Convergence of an Algorithm for Mean Curvature Motion

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ABSTRACT. Bence, Merriman and Osher [BMO] have proposed a new numerical algorithm for computing mean curvature flow, in terms of solutions of the usual heat equation, continually reinitialized after short time steps. This paper employs nonlinear semigroup theory to reconcile their algorithm with the "level-set" approach to mean curvature flow of Osher–Sethian [OS], Evans–Spruck [ES], and Chen–Giga–Goto [CGG].

Introduction. An interesting recent paper by Bence, Merriman and Osher [BMO] proposes a new computational algorithm for tracking the evolution in time of a set in $\mathbb{R}^n$ whose boundary moves with normal velocity equaling ($(n-1)$ times) its mean curvature. The procedure is this. Given a compact set $C_0 \subset \mathbb{R}^n$ we solve the heat equation, with initial data the indicator function of C_0:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = \chi_{C_0} & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{1}$$

Fix now a time step $t > 0$ and define the new set

$$C_t \equiv \left\{ x \in \mathbb{R}^n \mid u(x,t) \geq \frac{1}{2} \right\}. \tag{2}$$

We hereafter write

$$C_t = \mathcal{H}(t) C_0 \qquad (t \geq 0) \tag{3}$$

and regard $\mathcal{H}(t)$ for each time $t > 0$ as a mapping on the collection $\mathcal{K}$ of compact subsets of $\mathbb{R}^n$. We call $\{\mathcal{H}(t)\}_{t \geq 0}$ the *heat diffusion flow.* As explained heuristically in [BMO] the evolution of C_0 into $C_t = \mathcal{H}(t)C$ approximates for small times t the mean curvature motion of the boundary Γ_0 of C_0, at least if Γ_0 is smooth. Hence our continually reinitiating the procedure over short time steps should yield an approximation to mean curvature flow, valid even for large times. The interested reader should consult [BMO] for more explanation of the

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heuristics, and for a preliminary analysis of the numerically optimal time step versus grid resolution. See also Mascarenhas [M].

The purpose of this paper is to provide a rigorous analysis of the Bence–Merriman–Osher algorithm, within the context of the generalized mean curvature flow defined by Evans–Spruck [ES] and Chen–Giga–Goto [CGG]. (Osher and Sethian had introduced this approach for numerical computations in [OS].) This evolution is as follows defined in terms of the level sets of an "order parameter" u. Given a compact set $\Gamma_0 \subset \mathbb{R}^{n-1}$ we choose a bounded, smooth function $g : \mathbb{R}^n \to \mathbb{R}$, such that

$$\Gamma_0 = \{x \in \mathbb{R}^n \mid g(x) = 0\}. \tag{4}$$

We consider next the *mean curvature evolution* PDE

$$\begin{cases} u_t - \left(\delta_{ij} - \dfrac{u_{x_i} u_{x_j}}{|Du|^2}\right) u_{x_i x_j} = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{5}$$

where $Du = D_x u = (u_{x_1}, \dots, u_{x_n})$. As explained in [CGG], [ES], etc. this PDE asserts that each level set of u evolves via mean curvature motion, at least in regions where u is smooth and $|Du| \neq 0$. It turns out that (5) has a unique, continuous, weak solution: see [CGG], [ES], etc. for definitions and details. Consequently we may for each $t > 0$ *define* the compact level set

$$\Gamma_t \equiv \{x \in \mathbb{R}^n \mid u(x,t) = 0\}. \tag{6}$$

As in [ES] we write

$$\Gamma_t = \mathcal{M}(t)\Gamma_0 \qquad (t \geq 0), \tag{7}$$

thereby regarding $\mathcal{M}(t)$ as a mapping on $\mathcal{K}$. Since the weak solution u of the mean curvature PDE is unique, it follows that $\{\mathcal{M}(t)\}_{t\geq 0}$ satisfies the semigroup property:

$$\mathcal{M}(t+s) = \mathcal{M}(t)\mathcal{M}(s) = \mathcal{M}(s)\mathcal{M}(t) \qquad (t,\ s \geq 0) \tag{8}$$

on $\mathcal{K}$. We consequently call $\{\mathcal{M}(t)\}_{t\geq 0}$ the (generalized) *mean curvature flow semigroup.* The papers [5], [7–10], [14], etc. are devoted to detailed analysis of this evolution. A principal assertion is that $\Gamma_t = \mathcal{M}(t)\Gamma_0$ agrees with the classical differential geometric flow by mean curvature starting with Γ_0, if and for so long as the latter exists. Furthermore, $\Gamma_t = \mathcal{M}(t)\Gamma_0$ is defined for all times $t \geq 0$, and so allows for the onset of singularities, changes of topological type, etc.

There is by now a considerable body of numerics and rigorous analysis, all substantiating the belief that the level set approach (4)–(6) gives the "correct notion" of generalized mean curvature flow. It therefore seems appropriate to try to understand the new Bence–Merriman–Osher algorithm within this framework.

In symbolic terms the issue is to compare the mean curvature flow $\{\mathcal{M}(t)\}_{t\geq 0}$ and the heat diffusion flow $\{\mathcal{H}(t)\}_{t\geq 0}$. Given a smooth, bounded open set $U_0 \subset \mathbb{R}^n$, with boundary Γ_0, we set $C_0 = \bar{U}_0$ and can immediately define

$$\mathcal{M}(t)C_0$$

by everywhere replacing Γ_0 with C_0 in (4)–(6). (In other words, we can shift our attention from evolving the hypersurface Γ_0 by curvature flow, to evolving the entire set C_0.) The question is how to compare $\mathcal{M}(\,\cdot\,)C_0$ and the repeated application of $\mathcal{H}(\,\cdot\,)$ over short time intervals. The natural conjecture would be

$$\text{(9)} \qquad \text{“}\lim_{m\to\infty} \mathcal{H}(t/m)^m C_0 = \mathcal{M}(t)C_0\text{”} \qquad (C_0 \in \mathcal{K},\, t > 0).$$

However, I do not know how to make this assertion precise in any sense, principally because the time mapping $t \mapsto \mathcal{M}(t)C_0$ is so poorly behaved.

The idea instead will be to turn from flows of sets to flows of functions. This is already the viewpoint of the level-set method (4)–(6): the evolution $\Gamma_0 \mapsto \mathcal{M}(t)\Gamma_0$ is unique, but highly unstable, whereas the mapping $g \mapsto u(\cdot,t)$ is a contraction in the sup-norm. We will invoke the theory of nonlinear semigroups, which turns out to apply to the mean curvature evolution PDE (5). We will consequently be able to write

$$u(x,t) = [M(t)g](x) \qquad (x \in \mathbb{R}^n,\, t > 0),$$

where, informally speaking, $\{M(t)\}_{t\geq 0}$ is the semigroup on $X = C(\mathbb{R}^n)$ generated by the nonlinear operator

$$Au = -\left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|Du|^2}\right)u_{x_ix_j}.$$

The flow $g \mapsto M(t)g$ is computed by evolving each of the level sets of g by $\mathcal{M}(t)$.

By analogy, we will define a family of nonlinear mappings $\{H(t)\}_{t\geq 0}$ on X, in such a way that $g \mapsto H(t)g$ is had by evolving each superlevel (or sublevel) set of g by $\mathcal{H}(t)$. Having thereby shifted our viewpoint to flows on the space of functions $X = C(\mathbb{R}^n)$, we propose in place of (9) the assertion

$$\text{(10)} \qquad \lim_{m\to\infty} H(t/m)^m g = M(t)g \qquad (g \in X,\, t > 0).$$

This is in fact our main theorem, to be proved in Section 5. We will obtain (10) as a consequence of the “nonlinear Chernoff formula” for nonlinear semigroups, proved by Brezis and Pazy [BP].

In Section 2 below we review the rudiments of abstract nonlinear semigroup theory, and then recast the results of Evans–Spruck [ES], Chen–Giga–Goto [CGG] into this setting. It will be technically convenient to work in the space of Q-periodic functions, Q denoting the unit cube in $\mathbb{R}^n$. In Section 3 we study the flows $\{\mathcal{H}(t)\}_{t\geq 0}$ and $\{H(t)\}_{t\geq 0}$, and again it is convenient to reinterpret $\mathcal{H}(t)$ as acting on the collection $\mathcal{C}$ of closed subsets of $\mathbb{R}^n$.

