$$

$$- w(x_0) \int_{(\partial E_1) \cap \mathcal{G}'} \left(1 - \nu(x_0) \cdot \nu(x) \right) K_2(x, x_0) d\mathcal{H}_x^{n-1}$$

$$+ \frac{1}{2} \int_{\mathbb{R}^n \setminus \mathcal{G}'} \frac{\widetilde{\chi}_{E_2}(X) - \widetilde{\chi}_{E_1}(X)}{|X - X_0|^{n+s}} dX + O\left(w(x_0) \eta \right) + O\left(\frac{w(x_0)}{R^{1+s}} \right),$$

where $\nu = \nu_k$ is the outer unit normal of $\partial E_1 = \partial E_{1,k}$ at its regular points and $X_0 := x_0 + w(x_0)\nu(x_0)$.

Strictly speaking, the function w does not need to be defined everywhere in \mathbb{R}^n for the validity of (8.7), since such an equation does not consider values of w outside certain sets, but we will implicitly suppose that w is defined everywhere for definiteness.

We now take $\mathcal{G}'' \in \mathcal{G}'$ and $x_0 \in (\partial E_1) \cap B_{R/2} \cap \mathcal{G}''$. In this setting, owing to Lemma 7.2 and to the fact that $E_1 \subseteq E_2$, we see that, when the size of w is small,

(8.8)
$$0 \leqslant \int_{\mathbb{R}^{n \setminus G'}} \frac{\widetilde{\chi}_{E_1}(X) - \widetilde{\chi}_{E_2}(X)}{|X - X_0|^{n+s}} dX = O(\|w\|_{L^{\infty}(B_R \cap \mathcal{G})}).$$

Now, since $E_1 \neq E_2$, we have that w does not vanish identically, and in fact w > 0 on the regular part of ∂E_1 (otherwise, we could compute the nonlocal mean curvature of E_1 at that point in the smooth sense, as well as the one of E_2 in the viscosity sense and get a contradiction). Therefore we can normalize w to take value 1 at a suitable point. Roughly speaking, it would be desirable to pick this point to reach the supremum of $\frac{w(x)}{|x|}$, so to obtain a "linear" separation about the minimal sheets at the origin and thus contradict (8.1): however, this choice of the normalizing point may be impossible, since the supremum of $\frac{w(x)}{|x|}$ might well be on the singular set (where w is not even defined) or dangerously close to it. This is the reason for which we have chosen an appropriate sequence of blow-up radii r_k in (8.2) and (8.5).

In this setting, we define $x_{\star,k} := \frac{z_{\star,k}}{|z_{\star,k}|} = \frac{z_{\star,k}}{r_k}$ and, in light of (8.3), we stress that the distance of $x_{\star,k}$ from the singular set is at least θ_0 . This ensures that, up to a subsequence, $x_{\star,k}$ converges⁴ to a regular point $x_{\star,\infty}$ of the limit cone as $k \to +\infty$.

 $x_{\star,k}$ converges⁴ to a regular point $x_{\star,\infty}$ of the limit cone as $k \to +\infty$. Furthermore, by (8.4), given $\delta \in \left(0, \frac{1}{2}\right]$, to be chosen conveniently small in what follows, we can pick a point $y_k \in \mathcal{M}_0 \cap (\partial B_{\delta r_k})$. In this way, by (8.3), we also have that the distance of $y_{\star,k} := \frac{y_k}{r_k}$ from the singular set is at least $\delta \theta_0$ and therefore, up to a subsequence, $y_{\star,k}$ converges to a regular point $y_{\star,\infty}$ of the limit cone as $k \to +\infty$.

We also recall (8.2) and write that

$$\frac{w_0(y_k)}{\delta r_k} = \frac{w_0(y_k)}{|y_k|} \leqslant \frac{2w_0(z_{\star,k})}{|z_{\star,k}|} = \frac{2w_0(z_{\star,k})}{r_k}$$

and accordingly

$$(8.9) w_k(y_{\star,k}) \leqslant 2\delta w_k(x_{\star,k}).$$

We now define

$$v(x) = v_k(x) := \frac{w_k(x)}{w(x_{\star,k})}$$

and we divide (8.7) by $w(x_{\star,k}) = w_k(x_{\star,k})$, recalling also (8.8), to find that, for all $x_0 \in (\partial E_1) \cap B_{R/2} \cap \mathcal{G}''$,

$$0 = \int_{(\partial E_1) \cap \mathcal{G}'} \left(v(x_0) - v(x) \right) K_1(x, x_0) d\mathcal{H}_x^{n-1}$$

$$- v(x_0) \int_{(\partial E_1) \cap \mathcal{G}'} \left(1 - \nu(x_0) \cdot \nu(x) \right) K_2(x, x_0) d\mathcal{H}_x^{n-1}$$

$$- \psi(x_0) + O(\eta) + O\left(\frac{1}{R^{1+s}}\right),$$

where $0 \leqslant \psi = O(1)$.

Also, if $\mathcal{G}''' \in \mathcal{G}''$, we have that, in $(\partial E_1) \cap B_{R/4} \cap \mathcal{G}'''$,

$$(8.11) 0 \leqslant v \leqslant O(1).$$

Indeed, we first observe that $v \ge 0$, since $w \ge 0$. The upper bound in (8.11) follows from the previously developed regularity theory. Specifically, one first employs Lemma 7.1 and finds that

$$\int_{(\partial E_1)\cap \mathcal{G}'} v(x) \, d\mathcal{H}_x^{n-1} \leqslant O(1).$$

This inequality provides the uniform bound corresponding to (6.3) in the present setting. Now, the upper bound in (8.11) is a consequence of Lemma 6.1. The proof of (8.11) is thereby complete.

Accordingly, by the Hölder estimates in Lemma 5.1, we deduce that the C^{α} -norm of v is bounded locally uniformly in k (and we stress that, since $w = w_k$, also $v = v_k$).

