Counterexample to $Per_s(E_0\cup E_1,\Omega) \geq Per_s(E_0,\Omega)$ if $d(E_0,\Omega)>0$.

Consider $n=2$, $E_0 = B_2^c$, $\Omega = B_{\frac32}$, $E_1 = B_1$, then to prove $Per_s(E_0\cup E_1,\Omega) - Per_s(E_0,\Omega) < 0$ for some s

Idea: For all Ω, E_0 s.t. $d(E_0,\Omega)>0$ $\exists\, s_0$ s.t. E_0 not minimal for $Per_s(\cdot,\Omega)$ $\forall s<s_0$?
Could be a tool to find minimal surfaces ...

$$Per_s(E_0\cup E_1,\Omega) - Per_s(E_0,\Omega) = Per_s(E_1) - 2L(E_1,E_0)$$

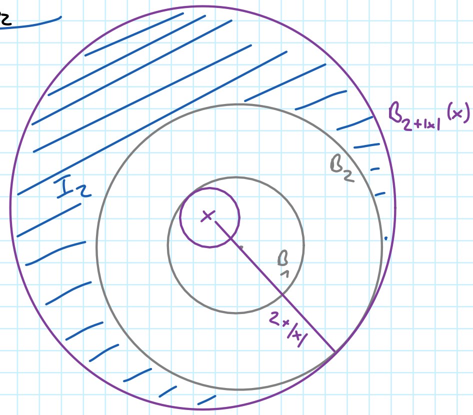
2022 Haddad-Ludwig Eq(11)

$$P_s(B^n) = \frac{2^{1-s}\pi^{(n-1)/2} n\omega_n \,\Gamma\left(\frac{1-s}{2}\right)}{s(n-s)\,\Gamma\left(\frac{n-s}{2}\right)} = \frac{2^{1-s} n\omega_n\omega_{n-s}}{s(1-s)\omega_{1-s}}$$

•) $$Per_s(E_1) = Per_s(B_1) = \frac{2^{1-s}\pi^{\frac12}\, 2\,\omega_2}{s(2-s)}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{2-s}{2})} = \frac{2^{2-s}\pi^{\frac32}}{s(2-s)}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{2-s}{2})}$$

•) $$L(E_1,E_0) = \int_{B_1}\int_{B_2^c}\frac{1}{|x-y|^{2+s}} = \underbrace{\int_{B_1}\int_{B^c_{2+|x|}(x)} \frac{1}{|x-y|^{2+s}}}_{I_1} + \underbrace{\int_{B_1}\int_{B_{2+|x|}(x)\setminus B_2}\frac{1}{|x-y|^{2+s}}}_{I_2}$$

see figure



$$I_1 = \int_{B_1}\int_{B^c_{2+|x|}(x)} \frac{1}{|x-y|^{2+s}} = \int_{B_1}\int_{B^c_{2+|x|}}\frac{1}{|y|^{2+s}} = 4\pi^2\int_0^1\int_{2+r_1}^\infty \frac{r_1}{r_2^{1+s}}$$

$$= \frac{4\pi^2}{s}\int_0^1 \frac{r_1}{(2+r_1)^s} = \frac{4\pi^2}{s}\left(\left[\frac{r_1}{1-s}(2+r_1)^{1-s}\right]_0^1 - \frac{1}{1-s}\int_0^1 (2+r_1)^{1-s}\right)$$

$$= \frac{4\pi^2}{s(1-s)}\left(3^{1-s} - \frac{1}{2-s}\left(3^{2-s}-2^{2-s}\right)\right) = \boxed{\frac{4\pi^2}{s(1-s)(2-s)}\left(2^{2-s}-(s+1)3^{1-s}\right)}$$

$$I_2 = \int_{B_1}\int_{B_{2+|x|}(x)\setminus B_2}\frac{1}{|x-y|^{2+s}} = \int_{B_1}\underbrace{\int_{B_{2+|x|}\setminus B_2(-x)}}_{\text{radial symmetric}}\frac{1}{|y|^{2+s}} = 2\pi\int_0^1\int_{2-r_1}^{2+r_1}\frac{r_1}{r_2^{1+s}}\int_{-\tilde\theta(r_1,r_2)}^{\tilde\theta(r_1,r_2)} d\theta\, dr_2\, dr_1 \qquad (1)$$

Find domain of $\tilde\theta(r_1,r_2)$:

We have following restrictions on x,y: $4\le|x-y|^2\le(2+2r_1)^2$, $(2-r_1)\le|y|\le(2+r_1)$

