PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 150, Number 5, May 2022, Pages 2223–2237
https://doi.org/10.1090/proc/15796
Article electronically published on February 7, 2022

(DIS)CONNECTEDNESS OF NONLOCAL MINIMAL SURFACES IN A CYLINDER AND A STICKINESS PROPERTY

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(Communicated by Guofang Wei)

Abstract. We consider nonlocal minimal surfaces in a cylinder with prescribed datum given by the complement of a slab. We show that when the width of the slab is large the minimizers are disconnected and when the width of the slab is small the minimizers are connected. This feature is in agreement with the classical case of the minimal surfaces.

Nevertheless, we show that when the width of the slab is large the minimizers are not flat discs, as it happens in the classical setting, and, in particular, in dimension 2 we provide a quantitative bound on the stickiness property exhibited by the minimizers.

Moreover, differently from the classical case, we show that when the width of the slab is small then the minimizers completely adhere to the side of the cylinder, thus providing a further example of stickiness phenomenon.

1. Introduction

Nonlocal minimal surfaces were introduced in [8] and constitute one of the most fascinating, and challenging, research topics in the realm of fractional equations. Roughly speaking, the problem is that of minimizing an energy functional built by the pointwise interaction of a set versus its complement (this energy functional can also be conveniently "localized" in a given domain by taking into account the interactions in which at least one point lies in the domain). The prototype interaction taken into account is scaling and translation invariant and with polynomial decay (but we mention that other versions of the problem considered also interactions via integrable kernels, see [13, 28, 29]).

The nonlocal minimal surfaces constructed by this minimization procedure have relevant features in terms of differential geometry and geometric measure theory, since their energy functional can be considered as a nonlocal approximation of the classical perimeter functional and the nonlocal minimal surfaces as a fractional variant of the classical minimal surfaces; see [2, 3, 10, 11, 17, 30]. Critical points of the nonlocal perimeter energy functional satisfy an integral relation that can be seen as a vanishing nonlocal mean curvature prescription (see [1, 8, 12, 19]) and accordingly the study of volume prescribed minimizers leads to the analysis of surfaces with constant nonlocal mean curvature (see [6, 7, 14, 18]). Moreover, nonlocal minimal surfaces arise as the large-scale limit of long-range phase coexistence models (see

Received by the editors October 3, 2020, and, in revised form, August 11, 2021.
2020 *Mathematics Subject Classification.* Primary 58E12, 35R11.
This work was supported by the Australian Research Council DECRA DE180100957 "PDEs, free boundaries and applications" and the Australian Laureate Fellowship FL190100081 "Minimal surfaces, free boundaries and partial differential equations".

[31]), as discrete iterations of fractional heat equations (see [9]) and as continuous approximations of interfaces of long-range Ising models (see [16]).

Given the importance of nonlocal minimal surfaces from all these perspectives, it is desirable to develop some intuition about their basic geometric features. For this, since it is very rare to have explicit solutions and precise formulas which entirely describe nonlocal minimal surfaces, it is often convenient to focus on some simplified cases in which the reference domain and the external datum possess some special characteristics which lead to a deep understanding of at least some cardinal aspects of the object under investigation.

This note follows precisely in this line of research, namely we will consider a very simple domain, that is a vertical cylinder in $\mathbb{R}^n$, and a very special external datum, that is the complement of a horizontal slab, and detect how the minimizers of the nonlocal perimeter functional change when the width of the slab varies.

On the one hand, when the width of the slab is large, we will show that these minimizers are disconnected, and this is somehow the nonlocal counterpart of the fact that the classical perimeter gets minimized by far-away parallel and co-axial discs.

On the other hand, when the width of the slab is small, we will show that these minimizers become connected. This change of topology is in agreement with the classical case, since perimeter minimizers constrained to two nearby parallel and co-axial circumferences are connected necks of catenoids. Nonetheless, the specific geometry exhibited in this case by nonlocal minimal surfaces is rather different from that of catenoids, since we will additionally show that when the width of the slab is small the nonlocal minimal surface obtained with this procedure actually coincides inside the cylinder with the cylinder itself.

More precisely, and in further detail, the mathematical framework that we use in this paper is the one introduced in [8] and can be summarized as follows. Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be an open subset with Lipschitz boundary. Then we define the nonlocal perimeter or s-perimeter $P_s(E\,;\Omega)$ for a measurable set $E \subset \mathbb{R}^n$ by

$$P_s(E\,;\Omega) := \int_{E\cap\Omega}\int_{E^c} \frac{dx\,dy}{|x-y|^{n+s}} + \int_{E\cap\Omega^c}\int_{\Omega\cap E^c} \frac{dx\,dy}{|x-y|^{n+s}}, \tag{1.1}$$

where we denote by E^c the complement of a set E. We say that a set $E \subset \mathbb{R}^n$ is a s-minimizer or s-minimal set in Ω if it holds that $P_s(E\,;\Omega') \le P_s(F\,;\Omega')$ for any open, bounded, and Lipschitz set Ω' contained in Ω and any $F \subset \mathbb{R}^n$ with $F \setminus \Omega' = E \setminus \Omega'$. See also [27] for additional details regarding the minimization procedure in bounded or unbounded domains.