⁴Let us stress that the fact that the limit point $x_{\star,\infty}$ remains bounded and bounded away from the singular set is essential to apply the Harnack Inequality in [CC19]: this is the reason for which we can well allow intermediate constants to depend on the radius of the large ball that we are considering, on the set of small balls chosen to cover the singular set, as well as on the regularity of the s-minimal sheets away of this cover, since we will pass $k \to +\infty$ before removing this large ball and this small cover, but we will employ the normalization at the limit point $x_{\star,\infty}$ to bound from below the last term in (8.13).

Therefore, up to a subsequence, v_k converges locally uniformly in $B_{R/4} \cap \mathcal{G}'''$ to a function v_{∞} . In view of (8.10), v_{∞} satisfies, for all $x_0 \in (\partial E_1) \cap B_{R/4} \cap \mathcal{G}'''$,

$$0 \leqslant \int_{(\partial \mathcal{C}) \cap \mathcal{G}'} \frac{v_{\infty}(x_0) - v_{\infty}(x)}{|x - x_0|^{n+s}} d\mathcal{H}_x^{n-1} - v_{\infty}(x_0) \int_{(\partial \mathcal{C}) \cap \mathcal{G}'} \frac{1 - \nu(x_0) \cdot \nu(x)}{|x - x_0|^{n+s}} d\mathcal{H}_x^{n-1} + O(\eta) + O\left(\frac{1}{R^{1+s}}\right).$$

Actually, we can now take R as large as we wish and invade all the space outside the singular set by the "good" domains \mathcal{G} , \mathcal{G}' and \mathcal{G}'' . In this way, we find that, on the regular part $\operatorname{Reg} \mathcal{C}$ of $\partial \mathcal{C}$,

$$0 \leqslant \int_{\operatorname{Reg} \mathcal{C}} \left(v_{\infty}(x_0) - v_{\infty}(x) \right) K_{1,\infty}(x, x_0) d\mathcal{H}_x^{n-1}$$
$$- v_{\infty}(x_0) \int_{\operatorname{Reg} \mathcal{C}} \left(1 - \nu(x_0) \cdot \nu(x) \right) K_{2,\infty}(x, x_0) d\mathcal{H}_x^{n-1}.$$

Now, we wish to apply Lemma 4.1. To this end, we observe that (4.1) is satisfied in our case by $\mathcal{R} := \text{Reg } \mathcal{C}$, thanks to the perimeter estimates for nonlocal minimal surfaces, see [CSV19, equation (1.16)].

Hence, from Lemma 4.1, we deduce that

$$\iint_{(\operatorname{Reg} \mathcal{C} \cap B_2) \times \operatorname{Reg} \mathcal{C}} \frac{(v_Q(x) - v_Q(y))^2}{|x - y|^{n+s}} d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} < +\infty$$

and, for every $\phi \in C_0^{\infty}(\mathbb{R}^n, [0, +\infty))$,

$$\iint_{\operatorname{Reg} \mathcal{C} \times \operatorname{Reg} \mathcal{C}} \frac{(v_Q(x) - v_Q(y))(\phi(x) - \phi(y))}{|x - y|^{n+s}} d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \geqslant 0,$$

where $v_Q := \min\{v_\infty, Q\}$.

Moreover,

$$(8.12) v_{\infty}(x_{\star,\infty}) = \lim_{k \to +\infty} v_k(x_{\star,k}) = 1.$$

It follows from the Harnack Inequality in [CC19, Theorem 1.7] that, for every compact subset \mathcal{K} of Reg \mathcal{C} ,

$$\inf_{\mathcal{K} \cap B_1} v_Q \geqslant c \left(\int_{\mathcal{K} \cap B_1} v_Q(x) \, d\mathcal{H}_x^{n-1} + \int_{\mathcal{K} \setminus B_1} \frac{v_Q(x)}{|x|^{n+s}} \, d\mathcal{H}_x^{n-1} \right),$$

for some c > 0, depending only on n and s.

Sending $Q \to +\infty$ and using (8.12), we thus obtain

(8.13)
$$m := \inf_{\mathcal{K} \cap B_1} v_{\infty} \geqslant c \int_{\mathcal{K} \cap B_1} v_{\infty}(x) d\mathcal{H}_x^{n-1} > 0.$$

As a result, for every compact subset K of Reg $C \cap B_{3/4}$, we have that

$$\inf_{x \in \mathcal{K}} \frac{w_k(x)}{w_k(x_{\star,k})} = \inf_{x \in \mathcal{K}} v_k(x) \geqslant \frac{m}{2},$$

as long as k is large enough (possibly in dependence of K).

In particular,

$$\frac{w_k(y_{\star,k})}{w_k(x_{\star,k})} \geqslant \frac{m}{2},$$

which gives a contradiction with (8.9) when δ is sufficiently small.

Appendix A. Proof of Lemma 4.1

Here we give a self-contained energy argument to prove Lemma 4.1.

Proof of Lemma 4.1. For the convenience of the reader, we split this proof into independent steps.

Step 1. Proof of (4.3). First of all, we have that, for every $x \in \mathcal{R}$,

(A.1)
$$\int_{\mathcal{R}} \frac{u_Q(x) - u_Q(y)}{|x - y|^{n+s}} d\mathcal{H}_y^{n-1} \geqslant 0.$$

Indeed, if $u(x) \ge Q$, the claim is obvious, since in this case $u_Q(x) = Q \ge u_Q(y)$. Therefore, we may suppose that u(x) < Q. In this case,

