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I hereby declare that this thesis is entirely the result of my own work except where otherwise indicated. I have only used the resources given in the list of references.

Munich, Date of submission Mohamed Noah Abdel Wahab

Abstract

Abstract

Nonlocal minimal surfaces confined within a cylinder exhibit unique behaviors dependent on external data. This thesis delves into these surfaces, which incorporate long-range spatial interactions compared to classical minimal surfaces. We consider two variations of the model discussed in [5], a minimal surfaces confined within a cylinder.

We investigate two scenarios: varying the height and width of data outside a separating slab. The results show that when the slab is wide, the minimal surface becomes disconnected from the data, while a narrow slab allows connection. This allows us to predict the behavior of similar models with symmetrically placed data. Additionally, the research reveals that for sufficiently narrow slabs, the surface "sticks" to the cylinder.

Finally, we present an example where the minimizer is completely disconnected from the external data, a phenomenon unique to nonlocal minimal surfaces. This work provides valuable insights into the behavior of these emerging mathematical objects and their interaction with external data.

Zusammenfassung

In Zylindern eingeschlossene nichtlokale Minimalflächen zeigen ein einzigartiges Verhalten, das von externen Daten abhängt. Diese Arbeit befasst sich mit diesen Flächen, die im Vergleich zu klassischen Minimalflächen weitreichende räumliche Wechselwirkungen berücksichtigen. Wir betrachten zwei Varianten des in [5] diskutierten Modells, einer in einem Zylinder eingeschlossenen Minimalfläche.

Dabei untersuchen wir zwei Szenarien: die Variation der Höhe und der Breite von Daten außerhalb einer trennenden Platte. Die Ergebnisse zeigen, dass die Minimalfläche bei breiter Platte von den Daten getrennt wird, während eine schmale Platte eine Verbindung ermöglicht. Dies erlaubt uns, das Verhalten ähnlicher Modelle mit symmetrisch angeordneten Daten vorherzusagen. Darüber hinaus zeigt die Forschung, dass die Fläche bei ausreichend schmalen Platten am Zylinder "haftet".

Schließlich präsentieren wir ein Beispiel, bei dem der Minimierer vollständig von den externen Daten getrennt ist, ein Phänomen, das für nichtlokale Minimalflächen einzigartig ist. Diese Arbeit liefert wertvolle Erkenntnisse über das Verhalten dieser neuen mathematischen Objekte und ihre Wechselwirkung mit externen Daten.

List of symbols

$\mathbb{R}^n$	Euclidean space of dimension n
$\mathrm{dist}(A, B)$	Distance between sets A and B

Contents

1 Introduction

IDEA. Idea: Start with short historical background
18th century: Lagrange, Euler
20th Century: DeGiorgi Perimeter and localized entity
2009 Cafarelli, Roquejoffre, Savin: Nonlocal minimal surfaces
Perimeter and nonlocal perimeter as the (semi)norm of an indicator function
Define the usual problem considered
Better regularity than classical minimal surfaces

Chapter 02
Model 01
Model 02
Combination of both

Chapter 03
Fully disconnected minimizer

Use the introduction in [9] as inspiration.

TODO. Add sources
What does "locally" mean here? And what do we minimize? The surface or the area the encompassed?
Rewrite the text

Minimal surfaces, characterized by locally minimizing their surface area, have captivated mathematicians for centuries. Dating back to the 18th century, mathematicians like *Euler* and *Lagrange* laid the foundation for the field. In an effort to describe these surfaces mathematically, they formulated the *Euler - Lagrange equations* in the late 18th century. These equations provide a powerful framework for identifying and characterizing minimal surfaces. Since the 19th century, many mathematicians contributed to the study of minimal surfaces, uncovering profound insights. Since then minimal surfaces found many applications in various fields beyond pure mathematics. From understanding physical phenomena like soap films and black holes to informing the design of optimal structures in engineering and architecture, the versatility of minimal surfaces continues to inspire exploration.

In this thesis, we want to explore a rather recent concept of minimal surfaces, namely *Nonlocal Minimal Surfaces*, which were first introduced by *Cafarelli*, *Roquejoffre*, and *Savin* in 2009. For that purpose, we will first give a short introduction to the theory of minimal surfaces in the context of this work.

1.1 Classical Minimal Surfaces

CHECK. Is this introduction enough and complete/correct?

The study of minimal surfaces concerns itself with finding the set with least surface area under certain constrains. But before we can formulate the usual problem, we have to define some tools.

CHECK. Do I need to cite this definition from [4]? Give a justification?

Definition 1.1. Let $A \subset \mathbb{R}^n$ with smooth boundary, then the surface area or *perimeter* of A is given by

$$\operatorname{Per}(A) := \sup\left\{\int_{\partial A} \varphi \cdot \nu_A \, \mathrm{d}\mathcal{H}^{n-1} \mid \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), |\varphi| \leq 1\right\},$$

where ν_A is the outer normal to A.

To extend this definition to general measurable sets, we can use the divergence theorem and rewrite the integration over the boundary as an integration over the set itself. This removes the need for a smooth boundary and allows us to define the surface area for general sets.

Definition 1.2. Let $A \subset \mathbb{R}^n$ be a Borel set, then the perimeter of A is given by

$$\operatorname{Per}(A) := \sup\left\{\int_A \operatorname{div}\varphi \mid \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), |\varphi| \leq 1\right\}.$$

TODO. Rewrite this text
Not just for Minimization problem, but Perimeter of a set in some other set

In the minimization problem, we want to find some set E which minimizes the surface of some external data E_0. Since the surface area may be infinite, if E_0 is unbounded, we can "localize"[1] the problem by just considering the part of ∂E in some bounded set Ω.

Definition 1.3. Let $A \subset \mathbb{R}^n$ be a Borel set and $\Omega \subset \mathbb{R}^n$ bounded, then the perimeter of A relative to Ω is given by

$$\operatorname{Per}(A, \Omega) := \sup\left\{\int_A \operatorname{div}\varphi \mid \varphi \in C_c^1(\Omega, \mathbb{R}^n), |\varphi| \leq 1\right\}.$$

Now we can formulate the usual problem.

Definition 1.4 (Minimal Surface Problem)**.** Let $\Omega \subset \mathbb{R}^n$ bounded and $E_0 \subset \mathbb{R}^n$, then we want to find $E \subset \mathbb{R}^n$ such that E minimizes the perimeter of E_0 relative to Ω, i.e.

$$\operatorname{Per}(E, \Omega) = \min\left\{\operatorname{Per}(A, \Omega) \mid A \setminus \Omega = E_0 \setminus \Omega\right\}.$$

This set E is then called a *minimal surface*.

TODO. Complete note
Case that $E_0 \cap \Omega \neq \emptyset$..
Give sources, that minimizer exists, thus minimal surfaces exist and say something about uniqueness
Note that in classical theory often one just has a contour over which one minimizes

Note. Usually E_0 is chosen such that $E_0 \cap \Omega = \emptyset$, then we minimize over the set E such that $E \setminus \Omega = E_0$. If $E_0 \cap \Omega \neq \emptyset$, then we can minimize over..

TODO. Standard example of minimal surfaces (Plateau's problem, soap bubble)

[1] Here "local" refers to the area in which we minimize

1.2 Nonlocal Minimal Surfaces

TODO. Do again, but start this time from the example of a soap bubble
Soap bubble, classical example, 2 dim surface..
Nanoscale, 3 dim, classical theory doesn't suffice anymore
Short construction of fractional perimeter a la Cafarelli [2]

Rewrite the text
Is the example fitting?
Emphasize that we are no longer just minimizing boundary but the set as well

Let us for now consider some set $A \subset \mathbb{R}^n$ with smooth boundary, then to get its perimeter we have to take the supremum of

$$\int_{\partial A} \varphi \cdot \nu_A.$$

This is a local quantity, i.e. it only depends on the boundary of A. Thus if we want to minimize the perimeter of some set E with external data E_0, we are only interested in the behavior of the boundary of E close to and in Ω and not in the contribution or the size of the external data. In many cases, this is enough to describe the behavior of the minimizer, but in some cases, this is not enough anymore. Take a soap bubble as an example, a standard example for a classical minimal surfaces. In our normal scaling, we can see the film of the soap bubble as a 2- dimensional object. But if we go to the molecular level, we see that the film is a 3-dimensional object. Thus we need to incorporate long - range correlation into our definition of perimeter and minimal surfaces. *Cafarelli*, *Roquejoffre*, and *Savin* did exactly that in 2009, when they introduced the concept of *nonlocal minimal surfaces* and *fractional perimeter* in [2].

