Scuola Normale Superiore

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Some Variational Problems Involving Nonlocal Perimeters and Applications

TESI DI PERFEZIONAMENTO IN MATEMATICA

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Abstract

The thesis is on several minimization problems involving nonlocal perimeters. The nonlocal perimeter is a nonlocal extension of the classical perimeter. The thesis is written on my contributions with several collaborators and these contributions can be found in the articles in [49, 96, 97, 99]. We have mainly investigated three problems: nonlocal minimal surfaces, nonlocal denoising problems, and nonlocal liquid drop models.

After giving a brief introduction on the nonlocal perimeters and its motivation in Chapter 1, we give the definition of nonlocal(fractional) perimeters in Chapter 2. Moreover, in Chapter 2, we collect several properties of the nonlocal(fractional) perimeters and also give some of their proofs. We also mention the regularity of sets which are (almost) minimizers of the nonlocal(fractional) perimeter.

In Chapter 3, we study the topology of nonlocal(fractional) minimal surfaces in a specific situation. The study of the nonlocal(fractional) minimal surfaces has been initiated by L. Caffarelli, J.M. Roquejoffre, and O. Savin [22] and, since then, this topic has attracted many authors. In particular, S. Dipierro, O. Savin, and E. Valdinoci [50, 51, 52] discovered the "stickiness" property of the nonlocal minimal surfaces, which is not true in the case of the classical minimal surfaces. Motivated by these works, we study how the shape of the nonlocal minimal surfaces look like in a specific cylinder with an initial data given by the complement of a slab perpendicular to the cylinder. In this setting, we prove that, if the width of the slab is small enough, then the nonlocal minimal surfaces coincide with the cylinder and, if the width is large enough, then the nonlocal minimal surfaces tend to stick to the boundary of the cylinder. The first result implies that nonlocal minimal surfaces cannot develop catenoids in some situation. This is not the case in the classical minimal surfaces.

In Chapter 4, we consider a nonlocal extension of the denoising model which was introduced by L. Rudin, S. Osher, and E. Fatemi [104]. Our denoising model is formulated as the minimization problem of the energy consisting of the nonlocal(fractional) total variation and L^2-fidelity term. The denoising model can be applied to remove noises from given images and recover the original images. In this thesis, we are particularly interested in the regularity of the (unique) minimizer of the energy. We obtain that, in 2 dimension, the minimizer is as regular as the given data (of class $C^{0,\alpha}$). This result can be regarded as a nonlocal version of the result by V. Caselles, A. Chambolle, and M. Novaga [30].

In Chapter 5, we consider a nonlocal extension of the liquid drop model which was introduced by G. Gamow [63] in 1930s. Our model is formulated as the minimization problem, with volume constraint, of the energy consisting of the nonlocal(fractional) perimeter and generalized Riesz potential term. The classical model was studied in order to explain the behaviour of atomic nuclei and predict nuclear fission. Heuristically, one can see that, if the volume is large, then the Riesz term dominates the perimeter term and, if the volume is small, then the perimeter term dominates the Riesz term. The former implies the nonexistence of minimizers (nuclear fission) and the latter implies the existence of minimizers (stability of atomic nuclei). In the classical case, there are a lot of works on the model [71, 72, 69, 83, 12, 100, 93] (not exhausted); however, the nonlocal case is not

well-understood (see [56, 27] for small mass regime). In this thesis, we are interested in the minimizers for large volumes. We obtain that, if the kernel of the Riesz term decays much faster than that of the nonlocal perimeter, then there exists a minimizer for any volume. On the other hand, if the kernel of the Riesz term is "properly" controlled by that of the nonlocal perimeter, then there exists no minimizer for large volumes. Moreover, if the Riesz term strongly dominates the perimeter term, then each minimizer converges to a ball as the volume diverges.

In Appendix A and B, we give several properties of the nonlocal(fractional) perimeter. In Appendix A, we state the compactness of sets of finite nonlocal perimeters with a general kernel. The proof is based on the results by E. Di Nezza, G. Palatucci, and E. Valdinoci [46]. In Appendix B, we show the Euler-Lagrange equations for minimizers of our functional studied in Chapter 4. The proof is based on the results by M.C. Caputo and N. Guillen [26].

Drizza la testa;
non è più tempo di gir sì sospeso.
Vedi colà un angel che s'appresta
per venir verso noi; vedi che torna
dal servigio del dì l'ancella sesta.
Di reverenza il viso e li atti addorna,
sì che i diletti lo 'nviarci in suso;
pensa che questo dì mai non raggiorna!

