On the Connectedness of Nonlocal Minimal Surfaces in a Cylinder with (un)bounded Boundary Data

Mohamed Noah Abdel Wahab

Master Thesis supervised by: Prof. Dr. Marco Cicalese, Dr. Fumihiko Onoue

Department of Mathematics
School of Computation, Information, and Technology
Technical University of Munich

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Introduction

Historical Background

- *Euler* and *Lagrange* were among the first to study minimal surfaces in the 18th century

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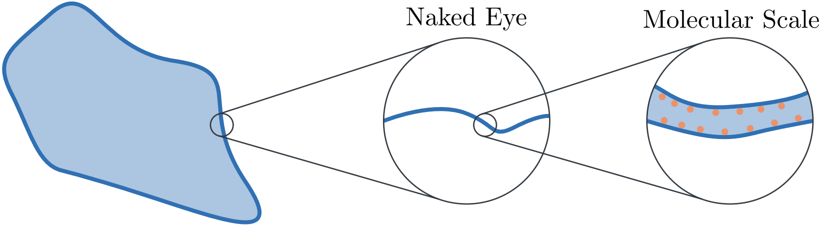
- The concept of *Perimeter* is nowadays used to study minimal surfaces
- Minimal surfaces can be defined as the set with the least perimeter given some constraints
- We consider a rather new concept of minimal surfaces, the *Nonlocal Minimal Surfaces* as defined in the seminal work of Cafarelli, Roquejoffre and Savin [1]

Introduction

Soap Film



Introduction

Nonlocal Perimeter

Definition (Nonlocal Perimeter)

Let $E \subset \mathbb{R}^n$ and $s \in (0,1)$, then the s-perimeter or fractional perimeter of E is given by

$$\mathrm{Per}_s(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{n+s}} \,\mathrm{d}y \,\mathrm{d}x.$$

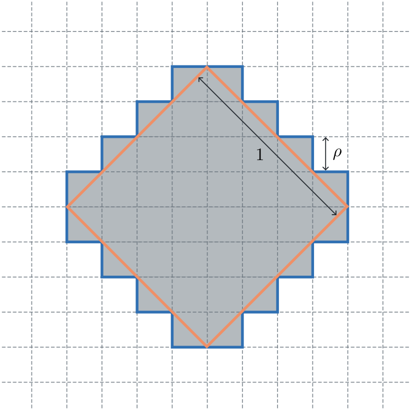
Definition (Relative Nonlocal Perimeter)

Let $E \subset \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ bounded and $s \in (0,1)$, then the s-perimeter of E relative to Ω is given by

$$\mathrm{Per}_s(E,\Omega) := \int_{E\cap\Omega} \int_{E^c} \frac{1}{|x-y|^{n+s}} \,\mathrm{d}y \,\mathrm{d}x + \int_{E\setminus\Omega} \int_{\Omega\setminus E} \frac{1}{|x-y|^{n+s}} \,\mathrm{d}y \,\mathrm{d}x.$$

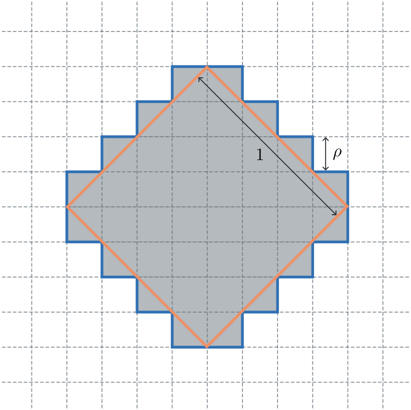
Introduction

Rotated pixelated square



Pixelsize: ρ, Square Sidelength: 1

Introduction
Rotated pixelated square



Pixelsize: ρ, Square Sidelength: 1

- Actual Perimeter: 4

Introduction

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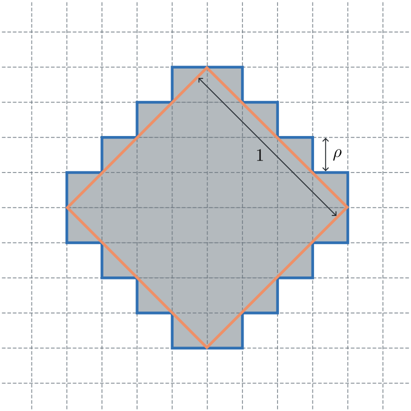