TODO. What is the effect of s?
Which definition is standard? Add note about other definitions
Maybe use the definition from [18], but it's without s, just with $(1-s)$ and with 2 in front. The 2 is just convention for the relation to the Gagliardo seminorm. Why not with s? Can I define it with s?
For the limiting behavior of $\text{Per}_s()$ for $s \to 0/1$ see [16]

Definition 1.5 (Fractional Perimeter)**.** Let $A \subset \mathbb{R}^n$ be a Borel set, $s \in (0,1)$, then the s- perimeter of A to is given by

$$\text{Per}_s(A) := \int_A \int_{A^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x.$$

Intuitively, we can understand the parameter s as the grade of nonlocality. For s big, we have a more local…

Just as in the classical case, we can define a relative fractional perimeter by removing the integration over the constant part..

$$\begin{aligned}
&\int_A \int_{A^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_{A\cap\Omega} \int_{A^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x + \int_{A\setminus\Omega} \int_{A^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_{A\cap\Omega} \int_{A^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x + \int_{A\setminus\Omega} \int_{\Omega\setminus A} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x + \int_{A\setminus\Omega} \int_{A^c\setminus\Omega} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x
\end{aligned}$$

While minimizing A relative to Ω we can ignore the last term as it is constant and thus does not affect the minimization.

Definition 1.6. Let $A, B \subset \mathbb{R}^n$ be Borel sets, $s \in (0,1)$, then the interaction of A and B is given by

$$\mathscr{L}(A,B) := \int_A \int_{B^c} \frac{1}{|x-y|^{n+s}} \,\mathrm{d}y \,\mathrm{d}x.$$

Definition 1.7 (Relative Fractional Perimeter)**.** Let $A \subset \mathbb{R}^n$ be a Borel set, $\Omega \subset \mathbb{R}^n$ bounded and $s \in (0,1)$, then the s- perimeter of A relative to Ω is given by

$$\operatorname{Per}_s(A,\Omega) := \mathscr{L}(A \cap \Omega, A^c) + \mathscr{L}(A \setminus \Omega, \Omega \setminus A).$$

TODO. Rewrite.. very bad
Not precise enough

Note. In some literature, the fractional perimeter is sometimes defined with the factor 2 in front of the integral. This is just a convention to relate the Gagliardo seminorm

$$\|f\|_{W^{s,1}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^{n+s}} \,\mathrm{d}y \,\mathrm{d}x$$

to the fractional perimeter. Notice that

$$\operatorname{Per}_s(A) = \int_A \int_{A^c} \frac{1}{|x-y|^{n+s}} \,\mathrm{d}y \,\mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\chi_A(x)-\chi_A(y)|}{|x-y|^{n+s}} \,\mathrm{d}y \,\mathrm{d}x = \frac{1}{2} \|\chi_A\|_{W^{s,1}(\mathbb{R}^n)},$$

i.e. the fractional perimeter is the seminorm of the indicator function of A up to a multiplicative constant.
In more recent literature like [18], the fractional perimeter is defined with the factor $(1-s)$ in front of the integral. This is based on the limiting behavior of the fractional perimeter for $s \to 1^-$ as shown in [1] and [3]. In the latter, the authors have shown that for sets of finite *classical* perimeter, we have that $(1-s)\operatorname{Per}_s(A,\Omega) \to c \operatorname{Per}(A,\Omega)$ as $s \to 1^-$ for some constant c depending on the dimension. In the former paper, the authors have shown the same behavior in sense of Γ- convergence and general measurable sets. In [18], the factor $(1-s)$ is used, to justify the fractional perimeter as a generalization of the classical perimeter.
In [9], the authors analyzed the behavior of $s \operatorname{Per}_s(A,\Omega)$ for $s \to 0^+$. They showed that not all sets have a limit for $s \to 0^+$ and if a limit exits, then it relates to the volume of the sets.
Thus if we are interested in the limiting behavior, it would make sense to define Perimeter with the factor $s(1-s)$ in front of the integral. We will stick to the usual definition for convenience and add the factor $s(1-s)$, when are interested in the limiting behavior.

TODO. give an example where classical theory doesn't suffice (cube rotated by 45 degree)
Serra 2023 pixelled square

TODO. Add note about advantages/properties (e.g. Euler - Lagrange Viscos) of nonlocal minimal surfaces like better regularity properties and..
Add some sentences about stickiness property and that we are looking at a model precisely about that property.
Give some justification, why fractional perimeter can be seen as a generalization of the classical perimeter.

With these tools we can now define the nonlocal minimal surface problem.

Definition 1.8 (Nonlocal Minimal Surface Problem)**.** Let $\Omega \subset \mathbb{R}^n$ bounded and $E_0 \subset \mathbb{R}^n$, then we want to find $E \subset \mathbb{R}^n$ such that E minimizes the s- perimeter of E_0 relative to Ω, i.e.

$$\mathrm{Per}_s(E, \Omega) = \min \left\{ \mathrm{Per}_s(A, \Omega) \mid A \setminus \Omega = E_0 \setminus \Omega \right\}.$$

Over the last few years these nonlocal minimal surfaces have been an are of great interest. Various properties have been studied and many results have been obtained. Next to the better regularity properties, Euler - Lagrange equations, stickiness property,..

TODO. Quick summary of Chapter 2 Quick summary of Chapter 3

In this thesis, we want to explore more on these surfaces and their properties. In Chapter 2, we will consider two models, analyze them on connectedness and try to understand the stickiness property and where the contribution lies to achieve stickiness. We will then derive the behavior of models similar to both, to get an understanding of general models of that form. In Chapter 3, we will discuss a natural question coming up while analyzing the models, namely the existence of a nontrivial minimizer in the case that the external data and the prescribed set are disconnected. We will provide an example where such a minimizer exists. This behavior is unique to nonlocal minimal surfaces.

2 Models

TODO. Instead of considering both models seperately consider them together

TODO. Rewrite the text
Add discussion about variation of models and why we are considering that

In this chapter we will consider a generalizastion of the model considered by Dipierro et al. in [5], where they considered the external data E_0 as the complement of a slab in $\mathbb{R}^n$ of width $2M$ and the prescribed data Ω as the cylinder of radius 1 and height $2M$. The showed that for M big enough the minimizer is disconnected which is consistent with the classical theory of minimal surfaces. When M is small enough, the minimizer is connected and even sticks to the boundary. The latter being a unique property of nonlocal minimal surfaces.

Here we will schow that for any w.r.t. e_n symmetric external data E_0 such that

$$E_R := \{(x', x_n \mid |x'| < 1, M < |x_n| < M + R)\} \subset E_0 \subset \{(x', x_n \mid |x_n| > M)\}$$

and prescribed data $\Omega := \{(x', x_n \mid |x'| < 1, |x_n| < M)\}$, for some $M, R > 0$ displays the same behavior as the model considered in [5]. That is, for M big enough the minimizer is disconnected and for M small enough the minimizer is connected and sticks to the boundary.

For $n \geq 2$ consider any external data E_0 and prescribed data as above. The Figure 2.1 illustrates the setting.

TODO. Add caption

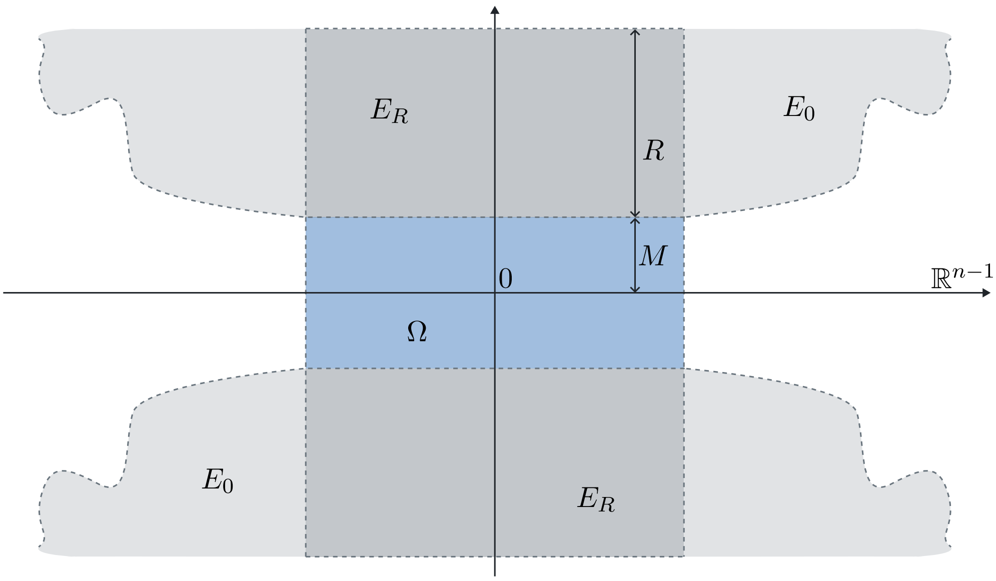

Figure 2.1 caption

We state the following two results, which we will prove afterwards.

Theorem 2.1. For E_0 and Ω as above and any $R > 0$ there exists an $M_0 \in (0, 1)$ depending on the dimension, R and s, such that for any $M \in (0, M_0)$ the minimizer E_M is given by $E_M = E_0 \cup \Omega$.