I also call the reader's attention to a recent paper by Mascarenhas [M], which provides a proof of the convergence of the Bence–Merriman–Osher algorithm in regions when the moving interface is smooth: his analysis is similar to that in Section 4 of this paper. I have also very recently received a preprint [BG] by Barles and Georgelin with results quite similar to those derived here.

Finally, let me note that the real interest in the Bence–Merriman–Osher algorithm is its extension to allow for triple and higher order junctions, that is, to the geometric evolution of the boundaries between three or more phases. This paper has nothing to say about the convergence of the algorithm in such a situation: there is no general, rigorous model for multiphase geometric flow currently available.

Nonlinear semigroups and mean curvature flow. This section rapidly recounts the basics of abstract nonlinear semigroup theory and then explains the connections with generalized mean curvature flow.

Nonlinear semigroup theory. Let X denote a real Banach space with norm $\| \quad \|$. Suppose A is a nonlinear, possibly discontinuous, possibly multi-valued operator, mapping its domain $D(A) \subset X$ into 2^X, the collection of all subsets of X. We say $-A$ is *dissipative* provided

$$\|x-\hat{x}\| \leq \|x-\hat{x}+\lambda(y-\hat{y})\| \tag{11}$$

for each $\lambda > 0$ and all x, $\hat{x} \in D(A)$, $y \in Ax$, $\hat{y} \in A\hat{x}$. (There is equivalent, but cacophonous, nomenclature: A is called *accretive* if $-A$ is dissipative.) The operator $-A$ is *m-dissipative* if, additionally,

$$R(I+\lambda A) = X \tag{12}$$

for each $\lambda > 0$, R denoting the range

When $-A$ is m-dissipative, it follows from (11) and (12) that the nonlinear *resolvents*

$$J_\lambda \equiv (I+\lambda A)^{-1} \qquad (\lambda > 0) \tag{13}$$

are single-valued contractions defined on all of X:

$$\|J_\lambda y - J_\lambda \hat{y}\| \leq \|y-\hat{y}\| \qquad (\lambda > 0,\ y,\ \hat{y} \in X). \tag{14}$$

Nonlinear semigroup theory (cf. Barbu [1; Chapter 3]) addresses the solvability of the evolution equation

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) \ni 0 \quad (t>0), \\ u(0) = x, \end{cases} \tag{15}$$

for some given initial point $x \in \overline{D(A)} \subset X$. Replacing the t-derivative by a difference quotient of size $h > 0$, we may formally at least approximate (15) by the sequence of equations

$$\begin{cases} \dfrac{u_h^{k+1} - u_h^k}{h} + Au_h^{k+1} \ni 0 \quad (k = 0,1,\dots) \\ u_h^0 = x. \end{cases}$$

Then

$$u_h^{k+1} = J_h u_h^k = \cdots = J_h^{k+1} x. \tag{17}$$

Fix $t > 0$, $m \in \mathbb{Z}^+$, and set $h = t/m$, $k = m-1$ in the term on the right hand side of (17), to obtain

$$J_{t/m}^m x \tag{18}$$

as an approximation to a solution of (15).

The fundamental Generation Theorem of Crandall–Liggett [CL] asserts that these approximations do indeed converge:

Theorem 2.1. *([CL]) Let $-A$ be m-dissipative. Then for each $x \in \overline{D(A)}$, the limit*

$$\lim_{\substack{m\to\infty \\ \lambda m \to t}} J_\lambda^m x \equiv u(t) \tag{19}$$

exists in X, uniformly for times t belonging to compact subsets of $[0,\infty)$.

We usually write

$$u(t) = S(t)x \qquad (t \geq 0, x \in \overline{D(A)}) \tag{20}$$

to display the dependence of $u(\,\cdot\,)$ on the initial condition x. Then $\{S(t)\}_{t\geq 0}$ is a one-parameter family of nonlinear operators on $\overline{D(A)}$, called the *nonlinear semigroup generated by $-A$*. It is easy to check the following properties obtain:

$$\begin{cases} \text{(i)} & S(t+s) = S(t)S(s) = S(s)S(t) & (t,\ s \geq 0); \\ \text{(ii)} & \|S(t)x - S(t)\hat{x}\| \leq \|x - \hat{x}\| & (t \geq 0,\ x,\ \hat{x} \in \overline{D(A)}); \\ \text{(iii)} & t \mapsto S(t)x \text{ is continuous on } [0,\infty) & \text{for each } x \in \overline{D(A)}. \end{cases} \tag{21}$$

In view of (16)–(20) it is reasonable to try to interpret $u(t) = S(t)x$ as a kind of weak or generalized solution to the evolution equation (15). This can be done, although at the current level of abstraction there are a number of subtleties.

Our application of nonlinear semigroup theory to prove convergence of a variant of the Bence–Merriman–Osher algorithm depends upon the following theorem of Brezis and Pazy [BP], which amounts to a nonlinear version of Chernoff's formula for linear semigroups:

Theorem 2.2. ([BP]) *Suppose $-A$ is m-dissipative on X, and so generates a nonlinear semigroup $\{S(t)\}_{t\geq 0}$. Assume also $\{F(t)\}_{t\geq 0}$ is a family of nonlinear operators, mapping $X \to X$ satisfying*

(i) $$\|F(t)x - F(t)\hat{x}\| \leq \|x - \hat{x}\| \qquad (t \geq 0,\ x,\ \hat{x} \in X)$$

and

(ii) $$\lim_{t\to 0^+} \left(I + \lambda \frac{(I - F(t))}{t}\right)^{-1} x \to (I + \lambda A)^{-1} x \qquad (x \in \overline{D(A)},\ \lambda > 0).$$

Then

$$\lim_{m\to\infty} F(t/m)^m x \to S(t)x \tag{22}$$

for all $t \geq 0$, $x \in \overline{D(A)}$, uniformly for times t belonging to compact subsets of $[0, \infty)$.

2.2. Mean curvature flow. We turn now to the mean curvature evolution problem discussed in Section 1. Our goal is to recast the approach of Evans–Spruck [ES] and Chen–Giga–Goto [CGG] into the nonlinear semigroup framework discussed above.

For this, let us fix some bounded, closed cube Q in $\mathbb{R}^n$ and hereafter take X to be the space of real-valued, continuous, Q-periodic functions on $\mathbb{R}^n$, taken with the supremum norm.

Fix $f \in X$, $t > 0$, $\lambda > 0$, and consider the nonlinear PDE

$$\begin{cases} u - \lambda \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}\right) u_{x_i x_j} = f \text{ in } \mathbb{R}^n, \\ u \text{ is } Q-\text{periodic}. \end{cases} \tag{23}$$

Definitions. (1) A bounded, continuous, Q-periodic function $u : \mathbb{R}^n \to \mathbb{R}$ is a *weak subsolution* of (23) provided for each $\varphi \in C^\infty(\mathbb{R}^n)$, if $u - \varphi$ has a maximum at a point $x_0 \in \mathbb{R}^n$, then

$$u(x_0) - \lambda \left(\delta_{ij} - \frac{\varphi_{x_i}(x_0)\varphi_{x_j}(x_0)}{|D\varphi(x_0)|^2}\right) \varphi_{x_i x_j}(x_0) \leq f(x_0) \tag{24}$$

provided $D\varphi(x_0) \neq 0$, and

$$u(x_0) \leq f(x_0) \tag{25}$$

provided $D\varphi(x_0) = 0$, $D^2\varphi(x_0) = 0$.