Pixelsize: ρ, Square Sidelength: 1

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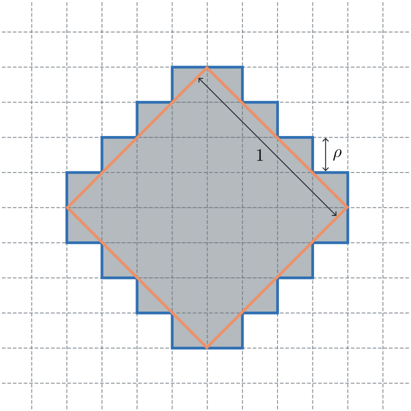
- Classical Perimeter of pixelated Square: $4\sqrt{2}$

Introduction

Rotated pixelated square



Pixelsize: ρ, Square Sidelength: 1

- Actual Perimeter: 4
- Classical Perimeter of pixelated Square: $4\sqrt{2}$
- Nonlocal Perimeter of pixelated Square: $\sim 4 + \rho^{1-s}$

Introduction

Definition (Nonlocal Minimal Surface)

Let $\Omega \subset \mathbb{R}^n$ bounded and $E_0 \subset \mathbb{R}^n$, then we want to find $E \subset \mathbb{R}^n$ such that E minimizes the fractional perimeter of E_0 relative to Ω, i.e.

$$\mathrm{Per}_s(E, \Omega) = \min \left\{ \mathrm{Per}_s(F, \Omega) \mid F \setminus \Omega = E_0 \setminus \Omega \right\}.$$

This set E is then called *Nonlocal Minimal Surface*.

Aim

Generalization of the result of Dipierro, Onoue and Valdinoci [2].

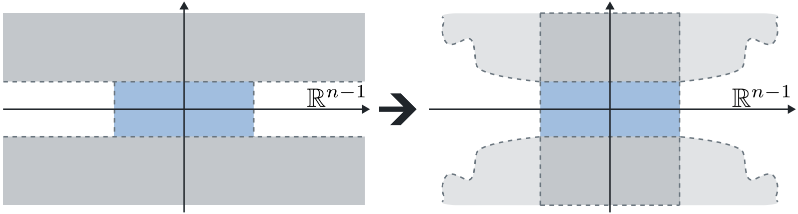


Figure: Left: Model in [2]. Right: Generalization.

Preliminaries

Properties of Nonlocal Perimeter

- $\lim_{s \to 1^-} (1-s)\,\mathrm{Per}_s(E,\Omega) = c\,\mathrm{Per}(E,\Omega)$ for any E with finite classical Perimeter
- $\lim_{s \to 0^+} s \int_A \int_B \frac{1}{|x-y|^{n+s}}\,\mathrm{d}y\,\mathrm{d}x = 0$ for any A, B such that $\mathrm{dist}(A,B) > 0$

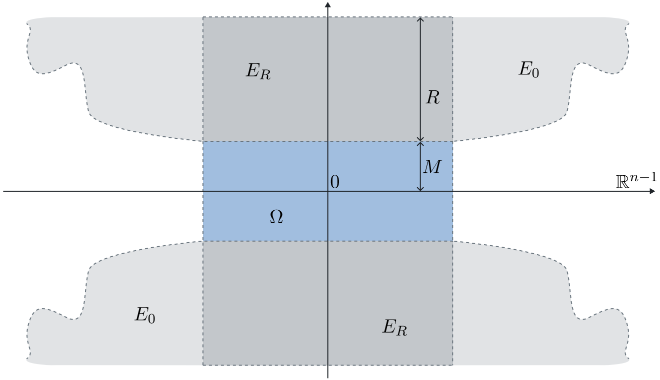
Theorem (Euler-Lagrange Equation)

