Goal: Let $E, \Omega \subset \mathbb{R}^n$, Ω bounded, Lipschitz and $d(E_0, \Omega) = d > 0$, then $Per_s(E_0, \Omega) \leq Per(E_0 \cup E_1, \Omega)$ $\forall E_1 \subset \Omega$

1. $E_1 = B_\varepsilon$ 2. E_1 smooth 3. E_1 Lipschitz

$$Per_s(E, \Omega) = L(E \cap \Omega, E^c) + L(E \setminus \Omega, \Omega \setminus E)$$

1.
$$
\begin{aligned}
& Per_s(E_0 \cup B_\varepsilon, \Omega) - Per_s(E_0, \Omega) \\
= \; & L(B_\varepsilon, B_\varepsilon^c \setminus E_0) + L(E_0, \Omega \setminus B_\varepsilon) - L(E_0, \Omega) \\
= \; & L(B_\varepsilon, B_\varepsilon^c) - 2L(E_0, B_\varepsilon)
\end{aligned}
$$

As stated in 2022 Figalli-Fusco-Maggi

a) $L(B_\varepsilon, B_\varepsilon^c) = Per_s(B_\varepsilon) = \dfrac{Per_s(B)}{|B|^{\frac{n-s}{n}}} |B_\varepsilon|^{\frac{n-s}{n}} = Per_s(B)\, \varepsilon^{n-s}$

b) $L(E_0, B_\varepsilon) = \displaystyle\int_{E_0}\int_{B_\varepsilon} \frac{1}{|x-y|^{n+s}} \leq c(n)|B_\varepsilon| \int_d^\infty \frac{r^{n-1}}{r^{n+s}}\, dr = \underbrace{c(n,s)}_{>0}\, \varepsilon^n d^{-s}$

$E_0 \subset B_d^c(x)$ $\forall x \in B_\varepsilon$

Thus
$$
\begin{aligned}
& Per_s(E_0 \cup E_1, \Omega) - Per_s(E_0, \Omega) = L(E_1, E_1^c) - 2L(E_1, E_0) \\
\geq \; & c_1 \varepsilon^{n-s} - c_2 \varepsilon^n d^{-s} = c_1 \varepsilon^n \left(\varepsilon^{-s} - \frac{c_2}{c_1} d^{-s}\right) > 0 \quad \forall \varepsilon \text{ small enough}
\end{aligned}
$$

2. E_1 smooth

a) $L(E_0, E_1)$

Since $E_1 \subset \Omega$ and Ω precompact $\Rightarrow$ $\bar{E}_1$ compact

Thus $\forall \varepsilon > 0$ $\exists$ finite subcover $E_1 \subset \bar{E}_1 \subset \bigcup^{N(\varepsilon)} B_\varepsilon(x_i)$

Then write $E_1 = \bigcup^{N(\varepsilon)} \underbrace{E_1 \cap B_\varepsilon(x_i)}_{=: E_{\varepsilon,i}}$

$$
\begin{aligned}
L(E_0, E_1) &= L\Big(E_0, \bigcup^{N(\varepsilon)} E_{\varepsilon,i}\Big) \leq \sum^{N(\varepsilon)} L(E_0, E_{\varepsilon,i}) \\
&\leq \sum^{N(\varepsilon)} \int_{E_0}\int_{E_{\varepsilon,i}} \frac{1}{|x-y|^{n+s}} \underset{\text{as in 1b)}}{\leq} \sum^{N(\varepsilon)} c(n)\, |E_{\varepsilon,i}| \int_d^\infty \frac{r^{n-1}}{r^{n+s}}\, dr \\
&\leq c(n,s)\, N(\varepsilon)\, \varepsilon^n d^{-s}
\end{aligned}
$$

$E_{\varepsilon,i} \subset B_\varepsilon(x_i)$

b) To bound $L(E_1, E_1^c) = Per_s(E_1)$ from below, we use the fractional Boxing inequality (Theorem 1.2 | 2018 Ponce-Spector)

W.l.o.g we can assume E_1 to be open since ∂E smooth. (?)