(2) Similarly, u is a *weak supersolution* provided for each $\varphi \in C^\infty(\mathbb{R}^n)$, if $u - \varphi$ has a minimum at a point $x_0 \in \mathbb{R}^n$, then

$$u(x_0) - \lambda \left(\delta_{ij} - \frac{\varphi_{x_i}(x_0)\varphi_{x_j}(x_0)}{|D\varphi(x_0)|^2}\right) \varphi_{x_i x_j}(x_0) \geq f(x_0) \tag{26}$$

provided $D\varphi(x_0) \neq 0$, and

$$u(x_0) \geq f(x_0) \tag{27}$$

provided $D\varphi(x_0) = 0$, $D^2\varphi(x_0) = 0$.

Remark. There is no assertion if $D\varphi(x_0) = 0$, $D^2\varphi(x_0) \neq 0$.

Definition. We say u is a *weak solution* if u is both a weak subsolution and a weak supersolution.

Our intention now is to define a nonlinear operator A associated with the PDE (23). Given $u \in X$, we say u belongs to the domain $D(A)$, and write

$$u + \lambda Au \ni f, \tag{28}$$

provided u is a weak solution of the PDE (23) for some function $f \in X$. Note that our definition of the operator A is independent of λ.

Theorem 2.3. *The operator $-A$ so defined is m-dissipative on X.*

Proof. 1. Select u, $\hat{u} \in D(A)$, $f \in u + \lambda Au$, $\hat{f} \in \hat{u} + \lambda A\hat{u}$. We must first show

$$\|u - \hat{u}\| \leq \|f - \hat{f}\|, \tag{29}$$

$\| \ \|$ denoting the sup-norm. However this estimate follows *exactly* as in the proof of Theorem 9.1 in Crandall–Ishii–Lions [5, p. 51–52].

2. Next we verify $R(I + \lambda A) = X$. Fix a smooth function $f \in X$, $\lambda > 0$, and consider then the PDE (23). We approximate by taking $\varepsilon > 0$ and introducing the PDE

$$\begin{cases} u^\varepsilon - \lambda \left(\delta_{ij} - \dfrac{u^\varepsilon_{x_i} u^\varepsilon_{x_j}}{|Du^\varepsilon|^2 + \varepsilon^2} \right) u^\varepsilon_{x_i x_j} = f \quad \text{in } \mathbb{R}^n \\ u^\varepsilon \ Q\text{-periodic.} \end{cases} \tag{30}$$

By the maximum principle we have

$$\sup_{\varepsilon > 0} \|u^\varepsilon\| \leq \|f\|,$$

$$\sup_{\varepsilon > 0} \|u^\varepsilon(\cdot + y) - u^\varepsilon(\,\cdot\,)\| \leq \|f(\cdot + y) - f(\,\cdot\,)\|$$

for each $y \in \mathbb{R}^n$. The sequence $\{u^\varepsilon\}_{0<\varepsilon\leq 1}$ is consequently bounded and equicontinuous; whence there exists a subsequence $\{u^{\varepsilon_k}\}_{k=1}^{\infty}$ and a function $u \in X$, with

$$u^{\varepsilon_k} \to u \text{ uniformly on } \mathbb{R}^n. \tag{31}$$

We assert

$$u \in D(A), \quad u + \lambda Au \ni f. \tag{32}$$

To see this, fix $\varphi \in C^\infty(\mathbb{R}^n)$ and suppose $u - \varphi$ has a *strict* maximum at $x_0 \in \mathbb{R}^n$. In view of (31) we note

$$u^{\varepsilon_k} - \varphi \text{ has a maximum at a point } x_k \in \mathbb{R}^n, \tag{33}$$

with $x_k \to x_0$ as $k \to \infty$. By the maximum principle

$$u^{\varepsilon}(x_k) - \lambda \left(\delta_{ij} - \frac{\varphi_{x_i}(x_k)\varphi_{x_j}(x_k)}{|D\varphi(x_k)|^2 + \varepsilon_k^2} \right) \varphi_{x_i x_j}(x_k) \leq f(x_k). \tag{34}$$

Now if $D\varphi(x_0) \neq 0$, we can send $k \to \infty$ to deduce (24). If $D\varphi(x_0) = 0$ and $D^2\varphi(x_0) = 0$, we deduce instead (25). A similar argument applies if $u - \varphi$ has a strict minimum at some point x_0. If $u - \varphi$ has a nonstrict maximum or minimum at a point x_0, we replace φ with $\varphi(x) \pm |x - x_0|^4$ to reduce to the previous cases. Assertion (31) is proved. Thus $R(I + \lambda A)$ is dense in X. In view of (29) however $R(I + \lambda A)$ is closed, and so $R(I + \lambda A) = X$ for each $\lambda > 0$. □

Remark. The proof above also provides some information if $u - \varphi$ has a maximum at a point x_0, with $D\varphi(x_0) = 0$, $D^2\varphi(x_0) \neq 0$. In this case we have, as above,

$$u^{\varepsilon_k}(x_k) - \lambda \left(\delta_{ij} - \frac{\varphi_{x_i}(x_k)\varphi_{x_j}(x_k)}{|D\varphi(x_k)|^2 + \varepsilon_k^2} \right) \varphi_{x_i x_j}(x_k) \leq f(x_k). \tag{35}$$

Now

$$\eta_k \equiv \frac{D\varphi(x_k)}{(|D\varphi(x_k)|^2 + \varepsilon_k^2)^{1/2}}$$

satisfies the bound $|\eta_k| \leq 1$, and thus we may assume, passing to a further subsequence if necessary, that $\eta_k \to \eta$ in $\mathbb{R}^n$. Passing then to limits in (34) we deduce

$$u(x_0) - \lambda(\delta_{ij} - \eta_i \eta_j)\varphi_{x_i x_j}(x_0) \leq f(x_0), \qquad |\eta| \leq 1. \tag{36}$$

The opposite inequality obtains for some η if $u - \varphi$ has a minimum at x_0. □

Theorem 2.4. *The domain $D(A)$ is dense in X.*

Proof. Fix $\varphi \in C^\infty(\mathbb{R}^n) \cap X$. Take $\lambda > 0$ and let $u^\lambda \in D(A)$ be the weak solution of

$$\begin{cases} u^\lambda - \lambda \left(\delta_{ij} - \dfrac{u^\lambda_{x_i} u^\lambda_{x_j}}{|Du^\lambda|^2} \right) u^\lambda_{x_i x_j} = \varphi \text{ in } \mathbb{R}^n \\ u^\lambda \ Q\text{-periodic.} \end{cases} \tag{37}$$

Assume $u^\lambda - \varphi$ attains a maximum at x_0. Then if $D\varphi(x_0) \neq 0$,

$$u^\lambda(x_0) - \lambda\left(\delta_{ij} - \frac{\varphi_{x_i}(x_0)\varphi_{x_j}(x_0)}{|D\varphi(x_0)|^2}\right)\varphi_{x_ix_j}(x_0) \leq \varphi(x_0).$$

Thus

$$u^\lambda(x_0) - \varphi(x_0) \leq O(\lambda). \tag{38}$$

If $D\varphi(x_0) = 0$, then according to (36),

$$u^\lambda(x_0) - \lambda(\delta_{ij} - \eta_i\eta_j)\varphi_{x_ix_j}(x_0) \leq \varphi(x_0)$$

for some $|\eta| \leq 1$, and again (38) holds. If $u - \varphi$ instead has a minimum at x_0, we similarly note

$$u^\lambda(x_0) - \varphi(x_0) \geq -O(\lambda).$$

Thus

$$\|u^\lambda - \varphi\| \leq O(\lambda) \qquad (\lambda > 0).$$

Since $u^\lambda \in D(A)$ and $C^\infty(\mathbb{R}^n) \cap X$ is dense in X, we conclude $D(A)$ is dense in X. □

Next we verify that semigroup techniques generate the unique, weak solution of (5).

