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TBD

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I hereby declare that this thesis is entirely the result of my own work except where otherwise indicated. I have only used the resources given in the list of references.

Munich, Date of submission Mohamed Noah Abdel Wahab

Abstract

TODO (GENERAL STUFF)**.**
1. Clean up Bibliography
2. ..
3. Replace s.t. with `\mid`
4. Complete List of symbols
5. Edit colors to fit TUMColors
6. Edit figures to smaller and increase font
7. ..

Abstract

Nonlocal minimal surfaces confined within a cylinder exhibit unique behaviors dependent on external data. This thesis delves into these surfaces, which incorporate long-range spatial interactions compared to classical minimal surfaces. We consider two variations of the model discussed in [5], a minimal surfaces confined within a cylinder.

We investigate two scenarios: varying the height and width of data outside a separating slab. The results show that when the slab is wide, the minimal surface becomes disconnected from the data, while a narrow slab allows connection. This allows us to predict the behavior of similar models with symmetrically placed data. Additionally, the research reveals that for sufficiently narrow slabs, the surface "sticks" to the cylinder.

Finally, we present an example where the minimizer is completely disconnected from the external data, a phenomenon unique to nonlocal minimal surfaces. This work provides valuable insights into the behavior of these emerging mathematical objects and their interaction with external data.

Zusammenfassung

In Zylindern eingeschlossene nichtlokale Minimalflächen zeigen ein einzigartiges Verhalten, das von externen Daten abhängt. Diese Arbeit befasst sich mit diesen Flächen, die im Vergleich zu klassischen Minimalflächen weitreichende räumliche Wechselwirkungen berücksichtigen. Wir betrachten zwei Varianten des in [5] diskutierten Modells, einer in einem Zylinder eingeschlossenen Minimalfläche.

Dabei untersuchen wir zwei Szenarien: die Variation der Höhe und der Breite von Daten außerhalb einer trennenden Platte. Die Ergebnisse zeigen, dass die Minimalfläche bei breiter Platte von den Daten getrennt wird, während eine schmale Platte eine Verbindung ermöglicht. Dies erlaubt uns, das Verhalten ähnlicher Modelle mit symmetrisch angeordneten Daten vorherzusagen. Darüber hinaus zeigt die Forschung, dass die Fläche bei ausreichend schmalen Platten am Zylinder "haftet".

Schließlich präsentieren wir ein Beispiel, bei dem der Minimierer vollständig von den externen Daten getrennt ist, ein Phänomen, das für nichtlokale Minimalflächen einzigartig ist. Diese Arbeit liefert wertvolle Erkenntnisse über das Verhalten dieser neuen mathematischen Objekte und ihre Wechselwirkung mit externen Daten.

List of symbols

$\mathbb{R}^n$	Euclidean space of dimension n
$\mathrm{dist}(A, B)$	Distance between sets A and B

Contents

1 Introduction

IDEA. Idea: Start with short historical background
18th century: Lagrange, Euler
20th Century: DeGiorgi Perimeter and localized entity
2009 Cafarelli, Roquejoffre, Savin: Nonlocal minimal surfaces
Perimeter and nonlocal perimeter as the (semi)norm of an indicator function
Define the usual problem considered
Better regularity than classical minimal surfaces

Chapter 02
Model 01
Model 02
Combination of both

Chapter 03
Fully disconnected minimizer

Use the introduction in [9] as inspiration.

TODO. Add sources
What does "locally" mean here? And what do we minimize? The surface or the area the encompassed?
Rewrite the text

Minimal surfaces, characterized by locally minimizing their surface area, have captivated mathematicians for centuries. Dating back to the 18th century, mathematicians like *Euler* and *Lagrange* laid the foundation for the field. In an effort to describe these surfaces mathematically, they formulated the *Euler-Lagrange equations* in the late 18th century. These equations provide a powerful framework for identifying and characterizing minimal surfaces. Since the 19th century, many mathematicians contributed to the study of minimal surfaces, uncovering profound insights. Since then minimal surfaces found many applications in various fields beyond pure mathematics. From understanding physical phenomena like soap films and black holes to informing the design of optimal structures in engineering and architecture, the versatility of minimal surfaces continues to inspire exploration.

In this thesis, we want to explore a rather recent concept of minimal surfaces, namely *Nonlocal Minimal Surfaces*, which were first introduced by *Cafarelli*, *Roquejoffre*, and *Savin* in 2009. For that purpose, we will first give a short introduction to the theory of minimal surfaces in the context of this work.

1.1 Classical Minimal Surfaces

CHECK. Is this introduction enough and complete/correct?

The study of minimal surfaces concerns itself with finding the set with least surface area under certain constrains. But before we can formulate the usual problem, we have to define some tools.

CHECK. Do I need to cite this definition from [4]? Give a justification?

Definition 1.1. Let $A \subset \mathbb{R}^n$ with smooth boundary, then the surface area or *perimeter* of A is given by

$$\mathrm{Per}(A) := \sup \left\{ \int_{\partial A} \varphi \cdot \nu_A \, \mathrm{d}\mathcal{H}^{n-1} \mid \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), |\varphi| \leq 1 \right\},$$

where ν_A is the outer normal to A.

To extend this definition to general measurable sets, we can use the divergence theorem and rewrite the integration over the boundary as an integration over the set itself. This removes the need for a smooth boundary and allows us to define the surface area for general sets.

Definition 1.2. Let $A \subset \mathbb{R}^n$ be a Borel set, then the perimeter of A is given by

$$\mathrm{Per}(A) := \sup \left\{ \int_{A} \mathrm{div}\varphi \mid \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), |\varphi| \leq 1 \right\}.$$

TODO. Rewrite this text
Not just for Minimization problem, but Perimeter of a set in some other set

In the minimization problem, we want to find some set E which minimizes the surface of some external data E_0. Since the surface area may be infinite, if E_0 is unbounded, we can "localize"[1] the problem by just considering the part of ∂E in some bounded set Ω.

Definition 1.3. Let $A \subset \mathbb{R}^n$ be a Borel set and $\Omega \subset \mathbb{R}^n$ bounded, then the perimeter of A relative to Ω is given by

$$\mathrm{Per}(A, \Omega) := \sup \left\{ \int_{A} \mathrm{div}\varphi \mid \varphi \in C_c^1(\Omega, \mathbb{R}^n), |\varphi| \leq 1 \right\}.$$

Now we can formulate the usual problem.

Definition 1.4 (Minimal Surface Problem)**.** Let $\Omega \subset \mathbb{R}^n$ bounded and $E_0 \subset \mathbb{R}^n$, then we want to find $E \subset \mathbb{R}^n$ such that E minimizes the perimeter of E_0 relative to Ω, i.e.

$$\mathrm{Per}(E, \Omega) = \min \left\{ \mathrm{Per}(A, \Omega) \mid A \setminus \Omega = E_0 \setminus \Omega \right\}.$$

This set E is then called a *minimal surface.*

TODO. Complete note
Case that $E_0 \cap \Omega \neq \emptyset$..
Give sources, that minimizer exists, thus minimal surfaces exist and say something about uniqueness
Note that in classical theory often one just has a contour over which one minimizes

Note. Usually E_0 is chosen such that $E_0 \cap \Omega = \emptyset$, then we minimize over the set E such that $E \setminus \Omega = E_0$. If $E_0 \cap \Omega \neq \emptyset$, then we can minimize over..

TODO. Standard example of minimal surfaces (Plateau's problem, soap bubble)

[1] Here "local" refers to the area in which we minimize

1.2 Nonlocal Minimal Surfaces

TODO. Do again, but start this time from the example of a soap bubble
Soap bubble, classical example, 2 dim surface..
Nanoscale, 3 dim, classical theory doesn't suffice anymore
Short construction of fractional perimeter a la Cafarelli [2]

Rewrite the text
Is the example fitting?
Emphasize that we are no longer just minimizing boundary but the set as well

Let us for now consider some set $A \subset \mathbb{R}^n$ with smooth boundary, then to get its perimeter we have to take the supremum of

$$\int_{\partial A} \varphi \cdot \nu_A.$$

This is a local quantity, i.e. it only depends on the boundary of A. Thus if we want to minimize the perimeter of some set E with external data E_0, we are only interested in the behavior of the boundary of E close to and in Ω and not in the contribution or the size of the external data. In many cases, this is enough to describe the behavior of the minimizer, but in some cases, this is not enough anymore. Take a soap bubble as an example, a standard example for a classical minimal surfaces. In our normal scaling, we can see the film of the soap bubble as a 2-dimensional object. But if we go to the molecular level, we see that the film is a 3-dimensional object. Thus we need to incorporate long-range correlation into our definition of perimeter and minimal surfaces. *Cafarelli*, *Roquejoffre*, and *Savin* did exactly that in 2009, when they introduced the concept of *nonlocal minimal surfaces* and *fractional perimeter* in [2].

TODO. What is the effect of s?
Which definition is standard? Add note about other definitions
Maybe use the definition from [18], but it's without s, just with $(1-s)$ and with 2 in front. The 2 is just convention for the relation to the Gagliardo seminorm. Why not with s? Can I define it with s?
For the limiting behavior of $\mathrm{Per}_s()$ for $s \to 0/1$ see [16]

Definition 1.5 (Fractional Perimeter)**.** Let $A \subset \mathbb{R}^n$ be a Borel set, $s \in (0,1)$, then the s-perimeter of A to is given by

$$\mathrm{Per}_s(A) := \int_A \int_{A^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x.$$

Intuitively, we can understand the parameter s as the grade of nonlocality. For s big, we have a more local…

Just as in the classical case, we can define a relative fractional perimeter by removing the integration over the constant part..

$$\begin{aligned}
&\int_A \int_{A^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_{A\cap\Omega} \int_{A^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x + \int_{A\setminus\Omega} \int_{A^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_{A\cap\Omega} \int_{A^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x + \int_{A\setminus\Omega} \int_{\Omega\setminus A} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x + \int_{A\setminus\Omega} \int_{A^c\setminus\Omega} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x
\end{aligned}$$

While minimizing A relative to Ω we can ignore the last term as it is constant and thus does not affect the minimization.