Thus by Thm 1.2 $\exists$ $c(n) > 0$ s.t. $\exists$ open cover of E_1
$$E_1 \subset \bigcup_{i=0}^\infty B_{r_i}(x_i) \quad \text{and}$$

Theorem 1.2. *There exists a constant $C = C(d) > 0$ such that for every bounded open set $U \subset \mathbb{R}^d$ one can find a covering*
$$U \subset \bigcup_{i=0}^\infty B_{r_i}(x_i)$$
by open balls of radii r_i for which
$$\sum_{i=0}^\infty r_i^{d-\alpha} \leq C\alpha(1-\alpha)P_\alpha(U),$$
for every $\alpha \in (0,1)$.

$$L(E_1, E_1^c) = Per_s(E_1) \geq \frac{1}{c(n)\, s(1-s)} \sum_{i=0}^\infty r_i^{n-s} \geq c(n,s) \sum^{N(\varepsilon)} r_i^{n-s}$$

Now restrict the sum to $N(\varepsilon)$ terms ($N(\varepsilon)$ as in 2a))

$$\geq c(n,s) \sum^{N(\varepsilon)} z^{n-s} \geq c(n,s)\, N(\varepsilon)\, \varepsilon^{n-s}$$

- If $z \geq \varepsilon$, then set $z = \varepsilon$
- If $z < \varepsilon$, then $\exists$ some $m \in \mathbb{N}$ s.t. $z \geq \frac{\varepsilon}{m}$, then set $z = \frac{\varepsilon}{m}$

$$
\begin{aligned}
\Rightarrow \quad & Per_s(E_0 \cup E_1, \Omega) - Per_s(E_0, \Omega) = L(E_1, E_1^c) - 2L(E_1, E_0) \\
\geq \; & c_1 N(\varepsilon)\, \varepsilon^{n-s} - c_2 N(\varepsilon)\, \varepsilon^n d^{-s} = c_1 N(\varepsilon)\, \varepsilon^n \left(\varepsilon^{-s} - \frac{c_2}{c_1} d^{-s}\right) > 0 \quad \text{for } \forall \varepsilon \text{ small enough}
\end{aligned}
$$

3. We use Approximation theorem as stated in Theorem 1.3 | 2016 Lombardini

Take $E_1 \subset \Omega$, then we can assume $\mathrm{Per}_s(E_0 \cup E_1, \Omega) < \infty$,
since $\mathrm{Per}_s(E_0, \Omega) = L(E_0, \Omega) \leq c\, d^{-s} < \infty$ and if $\mathrm{Per}_s(E_0 \cup E_1, \Omega) = \infty$
there is nothing to show.

Since $\mathrm{Per}_s(E_0 \cup E_1, \Omega) < \infty$ we can use Theorem 1.3. and find
a sequence $\{E_h\}$ of smooth open sets and $\varepsilon_h \to 0^+$ (Choose $\varepsilon_h < \frac{d}{4}$)
s.t.

$$\lim_{h\to\infty} \mathrm{Per}_s(E_h, \Omega) = \mathrm{Per}_s(E, \Omega) \quad \text{and} \quad \partial E_h \setminus N_{\varepsilon_h}(\partial\Omega) \subset N_{\varepsilon_h}(\partial E)$$

$N_\varrho(A) := \{x \mid d(x, A) < \varrho\}$

Notice that $d(E_h, E_0) > 0$, thus we can use 2.

Thus $$\mathrm{Per}_s(E, \Omega) = \lim_{h\to 0} \mathrm{Per}_s(E_h, \Omega) \geq \lim_{h \to 0} \mathrm{Per}_s(E_0, \Omega) = \mathrm{Per}_s(E_0, \Omega).$$