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Calculus of Variations

Uniform estimates and limiting arguments for nonlocal minimal surfaces

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Abstract We consider nonlocal minimal surfaces obtained by a fractional type energy functional, parameterized by $s \in (0, 1)$. We show that the s-energy approaches the perimeter as $s \to 1^-$. We also provide density properties and clean ball conditions, which are uniform as $s \to 1^-$, and optimal lower bounds obtained by a rearrangement result. Then, we show that s-minimal sets approach sets of minimal perimeter as $s \to 1^-$.

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1 Introduction

In [3] the following problem arising in geometric analysis was analyzed.

Given $n \geqslant 2$ and $s \in (0, 1)$, a bounded open set $\Omega \subset \mathbb{R}^n$ and a set $E \subseteq \mathbb{R}^n$, let

$$\mathcal{J}_s(E, \Omega) := \int_{E\cap\Omega} \int_{(\complement E)\cap\Omega} \frac{1}{|x-y|^{n+s}}\, dy\, dx + \int_{E\cap\Omega} \int_{(\complement E)\cap(\complement\Omega)} \frac{1}{|x-y|^{n+s}}\, dy\, dx + \int_{E\cap(\complement\Omega)} \int_{(\complement E)\cap\Omega} \frac{1}{|x-y|^{n+s}}\, dy\, dx.$$

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We say that E is s-minimal in Ω if for any $\tilde{E} \subseteq \mathbb{R}^n$ for which $\tilde{E} \cap (\mathcal{C}\Omega) = E \cap (\mathcal{C}\Omega)$ one has that

$$\mathcal{J}_s(E, \Omega) \leqslant \mathcal{J}_s(\tilde{E}, \Omega).$$

That is, E minimizes the functional among competitors which agree outside Ω.

We also say that E is s-minimal if it is s-minimal in any bounded and open $\Omega \subset \mathbb{R}^n$: in this sense, s-minimal sets may be also motivated by the minimization of fractional norms of Gagliardo type. Also, nonlocal minimal surfaces naturally arise as interfaces of phase transition problems with longrange interactions, see [6].

The main difficulty in the framework of s-minimal sets comes from the nonlocality of the contributions in the functional. The counterpart of this difficulty, however, is given by the fact that, in order to minimize the nonlocal minimal area, it is enough to deal with fractional norms of step functions, i.e. the functional setting simply involves Hilbert norms and measurable sets—in particular, there is no need to introduce Caccioppoli sets when minimizing nonlocal area and the existence result are usually straightforward.

The purpose of this paper is to study the limits as $s \to 1^-$ and to relate such limit with the theory of classical minimal surfaces (i.e., sets with minimal perimeter, see [4,5]).

First, we obtain an expansion of $\mathcal{J}_s$ as $s \to 1^-$, for which it is useful to introduce the following renormalization constant: for any $n \in \mathbb{N}$, $s \in (0, 1]$ and $\tau \in [0, +\infty]$, we define

$$\nu(n, s, \tau) := 2\left(1 - \frac{1}{2^s}\right)(n-1)\,\omega_{n-1}\int_0^\tau \frac{t^{n-2}\,dt}{(1+t^2)^{(n+s)/2}}\,.$$

Then, we have that the s-energy approaches the perimeter as $s \to 1^-$, as the following result states:

Theorem 1 *Let $R > 0$ and let $E \subset \mathbb{R}^n$.*

Suppose that $(\partial E) \cap B_R$ is $C^{1,\alpha}$, for some $\alpha \in (0, 1)$.

Then, for any $r \in (0, R)$,

$$\lim_{s \to 1^-}\left|\frac{s(1-s)}{\nu(n, s, +\infty)}\mathcal{J}_s(E, B_r) - \operatorname{Per}(E, B_r)\right| \leqslant 2\mathcal{H}^{n-1}(\partial E \cap \partial B_r). \tag{1.1}$$

In particular, there exists a set $\mathcal{R} \subseteq (0, R)$, which is dense in $(0, R)$, such that

$$\lim_{s \to 1^-}(1-s)\,\mathcal{J}_s(E, B_r) = \nu(n, 1, +\infty)\operatorname{Per}(E, B_r) \quad \textit{for any } r \in \mathcal{R}. \tag{1.2}$$

Dealing with the limit $s \to 1^-$, it is quite important to have geometric estimates that are independent of s. In particular, we obtain a uniform clean ball condition and uniform bounds on the $(n-1)$-dimensional Hausdorff measure and on the s-energy, as the following result states:

Theorem 2 *Let $s \in (1/2, 1)$, $r > 0$ and E be an s-minimal set, with $B_{r/2} \cap \partial E \neq \emptyset$. Then, there exist $x_1, x_2 \in \mathbb{R}^n$ such that $B_{cr}(x_1) \subseteq E \cap B_r$ and $B_{cr}(x_2) \subseteq (\mathcal{C}E) \cap B_r$,*

$$\mathcal{H}^{n-1}(\partial E \cap B_r) \geqslant cr^{n-1}$$

and

$$\frac{s(1-s)}{\nu(n, s, +\infty)}\int_{E\cap B_r}\int_{(\mathcal{C}E)\cap B_r}\frac{1}{|x-y|^{n+s}}\,dx\,dy \geqslant cr^{n-s},$$

for a suitable $c \in (0, 1)$, independent of s.

With the above estimate, we obtain a uniform perimeter density of the limit sets:

Theorem 3 *Let $s_k \in (0,1)$ be such that $s_k \to 1^-$ when $k \to +\infty$. Let E_k be a sequence of s_k-minimal sets converging to a set E locally uniformly*[1] *as $k \to +\infty$. Suppose $0 \in \partial E$. Then, for any $r > 0$, there exists $a, b \in \mathbb{R}^n$ such that*

$$B_{cr}(a) \subseteq E \cap B_r \text{ and } B_{cr}(b) \subseteq (\mathcal{C}E) \cap B_r \tag{1.3}$$

and also

$$\mathcal{H}^{n-1}(\partial E \cap B_r) \in \left[cr^{n-1},\, Cr^{n-1}\right], \tag{1.4}$$

for suitable $C > c > 0$.

Now, we state a useful rearrangement result—namely, if a set is trapped in a strip its s-energy is larger than the one of the cylinder:

Theorem 4 *Let $a_- \leqslant a_+ \in \mathbb{R}$, U be a bounded open subset of $\mathbb{R}^{n-1}$ and $K := U \times \mathbb{R}$. Let $A \subset \mathbb{R}^n$ be such that*

$$\{x_n < a_-\} \cap K \subseteq A \cap K \subseteq \{x_n < a_+\} \cap K. \tag{1.5}$$

Then

$$\int_{K\cap A}\int_{K\cap\mathcal{C}A} \frac{1}{|x-y|^{n+s}}\,dx\,dy \geqslant \int_{K\cap\{x_n<0\}}\int_{K\cap\{x_n\geqslant 0\}} \frac{1}{|x-y|^{n+s}}\,dx\,dy.$$

As a consequence of Theorem 4, we obtain a lower bound on the energy of s-minimizers:

Theorem 5 *Let $s_k \in (0,1)$ be such that $s_k \to 1^-$ when $k \to +\infty$. Let E_k be a sequence of s_k-minimal sets converging to a set E uniformly in B_R as $k \to +\infty$.*

Then,

$$\lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n,1,+\infty)} \int_{E_k\cap B_R}\int_{(\mathcal{C}E_k)\cap B_R} \frac{1}{|x-y|^{n+s}}\,dx\,dy \geqslant \mathrm{Per}\,(E, B_R).$$

The results above have the interesting consequence that s-minimizers approach, as $s \to 1^-$, classical minimal surfaces, as next result points out:

Theorem 6 *Let $R, \beta > 0$.*
Let E_k be a sequence of s_k-minimal sets in $B_{(1+\beta)R}$ converging to a set E uniformly in $B_{(1+\beta)R}$.

Then, E is a set of minimal perimeter in B_R.

Finally, it is worth to remark that s_k-minimizers are precompact when $s_k \to 1^-$, according to the following result:

Theorem 7 *Fix $R > 0$. Let $s_k \in (0,1)$, with*

$$\lim_{k\to+\infty} s_k = 1.$$

Let E_k be an s_k-minimal set. Then, the sequence $\{\chi_{E_k}\}_{k\in\mathbb{N}}$ is precompact in $L^1(B_R)$.

[1] I.e., for any ball B and any $\varepsilon > 0$ there exists $k(B,\varepsilon) > 0$ such that if $k \geqslant k(B,\varepsilon)$ then $F_k \cap B$ (resp., $(\mathcal{C}F_k) \cap B$) lies in an ε-neighborhood of F (resp., $\mathcal{C}F$).

2 Notation

Following is the basic notation we use:

- $x = (x', x_n)$ denotes a point in $\mathbb{R}^n$, with $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$ and $n \geqslant 2$,
- $B_r(p)$ denotes the standard Euclidean open ball of radius r centered at p, and $B_r := B_r(0)$,
- we also write $B^n_r(p)$ if we want to emphasize that we are dealing with the n-dimensional ball $B_r(p) \subset \mathbb{R}^n$,
- the standard Euclidean basis of $\mathbb{R}^n$ is denoted by $\{e_1, \dots, e_n\}$,
- if $\xi \in \mathrm{S}^{n-1}$, we denote $\tau_\xi x := x - (x \cdot \xi)\xi$; notice that $(x', 0) = \tau_{e_n} x$,
- for any $\Omega \subseteq \mathbb{R}^n$, which is Lebesgue measurable, we denote by $|\Omega|$ its Lebesgue measure (in fact, in the rest of the paper, all sets will be always implicitly assumed to be Lebesgue measurable),
- we set $\omega_n := |B_1|$,
- given a set $\Omega \subseteq \mathbb{R}^n$, we denote by $\mathcal{C}\Omega$ its complement, i.e. $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$,
- the n-dimensional Hausdorff measure will be denoted by $\mathcal{H}^n$,
- the perimeter of a Caccioppoli set E in an open set Ω will be denoted by $\operatorname{Per}(E, \Omega)$ and we refer to [4,5] for further details about Caccioppoli sets,
- all the constants in the computations are allowed to depend on the dimension n, and such dependence will be omitted,
- if Ω and $\Omega^\star$ are bounded open subsets of $\mathbb{R}^n$, with $\overline{\Omega} \subset \Omega^\star$, and $E \subseteq \mathbb{R}^n$ is such that $(\partial E) \cap \Omega^\star$ is locally $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, we have that, by compactness, $(\partial E) \cap \Omega$ may be covered by a finite number of balls $B^{(1)}, \dots, B^{(\ell)}$ such that $(\partial E) \cap B^{(j)}$ is the graph of a $C^{1,\alpha}$-function, say $\Psi^{(j)}$, for $j = 1, \dots, \ell$. If a quantity C depends on $\|\Psi^{(1)}\|_{C^{1,\alpha}}$, ..., $\|\Psi^{(\ell)}\|_{C^{1,\alpha}}$ we say that it depends on the $C^{1,\alpha}$-size of $(\partial E) \cap \Omega$. We remark that there is a slight abuse of terminology here, since the balls $B^{(j)}$ and the functions $\Psi^{(j)}$ are not uniquely determined.

3 Preliminaries on Caccioppoli sets

Given a Caccioppoli set E, we denote its reduced boundary by $\partial^\star E$ (see Definition 3.3 of [4]).

For any point $x \in \partial^\star E$, its affine tangent hyperplane will be denoted by $T(x)$ and $T^+(x)$ and $T^-(x)$ represent the upper and lower half space in the direction of $\nu(x)$ (see Definition 3.6 of [4]). For any $x \in \partial^\star E$, $\rho > 0$ and $\epsilon \in (0, 1)$, we consider the slab around $T(x) \cap B_\rho(x)$ of size $\epsilon\rho$, namely

$$T_{\rho,\epsilon}(x) := \left\{ y \in B_\rho(x) \text{ s.t. } |\nu(x) \cdot y| \leqslant \epsilon\rho \right\}.$$

With this notation, we have that these tangent hyperplanes may be shadowed with arbitrarily flat cylinders, according to the following result:

Lemma 8 *Fix $\epsilon \in (0, 1)$. Let $E \subset \mathbb{R}^n$ Suppose that there exists $c \in (0, 1)$ for which, for any $x \in \partial E$ there exists a, $b \in \mathbb{R}^n$ such that*

$$B_{cr}(a) \subseteq E \cap B_r(x) \text{ and } B_{cr}(b) \subseteq (\mathcal{C}E) \cap B_r(x) \tag{3.1}$$

for any $r \in (0, c)$.

Then, for any $\eta > 0$, there exist $r_\eta > 0$ and a collection of balls $\left\{B_{\rho_j}(a_j)\right\}_{j \in \mathbb{N}}$ such that

$$\sum_{j=0}^{+\infty} \rho_j^{n-1} \leqslant \eta \tag{3.2}$$

and if $x \in (\partial E) \setminus \bigcup_{j=1}^{+\infty} B_{\rho_j}(a_j)$*, we have that* $x \in \partial^\star E$ *and*

$$\partial E \cap B_r(x) \subseteq T_{r,\epsilon}(x)$$

for any $r \in (0, r_\eta]$.

Remark The reason for looking at condition (3.1) in this paper comes from the work in [3]. Indeed, it is shown in [3] that the s-minimizers satisfy the clean ball condition (3.1)—and it holds uniformly in s for $s \in (s_o, 1)$, for any fixed $s_o \in (0, 1)$, see Appendix A. Thus, so does any limiting set as $s \to 1^-$. In our setting, Lemma 8 thanks to Theorem 3.

Proof of Lemma 8 By (3.1) and the Isoperimetric Inequality, if $x \in \partial E$,

$$\mathcal{H}^{n-1}(\partial E \cap B_r(x)) \geqslant \bar{c} r^{n-1}, \tag{3.3}$$

for a suitable $\bar{c} > 0$. Hence, repeating verbatim the proof of Theorem 8.5 in [4] (but using (3.3) here instead of (5.16) of [4] and changing the constants after that), we see that

$$\mathcal{H}^{n-1}(\partial E \setminus \partial^\star E) = 0. \tag{3.4}$$

Now, we fix $\epsilon \in (0, 1)$ as in the statement of Lemma 8 and, for any $j \in \mathbb{N}$, $j \geqslant 1$, we define

$$\mathcal{N}_j := \left\{ x \in \partial^\star E \text{ s.t. } \exists r \in (0,\ 1/j] \text{ s.t. } \partial E \cap B_r(x) \not\subseteq T_{r,\epsilon}(x) \right\}.$$

Then, $\mathcal{N}_{j+1} \subseteq \mathcal{N}_j$ and therefore, by the Monotone Convergence Theorem,

$$\lim_{j\to+\infty} \mathcal{H}^{n-1}(\mathcal{N}_j) = \mathcal{H}^{n-1}\left(\bigcap_{j\in\mathbb{N}} \mathcal{N}_j \right). \tag{3.5}$$

We claim that

$$\bigcap_{j\in\mathbb{N}} \mathcal{N}_j = \emptyset. \tag{3.6}$$

Indeed, if there existed x in the above set, we would have that, for any $j \in \mathbb{N}$ there exists some $r_j \in (0,\ 1/j]$ for which

$$\partial E \cap B_{r_j}(x) \not\subseteq T_{r_j,\epsilon}(x).$$

That is, there exists

$$x_j \in \big(\partial E \cap B_{r_j}(x)\big) \setminus T_{r_j,\epsilon}(x).$$

Accordingly $B_{\epsilon r_j}(x_j)$ is all contained in either $T_+(x)$ or $T_-(x)$. Say, in $T_-(x)$, for definiteness.