Definition 1.6. Let $A, B \subset \mathbb{R}^n$ be Borel sets, $s \in (0,1)$, then the interaction of A and B is given by

$$\mathscr{L}(A,B) := \int_A \int_{B^c} \frac{1}{|x-y|^{n+s}} \,\mathrm{d}y \,\mathrm{d}x.$$

Definition 1.7 (Relative Fractional Perimeter)**.** Let $A \subset \mathbb{R}^n$ be a Borel set, $\Omega \subset \mathbb{R}^n$ bounded and $s \in (0,1)$, then the s-perimeter of A relative to Ω is given by

$$\mathrm{Per}_s(A,\Omega) := \mathscr{L}(A \cap \Omega, A^c) + \mathscr{L}(A \setminus \Omega, \Omega \setminus A).$$

TODO. Rewrite.. very bad
Not precise enough

Note. In some literature, the fractional perimeter is sometimes defined with the factor 2 in front of the integral. This is just a convention to relate the Gagliardo seminorm

$$\|f\|_{W^{s,1}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^{n+s}} \,\mathrm{d}y \,\mathrm{d}x$$

to the fractional perimeter. Notice that

$$\mathrm{Per}_s(A) = \int_A \int_{A^c} \frac{1}{|x-y|^{n+s}} \,\mathrm{d}y \,\mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\chi_A(x)-\chi_A(y)|}{|x-y|^{n+s}} \,\mathrm{d}y \,\mathrm{d}x = \frac{1}{2} \|\chi_A\|_{W^{s,1}(\mathbb{R}^n)},$$

i.e. the fractional perimeter is the seminorm of the indicator function of A up to a multiplicative constant.
In more recent literature like [18], the fractional perimeter is defined with the factor $(1-s)$ in front of the integral. This is based on the limiting behavior of the fractional perimeter for $s \to 1^-$ as shown in [1] and [3]. In the latter, the authors have shown that for sets of finite *classical* perimeter, we have that $(1-s)\,\mathrm{Per}_s(A,\Omega) \to c\,\mathrm{Per}(A,\Omega)$ as $s \to 1^-$ for some constant c depending on the dimension. In the former paper, the authors have shown the same behavior in sense of Γ-convergence and general measurable sets. In [18], the factor $(1-s)$ is used, to justify the fractional perimeter as a generalization of the classical perimeter.
In [9], the authors analyzed the behavior of $s\,\mathrm{Per}_s(A,\Omega)$ for $s \to 0^+$. They showed that not all sets have a limit for $s \to 0^+$ and if a limit exits, then it relates to the volume of the sets.
Thus if we are interested in the limiting behavior, it would make sense to define Perimeter with the factor $s(1-s)$ in front of the integral. We will stick to the usual definition for convenience and add the factor $s(1-s)$, when are interested in the limiting behavior.

TODO. give an example where classical theory doesn't suffice (cube rotated by 45 degree)
Serra 2023 pixelled square

TODO. Add note about advantages/properties (e.g. Euler-Lagrange Viscos) of nonlocal minimal surfaces like better regularity properties and..
Add some sentences about stickiness property and that we are looking at a model precisely about that property.
Give some justification, why fractional perimeter can be seen as a generalization of the classical perimeter.

With these tools we can now define the nonlocal minimal surface problem.

Definition 1.8 (Nonlocal Minimal Surface Problem)**.** Let $\Omega \subset \mathbb{R}^n$ bounded and $E_0 \subset \mathbb{R}^n$, then we want to find $E \subset \mathbb{R}^n$ such that E minimizes the s-perimeter of E_0 relative to Ω, i.e.

$$\mathrm{Per}_s(E, \Omega) = \min \{\mathrm{Per}_s(A, \Omega) \mid A \setminus \Omega = E_0 \setminus \Omega\}.$$

Over the last few years these nonlocal minimal surfaces have been an are of great interest. Various properties have been studied and many results have been obtained. Next to the better regularity properties, Euler-Lagrange equations, stickiness property, ..

TODO. Quick summary of Chapter 2 Quick summary of Chapter 3

In this thesis, we want to explore more on these surfaces and their properties. In Chapter 2, we will consider two models, analyze them on connectedness and try to understand the stickiness property and where the contribution lies to achieve stickiness. We will then derive the behavior of models similar to both, to get an understanding of general models of that form. In Chapter 3, we will discuss a natural question coming up while analyzing the models, namely the existence of a nontrivial minimizer in the case that the external data and the prescribed set are disconnected. We will provide an example where such a minimizer exists. This behavior is unique to nonlocal minimal surfaces.

2 Models

TODO. Instead of considering both models seperately consider them together

TODO. Rewrite the text
Add discussion about variation of models and why we are considering that

In this chapter we will consider two different models, which are variations of the model considered by Dipierro et al. in [5], where they considered the external date E_0 as the complement of a slab in $\mathbb{R}^n$ of width $2M$ and the prescribed data Ω as the cylinder of radius 1 and height $2M$. The showed that for M big enough the minimizer is disconnected which is consistent with the classical theory of minimal surfaces. When M is small enough, the minimizer is connected and even sticks to the boundary. The latter being a unique property of nonlocal minimal surfaces.
Here we will first consider a variation of the model, where we vary the width of the external data E_0. We observe similar behavior of the minimizer as in the original model. This is interesting in the sense of the stickiness property, since even for width of 1 we get stickiness to the boundary.
In the second model we will consider a variation of the height of the external data E_0. Again we observe similar behavior of the minimizer as in the original model, but for smaller heights, we cannot say a priori whether the minimizer is connected for small M as in the nonlocal case we could have a connected component of the minimizer which is fully disconnected from the rest of the minimizer. We will discuss this situation in Chapter 3.

2.1 Model 01

For $n \geq 2$ consider the model as follows:

$$E_0 := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{s.t.} |x'| \leq R, \ |x_n| \geq M\}$$
$$\Omega := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{s.t.} |x'| \leq 1, \ |x_n| \leq M\}$$

for $R \geq 1$ and $M > 0$. The Figure 2.1 illustrates the setting.

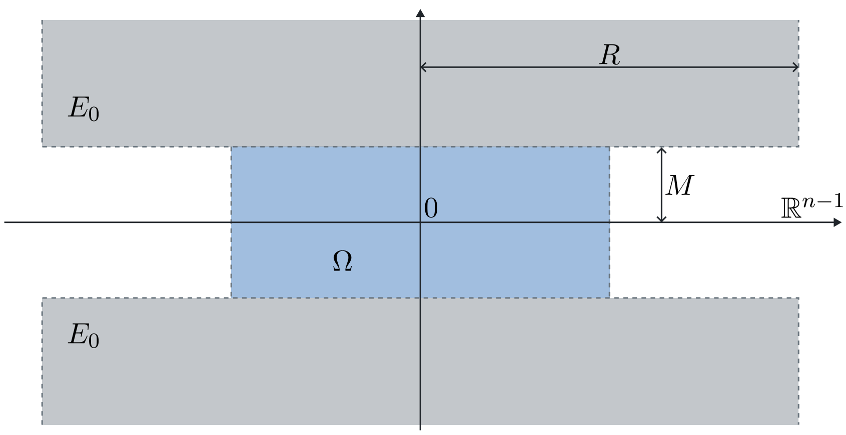

Figure 2.1

We state the following two results, which we will prove afterwards.

Theorem 2.1. For Ω and E_0 as given above and for all $R \geq 1$, then there exists $M_0 \in (0,1)$ depending only on the dimension and s, such that for any $M \in (0, M_0)$, the minimizer is $E_M = E_0 \cup \Omega$.

Theorem 2.2. For Ω and E_0 as given above and for all $R \geq 1$, then there exists $M_0 > 1$ depending only on the dimension and s, such that for any $M \geq M_0$, the minimizer E_M is disconnected.

TODO. Elaborate and add source
Connect to classical minimal surfaces by observing disconnectedness of the minimizer, but when connected, the minimizer may "stick" to the boundary. Whereas classical minimal surfaces cannot stick to the boundary.

TODO. Rewrite
Improve the figure

For the first proof, we will follow a similar construction as in [5].
In [2] the authors have shown that nonlocal minimizer satisfy the Euler - Lagrange equation in the viscosity sense, i.e. if E is a minimizer, there exists some such that $q \in \partial E$ and $B_r(q + r\nu) \subset E$ for some $r > 0$ and unit vector $\nu \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y-q|^{n+s}} \, \mathrm{d}y \geq 0. \tag{2.1}$$

In the proof we will assume that there exist a minimizer which is not $E_0 \cup \Omega$. To bring this assumption to a contradiction, we want to show that the left hand side of (2.1) is negative for M small enough. Thus, we have to construct some suitable ball such that we can apply the Euler - Lagrange equation. Constructing the ball by sliding it down from te_n. If the minimizer is not $E_0 \cup \Omega$, then at some point the ball will touch the minimizer for any $0 < r < 1$ and a point q, then exists. Then we will split the domain into four parts and estimate each part to get the contradiction.

TODO. Improve the proof

Proof of Theorem 2.1. Proof by contradiction. Assume E_M is not $E_0 \cup \Omega$, then we can slide a ball of radius r down and at some point it will touch E_M. We consider the ball $B_r(te_n)$. Since E_M not cylindrical, there exists $r_0 \in (0,1)$ and $t_0 > 0$ s.t. $\partial B_{r_0}(t_0 e_n) \cap \partial E_M \neq \emptyset$ and $B_{r_0}(te_n) \subset E_M$ for all $t > t_0$. See figure fig. 2.2.