Theorem 2.5. *Let $g \in X$ and let $\{M(t)\}_{t\geq 0}^\infty$ be the semigroup generated by $-A$. Then*

$$u(x,t) = [M(t)g](x) \qquad (x \in \mathbb{R}^n,\ t \geq 0)$$

is the unique weak solution of the mean curvature evolution PDE

$$(M)\begin{cases} u_t - \left(\delta_{ij} - \dfrac{u_{x_i}u_{x_j}}{|Du|^2}\right)u_{x_ix_j} = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ u = g \text{ on } \mathbb{R}^n \times \{t = 0\} \\ u\ Q\text{-periodic.} \end{cases}$$

Proof. 1. Fix $\varphi \in C^\infty(\mathbb{R}^n \times (0,\infty))$, φ bounded, and suppose $u - \varphi$ has a strict maximum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$. Let m be a positive integer and define

$$u^m(x,t) = [J^k_{1/m}g](x) \quad \text{if } \frac{k}{m} \leq t < \frac{k+1}{m}. \tag{39}$$

Then $u^m \to u$ uniformly on compact sets of $\mathbb{R}^n \times [0, \infty)$. Consequently for each m there exist points $(x_m, t_m) \in \mathbb{R}^n \times (0, \infty)$ such that

$$(u^m - \varphi)(x_m, t_m) + \frac{1}{m^2} \geq (u^m - \varphi)(x, t) \qquad (x \in \mathbb{R}^n,\ t > 0) \tag{40}$$

and

$$x \mapsto (u^m - \varphi)(x, t_m) \text{ has a maximum at } x_m. \tag{41}$$

Choose $k = k_m$ such that $k_m/m \leq t_m < (k_m + 1)/m$. Then $u^m(x, t_m) = [J^k_{1/m} g](x) = J_{1/m}[u^m(\cdot, t_m - 1/m)](x)$. Consequently,

$$u^m - \frac{1}{m}\left(\delta_{ij} - \frac{u^m_{x_i} u^m_{x_j}}{|Du^m|^2}\right) u^m_{x_i x_j} = v^m \tag{42}$$

in the weak sense, for $v^m(x) = u^m(x, t_m - 1/m)$. Now if $D\varphi(x_0, t_0) \neq 0$, then $D\varphi(x_m, t_m) \neq 0$ for large enough m; and (41), (42) imply

$$\begin{aligned} &u^m(x_m, t_m) - \frac{1}{m}\left(\delta_{ij} - \frac{\varphi_{x_i}(x_m, t_m)\varphi_{x_j}(x_m, t_m)}{|D\varphi(x_m, t_m)|^2}\right)\varphi_{x_i x_j}(x_m, t_m) \\ &\leq u^m\left(x_m, t_m - \frac{1}{m}\right). \end{aligned} \tag{43}$$

Since

$$u^m(x_m, t_m) - \varphi(x_m, t_m) + \frac{1}{m^2} \geq u^m\left(x_m, t_m - \frac{1}{m}) - \varphi(x_m, t_m - \frac{1}{m}\right),$$

according to (39), (43) yields the inequality

$$\begin{aligned} &\frac{\varphi(x_m, t_m) - \varphi\left(x_m, t_m - \frac{1}{m}\right)}{\frac{1}{m}} \\ &\quad - \left(\delta_{ij} - \frac{\varphi_{x_i}(x_m, t_m)\varphi_{x_j}(x_m, t_m)}{|D\varphi(x_m, t_m)|^2}\right)\varphi_{x_i x_j}(x_m, t_m) \leq \frac{1}{m}. \end{aligned} \tag{44}$$

Let $m \to \infty$:

$$\varphi_t(x_0, t_0) - \left(\delta_{ij} - \frac{\varphi_{x_i}(x_0, t_0)\varphi_{x_j}(x_0, t_0)}{|D\varphi(x_0, t_0)|^2}\right)\varphi_{x_i x_j}(x_0, t_0) \leq 0. \tag{45}$$

Suppose instead $D\varphi(x_0,t_0)=0$. If $D\varphi(x_m,t_m)\neq 0$, we again arrive at (44). If $D\varphi(x_m,t_m)=0$, then owing to (36),

$$\frac{\varphi(x_m,t_m)-\varphi\left(x_m,t_m-\dfrac{1}{m}\right)}{\dfrac{1}{m}}-(\delta_{ij}-\eta_i^m\eta_j^m)\varphi_{x_ix_j}(x_m,t_m)\leq\frac{1}{m},$$

for some $\eta^m\in\mathbb{R}^n$, $|\eta^m|\leq 1$. In either case, we deduce upon passing to limits that

$$\varphi_t(x_0,t_0)-(\delta_{ij}-\eta_i\eta_j)\varphi_{x_ix_j}(x_0,t_0)\leq 0 \tag{46}$$

for some $\eta\in\mathbb{R}^n$, $|\eta|\leq 1$.

A similar argument provides the reversed inequalities to (45), (46), should $u-\varphi$ have a minimum at (x_0,t_0). The validity of these inequalities is the definition that u be a weak solution of the mean curvature PDE. See Evans-Spruck [ES]. □

3. Evolving sets and functions. This section is devoted to a preliminary analysis of the flow $\mathcal{H}(\,\cdot\,)$ of closed sets by the heat equation. More precisely, given a closed set $C_0\subset\mathbb{R}^n$, consider the PDE

$$(H)\begin{cases}u_t-\Delta u=0 & \text{in } \mathbb{R}^n\times(0,\infty)\\ u=\chi_{C_0} & \text{on } \mathbb{R}^n\times\{t=0\},\end{cases} \tag{47}$$

the unique bounded solution of which is

$$u(x,t)=\frac{1}{(4\pi t)^{n/2}}\int_{C_0}e^{-|x-y|^2/4t}\,dy \qquad (x\in\mathbb{R}^n,\ t>0). \tag{48}$$

Fix $t>0$ and define the new closed set

$$C_t\equiv\left\{x\in\mathbb{R}^n\mid u(x,t)\geq\frac{1}{2}\right\}. \tag{49}$$

We will write

$$C_t=\mathcal{H}(t)C_0 \qquad (t>0) \tag{50}$$

as a shortened notation for (47)–(49). Then

$$\mathcal{H}(t):\mathcal{C}\to\mathcal{C} \tag{51}$$

for each time $t>0$, $\mathcal{C}$ denoting the collection of closed subsets of $\mathbb{R}^n$. We call $\{\mathcal{H}(t)\}_{t\geq 0}$ the *heat diffusion flow* on $\mathcal{C}$.

We first record some elementary properties:

Theorem 3.1. *Let $C_1, C_2 \in \mathcal{C}$, $t \geq 0$.*

(i) *If $C_1 \subseteq C_2$, then $\mathcal{H}(t)C_1 \subseteq \mathcal{H}(t)C_2$.*
(ii) *If $C_1 \cap C_2 = \emptyset$, then $\mathcal{H}(t)C_1 \cap \mathcal{H}(t)C_2 = \emptyset$.*
(iii) *If $C_1 \cup C_2 = \mathbb{R}^n$, then $\mathcal{H}(t)C_1 \cup \mathcal{H}(t)C_2 = \mathbb{R}^n$.*
(iv) *Furthermore,*

$$\operatorname{dist}(C_1, C_2) \leq \operatorname{dist}(\mathcal{H}(t)C_1, \mathcal{H}(t)C_2). \tag{52}$$

(*Here* $\operatorname{dist}(C_1, C_2) = \inf\{|x_1 - x_2| \mid x_1 \in C_1,\ x_2 \in C_2\}$.)

Proof. (i) Assume $C_1 \subseteq C_2$. Then, taking

$$\begin{cases} u^i_t - \Delta u^i = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^i = \chi_{C_i} & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

for $i = 1, 2$, we have $u^2 \geq u^1$ at time $t = 0$. By the maximum principle $u^2 \geq u^1$ in $\mathbb{R}^n \times [0,\infty)$. Thus if $x \in \mathcal{H}(t)C_1$, we have $u^2(x,t) \geq u^1(x,t) \geq \frac{1}{2}$, and so $x \in \mathcal{H}(t)C_2$. This proves (i).

(ii) Assume $C_1 \cap C_2 = \emptyset$, C_1, $C_2 \neq \emptyset$. Then $u = u^1 + u^2$ solves

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = \chi_{C_1 \cup C_2} & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Since C_1 and C_2 are both closed, $C_1 \cup C_2 \neq \mathbb{R}^n$. By the strong maximum principle, $0 < u(x,t) < 1$ for each $x \in \mathbb{R}^n$, $t > 0$. Thus if $x \in \mathcal{H}(t)C_1$, we have $u^1(x,t) \geq \frac{1}{2}$ and so $u^2(x,t) < \frac{1}{2}$, $x \notin \mathcal{H}(t)C_2$. Assertion (ii) is proved.