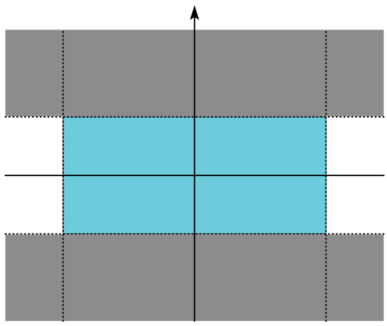
For our purposes, we will often denote coordinates in $\mathbb{R}^n$ by $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and we will focus here on the case of "cylindrical" domains of the form

$$\Omega := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x'| < 1\}. \tag{1.2}$$

We are interested in sets E whose exterior prescription outside Ω is the complement of a strip. Namely, given $M > 0$, we define

$$E_0 := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x_n| > M\} \tag{1.3}$$

and we consider s-minimal sets in Ω such that $E \setminus \Omega = E_0 \setminus \Omega$. See e.g. [27, Theorem 0.2.5] for existence results for this type of s-minimal sets.



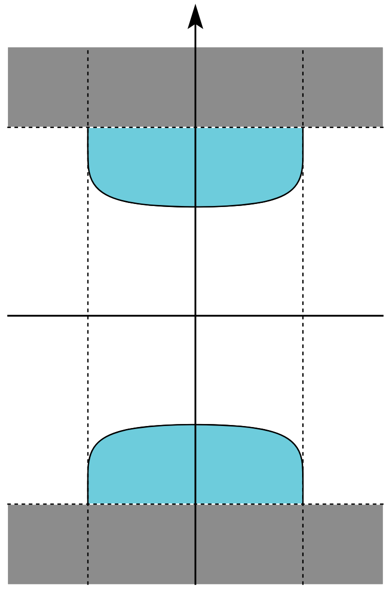


FIGURE 1. The minimizers in Theorem 1.1 (left) and Theorem 1.2 (right)

Our main concern in this note is how the variation of the parameter M affects the topological property of the s-minimizer and we will show that *for small values of M the s-minimizer is connected* while *for large values it is disconnected.*

Furthermore, we will show that *for small values of M the s-minimizer in Ω coincides with Ω itself*, and this is an interesting difference with respect to the classical case of minimal surfaces. Indeed, when $n \geq 3$ minimal surfaces in a cylinder do not coincide with the cylinder itself and, when connected, they develop a "neck" inside the cylinder, as exhibited by the classical example of the catenoid (as a matter of fact, when $n \geq 3$ the cylinder does not have vanishing mean curvature, hence it cannot be a minimizer for the classical perimeter functional).

Therefore, our construction of nonlocal minimal surfaces that coincide with the cylinder in their free domain heavily relies on the nonlocal character of the problem taken into consideration and can be seen as a new example of the *stickiness theory for nonlocal minimal surfaces* which was introduced in [22] and developed in [5, 21, 23, 24]. See also [20, 25] for surveys on nonlocal minimal surfaces discussing, among other topics, the stickiness phenomenon (and, for instance [26] to appreciate the structural differences with respect to the classical case).

In further detail, the precise result that we have concerning the connectedness of the s-minimizer and its stickiness properties for small values of M goes as follows:

Theorem 1.1. *Let Ω be as in* (1.2) *and let E_0 be defined by* (1.3)*. Then, there exists $M_0 \in (0, 1)$, depending only on n and s, such that, for any $M \in (0, M_0)$, the minimizer E_M in Ω of P_s coincides with Ω. In particular, E_M is connected.*

The minimizer described in Theorem 1.1 is depicted in Figure 1. As a counterpart of Theorem 1.1, the disconnectedness result for large values of M is the following:

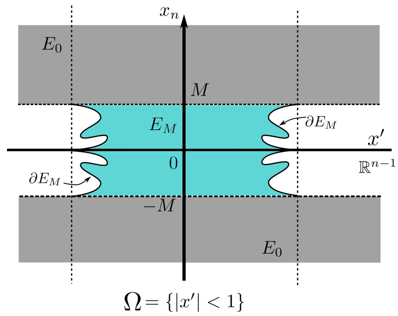


FIGURE 2. The situation in the proof of Theorem 1.1

Theorem 1.2. *Let Ω be as in (1.2) and let E_0 be defined by (1.3). Then, there exists $M_0 > 1$, depending only on n and s, such that, for any $M > M_0$, the minimizer E_M in Ω of P_s is disconnected.*

To favor the intuition, a sketch on how we believe the minimizer in Theorem 1.2 looks like is given in Figure 1.