$$\int_{\mathcal{R}} \frac{u_{Q}(x) - u_{Q}(y)}{|x - y|^{n+s}} d\mathcal{H}_{y}^{n-1} = \int_{\mathcal{R}} \frac{u(x) - u_{Q}(y)}{|x - y|^{n+s}} d\mathcal{H}_{y}^{n-1}
= \int_{\mathcal{R} \cap \{u \geqslant Q\}} \frac{u(x) - Q}{|x - y|^{n+s}} d\mathcal{H}_{y}^{n-1} + \int_{\mathcal{R} \cap \{u < Q\}} \frac{u(x) - u(y)}{|x - y|^{n+s}} d\mathcal{H}_{y}^{n-1}
= \int_{\mathcal{R} \cap \{u \geqslant Q\}} \frac{u(x) - Q}{|x - y|^{n+s}} d\mathcal{H}_{y}^{n-1} - \int_{\mathcal{R} \cap \{u \geqslant Q\}} \frac{u(x) - u(y)}{|x - y|^{n+s}} d\mathcal{H}_{y}^{n-1} + \int_{\mathcal{R}} \frac{u(x) - u(y)}{|x - y|^{n+s}} d\mathcal{H}_{y}^{n-1}
= \int_{\mathcal{R} \cap \{u \geqslant Q\}} \frac{u(y) - Q}{|x - y|^{n+s}} d\mathcal{H}_{y}^{n-1} + \int_{\mathcal{R}} \frac{u(x) - u(y)}{|x - y|^{n+s}} d\mathcal{H}_{y}^{n-1}
\geqslant \int_{\mathcal{R}} \frac{u(x) - u(y)}{|x - y|^{n+s}} d\mathcal{H}_{y}^{n-1}$$

and accordingly (A.1) follows from (4.2).

Now we prove (4.3). To this end, as a byproduct of (A.1), we find that

$$\iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x) - u_Q(y))(\zeta(x) - \zeta(y))}{|x - y|^{n+s}} d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}$$

$$= 2 \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x) - u_Q(y))\zeta(x)}{|x - y|^{n+s}} d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \geqslant 0,$$

which establishes (4.3).

Step 2. Covering arguments. Let now $\varepsilon \in (0,1)$, to be taken as small as we wish here below. From our assumption on S, if $D \in (d, n-2-s]$, we know that S is contained in the union of some balls $\{B_{r_j}(p_j)\}_{j\in\mathbb{N}}$, with

(A.2)
$$\sum_{j \in \mathbb{N}} r_j^D \leqslant \varepsilon.$$

We claim that we can reduce to the case in which

(A.3) the family of dilated balls $\{B_{32r_j}(p_j)\}_{j\in\mathbb{N}}$ has a finite intersection property.

For this, we start by noticing that, without loss of generality, there exists $q_j \in B_{r_j}(p_j) \cap S$ (if not, such a ball can be freely removed from the covering of S). Moreover, by the Besicovitch Covering Theorem, we may suppose that

(A.4) the original family of balls $\{B_{r_i}(p_j)\}_{j\in\mathbb{N}}$ has a finite intersection property.

Thus, to prove (A.3), we argue by contradiction and suppose, say, that there are infinitely many j's such that $B_{32r_j}(p_j) \cap B_{32r_1}(p_1)$. In particular, all these p_j 's remain at a bounded distance from p_1 , and so do the corresponding q_j 's.

Therefore, up to a subsequence, we can assume that there exists $q \in \mathbb{R}^n$ such that $q_j \to q$ as $j \to +\infty$. In fact, since S is closed, we know that $q \in S$. As a consequence, there exists $j_{\star} \in \mathbb{N}$ such that $q \in B_{r_{j_{\star}}}(p_{j_{\star}})$. This and the convergence of the q_j 's yield that $q_j \in B_{r_{j_{\star}}}(p_{j_{\star}})$ for infinitely many j's.

In particular, $q_j \in B_{r_{j_{\star}}}(p_{j_{\star}}) \cap B_{r_j}(p_j)$ for infinitely many j's, which is in contradiction with (A.4). The proof of (A.3) is thereby complete.

Step 3. Bump functions. Let now $\tau_j \in C_0^{\infty}(B_{4r_j}(p_j), [0, 1])$ be such that $\tau_j = 1$ in $B_{3r_j}(p_j)$ and

(A.5)
$$|\nabla \tau_j| \leqslant \frac{C}{r_j} \chi_{B_{4r_j}(p_j) \setminus B_{3r_j}(p_j)}.$$

Let $\phi \in C_0^{\infty}(\mathbb{R}^n, [0, +\infty))$ and assume that the support of ϕ is contained in some ball B_{R_0} . Since $S \cap \overline{B_{R_0}}$ is compact, we may suppose that

(A.6)
$$S \cap \overline{B_{R_0}} \subseteq \bigcup_{j=0}^N B_{r_j}(p_j).$$

Thus, we define

(A.7)
$$\varphi := \phi \, \tau_{\star}, \quad \text{where} \quad \tau_{\star} := \min_{j=0,\dots,N} \{1 - \tau_j\}.$$

We observe that φ actually depends on ε , since so does the family of balls for which (A.2) holds true, but, for short, we omit this dependence in the notation (yet, we will consider the limit in ε here below). Also, we note that $\varphi \in W_0^{1,\infty}(\mathbb{R}^n, [0, +\infty))$ and we also claim that

(A.8) the support of
$$\varphi|_{\mathcal{R}}$$
 is contained in \mathcal{R} .

For this, let $q_k \in \mathcal{R}$ be such that $\varphi(q_k) > 0$ and $q_k \to q$ as $k \to +\infty$. Then, $q_k \in B_{R_0}$, whence $q \in \overline{B_{R_0}}$. We point out that

(A.9)
$$|q - p_j| \geqslant 2r_j \text{ for all } j \in \{0, \dots, N\}.$$

Indeed, if not, say $|q-p_0| < 2r_0$, we would have that also $|q_k-p_0| < 2r_0$ for infinitely many k's and therefore $\tau_0(q_k) = 1$. But this would give that $\varphi(q_k) > 0$, which is a contradiction and (A.9) is proved.

From this, we obtain that q belongs to the complement of $B_{2r_0}(p_0) \cup \cdots \cup B_{2r_N}(p_N)$, and so to the complement of S, from which (A.8) follows.