Let $E \subset \mathbb{R}^n$ be a nonlocal minimal surface relative to some set Ω. If $E \cap \Omega$ has an interior tangent ball at some point $q \in \partial E \cap \Omega$, then

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y-q|^{n+s}}\,\mathrm{d}y \geq 0.$$

Problem Setting

$$E_R := \{(x', x_n) \mid |x'| < 1, M < |x_n| < M + R\} \subset E_0 \subset \{(x', x_n) \mid |x_n| > M\}$$
$$\Omega := \{(x', x_n) \mid |x'| < 1, |x_n| < M\}$$



Main Results

Theorem (Connectedness of Nonlocal Minimal Surface)

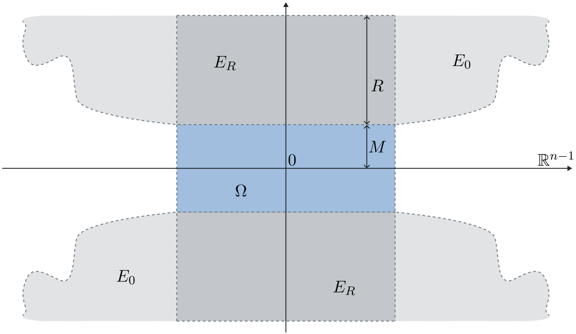
For E_0 and Ω as above and any $R > 0$ there exists an $M_0 \in (0,1)$ depending on the dimension, R and s, such that for any $M \in (0, M_0)$ the minimizer E_M is given by $E_M = E_0 \cup \Omega$.

Theorem (Disconectedness of Nonlocal Minimal Surface)

For E_0 and Ω as above and any $R > 0$ there exists an $M_0 > 1$ depending on the dimension, R and s, such that for any $M > M_0$ the minimizer E_M is disconnected.

Proof Idea of Main Results

Connectedness of Nonlocal Minimal Surface



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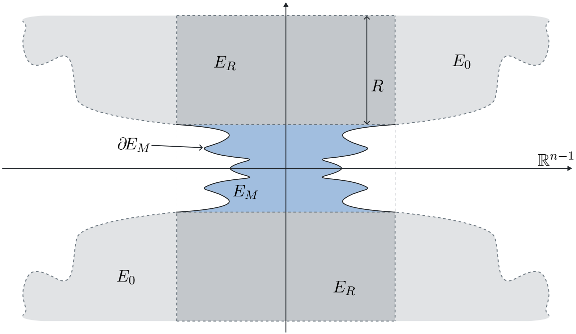
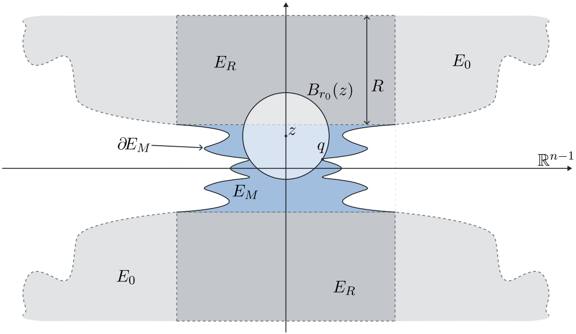


Proof Idea of Main Results

Connectedness of Nonlocal Minimal Surface

$$0 \leq \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y-q|^{n+s}} \, \mathrm{d}y$$



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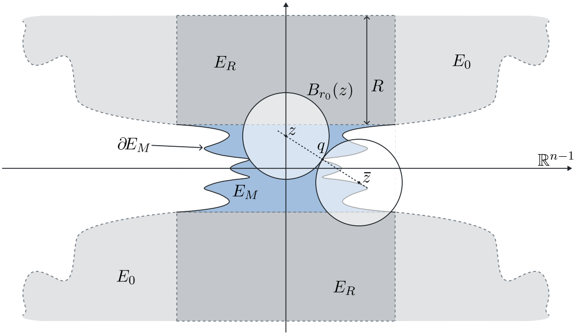