Interestingly, the situation described in Theorem 1.2 is similar, but structurally different from the one exhibited by classical minimal surfaces. Indeed, the analogy with the classical case is given by the disconnectedness of the minimizers. The difference in the pattern is that classical minimal surfaces in the framework of Theorem 1.2 are just flat disc, and this is not the case for their corresponding nonlocal counterpart (as we will make precise in Proposition 4.1).

The forthcoming Sections 2 and 3 contain the proofs of Theorems 1.1 and 1.2 respectively. In Section 4 we will present further similarities and differences with respect to the classical case in the framework of large M given by Theorem 1.2.

2. Proof of Theorem 1.1

Let E_M be the minimizer selected in Theorem 1.1, see Figure 2 (at this stage of the proof, we do not really know how this minimizer looks like, so the one depicted in Figure 2 will not be the "real" minimizer after all).

By [8, Corollary 5.3], we know that

$$\{x_n > M\} \cup \{x_n < -M\} \subset E_M. \tag{2.1}$$

Given $t \in \mathbb{R}$ and $r \in (0, 1)$, we consider the ball of radius r with center te_n, where $e_n = (0, \dots, 0, 1)$. By (2.1), we have that $B_r(te_n) \subset E_M$ for every $t > M + 1$. Hence, we can slide such a ball downwards till it touches ∂E_M inside Ω. The content of Theorem 1.1 is precisely that this touching does not occur, hence, by contradiction, we suppose instead that there exist $t_0 \in \mathbb{R}$ and $r_0 \in (0, 1)$ such that

$$B_{r_0}(te_n) \subset E_M \quad \text{for all } t > t_0 \tag{2.2}$$

with

$$\partial B_{r_0}(t_0 e_n) \cap \partial E_M \neq \varnothing. \tag{2.3}$$

Then, setting $z := t_0 e_n$, we can choose a point $q = (q', q_n) \in \partial B_{r_0}(z) \cap \partial E_M$.

Since E_M is a local minimizer of P_s in Ω, we obtain, by using the Euler-Lagrange equation in the viscosity sense[1] shown in [8, Theorem 5.1] (see also [5, Theorem B.9]), that

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{n+s}}\, dy \geq 0. \tag{2.4}$$

Our goal is now to produce a contradiction with (2.4) by showing that the left hand side is strictly negative. To this end, we let

$$S_M := \mathbb{R}^{n-1} \times [q_n - 2M, q_n + 2M].$$

We remark that

$$E_M^c \subset S_M \setminus B_{r_0}(z). \tag{2.5}$$

Indeed, by (2.1) we know that $q_n \in [-M, M]$ and $E_M^c \subset \{x_n \in [-M, M]\}$, whence $E_M^c \subset S_M$. This and (2.2) give (2.5).

[1]We recall that local minimizers E with boundary of class C^2 in a domain Ω possess zero nonlocal mean curvature, in the sense that

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y-q|^{n+s}}\, dy = 0 \qquad \text{for all } \ q \in \Omega \cap \partial E.$$

Given the singularity of the kernel, this equation is intended in the principal value sense and the smoothness of E is used precisely to employ cancellations for removing singularities. When the boundary of the set E is not, or is not known to be, sufficiently smooth, one cannot rely on the above equation in the pointwise sense. However, as pointed out in [8, Theorem 5.1] the above equation for local minimizers still holds true "in the viscosity sense", namely if a ball B is such that $q \in \partial B \cap \partial E$ and $B \subseteq E$ then

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y-q|^{n+s}}\, dy \geq 0$$

and if instead $B \subseteq E^c$ then

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y-q|^{n+s}}\, dy \leq 0.$$

This is clearly an interesting geometric information. In particular, we notice that the simple fact that a ball is contained in E (respectively, in E^c) makes the quantity $\chi_{E^c} - \chi_E$ accordingly small (respectively, large), regardless of the minimality of the set. The useful information encoded in the above inequalities is that for local minimizers one is also provided with a partial knowledge with respect to "the opposite sign": specifically, in this case the fact that a ball is contained contained in E (respectively, in E^c) makes the quantity $\chi_{E^c} - \chi_E$ accordingly large (respectively, small) after averaging with respect to the singular kernel.