By (3.1), there exists $y_j \in \mathbb{R}^n$ such that $B_{c\epsilon r_j}(y_j) \subseteq E \cap B_{\epsilon r_j}(x_j)$. By construction,

$$B_{c\epsilon r_j}(y_j) \subseteq E \cap B_{2r_j}(x) \cap T_-(x)$$

and therefore

$$\lim_{j\to+\infty} (2r_j)^{-n} \left| E \cap B_{2r_j} \cap T_-(x) \right| \geqslant \lim_{j\to+\infty} (2r_j)^{-n} \left| B_{c\epsilon r_j}(y_j) \right| > 0.$$

This is in contradiction with (3.18) in [4] and therefore (3.6) is proved.

Hence, from (3.5) and (3.6),

$$\lim_{j\to+\infty} \mathcal{H}^{n-1}(\mathcal{N}_j) = 0.$$

As a consequence of this, recalling (3.4), we conclude that, given any $\eta > 0$, there exists $J_\eta > 0$ such that we can cover $(\partial E \setminus \partial^\star E) \cup \mathcal{N}_{J_\eta}$ with a collection of balls $\left\{B_{\rho_j}(a_j)\right\}_{j\in\mathbb{N}}$, such that (3.2) holds.

Then, if x is outside these balls, we have that

$$x \in \partial^\star E \cap (\mathcal{C}\mathcal{N}_{J_\eta})$$

and so, in particular,

$$\partial E \cap B_r(x) \subseteq T_{r,\epsilon}(x)$$

for any $r \in (0, 1/J_\eta]$. □

4 Integral computations

The main difficulty given by the nonlocal minimal surfaces is that the integrand in the functional defining them diverges at the point. In addition, it takes into account the contribution coming from the whole space. Due to these technical complications, we need some careful estimates on the integrals, that we now perform. Namely, in the forthcoming results we estimate, first, the integral contribution of flat cylinders (see Lemma 9), and then, of smooth graphs (see Lemma 10). This will allow us to estimate the contribution to the total energy of the part coming from a ball where the transition surface is smooth (see Lemma 11).

Lemma 9 *Let* $\theta, \delta > 0$. *Let*

$$\begin{aligned} K_{\delta,\theta} &:= \left\{x \in \mathbb{R}^n \text{ s.t. } |x'| \leqslant \delta,\ |x_n| \leqslant \theta\right\}, \\ K^+_{\delta,\theta} &:= K_{\delta,\theta} \cap \{x_n > 0\}, \\ K^-_{\delta,\theta} &:= K_{\delta,\theta} \cap \{x_n < 0\}, \\ K^o_{\delta,\theta} &:= K_{\delta,\theta} \cap \{x_n = 0\} = \{|x'| \leqslant \delta\} \times \{0\} \end{aligned} \tag{4.1}$$

and

$$\mathcal{E}_{\delta,\theta,s} := \frac{s\,(1-s)}{|K^o_{\delta,\theta}|\,\theta^{1-s}} \int_{K^+_{\delta,\theta}} \int_{K^-_{\delta,\theta}} \frac{1}{|x-y|^{n+s}}\,dy\,dx.$$

Then, for any $\beta \in (0, 1)$,

$$\mathcal{E}_{\delta,\theta,s} \in \left[(1-\beta)^{n-1}\nu(n, s, \beta\delta/(2\theta)),\ \nu(n, s, +\infty)\right]. \tag{4.2}$$

Proof For short, we write $K := K_{\delta,\theta}$, $K^+ := K^+_{\delta,\theta}$, $K^- := K^-_{\delta,\theta}$ and $K^o := K^o_{\delta,\theta}$, unless differently specified.

Also, fixed any $x \in K^+$, we define

$$I(x) := \int_{K^-} \frac{1}{|x-y|^{n+s}}\,dy.$$

We have, with the change of variable $z := (y' - x')/|x_n - y_n|$,

$$I(x) = \int_{K^-} \frac{1}{\left(|x' - y'|^2 + |x_n - y_n|^2\right)^{(n+s)/2}}\, dy$$
$$= \int_{-\theta}^{0} \int_{z \in B^{n-1}_{\delta/|x_n - y_n|}(x'/|x_n - y_n|)} \frac{1}{|x_n - y_n|^{s+1}\, \left(1 + |z|^2\right)^{(n+s)/2}}\, dz\, dy_n. \tag{4.3}$$

Now, for any $x \in K^+$,

$$\int_{z \in B^{n-1}_{\delta/|x_n - y_n|}(x'/|x_n - y_n|)} \frac{1}{\left(1 + |z|^2\right)^{(n+s)/2}}\, dz \leqslant \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1 + |z|^2\right)^{(n+s)/2}}\, dz$$
$$= (n-1)\,\omega_{n-1} \int_0^{\infty} \frac{\rho^{n-2}\, d\rho}{(1 + \rho^2)^{(n+s)/2}} = \frac{\nu(n, s, +\infty)}{2 - 2^{1-s}}. \tag{4.4}$$

On the other hand, if $|x'| \leqslant (1 - \beta)\delta$, we have that

$$B^{n-1}_{\beta\delta/(2\theta)} \subseteq B^{n-1}_{\delta/|x_n - y_n|}(x'/|x_n - y_n|)$$

and so

$$\int_{z \in B^{n-1}_{\delta/|x_n - y_n|}(x'/|x_n - y_n|)} \frac{1}{\left(1 + |z|^2\right)^{(n+s)/2}}\, dz \geqslant \int_{z \in B^{n-1}_{\beta\delta/(2\theta)}} \frac{1}{\left(1 + |z|^2\right)^{(n+s)/2}}\, dz$$
$$= (n-1)\,\omega_{n-1} \int_0^{\beta\delta/(2\theta)} \frac{\rho^{n-2}\, d\rho}{(1 + \rho^2)^{(n+s)/2}} = \frac{\nu(n, s, \beta\delta/(2\theta))}{2 - 2^{1-s}}. \tag{4.5}$$

Moreover, for any $x \in K^+$,

$$\int_{-\theta}^{0} \frac{dy_n}{|x_n - y_n|^{s+1}} = \int_{-\theta}^{0} (x_n - y_n)^{-s-1}\, dy_n = \frac{(x_n)^{-s} - (x_n + \theta)^{-s}}{s},$$

and so (4.3) and (4.4) give that

$$I(x) \leqslant \frac{\nu(n, s, +\infty)}{(2 - 2^{1-s})\, s} \left[(x_n)^{-s} - (x_n + \theta)^{-s}\right] \quad \text{for any } |x'| \leqslant \delta, \tag{4.6}$$

while, using (4.5),

$$I(x) \geqslant \frac{\nu(n, s, \beta\delta/(2\theta))}{(2 - 2^{1-s})\, s} \left[(x_n)^{-s} - (x_n + \theta)^{-s}\right] \quad \text{for any } |x'| \leqslant (1 - \beta)\delta. \tag{4.7}$$

Furthermore, for any $\delta, \theta > 0$,

$$\int_{K^+_{\delta,\theta}} (x_n)^{-s} - (x_n + \theta)^{-s}\, dx = |K^o_{\delta,\theta}| \int_0^{\theta} (x_n)^{-s} - (x_n + \theta)^{-s}\, dx_n = |K^o_{\delta,\theta}|\, \frac{2 - 2^{1-s}}{1 - s}\, \theta^{1-s}. \tag{4.8}$$

We also notice that

$$\mathcal{E}_{\delta,\theta,s} = \frac{s(1-s)}{|K^o_{\delta,\theta}|\,\theta^{1-s}} \int\limits_{K^+_{\delta,\theta}} I(x)\,dx.$$

So, we obtain the upper bound in (4.2) directly from (4.6) and (4.8), while, using (4.7) and (4.8),

$$\begin{aligned}
\mathcal{E}_{\delta,\theta,s} &\geqslant \frac{s(1-s)}{|K^o_{\delta,\theta}|\,\theta^{1-s}} \int\limits_{K^+_{(1-\beta)\delta,\theta}} I(x)\,dx \\
&\geqslant \frac{(1-s)\,\nu\,(n,s,\beta\delta/(2\theta))}{|K^o_{\delta,\theta}|\,\theta^{1-s}\,(2-2^{1-s})} \int\limits_{K^+_{(1-\beta)\delta,\theta}} (x_n)^{-s} - (x_n+\theta)^{-s}\,dx \\
&\geqslant \frac{\nu\,(n,s,\beta\delta/(2\theta))\ |K^o_{(1-\beta)\delta,\theta}|}{|K^o_{\delta,\theta}|} \\
&= (1-\beta)^{n-1}\,\nu\,(n,s,\beta\delta/(2\theta))\,,
\end{aligned}$$

which is the lower bound in (4.2). □

Now, from the above integral bounds for flat cylinders, we obtain a bound for smooth graphs:

Lemma 10 *Let θ, δ and $K^o_{\delta,\theta}$ be as in Lemma 9.*

Let $\phi \in C^{1,\alpha}\left(K^o_{\delta,\theta}\right)$, for some $\alpha \in (0,1)$, with $\phi(0)=0$ and $\nabla\phi(0)=0$.

Let

$$\eta_{\delta,\theta}(\phi) := \frac{\|\phi\|_{C^{1,\alpha}(K^o_{\delta,\theta})}\,\delta^{1+\alpha}}{\theta} + \|\phi\|_{C^{1,\alpha}(K^o_{\delta,\theta})}\,\delta^{\alpha}. \tag{4.9}$$

Also, let

$$\begin{aligned}
K^+_{\delta,\theta}(\phi) &:= \left\{x\in\mathbb{R}^n \text{ s.t. } |x'|\leqslant\delta,\ x_n\in(\phi(x'),\theta]\right\}, \\
K^-_{\delta,\theta}(\phi) &:= \left\{x\in\mathbb{R}^n \text{ s.t. } |x'|\leqslant\delta,\ x_n\in[-\theta,\phi(x'),)\right\}
\end{aligned} \tag{4.10}$$

and

$$\mathcal{E}_{\delta,\theta,s}(\phi) := \frac{s\,(1-s)}{|K^o_{\delta,\theta}|\,\theta^{1-s}} \int\limits_{K^+_{\delta,\theta}(\phi)} \int\limits_{K^-_{\delta,\theta}(\phi)} \frac{1}{|x-y|^{n+s}}\,dy\,dx. \tag{4.11}$$

Then, there exists a constant $c_o \in (0,1)$, such that the following statement holds: if

$$\eta_{\delta,\theta}(\phi) \leqslant c_o \tag{4.12}$$

then, for any $\beta \in (0,1)$,

$$\begin{aligned}
\mathcal{E}_{\delta,\theta,s}(\phi) \in \Big[&\big(1-\ \mathrm{const}\,\eta_{\delta,\theta}(\phi)\big)\,(1-\beta)^{n-1}\nu(n,s,\beta\delta/(2\theta)), \\
&\times\big(1+\ \mathrm{const}\,\eta_{\delta,\theta}(\phi)\big)\,\nu(n,s,+\infty)\Big].
\end{aligned}$$

The above "const" is independent of θ, δ and β.

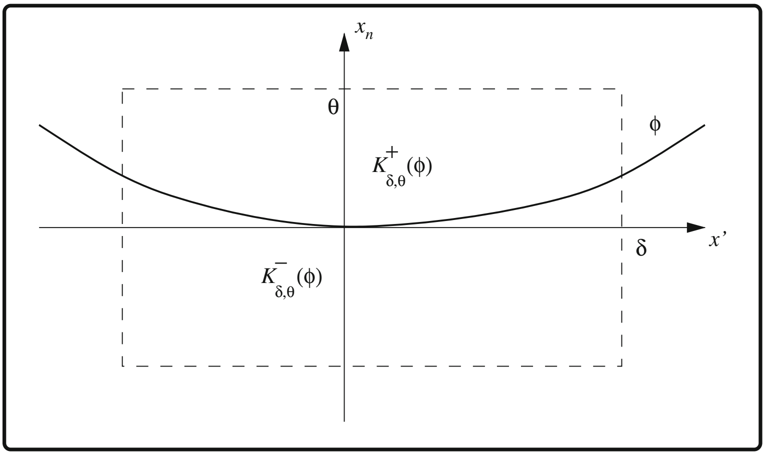

Fig. 1 The geometry involved in Lemma 10

Proof First of all, notice that the definition in (4.10) is well posed, because

$$\sup_{|x'|\leqslant\delta} |\phi(x')| \leqslant \|\phi\|_{C^{1,\alpha}}\delta^{1+\alpha} \leqslant \frac{\theta}{100},$$

thanks to (4.12) and we can find $\mathcal{T} \in C^\infty(\mathbb{R}^n)$ such that

$$\mathcal{T}(x) = \begin{cases} 1 & \text{if } |x_n - \phi(x')| \leqslant \theta/8, \\ 0 & \text{if } |x_n - \phi(x')| \geqslant \theta/4, \end{cases}$$

with

$$|\nabla\mathcal{T}| \leqslant \frac{10}{\theta}. \tag{4.13}$$

We observe that

$$\mathcal{T}(x', \pm\theta) = 0 \quad \text{for any } |x'| \leqslant \delta, \tag{4.14}$$

due to (4.12).

Then, we consider the transformation

$$\tilde{x} := \mathcal{T}^\star(x) := \big(x',\ x_n - \mathcal{T}(x)\phi(x')\big), \quad \tilde{y} := \mathcal{T}^\star(y) := \big(y',\ y_n - \mathcal{T}(y)\phi(y')\big).$$

Notice that

$$d\tilde{x}\,d\tilde{y} = \big(1 - \partial_n\mathcal{T}(x)\phi(x')\big)\big(1 - \partial_n\mathcal{T}(y)\phi(y')\big)\,dx\,dy = (1+\Xi)\,dx\,dy\,, \tag{4.15}$$

with

$$|\Xi| \leqslant \text{const}\,\eta_{\delta,\theta}(\phi),$$

thanks to (4.13) and so

$$dx\,dy = (1+\tilde{\Xi})\,d\tilde{x}\,d\tilde{y},$$

with

$$|\tilde{\Xi}| \leqslant \text{const}\,\eta_{\delta,\theta}(\phi). \tag{4.16}$$

Also, from (4.14),

$$\mathcal{T}^{\star}\left(K^{\pm}_{\delta,\theta}(\phi)\right) = K^{\pm}_{\delta,\theta}, \tag{4.17}$$

where the notation in (4.1) has been used.

Moreover, recalling (4.13),

$$|D\mathcal{T}^{\star} - \mathrm{Id}| \leqslant \text{ const } (|D\mathcal{T}|\,|\phi| + |D\phi|\,|\mathcal{T}|) \leqslant \text{ const } \eta_{\delta,\theta}(\phi),$$

which is small, as prescribed by (4.12).

Accordingly,

$$\big(1 - \text{ const } \eta_{\delta,\theta}(\phi)\big)\,|x-y| \leqslant |\tilde{x} - \tilde{y}| \leqslant \big(1 + \text{ const } \eta_{\delta,\theta}(\phi)\big)\,|x-y|.$$

Thus, since $s \in [0, 1]$,

$$\frac{1 - \text{ const } \eta_{\delta,\theta}(\phi)}{|x-y|^{n+s}} \leqslant \frac{1}{|\tilde{x}-\tilde{y}|^{n+s}} \leqslant \frac{1 + \text{ const } \eta_{\delta,\theta}(\phi)}{|x-y|^{n+s}}.$$

This, (4.2), (4.15), (4.16) and (4.17) give the desired result, by changing variable in the integral in (4.11) (Fig. 1). □

5 Nonlocal versus local minimal surfaces

Thanks to the integral computations performed in Sect. 4, now we can relate the nonlocal minimal surfaces to the classical ones.

Lemma 11 *Let Ω be a bounded open subset of $\mathbb{R}^n$.*

Suppose that there exist $\mu > 0$, $M \geqslant 1$ such that

$$\begin{aligned}&\textit{for any } x,\, y \in \Omega \textit{ with } |x-y| \leqslant \mu,\\ &\textit{there exists a curve of length less than } M\mu\\ &\textit{which joins } x \textit{ and } y \textit{ and lies in } \Omega.\end{aligned} \tag{5.1}$$

Let $F \subset \mathbb{R}^n$ and suppose that $(\partial F) \cap \Omega$ is $C^{1,\alpha}$.

