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ON THE VARIATION OF THE FRACTIONAL MEAN CURVATURE UNDER THE EFFECT OF $C^{1,\alpha}$ PERTURBATIONS

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Abstract. In this brief note we study how the fractional mean curvature of order $s \in (0,1)$ varies with respect to $C^{1,\alpha}$ diffeomorphisms. We prove that, if $\alpha > s$, then the variation under a $C^{1,\alpha}$ diffeomorphism Ψ of the s-mean curvature of a set E is controlled by the $C^{0,\alpha}$ norm of the Jacobian of Ψ. When $\alpha = 1$ we discuss the stability of these estimates as $s \to 1^-$ and comment on the consistency of our result with the classical framework.

1. **Introduction and statement of the result.** In the seminal work [9], Caffarelli, Roquejoffre and Savin introduced the concept of fractional perimeter of a measurable set $E \subset \mathbb{R}^n$ inside a fixed open bounded set $\Omega \subset \mathbb{R}^n$. More precisely, they defined

$$\mathrm{Per}_s(E,\Omega) := \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\Omega) + \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\mathcal{C}\Omega) + \mathcal{L}_s(E\cap\mathcal{C}\Omega, \mathcal{C}E\cap\Omega),$$

where $s \in (0,1)$ is a fixed parameter and $\mathcal{L}_s$ is the integral functional defined for any two non-overlapping measurable sets $A, B \subset \mathbb{R}^n$ as

$$\mathcal{L}_s(A,B) := \int_A \int_B \frac{dx\,dy}{|x-y|^{n+s}}.$$

In contrast with the classical notion of De Giorgi perimeter, this is *non-local*, as it also takes into account interactions with the complements of E and Ω in $\mathbb{R}^n$.

A non-local s-minimal surface in Ω is, hence, the boundary of a set E of finite s-perimeter for which

$$\mathrm{Per}_s(E,\Omega) \le \mathrm{Per}_s(F,\Omega) \quad \text{for any measurable } F \subset \mathbb{R}^n \text{ with } E\cap\mathcal{C}\Omega = F\cap\mathcal{C}\Omega.$$

In [9], the existence of such minimizers is proved, together with other results concerning their regularity, the Hausdorff dimension of the singular set and the relation

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with non-local equations. In particular, they proved that the rescaled characteristic function

$$\widetilde{\chi}_E(x) := \chi_E(x) - \chi_{\mathcal{C}E}(x) = \begin{cases} 1 & \text{if } x \in E, \\ -1 & \text{if } x \in \mathcal{C}E, \end{cases}$$

of a minimizer E satisfies the Euler-Lagrange equation

$$(-\Delta)^{s/2}\, \widetilde{\chi}_E = 0, \qquad \text{on } \partial E \cap \Omega,$$

in a suitable viscosity sense. Here, $(-\Delta)^\sigma$, for $\sigma \in (0,1)$, is the fractional Laplace operator, defined for a sufficiently smooth, bounded function u at a point $x \in \mathbb{R}^n$ as

$$(-\Delta)^\sigma u(x) := C_{n,\sigma}\, \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\sigma}}\, dy.$$

The symbol P.V. denotes the Cauchy principal value and $C_{n,\sigma}$ is a normalizing constant, depending only on n and σ. For more information about this operator we refer the interested reader for instance to [15, 10] and the classical [18].

Similarly to the local framework, a natural notion of fractional mean curvature has been introduced, so that s-minimal surfaces are precisely those having vanishing s-mean curvature. The result is the assignment, for $x \in \partial E$,

$$H_s[E](x) := \text{P.V.} \int_{\mathbb{R}^n} \frac{\widetilde{\chi}_E(y)}{|x-y|^{n+s}}\, dy. \tag{1.1}$$

Notice that this definition is well-posed if ∂E is of class C^2 at x (see, e.g., [2, Lemma 7] and also Corollary 3.5 in the present note).

Over the last few years, an increasing interest has risen around non-local minimal surfaces and the related fractional mean curvature operator. Nice surveys on the topic can be found in [25] and [2]. Moreover, the latter proposes a definition of non-local principal curvatures and establishes a relation with the s-mean curvature reminiscent to what happens in the classical setting. In the next few paragraphs we will give a brief overview of the main developments in the field of non-local minimal surfaces.

In [6, 24, 17] and [14] improvements concerning the regularity of s-minimal surfaces are obtained. See also [11, 12], again [6] and [3] for similar results for the fractional Laplacian.

Fractional perimeters arise naturally in phase separation models as Γ-limits of non-local Ginzburg-Landau energies. In [23] the authors propose an extension of the classical Modica-Mortola theory ([21, 20]) to fractional orders. See also [7, 8, 22] for relevant material on layer solutions and [4, 5], where these results are set in a slightly different non-local framework.

The problem of determining the asymptotic behaviours of the s-perimeter is successfully addressed in [16], as $s \to 0^+$, and in [1, 13], as $s \to 1^-$. We also mention [19], where the author presents analogous results obtained for a class of anisotropic non-local perimeters.

Finally, a Bernstein-type conjecture has been proposed for entire s-minimal graphs of $\mathbb{R}^{n+1}$. In [24] it has been proved to be true in the case $n = 1$ and for $n = 2$ the problem has been solved in [17]. In particular, in the latter contribution a De Giorgi-type lemma is stated: the validity of the conjecture in $n + 1$ dimensions is ensured by the non-existence of singular n-dimensional s-minimal cones. In higher dimensions the conjecture is still open, while in the classical case the result is true up to $n = 7$.

At a technical point of [17], the two authors needed to establish a relation between the s-mean curvature of a subgraph and that of its image under a C^2 graph diffeomorphism.

More in general, it is natural to conjecture that given a set E of class C^2 in a neighbourhood of a point $\bar{x} \in \partial E$ and a global C^2 diffeomorphism Ψ of $\mathbb{R}^n$, the difference between the non-local mean curvature of E and that of its transformed $\Psi(E)$, at $\bar{x}$ and $\Psi(\bar{x})$ respectively, can be controlled by means of the C^2 norm of Ψ.

In the present work we give a proof of this fact in full details. Indeed, we prove something slightly stronger, since we lower the regularity assumptions on both the sets and the diffeomorphism to $C^{1,\alpha}$, with $\alpha \in (s, 1]$.

The precise statement of the result is the content of the following

Theorem 1.1. *Let $\eta_0, R > 0$ and $s \in (0,1)$. Let E be an open subset of $\mathbb{R}^n$, take a point $\bar{x} \in \partial E$ and assume ∂E to be of class $C^{1,\alpha}$ in $B_R(\bar{x})$, for some $\alpha \in (s, 1]$. Let Ψ be a global diffeomorphism of $\mathbb{R}^n$ of class $C^1(\mathbb{R}^n, \mathbb{R}^n) \cap C^{1,\alpha}(B_R(\bar{x}), \mathbb{R}^n)$ and set*

$$F := \Psi(E), \qquad \bar{y} := \Psi(\bar{x}).$$

Decomposing Ψ and its inverse Ψ^{-1} as

$$\Psi(x) = x + \Phi(x), \qquad \textit{for any } x \in \mathbb{R}^n, \tag{1.2}$$

and

$$\Psi^{-1}(y) = y + \Xi(y), \qquad \textit{for any } y \in \mathbb{R}^n, \tag{1.3}$$

for suitable functions Φ and Ξ, suppose that

$$\|J\Phi\|_{L^\infty(\mathbb{R}^n)}, [J\Phi]_{C^{0,\alpha}(B_R(\bar{x}))}, \|J\Xi\|_{L^\infty(\mathbb{R}^n)}, [J\Xi]_{C^{0,\alpha}(\Psi(B_R(\bar{x})))} \le \eta, \tag{1.4}$$

for some $0 < \eta < \eta_0$. Then,

$$\big| H_s[E](\bar{x}) - H_s[F](\bar{y}) \big| \le C\eta, \tag{1.5}$$

for some constant $C > 0$ depending on n, s, η_0, R, α and the $C^{1,\alpha}$ norm of E at $\bar{x}$.