- Dante Alighieri, *Canto XII, PURGATORIO, Commedia*

Contents

Notations

- $\mathbb{R}$ and $\mathbb{N}$ are the sets of real numbers and natural numbers including 0, respectively.
- $\mathbb{R}_{>0} \coloneqq (0, +\infty) \subset \mathbb{R}$ and $\mathbb{R}_{\geq 0} \coloneqq [0, +\infty) \subset \mathbb{R}$.
- $\mathbb{R}^N$ is the Euclidean space of N-dimension with $N \in \mathbb{N}$.
- $\mathbb{S}^{N-1}$ is the $(N-1)$-dimensional sphere in $\mathbb{R}^N$.
- $\overline{\mathbb{R}} \coloneqq \mathbb{R} \cup \{+\infty\}$.
- $B_r(x)$ is an open ball in $\mathbb{R}^N$ of radius $r > 0$ centered at $x \in \mathbb{R}^N$. We often write B_r as the ball centered at the origin.
- We denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure for any $N \in \mathbb{N}$ and use the notation $|E| \coloneqq \mathcal{L}^N(E)$ for any set $E \subset \mathbb{R}^N$.
- We denote by $\mathcal{H}^N$ the N-dimensional Hausdorff measure for any $N \in \mathbb{N}$ and use the notation $|\partial B_1(x)| \coloneqq \mathcal{H}^{N-1}(\partial B_1(x))$ for any $x \in \mathbb{R}^N$.
- All sets and functions appearing in the dissertation are basically assumed to be Lebesgue measurable.
- For any $a, b \in \mathbb{R}$, $a \lesssim b$ means that there exists a constant $c > 0$ such that $a \leq c\, b$.

Chapter 1

Introduction

The mathematical concept of perimeter, for instance, in two dimension goes back, at least, to the ancient Greece, where mathematicians developed various geometric tools and ideas to measure the area or length of a mathematical object. In this context, the perimeter of a planar set was defined as the length of the curve enclosing a set. Nowadays, one has several modern concepts of perimeter that extend this intuitive idea and make it applicable to a wider class of sets. This modern point of view was developed by Renato Caccioppoli and Ennio De Giorgi in the 1950s. The standard concept of perimeter denoted by P is defined as

$$P(E) := \sup\left\{\int_E \operatorname{div} g(x)\,dx \mid g \in C_c^1(\mathbb{R}^N;\mathbb{R}^N), \quad |g| \le 1\right\} \tag{1.0.1}$$

for any measurable set $E \subset \mathbb{R}^N$. A fundamental result by Ennio De Giorgi and Herbert Federer shows that the perimeter defined in this way coincides with the $(N-1)$-dimensional Hausdorff measure of a suitable subset of the topological boundary. This means that the notion of the perimeter is consistent with the intuition of the measures such as areas and lengths.

Nonlocal perimeter, on the other hand, is a much newer concept than the classical perimeter and is defined by the double integral of some weight function over a set and its complement. Although the definition of the nonlocal perimeter does not seem to be relevant to the classical perimeter, one can actually observe that some of the nonlocal perimeters are closely related to the classical perimeter as we will see later. The study of the nonlocal perimeter has been initiated by L. Caffarelli, J.M. Roquejoffre, and O. Savin [22], who considered the local minimizers of the nonlocal (s-fractional) perimeter. Since then, enormous numbers of authors have been investigating various problems involving the nonlocal perimeter, as an analogy of the classical perimeter.

The nonlocal perimeter associated with a kernel $K : \mathbb{R}^N \to \mathbb{R}_{\ge 0} \cup \{+\infty\}$, $K \not\equiv +\infty$ is defined as the double integral of K over a set and its complement, namely,

$$P_K(E) := \int_E \int_{E^c} K(x-y)\,dx\,dy \tag{1.0.2}$$

for any set $E \subset \mathbb{R}^N$. Although the definition itself is simple and elementary, the nonlocal perimeter enjoys plenty of fruitful properties. For instance, the nonlocal perimeter is translation invariant. Precisely, one can observe that

$$P_K(E+h) = P_K(E)$$

for any set $E \subset \mathbb{R}^N$ and $h \in \mathbb{R}^N$. In addition, if K satisfies the homogeneous property, namely, K satisfies $K(\lambda\,x) = \lambda^\alpha K(x)$ for any $x \in \mathbb{R}^N$ and $\lambda > 0$ with some $\alpha \in \mathbb{R}$, then it is easy to see, by the change of variables, that

$$P_K(\lambda E) = \lambda^{2N+\alpha} P_K(E)$$

for any set $E \subset \mathbb{R}^N$. We discuss more the details on the nonlocal perimeter in Chapter 2.