Then, for any $\epsilon > 0$ there exists $K_\epsilon > 0$, possibly depending on the diameter of Ω, on the $C^{1,\alpha}$-size of $(\partial F) \cap \Omega$ and on μ and M such that

$$\left| \frac{s(1-s)}{\nu(n,s,+\infty)} \int\limits_{F\cap\Omega} \int\limits_{(\mathcal{C}F)\cap\Omega} \frac{1}{|x-y|^{n+s}}\,dx\,dy - \operatorname{Per}(F,\Omega) \right| \leqslant \epsilon + (1-s)K_\epsilon. \tag{5.2}$$

Proof We shadow the tangent hyperplane of ∂F with cylinders of the type

$$\{|\tau_\xi(x-x_o)| \leqslant \delta\} \times \{|(x-x_o)\cdot\xi| \leqslant \theta\},$$

with $x_o \in (\partial F) \cap \Omega$ and $\xi \in \mathrm{S}^{n-1}$ normal to ∂F at x_o.

Then, there exists $\epsilon_o(\epsilon) \in (0, 1)$ such that, when

$$\theta \leqslant \delta < \epsilon_o(\epsilon),$$

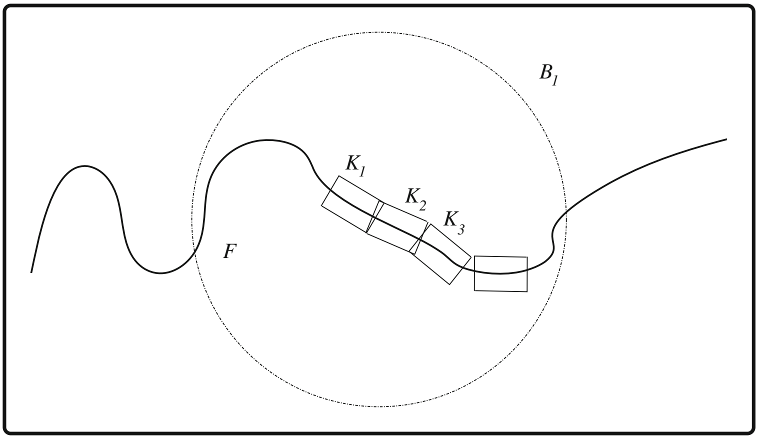

Fig. 2 The geometry involved in Lemma 11

we can select cylinders $K_1, \ldots, K_{N_{\delta,\theta,\epsilon}}$ for which

$$\begin{aligned}(\partial F) \cap \Omega \subset \bigcup_{i=1}^{N_{\delta,\theta,\epsilon}} K_i \text{ and} \\ \text{Per}\,(F, \Omega) + \epsilon^2 \geqslant \sum_{i=1}^{N_{\delta,\theta,\epsilon}} \left|K_i^o\right|,\end{aligned} \tag{5.3}$$

where K_i^o is the intersection of K_i and the tangent hyperplane of ∂F at the center of K_i—notice that such notation is compatible with the one introduced in (4.1).

For further use, we suppose, without loss of generality, that

$$\epsilon_o(\epsilon) \leqslant \min\left\{\frac{\mu}{2M}, \epsilon^{10/\alpha}\right\}. \tag{5.4}$$

From now on, we will suppose

$$\theta \leqslant \epsilon^3\delta \tag{5.5}$$

and we denote by $K_{i,\theta}$ the cylinder obtained from K_i by doubling its height (from θ to 2θ) and replacing the base of K_i, which is an $(n-1)$-dimensional ball of radius δ, with an $(n-1)$-dimensional ball of radius $\delta + \theta$ (Fig. 2).

Then,

$$\frac{|K_{i,\theta}^o|}{|K_i^o|} = \left(\frac{\delta+\theta}{\delta}\right)^{n-1}$$

and so, by (5.3) and (5.5),

$$\begin{aligned}\sum_{i=1}^{N_{\delta,\theta,\epsilon}} \left|K_{i,\theta}^o\right| &= \left(\frac{\delta+\theta}{\delta}\right)^{n-1} \sum_{i=1}^{N_{\delta,\theta,\epsilon}} \left|K_i^o\right| \\ &\leqslant (1+\epsilon^2)\left(\text{Per}\,(F,\Omega)+\epsilon^2\right),\end{aligned} \tag{5.6}$$

for small ϵ.

Notice also that

$$\begin{aligned}&\text{if } x \in F \cap \Omega \text{ and } y \in (\complement F) \cap \Omega \text{ with } |x-y| \leqslant \theta^2 \\ &\text{then } x \text{ belongs to some } K_i \text{ and } y \in K_{i,\theta}.\end{aligned} \tag{5.7}$$

To check that (5.7) holds true, just take the curve joining x and y whose existence is given by (5.1): on such arc, which lies in Ω due to (5.1), there will be a point $p \in \partial F$, with $|p-x| \leqslant M\mu < \theta^2$. Thus, x belongs to some K_i (roughly, the one centered at, or nearby, p). Then, since $|x-y| \leqslant \theta$, we obtain as a consequence that $y \in K_{i,\theta}$, proving (5.7).

Moreover, if ϕ_i is a local chart of ∂F in K_i and we take $\theta := \delta^{1+\alpha/2}$, we can define $\eta_{\delta,\theta}(\phi_i)$ in analogy with (4.9) and obtain that

$$\eta_{\delta,\theta}(\phi_i) \leqslant \frac{M\delta^{1+\alpha}}{\theta} + M\delta^\alpha \leqslant 2M\delta^{\alpha/2}, \tag{5.8}$$

where M is the $C^{1,\alpha}$-bound of $(\partial F)\cap\Omega$.

In particular, (5.8) says that (4.12) holds true and so we will be able to use Lemma 10 for ϕ_i in K_i.

Then, making use of (5.6), (5.7), (5.8) and Lemma 10, we obtain

$$\begin{aligned}&\frac{s(1-s)}{\nu(n,s,+\infty)} \int_{F\cap\Omega}\int_{(\complement F)\cap\Omega} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &\leqslant \frac{s(1-s)}{\nu(n,s,+\infty)}\left[\sum_{i=1}^{N_{\delta,\theta,\epsilon}} \int_{F\cap K_i}\int_{(\complement F)\cap K_{i,\theta}} \frac{1}{|x-y|^{n+s}}\,dx\,dy + \int_{F\cap\Omega}\int_{(\complement F)\cap\Omega} \frac{1}{\theta^{2(n+s)}}\,dx\,dy\right] \\ &\leqslant \theta^{1-s}\,(1+\mathrm{const}'\delta^{\alpha/2}) \sum_{i=1}^{N_{\delta,\theta,\epsilon}} |K^o_{i,\theta}| + \frac{\mathrm{const}'\,s(1-s)}{\theta^{2(n+s)}} \\ &\leqslant \theta^{1-s}\,(1+\mathrm{const}'\delta^{\alpha/2})\left(\mathrm{Per}\,(F,\Omega) + \mathrm{const}'\epsilon^2\right) + \frac{\mathrm{const}'\,s(1-s)}{\theta^{2(n+s)}},\end{aligned}$$

where we denoted by "const′" quantities depending only on the diameter of Ω, on the $C^{1,\alpha}$-size of $(\partial F)\cap\Omega$ and on Per (F,Ω).

Since $0<\theta<\delta<\epsilon_o(\epsilon)$ may be fixed in dependence of ϵ, this gives that, for a suitable $C_\epsilon > 0$,

$$\begin{aligned}&\frac{s(1-s)}{\nu(n,s,+\infty)} \int_{F\cap\Omega}\int_{(\complement F)\cap\Omega} \frac{1}{|x-y|^{n+s}}\,dx\,dy \leqslant \theta^{1-s}\,\mathrm{Per}\,(F,\Omega) + \epsilon + s(1-s)C_\epsilon \\ &\leqslant (1+(1-s)C_\epsilon\log\theta)\ \mathrm{Per}\,(F,\Omega) + \epsilon + s(1-s)C_\epsilon,\end{aligned}$$

which gives the upper bound for the integral in (5.2).

To obtain the lower bound for the integral in (5.2), we take smaller cylinders $\tilde K_i \subset K_i$, of the type

$$\{|\tau_\xi(x-x_o)| \leqslant \tilde\delta\} \times \{|(x-x_o)\cdot\xi| \leqslant \tilde\theta\} \tag{5.9}$$

with $\tilde\theta := \tilde\delta^{1+\alpha/2} \leqslant \epsilon^2\tilde\delta$ and $\tilde\delta \in (0,\delta)$, in such a way that the $\tilde K_i$'s are nonoverlapping and

$$\mathrm{Per}\,(F,\Omega) \leqslant \epsilon^2 + \sum_{i=1}^{N_{\delta,\theta,\epsilon}} \left|\tilde K_i^o\right|. \tag{5.10}$$

Then, employing again (5.8) and Lemma 10 (used here with $\beta := \epsilon^2$), we conclude that

$$\begin{aligned}
&\frac{s(1-s)}{\nu(n,s,+\infty)} \int_{F\cap\Omega} \int_{(\complement F)\cap\Omega} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\
&\geqslant \frac{s(1-s)}{\nu(n,s,+\infty)} \sum_{i=1}^{N_{\delta,\theta,\epsilon}} \int_{F\cap\tilde K_i} \int_{(\complement F)\cap\tilde K_i} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\
&\geqslant \tilde\theta^{1-s}\,(1-\,\mathrm{const}'\delta^{\alpha/2})\,(1-\epsilon^2)^{n-1}\,\frac{\nu(n,s,\epsilon\tilde\delta/(2\tilde\theta))}{\nu(n,s,+\infty)} \sum_{i=1}^{N_{\delta,\theta,\epsilon}} |\tilde K^o_{i,\theta}|.
\end{aligned}$$

Since $0<\tilde\theta<\tilde\delta<\epsilon_o(\epsilon)$ may be fixed in dependence of ϵ, this and (5.10) give that

$$\begin{aligned}
&\frac{s(1-s)}{\nu(n,s,+\infty)} \int_{F\cap\Omega} \int_{(\complement F)\cap\Omega} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\
&\geqslant \tilde\theta^{1-s}\,\frac{\nu(n,s,\epsilon^2\tilde\delta/(2\tilde\theta))}{\nu(n,s,+\infty)}\,(\,\mathrm{Per}\,(F,\Omega)-\,\mathrm{const}'\epsilon^2) \\
&\geqslant (1-(1-s)C_\epsilon\log\theta)\,\frac{\nu(n,s,\epsilon^2/(2\delta^{\alpha/2}))}{\nu(n,s,+\infty)}\,(\,\mathrm{Per}\,(F,\Omega)-\,\mathrm{const}'\epsilon^2). \qquad (5.11)
\end{aligned}$$

Now we observe that, by (5.4) and the fact that $\delta \leqslant \epsilon_o(\epsilon)$, we have

$$\frac{\epsilon}{2\delta^{\alpha/2}} \geqslant \frac{1}{\epsilon^2},$$

and so

$$\begin{aligned}
\big|\nu(n,s,\epsilon/(2\delta^{\alpha/2})) - \nu(n,s,+\infty)\big| &\leqslant \,\mathrm{const}' \int_{\epsilon/(2\delta^{\alpha/2})}^{+\infty} \frac{t^{n-2}\,dt}{(1+t^2)^{(n+s)/2}} \\
&\leqslant \,\mathrm{const}' \int_{1/\epsilon^2}^{+\infty} \frac{t^{n-2}\,dt}{(1+t^2)^{(n+s)/2}} \\
&\leqslant \,\mathrm{const}'\epsilon^{2(s+1)}.
\end{aligned}$$

This and (5.11) yield the lower bound for the integral in (5.2). □

Remark Condition (5.1) is obviously valid for convex domains, as well as for smooth ones, and it may be modified, weakened and adapted to other situations. For instance, without assuming condition (5.1) and dropping the dependence on μ and M, the proof of Lemma 11 gives that

$$\frac{s(1-s)}{\nu(n,s,+\infty)} \int_{F\cap\Omega_{-\epsilon}} \int_{(\complement F)\cap\Omega_{-\epsilon}} \frac{1}{|x-y|^{n+s}}\,dx\,dy \leqslant \,\mathrm{Per}\,(F,\Omega)+\epsilon+(1-s)K_\epsilon, \qquad (5.12)$$

where

$$\Omega_{-\epsilon} := \{x\in\Omega \text{ s.t. } B_\epsilon(x)\subseteq\Omega\}\,.$$

Indeed, one may repeat the arguments in the proof of Lemma 11, but assuming that the points x and y in (5.7) belong to $\Omega_{-\epsilon}$: then the segment joining them lies in $B_{\theta^2}(x) \subseteq B_\epsilon(x) \subseteq \Omega$ and the argument after (5.7) goes through.

6 Boundary contributions

Next result estimates the nonlocal contributions coming from the boundary:

Lemma 12 *Let $R, \beta > 0$ and $F \subseteq \mathbb{R}^n$, with $\partial F \cap (B_{(1+\beta)R} \setminus B_{(1-\beta)R})$ in $C^{1,\alpha}$. Let $s_k \in (0,1)$, with*

$$\lim_{k\to+\infty} s_k = 1.$$

Then,

$$\lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n,s_k,+\infty)} \int\limits_{F\cap B_R} \int\limits_{(\complement F)\cap(\complement B_R)} \frac{1}{|x-y|^{n+s_k}}\,dy\,dx \leqslant \mathcal{H}^{n-1}(\partial B_R \cap \partial F).$$

Proof Up to scaling, we set $R := 1$ and, without loss of generality, we take $s_k \in (1/2, 1)$. Fix $\eta > 0$ and define

$$U_\eta := B_{1+\eta} \setminus B_{1-\eta}.$$

In what follows, we denote by $c_i(\eta)$ suitable positive quantities, possibly depending on η but independent of k.

If $x \in F \setminus U_\eta$ and $y \in \complement F$, we have that $|x-y| \geqslant \eta$ and so

$$\begin{aligned}
&\int\limits_{(F\cap B_1)\setminus U_\eta} \int\limits_{(\complement F)\cap(\complement B_1)} \frac{1}{|x-y|^{n+s_k}}\,dy\,dx \\
&\quad\leqslant \eta^{-(n+s_k)}|B_1|\,|B_2\setminus B_1| + \int\limits_{B_1}\int\limits_{\complement B_2} \frac{1}{(|y|/2)^{n+s_k}}\,dx\,dy \leqslant c_1(\eta).
\end{aligned}$$

Analogously,

$$\int\limits_{F\cap B_1} \int\limits_{((\complement F)\cap(\complement B_1))\setminus U_\eta} \frac{1}{|x-y|^{n+s_k}}\,dy\,dx \leqslant c_2(\eta).$$

So, applying Lemma 11 (with $\Omega := U_\eta$—recall also the Remark before the statement of Lemma 12 and in particular (5.12)), we have

$$\begin{aligned}
&\frac{s_k(1-s_k)}{\nu(n,s_k,+\infty)} \int\limits_{F\cap B_1} \int\limits_{(\complement F)\cap(\complement B_1)} \frac{1}{|x-y|^{n+s_k}}\,dy\,dx \\
&\quad\leqslant \frac{s_k(1-s_k)}{\nu(n,s_k,+\infty)} \left[\int\limits_{F\cap U_\eta} \int\limits_{(\complement F)\cap U_\eta} \frac{1}{|x-y|^{n+s_k}}\,dy\,dx + c_1(\eta) + c_2(\eta) \right] \\
&\quad\leqslant \mathcal{H}^{n-1}\big((\partial F)\cap U_{2\eta}\big) + \eta + (1-s_k)c_3(\eta).
\end{aligned}$$

By taking the limit in k, we obtain

$$\lim_{k\to+\infty}\frac{s_k(1-s_k)}{\nu(n,s_k,+\infty)}\int_{F\cap B_1}\int_{(\mathcal{C}F)\cap(\mathcal{C}B_1)}\frac{1}{|x-y|^{n+s_k}}\,dy\,dx \leqslant \mathcal{H}^{n-1}\Big((\partial F)\cap U_{2\eta}\Big)+\eta.$$

The desired result follows by taking limit in η, via Monotone Convergence Theorem. □

7 Proof of Theorem 1

Let $s_k\in(0,1)$ be any sequence approaching 1 as $k\to+\infty$. We fix $\epsilon>0$ and we apply Lemma 11 and 12 (the latter used here with $F_k:=E$ and with $F_k:=\mathcal{C}E$): we obtain

$$\begin{aligned}&\lim_{k\to+\infty}\left|\frac{s_k(1-s_k)}{\nu(n,s_k,+\infty)}\mathcal{J}_{s_k}(E,B_r)-\operatorname{Per}(E,B_r)\right|\\ &\leqslant \lim_{k\to+\infty}\epsilon+(1-s_k)K_\epsilon+\int_{E\cap B_r}\int_{(\mathcal{C}E)\cap(\mathcal{C}B_r)}\frac{1}{|x-y|^{n+s_k}}\,dy\,dx\\ &\quad+\int_{(\mathcal{C}E)\cap B_r}\int_{E\cap(\mathcal{C}B_r)}\frac{1}{|x-y|^{n+s_k}}\,dy\,dx\\ &\leqslant \epsilon+2\mathcal{H}^{n-1}(\partial E\cap\partial B_r).\end{aligned}$$

By taking ϵ as small as we wish, we obtain (1.1). Then, elementary considerations[2] give (1.2). This completes the proof of Theorem 1.