Step 4. Integral estimates. Thanks to (A.8), we can exploit (4.3) with $\zeta := \frac{\varphi^2}{u_{Q,\ell}+1}$, being $u_{Q,\ell}$ a mollified sequence of u_Q , with $\ell \in \mathbb{N}$. In this way, symmetrizing the integrands

when necessary.

$$\begin{array}{ll} 0 &\leqslant& \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} (u_Q(x)-u_Q(y)) \left(\frac{\varphi^2(x)}{u_{Q,\ell}(x)+1} - \frac{\varphi^2(y)}{u_{Q,\ell}(y)+1}\right) \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}} \\ &=& \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} \varphi^2(x) (u_Q(x)-u_Q(y)) \left(\frac{1}{u_{Q,\ell}(x)+1} - \frac{1}{u_{Q,\ell}(y)+1}\right) \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}} \\ &&+ \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x)-u_Q(y))(\varphi^2(x)-\varphi^2(y))}{u_{Q,\ell}(y)+1} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}} \\ &=& - \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} \frac{\varphi^2(x)(u_Q(x)-u_Q(y))(u_{Q,\ell}(x)-u_{Q,\ell}(y))}{(u_{Q,\ell}(x)+1)(u_{Q,\ell}(y)+1)} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}} \\ &&+ \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x)-u_Q(y))(\varphi^2(x)-\varphi^2(y))}{u_{Q,\ell}(y)+1} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}} \\ &=& -\frac{1}{2} \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} \frac{(\varphi^2(x)+\varphi^2(y))(u_Q(x)-u_Q(y))(u_{Q,\ell}(x)-u_{Q,\ell}(y))}{(u_{Q,\ell}(x)+1)(u_{Q,\ell}(y)+1)} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}} \\ &&+ \frac{1}{2} \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x)-u_Q(y))(\varphi^2(x)-\varphi^2(y))}{u_{Q,\ell}(x)+1} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}} \\ &&+ \frac{1}{2} \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x)-u_Q(y))(\varphi^2(x)-\varphi^2(y))}{u_{Q,\ell}(x)+1} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}} \\ &&=& -\frac{1}{2} \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x)-u_Q(y))(\varphi^2(x)-\varphi^2(y))}{(u_{Q,\ell}(x)+1)(u_{Q,\ell}(y)-u_{Q,\ell}(y))} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}} \\ &&+ \frac{1}{2} \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x)-u_Q(y))(\varphi^2(x)-u_Q(y))(u_{Q,\ell}(x)-u_{Q,\ell}(y))}{(u_{Q,\ell}(x)+1)(u_{Q,\ell}(y)+1)} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}} \\ &&+ \frac{1}{2} \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x)-u_Q(y))(u_{Q,\ell}(x)-u_Q(y))(u_{Q,\ell}(x)-u_{Q,\ell}(y))}{(u_{Q,\ell}(x)+1)(u_{Q,\ell}(y)+1)} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}} \\ &&+ \frac{1}{2} \displaystyle \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x)-u_Q(y))(u_{Q,\ell}(x)+u_{Q,\ell}(y)+2)(\varphi(x)-\varphi(y))(\varphi(x)+\varphi(y))}{(u_{Q,\ell}(x)+1)(u_{Q,\ell}(y)+1)} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}}. \end{array}$$

Thus, given $\alpha \in (0,1)$, to be chosen suitably small in what follows, by a weighted Cauchy-Schwarz Inequality we deduce that

$$\iint_{\mathcal{R}\times\mathcal{R}} \frac{(\varphi^{2}(x) + \varphi^{2}(y))(u_{Q}(x) - u_{Q}(y))(u_{Q,\ell}(x) - u_{Q,\ell}(y))}{(u_{Q,\ell}(x) + 1)(u_{Q,\ell}(y) + 1)} \frac{d\mathcal{H}_{x}^{n-1} d\mathcal{H}_{y}^{n-1}}{|x - y|^{n+s}} \\
\leqslant \alpha \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_{Q}(x) - u_{Q}(y))^{2}(\varphi(x) + \varphi(y))^{2}}{(u_{Q,\ell}(x) + 1)(u_{Q,\ell}(y) + 1)} \frac{d\mathcal{H}_{x}^{n-1} d\mathcal{H}_{y}^{n-1}}{|x - y|^{n+s}} \\
+ C_{\alpha} \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_{Q,\ell}(x) + u_{Q,\ell}(y) + 2)^{2}(\varphi(x) - \varphi(y))^{2}}{(u_{Q,\ell}(x) + 1)(u_{Q,\ell}(y) + 1)} \frac{d\mathcal{H}_{x}^{n-1} d\mathcal{H}_{y}^{n-1}}{|x - y|^{n+s}}.$$

We also use the bound

$$\iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x) - u_Q(y))^2 (\varphi(x) + \varphi(y))^2}{(u_{Q,\ell}(x) + 1)(u_{Q,\ell}(y) + 1)} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}} \\
\leqslant 4 \iint_{\mathcal{R}\times\mathcal{R}} \frac{(u_Q(x) - u_Q(y))^2 (\varphi^2(x) + \varphi^2(y))}{(u_{Q,\ell}(x) + 1)(u_{Q,\ell}(y) + 1)} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}.$$

Hence, sending $\ell \to +\infty$, and reabsorbing one term to the left-hand side via a suitable choice of α , we conclude that

$$\iint_{\mathcal{R}\times\mathcal{R}} \frac{(\varphi^2(x) + \varphi^2(y))(u_Q(x) - u_Q(y))^2}{(u_Q(x) + 1)(u_Q(y) + 1)} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}$$

$$\leqslant C \iint_{\mathcal{R} \times \mathcal{R}} \frac{(u_Q(x) + u_Q(y) + 2)^2 (\varphi(x) - \varphi(y))^2}{(u_Q(x) + 1)(u_Q(y) + 1)} \, \frac{d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}.$$

Since $u_Q \in [0, Q]$, we have that $u_Q + 1 \in [1, Q + 1]$ and therefore

(A.10)
$$\iint_{\mathcal{R}\times\mathcal{R}} (\varphi^2(x) + \varphi^2(y)) (u_Q(x) - u_Q(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}} \\ \leqslant C_Q \iint_{\mathcal{R}\times\mathcal{R}} (\varphi(x) - \varphi(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}.$$

