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TBD

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I hereby declare that this thesis is entirely the result of my own work except where otherwise indicated. I have only used the resources given in the list of references.

Munich, Date of submission Mohamed Noah Abdel Wahab

Abstract

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Nonlocal minimal surfaces confined within a cylinder exhibit unique behaviors dependent on external data. This thesis delves into these surfaces, which incorporate long-range spatial interactions compared to classical minimal surfaces. We consider two variations of the model discussed in [4], a minimal surfaces confined within a cylinder.

We investigate two scenarios: varying the height and width of data outside a separating slab. The results show that when the slab is wide, the minimal surface becomes disconnected from the data, while a narrow slab allows connection. This allows us to predict the behavior of similar models with symmetrically placed data. Additionally, the research reveals that for sufficiently narrow slabs, the surface "sticks" to the cylinder.

Finally, we present an example where the minimizer is completely disconnected from the external data, a phenomenon unique to nonlocal minimal surfaces. This work provides valuable insights into the behavior of these emerging mathematical objects and their interaction with external data.

Zusammenfassung

In Zylindern eingeschlossene nichtlokale Minimalflächen zeigen ein einzigartiges Verhalten, das von externen Daten abhängt. Diese Arbeit befasst sich mit diesen Flächen, die im Vergleich zu klassischen Minimalflächen weitreichende räumliche Wechselwirkungen berücksichtigen. Wir betrachten zwei Varianten des in [4] diskutierten Modells, einer in einem Zylinder eingeschlossenen Minimalfläche.

Dabei untersuchen wir zwei Szenarien: die Variation der Höhe und der Breite von Daten außerhalb einer trennenden Platte. Die Ergebnisse zeigen, dass die Minimalfläche bei breiter Platte von den Daten getrennt wird, während eine schmale Platte eine Verbindung ermöglicht. Dies erlaubt uns, das Verhalten ähnlicher Modelle mit symmetrisch angeordneten Daten vorherzusagen. Darüber hinaus zeigt die Forschung, dass die Fläche bei ausreichend schmalen Platten am Zylinder "haftet".

Schließlich präsentieren wir ein Beispiel, bei dem der Minimierer vollständig von den externen Daten getrennt ist, ein Phänomen, das für nichtlokale Minimalflächen einzigartig ist. Diese Arbeit liefert wertvolle Erkenntnisse über das Verhalten dieser neuen mathematischen Objekte und ihre Wechselwirkung mit externen Daten.

Contents

1 Introduction

Idea: Start with short historical background
18th century: Lagrange, Euler
20th Century: DeGiorgi Perimeter and localized entity
2009 Cafarelli, Roquejoffre, Savin: Nonlocal minimal surfaces
Perimeter and nonlocal perimeter as the (semi)norm of an indicator function
Define the usual problem considered

Chapter 01
Model 01

Chapter 02
Model 02 and further discussion on the similar problems

Chapter 03
Fully disconnected minimize

In 2009 Cafarelli, Roquejoffre, and Savin [1] introduced a new concept of minimal surfaces. By incooperating long-range corelations into the classical perimeter, they defined the nonlocal perimeter as the (semi)norm of an indicator function. Minimizing over suitable set with given external data gives us *nonlocal minimal surfaces*. Instead…

Use the introduction in [8] as inspiration.

2 Model 01

For $n \geq 2$ consider the model as follows:

$$E_0 := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x'| \leq R,\ |x_n| \geq M\} \tag{2.1}$$

$$\Omega := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x'| \leq 1,\ |x_n| \leq M\} \tag{2.2}$$

for $R \geq 1$ and $M > 0$. The figure fig. 2.1 illustrates the setting.