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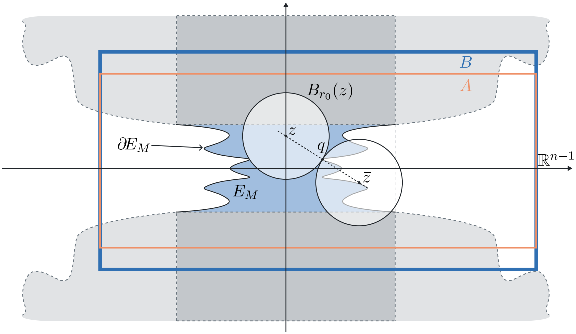
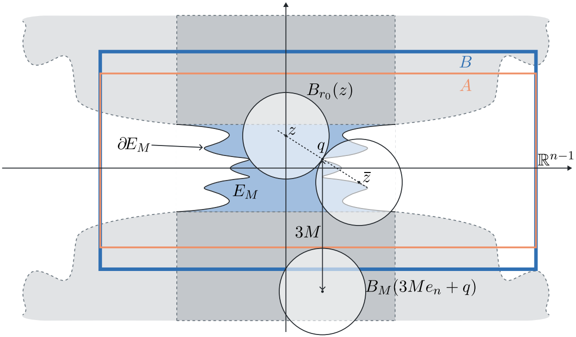


$$A := \{|x' - q'| < 2, |x_n - q_n| < 2M\}, \quad B := \{|x' - q'| < 2, |x_n - q_n| < R\}$$

Proof Idea of Main Results

Connectedness of Nonlocal Minimal Surface

$$0 \leq \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y-q|^{n+s}} \, \mathrm{d}y \leq -c_0 M^{-s}$$



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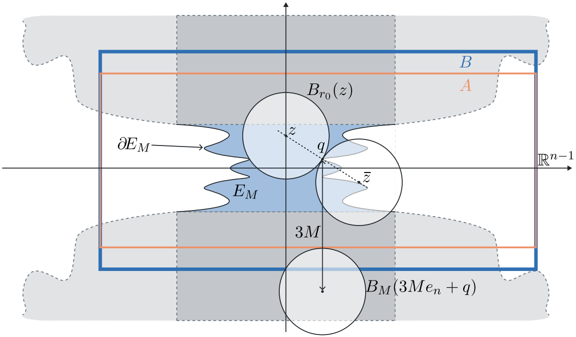
$$0 \leq \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y-q|^{n+s}} \, \mathrm{d}y \leq -c_0 M^{-s} + c_1 2^{-s}$$



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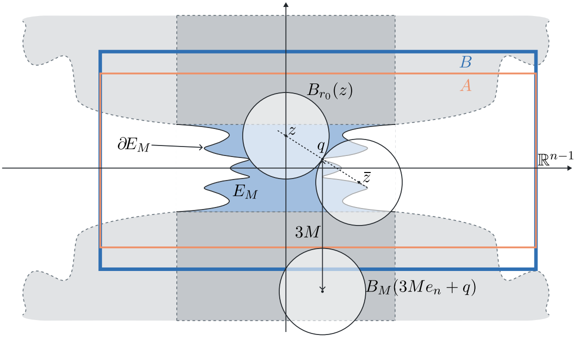

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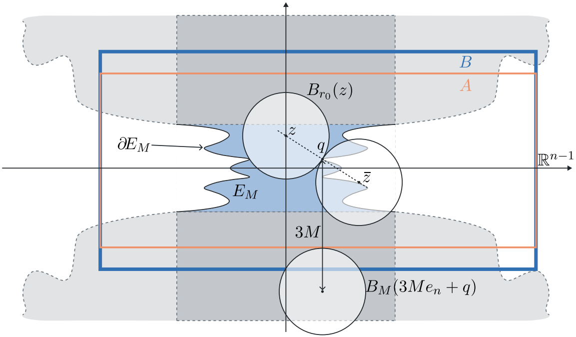