Notice that the s-mean curvature is well-defined not only for C^2 sets, but also for those being just $C^{1,\alpha}$ regular, provided $\alpha > s$. This fact is probably well-known to the experts but we nevertheless include a proof of it in Subsection 3.3.

Needless to say, the decompositions defined by formulae (1.2)-(1.3) are not restrictive at all. In fact, we employ this notation to the sole purpose of making more evident the role of Φ and Ξ as perturbations of the identity. The relation $\Xi = -\Phi \circ \Psi^{-1}$ clearly holds.

Notice that, if η_0 is suitably small, in dependence of n, then we can require condition (1.4) to hold a priori for $J\Phi$ only. Indeed, if this is the case, it can be shown that also the corresponding bound on $J\Xi$ is satisfied.

Finally, we stress that the hypotheses of the theorem are obviously satisfied by C^2 diffeomorphisms. In this case, one may be interested in the precise dependence of the constant C in (1.5) on s. To this scope, we took care of this dependence all along the proof and we finally made it explicit in formula (2.15).

As a result, one may observe that C diverges, while taking its limit as $s \to 1^-$. This is not surprising at all, since - at least regarding the asymptotic analysis with respect to the parameter s - the *right* normalization for the s-mean curvature is obtained by correcting the quantity described in (1.1) with the factor $1 - s$. Indeed, after this modification we see that the new constant C does not diverge anymore and, thus, the result is stable as s approaches 1 from below. Furthermore, by [2, Theorem 12] or [14, Lemma 9], we know that

$$(1-s)H_s[E](\bar{x}) \longrightarrow c_n H[E](\bar{x}), \qquad \text{as } s \to 1^-,$$

where $H[E](\bar{x})$ denotes the classical mean curvature of ∂E at $\bar{x}$ and c_n is some dimensional constant. Therefore, using estimate (2.15) we may recover the standard version of Theorem 1.1 for the classical mean curvature (see also Appendix A).

The heart of the proof of Theorem 1.1 is contained in Section 2, while we postpone some useful auxiliary computations to the subsequent Section 3. In the conclusive Appendix A we recall the corresponding well-known result for the classical mean curvature.

Notation. Next is a list of the less standard notations and conventions employed in the course of the work.

- Points of the $\mathbb{R}^n$ will be denoted with small letters, as x and y, while primed ones will indicate $(n-1)$-dimensional points. In general, we will make no difference between elements and sets of $\mathbb{R}^{n-1}$ and those of the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ of $\mathbb{R}^n$. Hence, we will often refer to a point of $\mathbb{R}^n$ as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.
  We will also use primed notations for differential operators applied to functions defined on subsets of $\mathbb{R}^{n-1}$. So, gradients of such functions will be denoted by ∇' and Laplacians by Δ'.
  No confusion should arise from the fact that the symbol Δ will also be used at times for the symmetric difference between two sets.
- The symbol $B_R(x)$ will indicate the open n-dimensional ball of radius R centered at a point $x \in \mathbb{R}^n$, while we will simply write B_R for that centered at the origin. Analogously, $B'_R(x')$ and B'_R will be used for $(n-1)$-dimensional balls.
  The $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ will be labeled as $S^{n-1} = \partial B_1$ and ω_{n-1} will denote its Hausdorff measure.
- Given a point $x \in \mathbb{R}^n$, a hyperplane $\pi \ni x$ orthogonal to $\nu \in S^{n-1}$ and two numbers $r, H > 0$, we will write $K_{\pi,r,H}(x)$ to denote the open cylinder of radius r and height $2H$, centered at x, directed along ν. In symbols,
  $$K_{\pi,r,H}(x) = \{ y \in \mathbb{R}^n : |y - x - [(y-x)\cdot\nu]\,\nu| < r,\ |(y-x)\cdot\nu| < H \}.$$
  We will use $K_{r,H}(x)$ to identify the cylinder directed along the n-th axis
  $$K_{r,H}(x) = B'_r(x') \times (-H, H),$$
  and set $K_{r,H} = K_{r,H}(0)$.
- The components of a vector valued function will be indicated with superscripts. Thus, if $F : \mathbb{R}^n \to \mathbb{R}^m$, we will write
  $$F(x) = \big(F^\ell(x)\big)^{\ell=1,\dots,m} = \big(F^1(x), \dots, F^m(x)\big).$$
  To avoid confusion, we will never use short notations for the derivatives of vector functions. Hence, the Jacobian matrix and Hessian tensor of F will be referred to as
  $$JF(x) = \big(\partial_i F^\ell(x)\big)_{i=1,\dots,n}^{\ell=1,\dots,m}, \qquad J^2F = \big(\partial^2_{ij} F^\ell(x)\big)_{i,j=1,\dots,n}^{\ell=1,\dots,m}.$$
- Latin letters, like i, j, k, will be used for indices running from 1 to n, while Greek letters, such as μ, ν, κ, identify those that range between 1 and $n-1$.

- We will understand the matrices as endowed with the Frobenius norm

$$\|A\|_F := \sqrt{A^TA} = \sqrt{\sum_{i,j=1}^n |A_{ij}|^2}, \qquad \text{for } A = [A_{ij}] \in \mathrm{Mat}_n(\mathbb{R}),$$

where A^T is the transpose of A. Any other norm works pretty much the same, but then some attention to the constants involved in the various computations should be paid.
- Sometimes we will use the big O notation. Indeed, saying that a function f is $O(\eta)$ will mean that there exists a constant $C > 0$ independent of η such that

$$|f(x)| \le C\eta,$$

for any x in the domain of f.

2. **Proof of Theorem 1.1.** First, denote by $\nu_F \in S^{n-1}$ the normal vector to the tangent hyperplane π_F to ∂F at $\bar{y}$ pointing inside F. Also, denote by L_F the half-space determined by π_F containing ν_F. We adopt the same notation with respect to E at the point $\bar{x}$.