We observe that for smooth sets one can obtain this Euler-Lagrange equation (or the corresponding Euler-Lagrange inequalities) simply by looking at energy perturbations under domain variations (see e.g. [25]), but without assuming any smoothness on the set suitable cancellations need to be detected (this is precisely the content of [8, Theorem 5.1] which leverages the presence of the ball B on one side of the set to remove part of the singularity by utilizing a reflection argument).

For completeness, we mention that the Euler-Lagrange equation obtained in the interior of the reference domain Ω in which E is a local minimizer can be carried through to the boundary of Ω by a limit process at the points of $\partial\Omega$ from which the boundary of E detaches (see [5, hypothesis (B.10) and Theorem B.9] for full details). For nonlocal minimal graphs in domains of $\mathbb{R}^2$, this can be strengthened, since the Euler-Lagrange equation is satisfied at all the boundary points that are accessible as limits of interior boundary points (see [23, Theorem 1.4] for more detailed information).

See [4,8,15,22,27] for further discussions on the notion of viscosity solutions related to fractional perimeter functionals.

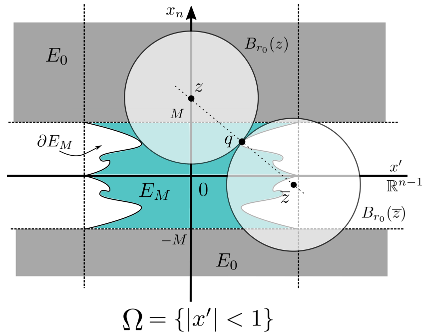


FIGURE 3. The touching between the ball $B_{r_0}(z)$ and the symmetric ball $B_{r_0}(\overline{z})$ at the point q

We also observe that $S_M \supset \{|x_n| \le M\}$, and therefore, in light of (2.1),

$$S_M^c \subset E_M. \tag{2.6}$$

Moreover, using the change of variable $y \mapsto y+q$,

$$\begin{aligned}\int_{S_M^c} \frac{dy}{|y-q|^{n+s}} &= \int_{\mathbb{R}^{n-1}\times((-\infty,-2M)\cup(2M,+\infty))} \frac{dy}{|y|^{n+s}} \\ &\ge \int_{B_M(3Me_n)} \frac{dy}{|y|^{n+s}} \ge cM^{-s},\end{aligned} \tag{2.7}$$

for a constant $c>0$ depending only on n.

Now we set $\overline{z} := z + 2(q-z)$ and we consider the symmetric ball $B_{r_0}(\overline{z})$ with respect to q; see Figure 3. Moreover, we take a free parameter $\Lambda \ge 4$, to be chosen conveniently large in what follows and we observe that, by symmetry,

$$\int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{dy}{|y-q|^{n+s}} = \int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{dy}{|y-q|^{n+s}}.$$

Also, by (2.5),

$$\int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy = -\int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{dy}{|y-q|^{n+s}},$$

and consequently

$$\begin{aligned}&\int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \\ &\qquad + \int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \\ &\le -\int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{dy}{|y-q|^{n+s}} + \int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{dy}{|y-q|^{n+s}} = 0.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_{S_M\cap B_{\Lambda M}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy\\
&= \int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy\\
&\qquad + \int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy\\
&\qquad + \int_{S_M\cap\left(B_{\Lambda M}(q)\setminus\left(B_{r_0}(z)\cup B_{r_0}(\overline{z})\right)\right)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy\\
&\le \int_{S_M\cap\left(B_{\Lambda M}(q)\setminus\left(B_{r_0}(z)\cup B_{r_0}(\overline{z})\right)\right)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy\\
&\le \int_{B_{\Lambda M}(q)\setminus\left(B_{r_0}(z)\cup B_{r_0}(\overline{z})\right)} \frac{dy}{|y-q|^{n+s}}\\
&\le C\Lambda^{1-s}M^{1-s},
\end{aligned}
\tag{2.8}
$$

for some $C>0$ depending only on n and s, where [21, Lemma 3.1] has been used in the last inequality (here with $R:=1$ and $\lambda:=\Lambda M$).

Furthermore,

$$
\begin{aligned}
\int_{S_M\setminus B_{\Lambda M}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy &\le \int_{S_M\setminus B_{\Lambda M}(q)} \frac{dy}{|y-q|^{n+s}}\\
&= \int_{(\mathbb{R}^{n-1}\times[-2M,2M])\setminus B_{\Lambda M}} \frac{dy}{|y|^{n+s}} \le \int_{(\mathbb{R}^{n-1}\times[-2M,2M])\setminus B_{\Lambda M}} \frac{dy}{|y'|^{n+s}}\\
&\le \int_{\{|y'|\ge \Lambda M/2,\ |y_n|\le 2M\}} \frac{dy}{|y'|^{n+s}} = \frac{C_0}{\Lambda^{1+s}M^s},
\end{aligned}
$$

for some $C_0>0$ depending only on n and s.