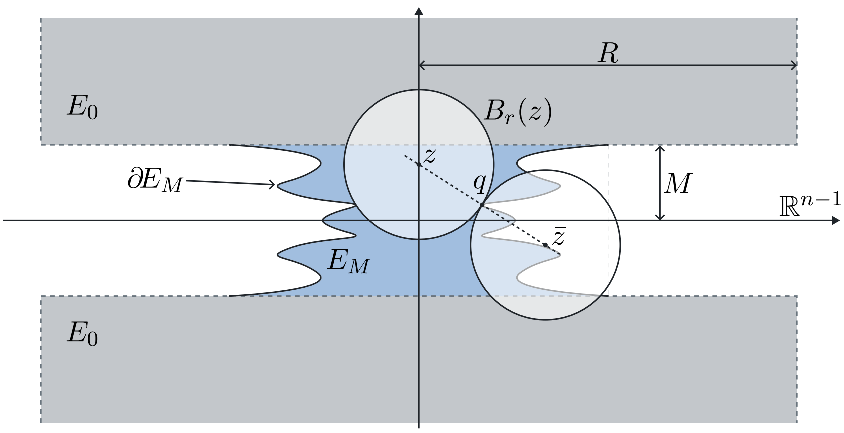

Figure 2.2

Since E_M is a minimizer it is also a variational solution and the inequality holds

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{n+s}} \, \mathrm{d}y \geq 0$$

whereas $q \in \partial B_{r_0}(t_0 e_n) \cap \partial E_M$.
We show that the left hand side is negative. Split the domain into four parts, as seen in the Figure fig. 2.3.

CHECK. Figure 2.3 looks good, but is this needed?

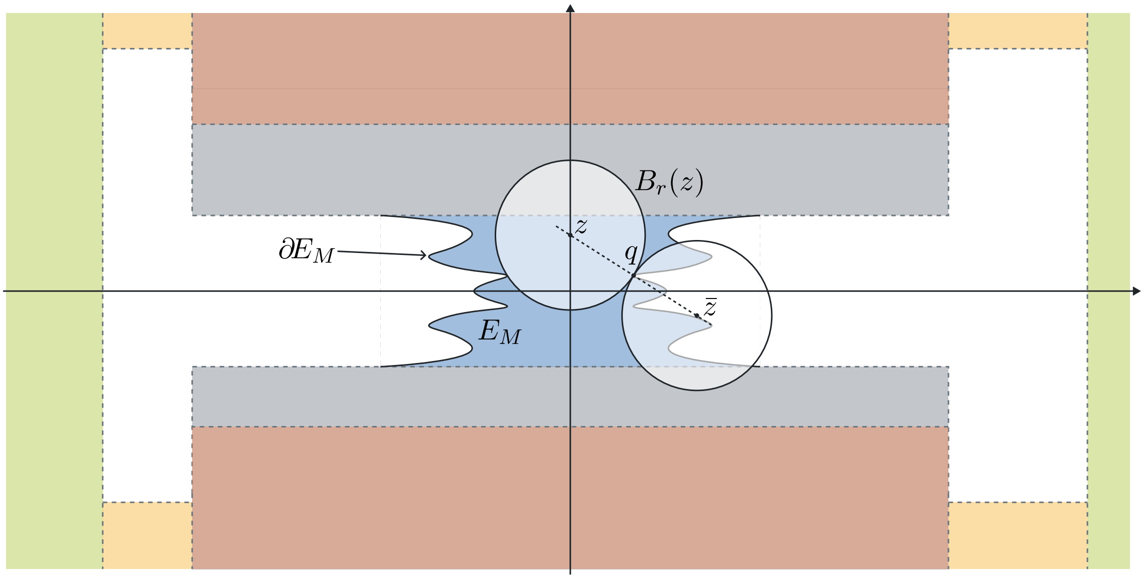

Figure 2.3

We define

$$
\begin{aligned}
A &:= \{(x', x_n) \text{s.t.} |x' - q'| \geq R + 1\} \text{Green Area} \\
B &:= \{(x', x_n) \text{s.t.} |x'| < R, |x_n - q_n| > 2M\} \\
C &:= \{(x', x_n) \text{s.t.} |x'| \geq R, |x' - q'| \leq R + 1, |x_n - q_n| > \Lambda M\} \\
\text{Everything else} \subset S &:= \{(x', x_n) \text{s.t.} |x' - q'| \leq R + 1, |x_n - q_n| \leq \Lambda M\}
\end{aligned}
$$

Integration over the first part:

$$
\int_A \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \, \mathrm{d}y \overset{A \subseteq E^c}{=} \int_{|y'| > R+1} \frac{1}{|y|^{n+s}} \, \mathrm{d}y \leq c(n) \int_{R+1}^{\infty} r^{-s-2} \, \mathrm{d}y \leq c(n,s) R^{-(1+s)}
$$

Integration over the second part:

$$
\int_B \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \, \mathrm{d}y \overset{B \subseteq E}{=} -\int_B \frac{1}{|y-q|^{n+s}} \, \mathrm{d}y \leq -c(n,s) M^{-s} \qquad \text{Idea: Consider ball with factor } 2^{-n}
$$

Integration over the third part:

$$
\begin{aligned}
&\int_C \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \, \mathrm{d}y \overset{C \subseteq E^C}{=} \int_C \frac{1}{|y-q|^{n+s}} \, \mathrm{d}y \leq c(n) \int_{R-1}^{R+1} \int_{\Lambda M}^{\infty} \frac{r^{n-2}}{(r^2 + y_n^2)^{\frac{n+s}{2}}} \, \mathrm{d}y_n \, \mathrm{d}r \\
&\overset{r^2 \leq r^2 + y_n^2}{\leq} c(n) \int_{R-1}^{R+1} \int_{2\Lambda M}^{\infty} \frac{1}{(r^2 + y_n^2)^{\frac{s+2}{2}}} \, \mathrm{d}y_n \, \mathrm{d}r \overset{\text{convexity}}{\leq} \int_{R-1}^{R+1} \int_{\Lambda M}^{\infty} \frac{1}{(r + y_n)^{s+2}} \, \mathrm{d}y_n \, \mathrm{d}r \\
&\leq c(n,s) \int_{R-1}^{R+1} \frac{1}{(r + \Lambda M)^{s+1}} \leq c(n,s)(R - 1 + \Lambda M)^{-s} \leq c(n,s)(\Lambda M)^{-s}
\end{aligned}
$$

Integration over the fourth part:
Justification that we can estimate with S: Only negative part of the integration is fully in the set we want to estimate and the rest in S is positive.
We split S into four parts:

i) $S \cap B_{\Lambda M}(q) \cap B_{r_0}(z)$

ii) $S \cap B_{\Lambda M}(q) \cap B_{r_0}(\overline{z})$

iii) $S \cap (B_{\Lambda M}(q) \setminus (B_{r_0}(z) \cup B_{r_0}(\overline{z})))$

iv) $S \setminus B_{\Lambda M}(q)$

where $\overline{z} := z + 2(q - z)$ and $\Lambda > 4$ chosen big enough and M chosen small enough s.t. $\Lambda M \leq 1$.
We estimate the first and second part:

$$\begin{aligned}&\int_{S\cap B_{\Lambda M}(q)\cap B_{r_0}(z)\cup S\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}}\,\mathrm{d}y \\ &\leq \int_{S\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y - \int_{S\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y \leq 0\end{aligned}$$

These two integrals cancel because of symmetry.
We estimate the third part:

$$\int_{S\cap(B_{\Lambda M}(q)\setminus(B_{r_0}(z)\cup B_{r_0}(z)))} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}}\,\mathrm{d}y \leq \int_{P_{1,\Lambda M}} \frac{1}{|y-q|^{n+s}}\,\mathrm{d}y \leq C\Lambda^{1-s}M^{1-s}$$

where we used lemma 3.1 in [6] with $R = r_0 = 1 and \lambda = \Lambda M$ (we can choose $r_0 = 1$, since if we can show the bound for $r_0 = 1$ then it holds for all smaller balls as well).
We estimate the fourth part:

$$\int_{S\setminus B_{\Lambda M}(q)} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}}\,\mathrm{d}y \leq \int_{B_{R+2}\setminus B_{\Lambda M}} \frac{1}{|y|^{n+s}}\,\mathrm{d}y = c(n,s)((\Lambda M)^{-s} - (R+2)^{-s})$$

since $S \subset B_{R+2}$ for $R \geq 1$ since $((\Lambda M)^2 + (R+1)^2)^{\frac{1}{2}} \leq (R^2 + 4R + 4)^{\frac{1}{2}} = R + 2$.

Thus in total we get:

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}}\,\mathrm{d}y &\leq -c_1 M^{-s} + c_0(R^{-(1+s)} + (\Lambda M)^{-s} + (\Lambda M)^{-s} - (R+2)^{-s} + \Lambda^{1-s}M^{1-s}) \\ &\leq -c_1 M^{-s}(1 - +\frac{c_0}{c_1}(R^{-(1+s)}M^s + 2\Lambda^{-s} - (R+2)^{-s}M^s + \Lambda^{1-s}M)\end{aligned}$$

Choose Λ large and M small enoguh

$$\leq -c_2 M^{-s} < 0$$

■

Interesting to see, that the contribution of the cylinder of radius 1 is enough to get connectedness of the minimizer and even stickiness to the boundary. Also see, that the model seems (maybe prove that) to converge to the problem, considered in [5].

TODO. Do I need to elaborate more? Yes..

Proof of Theorem 2.2. In theorem 1.2 of [5] the authors have shown that for ecternal data $F_0 = \{|x_n| > M\}$, that there exists $M_0 > 1$, such that for all $M \geq M_0$ the minimizer is disconnected. In particular, we have that $E_0 \subset F_0$ thus we can apply the same proof as in [5] to show that the minimizer is disconnected. ■

Whereas Theorem 2.2 is consistent with the classical therory of minimal surfaces, the behvavior of the minimizer in Theorem 2.1 is unique to nonlocal minimal surfaces. In [5] the authors have shown that the minimizer exhibits similar behavior as we found in Theorem 2.1 for the model considered in this chapter, however interesting to see is that even in the case $R = 1$ the minimizer is connected and even sticks to the boundary, for M small enough. This suggegests that the contribution of the external data E_0 above and below is enough to push the minimzer to the boundary of the prescribed set Ω. In the proof we have seen that the width R is with negative exponents in the upper bound, thus if we choose R large enough, M_0 increases.