Let $r > 0$ be some fixed number, whose value will be specified later. We begin with the computation inside the ball of radius r with center $\bar{x}$. We observe that, by symmetry,

$$\text{P.V.} \int_{B_r(\bar{y})} \frac{\widetilde{\chi}_{L_F}(y)}{|y-\bar{y}|^{n+s}}\, dy = 0. \tag{2.1}$$

Using (2.1) and applying the change of variables induced by Ψ, we compute

$$\begin{aligned}\text{P.V.} \int_{B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y-\bar{y}|^{n+s}}\, dy &= \int_{B_r(\bar{y})} \frac{\widetilde{\chi}_F(y) - \widetilde{\chi}_{L_F}(y)}{|y-\bar{y}|^{n+s}}\, dy \\ &= \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x) - \widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|\Psi(x)-\Psi(\bar{x})|^{n+s}} |\det J\Psi(x)|\, dx.\end{aligned}$$

Now, Lemma 3.2 tells us that

$$\begin{aligned} |\det J\Psi(x)| &= 1 + O(\eta), \\ |\Psi(x) - \Psi(\bar{x})|^{-n-s} &= |x-\bar{x}|^{-n-s}(1+O(\eta)). \end{aligned} \tag{2.2}$$

We remark that the functions defining the big O's only depend on n and η_0, besides x. Indeed, one can choose e.g. $\bar{\lambda} = n+1$, in the notation of Lemma 3.2, to obtain estimates independent of s. Thence, we obtain

$$\text{P.V.} \int_{B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y-\bar{y}|^{n+s}}\, dy = \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x) - \widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|x-\bar{x}|^{n+s}} (1+O(\eta))\, dx. \tag{2.3}$$

Now we prove that, up to choosing r small enough, it holds

$$\int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\left|\widetilde{\chi}_E(x) - \widetilde{\chi}_{\Psi^{-1}(L_F)}(x)\right|}{|x-\bar{x}|^{n+s}}\, dx \le C/(\alpha - s), \tag{2.4}$$

for some constant $C > 0$ depending only on n, η_0 and E.

To this scope, notice that we can select a radius $\tilde{r} > 0$ and a height $\tilde{H} > 0$, depending on n, α, R, η_0 and E, such that both

$$\partial E \cap K_{\pi_E, \tilde{r}, \tilde{H}}(\bar{x}) \qquad \text{and} \qquad \partial\Psi^{-1}(L_F) \cap K_{\pi_E, \tilde{r}, \tilde{H}}(\bar{x}),$$

can be written as graphs of $C^{1,\alpha}$ functions with respect to π_E. The assertion relative to ∂E is a direct consequence of its regularity properties in a neighbourhood of $\bar{x}$.

On the other hand, we may employ Proposition 3.3 to obtain that the same is true also for $\partial\Psi^{-1}(L_F)$. Furthermore, if $x \in \Psi^{-1}(B_r(\bar{y}))$, then

$$|x-\bar{x}| = |\Psi^{-1}(\Psi(x)) - \Psi^{-1}(\bar{y})| \le |\Psi(x)-\bar{y}| + |\Xi(\Psi(x)) - \Xi(\bar{y})| \le (1+\eta_0)r,$$

and so

$$\Psi^{-1}(B_r(\bar{y})) \subset B_{(1+\eta_0)r}(\bar{x}) \subset K_{\pi_E,(1+\eta_0)r,(1+\eta_0)r}(\bar{x}). \tag{2.5}$$

Thus, we take

$$r < \min\left\{\frac{\tilde{r}}{1+\eta_0}, \frac{\tilde{H}}{1+\eta_0}, 1\right\}. \tag{2.6}$$

Now, observe that both ∂E and $\partial\Psi^{-1}(L_F)$ are tangent to π_E at $\bar{x}$. We take advantage of this fact, together with Lemma 3.4 and (2.5), (2.6), to obtain that

$$\int_{\Psi^{-1}(B_r(\bar{y}))} \frac{|\widetilde{\chi}_E(x) - \widetilde{\chi}_{L_E}(x)|}{|x-\bar{x}|^{n+s}}\, dx \le C_1/(\alpha - s),$$

and

$$\int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\left|\widetilde{\chi}_{\Psi^{-1}(L_F)}(x) - \widetilde{\chi}_{L_E}(x)\right|}{|x-\bar{x}|^{n+s}}\, dx \le C_2\eta/(\alpha - s), \tag{2.7}$$

where $C_1 = C_1(n,E)$ and $C_2 = C_2(n,\eta_0)$ are positive constants. The combination of these two inequalities immediately leads to (2.4). Notice that we employed (3.10) to recover the bound for the $C^{1,\alpha}$ norm of $\Psi^{-1}(L_F)$ necessary to apply Lemma 3.4. Moreover, we simply controlled r with 1, since (2.6) is in force.

By this, (2.3) may be read as

$$\begin{aligned}\text{P.V.}\int_{B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y-\bar{y}|^{n+s}}\, dy = \int_{\Psi^{-1}(B_r(\bar{y}))} &\frac{\widetilde{\chi}_E(x) - \widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|x-\bar{x}|^{n+s}}\, dx \\ &+ (\alpha - s)^{-1} O(\eta).\end{aligned} \tag{2.8}$$

Now we only need to estimate the quantity

$$\text{P.V.}\int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|x-\bar{x}|^{n+s}}\, dx.$$

To do so, we first add and subtract $\widetilde{\chi}_{L_E}$ to the numerator. With the aid of (2.7), we compute

$$\begin{aligned}\left|\text{P.V.}\int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|x-\bar{x}|^{n+s}}\, dx\right| \le &\left|\text{P.V.}\int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{L_E}(x)}{|x-\bar{x}|^{n+s}}\, dx\right| \\ &+ C(\alpha - s)^{-1}\eta,\end{aligned} \tag{2.9}$$

with $C > 0$ depending only on n and η_0. Furthermore, by symmetry we have

$$\begin{aligned}\left|\text{P.V.}\int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{L_E}(x)}{|x-\bar{x}|^{n+s}}\, dx\right| &\le \int_{\Psi^{-1}(B_r(\bar{y}))\Delta B_r(\bar{x})} \frac{dx}{|x-\bar{x}|^{n+s}} \\ &\quad + \text{P.V.}\int_{B_r(\bar{x})} \frac{\widetilde{\chi}_{L_E}(x)}{|x-\bar{x}|^{n+s}}\, dx \\ &= \int_{\Psi^{-1}(B_r(\bar{y}))\Delta B_r(\bar{x})} \frac{dx}{|x-\bar{x}|^{n+s}}.\end{aligned}$$

Now, if $x \in \Psi^{-1}(B_r(\bar{y}))\Delta B_r(\bar{x})$, then we either have $x \notin B_r(\bar{x})$ or $x \notin \Psi^{-1}(B_r(\bar{y}))$. While in the first case it clearly holds $|x-\bar{x}| \ge r$, the latter yields

$$r \le |\Psi(x) - \Psi(\bar{x})| \le |x-\bar{x}| + |\Phi(x) - \Phi(\bar{x})| \le (1+\eta)|x-\bar{x}|.$$

That is

$$\text{if} \quad x \notin \Psi^{-1}(B_r(\bar{y})) \text{ or } x \notin B_r(\bar{x}), \quad \text{then} \quad |x-\bar{x}| \geq \frac{r}{1+\eta}. \tag{2.10}$$

A similar argument leads to the upper bound

$$|x-\bar{x}| \leq (1+\eta)r.$$

Thanks to these two inequalities, we compute

$$\begin{aligned}\left|\text{P.V.} \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{L_E}(x)}{|x-\bar{x}|^{n+s}}\, dx\right| &\leq \int_{B_{(1+\eta)r}(\bar{x})\setminus B_{r/(1+\eta)}(\bar{x})} \frac{dx}{|x-\bar{x}|^{n+s}} \\ &= \omega_{n-1}\int_{r/(1+\eta)}^{(1+\eta)r} \rho^{-1-s} d\rho \\ &= \frac{\omega_{n-1}}{sr^s(1+\eta)^s}\left[(1+\eta)^{2s}-1\right] \\ &\leq Cs^{-1}\eta,\end{aligned} \tag{2.11}$$

for some positive constant C depending on n, η_0, R, α and E. Notice that in the last line we used (2.6) and Lemma 3.1 with $\lambda = 2s$, $\bar{\lambda} = 2$. Combining (2.9) and (2.11) we get

$$\left|\text{P.V.} \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_{\Psi^{-1}(L_F)}(x)}{|x-\bar{x}|^{n+s}}\, dx\right| \leq C(s(\alpha-s))^{-1}\eta.$$

Consequently, (2.8) finally becomes

$$\begin{aligned}\text{P.V.} \int_{B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y-\bar{y}|^{n+s}}\, dy =& \,\text{P.V.} \int_{\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x)}{|x-\bar{x}|^{n+s}}\, dx \\ &+ (s(\alpha-s))^{-1}O(\eta).\end{aligned} \tag{2.12}$$

The computation outside $B_r(\bar{y})$ is much simpler. Here we do not have to deal with the singularity of the kernel and, indeed, the estimates are almost immediate. Nevertheless, we provide all the details.