Hence, combining this information with (2.8),

$$
\int_{S_M} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \le C\Lambda^{1-s}M^{1-s}+\frac{C_0}{\Lambda^{1+s}M^s}.
$$

This, (2.6) and (2.7) lead to

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy\\
&= -\int_{S_M^c} \frac{dy}{|y-q|^{n+s}} + \int_{S_M} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy\\
&\le -cM^{-s}+C\Lambda^{1-s}M^{1-s}+\frac{C_0}{\Lambda^{1+s}M^s}\\
&= -cM^{-s}\left(1-\frac{C\Lambda^{1-s}M}{c}-\frac{C_0}{c\Lambda^{1+s}}\right).
\end{aligned}
$$

Now we choose $\Lambda := \max\left\{4, \left(\frac{2C_0}{c}\right)^{\frac{1}{1+s}}\right\}$ and we thus obtain that

$$
\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \le -cM^{-s}\left(\frac{1}{2}-\frac{C\Lambda^{1-s}M}{c}\right).
$$

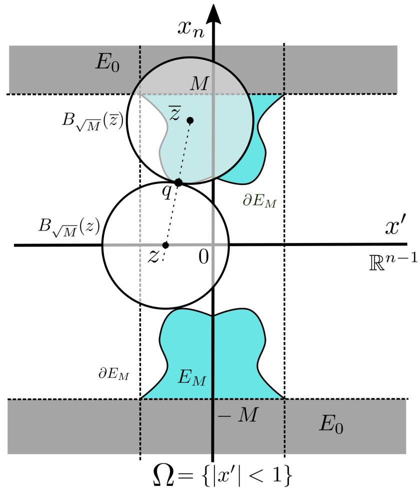


FIGURE 4. The touching between the ball $B_{\sqrt{M}}(z)$ and the symmetric ball $B_{\sqrt{M}}(\overline{z})$ at the point q

Taking now M conveniently small, we conclude that

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{n+s}}\, dy \le -\frac{cM^{-s}}{4} < 0,$$

which produces the desired contradiction with (2.4).

3. Proof of Theorem 1.2

We let $M > 1$ to be chosen conveniently large. Given $t \in \mathbb{R}$, we consider the ball $B_{\sqrt{M}}(te_1)$, where $e_1 = (1, 0, \dots, 0)$, and we slide it from left to right till it touches ∂E_M. Notice indeed that $B_{\sqrt{M}}(te_1) \subset E_0^c$ when $t < -\sqrt{M}$ and, to prove Theorem 1.2, we suppose by contradiction that there exists $t_0 \in \mathbb{R}$ such that $B_{\sqrt{M}}(te_1) \subset E_M^c$ for all $t < t_0$ with $\partial B_{\sqrt{M}}(t_0 e_1) \cap \partial E_M \neq \varnothing$.

We set $z := t_0 e_1$ and we pick a point $q = (q', q_n) \in \partial B_{\sqrt{M}}(z) \cap \partial E_M$. By the Euler-Lagrange equation in the viscosity sense (see [8, Theorem 5.1] and [5, Theorem B.9]), we know that

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{n+s}}\, dy \le 0. \tag{3.1}$$

We consider the symmetric ball with respect to q, by defining $\overline{z} := z + 2(q-z)$ and taking into account the ball $B_{\sqrt{M}}(\overline{z})$, see Figure 4.

We define

$$S := \big\{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x' - q'| \le 3 \big\}.$$

By symmetry,

$$\int_{S \cap B_{\sqrt{M}}(z)} \frac{dy}{|y-q|^{n+s}} = \int_{S \cap B_{\sqrt{M}}(\overline{z})} \frac{dy}{|y-q|^{n+s}}$$

and therefore

$$
\begin{aligned}
&\int_S \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \\
&= \int_{S\cap B_{\sqrt{M}}(z)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy + \int_{S\cap B_{\sqrt{M}}(\overline{z})} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \\
&\qquad + \int_{S\setminus\left(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z})\right)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \\
&\geq \int_{S\cap B_{\sqrt{M}}(z)} \frac{dy}{|y-q|^{n+s}} - \int_{S\cap B_{\sqrt{M}}(\overline{z})} \frac{dy}{|y-q|^{n+s}} \\
&\qquad + \int_{S\setminus\left(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z})\right)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \\
&\geq -\int_{S\setminus\left(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z})\right)} \frac{dy}{|y-q|^{n+s}}.
\end{aligned}
\tag{3.2}
$$