Step 5. Further integral estimates towards the proof of (4.4). We observe that $\varphi = \phi$ outside the set

$$U := \bigcup_{j=0}^{N} B_{4r_j}(p_j)$$

and, by (4.1) and (A.2),

$$(A.11) \mathcal{H}^{n-1}(\mathcal{R} \cap U) \leqslant \varepsilon.$$

This and (A.7) say that, as $\varepsilon \searrow 0$, τ_* approaches 1, and so φ approaches ϕ , up to negligible sets in \mathcal{R} and therefore, as $\varepsilon \searrow 0$, the left-hand side of (A.10) can be replaced (or minorized) by

(A.12)
$$\iint_{\mathcal{R}\times\mathcal{R}} (\phi^2(x) + \phi^2(y)) (u_Q(x) - u_Q(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}.$$

We claim that, as $\varepsilon \searrow 0$, the integral in the right-hand side of (A.10) can be majorized by

(A.13)
$$\iint_{\mathcal{R}\times\mathcal{R}} (\phi(x) - \phi(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}.$$

To check this, we remark that, given $\beta \in (0,1)$, to be taken as small as we wish in what follows,

$$(\varphi(x) - \varphi(y))^{2} = (\phi(x)\tau_{\star}(x) - \phi(y)\tau_{\star}(y))^{2}$$

$$= ((\phi(x) - \phi(y))\tau_{\star}(x) + \phi(y)(\tau_{\star}(x) - \tau_{\star}(y)))^{2}$$

$$\leq (1 + \beta)(\phi(x) - \phi(y))^{2}\tau_{\star}^{2}(x) + C_{\beta}\phi^{2}(y)(\tau_{\star}(x) - \tau_{\star}(y))^{2}.$$

The first term in the last line will produce the desired result in (A.13), after sending $\beta \searrow 0$, therefore, to prove (A.13), we need to check that, for a given $\beta > 0$, the second term produces an integral that vanishes as $\varepsilon \searrow 0$. To this end, without loss of generality we can assume that the balls selected in (A.6) satisfy $B_{r_j}(p_j) \subseteq B_{2R_0}$, since the others, for small ε would not intersect $\overline{B_{R_0}}$ anyway. As a result, if $x \in \mathbb{R}^n \setminus B_{2R_0}$, we have that $\tau_{\star}(x) = 1$ and so, if $y \in B_{R_0}$,

$$\tau_{\star}(x) - \tau_{\star}(y) = 1 - \tau_{\star}(y) = (1 - \tau_{\star}(y))\chi_{U}(y).$$

From this observation, (4.1) and (A.11), we infer that

$$\iint_{(\mathcal{R}\setminus B_{2R_0})\times\mathcal{R}} \phi^2(y) (\tau_{\star}(x) - \tau_{\star}(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}$$

$$\leqslant C \iint_{(\mathcal{R}\backslash B_{2R_0})\times(\mathcal{R}\cap B_{R_0})} (\tau_{\star}(x) - \tau_{\star}(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}$$

$$= C \iint_{(\mathcal{R}\backslash B_{2R_0})\times(\mathcal{R}\cap B_{R_0}\cap U)} (1 - \tau_{\star}(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}$$

$$\leqslant C \sum_{k=1}^{+\infty} \iint_{(\mathcal{R}\cap (B_{2^{k+1}R_0}\backslash B_{2^kR_0}))\times(\mathcal{R}\cap B_{R_0}\cap U)} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{\left((2^k - 1)R_0\right)^{n+s}}$$

$$\leqslant C\varepsilon \sum_{k=1}^{+\infty} \frac{(2^{k+1}R_0)^{n-1}}{\left((2^k - 1)R_0\right)^{n+s}}$$

$$\leqslant C\varepsilon.$$

Therefore, to estimate, as $\varepsilon \searrow 0$, the integral on the right-hand side of (A.10), we can focus on the computation below:

$$\iint_{(\mathcal{R} \cap B_{2R_0}) \times \mathcal{R}} \phi^2(y) (\tau_{\star}(x) - \tau_{\star}(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}} \\
\leqslant C \iint_{(\mathcal{R} \cap B_{2R_0}) \times (\mathcal{R} \cap B_{2R_0})} (\tau_{\star}(x) - \tau_{\star}(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}.$$

If both x and y lie outside U, we have that $\tau_{\star}(x) - \tau_{\star}(y) = 1 - 1 = 0$, hence, up to renaming C, we can actually reduce to

(A.14)
$$C \iint_{(\mathcal{R} \cap B_{2R_0} \cap U) \times (\mathcal{R} \cap B_{2R_0})} (\tau_{\star}(x) - \tau_{\star}(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}$$

$$\leq C \sum_{j=0}^{N} \iint_{(\mathcal{R} \cap B_{2R_0} \cap B_{4r_j}(p_j)) \times (\mathcal{R} \cap B_{2R_0})} (\tau_{\star}(x) - \tau_{\star}(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}.$$

Furthermore, using again (4.1) and (A.11),

$$\begin{split} &\sum_{j=0}^{N} \iint_{(\mathcal{R} \cap B_{2R_{0}} \cap B_{4r_{j}}(p_{j})) \times ((\mathcal{R} \cap B_{2R_{0}}) \backslash B_{16r_{j}}(p_{j}))} \frac{d\mathcal{H}_{x}^{n-1} d\mathcal{H}_{y}^{n-1}}{|x-y|^{n+s}} \\ &\leqslant \sum_{j=0}^{N} \sum_{k=2}^{+\infty} \iint_{(\mathcal{R} \cap B_{2R_{0}} \cap B_{4r_{j}}(p_{j})) \times ((\mathcal{R} \cap B_{2R_{0}}) \cap (B_{4^{k+1}r_{j}}(p_{j}) \backslash B_{4^{k}r_{j}}(p_{j})))} \frac{d\mathcal{H}_{x}^{n-1} d\mathcal{H}_{y}^{n-1}}{|x-y|^{n+s}} \\ &\leqslant \sum_{j=0}^{N} \sum_{k=2}^{+\infty} \iint_{(\mathcal{R} \cap B_{4r_{j}}(p_{j})) \times (\mathcal{R} \cap B_{4^{k+1}r_{j}}(p_{j}))} \frac{d\mathcal{H}_{x}^{n-1} d\mathcal{H}_{y}^{n-1}}{\left((4^{k}-4)r_{j}\right)^{n+s}} \\ &\leqslant C \sum_{j=0}^{N} \sum_{k=2}^{+\infty} \frac{(4r_{j})^{n-1} (4^{k+1}r_{j})^{n-1}}{\left((4^{k}-4)r_{j}\right)^{n+s}} \\ &\leqslant C \sum_{j=0}^{N} r_{j}^{n-2-s} \\ &\leqslant C\varepsilon. \end{split}$$