2.2 Model 02

For $n \geq 2$ consider the model as follows:

$$E_0 := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{s.t.} M|x_n| \geq R + M\}$$
$$\Omega := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{s.t.} |x'| \leq 1, \ |x_n| \leq M\}$$

for $R > 0$ and $M > 0$. The Figure 2.4 illustrates the setting.

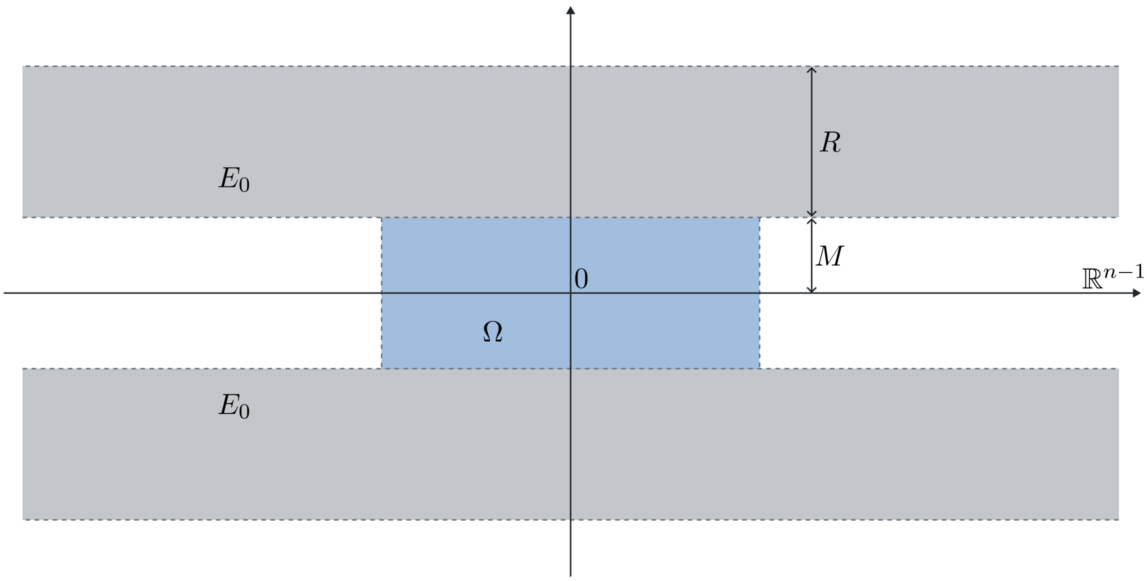

Figure 2.4

We state the following two results, which we will prove afterwards.

TODO. Specifiy R

Theorem 2.3. Let Ω and E_0 as given above and for all $R \geq 2$, then there exists $M_0 \in (0, 1)$ depending only on the dimension and s, such that for any $M \in (0, M_0)$, the minimizer is $E_M = E_0 \cup \Omega$. For $R < 2$, the cylinder $A := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{s.t.} \ |x'| \leq \frac{R}{2}, \ |x_n| \leq M\}$ is in the minimizer, i.e. $E_M \supset E_0 \cup A$.

Note. Bound on R depends on the construction of the proof.

TODO. Elaborate

Theorem 2.4. For Ω and E_0 as given above and for all $R > 0$, then there exists $M_0 >$ .. depending only on the dimension and s, such that for any $M \geq M_0$, the minimizer E_M is disconnected.

Again, similar proofs as in section 2.1.
Add some more discussion.

Proof of Theorem 2.3. We show that for every $R > 0$ at least the tube $\{|x_n| < r_0\}$ is in the minimizer for some $r_0 > 0$.

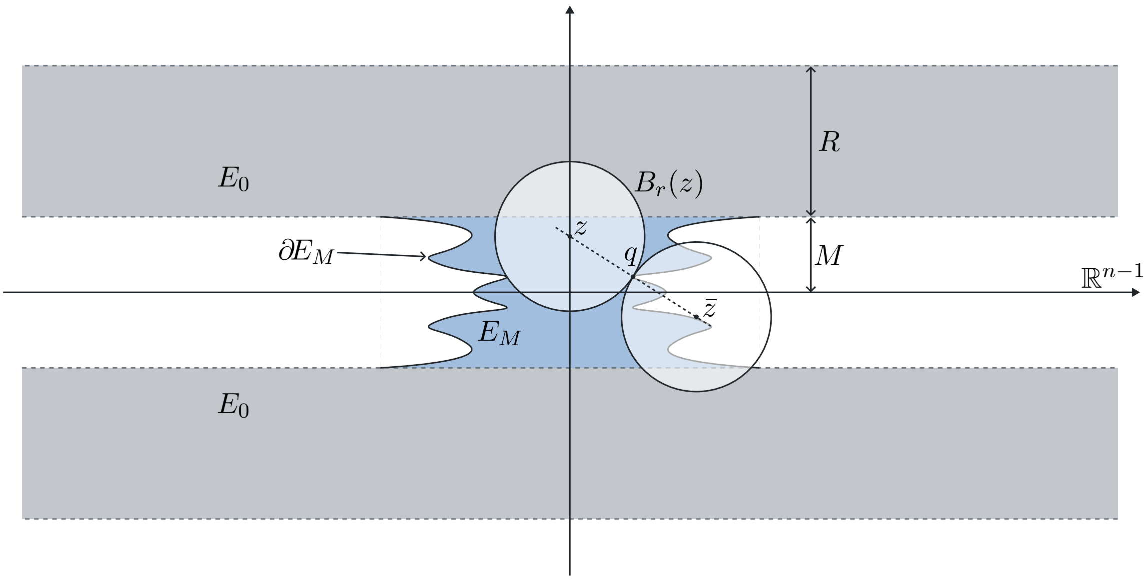

Figure 2.5

TODO. Edit proof to show cylinder is in minimizer and not assume that it's disconnected

We do that analogously to theorem theorem 2.1 by contradiction. We assume that E_M is disconnected, thus we can slide a ball of radius r down and for all $r_0 \in (0, 1)$ there exists a $t_0 > 0$ s.t. $\partial B_{r_0}(t_0 e_n) \cap \partial E_M \neq \emptyset$. If we can show that there exists a r_0 s.t. this conntradicts then the tube is in the minimizer. It is enough to show that for one r_0 since if we can contradict this for one r_0 then for all smaller there is no touching as well.
For that we split into four parts as seen in the figure:

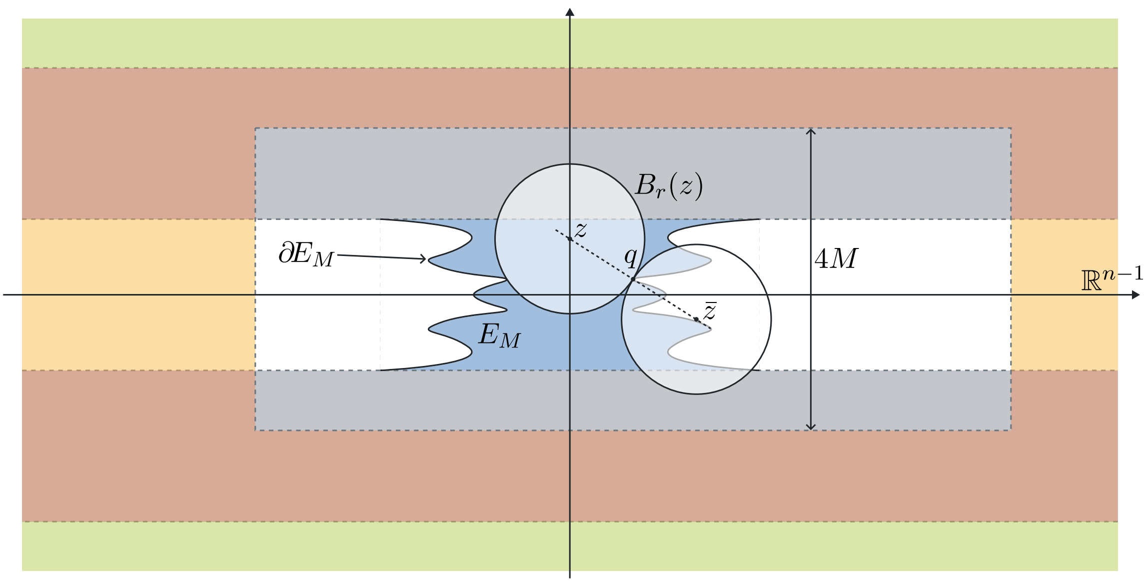

Figure 2.6

We define

$$
\begin{aligned}
&A := \{(x', x_n) \text{s.t.} |x_n| \geq M + R\} \\
&B := \{(x', x_n) \text{s.t.} |x_n| \leq M, |x' - q'| > 2\} \\
&C := E_0 \setminus S \\
&S := \{(x', x_n) \text{s.t.} |x_n - q_n| \leq M + R, |x' - q'| \leq 2\}
\end{aligned}
$$

Integration over the first part:

$$
\begin{aligned}
&\int_A \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y \overset{A \subset E^c}{\leq} \int_{|y_n| \geq R} \frac{1}{|y|^{n+s}} \,\mathrm{d}y \leq c(n) \int_0^\infty \int_R^\infty \frac{r^{n-2}}{(r^2 + y_n^2)^{\frac{n+s}{2}}} \,\mathrm{d}y_n \,\mathrm{d}r \\
&\leq c(n) \int_0^\infty \int_R^\infty \frac{1}{(r^2 + y_n^2)^{\frac{s+2}{2}}} \,\mathrm{d}y_n \,\mathrm{d}r \leq c(n) \int_0^\infty \int_R^\infty \frac{1}{(r + y_n)^{s+2}} \,\mathrm{d}y_n \,\mathrm{d}r \\
&= c(n,s) \int_0^\infty \frac{1}{(r+R)^{s+1}} \,\mathrm{d}r = c(n,s) R^{-s}
\end{aligned}
$$