Now, in view of [21, Lemma 3.1], used here with $R:=\sqrt{M}$ and $\lambda := 1/\sqrt[4]{M}$, we know that

$$
\int_{B_{\sqrt[4]{M}}(q)\setminus\left(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z})\right)} \frac{dy}{|y-q|^{n+s}} \leq CM^{-\frac{1+s}{4}},
$$

for some $C>0$ depending only on n and s. As a result,

$$
\begin{aligned}
&\int_{S\setminus\left(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z})\right)} \frac{dy}{|y-q|^{n+s}} \\
&\leq \int_{B_{\sqrt[4]{M}}(q)\setminus\left(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z})\right)} \frac{dy}{|y-q|^{n+s}} + \int_{S\setminus B_{\sqrt[4]{M}}(q)} \frac{dy}{|y-q|^{n+s}} \\
&\leq CM^{-\frac{1+s}{4}} + \int_{\mathbb{R}^n\setminus B_{\sqrt[4]{M}}(q)} \frac{dy}{|y-q|^{n+s}} = CM^{-\frac{1+s}{4}} + C_1M^{-\frac{s}{4}} \leq C_2 M^{-\frac{s}{4}},
\end{aligned}
$$

for some $C_1>0$ depending only on n and s, with $C_2 := C + C_1$.

This and (3.2) lead to

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \\
&= \int_S \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy + \int_{S^c} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \\
&\geq -C_2M^{-\frac{s}{4}} + \int_{S^c} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \\
&\geq -C_2M^{-\frac{s}{4}} - \int_{S^c\cap\{|y_n|\geq M\}} \frac{dy}{|y-q|^{n+s}} + \int_{S^c\cap\{|y_n|<M\}} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{n+s}}\,dy \\
&\geq -C_2M^{-\frac{s}{4}} - \int_{\{|y-q|\geq M/2\}} \frac{dy}{|y-q|^{n+s}} + \int_{S^c\cap\{|y_n|<M\}} \frac{dy}{|y-q|^{n+s}} \\
&= -C_2M^{-\frac{s}{4}} - C_3M^{-s} + \int_{S^c\cap\{|y_n|<M\}} \frac{dy}{|y-q|^{n+s}},
\end{aligned}
$$

for some $C_3>0$ depending only on n and s.

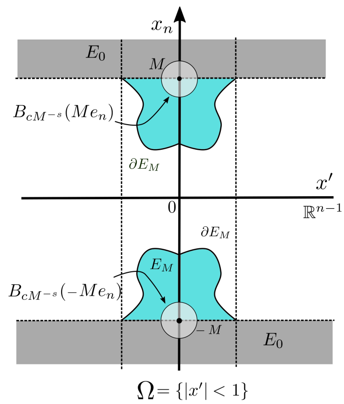


FIGURE 5. A sketch of an argument in Proposition 4.1

Thus, since $S^c \cap \{|y_n| < M\} \supset B_1(q + 5e_1)$, letting $C_4 := C_2 + C_3$ we have

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{n+s}}\, dy \geq -C_4 M^{-\frac{s}{4}} + \int_{B_1(q+5e_1)} \frac{dy}{|y-q|^{n+s}}$$
$$= -C_4 M^{-\frac{s}{4}} + \int_{B_1(5e_1)} \frac{dy}{|y|^{n+s}} = -C_4 M^{-\frac{s}{4}} + c,$$

for some $c > 0$ depending only on n and s. In particular, if M is sufficiently large, we deduce that the left hand side of (3.1) is strictly positive, thus reaching a contradiction with (3.1).

4. FURTHER REMARKS ON THEOREM 1.2

The goal of this section is to stress that the result in Theorem 1.2 is, on the one hand, related to the classical case of minimal surfaces, since both the classical and the nonlocal regimes exhibit disconnected minimizers for large values of M, but, on the other hand, presents some significant structural differences with respect to the classical scenario.

More precisely, differently from the classical case, the minimizer constructed in Theorem 1.2 exhibits the features listed below:

Proposition 4.1. *Let M and E_M be as in Theorem 1.2. Then,*

$$E_M \supsetneqq \{x_n > M\} \cup \{x_n < -M\}. \tag{4.1}$$

Moreover,

$$E_M \supset B_{cM^{-s}}(0, \dots, 0, -M) \cup B_{cM^{-s}}(0, \dots, 0, M), \tag{4.2}$$

for some $c > 0$ depending only on n and s.