Making the same substitution performed at the start of the proof and using (2.2) we recover

$$\begin{aligned}\int_{\mathcal{C}B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y-\bar{y}|^{n+s}}\, dy &= \int_{\Psi^{-1}(\mathcal{C}B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x)}{|\Psi(x)-\Psi(\bar{x})|^{n+s}} |\det JF(x)|\, dx \\ &= \int_{\mathcal{C}\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x)}{|x-\bar{x}|^{n+s}}(1+O(\eta))\, dx.\end{aligned} \tag{2.13}$$

Using now (2.10), we estimate

$$\begin{aligned}\int_{\mathcal{C}\Psi^{-1}(B_r(\bar{y}))} \frac{|\widetilde{\chi}_E(x)|}{|x-\bar{x}|^{n+s}}\, dx &\leq \int_{\mathcal{C}B_{r/(1+\eta)}(\bar{x})} \frac{dx}{|x-\bar{x}|^{n+s}} \\ &= \omega_{n-1}\int_{r/(1+\eta)}^{+\infty} \rho^{-1-s} d\rho \\ &= \frac{\omega_{n-1}(1+\eta)^s}{sr^s} \\ &\leq \frac{\omega_{n-1}(1+\eta_0)}{sr}.\end{aligned}$$

Thus, we can bring the big O in (2.13) out of the integral to write

$$\int_{\mathcal{C}B_r(\bar{y})} \frac{\widetilde{\chi}_F(y)}{|y-\bar{y}|^{n+s}}\,dy = \int_{\mathcal{C}\Psi^{-1}(B_r(\bar{y}))} \frac{\widetilde{\chi}_E(x)}{|x-\bar{x}|^{n+s}}\,dx + s^{-1}O(\eta). \tag{2.14}$$

Combining equations (2.12) and (2.14), we finally conclude that there exists a positive constant C, depending on n, η_0, R, α and E, such that

$$\big|H_s[F](\Psi(\bar{x})) - H_s[E](\bar{x})\big| \le C(s(\alpha-s))^{-1}\eta, \tag{2.15}$$

and hence (1.5) is proved.

3. Auxiliary results.

We collect here some minor results which have been used to prove Theorem 1.1. The section is divided into three parts. The first subsection contains an estimate for a one-dimensional function, the second is devoted to some general facts about diffeomorphisms of $\mathbb{R}^n$ and the third to singular integrals.

3.1. One-dimensional analysis.

In this short paragraph we include a technical computation involving a scalar function.

Lemma 3.1. *Fix $\eta_0 > 0$ and $\bar{\lambda} > 0$. Then, there exists a constant $C > 0$ depending only on $\bar{\lambda}$ and η_0 for which*

$$|(1+\eta)^\lambda - 1| \le C\eta, \tag{3.1}$$

for any $|\lambda| \le \bar{\lambda}$ and $\eta \in [0,\eta_0)$.

Proof. First notice that we can restrict to the case $\lambda > 0$. Indeed, when $\lambda = 0$ the result is obvious, while if $\lambda < 0$ we may recover it from the positive case, observing that

$$|(1+\eta)^\lambda - 1| = 1 - (1+\eta)^{-|\lambda|} = \frac{(1+\eta)^{|\lambda|}-1}{(1+\eta)^{|\lambda|}} \le |(1+\eta)^{|\lambda|}-1|.$$

Thus, assume $\lambda > 0$ and define

$$\varphi(t) := (1+t)^\lambda, \qquad \text{for any } t \in [0,\eta_0).$$

We have

$$\varphi'(t) = \lambda(1+t)^{\lambda-1}, \qquad \varphi''(t) = \lambda(\lambda-1)(1+t)^{\lambda-2}.$$

Then, we consider separately the two cases $\lambda \in (0,1)$ and $\lambda \ge 1$.

In the first situation, we have $\varphi'' < 0$ so that

$$\varphi'(t) \le \varphi'(0) = \lambda,$$

and thus

$$|(1+\eta)^\lambda - 1| = \varphi(\eta) - \varphi(0) = \int_0^\eta \varphi'(t)\,dt \le \lambda\eta \le \eta.$$

If $\lambda \ge 1$, then $\varphi'' \ge 0$ and hence

$$\varphi'(t) \le \varphi'(\eta_0) = \lambda(1+\eta_0)^{\lambda-1}.$$

By this we get

$$|(1+\eta)^\lambda - 1| = \varphi(\eta) - \varphi(0) = \int_0^\eta \varphi'(t)\,dt \le \lambda(1+\eta_0)^{\lambda-1}\eta \le \bar{\lambda}(1+\eta_0)^{\bar{\lambda}-1}\eta,$$

and in either case (3.1) is proved. □

3.2. **Facts concerning diffeomorphisms.** We collect here a pair of general results about diffeomorphisms of $\mathbb{R}^n$. In the first lemma we control some quantities related to a diffeomorphism with its C^1 norm.

Lemma 3.2. *Let $\eta_0 > 0$, U be a domain of $\mathbb{R}^n$ and $\Psi : U \to \mathbb{R}^n$ be a C^1 diffeomorphism. Decomposing Ψ and Ψ^{-1} as in (1.2)-(1.3), suppose that*

$$\|J\Phi\|_{L^\infty(U)}, \|J\Xi\|_{L^\infty(\Psi(U))} \leq \eta, \tag{3.2}$$

for some $0 < \eta < \eta_0$. Then,

$$\big| |\det J\Psi(x)| - 1 \big| \leq C\eta, \qquad \text{for any } x \in U, \tag{3.3}$$

for some constant $C > 0$ depending only on n and η_0. Moreover, given $0 < \lambda < \bar{\lambda}$, then

$$\left| \left[\frac{|\Psi(x) - \Psi(y)|}{|x-y|} \right]^{-\lambda} - 1 \right| \leq C\eta, \qquad \text{for any } x, y \in U \text{ such that } x \neq y, \tag{3.4}$$

for some constant $C > 0$ depending only on η_0 and $\bar{\lambda}$.