(iii) Suppose $C_1 \cup C_2 = \mathbb{R}^n$. Then $u = u^1 + u^2 \geq 1$ on $\mathbb{R}^n \times \{t = 0\}$, and so $u \geq 1$ on $\mathbb{R}^n \times (0,\infty)$. Thus for each $x \in \mathbb{R}^n$ and $t > 0$, $u^1(x,t) \geq \frac{1}{2}$, or $u^2(x,t) \geq \frac{1}{2}$, or both. Thus $x \in \mathcal{H}(t)C_1 \cup \mathcal{H}(t)C_2$. This is (iii).

(iv) Assume $0 < \alpha < \operatorname{dist}(C_1, C_2)$, and let ξ denote any vector in $\mathbb{R}^n$ with $|\xi| \leq 1$. Then $\tilde{C}_1 = C_1 + \alpha\xi$ satisfies $\tilde{C}_1 \cap C_2 = \emptyset$. By (ii), $\mathcal{H}(t)\tilde{C}_1 \cap \mathcal{H}(t)C_2 = \emptyset$. But $\mathcal{H}(t)\tilde{C}_1 = \mathcal{H}(t)C_1 + \alpha\xi$. Thus

$$(\mathcal{H}(t)C_1 + \alpha\xi) \cap \mathcal{H}(t)C_2 = \emptyset$$

for each $|\xi| \leq 1$. Consequently $\operatorname{dist}(\mathcal{H}(t)C_1, \mathcal{H}(t)C_2) \geq \alpha$. This conclusion obtains for all $0 < \alpha < \operatorname{dist}(C_1, C_2)$, and so assertion (iv) follows. □

Now we extend the heat diffusion flow of sets $\{\mathcal{H}(t)\}_{t \geq 0}$ to a flow $\{H(t)\}_{t \geq 0}$ of functions. To accomplish this, take any function $f \in X$, and define for each $t > 0$ a new function $H(t)f$ by

$$[H(t)f](x) = \sup\{\lambda \in \mathbb{R} \mid x \in \mathcal{H}(t)[f \geq \lambda]\}. \tag{53}$$

We are now writing "$[f \geq \lambda]$" to denote the superlevel set $\{x \in \mathbb{R}^n \mid f(x) \geq \lambda\}$.

Lemma 3.2. *We also have*

$$[H(t)f](x) = \inf\{\lambda \in \mathbb{R} \mid x \in \mathcal{H}(t)[f \leq \lambda]\}. \tag{54}$$

Proof. Define $H(t)f$ by (53), fix $x \in \mathbb{R}^n$, and set $\mu = [H(t)f](x)$. Then, since Theorem 3.1, (i) implies

$$\mathcal{H}(t)[f \geq \lambda_1] \subseteq \mathcal{H}(t)[f \geq \lambda_2] \text{ if } \lambda_1 \geq \lambda_2,$$

we have $x \in \mathcal{H}(t)[f \geq \mu-\varepsilon]$, $x \notin \mathcal{H}(t)[f \geq \mu+\varepsilon]$ for each $\varepsilon > 0$. Since $[f \geq \mu-\varepsilon] \cap [f \leq \mu-2\varepsilon] = \emptyset$, Theorem 3.1, (ii) implies $x \notin \mathcal{H}(t)[f \leq \mu-2\varepsilon]$. Similarly, since $[f \geq \mu+\varepsilon] \cup [f \leq \mu+2\varepsilon] = \mathbb{R}^n$, Theorem 3.1, (iii) forces $x \in \mathcal{H}(t)[f \leq \mu+2\varepsilon]$. Thus

$$\mu = [H(t)f](x) \in \mathcal{H}(t)[f \leq \mu+2\varepsilon] - \mathcal{H}(t)[f \leq \mu-2\varepsilon]$$

for each $\varepsilon > 0$, and the Lemma follows. □

We next deduce from Theorem 3.1 corresponding properties of the flow $\{H(t)\}_{t\geq 0}$.

Theorem 3.2.

(i) *$H(t) : X \to X$ for each $t \geq 0$.*

(ii) *If $f \geq \hat{f}$, then $H(t)f \geq H(t)\hat{f}$.*

(iii) *Furthermore, $H(t)$ is a contraction on X:*

$$\|H(t)f - H(t)\hat{f}\| \leq \|f - \hat{f}\| \qquad (t > 0;\ f,\ \hat{f} \in X). \tag{55}$$

(iv) *In fact,*

$$\|(H(t)f - H(t)\hat{f})^{\pm}\| \leq \|(f - \hat{f})^{\pm}\| \qquad (t > 0;\ f,\ \hat{f} \in X). \tag{56}$$

Proof. 1. We first prove (iv). Fix $t > 0$, $f, \hat{f} \in X$, $\delta > 0$. We may assume

$$0 < \|(H(t)f - H(t)\hat{f})^{+}\|_{L^\infty} \leq [H(t)f](x_0) - [H(t)\hat{f}](x_0) + \delta,$$

for some point $x_0 \in Q$. Write

$$\mu = [H(t)f](x_0),\ \hat{\mu} = [H(t)\hat{f}](x_0).$$

Then for each $\varepsilon > 0$, $x_0 \in \mathcal{H}(t)[f \geq \mu-\varepsilon]$, according to (53) and Theorem 3.1, (i). Similarly $x_0 \in \mathcal{H}(t)[\hat{f} \leq \hat{\mu}+\varepsilon]$, according to (54). In view of Theorem 3.1, (ii), there must exist a point

$$x_1 \in [f \geq \mu-\varepsilon] \cap [\hat{f} \leq \hat{\mu}+\varepsilon].$$

Thus $f(x_1) \geq \mu - \varepsilon$, $\hat{f}(x_1) \leq \hat{\mu} + \varepsilon$, and accordingly

$$\begin{aligned} \|(H(t)f - H(t)\hat{f})^+\|_{L^\infty} &\leq \mu - \hat{\mu} + \delta \\ &\leq f(x_1) - \hat{f}(x_1) + 2\varepsilon + \delta \\ &\leq \|(f-\hat{f})^+\| + 2\varepsilon + \delta. \end{aligned}$$

Estimate (56) (for the + sign) follows, and the argument for the − sign is similar. Inequality (55) follows from (56).

2. Applying (55) to f and $\hat{f}(\,\cdot\,) = f(\cdot + y)$ ($y \in \mathbb{R}^n$), we deduce

$$\|[H(t)f](\,\cdot\,) - [H(t)f](\cdot + y)\|_{L^\infty} \leq \|f(\,\cdot\,) - f(\cdot + y)\| \tag{57}$$

for each $t > 0$, $f \in X$. In particular $H(t)f$ is Q-periodic. As the right hand side of (57) goes to zero as $|y| \to 0$, $H(t)f$ is continuous, and so belongs to X.

3. Assertion (ii) is an immediate consequence of (iv). □

4. Diffusing smooth sets. Next we refine our analysis of the heat diffusion flow and carefully study the effects of $\mathcal{H}(\,\cdot\,)$ upon smooth sets.

Assume now $C = \bar{U}$, where U is a smooth open subset of $\mathbb{R}$. Take $x_0 \in \partial U$ and let $\nu = \nu(x_0)$ denote the outward unit normal to U at x_0. Consider then the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = \chi_C & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{58}$$

For small times $t > 0$, the spatial gradient $Du = D_x u = (u_{x_1}, \ldots, u_{x_n})$ is nonvanishing near x_0, and so $\partial[\mathcal{H}(t)C]$ is a smooth hypersurface near x_0.

Select $v \in \mathbb{R}$ so that

$$x_0 + vt\nu \in \partial[\mathcal{H}(t)C]; \tag{59}$$

then vt is the normal distance from $x_0 \in \partial C$ to $\partial[\mathcal{H}(t)C]$.