Our main interest in the present thesis is to minimize the nonlocal perimeter, or to minimize the functional which is the sum of the nonlocal perimeter and another energy such as the Riesz potential energy. The minimization problem of the nonlocal perimeter has been studied by an enormous number of authors since the pioneering work by L. Caffarelli, J.M. Roquejoffre, and O. Savin [22]. They treated the so-called *s-fractional perimeter*, whose kernel in the nonlocal perimeter is given by the function $x \mapsto |x|^{-(N+s)}$ with $s \in (0,\, 1)$. They considered the sets that locally [1] minimizes the s-fractional perimeter. There are also other versions of the minimization problem of nonlocal perimeters whose kernel is given by an integrable function, and these kinds of problems have been considered, for instance, by the group of J. Mazón [89, 90].

As we mentioned, the s-fractional perimeter is strongly related to the classical perimeter defined in (1.0.1) when s is close to 1. Noticing that a “localized” version of the perimeter can be also defined in the same way as (1.0.1), one obtains that, up to multiplying by a constant,

$$\lim_{s \uparrow 1} (1-s)\, P_s(E; \Omega) = P(E; \Omega)$$

for a set $E \subset \mathbb{R}^N$ and a given domain $\Omega \subset \mathbb{R}^N$ with a smooth boundary. Heuristically, we can observe this phenomenon in the following way (see also [54, Appendix A]): we assume that ∂E is smooth (at least $C^{1,1}$). From the divergence theorem, the s-fractional perimeter is described as

$$P_s(E) = \frac{1}{s(N+s-2)} \int_{\partial E} \int_{\partial E} \frac{\nu_E(x) \cdot \nu_E(y)}{|x-y|^{N+s-2}}\, d\mathcal{H}^{N-1}(x) d\mathcal{H}^{N-1}(y). \tag{1.0.3}$$

where ν_E is the outer unit normal of ∂E. Fixing any point $x \in \partial E$, we see that, for small $\delta > 0$, $\nu_E(y) = \nu_E(x) + g(\delta)$ for any $y \in \partial E \cap B_\delta(x)$ where $|g(\delta)| \to 0$ as $\delta \to 0$. Then, by multiplying by the factor $1-s$ and taking the limit as $s \uparrow 1$, we obtain

$$\begin{aligned}
(1-s)P_s(E) &= \frac{1-s}{s(N+s-2)} \int_{\partial E} (1 + g(\delta)\cdot \nu_E(x)) \int_{\partial E \cap B_\delta(x)} \frac{1}{|x-y|^{N+s-2}} \\
&\quad + \frac{1-s}{s(N+s-2)} \int_{\partial E} \int_{\partial E \cap B^c_\delta(x)} \frac{\nu_E(x)\cdot \nu_E(y)}{|x-y|^{N+s-2}} \\
&= \frac{1-s}{s(N+s-2)} \int_{\partial E} (1 + g(\delta)\cdot \nu_E(x))\, \frac{\omega_{N-2}\delta^{1-s}}{1-s}\, d\mathcal{H}^{N-1}(x) \\
&\quad + \frac{1-s}{s(N+s-2)} O(\delta^{-N-s+2}) \\
&\xrightarrow[s\uparrow 1]{} \frac{\omega_{N-2}}{N-1} \mathcal{H}^{N-1}(\partial E) + \frac{\omega_{N-2}}{N-1} \int_{\partial E} g(\delta)\cdot \nu_E(x)\, d\mathcal{H}^{N-1}(x)
\end{aligned}$$

for sufficiently small $\delta > 0$. See, for instance, [14, 39, 24, 4] for further discussions.

On the other hand, when s is close to 0, the s-fractional perimeter is also related to the volume measure in $\mathbb{R}^N$. Indeed, one can obtain that, up to multiplying a constant,

$$\lim_{s \downarrow 0} s P_s(E) = |E| \tag{1.0.4}$$

[1] The nonlocal perimeter can also be conveniently “localized” in a given domain by taking into account the interactions in which at least one point lies in the domain

for a bounded set $E \subset \mathbb{R}^N$. If $N \geq 3$, we can heuristically confirm this phenomenon by using again the expression (1.0.3) and the fundamental solution Γ of Laplacian (see also [54]). Indeed, up to constants, we have

$$\begin{aligned}\lim_{s\downarrow 0} sP_s(E) &= \frac{1}{N-2}\int_{\partial E}\int_{\partial E}\frac{\nu_E(x)\cdot\nu_E(y)}{|x-y|^{N-2}}\,d\mathcal{H}^{N-1}(x)d\mathcal{H}^{N-1}(y)\\ &= \frac{1}{N-2}\int_{\partial E}\int_{\partial E}(\nu_E(x)\cdot\nu_E(y))\,\Gamma(x-y)\,d\mathcal{H}^{N-1}(x)d\mathcal{H}^{N-1}(y).\end{aligned}$$