8 Scaling properties and uniform estimates

Now, we perform some estimates in order to control suitable quantities of s-minimizers, uniformly as $s\to1^-$. For this we will borrow some results and techniques from [3]. In particular, some of the main results of [3] are that minimizers possess positive density and a clean ball condition, and we show that these properties are independent of s for $s\in(s_o,1)$, once s_o is fixed. These properties are inherited by the limit, and therefore they will conveniently show up in the computations here below.

We start by computing the interaction between a ball and its complement:

[2] Indeed, if a set $D\subset\mathbb{R}^n$ is such that $\mathcal{H}^{n-1}(D\cap(B_b\setminus B_a))$ is finite, for some $b>a\geqslant0$, we have that, for any $k\in\mathbb{N}$, $k\geqslant1$, the set

$$\sigma_k:=\Big\{r\in(a,b)\text{ s.t. }\mathcal{H}^{n-1}(D\cap(\partial B_r))>1/k\Big\}$$

cannot have more than $k\mathcal{H}^{n-1}(D\cap(B_b\setminus B_a))$ elements. And therefore the set

$$\bigcup_{k\in\mathbb{N}\cap[1,+\infty)}\sigma_k=\Big\{r\in(a,b)\text{ s.t. }\mathcal{H}^{n-1}(D\cap(\partial B_r))>0\Big\}$$

is countable.

Lemma 13 *There exists $c \in (0, 1)$ such that, for any $s \in (1 - c, 1)$, we have that*

$$cr^{n-s} \leqslant \frac{s(1-s)}{\nu(n,s,+\infty)} \int_{B_r} \int_{\complement B_r} \frac{1}{|x-y|^{n+s}}\,dx\,dy \leqslant \frac{r^{n-s}}{c},$$

for any $r > 0$.

Proof Up to a scaling argument, we may and do suppose $r := 1$. We also assume $s \in (1/2, 1)$. Then, using Lemma 11, with $\Omega := B_2$ $F := B_1$ and $\mathcal{H}^{n-1}(\partial B_1)/4$, we have that

$$\left| \frac{s(1-s)}{\nu(n,s,+\infty)} \int_{B_1} \int_{(\complement B_1)\cap B_2} \frac{1}{|x-y|^{n+s}}\,dx\,dy - \mathcal{H}^{n-1}(\partial B_1) \right| \leqslant \frac{\mathcal{H}^{n-1}(\partial B_1)}{4} + (1-s)K, \tag{8.1}$$

for a suitable, universal, $K > 0$. On the other hand,

$$\int_{B_1} \int_{\complement B_2} \frac{1}{|x-y|^{n+s}}\,dx\,dy \leqslant \int_{B_1} \int_{\complement B_2} \frac{2^{n+s}}{|y|^{n+s}}\,dx\,dy \leqslant \frac{\tilde{K}}{s},$$

for a suitable $\tilde{K} > 0$ and so

$$\begin{aligned} &\frac{s(1-s)}{\nu(n,s,+\infty)} \left| \int_{B_1} \int_{\complement B_1} \frac{1}{|x-y|^{n+s}}\,dx\,dy - \int_{B_1} \int_{(\complement B_1)\cap B_2} \frac{1}{|x-y|^{n+s}}\,dx\,dy \right| \\ &= \frac{s(1-s)}{\nu(n,s,+\infty)} \int_{B_1} \int_{\complement B_2} \frac{1}{|x-y|^{n+s}}\,dx\,dy \leqslant \bar{K}(1-s) \end{aligned}$$

for a suitable $\bar{K} > 0$ and so, from (8.1), if

$$1 - \frac{\mathcal{H}^{n-1}(\partial B_1)}{4(K+\bar{K})} \leqslant s < 1,$$

we have

$$\begin{aligned} &\left| \frac{s(1-s)}{\nu(n,s,+\infty)} \int_{B_1} \int_{\complement B_1} \frac{1}{|x-y|^{n+s}}\,dx\,dy - \mathcal{H}^{n-1}(\partial B_1) \right| \\ &\leqslant \frac{\mathcal{H}^{n-1}(\partial B_1)}{4} + (1-s)(K+\bar{K}) \leqslant \frac{\mathcal{H}^{n-1}(\partial B_1)}{2}, \end{aligned}$$

as desired. □

Next result generalizes Lemma 13 to the intersection of a ball with a ring, which will be the case deal with in Sect. 15.

Corollary 14 *Let $p \in \mathbb{R}^n$, $r > 0$, $R_+ \geqslant R_- \geqslant 0$ and $A := B_{R_+}(p) \setminus B_{R_-}(p)$. Then, there exists $C \geqslant 1$ such that*

$$\lim_{s\to 1^-} \frac{s(1-s)}{\nu(n,s,+\infty)} \int_{B_r\cap A} \int_{\complement(B_r\cap A)} \frac{1}{|x-y|^{n+s}}\,dx\,dy \leqslant Cr^{n-1}.$$

Proof We observe that $\mathcal{C}(B_r \cap A) = (\mathcal{C}B_r) \cup (B_r \cap (\mathcal{C}A))$ and therefore

$$\begin{aligned}\int_{B_r\cap A}\int_{\mathcal{C}(B_r\cap A)} \frac{1}{|x-y|^{n+s}}\,dx\,dy &\leqslant \int_{B_r\cap A}\int_{\mathcal{C}B_r} \frac{1}{|x-y|^{n+s}}\,dx\,dy + \int_{B_r\cap A}\int_{B_r\cap(\mathcal{C}A)} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &\leqslant \int_{B_r}\int_{\mathcal{C}B_r} \frac{1}{|x-y|^{n+s}}\,dx\,dy + \int_{B_r\cap A}\int_{B_r\cap(\mathcal{C}A)} \frac{1}{|x-y|^{n+s}}\,dx\,dy. \qquad (8.2)\end{aligned}$$

On the other hand, making use of Lemma 11 with $\epsilon := r^{n-1}$, $\Omega := B_r$ and $F := A$, we obtain that

$$\frac{s(1-s)}{\nu(n,s,+\infty)}\int_{B_r\cap A}\int_{B_r\cap(\mathcal{C}A)} \frac{1}{|x-y|^{n+s}}\,dx\,dy \leqslant \operatorname{Per}(A, B_r) + r^{n-1} + (1-s)K, \quad (8.3)$$

for a suitable $K > 0$, possibly depending on d, R_- and R_+, but independent of s.

Since $\operatorname{Per}(A, B_r) \leqslant Cr^{n-1}$ for some $C > 0$, we deduce from (8.2) and (8.3) that

$$\begin{aligned}&\lim_{s\to 1^-}\frac{s(1-s)}{\nu(n,s,+\infty)}\int_{B_r\cap A}\int_{\mathcal{C}(B_r\cap A)} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &\leqslant (C+1)r^{n-1} + \lim_{s\to 1^-}\frac{s(1-s)}{\nu(n,s,+\infty)}\int_{B_r}\int_{\mathcal{C}B_r} \frac{1}{|x-y|^{n+s}}\,dx\,dy.\end{aligned}$$

So, the desired result follows from Lemma 13, up to renaming C. □

With Lemma 13, we can also estimate from above the energy of a minimizer in a ball, as next result shows:

Corollary 15 *There exists $c \in (0, 1)$ such that, for any $s \in (1-c, 1)$, the following holds true.*

Let $r > 0$ and E be an s-minimal set in B_r.

Then,

$$\frac{s(1-s)}{\nu(n,s,+\infty)}\mathcal{J}_s(E, B_r) \leqslant \frac{r^{n-s}}{c}.$$

Proof By minimality, the energy of E is smaller than the one obtained replacing $E \cap B_r$ by B_r, that is $\mathcal{J}_s(E, B_r) \leqslant \mathcal{J}_s(F, B_r)$ if $F := B_r \cup (E \cap (\mathcal{C}B_r))$. But

$$\frac{s(1-s)}{\nu(n,s,+\infty)}\mathcal{J}_s(F, B_r)$$

is bounded by

$$\begin{aligned}&\frac{s(1-s)}{\nu(n,s,+\infty)}\left[\int_{F\cap B_r}\int_{(\mathcal{C}F)\cap(\mathcal{C}B_r)} \frac{1}{|x-y|^{n+s}}\,dy\,dx + \int_{F\cap(\mathcal{C}B_r)}\int_{(\mathcal{C}F)\cap B_r} \frac{1}{|x-y|^{n+s}}\,dy\,dx\right] \\ &\leqslant \frac{2s(1-s)}{\nu(n,s,+\infty)}\int_{B_r}\int_{\mathcal{C}B_r} \frac{1}{|x-y|^{n+s}}\,dy\,dx \leqslant \frac{2r^{n-s}}{c},\end{aligned}$$

thanks to Lemma 13. □

Next result will estimate the part of the energy of a set in a ball which is concentrated near the boundary of the set.

Lemma 16 *Let $\Omega \subseteq \mathbb{R}^n$, $x \in \Omega$ and $d(x) := \operatorname{dist}(x, \partial\Omega)$. Then, there exists a constant $c \in (0, 1)$ such that*

$$c\, d(x)^{-(n+s)} \left|(\mathcal{C}\Omega) \cap B_{2d(x)}(x)\right| \leqslant \int_{(\mathcal{C}\Omega)\cap B_{2d(x)}(x)} \frac{1}{|x-y|^{n+s}}\, dy$$
$$\leqslant c^{-1} d(x)^{-(n+s)} \left|(\mathcal{C}\Omega) \cap B_{2d(x)}(x)\right|. \tag{8.4}$$

Moreover, if $|(\mathcal{C}\Omega) \cap B_{2d(x)}(x)| \geqslant c' d(x)^n$ for some $c' \in (0, 1)$, then there exists $c_* \in (0, 1)$, possibly depending on c', such that

$$\int_{\mathcal{C}\Omega} \frac{1}{|x-y|^{n+s}}\, dy \leqslant c_*^{-1} d(x)^{-(n+s)} \left|(\mathcal{C}\Omega) \cap B_{2d(x)}(x)\right|. \tag{8.5}$$

Proof We suppose $d(x) > 0$, otherwise the result is obvious.

We observe that if $y \in \mathcal{C}\Omega$ then

$$|y - x| \geqslant d(x). \tag{8.6}$$

On the other hand, if $y \in B_{2d(x)}(x)$, we have

$$|y - x| \leqslant 2d(x). \tag{8.7}$$

From (8.6) and (8.7) we easily bound the integral in (8.4) as desired.

Now, notice that

$$\int_{(\mathcal{C}\Omega)\cap(\mathcal{C}B_{2d(x)}(x))} \frac{1}{|x-y|^{n+s}}\, dy \leqslant \int_{\mathcal{C}B_{2d(x)}} \frac{1}{|\zeta|^{n+s}}\, d\zeta = \frac{n\omega_n}{s\,(2d(x))^s},$$

which, together with (8.4), yields (8.5). □

Now, we can apply the previous estimates to the s-minima:

Corollary 17 *Let $s \in (1/2, 1)$. Let E be an s-minimal set. Fix $x \in E$ and let $d(x) := \operatorname{dist}(x, \partial E)$. Then,*

$$\int_{(\mathcal{C}E)\cap B_{Cd(x)}(x)} \frac{1}{|x-y|^{n+s}}\, dy \geqslant \frac{d(x)^{-s}}{C}, \tag{8.8}$$

for a suitable $C \geqslant 1$, independent of s.

Proof By construction, there exists $p \in \overline{B_{d(x)}(x)} \cap \partial E$.

Thus, from Corollary 4.3 of [3], we have that[3] there exists a universal $C \geqslant 1$ and a point $q \in \mathbb{R}^n$ such that

$$B_{d(x)}(q) \subseteq (\mathcal{C}E) \cap B_{Cd(x)}(p).$$

[3] The fact that the constant of Corollary 4.3 of [3] is uniform in s when $s \to 1^-$ relies on the proof in [3] and in the asymptotics of the fractional Sobolev Embedding, which is discussed here in Appendix A. See, in particular, Corollary 25.

In particular, if we set $C_o := C + 2$, we have that

$$B_{d(x)}(q) \subseteq (\mathcal{C}E) \cap B_{C_o d(x)}(x). \tag{8.9}$$

From Lemma 16, we know that

$$\int_{(\mathcal{C}E)\cap B_{C_o d(x)}(x)} \frac{1}{|x-y|^{n+s}}\, dy \geqslant c_1 d(x)^{-(n+s)} \left|(\mathcal{C}E) \cap B_{C_o d(x)}(x)\right|,$$

for a suitable $c_1 > 0$. This and (8.9) give (8.8). □

Remark A useful consequence of the clean ball condition in Corollary 4.3 of [3] (and of the fact that the constant there is uniform, due to Appendix A here) is that if E_k is s_k-minimal and χ_{E_k} converges to χ_E in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then E_k converges to E locally uniformly. To check this, suppose the contrary: then there exist $R > 0$, $\delta \in (0, 1)$, a sequence $k_j \to +\infty$ and points

$$p_j \in E_{k_j} \cap B_R \tag{8.10}$$

for which

$$B_\delta(p_j) \subseteq \mathcal{C}E. \tag{8.11}$$

The case in which $p_j \in (\mathcal{C}E_{k_j}) \cap B_R$ and $B_\delta(p_j) \subseteq E$ is analogous. We observe that, for j large,

$$B_{\delta/2}(p_j) \cap B_R \not\subseteq E_{k_j}, \tag{8.12}$$

otherwise

$$\int_{B_R} |\chi_{E_{k_j}}(x) - \chi_E(x)|\, dx \geqslant \int_{B_{\delta/2}(p_j)\cap B_R} |\chi_{E_{k_j}}(x) - \chi_E(x)|\, dx = \left|B_{\delta/2}(p_j) \cap B_R\right| \geqslant c(\delta, R),$$

for a suitable $c(\delta, R) > 0$, and this is in contradiction with the $L^1(B_R)$-convergence of χ_{E_k}.