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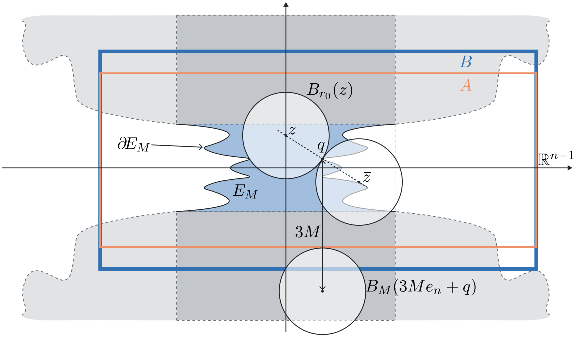

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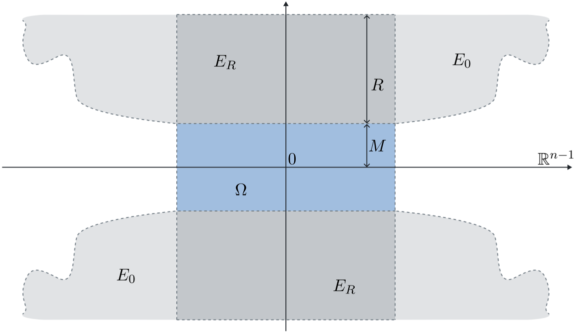
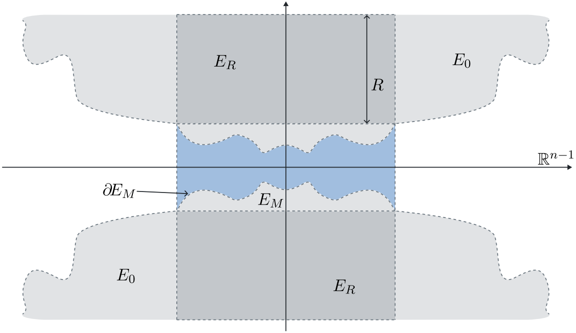
Proof Idea of Main Results

Disconnectedness of Nonlocal Minimal Surface



Proof Idea of Main Results

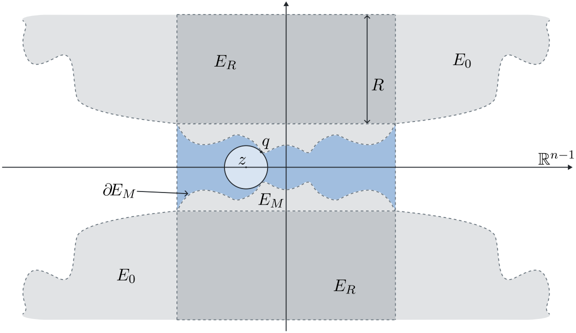
Disconnectedness of Nonlocal Minimal Surface



Proof Idea of Main Results

Disconnectedness of Nonlocal Minimal Surface

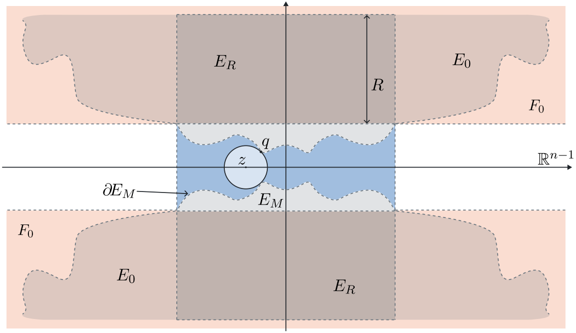
$$0 \geq \int_{\mathbb{R}^n} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n+s}} \, \mathrm{d}y$$