Integration over the second part:

$$
\begin{aligned}
&\int_B \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y \overset{B \subset E^c}{\leq} c(n) \int_0^M \int_2^\infty \frac{r^{n-2}}{(r^2 + y_n^2)^{\frac{n+s}{2}}} \,\mathrm{d}r \,\mathrm{d}y_n \\
&\leq c(n) \int_0^M \int_2^\infty \frac{1}{(r + y_n)^{s+2}} \,\mathrm{d}r \,\mathrm{d}y_n = c(n,s) \int_0^M \frac{1}{(2 + y_n)^{s+1}} \,\mathrm{d}r \\
&= c(n,s)(2^{-s} - (2+M)^{-s}) \leq c(n,s) 2^{-s}
\end{aligned}
$$

Integration over the third part (here we need $R > M$):

$$
\int_C \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y = -\int_C \frac{1}{|y-q|^{n+s}} \,\mathrm{d}y \leq -c(n) \int_{B_M(\dots)} \frac{1}{|y|^{n+s}} \,\mathrm{d}y \leq -c(n,s) M^{-s}
$$

Idea: Move part of the stripe outside, restrict to ball with radius M and multiply with $\frac{1}{2}$ since not whole ball may be in the set. See Figure 2.7.

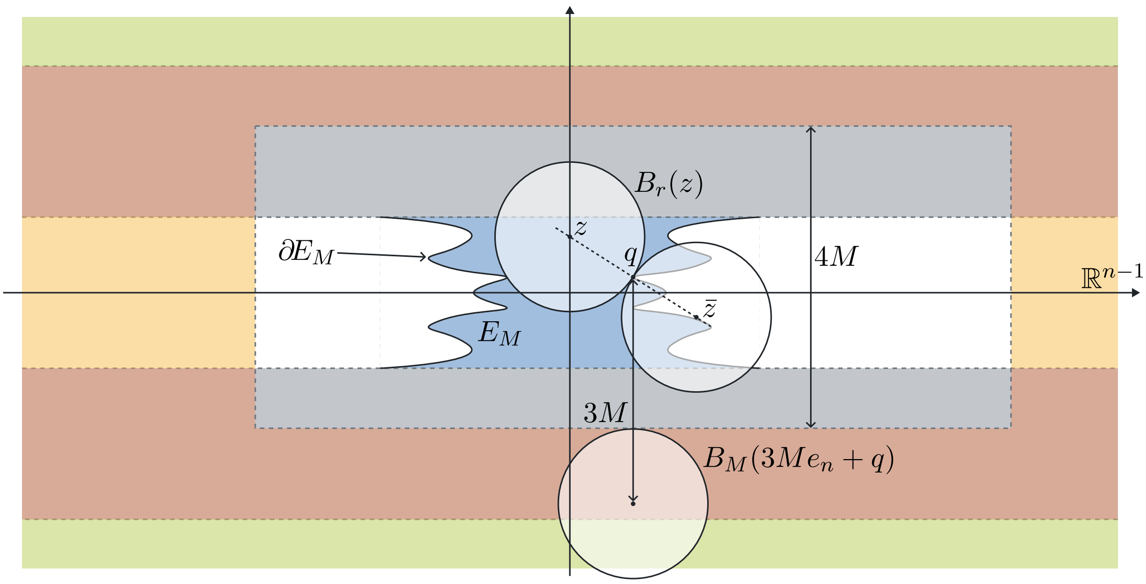

Figure 2.7 caption

Integration over the fourth part:
We split S into four parts:

i) $S \cap B_{\Lambda M}(q) \cap B_{r_0}(z)$

ii) $S \cap B_{\Lambda M}(q) \cap B_{r_0}(\overline{z})$

iii) $S \cap (B_{\Lambda M}(q) \setminus (B_{r_0}(z) \cup B_{r_0}(z)))$

iv) $S \setminus B_{\Lambda M}(q)$

where $\overline{z} := z + 2(q-z)$ and $\Lambda > 4$ chosen big enough and M chosen small enough s.t. $\Lambda M \leq 1$.
Again the first and second part are in sum smaller than zero.
We estimate the third part:

$$\begin{aligned}
&\int_{S \cap (B_{\Lambda M}(q) \setminus (B_{r_0}(z) \cup B_{r_0}(\overline{z})))} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y \\
&\leq \int_{P_{r_0,1}} \frac{1}{|y|^{n+s}} \,\mathrm{d}y + \int_{B_{\Lambda M} \setminus B_{r_0}} \frac{1}{|y|^{n+s}} \,\mathrm{d}y \leq c(n,s)(r_0^{-s} - (\Lambda M)^{-s})
\end{aligned}$$

We estimate the fourth part:

$$\begin{aligned}
&\int_{S \setminus B_{\Lambda M}(q)} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y \\
&\leq c(n) \int_{\Lambda M}^{R+3} \frac{1}{r^{s+1}} \,\mathrm{d}r \leq c(n,s)((\Lambda M)^{-s} - (R+3)^{-s})
\end{aligned}$$

Thus we estimate the domain S with

$$\int_S \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y \leq c(n,s)(r_0^{-s} - (R+3)^{-s}) \leq c(n,s) r_0^{-s}$$

Thus in total we get:

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|y-q|^{n+s}} \,\mathrm{d}y &\leq -c_0 M^{-s} + c_1(R^{-s} + 2^{-s} + r_0^{-s}) \\
&\leq -c_0 M^{-s}(1 - \frac{c_1}{c_0}(R^{-s}M^s + 2^{-s}M^s + r_0^{-s}M^s))
\end{aligned}$$

Now choose $r_0 = \frac{R}{2}$ and at most 2

$$\leq -c_0 M^{-s}(1 - \frac{c_1}{c_0}(R^{-s}M^s + 2^{-s}M^s + \left(\frac{2M}{R}\right)^s))$$

Choose Λ large and M small enoguh

$$\leq -c_2 M^{-s} < 0$$

■

TODO. Elaborate

Disscussion about connectedness in case of small R and refer to next chapter. Behavior unique to nonlocal minimal surfaces.
Talk about the contribution of the complement.

TODO. Complete proof of Theorem 2.4

Proof of Theorem 2.4. Analogously to Theorem 2.2. ■

TODO. Ignore the existence of a disconnected part of the minimizer for now and come back to that in Chapter 3

Theorem 2.5. For Ω and E_0 as given above and for all $R \geq 2$, then there exists $M_0 \in (0,1)$ depending only on the dimension and s, such that for any $M \in (0, M_0)$, the minimizer is $E_M = E_0 \cup \Omega$.

We will prove this in a somehow similar manner than before, but this time we consider a ball of fixed radius $R/2$ and slide it on the x_1 direction. Since symmtries are preserved it is enough to consider x_1. We will push the ball outside from the origin and contradict the assumption, that if the minimizer is not $E_0 \cup \Omega$, by assuming the exitence of the ball of radius $R/2$ at $(1 - R/2 - t)e_1 + he_n$ for some $t \in (0, 1 - R/2)$ and all $h \in (-M, M)$. Then since the cylinder is in the minimizer, we can conclude that $(1-t)e_1 + he_n$ is in the boundary for any h, thus the minimzer is $E_M = E_0 + \Omega$.

Proof of Theorem 2.5. Proof by contradiction. Assume $E_M \neq E_0 \cup \Omega$. By Theorem 2.3 we know that at least the cylinder $B'_{\frac{R}{2}} \times (-M, M)$ is in the minimizer, thus we know that the ball $B_{\frac{R}{2}}$ is in the minimizer. We slide the ball out in e_1 direction. It is enough to consider the e_1 direction as by (source) symmteies are preserved. Thus we consider the ball $B_{th} := B_{\frac{R}{2}}((1 - \frac{R}{2} - t)e_1 + he_n)$ for $t \in (0, 1 - \frac{R}{2})$ and $h \in (-M, M)$. Since $E_M \neq E_0 \cup \Omega$, there exists some t and h such that the ball touches the minimizer, say in $q \in \partial E_M \cap \partial B_{th}$. Then since $B_{th} \subset E_M$ we have by the Euler - Lagrange equation

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \,\mathrm{d}y \geq 0.$$

We will contradict this by splitting the domain into three parts, see (figure).

TODO. Add somewhere that $M < \frac{R}{2}$

Consider the following three parts:

$$\begin{aligned} A &:= \{(x', x_n) \mid |x' - q'| < R, |x_n - q_n| < 2M\} \\ B &:= E_M^c \setminus A \\ C &:= E_M \setminus A. \end{aligned}$$

We estimate the integrals over the three parts.

TODO. Add argument for first inequality

First we consider the integral over B:

$$\begin{aligned} \int_B \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \,\mathrm{d}y &= \int_B \frac{1}{|y-q|^{n-s}} \,\mathrm{d}y \\ &\leq \int_{|y|>R} \frac{1}{|y|^{n-s}} \,\mathrm{d}y \\ &\leq c(n,s) R^{-s}. \end{aligned}$$

TODO. Add argument for first inequality

Next we consider the integral over C:

$$\begin{aligned} \int_C \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}} \,\mathrm{d}y &= -\int_C \frac{1}{|y-q|^{n-s}} \,\mathrm{d}y \\ &\leq -c(n) \int_{B_M(4Me_n)} \frac{1}{|y|^{n-s}} \,\mathrm{d}y \\ &\leq -c(n,s) M^{-s}. \end{aligned}$$

TODO. Add argument for first inequality
Add context for $P_{\frac{R}{2},1}$

Finally we consider the integral over A:

$$\begin{aligned}\int_A \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}}\,\mathrm{d}y &\le \int_{B_4 \setminus B_{\frac{R}{2}}} \frac{1}{|y|^{n-s}}\,\mathrm{d}y + \int_{P_{\frac{R}{2},1}} \frac{1}{|y|^{n-s}}\,\mathrm{d}y \\ &\le c(n) \int_{\frac{R}{2}}^{4} \frac{1}{r^{1+s}}\,\mathrm{d}r + c(n)\frac{(\frac{R}{2})^{-s}}{1-s} \\ &\le c(n,s)(R^{-s} - 4^{-s}).\end{aligned}$$

Adding the three integrals we get

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c} - \chi_{E_M}}{|y-q|^{n-s}}\,\mathrm{d}y \le -c_0 M^{-s} + c_1(R^{-s} - 4^{-s})$$

now if we choose M small enough, then the right hand side is strictly smaller than zero, which is a contradiction to the assumption that the ball touches the minimizer.