Theorem 4.1. *We have*

$$v = H + O(t^{1/2}) \text{ as } t \to 0, \tag{60}$$

where H is ($(n-1)$ times) the mean curvature of ∂C at x_0, computed with respect to ν.

Consequently the normal velocity v of the boundary $\partial[\mathcal{H}(t)C]$ near x_0 is the mean curvature, plus a small correction.

Proof. 1. We may assume $x_0 = 0$, $\nu = e_n = (0,\dots,1)$, and boundary ∂C near 0 is the graph $\{x_n = \gamma(x'), x' \in \mathbb{R}^{n-1}\}$, for some smooth function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, $x' = (x_1,\dots,x_{n-1})$. We then have

$$\gamma(0) = 0,\ D\gamma(0) = 0,\ H = \Delta\gamma(0). \tag{61}$$

2. In view of (59),

$$\frac{1}{2} = u(0,\dots,0,vt),$$

and so

$$\frac{1}{2} = \frac{1}{(4\pi t)^{n/2}} \int_C e^{-(|x'|^2+|x_n-vt|^2)/4t} dx.$$

We may assume that within the unit cube $Q = \{|x_i| \leq 1,\ i = 1,\dots,n\}$ ∂C is represented by the graph of γ, and $|\gamma(x')| < 1$ if $|x'| \in Q'$, the unit cube in $\mathbb{R}^{n-1}$. Thus

$$\frac{1}{2} = \frac{1}{(4\pi t)^{n/2}} \left(\int_{Q'} e^{-|x'|^2/4t} \int_{-1}^{\gamma(x')} e^{-|x_n-vt|^2/4t} dx_n \right) dx' + \frac{1}{(4\pi t)^{n/2}} \int_{C-Q} e^{-(|x'|^2+|x_n-vt|^2)/4t} dx. \tag{62}$$

Changing variables by setting $y = t^{-1/2}x$ in the first term on the right hand side of (62), we compute

$$\frac{1}{2} = \frac{1}{(4\pi)^{n/2}} \int_{t^{-1/2}Q'} e^{-|y'|^2/4} \left(\int_{-t^{-1/2}}^{t^{-1/2}\gamma(t^{1/2}y')} e^{-|y_n-vt^{1/2}|^2/4} dy_n \right) dy' + O(e^{-\alpha/t}),$$

for some $\alpha > 0$. Now set $z' = y'$, $z_n = y_n - vt^{1/2}$:

$$\begin{aligned}\frac{1}{2} &= \frac{1}{(4\pi)^{n/2}} \int_{t^{-1/2}Q'} \left(\int_{-t^{-1/2}-vt^{1/2}}^{t^{-1/2}\gamma(t^{1/2}z')-vt^{1/2}} e^{-|z|^2/4} dz_n \right) dz' + O(e^{-\alpha t}) \\ &= \frac{1}{(4\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{t^{-1/2}[\gamma(t^{1/2}z')-tv]} e^{-|z|^2/4} dz_n \right) dz' + O(e^{-\alpha/t}).\end{aligned}$$

Since

$$\frac{1}{(4\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} e^{-|z|^2/4} dz = \frac{1}{2},$$

we deduce:

$$\int_{\mathbb{R}^{n-1}} \left(\int_0^{t^{-1/2}[\gamma(t^{1/2}z')-tv]} e^{-|z|^2/4} dz_n \right) dz' = O(e^{-\alpha/t}). \tag{63}$$

Next, using (61) we calculate:

$$\begin{aligned} &t^{-1/2}[\gamma(t^{1/2}z') - tv] \\ &= t^{-1/2}\left[\gamma(0) + t^{1/2}\gamma_{x_i}(0)z_i + \frac{t}{2}\gamma_{x_ix_j}(0)z_iz_j - tv + O(t^{3/2}|z'|^3)\right] \\ &= t^{1/2}[\frac{1}{2}\gamma_{x_ix_j}(0)z_iz_j - v] + O(t|z'|^3). \end{aligned}$$

Thus (63) implies

$$\int_{\mathbb{R}^{n-1}} e^{-|z'|^2/4} \left(\frac{1}{2}\gamma_{x_ix_j}(0)z_iz_j - v\right) dz' = O(t^{1/2}). \tag{64}$$

But

$$\int_{\mathbb{R}^{n-1}} z_iz_je^{-|z'|^2/4}dz' = 2\delta_{ij}\int_{\mathbb{R}^{n-1}} e^{-|z'|^2/4}dz',$$

and so (64) yields

$$v = \Delta\gamma(0) + O(t^{1/2}) = H + O(t^{1/2}),$$

as $t \to 0$. □

For use later in Section 5, we now recast Theorem 4.1, changing our attention from the evolution of a given set to the evolution of a family of sets. More precisely, assume $\varphi : \mathbb{R}^n \to \mathbb{R}$ is smooth, and

$$D\varphi(0) \neq 0. \tag{65}$$

We will consider the family of superlevel sets

$$C_\mu = \{\varphi \geq \mu\} \qquad (\mu \in \mathbb{R}). \tag{66}$$

In view of (65), ∂C_μ and so also $\partial[\mathcal{H}(t)C_\mu]$ are smooth hypersurfaces near 0, for small μ and $t > 0$. Fix $t > 0$ and then select $\mu = \mu(t)$ such that

$$0 \in \partial[\mathcal{H}(t)C_\mu]. \tag{67}$$

In other words, we choose that μ characterizing the level set of φ which is flowing through the point 0 at time t.

Theorem 4.2. *We have*

$$\mu = \varphi(0) + t\left(\delta_{ij} - \frac{\varphi_{x_i}(0)\varphi_{x_j}(0)}{|D\varphi(0)|^2}\right)\varphi_{x_ix_j}(0) + o(t) \tag{68}$$

as $t \to 0$.

Proof. 1. We introduce the one-parameter family of solutions to the heat equation

$$\begin{cases} u_t^\mu - \Delta u^\mu = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\mu = \chi_{C_\mu} & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

corresponding to the different superlevel sets C_μ of φ. According to (67),

$$\frac{1}{2} = u^\mu(0,t). \tag{69}$$

Our task is to solve for $\mu = \mu(t)$ in this expression.

2. We may as well assume

$$D\varphi(0) = \beta e_n \tag{70}$$

for some $\beta > 0$. Owing to the Implicit Function Theorem, near 0 the set $\{\varphi = \mu\}$ is the graph of a function $\gamma^\mu : \mathbb{R}^{n-1} \to \mathbb{R}$. Thus

$$\varphi(x', \gamma^\mu(x')) = \mu, \tag{71}$$

for small $|x'|$ and for μ near $\varphi(0)$. Differentiating gives

$$\begin{cases} \varphi_{x_i} + \varphi_{x_n}\gamma^\mu_{x_i} = 0 & (i = 1,\dots,n-1) \\ \varphi_{x_ix_j} + \varphi_{x_ix_n}\gamma^\mu_{x_j} + \varphi_{x_nx_j}\gamma^\mu_{x_i} + \\ \qquad \varphi_{x_nx_n}\gamma^\mu_{x_i}\gamma^\mu_{x_j} + \varphi_{x_n}\gamma^\mu_{x_ix_j} = 0 & (i,j = 1,\dots,n-1). \end{cases} \tag{72}$$

Now (69) implies

$$\begin{aligned} \frac{1}{2} &= \frac{1}{(4\pi t)^{n/2}}\int_{C_\mu} e^{-|x|^2/4t}\,dx \\ &= \frac{1}{(4\pi t)^{n/2}}\int_{Q'}\left(\int_{\gamma^\mu(x')}^1 e^{-|x|^2/4t}\,dx_n\right)dx' + \frac{1}{(4\pi t)^{n/2}}\int_{C_\mu - Q'} e^{-|x|^2/4t}\,dx \\ &= \frac{1}{(4\pi)^{n/2}}\int_{t^{-1/2}Q'}\left(\int_{t^{-1/2}\gamma^\mu(t^{1/2}y')}^{t^{-1/2}} e^{-|y|^2/4}\,dy_n\right)dy' + 0(e^{-\alpha/t}) \\ &= \frac{1}{(4\pi)^{n/2}}\int_{\mathbb{R}^{n-1}}\left(\int_{t^{-1/2}\gamma^\mu(t^{1/2}y')}^{\infty} e^{-|y|^2/4}\,dy_n\right)dy' + 0(e^{-\alpha/t}). \end{aligned}$$