Then, by (8.10) and (8.12), there exists $P_j \in \partial E_{k_j} \cap B_{\delta/2}(p_j) \cap B_R$. So, by Corollary 4.3 of [3], there exists $q_j \in \mathbb{R}^n$ such that $B_{c\delta/2}(q_j) \subseteq B_{\delta/2}(P_j) \cap E_{k_j}$, for a suitable $c \in (0, 1)$. Hence, by (8.11),

$$B_{c\delta/2}(q_j) \subseteq B_\delta(p_j) \cap E_{k_j} \subseteq (\mathcal{C}E) \cap E_{k_j}.$$

Accordingly

$$\int_{B_{R+1}} |\chi_{E_{k_j}}(x) - \chi_E(x)|\, dx \geqslant \int_{B_{c\delta/2}(q_j)} |\chi_{E_{k_j}}(x) - \chi_E(x)|\, dx = |B_{c\delta/2}|,$$

which is in contradiction with the $L^1(B_R)$-convergence of χ_{E_k}.

The density estimates of [3] are also useful to estimate the measure of the points at a given distance to the boundary, as next result shows:

Lemma 18 *Let $s \in (1/2, 1)$ and $r > 0$. Let E be an s-minimal set, with $p_o \in \partial E$. Then, there exist $C_* \geqslant 1$ and $c \in (0, 1)$, independent of μ, r and s such that the following estimate holds.*

For any $\mu > 0$, define

$$A_\mu(B_r(p_o)) := \{x \in E \cap B_r(p_o) \text{ s.t. } \mu < d(x) \leqslant C_*\mu\}.$$

Then, if $\mu \in (0, r/C_*]$, there exists $\bar{x} \in \mathbb{R}^n$ such that

$$B_\mu(\bar{x}) \subseteq A_\mu(B_r(p_o)).$$

Proof Without loss of generality, we take $p_o = 0$, and we write $A_\mu := A_\mu(B_r)$. Then, we use Corollary 4.3 of [3] to see that there exists $\bar{x} \in \mathbb{R}^n$ such that $B_{2\mu}(\bar{x}) \subseteq E \cap B_{2C\mu}$. We choose $C_* := 2C + 1$. Notice that, if $\zeta \in B_\mu(\bar{x})$ we have that

$$|\zeta| \leqslant |\zeta - \bar{x}| + |\bar{x}| < \mu + 2C\mu = C_*\mu, \tag{8.13}$$

and so, since $0 \in \partial E$,

$$d(\zeta) \leqslant C_*\mu. \tag{8.14}$$

Furthermore, for any $Q \in \partial E$,

$$|Q - \zeta| \geqslant |Q - \bar{x}| - |\bar{x} - \zeta| > 2\mu - \mu = \mu \tag{8.15}$$

From (8.14) and (8.15), we conclude that

$$\text{if } \zeta \in B_\mu(\bar{x}), \text{ then } d(\zeta) \in (\mu, C_*\mu]. \tag{8.16}$$

Moreover, (8.13) says that

$$B_\mu(\bar{x}) \subseteq B_{C_*\mu} \subseteq B_r.$$

This and (8.16) give the desired result. □

With the above estimates, we bound from below the energy by using a collection of balls shadowing the boundary:

Proposition 19 *Let $s \in (1/2, 1)$, $r > 0$ and E be an s-minimal set, with $0 \in \partial E$. Then,*

$$\text{there exist } x_1, x_2 \in \mathbb{R}^n \text{ such that } B_{cr}(x_1) \subseteq E \cap B_r \text{ and } B_{cr}(x_2) \subseteq (\mathcal{C}E) \cap B_r, \tag{8.17}$$

$$\mathcal{H}^{n-1}(\partial E \cap B_r) \geqslant cr^{n-1} \tag{8.18}$$

and

$$\frac{s(1-s)}{\nu(n, s, +\infty)} \int_{E \cap B_r} \int_{(\mathcal{C}E) \cap B_r} \frac{1}{|x - y|^{n+s}}\, dx\, dy \geqslant cr^{n-s}, \tag{8.19}$$

for a suitable $c \in (0, 1)$, independent of s.

Proof We will denote by c_i suitable positive constants independent of s. We observe that, by Corollary 4.3 of [3], there exist $x_1, x_2 \in \mathbb{R}^n$ such that

$$B_{c_1}(x_1) \subseteq E \cap B_1 \quad \text{and} \quad B_{c_1}(x_2) \subseteq (\mathcal{C}E) \cap B_1, \tag{8.20}$$

and therefore, by Isoperimetric Inequality,

$$\begin{aligned} \mathcal{H}^{n-1}(\partial E \cap B_1) &\geqslant c_2 \min\{|E \cap B_1|,\ |(\mathcal{C}E) \cap B_1|\}^{(n-1)/n} \\ &\geqslant c_2 \min\left\{|B_{c_1}(x_1)|,\ |B_{c_1}(x_2)|\right\}^{(n-1)/n} \geqslant c_3. \end{aligned} \tag{8.21}$$

This proves (8.17) and (8.18) for $r := 1$ and the general case follows from scaling.

Now, we fix a constant $C \geqslant 1$, to be taken appropriately large in the sequel (with respect to the other constants $c_1, \dots, c_8$ and to the constants given in Corollary 17 and Lemma 18, but independently of s) and, for any $k \in \mathbb{N}$, we consider the collection of balls $B_{C^{-k}}(p)$, for any $p \in \partial E \cap \overline{B_1}$. Then (see, e.g., Lemma 2.2 in [4]), we take a countable subcovering of $\partial E \cap \overline{B_1}$ by balls $\left\{B_{C^{-k}}(p_{j,k})\right\}_{j\in\mathbb{N}}$ such that

$$B_{C^{-k}/3}(p_{j,k}) \cap B_{C^{-k}/3}(p_{j',k}) = \emptyset \quad \text{if } j \neq j'. \tag{8.22}$$

By compactness, we may take a finite subcover of $\partial E \cap \overline{B_1}$, say $\left\{B_{C^{-k}}(p_{j,k})\right\}_{j\in\{1,\dots,N_k\}}$, for a suitable $N_k \in \mathbb{N}$.

We estimate N_k from below via the following covering argument. We define

$$\beta_k := \bigcup_{j=1}^{N_k} B_{C^{-k}}(p_{j,k}).$$

Notice that $\partial E \cap \overline{B_1} \subseteq \beta_k$. Also,

$$\begin{aligned} |\beta_k| &\leqslant \sum_{j=1}^{N_k} |B_{C^{-k}}(p_{j,k})| \leqslant c_4 N_k C^{-kn} \text{ and} \\ \mathcal{H}^{n-1}(\partial\beta_k) &\leqslant \sum_{j=1}^{N_k} \mathcal{H}^{n-1}(\partial B_{C^{-k}}(p_{j,k})) \leqslant c_4 N_k C^{-k(n-1)}. \end{aligned} \tag{8.23}$$

We claim that

$$N_k \geqslant c_5 C^{k(n-1)}. \tag{8.24}$$

To prove (8.24), we may suppose that $N_k \leqslant \left(\omega_n c_1^n/(2c_4)\right) C^{kn}$, otherwise we are done, and so, if we define $F_k := E \cup \beta_k$, we have that

$$\begin{aligned} &|F_k \cap B_1| \geqslant |E \cap B_1| \geqslant \omega_n c_1^n \text{ and} \\ &|(\mathcal{C}F_k) \cap B_1| \geqslant |(\mathcal{C}E) \cap B_1| - |\beta_k \cap B_1| \\ &\geqslant \omega_n c_1^n - c_4 N_k C^{-kn} \geqslant \frac{\omega_n c_1^n}{2}, \end{aligned}$$

thanks to (8.20) and (8.23). As a consequence, by Isoperimetric Inequality,

$$\mathcal{H}^{n-1}(\partial\beta_k) \geqslant \mathcal{H}^{n-1}(\partial F_k \cap B_1) \geqslant c_6$$

and so, by (8.23),

$$c_6 \leqslant c_4 N_k C^{-k(n-1)},$$

which proves (8.24).

Now, using the notation of Lemma 18, we define

$$A_{j,k} := A_{C^{-(k+3)}}\left(B_{C^{-(k+1)}}(p_{j,k})\right).$$

By Lemma 18, taking C large enough, we have that there exists $x_{j,k} \in \mathbb{R}^n$ such that

$$B_{j,k} := B_{C^{-(k+3)}}(x_{j,k}) \subseteq A_{j,k}.$$

By construction,

$$\text{if } x \in B_{j,k}, \text{ then } \operatorname{dist}(x, \partial E) \in \left(C^{-(k+3)}, C^{-(k+2)}\right] \tag{8.25}$$

if C is large. Consequently, by Corollary 17, for any $x \in B_{j,k}$,

$$c_7 C^{ks} \leqslant \int_{(\complement E)\cap B_{c_8 C^{-(k+1)}}(x)} \frac{1}{|x-y|^{n+s}}\,dy. \tag{8.26}$$

Now, we observe that, if $x \in B_{j,k}$, then

$$B_{c_8 C^{-(k+1)}}(x) \subseteq B_{C^{-k}/3}(p_{j,k}) \tag{8.27}$$

if C is large enough (and this will fix C once and for all, so in the sequel c_i may also depend on C): indeed, if $z \in B_{c_7 C^{-(k+1)}}(x)$, we have that

$$|z-p_{j,k}| \leqslant |z-x|+|x-x_{j,k}|+|x_{j,k}-p_{j,k}| \leqslant c_7 C^{-(k+1)}+C^{-(k+3)}+C^{-(k+1)} \leqslant C^{-k}/3,$$

as desired.

Thus, from (8.26) and (8.27),

$$c_7 C^{ks} \leqslant \int_{(\complement E)\cap B_{C^{-k}/3}(p_{j,k})} \frac{1}{|x-y|^{n+s}}\,dy,$$

for any $x \in B_{j,k}$ and therefore

$$c_9 C^{k(s-n)} \leqslant \int_{B_{j,k}} \int_{(\complement E)\cap B_{C^{-k}/3}(p_{j,k})} \frac{1}{|x-y|^{n+s}}\,dx\,dy.$$

By summing up in $k \in \mathbb{N}$ and $j = 1, \ldots, N_k$, we obtain

$$\begin{aligned} c_9 \sum_{k\in\mathbb{N}} N_k C^{k(s-n)} &\leqslant \sum_{\substack{1\leqslant j\leqslant N_k\\ k\in\mathbb{N}}} \int_{B_{j,k}} \int_{(\complement E)\cap B_{C^{-k}/3}(p_{j,k})} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &= \sum_{\substack{1\leqslant j\leqslant N_k\\ k\in\mathbb{N}}} \int_{E_{j,k}} \frac{1}{|x-y|^{n+s}}\,d(x,y), \end{aligned} \tag{8.28}$$

with

$$E_{j,k} := B_{j,k} \times \left((\complement E)\cap B_{C^{-k}/3}(p_{j,k})\right) \subseteq \mathbb{R}^n \times \mathbb{R}^n.$$

We claim that

$$E_{j,k} \cap E_{j',k'} = \emptyset \quad \text{unless } j = j' \text{ and } k = k'. \tag{8.29}$$

Indeed, if there exists $(z, w) \in \mathbb{R}^n \times \mathbb{R}^n$, with $(z, w) \in E_{j,k} \cap E_{j',k'}$ we have that $z \in B_{j,k} \cap B_{j',k'} \subseteq A_{j,k} \cap A_{j',k'}$ and so, recalling (8.25),

$$\operatorname{dist}(z, \partial E) \in \left(C^{-(k+3)}, C^{-(k'+2)}\right].$$

This gives that $k \geqslant k'$, hence, by exchanging their roles, we see that $k = k'$. Then, $w \in B_{C^{-k}/3}(p_{j,k}) \cap B_{C^{-k}/3}(p_{j',k})$ and so $j = j'$, thanks to (8.22). This proves (8.29).

Then, by (8.28) and (8.29),

$$c_9 \sum_{k\in\mathbb{N}} N_k C^{k(s-n)} \leqslant \int_{B_2\times B_2} \frac{1}{|x-y|^{n+s}}\,d(x,y).$$

Hence, recalling (8.24),

$$\int_{B_2\times B_2} \frac{1}{|x-y|^{n+s}}\, d(x,y) \geqslant \frac{c_{10}}{C^{1-s}-1} \geqslant \frac{c_{11}}{1-s}.$$

This proves (8.19) when $r := 2$ and the general case follows from scaling. □

9 Proof of Theorem 2

Take $p \in B_{r/2}\cap\partial E$. Then $B_{r/2}(p) \subseteq B_r$ and so Proposition 19 gives the claims in Theorem 2.

10 Proof of Theorem 3

We will take $r := 1$, and the general case will follow from scaling. We will denote by c_i suitable positive constants. By the uniform convergence, we have that, for large k, there exists $p_k \in \partial E_k \cap B_1$. Consequently, by Corollary 4.3 of [3], we have that there exist $a_k, b_k \in \mathbb{R}^n$ such that $B_{c_1}(a_k) \subseteq E_k \cap B_1$ and $B_{c_1}(b_k) \subseteq (\complement E_k)\cap B_1$. Up to subsequence, we may suppose that a_k and b_k converge, respectively, to some a and $b \in B_1$, and so $B_{c_1/2}(a) \subseteq E \cap B_1$ and $B_{c_1/2}(b) \subseteq (\complement E) \cap B_1$, which proves (1.3). Accordingly, by Isoperimetric Inequality, $\mathcal{H}^{n-1}(\partial E \cap B_1) \geqslant c_2$, which proves the lower bound in (1.4).

We now prove the upper bound in (1.4). For this, we take $\rho \in (0, 1/2)$ and we cover $\partial E \cap \overline{B_1}$ with N_ρ balls $B_\rho(p_1), \ldots, B_\rho(p_{N_\rho})$ such that $B_{\rho/3}(p_i) \cap B_{\rho/3}(p_j) = \emptyset$ if $i \neq j$ (see, e.g., Lemma 2.2 in [4]).

By the uniform convergence of E_k, if k is sufficiently large (possibly in dependence of ρ), we have that $B_{\rho/6}(p_j) \cap \partial E_k \neq \emptyset$ for any $j \in \{1, \ldots, N_\rho\}$. Then, by Theorem 2,

$$\begin{aligned} c_3 N_\rho \rho^{n-s_k} &\leqslant \frac{s_k(1-s_k)}{\nu(n, s_k, +\infty)} \sum_{j=1}^{N_\rho} \int_{E_k\cap B_{\rho/3}(p_j)} \int_{(\complement E_k)\cap B_{\rho/3}(p_j)} \frac{1}{|x-y|^{n+s_k}}\, dx\, dy \\ &\leqslant \frac{s_k(1-s_k)}{\nu(n, s_k, +\infty)} \int_{E_k\cap B_2} \int_{(\complement E_k)\cap B_2} \frac{1}{|x-y|^{n+s_k}}\, dx\, dy. \end{aligned}$$

Hence, since the latter quantity is bounded in the light of Corollary 15, we conclude that $N_\rho \leqslant c_4 \rho^{s_k-n}$. Since this was valid, fixed ρ, for any k sufficiently large, we conclude that

$$N_\rho \leqslant \lim_{k\to+\infty} c_4 \rho^{s_k-n} = c_4 \rho^{1-n}.$$

Hence,

$$\mathcal{H}^{n-1}(\partial E \cap B_1) \leqslant \lim_{\rho\to 0^+} \omega_{n-1} \sum_{j=1}^{N_\rho} \rho^{n-1} \leqslant c_5.$$

This gives the upper bound in (1.4) and so Theorem 3.