■

TODO. Elaborate

Discussion about extending the model to arbitrary models with symmetric external data. Enough to consider discs of radius.. and heigth.. to have connectedness and even stickiness at some point.
New idea: If there is a minimizer E_M, can it ever be non sticky to the boundary?
Maybe able to give own interpretation of nonlocal minimal surfaces. Idea about Volume or Gravity? Gravity?

3 Disconnected Minimizer

IDEA. Open this chapter with the train of thought motivated by model02
For the unbounded case, consider all dimensions and general r, R and just the upper bound.
For the bounded case consider $n = 2$ to show, that even though we are positive at $s = 0, 1$ we could still have negative values somewhere in between
Then give some interpretation if or how that helps or the consequences of that.

When discussing the connectedness of the second model in Chapter 2, we first just stated, that if $R \geq 2$ the minimizer is connected and that if $R < 2$ then at least the cylinder $Z_R := B''_{R/2} \times (-M, M)$ is in the minimizer. This fact is not enough to for connectedness of the minimizer. To show connectedness, we would still need to show, that if there cannot exist a part of the minimizer that is fully detached from the cylinder and the external data.
Motivated by the fact, that in the classical case, if we have some external data E_0 and some prescribed set Ω that are fully disconnedted, i.e. $\operatorname{dist}(E_0, \Omega) =: d > 0$, then the minimizer is the external data itself, we wanted to prove the same thing for the nonlocal case as well.

Indeed, if we could show that, then the minimizer in the second model would be connected. Assume there exists a part of the minimizer that is not connected to the cylinder and the external data, i.e. there exists a set E_1 such that $\operatorname{dist}(E_1, E_0 \cup Z_R) > 0$. Then we can rewrite the fractional perimeter of $E_M := E_0 \cup E_1$ relative to Ω as follows:

$$\begin{aligned}
\operatorname{Per}_s(E_M, \Omega) &= \mathscr{L}(E_M \cap \Omega, E_M^c) + \mathscr{L}(E_M \setminus \Omega, \Omega \setminus E_M) \\
&= \mathscr{L}(E_1 \cup Z_R, E_M^c) + \mathscr{L}(E_0, \Omega \setminus (E_1 \cup Z_R)) \\
&= \mathscr{L}(E_1, E_M^c) + \mathscr{L}(Z_R, E_M^c) + \mathscr{L}(E_0 \cup Z_R, \Omega \setminus (E_1 \cup Z_R)) - \mathscr{L}(Z_R, \Omega \setminus (E_1 \cup Z_R)) \\
&= \operatorname{Per}_s(E_M, \Omega \setminus Z_R) + \mathscr{L}(Z_R, (E_0 \cup Z_R)^c). \qquad (3.1)
\end{aligned}$$

Notice that the second term in (3.1) is now independent of E_1, thus to minimize $\operatorname{Per}_s(E_M, \Omega)$ we can minimize $\operatorname{Per}_s(E_M, \Omega \setminus Z_R)$ instead.
We define a sequence of prescribed sets Ω_n such that $\operatorname{dist}(E_0 \cup Z_R, \Omega_n) = d/n$, where $d = \operatorname{dist}(E_0 \cup Z_R, E_1)$. Then for each n we are in the situation of fully disconnected external data, here $E_0 \cup Z_R$ and prescribed set, here Ω_n, and we could conclude

$$\operatorname{Per}_s(E_M, \Omega \setminus Z_R) \leq \operatorname{Per}_s(E_M, \Omega_n) \geq \operatorname{Per}_s(E_0, \Omega_n) \nwarrow \operatorname{Per}_s(E_0, \Omega \setminus Z_R).$$

As it turns out, this is not true in general and thus we cannot state connectedness just with the existence of the cylinder in the minimizer.
In the following, we will consider an example where depending on s the minimizer is not the external data itslef, even though the external data and the prescribed set have nonzero distance.

Example 3.1. Let $E_0 = B_2^c$ and $\Omega = B_1$ in $\mathbb{R}^2$. Then we compare the fractional perimeter of E_0 relative to Ω with the fractional perimeter of $E_0 \cup \Omega$ relative to Ω

$$\operatorname{Per}_s(E_0 \cup \Omega, \Omega) - \operatorname{Per}_s(E_0, \Omega) = \operatorname{Per}_s(B_1) - 2L(B_2^c, B_1) \qquad (3.2)$$

and show that this difference is negative for s small enough.

For the first term we have by [14, Eq. (11)]

$$\operatorname{Per}_s(B_1) = \frac{2^{2-s}\pi^{\frac{3}{2}}}{s(2-s)} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{2-s}{2})}.$$

We want to bound the second term from above and below. For that we will split the domain depending on x, see Figure 3.1.

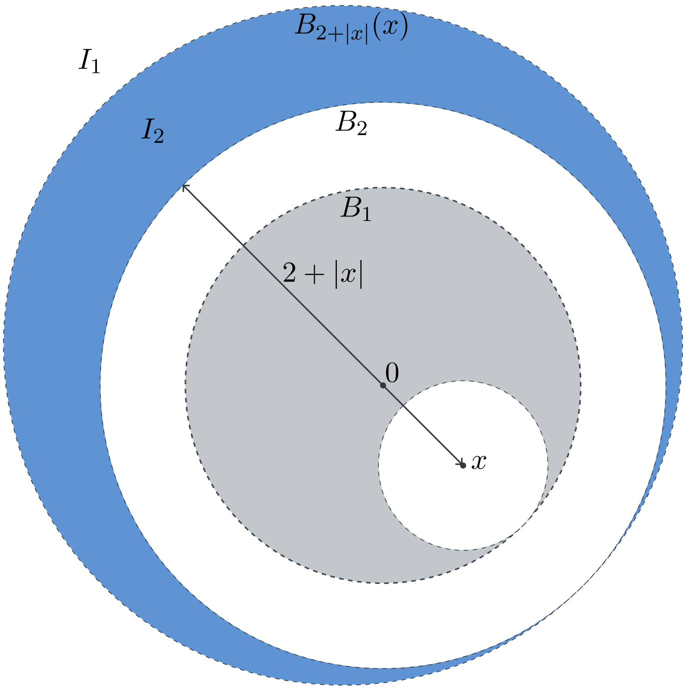

Figure 3.1 Split B_2^c depending on x

Thus we have

$$\mathscr{L}(B_2^c, B_1) = \int_{B_1}\int_{B_2^c} \frac{1}{|x-y|^{2-s}}\,\mathrm{d}y\,\mathrm{d}x = \underbrace{\int_{B_1}\int_{B^c_{2+|x|}(x)} \frac{1}{|x-y|^{2-s}}\,\mathrm{d}y\,\mathrm{d}x}_{I_1} + \underbrace{\int_{B_1}\int_{B_{2+|x|}(x)\setminus B_2} \frac{1}{|x-y|^{2-s}}\,\mathrm{d}y\,\mathrm{d}x}_{I_2}.$$

We start with I_1:

$$\begin{aligned}
I_1 &= \int_{B_1}\int_{B^c_{2+|x|}(x)} \frac{1}{|x-y|^{2+s}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{B_1}\int_{B^c_{2+|x|}} \frac{1}{|y|^{2+s}}\,\mathrm{d}y\,\mathrm{d}x \\
&= 4\pi^2 \int_0^1\int_{2+r_1}^{\infty} \frac{r_1}{r_2^{1+s}}\,\mathrm{d}r_2\,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s}\int_0^1 \left[-\frac{r_1}{r_2^s}\right]_{2+r_1}^{\infty}\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s}\int_0^1 \frac{r_1}{(2+r_1)^s}\,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s}\int_2^3 \frac{r_1-2}{r_1^s}\,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s}\left[\frac{r_1^{2-s}}{2-s} - 2\frac{r_1^{1-s}}{1-s}\right]_2^3 \\
&= \frac{4\pi^2}{s(1-s)(2-s)}\left(2^{2-s} - (s+1)3^{1-s}\right).
\end{aligned}$$

Thus for I_1 we have

$$I_1 = \frac{4\pi^2}{s(1-s)(2-s)} \left(2^{2-s} - (s+1)3^{1-s}\right). \tag{3.3}$$

Now to I_2. Here the idea is to use radial coordinates again. Since the ntegral is radial symmetric with respect to x, we can fix x such that $x = (r, 0)$ for $r = |x|$. Now for fixed x the domain of y is not radial symmetric anymore, thus we first have to compute the domain of $\vartheta := \vartheta(r_1, r_2)$.
We have two restrictions on y:

> **TODO.** Give justifications of bounds

(1) $4 \leq |x-y|^2 \leq (2 + 2|x|)^2$

(2) $2 - |x| \leq |y| \leq 2 + |x|$

From the first restriction with $|x| = r_1$, $|y| = r_2$ and ϑ the angle between x and y we get

$$\begin{aligned} 4 \leq |x-y|^2 &\leq (2+2r_1)^2 \\ \Leftrightarrow 4 \leq r_1^2 + r_2^2 - 2r_1 r_2 \cos(\vartheta) &\leq 4(1+r_1)^2 \\ \Leftrightarrow \frac{r_1^2 + r_2^2 - 4}{2r_1 r_2} \geq \cos(\vartheta) &\geq \frac{r_1^2 + r_2^2 - 4(1+r_1)^2}{2r_1 r_2}. \end{aligned} \tag{3.4}$$