Theorem 2.2. For E_0 and Ω as above and any $R > 0$ there exists an $M_0 > 1$ depending on the dimension, R and s, such that for any $M > M_0$ the minimizer E_M is disconnected.

TODO. Elaborate and add source
Connect to classical minimal surfaces by observing disconnectedness of the minimizer, but when connected, the minimizer may "stick" to the boundary. Whereas classical minimal surfaces cannot stick to the boundary.

For the first proof, we will follow a similar construction as in [5].
In [2] the authors have shown that nonlocal minimizer satisfy the Euler-Lagrange equation in the viscosity sense, i.e. if E is a minimizer, there exists some such that $q \in \partial E$ and $B_r(q + r\nu) \subset E$ for some $r > 0$ and unit vector $\nu \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y-q|^{n+s}} \,\mathrm{d}y \geq 0. \tag{2.1}$$

In the proof we will assume that there exist a minimizer which is not $E_0 \cup \Omega$. To bring this assumption to a contradiction, we want to show that the left hand side of (2.1) is negative for M small enough. Thus, we have to construct some suitable ball such that we can apply the Euler-Lagrange equation. Constructing the ball by sliding it down from te_n. If the minimizer is not $E_0 \cup \Omega$, then at some point the ball will touch the minimizer for any $0 < r < 1$ and a point q, then exists. Then we will split the domain into four parts and estimate each part to get the contradiction.

Proof of Theorem 2.1. Proof by contradiction. Assume the minimizer E_M is not $E_0 \cup \Omega$. Then there we can slide a ball of radius r down in e_n direction and at some point it will touch the minimizer. We consider the ball $B_r(te_n)$. Since E_M is not $E_0 \cup \Omega$, there exists $r_0 \in (0,1)$ and $t_0 > 0$ such that $\partial B_{r_0}(t_0 e_n) \cap \partial E_M \neq \emptyset$ and $B_{r_0}(te_n) \subset E_M$ for all $t > t_0$, see Figure 2.2. In the following we define $z := t_0 e_n$.

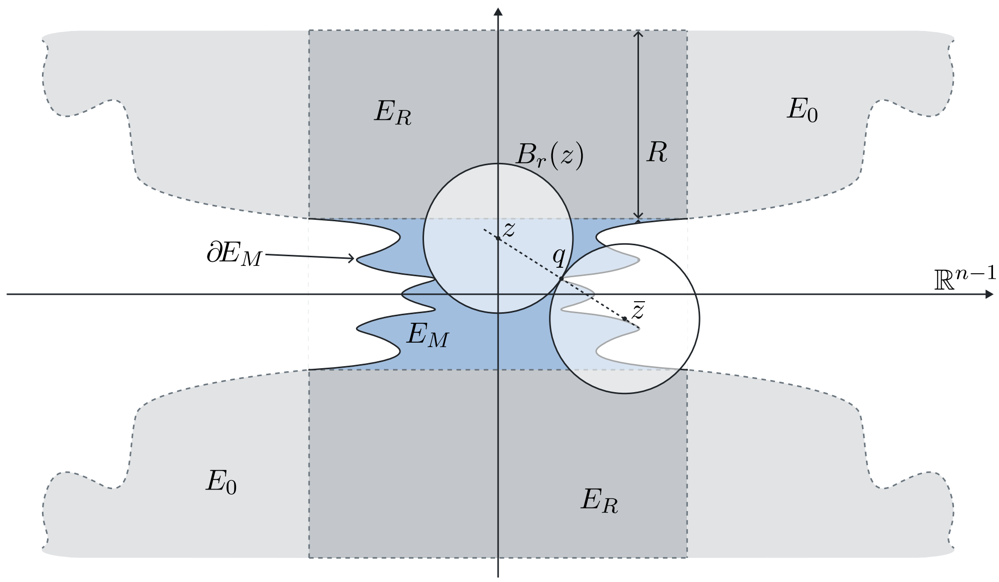

Figure 2.2 Sliding Ball down and reflecting at the touching point q

Since E_M is a minimizer it is also a viscosity solution of the Euler-Lagrange equation and the inequality

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{n+s}} \,\mathrm{d}y \geq 0 \tag{2.2}$$

holds, whereas $q \in \partial B_{r_0}(t_0 e_n) \cap \partial E_M$.

We will bring this to a contradiction by showing that the left hand side is negative for M small enough. We will split the domain into three parts and estimate each part, see Figure 2.3.

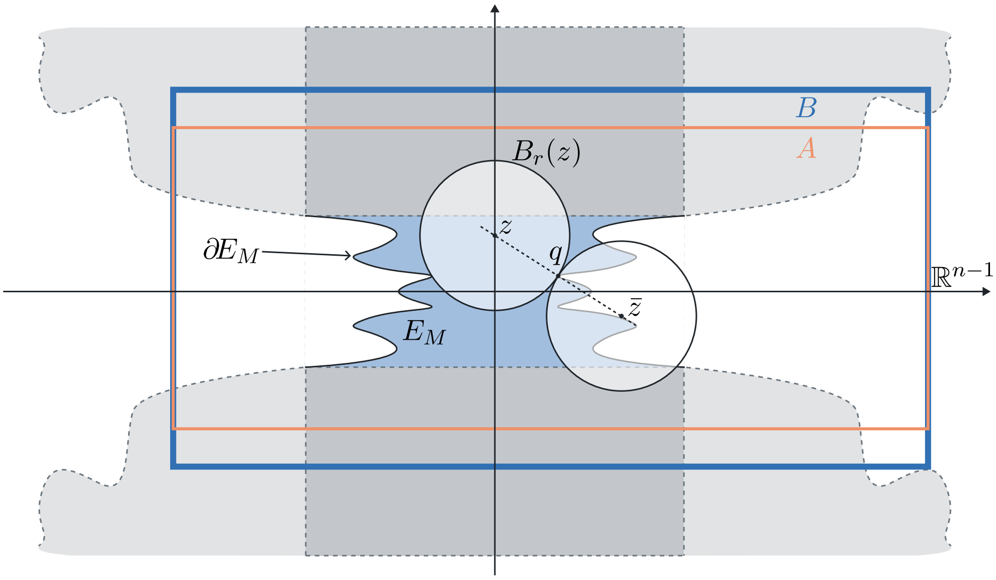

Figure 2.3 Splitting of the domain

We define the following sets:

$$\begin{aligned}
A &:= \{(x', x_n) \mid |x' - q'| < 2, |x_n - q_n| < 2M\} \\
B &:= \{(x', x_n) \mid |x' - q'| < 2, |x_n - q_n| < R\} \\
C &:= E_M^c \setminus B \\
D &:= E_M \setminus A,
\end{aligned}$$

where R is chosen such that $E_R \subset E_0$ and we chose M such that $2M < R$ for technical reasons, which will become clear later.

First let us assume that $R \geq 2$.
We start by estimating the integral over C. Notice that $C \subset E_M$ and that $C \subset \{|y| \geq 2\}$. Thus we can bound

$$\int_C \frac{\chi_{E_M^c} - \chi_{E_M}}{|y - q|^{n-s}} \, \mathrm{d}y = \int_C \frac{1}{|y - q|^{n-s}} \, \mathrm{d}y \leq \int_{|y| \geq 2} \frac{1}{|y|^{n+s}} \, \mathrm{d}y \leq c(n, s) 2^{-s},$$

where $c(n, s)$ is some positive constant depending on the dimension and the parameter s.