Proof. Recalling Leibniz formula for the determinant of a matrix, we compute for any $x \in U$

$$\begin{aligned} \det J\Psi(x) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n \partial_{\sigma(i)} \Psi^i(x) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n \left(\delta_{i\sigma(i)} + \partial_{\sigma(i)} \Phi^i(x) \right), \end{aligned} \tag{3.5}$$

where S_n is the symmetric group on $\{1, \ldots, n\}$ and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. Notice now that if $\sigma \neq I$ - the identical permutation - then there exists an index j for which $\sigma(j) \neq j$ and so, with the aid of (3.2),

$$\begin{aligned} \left| \operatorname{sgn}(\sigma) \prod_{i=1}^n \left(\delta_{i\sigma(i)} + \partial_{\sigma(i)} \Phi^i(x) \right) \right| &= \left| \partial_{\sigma(j)} \Phi^j(x) \prod_{\substack{i=1 \\ i \neq j}}^n \left(\delta_{i\sigma(i)} + \partial_{\sigma(i)} \Phi^i(x) \right) \right| \\ &\leq \left| \partial_{\sigma(j)} \Phi^j(x) \right| \prod_{\substack{i=1 \\ i \neq j}}^n \left(1 + \left| \partial_{\sigma(i)} \Phi^i(x) \right| \right) \\ &\leq \eta (1+\eta)^{n-1} \\ &\leq (1+\eta_0)^{n-1} \eta, \end{aligned} \tag{3.6}$$

On the other hand, the term relative to the identical permutation I can be written as

$$\begin{aligned} \operatorname{sgn}(I) \prod_{i=1}^n \left(\delta_{iI(i)} + \partial_{I(i)} \Phi^i(x) \right) &= \prod_{i=1}^n \left(1 + \partial_i \Phi^i(x) \right) \\ &= 1 + \sum_{j=1}^n \sum_{1 \leq i_1 < \ldots < i_j \leq n} \prod_{k=1}^j \partial_{i_k} \Phi^{i_k}(x). \end{aligned}$$

Since, using (3.2) and Lemma 3.1, it holds

$$\left|\sum_{j=1}^{n}\sum_{1\le i_1<\ldots<i_j\le n}\prod_{k=1}^{j}\partial_{i_k}\Phi^{i_k}(x)\right|\le\sum_{j=1}^{n}\binom{n}{j}\eta^j=(1+\eta)^n-1\le C\eta,$$

we are then able to deduce that

$$\left|\left|\operatorname{sgn}(I)\prod_{i=1}^{n}\left(\delta_{iI(i)}+\partial_{I(i)}\Phi^i(x)\right)\right|-1\right|\le C\eta, \tag{3.7}$$

for some constant $C>0$ depending only on n and η_0. Putting together inequalities (3.6) and (3.7), recalling (3.5) we finally conclude that

$$\big||\det J\Psi(x)|-1\big|\le C\eta, \qquad \text{for any } x\in U,$$

for some constant $C>0$ depending only on n and η_0, which is (3.3).

Now we turn to (3.4). Notice that, for any $x,y\in U$,

$$\frac{|\Psi(x)-\Psi(y)|}{|x-y|}\le\frac{|x-y|+|\Phi(x)-\Phi(y)|}{|x-y|}\le 1+\eta,$$

and

$$\begin{aligned}\frac{|\Psi(x)-\Psi(y)|}{|x-y|}&=\frac{|\Psi(x)-\Psi(y)|}{|\Psi^{-1}(\Psi(x))-\Psi^{-1}(\Psi(y))|}\\&\ge\frac{|\Psi(x)-\Psi(y)|}{|\Psi(x)-\Psi(y)|+|\Xi(\Psi(x))-\Xi(\Psi(y))|}\\&\ge\frac{1}{1+\eta},\end{aligned}$$

by (3.2). Furthermore, by Lemma 3.1 there exists a constant $C>0$ depending only on η_0 and $\bar\lambda$ for which

$$\big|(1+\eta)^{\pm\lambda}-1\big|\le C\eta.$$

Hence, we deduce that

$$\left|\left[\frac{|\Psi(x)-\Psi(y)|}{|x-y|}\right]^{-\lambda}-1\right|\le C\eta,$$

and the proof is complete. □

Next is the following proposition, where we address the problem of estimating the size of the domain over which the perturbation of a hyperplane is a graph. Moreover, we give an estimate of its norm as a graph in terms of the norm of the diffeomorphism.

Proposition 3.3. *Fix $\eta_0,R>0$, $\alpha\in(0,1]$, $\bar x\in\mathbb{R}^n$ and $e\in S^{n-1}$. Denote by π the hyperplane orthogonal to e which passes through $\bar x$. Let $\Psi:B_R(\bar x)\subset\mathbb{R}^n\to\mathbb{R}^n$ be a $C^{1,\alpha}$ diffeomorphism and, decomposing Ψ and Ψ^{-1} as in (1.2)-(1.3), assume that, for some $0<\eta<\eta_0$,*

$$\|J\Phi\|_{C^{0,\alpha}(B_R(\bar x))},\|J\Xi\|_{C^{0,\alpha}(\Psi(B_R(\bar x)))}\le\eta. \tag{3.8}$$

Then, there exists a radius $r_\star>0$ and a height $H_\star>0$, depending only on n, α, η_0 and R, such that the hypersurface

$$\Psi\left(\pi\cap B_R(\bar x)\right)\cap K_{\Psi_*\pi,r_\star,H_\star}(\Psi(\bar x)), \tag{3.9}$$

is a $C^{1,\alpha}$ graph with respect to the tangent hyperplane $\Psi_*\pi$ to $\Psi\left(\pi\cap B_R(\bar x)\right)$ at $\Psi(\bar x)$.

Moreover, denoting by h the $C^{1,\alpha}$ function defining (3.9) as a graph and by B' the $(n-1)$-dimensional ball of center $\Psi(\bar{x})$ and radius $r_\star$ contained in $\Psi_*\pi$ on which h is defined, we have

$$\|\nabla' h\|_{C^{0,\alpha}(B')} \leq C\eta, \tag{3.10}$$

for some constant $C > 0$ depending only on n and η_0.

Proof. We remark that it is enough to prove the proposition for $e = e_n$ and $\bar{x} = 0$. Moreover, by composing Ψ with a translation, we may also assume $\Psi(0) = 0$.

We restrict for the moment to prove the result under the additional hypothesis

$$\Psi_*\pi = \pi \qquad \text{and} \qquad \langle J\Psi(0)e_n, e_n\rangle > 0. \tag{3.11}$$

At a second stage we will show that the general case boils down to this one.

First, observe that (3.11) is equivalent to asking

$$\partial_\mu \Psi^n(0) = \partial_\mu (\Psi^{-1})^n(0) = 0, \qquad \text{for any } \mu = 1, \dots, n-1,$$

and

$$\partial_n \Psi^n(0), \partial_n (\Psi^{-1})^n(0) > 0.$$

By this and (3.8) we then obtain

$$\partial_n (\Psi^{-1})^n(0) = \frac{1}{\partial_n \Psi^n(0)} = \frac{1}{1 + \partial_n \Phi^n(0)} \geq \frac{1}{1+\eta_0}. \tag{3.12}$$

Now, we claim that there exists a radius $R_\star \in (0, R]$, depending only on α, η_0 and R, such that

$$\partial_n (\Psi^{-1})^n(y) \geq \frac{1}{2(1+\eta_0)}, \qquad \text{for any } y \in B_{R_\star}. \tag{3.13}$$

Indeed, by (3.12) and (3.8) we get

$$\partial_n (\Psi^{-1})^n(y) \geq \partial_n (\Psi^{-1})^n(0) - \eta|y|^\alpha \geq \frac{1}{1+\eta_0} - \eta_0 |y|^\alpha,$$

which gives (3.13), by taking $R_\star = \min\{[2\eta_0(1+\eta_0)]^{-1/\alpha}, R\}$.