Since

$$\frac{1}{(4\pi)^{n/2}}\int_{\mathbb{R}^{n-1}}\int_0^\infty e^{-|y|^2/4}dy = \frac{1}{2},$$

we compute

$$\int_{\mathbb{R}^{n-1}}\left(\int_0^{t^{-1/2}\gamma^\mu(t^{1/2}y')} e^{-|y|^2/4}dy_n\right)dy' = O(e^{-\alpha/t}).$$

Observe

$$t^{-1/2}\gamma^\mu(t^{1/2}y') = t^{-1/2}\left[\gamma^\mu(0)+t^{1/2}\gamma^\mu_{x_i}(0)y_i+\frac{t}{2}\gamma^\mu_{x_ix_j}(0)y_iy_j+O(t^{3/2}|y'|^3)\right].$$

Consequently,

$$\int_{\mathbb{R}^{n-1}} e^{-|y'|^2/4}\left[\gamma^\mu(0)+t^{1/2}\gamma^\mu_{x_i}(0)y_i+\frac{t}{2}\gamma^\mu_{x_ix_j}(0)y_iy_j\right]dy' = O(t^{3/2}).$$

Next we note

$$\int_{\mathbb{R}^{n-1}} y_ie^{-|y'|^2/4}dy' = 0 \qquad (i=1,\dots,n)$$

and

$$\int_{\mathbb{R}^{n-1}} y_iy_je^{-|y'|^2/4}dy' = 2\delta_{ij}\int_{\mathbb{R}^{n-1}} e^{-|y'|^2/4}dy'.$$

Thus

$$\begin{aligned}\gamma^\mu(0) &= -t\Delta\gamma^\mu(0)+O(t^{3/2})\\ &= \frac{t}{\beta}\Delta'\varphi(0)+o(t),\end{aligned}$$

according to (72), where Δ' denotes the Laplacian in the variables $x' = (x_1,\dots,x_{n-1})$. Consequently (71) implies

$$\begin{aligned}\mu = \varphi(0,\gamma^\mu(0)) &= \varphi(0)+\beta\gamma^\mu(0)+O(t^2)\\ &= \varphi(0)+t\Delta'\varphi+o(t)\\ &= \varphi(0)+t\left(\delta_{ij}-\frac{\varphi_{x_i}(0)\varphi_{x_j}(0)}{|D\varphi(0)|^2}\right)\varphi_{x_ix_j}(0)+o(t). \qquad \square\end{aligned}$$

Remark. If we replace 0 by any nearby point x_0, and choose μ so that

$$x_0\in\partial[\mathcal{H}(t)C_\mu],$$

then analogously

$$\mu = \varphi(x_0)+t\left(\delta_{ij}-\frac{\varphi_{x_i}(x_0)\varphi_{x_j}(x_0)}{|D\varphi(x_0)|^2}\right)\varphi_{x_ix_j}(x_0)+o(t),$$

where

$$\lim_{t\to 0+}\frac{o(t)}{t} = 0,$$

uniformly for x_0 near 0. $\square$

5. Convergence. Our main result is this:

Theorem 5.1. *Let $g \in X$ and define the family of nonlinear operators $\{H(t)\}_{t\geq 0}$ as in Section 3. Then*

$$\lim_{m\to\infty} H\left(\frac{t}{m}\right)^m g = u \text{ in } X, \tag{73}$$

uniformly for t in compact subsets of $[0,\infty)$, *where u is the unique weak solution of the mean curvature evolution* PDE (5).

Proof. 1. We verify the hypotheses (i) and (ii) of Theorem 2.2, the nonlinear Chernoff formula, with $\{H(t)\}_{t\geq 0}$ replacing $\{F(t)\}_{t\geq 0}$. Note first that each operator $H(t)$ is a contraction on X, according to Theorem 3.3, (iii). This establishes (i).

2. We must prove (ii); that is,

$$\lim_{t\to 0^+} \left(I + \lambda\left(\frac{I - H(t)}{t}\right)^{-1}\right) f \to (I+\lambda A)^{-1} f \text{ in } X. \tag{5.2}$$

for each $f \in X$, $\lambda > 0$. It is enough to take $\lambda = 1$. Write

$$u^t \equiv \left(I + \left(\frac{I - H(t)}{t}\right)^{-1}\right) f. \tag{5.3}$$

Then

$$u^t + \left(\frac{u^t - H(t)u^t}{t}\right) = f. \tag{5.4}$$

We also set

$$A^t u \equiv \frac{u - H(t)u}{t} \qquad (t > 0,\ u \in X). \tag{5.5}$$

We note $-A^t$ is m-dissipative on X (cf. [BP]) and so

$$\|u - \hat{u}\| \leq \|u - \hat{u} + (A^t u - A^t \hat{u})\| \tag{5.6}$$

for all $u, \hat{u} \in X$. Setting $u = u^t(\,\cdot\,)$, $\hat{u} = u^t(\cdot + y)$ for $y \in \mathbb{R}^n$, we deduce

$$\|u^t(\,\cdot\,) - u^t(\cdot + y)\| \leq \|f(\,\cdot\,) - f(\cdot + y)\|.$$

Thus

$$\{u^t\}_{0<t\leq 1} \text{ is bounded and equicontinuous on } Q. \tag{5.7}$$

Consequently there exists a sequence $t_k \to 0$ and a function $u \in X$ such that

$$u^{t_k} \to u \text{ in } X. \tag{5.8}$$

We must show $u = (I + A)^{-1} f$; that is,

$$u - \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}\right) u_{x_i x_j} = f \text{ in } Q \tag{5.9}$$

in the weak sense.

So fix any smooth test function $\varphi \in C^\infty(\mathbb{R}^n)$ and suppose first $u - \varphi$ has a *strict* positive maximum at a point $x_0 \in \mathbb{R}^n$. According to the definitions in Section 2 we must verify

$$u(x_0) - \left(\delta_{ij} - \frac{\varphi_{x_i}(x_0)\varphi_{x_j}(x_0)}{|D\varphi(x_0)|^2}\right) \varphi_{x_i x_j}(x_0) \le f(x_0) \tag{5.10}$$

if $D\varphi(x_0) \ne 0$, and

$$u(x_0) \le f(x_0) \tag{5.11}$$

if $D\varphi(x_0) = 0$, $D^2\varphi(x_0) = 0$. (We need not consider the possibility $D\varphi(x_0) = 0$, $D^2\varphi(x_0) \ne 0$.)

3. Assume first

$$D\varphi(x_0) \ne 0. \tag{5.12}$$

Now since $u^{t_k} \to u$ uniformly and $u - \varphi$ has a strict maximum at x_0, there exist points $x_k \to x_0$ such that

$$u^{t_k} - \varphi \text{ has a positive maximum at } x_k. \tag{5.13}$$

We next observe using Theorem 3.3, (iv) and (85) that

$$[H(t_k)u^{t_k}](x_k) - [H(t_k)\varphi](x_k) \le u^{t_k}(x_k) - \varphi(x_k).$$

Hence

$$A^{t_k}\varphi(x_k) \le A^{t_k} u^{t_k}(x_k),$$

and so

$$u(x_k) + \frac{\varphi(x_k) - [H(t_k)\varphi](x_k)}{t_k} \le f(x_k). \tag{5.14}$$

We must now let $x_k \to x_0$, $t_k \to 0$ in this inequality.