For this reason, as $\varepsilon \searrow 0$, we can reduce the calculation in (A.14) to

(A.15)
$$C \sum_{j=0}^{N} \iint_{(\mathcal{R} \cap B_{2R_0} \cap B_{16r_j}(p_j)) \times (\mathcal{R} \cap B_{2R_0} \cap B_{16r_j}(p_j))} (\tau_{\star}(x) - \tau_{\star}(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}.$$

Now we observe that, given real numbers $\{a_j\}_{j\in\{0,\dots,N\}}$ and $\{b_j\}_{j\in\{0,\dots,N\}}$, we have that

(A.16)
$$\left| \min_{j \in \{0, \dots, N\}} a_j - \min_{j \in \{0, \dots, N\}} b_j \right| \leqslant \max_{j \in \{0, \dots, N\}} |a_j - b_j|.$$

To check this, one can assume, up to swapping the two classes of numbers, that

$$a_{j_a} = \min_{j \in \{0, \dots, N\}} a_j \geqslant \min_{j \in \{0, \dots, N\}} b_j = b_{j_b},$$

for suitable $j_a, j_b \in \{0, \dots, N\}$.

Therefore,

$$\left| \min_{j \in \{0, \dots, N\}} a_j - \min_{j \in \{0, \dots, N\}} b_j \right| = a_{j_a} - b_{j_b} \leqslant a_{j_b} - b_{j_b} \leqslant |a_{j_b} - b_{j_b}| \leqslant \max_{j \in \{0, \dots, N\}} |a_j - b_j|,$$

which proves (A.16).

As a byproduct of (A.16), we see that

$$|\tau_{\star}(x) - \tau_{\star}(y)| = \left| \min_{j=0,\dots,N} \{1 - \tau_{j}(x)\} - \min_{j=0,\dots,N} \{1 - \tau_{j}(y)\} \right|$$

$$\leq \max_{j=0,\dots,N} \left| (1 - \tau_{j}(x)) - (1 - \tau_{j}(y)) \right| = \max_{j=0,\dots,N} |\tau_{j}(x) - \tau_{j}(y)|.$$

Utilizing this information, we majorize (A.15) by

$$(A.17) \quad C \sum_{i=0}^{N} \iint_{(\mathcal{R} \cap B_{2R_0} \cap B_{16r_i}(p_j)) \times (\mathcal{R} \cap B_{2R_0} \cap B_{16r_i}(p_j))} \max_{m=0,\dots,N} (\tau_m(x) - \tau_m(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}}.$$

Now we observe that, for each $x, y \in B_{16r_j}(p_j)$, we have that $\tau_m(x) - \tau_m(y) = 0$ if both x and y lie in the complement of $B_{4r_m}(p_m)$. Accordingly, (A.17) reduces to

$$C\sum_{j=0}^{N} \iint_{(\mathcal{R} \cap B_{2R_0} \cap B_{16r_j}(p_j)) \times (\mathcal{R} \cap B_{2R_0} \cap B_{16r_j}(p_j))} \max_{\substack{m=0,\dots,N \\ B_{4r_m}(p_m) \cap B_{16r_i}(p_j) \neq \varnothing}} (\tau_m(x) - \tau_m(y))^2 \, \frac{d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1}}{|x-y|^{n+s}}.$$

Thus, recalling the gradient bound in (A.5), we majorize this quantity by

$$C \sum_{j,m=0}^{N} \iint_{\substack{(\mathcal{R} \cap B_{2R_{0}} \cap B_{16r_{j}}(p_{j})) \times (\mathcal{R} \cap B_{2R_{0}} \cap B_{16r_{j}}(p_{j})) \\ B_{16r_{j}}(p_{j}) \cap B_{16r_{m}}(p_{m}) \neq \emptyset}} \frac{d\mathcal{H}_{x}^{n-1} d\mathcal{H}_{y}^{n-1}}{r_{m}^{2} |x-y|^{n+s-2}}$$

$$\leqslant C \sum_{j,m=0}^{N} \iint_{\substack{(\mathcal{R} \cap B_{2R_{0}} \cap B_{32r_{m}}(p_{m})) \times (\mathcal{R} \cap B_{2R_{0}} \cap B_{32r_{m}}(p_{m})) \\ B_{16r_{j}}(p_{j}) \cap B_{16r_{m}}(p_{m}) \neq \emptyset}} \frac{d\mathcal{H}_{x}^{n-1} d\mathcal{H}_{y}^{n-1}}{r_{m}^{2} |x-y|^{n+s-2}}$$

and consequently, in light of the finite intersection property in (A.3), up to renaming constants, by

(A.18)
$$C \sum_{m=0}^{N} \iint_{(\mathcal{R} \cap B_{2R_0} \cap B_{32r_m}(p_m)) \times (\mathcal{R} \cap B_{2R_0} \cap B_{32r_m}(p_m))} \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{r_m^2 |x-y|^{n+s-2}}.$$

Now, given $x \in \mathcal{R} \cap B_{2R_0} \cap B_{32r_m}(p_m)$, we utilize (4.1) to see that

$$\begin{split} & \int_{\mathcal{R} \cap B_{2R_0} \cap B_{32r_m}(p_m)} \frac{d\mathcal{H}_y^{n-1}}{r_m^2 \, |x-y|^{n+s-2}} \leqslant \sum_{k=0}^{+\infty} \int_{\mathcal{R} \cap (B_{64r_m/2^k}(x) \backslash B_{64r_m/2^{k+1}}(x))} \frac{d\mathcal{H}_y^{n-1}}{r_m^2 \, |x-y|^{n+s-2}} \\ & \leqslant \sum_{k=0}^{+\infty} \frac{(r_m/2^k)^{n-1}}{r_m^2 \, (r_m/2^k)^{n+s-2}} \leqslant \frac{C}{r_m^{1+s}} \sum_{k=0}^{+\infty} \frac{1}{2^{(1-s)k}} \leqslant \frac{C}{r_m^{1+s}}. \end{split}$$