Consequently, we may apply the Implicit Function Theorem to deduce the existence of two numbers $r, H \in (0, R_\star]$ and a C^1 function $h : B'_r \to [-H, H]$ for which

$$(y', h(y')) \in B_{R_\star}, \qquad \text{for any } y' \in B'_r, \tag{3.14}$$

and

$$(\Psi^{-1})^n(y', h(y')) = 0, \qquad \text{for any } y' \in B'_r. \tag{3.15}$$

We recover the $C^{0,\alpha}$ bound on the gradient of h. By differentiating (3.15) we get

$$\partial_\mu (\Psi^{-1})^n(y', h(y')) + \partial_n (\Psi^{-1})^n(y', h(y')) h_\mu(y') = 0,$$

for any $\mu = 1, \dots, n-1$, and so

$$h_\mu(y') = -\frac{\partial_\mu (\Psi^{-1})^n(y', h(y'))}{\partial_n (\Psi^{-1})^n(y', h(y'))}, \qquad \text{for any } y' \in B'_r. \tag{3.16}$$

Then, combining (3.8) and (3.13), we have

$$\|\nabla' h\|_{L^\infty(B'_r)} \leq \left\| \frac{\nabla' \Xi^n}{\partial_n (\Psi^{-1})^n} \right\|_{L^\infty(B_{R_\star})} \leq 2(1+\eta_0)\eta. \tag{3.17}$$

From this bound and (3.14), we see that the choice

$$r = \frac{R_\star}{\sqrt{1 + 4(1+\eta_0)^2 \eta_0^2}}, \qquad H = \frac{2(1+\eta_0)\eta_0 R_\star}{\sqrt{1 + 4(1+\eta_0)^2 \eta_0^2}},$$

is admissible. Moreover, by the Implicit Function Theorem there exists a number $\kappa \in (0,1]$ such that $\Psi(\pi \cap B_R) \cap K_{\kappa r, \kappa H}$ is *entirely* parametrized by the graph of h restricted to $B'_{\kappa r}$. We claim that κ may be chosen to depend only on η_0 and α. Indeed, assume that there exists $y \in K_{\kappa r, H}$ such that $(\Psi^{-1})^n(y) = 0$, but y does not belong to the graph of h. Hence, by (3.8) and the fact that $\Psi^{-1}(y)$ should lay outside of the set $\Psi^{-1}(\{(z', h(z')) : z' \in B'_r\})$, we have

$$|\Psi^{-1}(y)| \geq \inf_{z' \in \partial B'_r} |\Psi^{-1}(z', h(z'))| \geq \inf_{z' \in \partial B'_r} \frac{\sqrt{|z'|^2 + h(z')^2}}{1+\eta_0} \geq \frac{r}{1+\eta_0},$$

so that

$$|y_n|^2 = |y|^2 - |y'|^2 \geq \frac{|\Psi^{-1}(y)|^2}{(1+\eta_0)^2} - |y'|^2 \geq \left[\frac{1}{(1+\eta_0)^4} - \kappa^2\right] r^2.$$

In order to have $|y_n| > \kappa H$ it is enough to take $\kappa < (1+\eta_0)^{-2}\left(1 + 4(1+\eta_0)^2\eta_0^2\right)^{-1/2}$. Note that, if we set $r_\star := \kappa r$ and $H_\star := \kappa H$, then h defines $\Psi(\pi \cap B_R)$ as a graph in the cylinder $K_{r_\star, H_\star}$.

Finally, we turn to the $C^{0,\alpha}$ seminorm of h. In order to simplify the exposition, we will adopt the shorter notation

$$\psi_i(y') := \partial_i (\Psi^{-1})^n(y', h(y')).$$

We stress that (3.16) now reads as

$$h_\mu(y') = -\frac{\psi_\mu(y')}{\psi_n(y')}.$$

Moreover, we have that

$$\frac{1}{2(1+\eta_0)} \leq |\psi_n(y')| \leq 1 + \eta \quad \text{and} \quad |\psi_\mu(y')| \leq \eta, \qquad \text{for any } y' \in B'_{r_\star}.$$

Given $y', z' \in B'_{r_\star}$, we also notice that, using (3.17), we may estimate

$$\begin{aligned}
|\psi_i(y') - \psi_i(z')| &= |\partial_i(\Psi^{-1})^n(y', h(y')) - \partial_i(\Psi^{-1})^n(z', h(z'))| \\
&\leq \left[\partial_i(\Psi^{-1})^n\right]_{C^{0,\alpha}(B_{r_\star})} \left(|y'-z'|^2 + |h(y') - h(z')|^2\right)^{\alpha/2} \\
&\leq (\delta_{in} + \eta)\left(1 + 4(1+\eta_0)^2\eta^2\right)^{\alpha/2} |y'-z'|^\alpha.
\end{aligned}$$

Using these inequalities we compute

$$\begin{aligned}
|h_\mu(y') - h_\mu(z')| &= \frac{|\psi_n(z')\psi_\mu(y') - \psi_n(y')\psi_\mu(z')|}{|\psi_n(y')||\psi_n(z')|} \\
&\leq \frac{|\psi_\mu(y')||\psi_n(z') - \psi_n(y')| + |\psi_n(y')||\psi_\mu(y') - \psi_\mu(z')|}{|\psi_n(y')||\psi_n(z')|} \\
&\leq 4\left[\eta(1+\eta) + (1+\eta)\eta\right]\left(1 + 4(1+\eta_0)^2\eta^2\right)^{\alpha/2}(1+\eta_0)^2|y'-z'|^\alpha \\
&\leq 8(1+\eta_0)^3\left(1 + 4(1+\eta_0)^2\eta_0^2\right)^{1/2}\eta\,|y'-z'|^\alpha,
\end{aligned}$$

that is $h \in C^{1,\alpha}(B'_{r_\star})$ and

$$[\nabla' h]_{C^{0,\alpha}(B'_{r_\star})} \leq C\eta, \tag{3.18}$$

for some constant $C > 0$ depending only on η_0. The combination of (3.17) and (3.18) leads to (3.10).

To conclude, we show that hypothesis (3.11) may be dropped.

Let $v \in S^{n-1}$ be a vector orthogonal to $\Psi_* \pi$ and consider a rotation $Q \in \mathrm{SO}(n)$ such that $Qv = e_n$. Up to an orthogonal change of variables in the hyperplane π, we may indeed assume v to be spanned by e_{n-1} and e_n. Hence, we write

$$v = \frac{1}{\sqrt{1+t^2}}\left(e_{n-1} + te_n\right),$$

for some $t \in \mathbb{R}$. Moreover we may take Q of the form

$$Q = \begin{pmatrix} I_{n-2} & 0 \\ 0 & R \end{pmatrix}, \tag{3.19}$$

where I_{n-2} is the identity matrix of $\mathrm{Mat}_{n-2}(\mathbb{R})$ and $R \in SO(2)$ is defined by

$$R = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} t & -1 \\ 1 & t \end{pmatrix}. \tag{3.20}$$

Then, we introduce the function

$$\Psi_Q(x) := Q\Psi(x), \qquad \text{for any } x \in B_R.$$

Notice that Ψ_Q is a $C^{1,\alpha}$ diffeomorphism. In addition, for any $w \in \pi$ we have

$$\langle J\Psi_Q(0)w, e_n\rangle = \langle QJ\Psi(0)w, e_n\rangle = \langle J\Psi(0)w, Q^T e_n\rangle = \langle J\Psi(0)w, v\rangle = 0,$$

since $J\Psi(0)w \in \Psi_*\pi$ and v is orthogonal to $\Psi_*\pi$ by definition. Hence, $(\Psi_Q)_*\pi = \pi$. Furthermore, we can choose v in a way that

$$\langle J_Q\Psi(0)e_n, e_n\rangle = \langle J\Psi(0)e_n, v\rangle > 0.$$

Thus, assumption (3.11) holds true for Ψ_Q.