Next we estimate the integral over D. This part is the negative part of our integral and we want it to increase as we make M smaller. First notice that $D \subset E_M$, thus the integral is negative. To get an upper bound we can restrict the domain to something smaller. We choose the ball $B_M(3Me_n + q)$ if q_n is negative or $B_M(3Me_n + q)$ if q_n is positive, see Figure 2.4. If needed we also shift the ball closer to the origin by shifting it by M in $\frac{-q'}{|q'|}$ if needed. Important here is that we the distance between the ball and q decreases with M. Finally we multiply the intgeral by $\frac{1}{2}$, since it may be that not the whole ball is in D but since we chose $2M < R$ half of the ball (the part closer to Q) is in D for sure. Thus we have

$$\int_D \frac{\chi_{E_M^c} - \chi_{E_M}}{|y - q|^{n-s}} \, \mathrm{d}y = -\int_D \frac{1}{|y - q|^{n-s}} \, \mathrm{d}y \leq -c(n) \int_{B_M(3Me_n)} \frac{1}{|y|^{n+s}} \, \mathrm{d}y \leq -c(n, s) M^{-s}.$$

Finally we estimate the integral over the rest of the domain $S := \mathbb{R}^n \setminus (C \cup D)$. First notice that

$$\int_{S \cap B_{r_0}(q) \cap B_{r_0}(z)} \frac{1}{|y - q|^{n+s}} \, \mathrm{d}y = \int_{S \cap B_{r_0}(q) \cap B_{r_0}(\bar{z})} \frac{1}{|y - q|^{n+s}} \, \mathrm{d}y$$

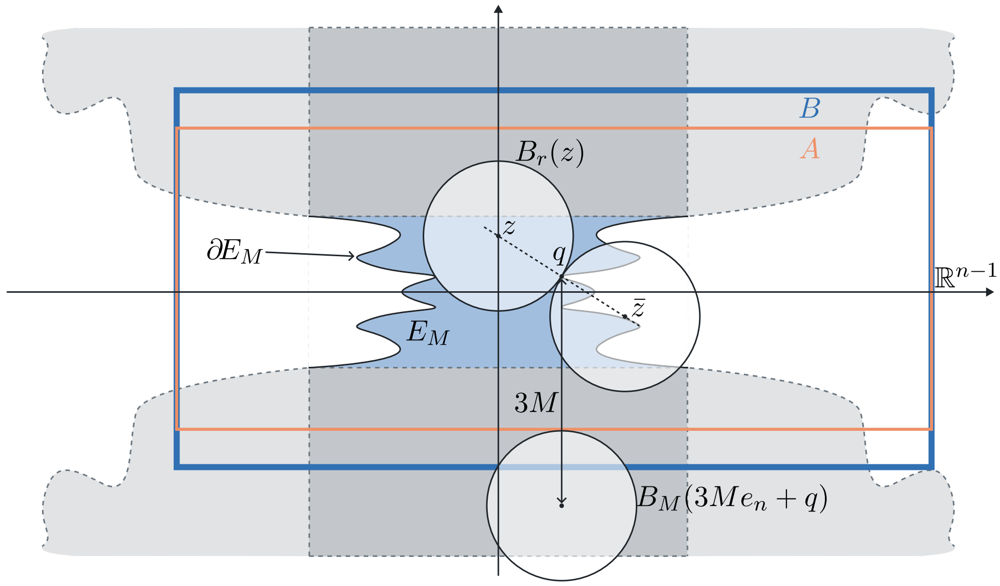

Figure 2.4 caption

since the integrand is pointsymmetric in q. Since $B_{r_0}(z) \subset E_M$ we have that

$$\begin{aligned}
&\int_{S\cap B_{r_0}(q)\cap B_{r_0}(z)} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n+s}} \,\mathrm{d}y + \int_{S\cap B_{r_0}(q)\cap B_{r_0}(\bar{z})} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n+s}} \,\mathrm{d}y \\
&\leq \int_{S\cap B_{r_0}(q)\cap B_{r_0}(z)} \frac{1}{|y-q|^{n+s}} \,\mathrm{d}y - \int_{S\cap B_{r_0}(q)\cap B_{r_0}(\bar{z})} \frac{1}{|y-q|^{n+s}} \,\mathrm{d}y = 0. \qquad (2.3)
\end{aligned}$$

Thus we have

$$\int_S \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \,\mathrm{d}y \overset{(2.3)}{\leq} \int_{S\setminus B_{r_0}(q)} \frac{1}{|y-q|^{n-s}} \,\mathrm{d}y + \int_{S\cap(B_{r_0}(q)\setminus(B_{r_0}(z)\cup B_{r_0}(\bar{z})))} \frac{1}{|y-q|^{n-s}} \,\mathrm{d}y$$

For the first term we will use that $S \subset B \subset B_{R+2}$ and for the second term we will use Lemma 3.1 from [6] with $R = r_0$ and $\lambda = 1$. We then get

$$\begin{aligned}
&\leq \int_{B_{R+2}\setminus B_{r_0}} \frac{1}{|y|^{n+s}} \,\mathrm{d}y + \int_{P_{r_0,1}} \frac{1}{|y|^{n+s}} \,\mathrm{d}y \\
&\leq c(n,s)(r_0^{-s} - (R+2)^{-s}) + c'(n,s) r_0^{-s} \\
&\leq c(n,s)(r_0^{-s} - (R+2)^{-s}).
\end{aligned}$$

Since we want to contradict that $\partial B_r(te_n) \cap \partial E_M \neq \emptyset$ for all $r \in (0,1)$ and t it is enough to consider r_0 large. Since if we have that there exists some t_0 such that $\partial B_{r_0}(t_0 e_n) \cap \partial E_M \neq \emptyset$ for some r_0, then for all $r \in (r_0, 1)$ the same holds as well. Conversly, if we have that $\partial B_{r_0}(te_n) \cap \partial E_M = \emptyset$, then for all $r \in (0, r_0)$ the same holds as well. In particular, we can choose $r_0 = 1$ (notice that we can choose $r_0 = 1$, since $B_1(z) \subset E_M$ and (2.2) still holds).

Thus in total we have that

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \,\mathrm{d}y &\leq -c_0 M^{-s} + c_1(2^{-s} + 1 - (R+2)^{-s}) \\
&= -c_0 M^{-s}\Big(1 - \frac{c_1}{c_0}\Big(\Big(\frac{M}{2}\Big)^s + M^s - \Big(\frac{M}{R+2}\Big)^2\Big)\Big).
\end{aligned}$$

Now we can choose M small enough such that the right hand side is negative. Thus we have contradicted our assumption, that the minimizer is not $E_0 \cup \Omega$ for $R \geq 2$.

Let us now consider the case that $R < 2$, then for the integral over $C \subset \{|y| \geq R\}$ we have

$$\int_C \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \, \mathrm{d}y = \int_C \frac{1}{|y-q|^{n-s}} \, \mathrm{d}y \leq \int_{|y| \geq R} \frac{1}{|y|^{n+s}} \, \mathrm{d}y \leq c(n,s) R^{-s}.$$

The integral over D and S are the same as before, but this time we can't chose $r_0 = 1$ since it could be that $B_1(z) \not\subset E_M$. Still we can choose $r_0 = \frac{R}{2}$, since $B_{\frac{R}{2}}(z) \subset E_M$. Thus we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \, \mathrm{d}y &\leq -c_0 M^{-s} + c_1 (R^{-s} + r_0^{-s} - (R+2)^{-s}) \\ &= -c_0 M^{-s} (1 - \frac{c_1}{c_0} ((\frac{M}{R}^s) + (\frac{2M}{R})^s - (\frac{M}{R+2})^s)).\end{aligned}$$

Again we choose M small enough such that the right hand side is negative.

Notice that in this case we have only shown for know that the cylinder $Z_R := B'_{\frac{R}{2}} \times (-M, M)$ is part of the minimizer.

To prove that the minimizer is $E_0 \cup \Omega$ we will proceed in a similar manner, but instead of sliding the ball down, we will push a ball outwards from inside the cylinder. Since we have shown that the cylinder Z_R is part of the minimizer, the ball $B_{\frac{R}{2}}(he_n)$ for any $h \in (-M, M)$ is part of the minimizer. We will push this ball outwards (in any direction, w.l.o.g. we can choose e_1). We assume again, that $E_M \neq E_0 \cup \Omega$, then for the ball $B_{\frac{R}{2}}((1 - \frac{R}{2} - t) + he_n)$ with $t \in (0, 1 - \frac{R}{2})$ and $h \in (-M, M)$ there exists some t_0 and h_0 such that the ball touches the minimizer, i.e. $\partial B_{\frac{R}{2}}((1 - \frac{R}{2} - t_0) + h_0 e_n) \cap \partial E_M \neq \emptyset$ and for all $t \in (t_0, 1 - \frac{R}{2})$ we have that $B_{\frac{R}{2}}((1 - \frac{R}{2} - t) + h_0 e_n) \subset E_M$. Again we define $z := (1 - \frac{R}{2} - t_0) e_1 + h_0 e_n$.

TODO. Should I add some figures here too?