In addition, if $n = 2$, given any $\epsilon_0 > 0$ there exists $c_\star > 0$, depending only on s and ϵ_0, such that

$$\begin{aligned} E_M \supset & \left((-1,1) \times \left(-\infty, -M + c_\star\, M^{-\frac{(2+\epsilon_0)s}{1-s}} \right) \right) \\ & \cup \left((-1,1) \times \left(M - c_\star\, M^{-\frac{(2+\epsilon_0)s}{1-s}}, +\infty \right) \right). \end{aligned} \tag{4.3}$$

We remark that (4.2) and (4.3) are quantitative versions of (4.1) and a sketch of an argument used in the proof of Proposition 4.1 is depicted in Figure 5. Though (4.2) and (4.3) provide a stronger result than (4.1), we give an independent proof of (4.1) based on a simple symmetry argument, while the proofs of (4.2) and (4.3) rely on finer quantitative arguments. We also point out that (4.3) provides an explicit quantitative bound on the stickiness property in dimension 2.

Proof of Proposition 4.1. To prove (4.1), we need to show that the inclusion in (2.1) is strict. For this, we argue by contradiction and suppose that $E_M = \{x_n > M\} \cup \{x_n < -M\}$. Then we can use the Euler-Lagrange equation in the viscosity sense shown in [8, Theorem 5.1] at the point $q := (0, \dots, 0, -M) \in \partial E_M$, thus finding that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{n+s}}\, dy \\ &= \int_{\{|y_n|<M\}} \frac{dy}{|y-q|^{n+s}} - \int_{\{|y_n|\geq M\}} \frac{dy}{|y-q|^{n+s}} \\ &= \int_{\{z_n \in (0,2M)\}} \frac{dz}{|z|^{n+s}} - \int_{\{z_n \in (-\infty,0] \cup [2M,+\infty)\}} \frac{dz}{|z|^{n+s}}. \end{aligned} \tag{4.4}$$

Also, by the transformation $(z', z_n) \mapsto (z', -z_n)$, we see that

$$\int_{\{z_n \in (0,2M)\}} \frac{dz}{|z|^{n+s}} = \int_{\{z_n \in (-2M,0)\}} \frac{dz}{|z|^{n+s}},$$

and therefore (4.4) gives that

$$0 = -\int_{\{z_n \in (-\infty,-2M] \cup [2M,+\infty)\}} \frac{dz}{|z|^{n+s}} < 0.$$

This contradiction proves (4.1), and we now deal with the proof of (4.2). To this end, we let $\phi \in C_0^\infty(\mathbb{R}^{n-1}, [0,1])$ with $\phi(x') = 1$ if $|x'| \leq 1/2$ and $\phi(x') = 0$ if $|x'| \geq 3/4$. Given $\eta > 0$, we define

$$F := \{x_n < \eta\phi(x')\}$$

and we claim that, for every $p \in \partial F$,

$$\int_{\mathbb{R}^n} \frac{\chi_{F^c}(y) - \chi_F(y)}{|y-p|^{n+s}}\, dy \leq C_0 \eta, \tag{4.5}$$

for some $C_0 > 0$ depending only on n, s and ϕ. To prove this, we let

$$\Psi(x', x_n) := (x', x_n + \eta\phi(x')) \quad \text{and} \quad \Phi(x) := \Psi(x) - x = (0, \dots, 0, \eta\phi(x')).$$

Notice that $F = \Psi(\{x_n < 0\})$ and the Jacobian of Φ is bounded by $C\eta$, together with its derivatives, for some $C > 0$ depending only on n and η. Furthermore, the inverse of Ψ is given by

$$\Psi^{-1}(x) = (x', x_n - \eta\phi(x'))$$

and, setting $\Xi(x) := \Psi^{-1}(x) - x = -(0, \dots, 0, \eta\phi(x'))$, we find that also the Jacobian of Ξ is bounded by $C\eta$. Consequently, we are in the position of exploiting [15, Theorem 1.1] and deduce that

$$\int_{\mathbb{R}^n} \frac{\chi_{F^c}(y) - \chi_F(y)}{|y-p|^{n+s}}\, dy \leq \int_{\mathbb{R}^n} \frac{\chi_{\{y_n>0\}}(y) - \chi_{\{y_n<0\}}(y)}{|y-\Psi^{-1}(p)|^{n+s}}\, dy + C_0\eta = C_0\eta,$$

for some $C_0 > 0$ depending only on n, s and ϕ, thus completing the proof of (4.5).