Proof Idea of Main Results

Disconnectedness of Nonlocal Minimal Surface

$$0 \geq \int_{\mathbb{R}^n} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n+s}}\, \mathrm{d}y \geq \int_{\mathbb{R}^n} \frac{\chi_{F_M^c} - \chi_{F_M}}{|y-q|^{n+s}}\, \mathrm{d}y$$



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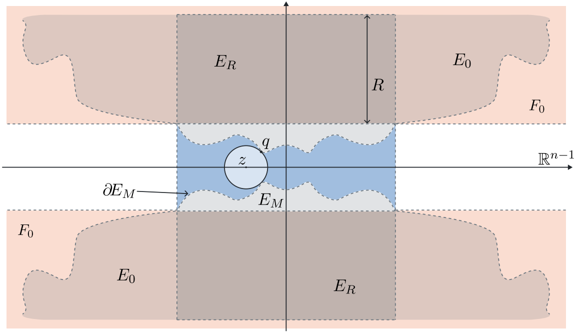

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Related Results

Theorem (Existence of Disconnected Minimizer for unbounded Data)

Let $n \geq 2$ and $0 < r < R$. Let $E_0 = B_R^c$ and $\Omega = B_r$, then there exists an $s_0 \in (0,1)$ such that for all $s \in (0, s_0)$ the minimizer is not the external data E_0 itself.

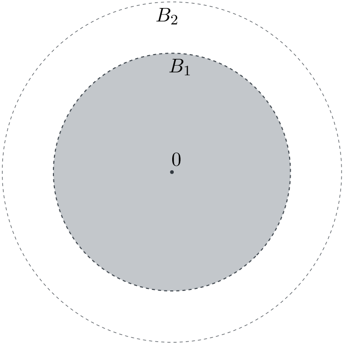
Theorem (Existence of Disconnected Minimizer for bounded Data)

Let $n \geq 2$ and $0 < r < R$ and $T > 0$. Let $E_0 = B_{R+T} \setminus B_R$ and $\Omega = B_r$, then for any T large enough there exists $s_0, s_1 \in (0,1)$ such that for all $s \in (s_0, s_1)$ the minimizer is not the external data E_0 itself.

Example

$$\Omega = B_1, \quad E_0 = B_2^c$$

$$\mathrm{Per}_s(B_2^c \cup B_1, B_1) - \mathrm{Per}_s(B_2^c, B_1)$$



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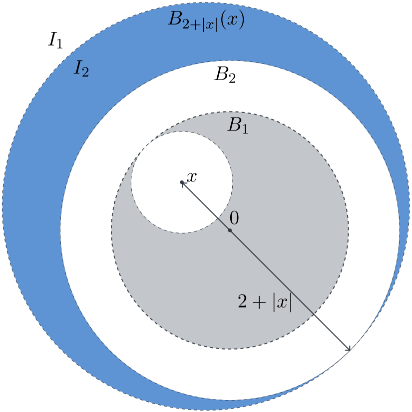

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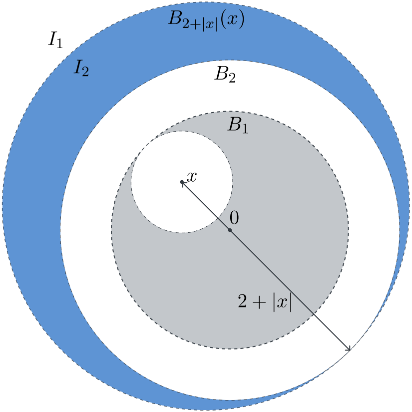