Then

$$4\le|x-y|^2\le(2+2r)^2 \iff 4\le r_1^2+r_2^2-2r_1r_2\cos\theta \le 4(1+r_1)^2$$

$$\iff \frac{r_1^2+r_2^2-4}{2r_1r_2}\ge\cos\theta\ge\frac{r_1^2+r_2^2-4(1+r_2)^2}{2r_1r_2}\ge -1 \quad \text{for } r_2\ge 2-r_1$$

$$\Rightarrow \frac{r_1^2+r_2^2-4}{2r_1r_2}\ge\cos\theta\ge-1$$

Thus $$\int_{-\tilde\theta(r_1,r_2)}^{\tilde\theta(r_1,r_2)} d\theta = 2\pi - 2\arccos\left(\frac{r_1^2+r_2^2-4}{2r_1r_2}\right)$$

$$\Rightarrow (1) = 2\pi\int_0^1\int_{2-r_1}^{2+r_1}\frac{r_1}{r_2^{1+s}}\left(2\pi - 2\arccos\left(\frac{r_1^2+r_2^2-4}{2r_1r_2}\right)\right)dr_2\,dr_1$$

$$= 4\pi^2\int_0^1\int_{2-r_1}^{2+r_1}\frac{r_1}{r_2^{1+s}} - 4\pi\int_0^1\int_{2-r_1}^{2+r_1}\frac{r_1}{r_2^{1+s}}\arccos\left(\frac{r_1^2+r_2^2-4}{2r_1r_2}\right)$$

$$= \frac{4\pi^2}{s(1-s)(2-s)}\left((1+s)3^{1-s}-3+s\right) - \frac{4\pi^2}{s}\int_0^1\frac{r_1}{(2-r_1)^s} + \frac{4\pi}{s}\int_0^1\int_{2-r_1}^{2+r_1}\frac{r_1}{r_2^{1+s}}\frac{r_2^2-r_1^2+4}{\sqrt{4r_1^2r_2^2-(r_1^2+r_2^2-4)^2}}$$

$$\int_{2-r_1}^{2+r_1}\frac{r_1}{r_2^{1+s}}\arccos\left(\frac{r_1^2+r_2^2-4}{2r_1r_2}\right) = \left[-\frac1s\frac{r_1}{r_2^s}\arccos\left(\frac{r_1^2+r_2^2-4}{2r_1r_2}\right)\right]_{2-r_1}^{2+r_1} + \frac1s\int_{2-r_1}^{2+r_1}\frac{r_1}{r_2^s}\frac{-1}{\sqrt{1-\left(\frac{r_1^2+r_2^2-4}{2r_1r_2}\right)^2}}\;\frac{2r_2(2r_1r_2)-(r_1^2+r_2^2-4)\,2r_1}{4r_1^2r_2^2}$$

$$= \frac1s\frac{r_1}{(2-r_1)^s}\pi - \frac1s\int_{2-r_1}^{2+r_1}\frac{r_1}{r_2^{1+s}}\frac{r_2^2-r_1^2+4}{\sqrt{4r_1^2r_2^2-(r_1^2+r_2^2-4)^2}}$$

$$= \boxed{\frac{4\pi^2}{s(1-s)(2-s)}\left((1+s)3^{1-s}-2^{2-s}\right)} + \frac{4\pi}{s}\int_0^1\int_{2-r_1}^{2+r_1}\frac{r_1}{r_2^{1+s}}\frac{r_2^2-r_1^2+4}{\sqrt{4r_1^2r_2^2-(r_1^2+r_2^2-4)^2}}$$

$= -I_1$

Thus $\quad L(E_1,E_0) = \frac{4\pi}{s}\int_0^1\int_{2-r_1}^{2+r_1} \frac{r_1}{r_2^{1+s}} \frac{r_2^2-r_1^2+4}{\sqrt{4r_1^2r_2^2-(r_1^2+r_2^2-4)^2}}$

We bound $L(E_1,E_0)$ from above and below

- $L(E_1,E_0) \overset{(2-r_1)^s \le r_2^s}{\le} \frac{4\pi}{s}\int_0^1\int_{2-r_1}^{2+r_1} \frac{r_1}{(2-r_1)^s}\,\frac{1}{r_2}\,\frac{r_2^2-r_1^2+4}{\sqrt{4r_1^2r_2^2-(r_1^2+r_2^2-4)^2}} = \frac{4\pi}{s}\int_0^1 \frac{r_1}{(2-r_1)^s}\left(-\arccos\left(\frac{r_2^2+r_1^2-4}{2r_1r_2}\right)\right)\Big|_{2-r_1}^{2+r_1}$

$$= \frac{4\pi^2}{s}\int_0^1 \frac{r_1}{(2-r_1)^s} = \frac{4\pi^2}{s(1-s)(2-s)}\left(2^{2-s}-3+s\right)$$

- $L(E_1,E_0) \overset{r_2^s \le (2+r_1)^s}{\ge} \frac{4\pi^2}{s}\int_0^1 \frac{r_1}{(2+r_1)^s}\,\frac{1}{r_2}\,\frac{r_2^2-r_1^2+4}{\sqrt{4r_1^2r_2^2-(r_1^2+r_2^2-4)^2}} = \frac{4\pi^2}{s}\int_0^1 \frac{r_1}{(2+r_1)^s} = \frac{4\pi^2}{s(1-s)(2-s)}\left(2^{2-s}-(1+s)3^{1-s}\right)$

Thus

$$\frac{2^{2-s}\pi^{\frac{3}{2}}}{s(2-s)}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)} - \frac{8\pi^2}{s(1-s)(2-s)}\left(2^{2-s}-3+s\right) \le Per_s(E_1) - 2L(E_1,E_0) \le \frac{2^{2-s}\pi^{\frac{3}{2}}}{s(2-s)}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)} - \frac{8\pi^2}{s(1-s)(2-s)}\left(2^{2-s}-(1+s)3^{1-s}\right)$$

In conclusion: $\exists\, s_0$ s.t. $\forall s \in (0,s_0)$ E_0 is not s-minimal in Ω