Now we define

$$G := F \cup \{x_n > 4M\},$$

we point out that this union is disjoint for large M and small η, and we claim that there exists $c > 0$, depending only on n, s and ϕ, such that if $\eta \in (0, cM^{-s}]$ then, for every $p \in \partial F$,

$$\int_{\mathbb{R}^n} \frac{\chi_{G^c}(y) - \chi_G(y)}{|y-p|^{n+s}}\, dy < 0. \tag{4.6}$$

Indeed, we have that $\chi_G = \chi_F + \chi_{\{x_n > 4M\}}$, whence $\chi_{G^c} = 1 - \chi_G = 1 - \chi_F - \chi_{\{x_n>4M\}} = \chi_{F^c} - \chi_{\{x_n>4M\}}$. Accordingly, we have that $\chi_{G^c} - \chi_G = \chi_{F^c} - \chi_F - 2\chi_{\{x_n>4M\}}$ and therefore, using (4.5),

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\chi_{G^c}(y) - \chi_G(y)}{|y-p|^{n+s}}\, dy &= \int_{\mathbb{R}^n} \frac{\chi_{F^c}(y) - \chi_F(y)}{|y-p|^{n+s}}\, dy - 2\int_{\{y_n>4M\}} \frac{dy}{|y-p|^{n+s}} \\ &\le C_0\eta - 2\int_{(-M,M)^{n-1}\times(4M,5M)} \frac{dy}{|y-p|^{n+s}} \le C_0\eta - c_0 M^{-s},\end{aligned}$$

for some $c_0 > 0$ depending only on n and s, which plainly leads to (4.6).

By means of (4.6), we can thus use the set G as a sliding barrier from below with $\eta := cM^{-s}$ (starting the sliding from a vertical translation of the set G equal to $-2M$) and find that $E_M \supset \{x_n < -M + cM^{-s}\phi(x')\}$. In particular, we see that $E_M \supset \left[-\frac{1}{2}, \frac{1}{2}\right]^{n-1} \times (-\infty, -M + cM^{-s}] \supset B_{cM^{-s}}(0, \dots, 0, -M)$.

Similarly, one proves that $E_M \supset B_{cM^{-s}}(0, \dots, 0, M)$, thus completing the proof of (4.2).

Now we suppose that $n = 2$ and we establish (4.3). For this, we fix $\epsilon_0 > 0$, we consider a suitably small $\delta > 0$ and we exploit [22, Corollary 7.2] to construct a set $H \subset \mathbb{R}^2$ such that

$$\begin{aligned}& H \subset \{x_2 < \delta\}, \\ & H \cap \{x_1 < -1\} = (-\infty, -1) \times (-\infty, 0), \\ & H \cap \{x_1 > 1\} = (1, +\infty) \times (-\infty, 0), \\ & H \supset (-1,1) \times \left(-\infty, \delta^{\frac{2+\epsilon_0}{1-s}}\right) \\ \text{and } & \int_{\mathbb{R}^2} \frac{\chi_{H^c}(y) - \chi_H(y)}{|y-p|^{2+s}}\, dy \le \bar{C}\delta\end{aligned}$$

for every $p = (p_1, p_2) \in \partial H$ with $|p_1| < 1$, where $\bar{C} > 0$ depends only on s and ϵ_0.

We define

$$L := H \cup \{x_2 > 4M\},$$

and we see that $\chi_{L^c} - \chi_L = \chi_{H^c} - \chi_H - 2\chi_{\{x_2>4M\}}$ and thus

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{\chi_{L^c}(y) - \chi_L(y)}{|y-p|^{2+s}}\, dy &\le \int_{\mathbb{R}^2} \frac{\chi_{H^c}(y) - \chi_H(y)}{|y-p|^{2+s}}\, dy - 2\int_{\{y_2>4M\}} \frac{dy}{|y-p|^{2+s}} \\ &\le \bar{C}\delta - 2\int_{(-M,M)\times(4M,5M)} \frac{dy}{|y-p|^{2+s}} \le \bar{C}\delta - \bar{c}M^{-s} < 0\end{aligned}$$

for every $p = (p_1, p_2) \in \partial H$ with $|p_1| < 1$, where $\bar{c} > 0$ depends only on s, and $\delta := \frac{\bar{c}M^{-s}}{2\bar{C}}$.

In this way, we can use L as sliding barrier from below (starting the sliding from a vertical translation of the set L equal to $-2M$) and deduce that

$$E_M \cap \{|x_1| < 1\} \supset \left(-\infty, -M + \delta^{\frac{2+\epsilon_0}{1-s}}\right) = \left(-\infty, -M + c_\star\, M^{-\frac{(2+\epsilon_0)s}{1-s}}\right)$$

for some $c_\star > 0$. Similarly, one finds that

$$E_M \cap \{|x_1| < 1\} \supset \left(M - c_\star\, M^{-\frac{(2+\epsilon_0)s}{1-s}}, +\infty\right).$$

The proof of (4.3) is thereby complete. $\square$

Acknowledgments

The first and third authors are members of INdAM and AustMS.

This work has been carried out during a very fruitful visit of the second author to the University of Western Australia, whose warm hospitality is a pleasure to acknowledge.

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