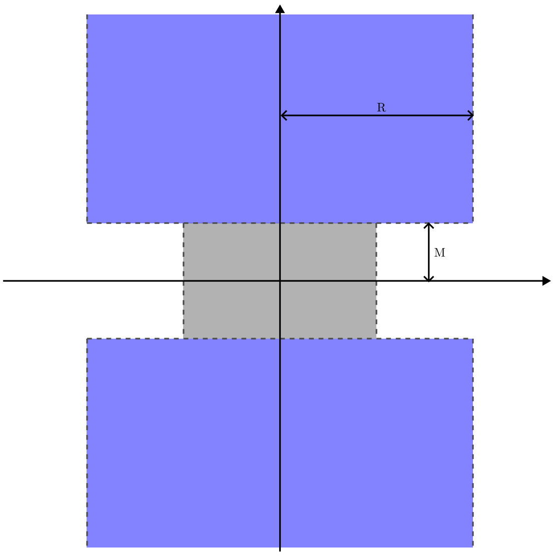

Figure 2.1

We state the following two results, which we will prove afterwards.

Theorem 2.1. *For Ω and E_0 as given above and for all $R \geq 1$, then there exists $M_0 \in (0,1)$ depending only on the dimension and s, such that for any $M \in (0, M_0)$, the minimizer is $E_M = E_0 \cup \Omega$.*

Theorem 2.2. *For Ω and E_0 as given above and for all $R \geq 1$, then there exists $M_0 > 1$ depending only on the dimension and s, such that for any $M \geq M_0$, the minimizer E_M is disconnected.*

For both proofs, we follow a similar construction as in [4].
Prove per Contradiciton, explain idea of the proof and explain viscosity solution…

Proof of definition 2.1. Let □

Intersting to see, that the contribution of the cylinder of radius 1 is enough to get connectedness of the minimizer and even stickiness to the boundary. Also see, that the model seems (maybe prove that) to converge to the problem, considered in [4].

Proof is almost identical to the one in [4], but is included for completeness.

Proof of definition 2.2. Let □

3 Model 02

For $n \geq 2$ consider the model as follows:

$$E_0 \coloneqq \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } M|x_n| \geq R + M\} \tag{3.1}$$
$$\Omega \coloneqq \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x'| \leq 1,\ |x_n| \leq M\} \tag{3.2}$$

for $R > 0$ and $M > 0$. The figure.. illustrates the setting.
We state the following two results, which we will prove afterwards.

Theorem 3.1. *Let Ω and E_0 as given above and for all $R > ..$, then there exists $M_0 \in (0, 1)$ depending only on the dimension and s, such that for any $M \in (0, M_0)$, the minimizer is $E_M = E_0 \cup \Omega$. For $R \leq ..$, the cylinder $A \coloneqq \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x'| \leq ..,\ |x_n| \leq M\}$ is in the minimizer, i.e. $E_M \supset E_0 \cup A$.*

Theorem 3.2. *For Ω and E_0 as given above and for all $R > 0$, then there exists $M_0 > ..$ depending only on the dimension and s, such that for any $M \geq M_0$, the minimizer E_M is disconnected.*

Again, similar proofs as in chapter 2.
Add some

Proof of definition 3.1. Let □

Disscussion about connectedness in case of small R and refer to next chapter. Behavior unique to nonlocal minimal surfaces.
Talk about the contribution of the complement.

Proof of definition 3.2. Let □

Discussion about extending the model to arbitrary models with symmetric external data. Enough to consider discs of radius.. and heigth.. to have connectedness and even stickiness at some point.
New idea: If there is a minimizer E_M, can it ever be non sticky to the boundary?
Maybe able to give own interpretation of nonlocal minimal surfaces. Idea about Volume or Gravity?

4 Disconnected Minimizer

Example of a minimizer that has a non-empty set in Ω, while $d(E_0, \Omega) =: d > 0$.
Compare to classical case, where this cannot happen. Refer to .. and .. where discussion about the behavior of the perimeter for $s \to 1^-$ and $s \to 0^+$ was done.
Connect to the discussion in chapter 3..

Idea: If $d(E_0, \Omega) = 0$, does there exist a connected component $F \subset E$ s.t. $d(E_0, F) > 0$?

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