11 Rearrangements on the lines

The rearrangement result of Theorem 4 will be obtained after performing a geometric analysis on straight lines. Namely, we discuss how the energy corresponding to the interaction between parallel lines is minimized for particular configurations: for compact perturbations of half lines, the energy is minimal for the half lines with same end points, and for compact sets of prescribed measure the energy is minimized by the two parallel segments with the same mid points:

Lemma 20 *Let L_1 and L_2 be straight lines of direction e_n. Let $a_- \leqslant a_+ \in \mathbb{R}$ and $A \subset \mathbb{R}^n$. Suppose that, for $i = 1, 2$,*

$$\{x_n < a_-\} \cap L_i \subseteq A \cap L_i \subseteq \{x_n < a_+\} \cap L_i. \tag{11.1}$$

Then

$$\begin{aligned} &\int_{A\cap L_1} \int_{(\complement A)\cap L_2} \frac{1}{|x-y|^{n+s}} dx_n\, dy_n + \int_{(\complement A)\cap L_1} \int_{A\cap L_2} \frac{1}{|x-y|^{n+s}} dx_n\, dy_n \\ &\geqslant \int_{\{x_n<0\}\cap L_1} \int_{\{x_n\geqslant 0\}\cap L_2} \frac{1}{|x-y|^{n+s}} dx_n\, dy_n + \int_{\{x_n\geqslant 0\}\cap L_1} \int_{\{x_n<0\}\cap L_2} \frac{1}{|x-y|^{n+s}} dx_n\, dy_n. \end{aligned} \tag{11.2}$$

Proof The rough idea for the proof of the rearrangement is the following: if the endpoint of one of the lines is strictly above (below) the other, one lowers (raises) that set: in this way, one looses the interaction with two opposite halflines, and gains the interaction with less than that—so the energy decreases. Then, when the end points are at the same height, one lowers both a little bit from the top (raises both a little bit from the bottom), and again, one looses the interaction with the two opposite halflines etc.—and the energy still decreases. Now, let us go into the technical details of the argument.

First, we observe that we may assume that

$$A \cap L_1 \text{ and } A \cap L_2 \text{ are the countable union of disjoint open intervals.} \tag{11.3}$$

Indeed, by possibly changing $A \cap L_i$ by a zero one-dimensional Lebesgue measure set, which does not affect the integral, we may suppose that, for $i = 1, 2$,

$$A \cap L_i = \bigcap_{j=1}^{+\infty} O_{j,i},$$

for suitable open sets $O_{j,i} \subset \mathbb{R}$, which we may take to be disjoint intervals (see, e.g., Theorem 3.29 in [7]). Then, for any $M \in \mathbb{N}$, we consider the set

$$A_M := \bigcup_{i=1}^{2} \bigcap_{j=1}^{M} O_{j,i},$$

which satisfies (11.3). Then, if (11.2) holds under assumption (11.3), it holds for A_M and so, by taking M arbitrarily large and using the Dominated Convergence Theorem, for A as well.

Hence, we assume (11.3). In fact, we may reduce to a finite collection of intervals, i.e. to the case in which

$$A \cap L_1 \text{ and } A \cap L_2 \text{ are the union of finitely many disjoint open intervals.} \tag{11.4}$$

Indeed, if (11.2) holds under assumption (11.4), we fix M and consider the union of M intervals in (11.3); then (11.2) holds for such union, and so, by Dominated Convergence Theorem, for A as well.

Moreover, if two intervals of (11.4) are adjacent, we may just add one point without changing the integral. This would merge together all the adjacent integrals, so we may also assume that

$$\text{the intervals in (11.4) are not adjacent.} \tag{11.5}$$

Now, we introduce the short notations

$$\int_X \int_Y := \int_{X\cap L_1} \int_{Y\cap L_2} \frac{1}{|x-y|^{n+s}}\, dx_n\, dy_n$$

and

$$\Lambda(A) := \int_A \int_{\complement A} + \int_{\complement A} \int_A .$$

Notice that, in this notation, the game is to make $\Lambda(A)$ as small as possible. For this, we first observe that we can reduce to the case in which $A \cap L_1$ and $A \cap L_2$ have the same halflines at $\pm\infty$, that is we may suppose that

(**1**) either $A \cap L_1 = \{x_n < a\} \cap L_1$ and $A \cap L_2 = \{x_n < a\} \cap L_2$, for some $a \in \mathbb{R}$,

(**2**) or there exists $b_- < b_+ \in \mathbb{R}$ and $c \in (0, (b_+ - b_-)/2)$ such that

$$\begin{aligned}
&\{x_n < b_-\} \cap L_i \subseteq A \cap L_i,\\
&\{x_n \geqslant b_+\} \cap L_i \subseteq (\complement A) \cap L_i,\\
&\{b_- \leqslant x_n < b_- + c\} \cap L_i \subseteq (\complement A) \cap L_i \ \text{ and}\\
&\{b_+ - c < x_n < b_+\} \cap L_i \subseteq A \cap L_i,\\
&\text{for } i = 1, 2.
\end{aligned} \tag{11.6}$$

To prove (11.6), with no loss of generality, thanks to the discussions above (recall (11.4) and (11.5)), we can consider the case in which the bottom halfline of L_2 is higher than the one of L_1, that is, for some $\alpha > \beta \geqslant 0$,

$$\begin{aligned}
&\{x_n < a_-\} \cap L_1 \subseteq A \cap L_1\\
&\{a_- \leqslant x_n < a_- + \alpha\} \cap L_1 \subseteq (\complement A) \cap L_1 \text{ and}\\
&\{x_n < a_- + \beta\} \cap L_2 \subseteq A \cap L_2.
\end{aligned} \tag{11.7}$$

Then, we show that we can decrease Λ by adding the additional interval on the bottom halfline of L_1 too. That is, we set

$$J_i := \{a_- \leqslant x_n < a_- + \beta\} \cap L_i$$

and $A := A \cup J_1$ (notice that this is a disjoint union). We have

$$\begin{aligned}
\Lambda(A) - \Lambda(\tilde{A}) &= \int_{J_1} \int_A - \int_{J_1} \int_{\complement A}\\
&\geqslant \int_{J_1} \int_{\{x_n < a_-\}} - \int_{J_1} \int_{\{x_n > a_- + \beta\}} .
\end{aligned}$$

Since the latter quantity vanishes by symmetry, it follows that $\Lambda(\tilde{A}) \leqslant \Lambda(A)$. That is, the energy decreases by adding the additional interval on L_1.

Repeating this argument with the roles of A and $\complement A$ interchanged, we obtain (11.6).

Now, if we are in the first alternative of (11.6) we are done. So we have to show that if we are in the second alternative of (11.6) the energy decreases if we raise b_- towards b_+ on both the lines. Namely, if we are in the second alternative of (11.6), we define

$$J := \{b_- \leqslant x_n < b_- + c\}$$

and $A := A \cup J$ (notice that this union is disjoint when considered on L_1 and on L_2). Then,

$$\begin{aligned}
&\Lambda(A) - \Lambda(\tilde{A}) \\
&= \int\limits_J \int\limits_A + \int\limits_A \int\limits_J - \int\limits_J \int\limits_{(\complement A)\setminus J} - \int\limits_{(\complement A)\setminus J} \int\limits_J \\
&\geqslant \int\limits_J \int\limits_{\{x_n < b_-\}} + \int\limits_{\{x_n < b_-\}} \int\limits_J - \int\limits_J \int\limits_{\{x_n \geqslant b_- + c\}} - \int\limits_{\{x_n \geqslant b_- + c\}} \int\limits_J \\
&= \left(\int\limits_J \int\limits_{\{x_n < b_-\}} - \int\limits_J \int\limits_{\{x_n \geqslant b_- + c\}} \right) + \left(\int\limits_{\{x_n < b_-\}} \int\limits_J - \int\limits_{\{x_n \geqslant b_- + c\}} \int\limits_J \right),
\end{aligned}$$

which, again, vanishes by symmetry. This proves that we can reduce to the first alternative of (11.6), as desired. □

Integrating line by line, we can use the above result to estimate the full double integral of Theorem 4, as we now discuss.

12 Proof of Theorem 4

Fix $x'_o\ y'_o \in U \subset \mathbb{R}^{n-1}$. Then, we can use Lemma 20 with $L_1 := \{x \in \mathbb{R}^n \text{ s.t. } x' = x'_o\}$ and $L_2 := \{x \in \mathbb{R}^n \text{ s.t. } x' = y'_o\}$, since condition (11.1) is granted here by (1.5). After this, by integrating over x'_o, $y'_o \in U$, and by dividing by 2, we obtain the claim of Theorem 4.

13 A lower bound as a consequence of Theorem 4

With the above rearrangement result, one obtains that the s-energy on an infinite cylinder is bounded from below by the perimeter of the disc.

Corollary 21 *Let $s \in [1/2, 1)$. There exists a universal $\eta_o \in (0, 1/10)$, independent of s, for which the following holds. Let $r > 0$, $\eta \in (0, \eta_o)$. Let $A \subset \mathbb{R}^n$ be such that*

$$B_r \cap \{x_n < -r\eta\} \subseteq A \quad \textit{and} \quad B_r \cap \{x_n > r\eta\} \subseteq \complement A. \tag{13.1}$$

Then, there exists $C \geqslant 1$, possibly depending on r and η but independent of s, such that

$$\frac{s(1-s)}{\nu(n, s, +\infty)} \int\limits_{B_r \cap A} \int\limits_{B_r \cap \complement A} \frac{1}{|x-y|^{n+s}} dx\, dy \geqslant \omega_{n-1} \left(r(1-\eta)\right)^{n-1} - \eta - C\,(1-s).$$

Proof Let

$$\begin{aligned} K &:= \{x \in \mathbb{R}^n \text{ s.t. } |x'| < (1-\eta)r\}, \\ K_+ &:= \{x \in K \text{ s.t. } x_n > 0\}, \\ K_- &:= \{x \in K \text{ s.t. } x_n < 0\}, \text{ and} \\ A' &:= (A \cup (K_- \setminus B_r)) \setminus (K_+ \setminus B_r). \end{aligned}$$

Notice that

$$A' \cap B_r = A \cap B_r.$$

This and (13.1) give that

$$\{x_n \leqslant -r\eta\} \cap K \subseteq A' \cap K \subseteq \{x_n \leqslant r\eta\} \cap K,$$

so assumption (1.5) in Theorem 4 holds for A' and therefore

$$\int_{K\cap A'}\int_{K\cap \complement A'} \frac{1}{|x-y|^{n+s}}\,dx\,dy \geqslant \int_{K\cap\{x_n<0\}}\int_{K\cap\{x_n\geqslant 0\}} \frac{1}{|x-y|^{n+s}}\,dx\,dy. \tag{13.2}$$

Now, notice that if $x \in A' \cap (K \setminus B_{(1-\eta/2)r})$ and $y \in \complement A' \cap B_{(1-\eta/2)r}$, or if $x \in \complement A' \cap (K \setminus B_{(1-\eta/2)r})$ and $y \in A' \cap B_{(1-\eta/2)r}$, or if $x \in A' \cap (K \setminus B_{(1-\eta/2)r})$ and $y \in \complement A' \cap (K \setminus B_{(1-\eta/2)r})$, we have $|x-y| \geqslant c$, for a suitable $c > 0$ possibly depending on r and η but independent of s, thanks to (13.1), if η_o is small enough. Therefore

$$\begin{aligned} \int_{K\cap A'}\int_{K\cap \complement A'} \frac{1}{|x-y|^{n+s}}\,dx\,dy &\leqslant \int_{B_{(1-\eta/2)r}\cap A'}\int_{B_{(1-\eta/2)r}\cap \complement A'} \frac{1}{|x-y|^{n+s}}\,dx\,dy + C_o \\ &\leqslant \int_{B_r\cap A'}\int_{B_r\cap \complement A'} \frac{1}{|x-y|^{n+s}}\,dx\,dy + C_o = \int_{B_r\cap A}\int_{B_r\cap \complement A} \frac{1}{|x-y|^{n+s}}\,dx\,dy + C_o \end{aligned}$$

for a suitable $C_o \geqslant 1$ possibly depending on r and η but independent of s.

This, (13.2) and Lemma 11 imply the desired result. □

14 Proof of Theorem 5

We fix $\mu > 0$, to be taken arbitrarily small in the sequel.

Thanks to (1.3) in Theorem 3, we may apply Lemma 8 to our limit set E, with $\epsilon := \mu/2$, and obtain that there exist $r_\mu > 0$ and a collection of balls $\left\{B_{\rho_j}(a_j)\right\}_{j\in\mathbb{N}}$ such that, if we set

$$\mathcal{A} := \bigcup_{j=1}^{+\infty} B_{\rho_j}(a_j),$$

we have that

$$\omega_{n-1}\sum_{j=1}^{+\infty}\rho_j^{n-1} \leqslant \mu \tag{14.1}$$

and if $x \in (\partial E) \setminus \mathcal{A}$ then $x \in \partial^\star E$ and

$$\partial E \cap B_r(x) \subseteq T_{r,\mu/2}(x) \tag{14.2}$$

for any $r \in (0, r_\mu]$.

Also, making use of (1.4) and (14.1),

$$\operatorname{Per}(E, B_{R-\mu}) \leqslant \mathcal{H}^{n-1}\left(\left((\partial E)\cap B_{R-\mu}\right)\setminus \mathcal{A}\right) + \mu < +\infty.$$

That is, there exists a suitable $\delta_\mu \in (0, r_\mu/2)$, independent of s_k, such that, for any $\delta \in (0, \delta_\mu)$,

$$\operatorname{Per}(E, B_{R-\mu}) \leqslant 2\mu + \inf\left\{\omega_{n-1}\sum_{i=1}^{+\infty} r_i^{n-1}\right\} < +\infty, \tag{14.3}$$

where the above inf is taken over all the coverings of $\left((\partial E)\cap B_{R-\mu}\right)\setminus\mathcal{A}$ made of balls $B_{r_i}(b_i)$ with $b_i \in \left((\partial E)\cap B_{R-\mu}\right)\setminus\mathcal{A}$ and $r_i \leqslant \delta$.

By the Vitali's Covering Theorem, we can take a disjoint subcovering of $\left((\partial E)\cap B_{R-\mu}\right)\setminus \mathcal{A}$, which we still denote by $B_{r_i}(b_i)$, and (14.3) becomes

$$\operatorname{Per}(E, B_{R-\mu}) \leqslant 2\mu + \omega_{n-1}\sum_{i=1}^{+\infty} r_i^{n-1} < +\infty. \tag{14.4}$$

Now, we take $N_\mu \in \mathbb{N}$ so large that

$$\omega_{n-1}\sum_{i=N_\mu+1}^{+\infty} r_i^{n-1} \leqslant \mu.$$

Then, (14.4) becomes

$$\operatorname{Per}(E, B_{R-\mu}) \leqslant 3\mu + \omega_{n-1}\sum_{i=1}^{N_\mu} r_i^{n-1}. \tag{14.5}$$

Now, we observe that $2r_i \leqslant 2\delta_\mu \leqslant r_\mu$; so, we recall (14.2) to point out that there exists $k_\mu \in \mathbb{N}$ such that, if $k \geqslant k_\mu$, then

$$\partial E_k \cap B_{2r_i}(b_i) \subseteq T_{2r_i,\mu}(b_i) \tag{14.6}$$

for any $i = 1, \ldots, N_\mu$.

Thanks to (14.6), we can apply Corollary 21 to E_k in any $B_{r_i}(b_i)$: fixed $\eta \in (0, 1)$, independent of μ, and to be taken arbitrarily small in the sequel, we obtain that there exists $C_{\eta,\mu} \geqslant 1$, independent of s_k, such that

$$\frac{s_k(1-s_k)}{\nu(n, s_k, +\infty)}\int_{B_{r_i}(b_i)\cap E_k}\int_{B_{r_i}(b_i)\cap \complement E_k}\frac{1}{|x-y|^{n+s}}dx\,dy \geqslant \omega_{n-1}\left(r_i(1-\eta)\right)^{n-1} - \eta - C_{\eta,\mu}(1-s_k),$$

for any $i = 1, \ldots, N_\mu$ and any $k \geqslant k_\mu$.