In view of (84) we have $D\varphi(x_k) \neq 0$ if k is large enough. According to Theorem 4.2 with t_k replacing t, x_k replacing 0, and $[H(t_k)\varphi](x_k)$ replacing μ, we have

$$\frac{\varphi(x_k) - [H(t_k)\varphi](x_k)}{t_k} = -\left(\delta_{ij} - \frac{\varphi_{x_i}(x_k)\varphi_{x_j}(x_k)}{|D\varphi(x_k)|^2}\right)\varphi_{x_ix_j}(x_k) + o(1) \text{ as } k \to \infty.$$

In view of the Remark at the end of Section 4, the error term $o(1)$ goes to zero, as k tends to infinity. Thus from (86) we conclude

$$u(x_0) - \left(\delta_{ij} - \frac{\varphi_{x_i}(x_0)\varphi_{x_j}(x_0)}{|D\varphi(x_0)|^2}\right)\varphi_{x_ix_j}(x_0) \leq f(x_0),$$

as required.

4. Next assume

$$D\varphi(x_0) = 0,\ D^2\varphi(x_0) = 0; \tag{5.15}$$

we must establish the inequality (83). Fix $\varepsilon > 0$ and define

$$\tilde{\varphi}(x) \equiv \varphi(x) + \frac{\varepsilon}{2}|x - x_0|^2. \tag{5.16}$$

Since $u - \varphi$ has a strict positive maximum at x_0, so also $u - \tilde{\varphi}$ has a strict positive maximum at x_0. As before, there consequently exist points $x_k \to x_0$, such that $u^{t_k} - \tilde{\varphi}$ has a maximum at x_k and

$$u(x_k) + \frac{\tilde{\varphi}(x_k) - [H(t_k)\tilde{\varphi}](x_k)}{t_k} \leq f(x_k). \tag{5.17}$$

We consider now two possibilities.

CASE 1: $x_k \neq x_0$ for all k large enough.

In this situation we consider the family of solutions u^μ of the PDE

$$\begin{cases} u_t^\mu - \Delta u^\mu = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\mu = \chi_{C_\mu} & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{5.18}$$

where $C_\mu = \{\tilde{\varphi} \geq \mu\}$.

Choose $\mu = \mu_k$ so that

$$\frac{1}{2} = u^{\mu_k}(x_k, t_k) = \frac{1}{(4\pi t_k)^{n/2}} \int_{C_{\mu_k}} e^{-|x_k - x|^2/4t_k}\, dx. \tag{5.19}$$

We rescale by setting

$$\sigma_k \equiv |x_k - x_0|, \quad y_k = \frac{x_k - x_0}{\sigma_k}, \quad y \equiv \frac{x - x_0}{\sigma_k}. \tag{5.20}$$

Upon rotating coordinates if needs be, we may also suppose

$$y_k = e_n \qquad (k = 1, \dots). \tag{5.21}$$

Utilizing (92), (93) we transform the equality (91) to read

$$\frac{1}{2} = \frac{1}{(4\pi\tau_k)^{n/2}} \int_{C_k} e^{-|y-e_n|^2/4\tau_k} dy, \tag{5.22}$$

for

$$\tau_k \equiv t_k/\sigma_k^2 \tag{5.23}$$

and

$$C_k \equiv \{y \in \mathbb{R}^n \mid \tilde{\varphi}(\sigma_k y + x_0) \geq \mu_k\}. \tag{5.24}$$

Write

$$\tilde{\varphi}_k(y) \equiv \frac{\tilde{\varphi}(\sigma_k y + x_0) - \tilde{\varphi}(x_0)}{\varepsilon \sigma_k^2} \tag{5.25}$$

and

$$\tilde{\mu}_k \equiv \frac{\mu_k - \tilde{\varphi}(x_0)}{\varepsilon \sigma_k^2}; \tag{5.26}$$

so that (96) becomes

$$C_k = \{y \in \mathbb{R}^n \mid \tilde{\varphi}_k(y) \geq \tilde{\mu}_k\}. \tag{5.30}$$

Now according to (87), (88) and (97) we have

$$\tilde{\varphi}_k(0) = 0, \ D\tilde{\varphi}_k(0) = 0,$$

and

$$D^2\tilde{\varphi}_k(y) = I + o(1) \text{ as } k \to \infty, \text{ uniformly on compact sets.}$$

In particular

$$D\tilde{\varphi}_k(e_n) = e_n + o(1) \text{ as } k \to \infty.$$

We are accordingly in the setting of Theorem 4.2, which we invoke to deduce

$$\tilde{\mu}_k = \tilde{\varphi}_k(e_n) + \tau_k \left(\delta_{ij} - \frac{\tilde{\varphi}_{k,x_i}(e_n)\tilde{\varphi}_{k,x_j}(e_n)}{|D\tilde{\varphi}_k(e_n)|^2} \right) \tilde{\varphi}_{k,x_i x_j}(e_n) + O(\tau_k).$$

Utilizing (97), (98) we can rewrite the preceeding to obtain

$$\begin{aligned}\mu_k &= \tilde{\varphi}(\sigma_k e_n + x_0) + O(\tau_k \varepsilon \sigma_k^2) \\ &= \tilde{\varphi}(x_k) + O(\varepsilon t_k).\end{aligned}$$

Since

$$[H(t_k)\tilde{\varphi}](x_k) = \mu_k$$

according to (90), (91), we have

$$\frac{\tilde{\varphi}(x_k) - [H(t_k)\tilde{\varphi}](x_k)}{t_k} = O(\varepsilon) \text{ as } k \to \infty.$$

Thus if Case 1 obtains, we conclude from (89) that

$$u(x_0) \le f(x_0) + O(\varepsilon).$$

This inequality is valid for each $\varepsilon > 0$, whence (83) holds.

CASE 2: $x_{k_j} = x_0$ for some subsequence $k_j \to \infty$.

In this situation, we may as well reindex to obtain $x_k = x_0$ for all $k = 1, \ldots$. Recalling (90), we select $\mu = \mu_k$ so that

$$\frac{1}{2} = u^{\mu_k}(x_0, t_k) = \frac{1}{(4\pi t_k)^{n/2}} \int_{C_{\mu_k}} e^{-|x_0 - x|^2/4t_k} dx. \tag{5.31}$$

Now

$$\begin{aligned}C_{\mu_k} &= \{x \in \mathbb{R}^n \mid \tilde{\varphi} \ge \mu_k\} \\ &= \left\{x \in \mathbb{R}^n \mid \varphi(x) + \frac{\varepsilon}{2}|x - x_0|^2 \ge \varphi(x_0) + \lambda_k\right\},\end{aligned}$$

for

$$\lambda_k = \mu_k - \varphi(x_0).$$

As $D\varphi(x_0) = 0$, $D^2\varphi(x_0) = 0$, we have

$$C_{\mu_k} \subseteq \{x \in \mathbb{R}^n \mid \varepsilon|x - x_0|^2 \ge \lambda_k\}.$$

Thus (100) implies

$$\begin{aligned}\frac{1}{2} &\le \frac{1}{(4\pi t_k)^{n/2}} \int_{\mathbb{R}^n - B(x_0, (\lambda_k/\varepsilon)^{1/2})} e^{-|x_0 - x|^2/4t_k} dx \\ &= \frac{1}{(4\pi)^{n/2}} \int_{\mathbb{R}^n - B(0, (\lambda_k/\varepsilon t_k)^{1/2})} e^{-|y|^2/4} dy \\ &\le \frac{1}{4},\end{aligned} \tag{5.32}$$

if

$$\frac{\lambda_k}{\varepsilon t_k} \geq M \tag{5.33}$$

for some constant M. As (101), (102) entail a contradiction, we conclude

$$\lambda_k = \mu_k - \varphi(x_0) \leq M\varepsilon t_k. \tag{5.34}$$

Since $[H(t_k)\tilde{\varphi}](x_k) = [H(t_k)\tilde{\varphi}](x_0) = \mu_k$, we deduce

$$\frac{\tilde{\varphi}(x_k) - [H(t_k)\tilde{\varphi}](x_k)}{t_k} = O(\varepsilon) \text{ as } k \to \infty.$$

Thus (89) implies

$$u(x_0) \leq f(x_0) + O(\varepsilon)$$

for each $\varepsilon > 0$, and so inequality (83) is valid.

The opposite inequalities similarly hold if $u - \varphi$ has a minimum at x_0. □

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