Now we can estimate the integral again by splitting the domain. We define the following sets:

$$\begin{aligned}A &:= \{(x', x_n) \mid |x' - q'| < R, |x_n - q_n| < 2M\} \\ B &:= \{(x', x_n) \mid |x' - q'| < R, |x_n - q_n| < R\} \\ C &:= E_M^c \setminus B \\ D &:= E_M \setminus A.\end{aligned}$$

The integral over C and D is estimated as before. We have that

$$\int_C \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \, \mathrm{d}y \leq \int_{|y| \geq R} \frac{1}{|y|^{n+s}} \, \mathrm{d}y \leq c(n,s) R^{-s}$$

and

$$\int_D \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \, \mathrm{d}y \leq -c(n) \int_{B_M(3Me_n)} \frac{1}{|y|^{n+s}} \, \mathrm{d}y \leq -c(n,s) M^{-s}.$$

For the integral over the rest of the domain S, we proceed as before. First we have again

$$\int_{S \cap B_{\frac{R}{2}}(q) \cap B_{\frac{R}{2}}(z)} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n+s}} \, \mathrm{d}y + \int_{S \cap B_{\frac{R}{2}}(q) \cap B_{\frac{R}{2}}(\bar{z})} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n+s}} \, \mathrm{d}y \leq 0.$$

Thus we have

$$\int_S \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \, \mathrm{d}y \leq \int_{S \setminus B_{\frac{R}{2}}(q)} \frac{1}{|y-q|^{n-s}} \, \mathrm{d}y + \int_{S \cap (B_{\frac{R}{2}}(q) \setminus (B_{\frac{R}{2}}(z) \cup B_{\frac{R}{2}}(\bar{z})))} \frac{1}{|y-q|^{n-s}} \, \mathrm{d}y$$

For the first term we will use that $S \subset B \subset B_4$ and for the second term we will use Lemma 3.1 from [6] with $R = \frac{R}{2}$ and $\lambda = 1$. We then get

$$\begin{aligned}
&\leq \int_{B_4 \setminus B_{\frac{R}{2}}} \frac{1}{|y|^{n+s}} \,\mathrm{d}y + \int_{P_{\frac{R}{2},1}} \frac{1}{|y|^{n+s}} \,\mathrm{d}y \\
&\leq c(n,s)((\frac{R}{2})^{-s} - 4^{-s}) + c'(n,s)(\frac{R}{2})^{-s} \\
&\leq c(n,s)(R^{-s} - 8^{-s}).
\end{aligned}$$

Thus in total we have that

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \,\mathrm{d}y &\leq -c_0 M^{-s} + c_1(R^{-s} - 8^{-s}) \\
&= -c_0 M^{-s}(1 - \frac{c_1}{c_0}((\frac{M}{R})^s - (\frac{M}{8})^{-s}).
\end{aligned}$$

Thus we can choose M small enough such that the right hand side is negative.

■

Interesting to see, that the contribution of the external data of the same width as the prescribed set is enough to get connectedness of the minimizer and even stickiness to the boundary. Also see, that the model seems (maybe prove that) to converge to the problem, considered in [5].

TODO. Is that enough? Should I elaborate more?

Proof of Theorem 2.2. We show that for M big enough, in particular we can choose $M > 1$, the minimizer is disconnected. We will slide a ball of radius $\sqrt{M}$ in e_1 direction and show, that this ball and the minimizer don't touch for M big enough. Assume that they are touching at some point q, then since the minimizer is a viscosity solution of the Euler-Lagrange equation, we have that

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \,\mathrm{d}y \leq 0. \tag{2.4}$$

Now notice that we have

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \,\mathrm{d}y \geq \int_{\mathbb{R}^n} \frac{\chi_{F_M^c} - \chi_{F_M}}{|y-q|^{n-s}} \,\mathrm{d}y$$

where $F_M = E_M \cup F_0$ and F_0 being the external data from the model considered in [5]. Thus we are in the same setting as in [5] and can conclude that for M large enough the left hand side of (2.4) is positive. Thus we have a contradiction and the minimizer is disconnected. ■

Whereas Theorem 2.2 is consistent with the classical therory of minimal surfaces, the behvavior of the minimizer in Theorem 2.1 is unique to nonlocal minimal surfaces. In [5] the authors have shown that the minimizer exhibits similar behavior as we found in Theorem 2.1 for the model considered in this chapter, however interesting to see is that even in the case $R = 1$ the minimizer is connected and even sticks to the boundary, for M small enough. This suggegests that the contribution of the external data E_0 above and below is enough to push the minimzer to the boundary of the prescribed set Ω. In the proof we have seen that the width R is with negative exponents in the upper bound, thus if we choose R large enough, M_0 increases.

3 Disconnected Minimizer

IDEA. Open this chapter with the train of thought motivated by the case $R < 2$
For the unbounded case, consider all dimensions and general r, R and just the upper bound.
For the bounded case consider $n = 2$ to show, that even though we are positive at $s = 0, 1$ we could still have negative values somewhere in between
Then give some interpretation if or how that helps or the consequences of that.

When discussing the connectedness of the model in Chapter 2 in the case that $R < 2$, we first just stated, that if $R < 2$ then at least the cylinder $Z_R := B'_{R/2} \times (-M, M)$ is in the minimizer. This fact is not enough for connectedness of the minimizer. To show connectedness, we would still need to show, that there cannot exist a part of the minimizer that is fully detached from the cylinder and the external data.

Motivated by the fact, that in the classical case, if we have some external data E_0 and some prescribed set Ω that are fully disconnected, i.e. $\operatorname{dist}(E_0, \Omega) =: d > 0$, then the minimizer is the external data itself, we wanted to prove the same thing for the nonlocal case as well. Indeed, if we could show that, then the existence of the cylinder Z_R is enough to conclude that the minimizer is connected.

Assume there exists a part of the minimizer that is not connected to the cylinder and the external data, i.e. there exists a set E_1 such that $\operatorname{dist}(E_1, E_0 \cup Z_R) > 0$. Then we can rewrite the fractional perimeter of $E_M := E_0 \cup E_1$ relative to Ω as follows:

$$\begin{aligned}\operatorname{Per}_s(E_M, \Omega) &= \mathscr{L}(E_M \cap \Omega, E_M^c) + \mathscr{L}(E_M \setminus \Omega, \Omega \setminus E_M) \\ &= \mathscr{L}(E_1 \cup Z_R, E_M^c) + \mathscr{L}(E_0, \Omega \setminus (E_1 \cup Z_R)) \\ &= \mathscr{L}(E_1, E_M^c) + \mathscr{L}(Z_R, E_M^c) + \mathscr{L}(E_0 \cup Z_R, \Omega \setminus (E_1 \cup Z_R)) - \mathscr{L}(Z_R, \Omega \setminus (E_1 \cup Z_R)) \\ &= \operatorname{Per}_s(E_M, \Omega \setminus Z_R) + \mathscr{L}(Z_R, (E_0 \cup Z_R)^c). \end{aligned} \tag{3.1}$$

Notice that the second term in (3.1) is now independent of E_1, thus to minimize $\operatorname{Per}_s(E_M, \Omega)$ we can minimize $\operatorname{Per}_s(E_M, \Omega \setminus Z_R)$ instead.
We define a sequence of prescribed sets Ω_n such that $\operatorname{dist}(E_0 \cup Z_R, \Omega_n) = \frac{d}{n}$, where $d := \operatorname{dist}(E_0 \cup Z_R, E_1)$. Then for each n we are in the situation of fully disconnected external data, here $E_0 \cup Z_R$, and prescribed set, here Ω_n. If our assumption is correct, then we could conclude

$$\operatorname{Per}_s(E_M, \Omega \setminus Z_R) \leq \operatorname{Per}_s(E_M, \Omega_n) \geq \operatorname{Per}_s(E_0, \Omega_n) \nwarrow \operatorname{Per}_s(E_0, \Omega \setminus Z_R).$$

Thus, there cannot exist a set E_1 fully detached from $E_0 \cup Z_R$.

As it turns out, this is not true in general and thus we cannot state connectedness just with the existence of the cylinder in the minimizer.
In the following, we will consider an example where depending on s the minimizer is not the external data itslef, even though the external data and the prescribed set have nonzero distance.

Example 3.1. Let $E_0 = B_2^c$ and $\Omega = B_1$ in $\mathbb{R}^2$. Then we compare the fractional perimeter of E_0 relative to Ω with the fractional perimeter of $E_0 \cup \Omega$ relative to Ω

$$\operatorname{Per}_s(E_0 \cup \Omega, \Omega) - \operatorname{Per}_s(E_0, \Omega) = \operatorname{Per}_s(B_1) - 2L(B_2^c, B_1) \tag{3.2}$$

and show that this difference is negative for s small enough. If the difference is negative we have found a competitor to the external data with smaller fractional perimeter and thus the external data cannot be the minimizer.

For the first term we have by [14, Eq. (11)]

$$\mathrm{Per}_s(B_1) = \frac{2^{2-s}\pi^{\frac{3}{2}}}{s(2-s)} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{2-s}{2})}.$$

We want to bound the second term from above and below. For that we will split the domain depending on x, see Figure 3.1.