From the second restriction we get that the right - hand - side of (3.4) is always greater or equal to -1, thus we have

$$\frac{r_1^2 + r_2^2 - 4}{2r_1 r_2} \geq \cos(\vartheta) \geq -1.$$

We will see, that for all r_1 and r_2 the argument is independent of ϑ, thus we can integrate over ϑ first. We then get

> **TODO.** Argument for symmetry and how domain was chosen

$$\int_{-\vartheta}^{\vartheta} \mathrm{d}\vartheta = 2\pi - 2\arccos\left(\frac{r_1^2 + r_2^2 - 4}{2r_1 r_2}\right).$$

For I_2 we get then

> **TODO.** Simplify computations?
> Add arguments about splitting, change of variables, computationsteps etc

$$
\begin{aligned}
I_2 &= \int_{B_1} \int_{B_{2+|x|}(x)\setminus B_2} \frac{1}{|x-y|^{2+s}} \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_{B_1} \underbrace{\int_{B_{2+|x|}\setminus B_2(-x)} \frac{1}{|y|^{2+s}} \,\mathrm{d}y}_{\text{radial symmetric w.r.t. } x} \,\mathrm{d}x \\
&= 2\pi \int_0^1 \int_{2-r_1}^{2+r_1} \frac{r_1}{r_2^{1+s}} \int_{-\vartheta}^{\vartheta} \mathrm{d}\vartheta \,\mathrm{d}r_2 \,\mathrm{d}r_1 \\
&= 2\pi \int_0^1 \int_{2-r_1}^{2+r_1} \frac{r_1}{r_2^{1+s}} \left(2\pi - 2\arccos\left(\frac{r_1^2+r_2^2-4}{2r_1 r_2}\right)\right) \mathrm{d}r_2 \,\mathrm{d}r_1 \\
&= 4\pi^2 \int_0^1 \int_{2-r_1}^{2+r_1} \frac{r_1}{r_2^{1+s}} \,\mathrm{d}r_2 \,\mathrm{d}r_1 - 4\pi \int_0^1 \int_{2-r_1}^{2+r_1} \frac{r_1}{r_2^{1+s}} \arccos\left(\frac{r_1^2+r_2^2-4}{2r_1 r_2}\right) \mathrm{d}r_2 \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s(1-s)(2-s)} \left((s+1)3^{1-s} - 3 + s\right) - \frac{4\pi^2}{s} \int_0^1 \frac{r_1}{(2-r_1)^s} \,\mathrm{d}r_1 + \frac{4\pi}{s} \int_0^1 \int_{2-r_1}^{2+r_2} \frac{r_1}{r_2^{1+s}} \frac{r_2^2 - r_1^2 + 4}{\sqrt{4r_1^2 r_2^2 - (r_1^2+r_2^2-4)^2}} \,\mathrm{d} \\
&= \underbrace{\frac{4\pi^2}{s(1-s)(2-s)} \left((s+1)3^{1-s} - 2^{2-s}\right)}_{-I_1} + \frac{4\pi}{s} \int_0^1 \int_{2-r_1}^{2+r_2} \frac{r_1}{r_2^{1+s}} \frac{r_2^2 - r_1^2 + 4}{\sqrt{4r_1^2 r_2^2 - (r_1^2+r_2^2-4)^2}} \,\mathrm{d}r_2 \,\mathrm{d}r_1.
\end{aligned}
$$

Thus we get for the second term in (3.2)

$$
\mathscr{L}(E_0, E_1) = \frac{4\pi}{s} \int_0^1 \int_{2-r_1}^{2+r_2} \frac{r_1}{r_2^{1+s}} \frac{r_2^2 - r_1^2 + 4}{\sqrt{4r_1^2 r_2^2 - (r_1^2+r_2^2-4)^2}} \,\mathrm{d}r_2 \,\mathrm{d}r_1.
$$

We can now bound this term without losing too much information. For the upper bound, we will use that $r_2 \geq 2 - r_1$ and for the lower bound we will use that $r_2 \leq 2 + r_1$. We then get

$$
\begin{aligned}
\bullet) \quad \mathscr{L}(E_0, E_1) &\leq \frac{4\pi}{s} \int_0^1 \int_{2-r_1}^{2+r_2} \frac{r_1}{(2-r_1)^s} \frac{1}{r_2} \frac{r_2^2 - r_1^2 + 4}{\sqrt{4r_1^2 r_2^2 - (r_1^2+r_2^2-4)^2}} \,\mathrm{d}r_2 \,\mathrm{d}r_1 \\
&= \frac{4\pi}{s} \int_0^1 \frac{r_1}{(2-r_1)^s} \left[\arccos\left(\frac{r_1^2+r_2^2-4}{2r_1 r_2}\right)\right]_{2-r_1}^{2+r_2} \mathrm{d}r_1 \\
&= \frac{4\pi^2}{s} \int_0^1 \frac{r_1}{(2-r_1)^s} \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s(1-s)(2-s)} \left(2^{2-s} - 3 + s\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\bullet) \quad \mathscr{L}(E_0, E_1) &\geq \frac{4\pi}{s} \int_0^1 \int_{2-r_1}^{2+r_2} \frac{r_1}{(2+r_1)^s} \frac{1}{r_2} \frac{r_2^2 - r_1^2 + 4}{\sqrt{4r_1^2 r_2^2 - (r_1^2+r_2^2-4)^2}} \,\mathrm{d}r_2 \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s} \int_0^1 \frac{r_1}{(2+r_1)^s} \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s(1-s)(2-s)} \left(2^{2-s} - (s+1)3^{1-s}\right).
\end{aligned}
$$

Thus we have that

$$
\frac{2^{2-s}\pi^{\frac{3}{2}}}{s(2-s)} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{2-s}{2})} - \frac{8\pi^2}{s(1-s)(2-s)} \left(2^{2-s} - 3 + s\right) \leq \mathrm{Per}_s(E_1) - 2\,\mathscr{L}(E_0, E_1) \leq \frac{2^{2-s}\pi^{\frac{3}{2}}}{s(2-s)} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{2-s}{2})} - \frac{8\pi^2}{s(1-s)(2-s)} \Big(
$$

TODO. Give justification, that both sides are continuous w.r.t. s and conclude
Maybe draw a picture

Example 3.2 (Continuoation of Example 3.1)**.** Let us now consider the same setting as in Example 3.1, but instead with the external data $E_1 = B_{2+T} \setminus B_2$ for $T > 0$ large enough. Notice, that this change just adds one additional term compared to before

$$\mathscr{L}(B_{2+T} \setminus B_2, B_1) = \mathscr{L}(B_2^c, B_1) - \underbrace{\int_{B_{2+T}^c} \int_{B_1} \frac{1}{|x-y|^{2-s}} \,\mathrm{d}x \,\mathrm{d}y}_{I_3}$$

We will bound I_3 from above and below.
The upper bound

$$\begin{aligned}
\int_{B_1} \int_{B_{2+T-|x|}^c(x)} \frac{1}{|x-y|^{2-s}} \,\mathrm{d}y \,\mathrm{d}x &\leq 4\pi^2 \int_0^1 \int_{2+T-r_1}^{\infty} \frac{r_1}{r_2^{1-s}} \,\mathrm{d}r_2 \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s} \int_0^1 \frac{r_1}{(2+T-r_1)^s} \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s(1-s)(2-s)} [(2+T)^{2-s} - (3-s+T)(1+T)^{1-s}]
\end{aligned}$$

and the lower bound

$$\begin{aligned}
\int_{B_1} \int_{B_{2+T+|x|}^c(x)} \frac{1}{|x-y|^{2-s}} \,\mathrm{d}y \,\mathrm{d}x &\geq 4\pi^2 \int_0^1 \int_{2+T+r_1}^{2+T+r_1} \frac{r_1}{r_2^{1-s}} \,\mathrm{d}r_2 \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s} \int_0^1 \frac{r_1}{(2+T+r_1)^s} \,\mathrm{d}r_1 \\
&= \frac{4\pi^2}{s(1-s)(2-s)} [(2+T)^{2-s} - (1+s+T)(3+T)^{1-s}]
\end{aligned}$$

Thus we have the bounds

$$\begin{aligned}
\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega) &\leq (\text{add upper bound}) + \frac{4\pi^2}{s(1-s)(2-s)} [(2+T)^{2-s} - (3-s+T)(1+T)^{1-s}] \\
\mathrm{Per}_s(E_0 \cup \Omega, \Omega) - \mathrm{Per}_s(E_0, \Omega) &\geq (\text{add lower bound}) + \frac{4\pi^2}{s(1-s)(2-s)} [(2+T)^{2-s} - (1+s+T)(3+T)^{1-s}]
\end{aligned}$$

TODO. Add picture and more explanation

Notice that the limits for $s \searrow 0$ and $s \nearrow 1$ are independent of T and that for $s \nearrow 1$ the limit is the same as in the unbounded case. For $s \searrow 0$ however we have a different limit than in the unbounded case.

TODO. Add limit for $s \searrow 0$, independent of T and show that for T large enough the difference is negative for some interval of s.

TODO. Go from the example to the general case

We will consider the following setting once with unbounded data and once with bounded data: Let $n \geq 1$ and $r, R, T > 0$, such that $r < R$. Take the external data $E_0 = B_R^c$ in the unbounded case and $E_0 = B_T \setminus B_R$ in the bounded case. Define the prescribed set $\Omega = B_r$.

CHECK. Is that correct?
Elaborate

In [3] the authors have shown that for $s \nearrow 1$ the fractional perimeter approaches the classical perimeter. Thus, we can expect that for s large enough the minimizer should be the external data itself. In [9] the authors have shown that for bounded sets with nonzero distance and s small enough the minimizer is the external data itself as well.