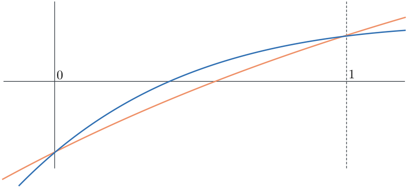
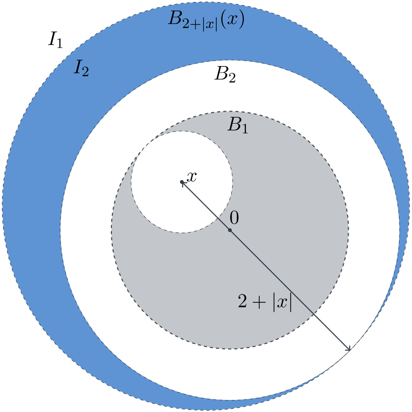


Figure: Difference multiplied with $s(1-s)$ and plotted for $s \in (0,1)$

Example

$$\Omega = B_1, \quad E_0 = B_{5000} \setminus B_2$$

$$\mathrm{Per}_s(B_1^c \cup (B_{5000} \setminus B_2), B_1) - \mathrm{Per}_s(B_{5000} \setminus B_2, B_1)$$



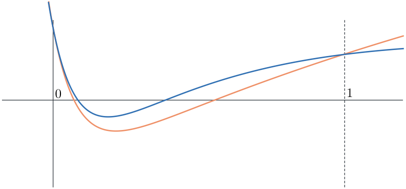


Figure: Difference multiplied with $s(1-s)$ and plotted for $s \in (0,1)$

Proof Idea of Related Results

$$\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega) = \mathrm{Per}_s(B_R^c \cup B_r, B_r) - \mathrm{Per}_s(B_R^c, B_r)$$

Proof Idea of Related Results

$$\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega) = \mathrm{Per}_s(B_R^c \cup B_r, B_r) - \mathrm{Per}_s(B_R^c, B_r)$$
$$= \mathrm{Per}_s(B_r) - 2\int_{B_r}\int_{B_R^c} \frac{1}{|x-y|^{n+s}}\,\mathrm{d}y\,\mathrm{d}x$$

Proof Idea of Related Results

$$
\begin{aligned}
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&\quad = \mathrm{Per}_s(B_r) - 2 \int_{B_r} \int_{B_R^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x
\end{aligned}
$$

$$
\lim_{s \to 0^+} s(1-s)(\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega)) = -\frac{4\pi^n}{n} \frac{1}{(\Gamma(\frac{n}{2}))^2} r^n < 0
$$

$$
\lim_{s \to 1^-} s(1-s)(\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega)) = \frac{4\pi^{n-\frac{1}{2}}}{n-1} \frac{1}{\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2})} r^{n-1} > 0
$$

Proof Idea of Related Results

$$\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega) = \mathrm{Per}_s(B_{R+T} \setminus B_R \cup B_r, B_r) - \mathrm{Per}_s(B_{R+T} \setminus B_R, B_r)$$
$$= \mathrm{Per}_s(B_r) - 2\int_{B_r}\int_{B_R^c} \frac{1}{|x-y|^{n+s}}\,\mathrm{d}y\,\mathrm{d}x + 2\int_{B_r}\int_{B_{R+T}^c} \frac{1}{|x-y|^{n-s}}\,\mathrm{d}y\,\mathrm{d}x$$

$$\lim_{s\to 0^+} s(1-s)(\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega)) = \frac{4\pi^n}{n}\frac{1}{(\Gamma(\frac{n}{2}))^2} r^n > 0$$

$$\lim_{s\to 1^-} s(1-s)(\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega)) = \frac{4\pi^{n-\frac{1}{2}}}{n-1}\frac{1}{\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2})} r^{n-1} > 0$$

Bibliography

[1] L. Caffarelli, J. Roquejoffre, and O. Savin. "Nonlocal Minimal Surfaces". In: *Communications on Pure and Applied Mathematics* 63 (Sept. 2010). DOI: 10.1002/cpa.20331.

[2] S. Dipierro, F. Onoue, and E. Valdinoci. "(Dis)connectedness of nonlocal minimal surfaces in a cylinder and a stickiness property". In: *Proceedings of the American Mathematical Society* (Feb. 2022). DOI: 10.1090/proc/15796.