Hence, by summing up the above estimate, recalling that the balls $B_{r_i}(b_i)$ are disjoint and centered at points of $B_{R-\eta}$, and so contained in B_R, and taking the limit, we conclude that

$$\lim_{k\to+\infty}\frac{s_k(1-s_k)}{\nu(n, s_k, +\infty)}\int_{B_R\cap E_k}\int_{B_R\cap \complement E_k}\frac{1}{|x-y|^{n+s}}dx\,dy \geqslant \omega_{n-1}\sum_{i=1}^{N_\mu}\left(r_i(1-\eta)\right)^{n-1} - \eta N_\mu.$$

Consequently, fixing μ and taking η arbitrarily small,

$$\lim_{k\to+\infty}\frac{s_k(1-s_k)}{\nu(n, s_k, +\infty)}\int_{B_R\cap E_k}\int_{B_R\cap \complement E_k}\frac{1}{|x-y|^{n+s}}dx\,dy \geqslant \omega_{n-1}\sum_{i=1}^{N_\mu} r_i^{n-1}.$$

Therefore, recalling (14.5),

$$\begin{aligned}\mathrm{Per}\,(E, B_{R-\mu}) - 3\mu &\leqslant \omega_{n-1}\sum_{i=1}^{N_\mu} r_i^{n-1}\\ &\leqslant \lim_{k\to+\infty}\frac{s_k(1-s_k)}{\nu(n,s_k,+\infty)}\int_{B_R\cap E_k}\int_{B_R\cap \mathcal{C}E_k}\frac{1}{|x-y|^{n+s}}dx\,dy.\end{aligned}$$

Thus, the desired result follows by taking now μ arbitrarily small.

Remark We point out that Theorem 5 holds (with the same proof) dropping the minimality assumptions on E_k but requiring E to be locally finite perimeter and satisfying (1.3).

15 Proof of Theorem 6

We argue by contradiction: we suppose that

$$E \text{ does not have minimal perimeter in } B_R. \tag{15.1}$$

We know that E has locally finite $(n-1)$-dimensional Hausdorff measure and perimeter, thanks to Theorem 3. So, by possibly taking R slightly larger, we can suppose that $\mathcal{H}^{n-1}(\partial E\cap\partial B_R)=0$ (recall footnote 2). We take $E^\sharp$ to be a set with minimal perimeter in B_R (see Theorem 1.20 in [4]), that is

$$\mathrm{Per}\,(E^\sharp,\mathbb{R}^n)\leqslant \mathrm{Per}\,(F,\mathbb{R}^n)\quad \text{if } F\cap(\mathcal{C}B_R)=E^\sharp\cap(\mathcal{C}B_R)=E\cap(B_{(1+\beta)R}\setminus B_R).$$

By (15.1),

$$\mathrm{Per}\,(E^\sharp,\mathbb{R}^n) < \mathrm{Per}\,(E,\mathbb{R}^n). \tag{15.2}$$

We remark that, since B_R is convex, we can write (15.2) as

$$\mathrm{Per}\,(E^\sharp, B_R) < \mathrm{Per}\,(E, B_R).$$

That is,

$$t + \mathrm{Per}\,(E^\sharp, B_R) < \mathrm{Per}\,(E, B_R) \tag{15.3}$$

for some $t\in(0,1)$.

The idea is now to interpolate between $E^\sharp$ and E_k across a suitable ring outside B_R, by estimating the boundary contributions with a counting argument. We take $d\in(0,t)$, to be chosen conveniently small with respect to t (as well as with respect to R and β, which are now considered as fixed). We denote by $c_i\in(0,1)$ suitable quantities independent of k, t and d (Fig. 3).

First of all, we claim that

$$\begin{aligned}&\text{there exist } M\leqslant d^{2-n}/c_1,\ j\in\mathbb{N}\cap[2,\ c_2/d]\\ &\text{and points } \{p_i\}_{i=1,\dots,M} \text{ with } |p_i|\in(R+10d(j-1),\,R+10dj)\\ &\text{such that } \partial E\cap\left(B_{R+10dj+(d/5)}\setminus B_{R+10d(j-1)-(d/5)}\right)\subseteq \textstyle\bigcup_{i=1}^M B_d(p_i).\end{aligned} \tag{15.4}$$

To prove this, we take c_2 such that

$$c_2\leqslant\frac{\beta R}{20} \tag{15.5}$$

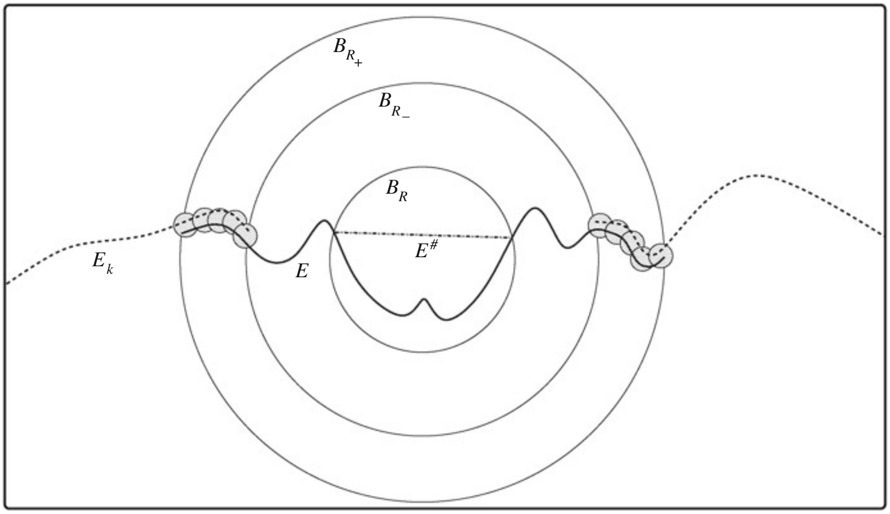

Fig. 3 The sets involved in Theorem 6

and we consider, for any $j \in \mathbb{N} \cap [2, c_2/d]$, a covering of $B_{R+10dj+(d/5)} \setminus B_{R+10d(j-1)-(d/5)}$ by balls of radius d centered in $B_{R+10dj} \setminus B_{R+10d(j-1)}$ and such that the dilations by a factor 2 of these balls overlap at most $1/c_3$ times. Let M_j be the number of these balls which intersect ∂E: we write $B_d(q_{i,j}) \cap \partial E \neq \emptyset$ for any $i = 1, \ldots, M_j$. By the lower bound in (1.4),

$$\mathcal{H}^{n-1}(\partial E \cap B_{d+(d/5)}(q_{i,j})) \geqslant c_4 d^{n-1}.$$

Since the above balls are finitely overlapping, we infer that

$$\mathcal{H}^{n-1}\left(\partial E \cap \left(B_{R+10dj+2d} \setminus B_{R+10d(j-1)-2d}\right)\right) \geqslant c_5 d^{n-1} N_j.$$

Now, if (15.4) were false, we would have $N_j \geqslant d^{2-n}/c_1$ for any $j \in \mathbb{N} \cap [2, c_2/d]$, and so

$$\mathcal{H}^{n-1}\left(\partial E \cap \left(B_{R+10dj+2d} \setminus B_{R+10d(j-1)-2d}\right)\right) \geqslant \frac{c_5 d}{c_1}. \tag{15.6}$$

So, by summing up (say, over the even j's, so that the rings in (15.6) do not overlap), recalling the upper bound in (1.4) and (15.5), we deduce that

$$c_7 \geqslant \mathcal{H}^{n-1}(\partial E \cap B_{(1+\beta)R}) \geqslant \frac{c_6}{c_1}.$$

This is a contradiction if c_1 is chosen suitably small in (15.4), and so (15.4) is proved.

With this, for any $\lambda > 0$, we define

$$\mathcal{B}_\lambda := \bigcup_{i=1}^{M} B_{\lambda d}(p_i).$$

By the uniform convergence of E_k, we deduce from (15.4) that

$$(\partial E_k \cup \partial E) \cap \left(B_{R+10dj+(d/10)} \setminus B_{R+10d(j-1)-(d/10)}\right) \subseteq \mathcal{B}_2 \tag{15.7}$$

if k is large enough.

Let also $R_- := R + 10d(j-1)$, $R_+ := R + 10dj$ and $A := B_{R_+} \setminus B_{R_-}$. The idea is now to interpolate E inside B_{R_-} with E_k outside B_{R_+} by adding $\mathcal{B}_4$ to the set E in A. Then, the interaction inside/inside would converge to something smaller than the perimeter inside, while the interaction inside/outside without $\mathcal{B}_4$ and the interaction of $\mathcal{B}_4$ with anything else are smaller than d, that we can make smaller than t, contradicting the s_k-minimality of E_k.

The details of this argument go as follows. Recalling (15.3), we consider a smooth set $E^\star$ with almost the same perimeter of $E^\sharp$ in B_{R_-}. In this way, recalling (15.7), we may write

$$(\partial E_k \cup \partial E \cup \partial E^\star) \cap \left(B_{R+10dj+(d/10)} \setminus B_{R+10d(j-1)-(d/10)}\right) \subseteq \mathcal{B}_3. \tag{15.8}$$

Also, the fact that $R_- > R$ and (15.3) imply

$$\frac{t}{2} + \operatorname{Per}(E^\star, B_{R_-}) < \operatorname{Per}(E, B_{R_-}). \tag{15.9}$$

We define

$$E_k^\star := \left(E^\star \cap B_{R_-}\right) \cup ((\mathcal{B}_4 \cup E) \cap A) \cup \left(E_k \cap (\mathcal{C}B_{R_+})\right).$$

By the minimality of E_k, we know that

$$\mathcal{J}_{s_k}(E_k, B_{R_+}) \leqslant \mathcal{J}_{s_k}(E_k^\star, B_{R_+}). \tag{15.10}$$

Moreover, from Corollary 14,

$$\begin{aligned} \lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n, s_k, +\infty)} \int_{\mathcal{B}_4\cap A} \int_{\mathcal{C}(\mathcal{B}_4\cap A)} &\leqslant \lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n, s_k, +\infty)} \sum_{i=1}^{M} \int_{B_{4d}(p_i)\cap A} \int_{\mathcal{C}(\mathcal{B}_4\cap A)} \\ &\leqslant \sum_{i=1}^{M} \lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n, s_k, +\infty)} \int_{B_{4d}(p_i)\cap A} \int_{\mathcal{C}(B_{4d}(p_i)\cap A)} \leqslant \frac{Md^{n-1}}{c_7} \leqslant \frac{d}{c_8}, \end{aligned} \tag{15.11}$$

where the integrand $1/|x-y|^{n+s_k}\,dx\,dy$ has been omitted for simplicity.

Moreover, by (15.8), we know that if $x \in (E_k^\star \cap A) \setminus \mathcal{B}_4$ and $y \in \mathcal{C}E_k^\star$ (as well if $x \in ((\mathcal{C}E_k^\star) \cap A) \setminus \mathcal{B}_4$ and $y \in E_k^\star$), then $|x-y| \geqslant c_9 d$ and therefore

$$\int_{(E_k^\star\cap A)\setminus\mathcal{B}_4} \int_{\mathcal{C}E_k^\star} + \int_{((\mathcal{C}E_k^\star)\cap A)\setminus\mathcal{B}_4} \int_{E_k^\star} \leqslant \frac{1}{c_{10} d^{n+s}},$$

that is

$$\lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n, s_k, +\infty)} \left[\int_{(E_k^\star\cap A)\setminus\mathcal{B}_4} \int_{\mathcal{C}E_k^\star} + \int_{((\mathcal{C}E_k^\star)\cap A)\setminus\mathcal{B}_4} \int_{E_k^\star} \right] = 0. \tag{15.12}$$

By (15.11) and (15.12), and noticing that $(\mathcal{C}E_k^\star) \cap A \cap \mathcal{B}_4 = \emptyset$, we have

$$\lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n, s_k, +\infty)} \left[\int_{E_k^\star\cap A} \int_{\mathcal{C}E_k^\star} + \int_{E_k^\star} \int_{(\mathcal{C}E_k^\star)\cap A} \right] \leqslant \frac{d}{c_{11}},$$

and, since $|x-y| \geqslant R_+ - R_-$ for any $x \in R_-$ and $y \in R_+$,

$$\lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n, s_k, +\infty)} \int_{B_{R_-}} \int_{\mathcal{C}B_{R_+}} \leqslant \lim_{k\to+\infty} \frac{s_k(1-s_k)}{c_{12}\,\nu(n, s_k, +\infty)} = 0.$$

Accordingly, using also Theorem 1,

$$\lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n,s_k,+\infty)} \mathcal{J}_{s_k}(E_k^\star, B_{R_+}) \leqslant \lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n,s_k,+\infty)} \int\limits_{E_k^\star\cap B_{R_-}} \int\limits_{(\complement E_k^\star)\cap B_{R_-}} + \frac{d}{c_{13}}$$

$$= \lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n,s_k,+\infty)} \int\limits_{E^\star\cap B_{R_-}} \int\limits_{(\complement E^\star)\cap B_{R_-}} + \frac{d}{c_{13}}$$

$$= \operatorname{Per}(E^\star, B_{R_-}) + \frac{d}{c_{13}}.$$

Therefore, if $d \leqslant c_{13}t/4$, recalling (15.10),

$$\lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n,s_k,+\infty)} \mathcal{J}_{s_k}(E_k, B_{R_+}) \leqslant \operatorname{Per}(E^\star, B_{R_-}) + \frac{t}{4}. \tag{15.13}$$

On the other hand, combining (15.9) and Theorem 5, we see that

$$\lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n,s_k,+\infty)} \mathcal{J}_{s_k}(E_k, B_{R_+}) \geqslant \lim_{k\to+\infty} \frac{s_k(1-s_k)}{\nu(n,1,+\infty)} \int\limits_{E_k\cap B_{R_-}} \int\limits_{(\complement E_k)\cap B_{R_-}}$$

$$\geqslant \operatorname{Per}(E, B_{R_-}) \geqslant \operatorname{Per}(E^\star, B_{R_-}) + \frac{t}{2}.$$

Since this is in contradiction with (15.13), the proof of Theorem 6 is complete.

16 Compactness of minimizers and fine cubes analysis

Our next target is to prove Theorem 7, i.e. that s_k-minimizers are precompact in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ when $s_k \to 1^-$. For this, we perform a measure theoretic analysis on fine cubes:

Lemma 22 *There exist $c \in (0, 1/2)$ and $C \geqslant 1$, independent of s, for which the following claim holds true.*

Let E be an s-minimal set, with $s \in (1-c, 1)$. Let $N \in \mathbb{N}$. Let Q be the unit cube, i.e.

$$Q := \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \text{ s.t. } \max_{i=1,\dots,n} |x_i| < 1 \right\}. \tag{16.1}$$

Let us split Q into a collection of nonoverlapping cubes $\{Q_j\}_{j\in\{1,\dots,N^n\}}$ of side $1/N$ covering Q.

Then,

$$\sum_{j=1}^{N^n} \frac{|E\cap Q_j|\,|Q_j\setminus E|}{|Q_j|} \leqslant C N^{-s}.$$

Proof We will denote by C_i suitable constants, independent of N. Let us define

$$G = \{j \in \{1,\dots,N^n\} \text{ s.t. } Q_j \cap \partial E = \emptyset\} \text{ and}$$
$$H = \{j \in \{1,\dots,N^n\} \text{ s.t. } Q_j \cap \partial E \neq \emptyset\}.$$

Of course, $|Q_j| = 1/N^n$, $\{1,\dots,N^n\} = G \cup H$ and

$$\sum_{j=1}^{N^n} \frac{|E\cap Q_j|\,|Q_j\setminus E|}{|Q_j|} = \sum_{j\in H} \frac{|E\cap Q_j|\,|Q_j\setminus E|}{|Q_j|}. \tag{16.2}$$

Let Q'_j be the dilation of Q_j with respect to its center of factor $2N$.