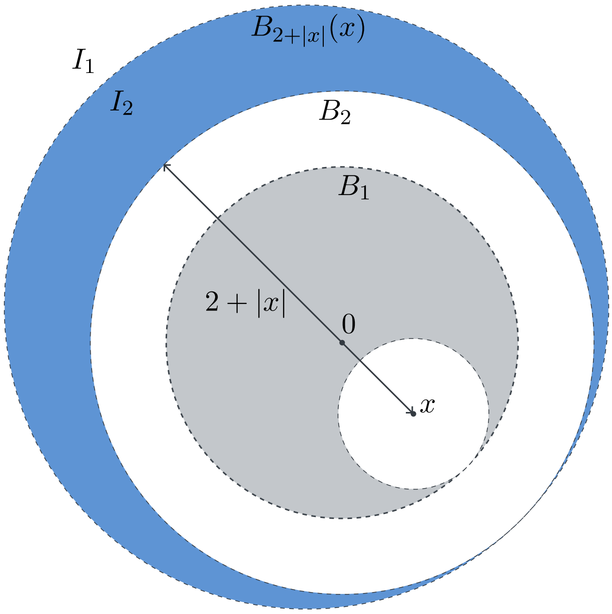

Figure 3.1 Splitting of B_2^c depending on x

Thus, we have

$$\mathscr{L}(B_2^c, B_1) = \int_{B_1}\int_{B_2^c} \frac{1}{|x-y|^{2-s}}\,\mathrm{d}y\,\mathrm{d}x = \underbrace{\int_{B_1}\int_{B^c_{2+|x|}(x)} \frac{1}{|x-y|^{2-s}}\,\mathrm{d}y\,\mathrm{d}x}_{I_1} + \underbrace{\int_{B_1}\int_{B_{2+|x|}(x)\setminus B_2} \frac{1}{|x-y|^{2-s}}\,\mathrm{d}y\,\mathrm{d}x}_{I_2}.$$

We start with I_1:

$$\begin{aligned} I_1 &= \int_{B_1}\int_{B^c_{2+|x|}(x)} \frac{1}{|x-y|^{2+s}}\,\mathrm{d}y\,\mathrm{d}x = \int_{B_1}\int_{B^c_{2+|x|}} \frac{1}{|y|^{2+s}}\,\mathrm{d}y\,\mathrm{d}x \\ &= 4\pi^2 \int_0^1\int_{2+r_1}^{\infty} \frac{r_1}{r_2^{1+s}}\,\mathrm{d}r_2\,\mathrm{d}r_1 = \frac{4\pi^2}{s}\int_0^1 \frac{r_1}{(2+r_1)^s}\,\mathrm{d}r_1 \\ &= \frac{4\pi^2}{s(1-s)(2-s)}\left(2^{2-s} - (s+1)3^{1-s}\right). \end{aligned} \tag{3.3}$$

Now to I_2. Here the idea is to use radial coordinates again. Since the integral is radial symmetric w.r.t. x, we can fix x such that $x = (r, 0)$ for $r = |x|$. Now for fixed x the domain of y is not radial symmetric, thus we first have to compute the domain of $\vartheta := \vartheta(r_1, r_2)$.
We have two restrictions on y:

(1) $4 \leq |x-y|^2 \leq (2+2|x|)^2$

(2) $2 - |x| \leq |y| \leq 2 + |x|$

From the first restriction with $|x| = r_1$, $|y| = r_2$ and ϑ the angle between x and y we get

$$\begin{aligned}
&4 \le |x-y|^2 \le (2+2r_1)^2 \\
\Leftrightarrow\ & 4 \le r_1^2 + r_2^2 - 2r_1r_2\cos(\vartheta) \le 4(1+r_1)^2 \\
\Leftrightarrow\ & \frac{r_1^2+r_2^2-4}{2r_1r_2} \ge \cos(\vartheta) \ge \frac{r_1^2+r_2^2-4(1+r_1)^2}{2r_1r_2}.
\end{aligned} \tag{3.4}$$

From the second restriction we get that the right hand side of (3.4) is always greater or equal to -1, thus we have

$$\frac{r_1^2+r_2^2-4}{2r_1r_2} \ge \cos(\vartheta) \ge -1.$$

We will see, that for all r_1 and r_2 the integrand is independent of ϑ, thus we can integrate over ϑ first with $\vartheta \in (0, \arccos\left(\frac{r_1^2+r_2^2-4}{2r_1r_2}\right) \cup (2\pi - \arccos\left(\frac{r_1^2+r_2^2-4}{2r_1r_2}\right), 2\pi)$

$$\int_{-\vartheta}^{\vartheta} \mathrm{d}\vartheta = 2\pi - 2\arccos\left(\frac{r_1^2+r_2^2-4}{2r_1r_2}\right).$$

For I_2 we get then get

TODO. Simplify computations?
Add arguments about splitting, change of variables, computation steps etc

$$I_2 = \int_{B_1}\int_{B_{2+|x|}(x)\setminus B_2} \frac{1}{|x-y|^{2+s}}\,\mathrm{d}y\,\mathrm{d}x = \int_{B_1}\underbrace{\int_{B_{2+|x|}\setminus B_2(-x)} \frac{1}{|y|^{2+s}}\,\mathrm{d}y}_{\text{radial symmetric w.r.t. } x}\,\mathrm{d}x$$

We use radial coordinates

$$\begin{aligned}
&= 2\pi \int_0^1 \int_{2-r_1}^{2+r_1} \frac{r_1}{r_2^{1+s}} \int_{-\vartheta}^{\vartheta} \mathrm{d}\vartheta\,\mathrm{d}r_2\,\mathrm{d}r_1 = 2\pi \int_0^1 \int_{2-r_1}^{2+r_1} \frac{r_1}{r_2^{1+s}} \left(2\pi - 2\arccos\left(\frac{r_1^2+r_2^2-4}{2r_1r_2}\right)\right)\mathrm{d}r_2\,\mathrm{d}r_1 \\
&= 4\pi^2 \int_0^1 \int_{2-r_1}^{2+r_1} \frac{r_1}{r_2^{1+s}}\,\mathrm{d}r_2\,\mathrm{d}r_1 - 4\pi \int_0^1 \int_{2-r_1}^{2+r_1} \frac{r_1}{r_2^{1+s}} \arccos\left(\frac{r_1^2+r_2^2-4}{2r_1r_2}\right)\mathrm{d}r_2\,\mathrm{d}r_1
\end{aligned}$$

Then partial integration

$$\begin{aligned}
&= 4\pi^2 \int_0^1 \int_{2-r_1}^{2+r_1} \frac{r_1}{r_2^{1+s}}\,\mathrm{d}r_2\,\mathrm{d}r_1 - \frac{4\pi^2}{s}\int_0^1 \frac{r_1}{(2-r_1)^s}\,\mathrm{d}r_1 + \frac{4\pi}{s}\int_0^1\int_{2-r_1}^{2+r_2} \frac{r_1}{r_2^{1+s}} \frac{r_2^2-r_1^2+4}{\sqrt{4r_1^2r_2^2-(r_1^2+r_2^2-4)^2}}\,\mathrm{d}r_2\,\mathrm{d}r_1 \\
&= \underbrace{\frac{4\pi^2}{s(1-s)(2-s)}\left((s+1)3^{1-s} - 2^{2-s}\right)}_{-I_1} + \frac{4\pi}{s}\int_0^1\int_{2-r_1}^{2+r_2} \frac{r_1}{r_2^{1+s}} \frac{r_2^2-r_1^2+4}{\sqrt{4r_1^2r_2^2-(r_1^2+r_2^2-4)^2}}\,\mathrm{d}r_2\,\mathrm{d}r_1.
\end{aligned}$$

Thus we get for the second term in (3.2)

$$\mathscr{L}(E_0, E_1) = \frac{4\pi}{s}\int_0^1\int_{2-r_1}^{2+r_2} \frac{r_1}{r_2^{1+s}} \frac{r_2^2-r_1^2+4}{\sqrt{4r_1^2r_2^2-(r_1^2+r_2^2-4)^2}}\,\mathrm{d}r_2\,\mathrm{d}r_1.$$

We can now bound this term without losing too much information. For the upper bound, we will use that $r_2 \geq 2 - r_1$ and for the lower bound we will use that $r_2 \leq 2 + r_1$. We then get

$$
\begin{aligned}
\bullet) \quad \mathscr{L}(E_0, E_1) &\leq \frac{4\pi}{s} \int_0^1 \int_{2-r_1}^{2+r_2} \frac{r_1}{(2-r_1)^s} \frac{1}{r_2} \frac{r_2^2 - r_1^2 + 4}{\sqrt{4r_1^2 r_2^2 - (r_1^2 + r_2^2 - 4)^2}} \, \mathrm{d}r_2 \, \mathrm{d}r_1 \\
&= \frac{4\pi}{s} \int_0^1 \frac{r_1}{(2-r_1)^s} \left[\arccos\left(\frac{r_1^2 + r_2^2 - 4}{2 r_1 r_2} \right) \right]_{2-r_1}^{2+r_2} \mathrm{d}r_1 \\
&= \frac{4\pi^2}{s} \int_0^1 \frac{r_1}{(2-r_1)^s} \, \mathrm{d}r_1 \\
&= \frac{4\pi^2}{s(1-s)(2-s)} \left(2^{2-s} - 3 + s \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\bullet) \quad \mathscr{L}(E_0, E_1) &\geq \frac{4\pi}{s} \int_0^1 \int_{2-r_1}^{2+r_2} \frac{r_1}{(2+r_1)^s} \frac{1}{r_2} \frac{r_2^2 - r_1^2 + 4}{\sqrt{4r_1^2 r_2^2 - (r_1^2 + r_2^2 - 4)^2}} \, \mathrm{d}r_2 \, \mathrm{d}r_1 \\
&= \frac{4\pi^2}{s} \int_0^1 \frac{r_1}{(2+r_1)^s} \, \mathrm{d}r_1 \\
&= \frac{4\pi^2}{s(1-s)(2-s)} \left(2^{2-s} - (s+1) 3^{1-s} \right).
\end{aligned}
$$