Now, we prove that the Jacobians $J\Phi_Q$ and $J\Xi_Q$, defined as in (1.2)-(1.3), satisfy a bound similar to (3.8). We claim that it is enough to show that there exists a dimensional constant $C > 0$ such that

$$\|Q - I\| = \|Q^T - I\| \le C\eta. \tag{3.21}$$

Indeed, we compute

$$J\Phi_Q = J\Psi_Q - I = QJ\Psi - I = QJ\Phi + Q - I,$$

and similarly

$$J\Xi_Q = J\Xi Q^T + Q^T - I.$$

Notice that formula $\Psi_Q^{-1} = \Psi^{-1} \circ Q^T$ has been used to recover the last identity. Thus, since $\|Q\| = \|Q^T\| = \sqrt{n}$, if (3.21) holds, then we immediately deduce that

$$\|J\Phi_Q\|_{C^{0,\alpha}(B_R)}, \|J\Xi_Q\|_{C^{0,\alpha}(B_R)} \le (\sqrt{n} + C)\eta.$$

Now we prove (3.21). Observe that we may restrict to consider $\eta \le 1/2$. Indeed, if this is not the case we simply estimate

$$\|Q - I\| \le \|Q\| + \|I\| = 2\sqrt{n} \le 4\sqrt{n}\eta.$$

Thus, we assume $\eta \le 1/2$ in what follows. By (3.19) and (3.20), we have

$$\|Q - I\|^2 = \|R - I\|^2 = 2\left(\frac{t}{\sqrt{1+t^2}} - 1\right)^2 + \frac{2}{1+t^2} = 4\left(1 - \frac{t}{\sqrt{1+t^2}}\right). \tag{3.22}$$

Note that, by (3.8) and the definition of v, we get

$$\begin{aligned} 0 < 1 - \eta &\le |v|^2 + \langle J\Phi(0)v, v\rangle = \langle J\Psi(0)v, v\rangle \\ &= \frac{1}{\sqrt{1+t^2}}\left[\langle J\Psi(0)e_{n-1}, v\rangle + t\langle J\Psi(0)e_n, v\rangle\right] = \frac{t}{\sqrt{1+t^2}}\langle J\Psi(0)e_n, v\rangle. \end{aligned}$$

Moreover, it holds $\langle J\Psi(0)e_n, v\rangle > 0$, so that $t > 0$. On the other hand,

$$\begin{aligned}0 = \langle J\Psi(0)e_{n-1}, v\rangle &= \frac{1}{\sqrt{1+t^2}} \left[\langle J\Psi(0)e_{n-1}, e_{n-1}\rangle + t\langle J\Psi(0)e_{n-1}, e_n\rangle\right]\\ &= \frac{1}{\sqrt{1+t^2}} \left[1 + \langle J\Phi(0)e_{n-1}, e_{n-1}\rangle + t\langle J\Phi(0)e_{n-1}, e_n\rangle\right].\end{aligned}$$

Hence, we obtain that

$$\frac{1}{2} \le 1 - \eta \le 1 + \langle J\Phi(0)e_{n-1}, e_{n-1}\rangle = -t\langle J\Phi(0)e_{n-1}, e_n\rangle \le \eta t,$$

that is, $t \ge 1/(2\eta)$. Then, after a simple computation, from (3.22) we finally deduce the bound

$$\|Q - I\|^2 \le 8\eta^2,$$

which immediately implies (3.21).

By the previous results, it is now clear that Ψ_Q satisfies the hypotheses of the proposition and (3.11). Consequently, the first part of the argument applies, yielding the thesis for Ψ_Q. But then the proof is complete, since $\Psi(\pi \cap B_R)$ is the rotation of $\Psi_Q(\pi \cap B_R)$ by means of Q^T. □

3.3. **Integral computations.** In this subsection we report a couple of straightforward results concerning singular integrals. The first one provides an estimate for the detachment of a $C^{1,\alpha}$ graph from its tangent hyperplane inside a ball.

Lemma 3.4. *Let $\eta, r > 0$, $s \in (0,1)$, $\alpha \in (s,1]$ and $\bar{x} \in \mathbb{R}^n$. Let $h : B'_r(\bar{x}') \to \mathbb{R}$ be a given $C^{1,\alpha}$ function, with $h(\bar{x}') = \bar{x}_n$ and*

$$[\nabla' h]_{C^{0,\alpha}(B'_r(\bar{x}'))} \le \eta.$$

Then, denoting by

$$G := \{(x', x_n) \in B'_r(\bar{x}') \times \mathbb{R} : x_n < h(x')\},$$

the subgraph of h, and by

$$L := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n < h(\bar{x}') + \nabla' h(\bar{x}') \cdot (x' - \bar{x}')\},$$

the lower half-space determined by the tangent hyperplane of h at $\bar{x}$, we have that

$$\int_{B_r(\bar{x})} \frac{|\widetilde{\chi}_G(x) - \widetilde{\chi}_L(x)|}{|x - \bar{x}|^{n+s}}\, dx \le C(\alpha - s)^{-1} r^{\alpha - s} \eta, \tag{3.23}$$

for some constant $C > 0$ depending only on n.

Proof. We assume without loss of generality that $\bar{x} = 0$, i.e. $h(0) = 0$, and $\nabla' h(0) = 0$. Observe that the function $\mathcal{P}$ defined by

$$\mathcal{P}(x') := \eta |x'|^{1+\alpha}, \qquad \text{for any } x' \in \mathbb{R}^{n-1}$$

is such that

$$-\mathcal{P}(x') \le h(x') \le \mathcal{P}(x'), \qquad \text{for any } x' \in B'_r.$$

Therefore, setting

$$P := \left\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_n| < \mathcal{P}(x')\right\},$$

we have

$$|\widetilde{\chi}_G - \widetilde{\chi}_L| \le 2\chi_{G\Delta L} \le 2\chi_P, \qquad \text{in } B_r.$$

Thus, we may conclude that

$$\begin{aligned}\int_{B_r} \frac{|\widetilde{\chi}_G(x) - \widetilde{\chi}_L(x)|}{|x|^{n+s}}\,dx &\le 2\int_{B_r} \frac{\chi_P(x)}{|x|^{n+s}}\,dx \\ &\le 4\int_{B_r'} \left(\int_0^{\eta|x'|^{1+\alpha}} \frac{dx_n}{|x|^{n+s}}\right) dx' \\ &\le 4\eta \int_{B_r'} \frac{|x'|^{1+\alpha}}{|x'|^{n+s}}\,dx' \\ &= \frac{4\omega_{n-2}}{\alpha - s} r^{\alpha-s}\eta,\end{aligned}$$

which yields (3.23). □

As a consequence, we deduce that the s-mean curvature is a well-defined quantity for $C^{1,\alpha}$ sets, if $\alpha > s$.