Therefore, using again (4.1) and (A.2), the quantity in (A.18) can be bounded from above by

$$C\sum_{m=0}^{N} \int_{(\mathcal{R} \cap B_{2R_0} \cap B_{32r_m}(p_m)} \frac{d\mathcal{H}_x^{n-1}}{r_m^{1+s}} \leqslant C\sum_{m=0}^{N} r_m^{n-2-s} \leqslant C\varepsilon,$$

as desired: this shows that, as $\varepsilon \searrow 0$, the integral in the right-hand side of (A.10) can be majorized by the quantity in (A.13).

Step 6. Completion of the proof of (4.4). In view of the bounds in (A.10), (A.12) and (A.13), we know that

$$\iint_{\mathcal{R}\times\mathcal{R}} (\phi^2(x) + \phi^2(y)) (u_Q(x) - u_Q(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}} \leqslant C \iint_{\mathcal{R}\times\mathcal{R}} (\phi(x) - \phi(y))^2 \frac{d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}}{|x - y|^{n+s}}.$$

Assuming $\phi = 1$ in $B_{R_0/2}$ (and possibly renaming R_0) we thereby conclude that (4.4) holds true, as desired.

Step 7. Proof of (4.5). We observe that the left-hand side of (4.5) is finite, thanks to (4.4). We employ (4.3) by taking ζ as the function defined in (A.7). In this way, we have that

$$(A.19) \qquad 0 \leq \iint_{\mathcal{R} \times \mathcal{R}} \frac{(u_{Q}(x) - u_{Q}(y))(\phi(x)\tau_{\star}(x) - \phi(y)\tau_{\star}(y))}{|x - y|^{n+s}} d\mathcal{H}_{x}^{n-1} d\mathcal{H}_{y}^{n-1}$$

$$= \iint_{\mathcal{R} \times \mathcal{R}} \frac{\tau_{\star}(x)(u_{Q}(x) - u_{Q}(y))(\phi(x) - \phi(y))}{|x - y|^{n+s}} d\mathcal{H}_{x}^{n-1} d\mathcal{H}_{y}^{n-1}$$

$$+ \iint_{\mathcal{R} \times \mathcal{R}} \frac{\phi(y)(u_{Q}(x) - u_{Q}(y))(\tau_{\star}(x) - \tau_{\star}(y))}{|x - y|^{n+s}} d\mathcal{H}_{x}^{n-1} d\mathcal{H}_{y}^{n-1}.$$

The latter term vanishes as $\varepsilon \searrow 0$, owing to (4.4), and the claim in (4.5) follows.

APPENDIX B. THE SET OF "GOOD" REGULAR POINTS

Here we describe the set of regular points of a nonlocal minimal set around which the boundary is a hypersurface of class C^2 with bounded curvatures in a conveniently scale invariant setting. In the classical case, this set was introduced in [Sim87, Lemma 1] and we provide here a nonlocal version of it.

Lemma B.1. Let E be s-minimal in B_1 and assume that the origin belongs to its boundary. Given $\theta > 0$, let $\mathcal{M}_{\theta,E}$ be the set of points x belonging to the regular part of ∂E such that

$$(\partial E) \cap B_{\theta|x|}(x)$$
 belongs to the regular part of ∂E

(B.1) and possesses curvatures bounded in absolute value by
$$\frac{1}{\theta|x|}$$
.

Then, there exist ρ_0 , $\theta_0 > 0$, depending only on E, n and s, such that, for every $\rho \in (0, \rho_0]$ and $\theta \in (0, \theta_0]$, we have that

(B.2)
$$\mathcal{M}_{\theta,E} \cap (\partial B_{\rho}) \neq \varnothing.$$

Proof. Suppose not. Then, there exist sequences $\theta_j \searrow 0$ and $\rho_j \searrow 0$ for which $\mathcal{M}_{\theta_j,E} \cap (\partial B_{\rho_j}) = \emptyset$. That is, if $x \in \partial B_{\rho_j}$ belongs to the regular part of ∂E and also $(\partial E) \cap B_{\theta_j|x|}(x)$ belongs to the regular part of ∂E , then necessarily its curvatures are not bounded in absolute value by $\frac{1}{\theta_j|x|} = \frac{1}{\theta_j\rho_j}$.

Let $F_j := E/\rho_j$. Up to a subsequence, we know that F_j converges locally uniformly to a cone \mathcal{C} which is s-minimal (see [CRS10]). In fact, thanks to the improvement of flatness of nonlocal minimal surfaces (see [CRS10, Corollary 4.4 and Theorem 6.1]), this convergence occurs in the $C^{1,\alpha}$ sense at the regular points of \mathcal{C} , and actually in the C^k sense for any $k \geq 2$ (see [BFV14]).

Hence, we pick a regular point $y_0 \in \mathcal{C} \cap (\partial B_1)$ (whose existence is guaranteed by the regularity theory of nonlocal minimal surfaces, see [SV13]). Then, we find a sequence $y_j \in \partial B_1$ of regular points of F_j approaching y_0 as $j \to +\infty$ and $(\partial F_j) \cap B_{r_0}(y_j)$ consists of regular points with curvatures bounded in absolute value by M_0 , for suitable r_0 , $M_0 > 0$.

Scaling back and setting $x_j := \rho_j y_j \in \partial B_{\rho_j}$, we find that $(\partial E) \cap B_{r_0 \rho_j}(x_j)$ lies in the regular set of ∂E , with curvatures bounded in absolute value by $\frac{M_0}{\rho_j}$, which, for large j, is strictly less than $\frac{1}{\theta_j \rho_j}$, contradiction.