Thus we have the bounds (3.2)

$$
\mathrm{Per}_s(E_1) - 2\mathscr{L}(E_0, E_1) \leq \frac{2^{2-s} \pi^{\frac{3}{2}}}{s(2-s)} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{2-s}{2})} - \frac{8\pi^2}{s(1-s)(2-s)} \left(2^{2-s} - (s+1) 3^{1-s} \right) \tag{3.5}
$$

and

$$
\mathrm{Per}_s(E_1) - 2\mathscr{L}(E_0, E_1) \geq \frac{2^{2-s} \pi^{\frac{3}{2}}}{s(2-s)} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{2-s}{2})} - \frac{8\pi^2}{s(1-s)(2-s)} \left(2^{2-s} - 3 + s \right). \tag{3.6}
$$

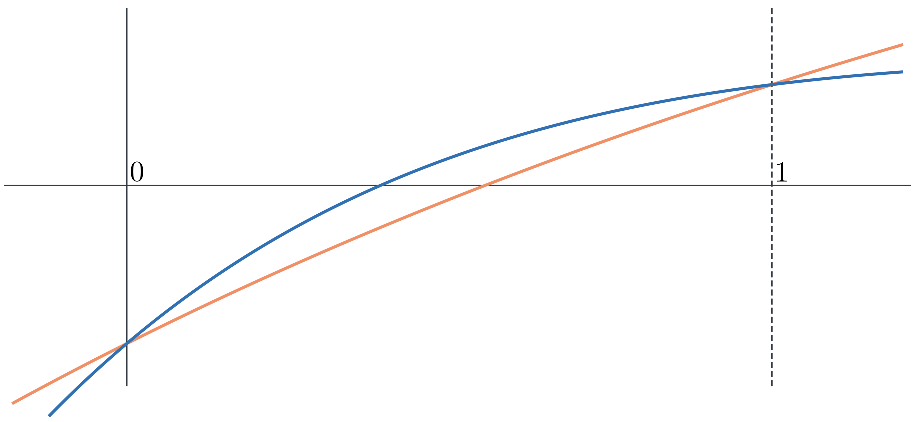

Figure 3.2 Upper and lower bound plotted for $s \in (0, 1)$

TODO. Give justification, that both sides are continuous w.r.t. s and conclude
Maybe draw a picture

Example 3.2 (Continuation of Example 3.1)**.** Let us now consider the same setting as in Example 3.1, but instead with the external data $E_1 = B_{2+T} \setminus B_2$ for $T > 0$ large enough. Notice, that this change just adds one additional term compared to before

$$\mathscr{L}(B_{2+T} \setminus B_2, B_1) = \mathscr{L}(B_2^c, B_1) - \underbrace{\int_{B_{2+T}^c} \int_{B_1} \frac{1}{|x-y|^{2-s}} \,\mathrm{d}x \,\mathrm{d}y}_{I_3}$$

We will bound I_3 from above and below.

The upper bound

$$\begin{aligned}
\int_{B_1} \int_{B^c_{2+T-|x|}(x)} \frac{1}{|x-y|^{2-s}} \,\mathrm{d}y \,\mathrm{d}x &\leq 4\pi^2 \int_0^1 \int_{2+T-r_1}^{\infty} \frac{r_1}{r_2^{1-s}} \,\mathrm{d}r_2 \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s} \int_0^1 \frac{r_1}{(2+T-r_1)^s} \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s(1-s)(2-s)} [(2+T)^{2-s} - (3-s+T)(1+T)^{1-s}]
\end{aligned}$$

and the lower bound

$$\begin{aligned}
\int_{B_1} \int_{B^c_{2+T+|x|}(x)} \frac{1}{|x-y|^{2-s}} \,\mathrm{d}y \,\mathrm{d}x &\geq 4\pi^2 \int_0^1 \int_{2+T+r_1}^{2+T+r_1} \frac{r_1}{r_2^{1-s}} \,\mathrm{d}r_2 \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s} \int_0^1 \frac{r_1}{(2+T+r_1)^s} \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s(1-s)(2-s)} [(2+T)^{2-s} - (1+s+T)(3+T)^{1-s}].
\end{aligned}$$

Thus we have the bounds

$$\begin{aligned}
\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega) &\leq (3.5) + \frac{8\pi^2}{s(1-s)(2-s)} [(2+T)^{2-s} - (3-s+T)(1+T)^{1-s}] \\
\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega) &\geq (3.6) + \frac{8\pi^2}{s(1-s)(2-s)} [(2+T)^{2-s} - (1+s+T)(3+T)^{1-s}]
\end{aligned}$$

TODO. Add picture and more explanation

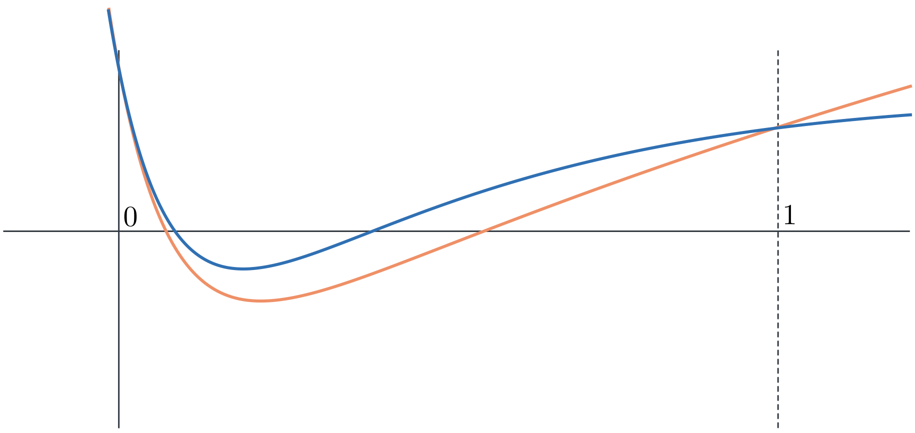

Figure 3.3 Upper and lower bound plotted for $s \in (0, 1)$

Notice that the limits for $s \searrow 0$ and $s \nearrow 1$ are independent of T and that for $s \nearrow 1$ the limit is the same as in the unbounded case. For $s \searrow 0$ however we have a different limit than in the unbounded case.

We now want to generalize this example to arbitrary dimensions $n \geq 2$ and radii $0 < r < R$. We will consider the following setting once with unbounded data and once with bounded data: Let $n \geq 1$ and $r, R, T > 0$, such that $r < R$. Take the external data $E_0 = B_R^c$ in the unbounded case and $E_0 = B_{R+T} \setminus B_R$ in the bounded case. Define the prescribed set $\Omega = B_r$.

CHECK. Is that correct?
Elaborate

In [3] the authors have shown that for $s \nearrow 1$ the fractional perimeter approaches the classical perimeter. Thus, we can expect that for s large enough the minimizer should be the external data itself. In [9] the authors have shown that for bounded sets with nonzero distance and s small enough the minimizer is the external data itself as well.

Theorem 3.3. Let $n \geq 2$ and $0 < r < R$. Let $E_0 = B_R^c$ and $\Omega = B_r$, then there exists a $s_0 \in (0,1)$ such that for all $s \in (0, s_0)$ the minimizer is not the external data itself.

Theorem 3.4. Let $n \geq 2$ and $0 < r < R$ and $T > 0$. Let $E_0 = B_{R+T} \setminus B_R$ and $\Omega = B_r$, then for any T large enough there exists $s_0, s_1 \in (0,1)$ such that for all $s \in (s_0, s_1)$ the minimizer is not the external data itself.