Theorem 3.3. Let $n \geq 2$ and $0 < r < R$. Let $E_0 = B_R^c$ and $\Omega = B_r$, then there exists a $s_0 \in (0,1)$ such that for all $s \in (0, s_0)$ the minimizer is not the external data itself.

Theorem 3.4. Let $n \geq 2$ and $0 < r < R$ and $R + T > 0$. Let $E_0 = B_{R+T} \setminus B_R$ and $\Omega = B_r$, then there exists $s_0, s_1 \in (0,1)$ such that for all $s \in (s_0, s_1)$ and T large enough the minimizer is not the external data itself and for s small and large enough the minimizer is the external data.

We will proof that the minimizer is not the external data itself by comparing the fractional perimeter of E_0 with the fractional perimeter of $E_0 \cup \Omega$.

Proof of Theorem 3.3.

$$\operatorname{Per}_s(E_0 \cup \Omega, \Omega) - \operatorname{Per}_s(E_0, \Omega) = \operatorname{Per}_s(\Omega) - 2L(E_0, \Omega) \tag{3.5}$$

For the first term we have by [14, Eq. (11)]

$$\operatorname{Per}_s(\Omega) = \operatorname{Per}_s(B_r) = \frac{2^{1-s}\pi^{\frac{n-1}{2}} n\omega_n}{s(n-s)} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{n-s}{2})} r^{n-s} = \frac{2^{2-s}\pi^{n-\frac{1}{2}}}{s(n-s)} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{n-s}{2})\Gamma(\frac{n}{2})} r^{n-s}$$

with $\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

The second term we will bound from below to get an upper bound for (3.5).

$$\begin{aligned}
\mathscr{L}(E_0, E_1) = \mathscr{L}(B_r, B_R^c) &= \int_{B_r}\int_{B_R^c} \frac{1}{|x-y|^{n-s}} \,\mathrm{d}y\,\mathrm{d}x \\
&\geq \int_{B_r}\int_{B^c_{R+|x|}(x)} \frac{1}{|x-y|^{n-s}} \,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{B_r}\int_{B^c_{R+|x|}} \frac{1}{|y|^{n-s}} \,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2} \int_0^r \int_{R+r_1}^\infty \frac{r_1^{n-1}}{r_2^{1+s}} \,\mathrm{d}r_2\,\mathrm{d}r_1 \\
&= \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2} \frac{1}{s} \int_0^r \frac{r_1^{n-1}}{(R+r_1)^s} \,\mathrm{d}r_1 \\
&= \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2} \frac{1}{ns} \frac{r^{n-1}}{R^s} {}_2F_1(s, n; n+1; -\frac{r}{R}).
\end{aligned}$$

TODO. Add source

In the last step we used the the following identity (source) for the hypergeometric function

$$B(b, c-b) {}_2F_1(a, b; c; z) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \,\mathrm{d}t \qquad \text{for} \Re(c) > \Re(b) > 0,$$

where B is the beta function. In our case we have $a = s$, $b = n$, $c = n + 1$ and $z = -\frac{r}{R}$, thus

$$\int_0^r \frac{r_1^{n-1}}{(R+r_1)^s}\,\mathrm{d}r_1 = \frac{r^n}{R^s}\int_0^1 r_1^{n-1}(1+\frac{r}{R}r_1)^{-s}\,\mathrm{d}r_1$$
$$= \frac{r^n}{R^s}B(n,1)_2F_1(s,n;n+1;-\frac{r}{R}) = \frac{r^n}{nR^s}{}_2F_1(s,n;n+1;-\frac{r}{R}).$$

Thus we can bound (3.5) from above by

$$\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega) \le \frac{2^{2-s}\pi^{n-\frac{1}{2}}}{s(n-s)}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{n-s}{2})\Gamma(\frac{n}{2})}r^{n-s} - \frac{8\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{r^n}{snR^s}{}_2F_1(s,n;n+1;-\frac{r}{R}). \tag{3.6}$$

TODO. Rewrite

Since we are interested in the behavior of (3.5) depending on s we multiply (3.6) by $s(1-s)$ to deal with the singularities at $s = 0$ and $s = 1$

$$s(1-s)(\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega)) \le \frac{2^{3-s}\pi^{n-\frac{1}{2}}}{n-s}\frac{\Gamma(\frac{3-s}{2})}{\Gamma(\frac{n-s}{2})\Gamma(\frac{n}{2})}r^{n-s} - \frac{8\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{r^n}{nR^s}(1-s){}_2F_1(s,n;n+1;-\frac{r}{R}). \tag{3.7}$$

TODO. Add source

Since Γ is continuous for all positive reals and ${}_2F_1$ is contiouous for $|z| \le 1$, the right hand side of (3.7) is continuous for all $s \in (0,1)$. Now take the limit for $s \searrow 0$ and $s \nearrow 1$

$$\lim_{s\searrow 0} s(1-s)(\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega)) = -\frac{4\pi^n}{n}\frac{1}{(\Gamma(\frac{n}{2}))^2}r^n < 0 \tag{3.8}$$

and

$$\lim_{s\nearrow 1} s(1-s)(\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega)) = \frac{4\pi^{n-\frac{1}{2}}}{n-1}\frac{1}{\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2})}r^{n-1} > 0.$$

Now we know for $s \nearrow 1$ that $(1-s)\,\mathrm{Per}_s(E,\Omega)$ is aproaching the classical perimeter, thus in (3.8) is actually an equality. Thus we can conclude that there exists a $s_0 \in (0,1)$ such that for all $s \in (0,s_0)$ the minimizer is not the external data itself.

■

Proof of Theorem 3.4. We consider again the difference

$$\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega) = \mathrm{Per}_s(\Omega) - 2\mathscr{L}(E_0,\Omega) = \mathrm{Per}_s(B_{R+T}\setminus B_R) - 2\mathscr{L}(B_R^c,B_r) + 2\mathscr{L}(B_{R+T}^c,B_r) \tag{3.9}$$

We can use the upper bound from the proof of Theorem 3.3 for the first 2 terms in (3.9). The third term we will bound from above

$$
\begin{aligned}
\mathscr{L}(B_{R+T}^c, B_r) &= \int_{B_r}\int_{B_{R+T}^c} \frac{1}{|x-y|^{n-s}}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \int_{B_r}\int_{B_{R+T-|x|}^c} \frac{1}{|x-y|^{n-s}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2}\int_0^r\int_{R+T-r_1}^\infty \frac{r_1^{n-1}}{r_2^{1+s}}\,\mathrm{d}r_2\,\mathrm{d}r_1 \\
&= \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{1}{s}\int_0^r \frac{r_1^{n-1}}{(R+T-r_1)^s}\,\mathrm{d}r_1 \\
&= \frac{4\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{1}{ns}\frac{r^n}{(R+T)^s}{}_2F_1(s,n;n+1;-\frac{r}{R+T}).
\end{aligned}
$$

Thus we can bound (3.9) from above by

$$
\begin{aligned}
&\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega) \\
&\leq \frac{2^{2-s}\pi^{n-\frac{1}{2}}}{s(n-s)}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{n-s}{2})\Gamma(\frac{n}{2})}r^{n-s} - \frac{8\pi^n}{(\Gamma(\frac{n}{2}))^2}\frac{r^n}{sn}\Big(\frac{1}{R^s}{}_2F_1(s,n;n+1;-\frac{r}{R}) - \frac{1}{(R+T)^s}{}_2F_1(s,n;n+1;-\frac{r}{R+T}\Big).
\end{aligned} \tag{3.10}
$$

TODO. Argument for continuity

Now we multiply by $s(1-s)$ to deal with the singularities at $s=0$, $s=1$ and let $s\searrow 0$ and $s\nearrow 1$

$$
\lim_{s\searrow 0} s(1-s)(\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega)) = \frac{4\pi^n}{n}\frac{1}{(\Gamma(\frac{n}{2}))^2}r^n > 0
$$

and

$$
\lim_{s\nearrow 1} s(1-s)(\mathrm{Per}_s(E_0\cup\Omega,\Omega) - \mathrm{Per}_s(E_0,\Omega)) = \frac{4\pi^{n-\frac{1}{2}}}{n-1}\frac{1}{\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2})}r^{n-1} > 0.
$$

Notice, that the limits are independent of R and T. Also both limits are positive. Nonetheless for T big enough, there will exist some interval in $(0,1)$ such that the difference is negative.

TODO. Elaborate or justify

The upper bound (3.10) is continuous in T for all $s\in(0,1)$ and for $T\to\infty$, the third term vanishes. Thus for s,T large enough, the upper bound behaves similar to the upper bound in the proof of Theorem 3.3. Thus there exists some $s\in(0,1)$ such that the difference is negative.
Since the limit for $s\searrow 0$ is independent of T and positive and by [9, Eq. (3.2)] we have that $s\,\mathscr{L}(B_{R+T}\setminus B_R, B_r)\to 0$ for $s\searrow 0$. Thus the limits in $s=0$ and $s=1$ are not only upper bounds but exact values.

Thus we can conclude, that there exists an interval (s_0,s_1) such that for all $s\in(s_0,s_1)$ the minimizer is not the external data itself.

■

TODO. Give some conclusion and interpretation of these results

When comparing both theorems and their proofs, we notice that the example with bounded exxternal data doesn't seem to converge to the example of unbounded external data. At least in the limit for $s\searrow 0$. This is interesting, as this entails, that if we want to analyze the limiting behavior of a minimizer for $s\searrow 0$ we cannot restrict the boundary data to be bounded or unbounded.

Conclusion

dicussion of the results, comparison to classical case, open problems, future work,..

1. Change of Topology in the models (barrier construction)
2. Cubic construction for arbitrary external data
3. Existence of s_0 for all external data and prescribed sets
4. Minimizer touching the boundary of the prescribed set (Calculations with of 3. with arbitrary parameter shows, no)
5. Can we give an estimate of the amount of connected components?

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