Since the collection $\{Q'_j\}_{j\in\{1,\dots,N^n\}}$ is finite overlapping, we have that

$$\sum_{j\in H}\int_{E\cap Q'_j}\int_{(\mathcal{C}E)\cap Q'_j}\frac{1}{|x-y|^{n+s}}\,dx\,dy\leqslant C_1\int_{E\cap B_{C_2}}\int_{(\mathcal{C}E)\cap B_{C_2}}\frac{1}{|x-y|^{n+s}}\,dx\,dy\leqslant\frac{C_3}{1-s}, \tag{16.3}$$

thanks to Corollary 15.

On the other hand, by Theorem 2, if $j\in H$, we have that

$$\int_{E\cap Q'_j}\int_{(\mathcal{C}E)\cap Q'_j}\frac{1}{|x-y|^{n+s}}\,dx\,dy\geqslant\frac{(1/N)^{n-s}}{C_4(1-s)}$$

and so, recalling (16.3),

$$\frac{h\,(1/N)^{n-s}}{C_4(1-s)}\leqslant\sum_{j\in H}\int_{E\cap Q'_j}\int_{(\mathcal{C}E)\cap Q'_j}\frac{1}{|x-y|^{n+s}}\,dx\,dy\leqslant\frac{C_3}{1-s},$$

where h is the cardinality of H, that is

$$h\leqslant C_5\,N^{n-s}.$$

Thus, from (16.2),

$$\sum_{j=1}^{N^n}\frac{|E\cap Q_j|\,|Q_j\setminus E|}{|Q_j|}\leqslant\sum_{j\in H}|Q_j|\leqslant h\,(1/N)^n\leqslant C_5\,N^{-s},$$

as desired. □

17 Proof of Theorem 7

We will show that $\{\chi_{E_k}\}_{k\in\mathbb{N}}$ is totally bounded, i.e., fixed $N\in\mathbb{N}$ arbitrarily large, there exist $M(N)\in\mathbb{N}$ and functions $\psi_1,\dots,\psi_{M(N)}\in L^1(B_R)$ such that, for any $k\in\mathbb{N}$, there exists $i_k\in\{1,\dots,\psi_{M(N)}\}$ for which

$$\|\chi_{E_k}-\psi_{i_k}\|_{L^1(B_R)}\leqslant\frac{1}{\sqrt[4]{N}}. \tag{17.1}$$

Without loss of generality, we take $s_k\in(1-c,1)$, where c is as in Lemma 22, $R:=1$ and we observe that $B_1\subseteq Q$, where Q is as in (16.1). Fixed $N\in\mathbb{N}$, we take the collection $\{Q_j\}_{j\in\{1,\dots,N^n\}}$, as in Lemma 22. For any $j\in\{1,\dots,N^n\}$, we define

$$p_j(E_k):=\frac{|E_k\cap Q_j|}{|Q_j|}\in[0,1]. \tag{17.2}$$

Notice that $p(E_k):=(p_1(E_k),\dots,p_{N^n}(E_k))$ belongs to a bounded set of $\mathbb{R}^{N^n}$, so there exist $M(N)\in\mathbb{N}$ and points $q^{(1)},\dots,q^{M(N)}\in\mathbb{R}^{N^n}$ such that

$$\text{for any } k \text{ there exists } i_k\in\{1,\dots,M(N)\} \text{ for which } |p(E_k)-q^{(i_k)}|\leqslant\frac{1}{N}. \tag{17.3}$$

For any $i \in \{1, \dots, M(N)\}$ we define

$$\psi_i(x) := \sum_{j=1}^{N^n} q_j^{(i)} \chi_{Q_j}(x).$$

We observe that, if i_k is as in (17.3),

$$\begin{aligned}
\|\chi_{E_k} - \psi_{i_k}\|_{L^1(Q)} &\leqslant \sum_{j=1}^{N^n} \int_{Q_j} |\chi_{E_k}(x) - \psi_{i_k}(x)|\, dx \\
&= \sum_{j=1}^{N^n} \int_{E_k \cap Q_j} |1 - \psi_{i_k}(x)|\, dx + \int_{Q_j \setminus E_k} |\psi_{i_k}(x)|\, dx \\
&= \sum_{j=1}^{N^n} |1 - q_j^{(i_k)}|\, |E_k \cap Q_j| + |q_j^{(i_k)}|\, |Q_j \setminus E_k| \\
&\leqslant \sum_{j=1}^{N^n} |1 - p_j(E_k)|\, |E_k \cap Q_j| + |p_j(E_k)|\, |Q_j \setminus E_k| + 2|p(E_k) - q^{(i_k)}|\, |Q_j| \\
&\leqslant \frac{2}{N} + \sum_{j=1}^{N^n} |1 - p_j(E_k)|\, |E_k \cap Q_j| + |p_j(E_k)|\, |Q_j \setminus E_k|.
\end{aligned}$$

Recalling (17.2), we see that

$$1 - p_j(E_k) = \frac{|Q_j \setminus E_k|}{|Q_j|} \in [0, 1],$$

and so the above computation reads

$$\|\chi_{E_k} - \psi_{i_k}\|_{L^1(Q)} \leqslant \frac{2}{N} + 2 \sum_{j=1}^{N^n} \frac{|E_k \cap Q_j|\, |Q_j \setminus E_k|}{|Q_j|} \leqslant \frac{2}{N} + \frac{2C}{N^{s_k}} \leqslant \frac{2(1+C)}{\sqrt{N}},$$

thanks to Lemma 22. This implies (17.1) for large N, as desired.

Remark We observe that the proof of Corollary 7 does not use explicitly the minimization property, but rather (8.18) and (8.19). In the lines of the argument above, the boundary of sets that satisfy a clean ball condition and have finite $\mathcal{H}^{n-\epsilon}$-measure are precompact in L^1.

Note added in proof After the acceptance of this paper, we have received the very interesting article [1], where some results related to ours have been obtained by different methods (in particular, a coarea formula is used there for the interpolation). Also, we notice that $\nu(n, 1, +\infty) = \omega_{n-1}$, see (24) in [1].

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Appendix A: Uniform s-Sobolev embedding

The purpose of this appendix is to remark that the fractional norm controls the area of a set up to a factor $(1-s)$, uniformly as $s \to 1^-$. This will be achieved in the forthcoming Corollary 25, after some rearranging results.

Lemma 23 *Let L_1 and L_2 be straight lines of direction e_n. Let A_1, A_2 be bounded subsets of $\mathbb{R}^n$. Then*

$$\begin{aligned}
&\int_{A_1\cap L_1}\int_{(\complement A_2)\cap L_2} \frac{1}{|x-y|^{n+s}}dx_n\,dy_n + \int_{(\complement A_1)\cap L_1}\int_{A_2\cap L_2} \frac{1}{|x-y|^{n+s}}dx_n\,dy_n\\
&\geqslant \int_{L_1\cap\{|x_n|<|A_1\cap L_1|/2\}}\int_{L_2\cap\{|y_n|\geqslant|A_2\cap L_2|/2\}} \frac{1}{|x-y|^{n+s}}dx_n\,dy_n\\
&\quad+\int_{L_1\cap\{|x_n|\geqslant|A_1\cap L_1|/2\}}\int_{L_2\cap\{|y_n|<|A_2\cap L_2|/2\}} \frac{1}{|x-y|^{n+s}}dx_n\,dy_n. \qquad (A.1)
\end{aligned}$$

Proof For $i=1,2$, we consider the segments

$$I_i := L_i \cap \left\{ x_n \in \left(-\frac{|A_i\cap L_i|}{2}, \frac{|A_i\cap L_i|}{2} \right) \right\}.$$

We observe that

$$\int_{A_1\cap L_1}\int_{L_2} \frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n = \int_{I_1}\int_{L_2} \frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n, \qquad (A.2)$$

and analogously with the indices 1 and 2 interchanged.

The idea to prove (A.2) is that when we integrate "a point" in $A_1\cap L_1$ against the whole line L_2, we just get a number, so when we "sum up" the whole $A_1\cap L_1$ we get a "constant" (depending on the distance between the lines) times the measure of the set $A_1\cap L_1$.

The details of the proof of (A.2) are the following. By possibly changing $A_1\cap L_1$ by a zero measure set, which does not affect the integral, we may suppose that

$$A_1\cap L_1 = \bigcap_{j=1}^{+\infty} O_j,$$

for suitable bounded open sets $O_j \subset L_1$ (see, e.g., Theorem 3.29 in [7]). For any $M\in\mathbb{N}$, let

$$J_M := \bigcap_{j=1}^{M} O_j.$$

Since J_M is open and bounded, we can write

$$J_M = \bigcup_{j=1}^{+\infty} J_{M,j},$$

for suitable open bounded disjoint segments $J_{M,j}\subset L_1$ of length $\ell_{M,j}\geqslant 0$. We also set $\ell_{M,0} := 0$.

By translating both the variables of integrations, we see that, for any fixed j,

$$\int_{J_{M,j}}\int_{L_2}\frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n=\int_{\{0\leqslant x_n\leqslant \ell_{M,j}\}\cap L_1}\int_{L_2}\frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n$$
$$=\int_{\{\ell_{M,j-1}\leqslant x_n\leqslant \ell_{M,j-1}+\ell_{M,j}\}\cap L_1}\int_{L_2}\frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n.$$

So, by summing up,

$$\begin{aligned}&\int_{J_M}\int_{L_2}\frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n\\&=\sum_{j=1}^{+\infty}\int_{J_{M,j}}\int_{L_2}\frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n\\&=\int_{\left\{0\leqslant x_n\leqslant\sum_{j=1}^{+\infty}\ell_{M,j}\right\}\cap L_1}\int_{L_2}\frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n\\&=\int_{\{0\leqslant x_n\leqslant |J_M|\}\cap L_1}\int_{L_2}\frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n\\&=\int_{\{|x_n|\leqslant |J_M|/2\}\cap L_1}\int_{L_2}\frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n.\end{aligned}$$

By the Dominated Convergence Theorem, we obtain (A.2).

Moreover, by applying the Riesz Convolution Inequality (see, e.g., [2]) that establishes that a double convolution is maximized at the symmetric rearrangements, we get

$$\int_{A_1\cap L_1}\int_{A_2\cap L_2}\frac{1}{|x-y|^{n+s}}dx_n\,dy_n\leqslant\int_{I_1}\int_{I_2}\frac{1}{|x-y|^{n+s}}dx_n\,dy_n. \tag{A.3}$$

Thus, noticing that

$$\begin{aligned}&\int_{A_1\cap L_1}\int_{(\complement A_2)\cap L_2}\frac{1}{|x-y|^{n+s}}dx_n\,dy_n+\int_{(\complement A_1)\cap L_1}\int_{A_2\cap L_2}\frac{1}{|x-y|^{n+s}}dx_n\,dy_n\\&=\int_{A_1\cap L_1}\int_{L_2}\frac{1}{|x-y|^{n+s}}dx_n\,dy_n+\int_{L_1}\int_{A_2\cap L_2}\frac{1}{|x-y|^{n+s}}dx_n\,dy_n\\&\quad-2\int_{A_1\cap L_1}\int_{A_2\cap L_2}\frac{1}{|x-y|^{n+s}}dx_n\,dy_n,\end{aligned}$$

we obtain from (A.2) and (A.3) that the left hand side of (A.1) is bounded from below by

$$\int_{I_1}\int_{L_2} \frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n + \int_{L_1}\int_{I_2} \frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n$$
$$-2\int_{I_1}\int_{I_2} \frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n$$
$$= \int_{I_1}\int_{L_2\setminus I_2} \frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n + \int_{L_1\setminus I_1}\int_{I_2} \frac{1}{|x-y|^{n+s}}\,dx_n\,dy_n,$$

as desired. □

Corollary 24 *For any bounded set $A \subset \mathbb{R}^n$, let $B(A)$ be the ball centered at the origin such that $|B(A)| = |A|$. Then*

$$\int_{A}\int_{\mathcal{C}A} \frac{1}{|x-y|^{n+s}}\,dx\,dy \geqslant \int_{B(A)}\int_{\mathcal{C}(B(A))} \frac{1}{|x-y|^{n+s}}\,dx\,dy. \tag{A.4}$$

Proof This proof is reminiscent of the classical Steiner Symmetrization. Given $x'_o \in \mathbb{R}^{n-1}$, we consider the one-dimensional measure of the x'_o-slices of A, that is

$$m_A(x'_o) := \big|\{x \in A \text{ s.t. } x' = x'_o\}\big|$$

and we set

$$\mathcal{O}_A := \left\{x \in \mathbb{R}^n \text{ s.t. } |x_n| < \frac{m_A(x')}{2}\right\}.$$

Now, we fix $x'_o, y'_o \in \mathbb{R}^{n-1}$. We apply Lemma 23 with $L_1 := \{x \in \mathbb{R}^n \text{ s.t. } x' = x'_o\}$, $L_2 := \{x \in \mathbb{R}^n \text{ s.t. } x' = x'_o\}$, $A_1 := A_2 := A$, so we obtain

$$\int_{A\cap\{x'=x'_o\}}\int_{(\mathcal{C}A)\cap\{y'=y'_o\}} \frac{1}{|x-y|^{n+s}}dx_n\,dy_n + \int_{(\mathcal{C}A)\cap\{x'=x'_o\}}\int_{A\cap\{y'=y'_o\}} \frac{1}{|x-y|^{n+s}}dx_n\,dy_n$$
$$\geqslant \int_{\{x'=x'_o,\ |x_n|<|A\cap\{x'=x'_o\}|/2\}}\int_{\{y'=y'_o,\ |y_n|\geqslant|A\cap\{y'=y'_o\}|/2\}} \frac{1}{|x-y|^{n+s}}dx_n\,dy_n$$
$$+ \int_{\{x'=x'_o,\ |x_n|\geqslant|A\cap\{x'=x'_o\}|/2\}}\int_{\{y'=y'_o,\ |y_n|<|A\cap\{y'=y'_o\}|/2\}} \frac{1}{|x-y|^{n+s}}dx_n\,dy_n$$
$$= \int_{\{x'=x'_o,\ |x_n|<m_A(x'_o)/2\}}\int_{\{y'=y'_o,\ |y_n|\geqslant m_A(y'_o)/2\}} \frac{1}{|x-y|^{n+s}}dx_n\,dy_n$$
$$+ \int_{\{x'=x'_o,\ |x_n|\geqslant m_A(x'_o)/2\}}\int_{\{y'=y'_o,\ |y_n|<m_A(y'_o)/2\}} \frac{1}{|x-y|^{n+s}}dx_n\,dy_n.$$

Consequently, integrating over $x'_o, y'_o \in \mathbb{R}^{n-1}$, and dividing by 2,

$$\int_{A}\int_{\mathcal{C}A} \frac{1}{|x-y|^{n+s}}dx\,dy \geqslant \int_{\mathcal{O}_A}\int_{\mathcal{C}\mathcal{O}_A} \frac{1}{|x-y|^{n+s}}dx_n\,dy_n.$$

Notice that $\mathcal{O}_A$ is symmetric with respect to the direction e_n and $|\mathcal{O}_A| = |A|$ by Fubini's Theorem: thus, by repeating the above argument in any direction, we obtain (A.4). □

Corollary 25 *There exists $c \in (0, 1)$, independent of s for which the following holds. Let $s \in (1 - c, 1)$ and A be a bounded subset of $\mathbb{R}^n$. Then*

$$(1-s)\int_A \int_{\mathcal{C}A} \frac{1}{|x-y|^{n+s}}\,dx\,dy \geqslant c|A|^{(n-s)/n}.$$

Proof The claim follows from Corollary 24 and Lemma 13 (applied here with r equal to the radius of $B(A)$). □

Remark We observe that for sets (i.e., for indicator functions of sets) the embedding constant is special, thanks to (A.4), which gives that the optimal indicator function is the one of the ball. It would be interesting to determine the optimal Sobolev constants for special families of critical functions.

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