Proof of Theorem 3.3. As in Example 3.1 and Example 3.2 we will compare the fractional perimeter of $E_0 \cup \Omega$ relative to Ω with the fractional perimeter of E_0 relative to Ω

$$\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega) = \mathrm{Per}_s(\Omega) - 2\,\mathscr{L}(E_0, \Omega) = \mathrm{Per}_s(B_r) - 2\,\mathscr{L}(B_R^c, B_r). \tag{3.7}$$

For the first term we have by [14, Eq. (11)]

$$\mathrm{Per}_s(B_r) = \frac{2^{1-s}\pi^{\frac{n-1}{2}} n\omega_n}{s(n-s)} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{n-s}{2})} r^{n-s} = \frac{2^{2-s}\pi^{n-\frac{1}{2}}}{s(n-s)} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{n-s}{2})\Gamma(\frac{n}{2})} r^{n-s}$$

with $\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

The second term we will bound from below to get an upper bound for (3.7).

$$\begin{aligned}
\mathscr{L}(B_r, B_R^c) &= \int_{B_r}\int_{B_R^c} \frac{1}{|x-y|^{n-s}}\,\mathrm{d}y\,\mathrm{d}x \geq \int_{B_r}\int_{B^c_{R+|x|}(x)} \frac{1}{|x-y|^{n+s}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{B_r}\int_{B^c_{R+|x|}} \frac{1}{|y|^{n+s}}\,\mathrm{d}y\,\mathrm{d}x = \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2}\int_0^r\int_{R+r_1}^\infty \frac{r_1^{n-1}}{r_2^{1+s}}\,\mathrm{d}r_2\,\mathrm{d}r_1 \\
&= \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{1}{s}\int_0^r \frac{r_1^{n-1}}{(R+r_1)^s}\,\mathrm{d}r_1 \\
&= \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{1}{ns}\frac{r^{n-1}}{R^s}\,{}_2F_1(s,n;n+1;-\frac{r}{R}).
\end{aligned}$$

TODO. Add source

In the last step we used the following identity (source) for the hypergeometric function

$$B(b, c-b)_2F_1(a,b;c;z) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t \qquad \text{for} \Re(c) > \Re(b) > 0,$$

where B is the beta function. In our case we have $a = s$, $b = n$, $c = n+1$ and $z = -\frac{r}{R}$, thus

$$\begin{aligned}\int_0^r \frac{r_1^{n-1}}{(R+r_1)^s}\,\mathrm{d}r_1 &= \frac{r^n}{R^s}\int_0^1 r_1^{n-1}(1+\frac{r}{R}r_1)^{-s}\,\mathrm{d}r_1 \\ &= \frac{r^n}{R^s}B(n,1)_2F_1(s,n;n+1;-\frac{r}{R}) = \frac{r^n}{nR^s}{}_2F_1(s,n;n+1;-\frac{r}{R}).\end{aligned}$$

Thus we can bound (3.7) from above by

$$\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega) \leq \frac{2^{2-s}\pi^{n-\frac{1}{2}}}{s(n-s)}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{n-s}{2})\Gamma(\frac{n}{2})}r^{n-s} - \frac{8\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{r^n}{snR^s}{}_2F_1(s,n;n+1;-\frac{r}{R}). \tag{3.8}$$

TODO. Rewrite

Since we are interested in the behavior of (3.7) depending on s we multiply (3.8) by $s(1-s)$ to deal with the singularities at $s=0$ and $s=1$

$$s(1-s)(\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega)) \leq \frac{2^{3-s}\pi^{n-\frac{1}{2}}}{n-s}\frac{\Gamma(\frac{3-s}{2})}{\Gamma(\frac{n-s}{2})\Gamma(\frac{n}{2})}r^{n-s} - \frac{8\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{r^n}{nR^s}(1-s)_2F_1(s,n;n+1;-\frac{r}{R}). \tag{3.9}$$

TODO. Add source

Since Γ is continuous for all positive reals and ${}_2F_1$ is absolutely contiouous for $|z| \leq 1$, the right hand side of (3.9) is continuous for all $s \in (0,1)$. First notice that we have the following limits

$$\lim_{s\searrow 0} {}_2F_1(s,n;n+1;z) = 1 \quad \text{and} \quad \lim_{s\nearrow 1}(1-s)_2F_1(s,n;n+1;z) = 0.$$

Now take the limit for $s \searrow 0$ and $s \nearrow 1$ in (3.9)

$$\lim_{s\searrow 0} s(1-s)(\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega)) = -\frac{4\pi^n}{n}\frac{1}{(\Gamma(\frac{n}{2}))^2}r^n < 0$$

and

$$\lim_{s\nearrow 1} s(1-s)(\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega)) = \frac{4\pi^{n-\frac{1}{2}}}{n-1}\frac{1}{\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2})}r^{n-1} > 0. \tag{3.10}$$

Now we know for $s \nearrow 1$ that $(1-s)\,\mathrm{Per}_s(E,\Omega)$ is approaching the classical perimeter, thus in (3.10) is actually an equality. Thus, we can conclude that there exists an $s_0 \in (0,1)$ such that for all $s \in (0,s_0)$ the minimizer is not the external data itself.

■

Proof of Theorem 3.4. We consider again the difference

$$\begin{aligned}\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega) &= \mathrm{Per}_s(\Omega) - 2\mathscr{L}(E_0,\Omega) \\ &= \mathrm{Per}_s(B_r) - 2\mathscr{L}(B_R^c,B_r) + 2\mathscr{L}(B_{R+T}^c,B_r)\end{aligned} \tag{3.11}$$

We can use the upper bound from the proof of Theorem 3.3 for the first 2 terms in (3.11). The third term we will bound from above

$$\begin{aligned}\mathscr{L}(B_{R+T}^c,B_r) &= \int_{B_r}\int_{B_{R+T}^c}\frac{1}{|x-y|^{n-s}}\,\mathrm{d}y\,\mathrm{d}x \leq \int_{B_r}\int_{B_{R+T-|x|}^c}\frac{1}{|x-y|^{n-s}}\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2}\int_0^r\int_{R+T-r_1}^\infty \frac{r_1^{n-1}}{r_2^{1+s}}\,\mathrm{d}r_2\,\mathrm{d}r_1 = \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{1}{s}\int_0^r\frac{r_1^{n-1}}{(R+T-r_1)^s}\,\mathrm{d}r_1 \\ &= \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{1}{ns}\frac{r^n}{(R+T)^s}{}_2F_1(s,n;n+1;-\frac{r}{R+T}).\end{aligned}$$

Thus we can bound (3.11) from above by

$$\begin{aligned}&\mathrm{Per}_s(E_0\cup\Omega,\Omega)-\mathrm{Per}_s(E_0,\Omega)\\&\le\frac{2^{2-s}\pi^{n-\frac{1}{2}}}{s(n-s)}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{n-s}{2})\Gamma(\frac{n}{2})}r^{n-s}-\frac{8\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{r^n}{sn}\Big(\frac{1}{R^s}{}_2F_1(s,n;n+1;-\frac{r}{R})-\frac{1}{(R+T)^s}{}_2F_1(s,n;n+1;-\frac{r}{R+T}\Big).\end{aligned}\tag{3.12}$$

Now we multiply by $s(1-s)$ to deal with the singularities at $s=0$, $s=1$ and let $s\searrow 0$ and $s\nearrow 1$

$$\lim_{s\searrow 0}s(1-s)(\mathrm{Per}_s(E_0\cup\Omega,\Omega)-\mathrm{Per}_s(E_0,\Omega))=\frac{4\pi^n}{n}\frac{1}{(\Gamma(\frac{n}{2}))^2}r^n>0$$

and

$$\lim_{s\nearrow 1}s(1-s)(\mathrm{Per}_s(E_0\cup\Omega,\Omega)-\mathrm{Per}_s(E_0,\Omega))=\frac{4\pi^{n-\frac{1}{2}}}{n-1}\frac{1}{\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2})}r^{n-1}>0.$$

Notice, that the limits are independent of R and T. Also both limits are positive. Nonetheless for T big enough, there will exist some interval in $(0,1)$ such that the difference is negative.

TODO. Elaborate or justify

The upper bound (3.12) is continuous in T for all $s\in(0,1)$ and for $T\to\infty$ the third term vanishes. Thus the upper bound converges pointwise to the upper bound in the proof of Theorem 3.3. Thus for T big enough there exists some $s\in(0,1)$ such that the difference is negative.

The limit for $s\searrow 0$ is independent of T, positive and by [9, Eq. (3.2)] we have that $s\mathscr{L}(B_{R+T}\setminus B_R,B_r)\to 0$ for $s\searrow 0$, thus the limits in $s=0$ and $s=1$ are not only upper bounds but exact values.

Thus we can conclude, that there exists an interval (s_0,s_1) such that for all $s\in(s_0,s_1)$ the minimizer is not the external data itself.

■

TODO. Give some conclusion and interpretation of these results

When comparing both theorems and their proofs, we notice that the example with bounded exxternal data doesn't seem to converge to the example of unbounded external data. At least in the limit for $s\searrow 0$. This is interesting, as this entails, that if we want to analyze the limiting behavior of a minimizer for $s\searrow 0$ we cannot restrict the boundary data to be bounded or unbounded.

Conclusion

dicussion of the results, comparison to classical case, open problems, future work,..

1. Change of Topology in the models (barrier construction)
2. Cubic construction for arbitrary external data
3. Existence of s_0 for all external data and prescribed sets
4. Minimizer touching the boundary of the prescribed set (Calculations with of 3. with arbitrary parameter shows, no)
5. Can we give an estimate of the amount of connected components?

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