With reference to Lemma B.1, we observe that, for all r > 0,

(B.3)
$$\mathcal{M}_{\theta,\frac{E}{r}} = \frac{\mathcal{M}_{\theta,E}}{r}.$$

Then, we have:

Corollary B.2. Let E_1 , E_2 be s-minimal sets in B_1 and assume that the origin belongs to their boundary. Assume also that E_1 and E_2 have the same tangent cone at the origin.

Let θ_0 , $\rho_0 > 0$ be as in Lemma B.1, used here with $E := E_1$. Let also \mathcal{M} be the set $\mathcal{M}_{\theta_0,E}$ in Lemma B.1 with $E = E_1$.

Suppose that

$$E_2 \setminus E_1 = \left\{ x + t\nu(x), \ x \in \mathcal{M}, \ t \in [0, w(x)) \right\},$$

for some function w of class C^2 , where ν denotes the external unit normal of E_1 at its regular points.

Then,

(B.4)
$$\lim_{r \searrow 0} \frac{\sup_{x \in (\partial B_r) \cap \mathcal{M}} |w(x)|}{r} = 0.$$

In addition, if $w(x) \neq 0$ for all $x \neq 0$, then there exists an infinitesimal sequence of points $z_{\star,k} \in \mathcal{M}$ such that

(B.5)
$$\frac{|w(x)|}{|x|} \leqslant \frac{2|w(z_{\star,k})|}{|z_{\star,k}|}$$

for all $x \in \mathcal{M}$ with $|x| \leq |z_{\star,k}|/2$.

Finally, the distance of $z_{\star,k}$ from the singular set of E_1 is at least $\theta_0|z_{\star,k}|$.

Proof. Before proving (B.4), we stress that the supremum in this equation makes sense, thanks to (B.2).

We now prove (B.4). Suppose not. Then, there exist c > 0, an infinitesimal sequence $r_j > 0$ and $x_j \in (\partial B_{r_j}) \cap \mathcal{M}$ such that

$$(B.6) |w(x_i)| \geqslant cr_i.$$

Thus, if $w_j := \frac{w}{r_j}$ and $E_{i,j} := \frac{E_i}{r_j}$ for $i \in \{1, 2\}$, we see that

$$E_{2,j} \setminus E_{1,j} = \left\{ x + t\nu_j(x), \ x \in \frac{\mathcal{M}}{r_j}, \ t \in [0, w_j(x)) \right\},$$

where ν_j denotes the external unit normal of $E_{1,j}$.

Combining this and (B.3), we obtain

$$E_{2,j} \setminus E_{1,j} = \left\{ x + t\nu_j(x), \ x \in \mathcal{M}_{\theta_0, E_{1,j}}, \ t \in [0, w_j(x)) \right\}.$$

So, setting $y_j := \frac{x_j}{r_j} \in (\partial B_1) \cap \mathcal{M}_{\theta_0, E_{1,j}}$, we have that

(B.7)
$$y_j + w_j(y_j)\nu_j(y_j) \in \partial E_{2,j}.$$

Notice in addition that, since the distance of x_j from the singular set of E_1 is at least $\theta_0|x_j| = \theta_0 r_j$, we have that the distance of y_j from the singular set of $E_{1,j}$ is at least θ_0 . The curvature bound thus allows us to pass to the limit as $j \to +\infty$, up to a subsequence, and find that $y_j \to y_\infty$, with y_∞ belonging to the regular part of the tangent cone of E_1 , and also $\nu_j(y_j) \to \nu_\infty(y_\infty)$, where ν_∞ denotes the normal of this tangent cone.

Hence, since E_2 shares the same tangent cone of E_1 , it follows from (B.7) that $w_j(y_j) \to 0$ as $j \to +\infty$. But this is in contradiction with (B.6) and the proof of (B.4) is thereby complete.

Now we prove (B.5). For this, let

$$\sigma(r) := \frac{\sup_{x \in (\partial B_r) \cap \mathcal{M}} |w(x)|}{r}.$$

Thanks to (B.4), we have that $\sigma(r) \to 0$ as $r \searrow 0$.

Also, we know that $\sigma(r) > 0$ for all $r \neq 0$. Therefore, given $\mu_0 > 1$ we pick an infinitesimal sequence $r_k > 0$ and choose $r_{\star,k} \in (0, r_k]$ such that

$$\sigma(r_{\star,k}) \geqslant \frac{\sup_{r \in (0,r_k]} \sigma(r)}{\mu_0}.$$

As a result, for all $\mu \in (0,1)$,

$$\frac{\sup_{x \in (\partial B_{\mu r_{\star,k}}) \cap \mathcal{M}} |w(x)|}{\mu r_{\star,k}} = \sigma(\mu r_{\star,k}) \leqslant \sup_{r \in (0,r_k]} \sigma(r) \leqslant \mu_0 \, \sigma(r_{\star,k}).$$

Besides, by (B.2), we can pick $z_{\star,k} \in (\partial B_{r_{\star,k}}) \cap \mathcal{M}$ with

$$\frac{|w(z_{\star,k})|}{r_{\star,k}} \geqslant \frac{\sup_{x \in (\partial B_{r_{\star,k}}) \cap \mathcal{M}} |w(x)|}{\mu_0 r_{\star,k}} = \frac{\sigma(r_{\star,k})}{\mu_0}.$$

This gives that, for all $x \in \mathcal{M}$ with $|x| \leq |z_{\star,k}|/2$, we have that

$$\frac{|w(x)|}{|x|} \leqslant \sup_{\mu \in (0,1/2]} \frac{\sup_{x \in (\partial B_{\mu r_{\star,k}}) \cap \mathcal{M}} |w(x)|}{\mu r_{\star,k}} \leqslant \mu_0 \, \sigma(r_{\star,k}) \leqslant \frac{\mu_0^2 |w(z_{\star,k})|}{|z_{\star,k}|},$$

which, choosing $\mu_0 := \sqrt{2}$, establishes (B.5), as desired.

Finally, the distance of $z_{\star,k}$ from the singular set is estimated from below by the first line in (B.1).

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