Corollary 3.5. *Let $s \in (0,1)$ and $\alpha \in (s,1]$. Let $E \subset \mathbb{R}^n$ be an open set and take $\bar{x} \in \partial E$. If ∂E is of class $C^{1,\alpha}$ at $\bar{x}$, then $H_s[E](\bar{x})$ is well-defined in the principal value sense.*

Proof. By definition, we know that E may be written as the subgraph of a $C^{1,\alpha}$ function, locally in $B_r(\bar{x})$, for some small radius $r > 0$. Thus, denoting by L the lower half-space determined by the tangent hyperplane to ∂E at $\bar{x}$, we may apply Lemma 3.4 to deduce that

$$\begin{aligned}\left|\mathrm{P.V.}\int_{B_r(\bar{x})} \frac{\widetilde{\chi}_E(x)}{|x-\bar{x}|^{n+s}}\,dx\right| &\le \int_{B_r(\bar{x})} \frac{|\widetilde{\chi}_E(x) - \widetilde{\chi}_L(x)|}{|x-\bar{x}|^{n+s}}\,dx \\ &\quad + \left|\mathrm{P.V.}\int_{B_r(\bar{x})} \frac{\widetilde{\chi}_L(x)}{|x-\bar{x}|^{n+s}}\,dx\right| \\ &= \int_{B_r(\bar{x})} \frac{|\widetilde{\chi}_E(x) - \widetilde{\chi}_L(x)|}{|x-\bar{x}|^{n+s}}\,dx \\ &< +\infty.\end{aligned}$$

Notice that the integral on the second line vanishes by symmetry, in the principal value sense. Furthermore, outside of $B_r(\bar{x})$ we simply estimate

$$\left|\int_{\mathcal{C}B_r(\bar{x})} \frac{\widetilde{\chi}_E(x)}{|x-\bar{x}|^{n+s}}\,dx\right| \le \omega_{n-1}\int_r^{+\infty} \rho^{-1-s}\,d\rho = \frac{\omega_{n-1}}{s r^s} < +\infty.$$

These two estimates yield the thesis. □

Appendix A. **The result in the classical framework.** In this appendix we present a straightforward computation showing the validity of the counterpart of Theorem 1.1 for the classical mean curvature. By so doing, we extend our result, formally including the case $s = 1$. Notice that this conclusion may be rigorously obtained as a limiting case of Theorem 1.1, as discussed in the introduction. Nevertheless, we provide here a direct proof.

Let E be an open set of $\mathbb{R}^n$ and $\bar{x} \in \partial E$. Assume E to have C^2 boundary at $\bar{x}$. Let $R > 0$ and $\Psi : B_R(\bar{x}) \to \mathbb{R}^n$ be a C^2 diffeomorphism. Define $F \subset \mathbb{R}^n$ and $\bar{y} \in \partial F$ by setting

$$F := \Psi(E \cap B_R(\bar{x})), \qquad \bar{y} := \Psi(\bar{x}).$$

Decomposing Ψ as in (1.2) and assuming

$$\|J\Psi(\bar{x})\|, \|J^2\Psi(\bar{x})\| \le \eta,$$

for some small $\eta > 0$, we will show that the mean curvatures of ∂E and ∂F, at $\bar{x}$ and $\bar{y}$ respectively, are linked by the relation

$$\big|H[E](\bar{x}) - H[F](\bar{y})\big| \le C\eta.$$

Notice that, without any loss of generality, we may assume both ∂E and ∂F to be smooth graphs with respect to the hyperplane $\{x_n = 0\}$, locally around $\bar{x}$ and $\bar{y}$ respectively. That is

$$\begin{aligned} E \cap B_\varepsilon(\bar{x}) &= \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : u(x') < x_n\big\} \cap B_\varepsilon(\bar{x}), \\ F \cap B_\varepsilon(\bar{y}) &= \big\{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : v(y') < y_n\big\} \cap B_\varepsilon(\bar{y}), \end{aligned}$$

for some C^2 functions $u : B'_\varepsilon(\bar{x}') \to \mathbb{R}$ and $v : B'_\varepsilon(\bar{y}') \to \mathbb{R}$, satisfying

$$\Psi^n(x', u(x')) = v(\Psi'(x', u(x'))), \qquad \text{for any } x' \in B'_\varepsilon(\bar{x}').$$

When we differentiate this equation we get

$$\partial_\mu \Psi^n + \partial_n \Psi^n u_\mu = v_\kappa \left(\partial_\mu \Psi^\kappa + \partial_n \Psi^\kappa u_\mu\right).$$

Taking one more derivative, we find

$$\begin{aligned} &\partial^2_{\mu\nu}\Psi^n + \partial^2_{\mu n}\Psi^n u_\nu + \left(\partial^2_{n\nu}\Psi^n + \partial^2_{nn}\Psi^n u_\nu\right) u_\mu + \partial_n \Psi^n u_{\mu\nu} \\ &\quad = v_{\kappa\xi}\Big(\partial_\mu \Psi^\kappa + \partial_n \Psi^\kappa u_\mu\Big)\Big(\partial_\nu \Psi^\xi + \partial_n \Psi^\xi u_\nu\Big) \\ &\qquad + v_\kappa \left(\partial^2_{\mu\nu}\Psi^\kappa + \partial^2_{\mu n}\Psi^\kappa u_\nu + \left(\partial^2_{n\nu}\Psi^\kappa + \partial^2_{nn}\Psi^\kappa u_\nu\right) u_\mu + \partial_n \Psi^\kappa u_{\mu\nu}\right). \end{aligned}$$

Supposing then for simplicity that $\bar{x} = \bar{y} = 0$, $u(0) = v(0) = 0$ and $\nabla' v(0) = 0$, we deduce from the above relations

$$u_\mu(0) = -\frac{\partial_\mu \Psi^n(0)}{1 + \partial_n \Phi^n(0)} = O(\eta),$$

and

$$\begin{aligned} u_{\mu\nu}(0) &= \Big[v_{\kappa\xi}(0)\Big(\partial_\mu \Psi^\kappa(0) + \partial_n \Psi^\kappa(0) u_\mu(0)\Big)\Big(\partial_\nu \Psi^\xi(0) + \partial_n \Psi^\xi(0) u_\nu(0)\Big) \\ &\qquad - \partial^2_{\mu\nu}\Psi^n(0) - \partial^2_{\mu n}\Psi^n(0) u_\nu(0) \\ &\qquad - \left(\partial^2_{n\nu}\Psi^n(0) + \partial^2_{nn}\Psi^n(0) u_\nu(0)\right) u_\mu(0)\Big]\left[1 + \partial_n \Phi^n(0)\right]^{-1} \\ &= v_{\mu\nu}(0) + O(\eta). \end{aligned}$$

Hence, we may finally conclude that

$$\begin{aligned} H[E](0) &= \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)(0) \\ &= \left(1 + |\nabla u|^2\right)^{-1/2} \Delta u(0) - \left(1 + |\nabla u|^2\right)^{-3/2} u_{\mu\nu}(0) u_\mu(0) u_\nu(0) \\ &= \Delta v(0) + O(\eta) \\ &= H[F](0) + O(\eta), \end{aligned}$$

which is what we wanted to prove.

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