Then, by applying the divergence theorem twice, we may obtain

$$\lim_{s\downarrow 0} sP_s(E) \approx \int_E\int_E \Delta\Gamma(x-y)\,dx\,dy = |E|.$$

See [91, 48] for the rigorous arguments.

Now, to capture a concrete intuition of the nonlocal perimeter, we give a practical application of the nonlocal perimeter. A simple application that we present here is related to image processing, which is one of the topics in the present thesis, and this topic is on a nonlocal version of the denoising model (see Chapter 4 for the detail). This application was also mentioned in [35].

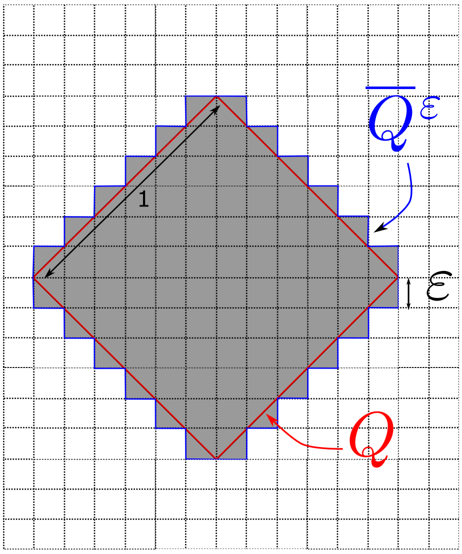

Figure 1.1: The real picture Q and the displayed picture $\overline{Q}^{\varepsilon}$ in a bitmap.

Let us consider the framework of BMP(bitmap) type images with square pixels of size $\varepsilon > 0$ and suppose that $\sqrt{2}\varepsilon^{-1} \in \mathbb{N}$ for simplicity. Moreover, let us consider a picture of a square of side 1, which is rotated by $\pi/4$ with respect to the origin, and let us compare the real picture (the region enclosed by the red line) with the picture displayed on the screen (the region enclosed by the blue line) as we show in Figure 1.1. In this configuration, the classical perimeter may provide a less accurate tool to analyse pictures on the screen than the nonlocal perimeter, no matter how the pixels are small (namely, ε is small). The smallness of the pixels corresponds to the resolution of the screen. Indeed, let $Q, \overline{Q}^{\varepsilon} \subset \mathbb{R}^2$ be the real picture and the displayed picture on the screen, respectively. The classical perimeter of the real picture Q is equal to 4, while the perimeter of the displayed picture $\overline{Q}^{\varepsilon}$ is always $4\sqrt{2}$. Hence the classical perimeter always produces an error ($|P(Q) - P(\overline{Q}^{\varepsilon})|$) by $\sqrt{2}-1$, even though the resolution of the screen is of high quality. On the other hand, the nonlocal perimeter with the kernel $K(x) = |x|^{-(2+s)}$ with $s \in (0, 1)$ in two dimension can be more sensitive to the resolution of the screen than the classical perimeter. Indeed, as shown in Figure 1.1, the displayed picture $\overline{Q}^{\varepsilon}$ is composed of the real picture Q and the disjoint $2\sqrt{2}\varepsilon^{-1}$ isosceles right triangles $\{T_i^{\varepsilon}\}_{i=1}^{2\sqrt{2}\varepsilon^{-1}}$ of short side ε. Then, by using some basic properties of the nonlocal perimeter (see in Chapter 2), we have

$$P_s(\overline{Q}^{\varepsilon}) = P_s(Q) + P_s(\cup_{i=1}^{M} T_i^{\varepsilon}) - 2\int_Q \int_{\cup_{i=1}^{M} T_i^{\varepsilon}} \frac{1}{|x-y|^{2+s}}\, dx\, dy \tag{1.0.5}$$

where we set $M := 2\sqrt{2}\varepsilon^{-1}$. By rescaling each triangle T_i^{ε} for i and from the translation invariance of the nonlocal perimeter, we may compute the quantity $P_s(\cup_{i=1}^{M} T_i^{\varepsilon})$ as follows:

$$P_s(\cup_{i=1}^{M} T_i^{\varepsilon}) \leq \sum_{i=1}^{M} \varepsilon^{2-s} P_s(T) = 2\sqrt{2} P_s(T)\, \varepsilon^{1-s} \tag{1.0.6}$$

where $T \subset \mathbb{R}^2$ is a isosceles right triangle of short side 1, of which the nonlocal perimeter is finite. Moreover, from the choice of $\{T_i^{\varepsilon}\}_i$, we have that

$$\sup_{\varepsilon>0} \int_Q \int_{\cup_{i=1}^{M} T_i^{\varepsilon}} \frac{dx\, dy}{|x-y|^{2+s}} \leq \int_Q \int_{Q^c} \frac{dx\, dy}{|x-y|^{2+s}} < \infty, \quad \chi_{\cup_{i=1}^{M} T_i^{\varepsilon}} \xrightarrow[\varepsilon\to 0]{} 0 \quad \text{a.e. in } \mathbb{R}^2$$

and thus, from the dominated convergence theorem, we obtain

$$\int_Q \int_{\cup_{i=1}^{M} T_i^{\varepsilon}} \frac{1}{|x-y|^{2+s}}\, dx\, dy \xrightarrow[\varepsilon\to 0]{} 0. \tag{1.0.7}$$

Therefore, from (1.0.5), we can obtain the following estimate of the discrepancy:

$$|P_s(\overline{Q}^{\varepsilon}) - P_s(Q)| \leq \max\left\{ P_s(\cup_{i=1}^{M} T_i^{\varepsilon}),\ 2\int_Q \int_{\cup_{i=1}^{M} T_i^{\varepsilon}} \frac{1}{|x-y|^{2+s}}\, dx\, dy \right\}. \tag{1.0.8}$$

From (1.0.6) and (1.0.7), we conclude that the nonlocal perimeter of the displayed picture is as close to that of the real picture as you want according to the size ε of the pixels.

Now let us state the main contributions of our works conducted during my doctoral studies. All the results that we show in the present thesis are included in the following list of papers.

- *(Dis)connectedness of nonlocal minimal surfaces in a cylinder and a stickiness property*, with S. Dipierro and E. Valdinoci, [49].
- *Local Hölder regularity of minimizers for nonlocal denoising problems*, with M. Novaga, [96].
- *Nonexistence of minimizers for a nonlocal perimeter functional with a Riesz and a background potential*, [99]
- *Existence of minimizers for a generalized liquid drop model with fractional perimeter*, with M. Novaga, [97].

In the sequel, we will briefly explain three topics of our works, and give the main results in each topic.

Shape of nonlocal minimal surfaces

A nonlocal minimal surface is defined as the boundary of a set which minimize the nonlocal perimeter, and is firstly studied by L. Caffarelli, J.M. Roquejoffre, and O. Savin [22]. The nonlocal minimal surfaces constitute one of the most fascinating, and challenging, research topics in the realm of fractional equations. The nonlocal minimal surfaces

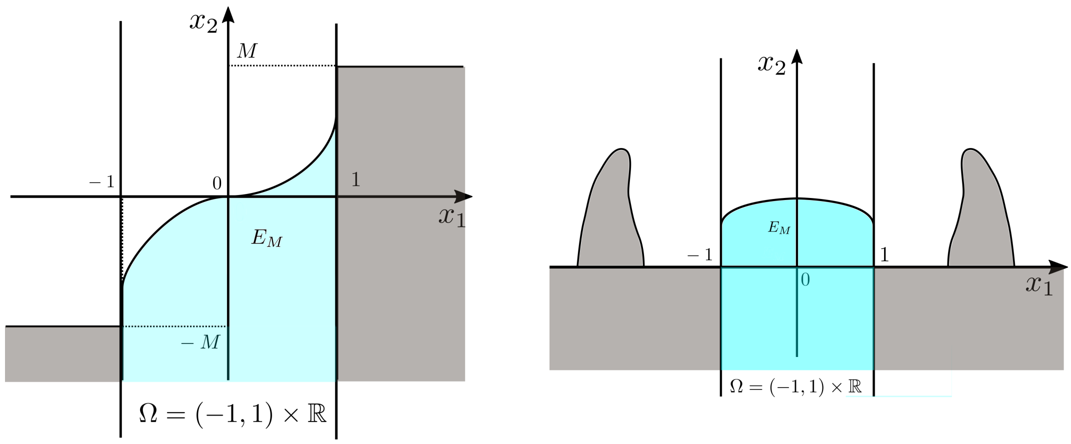

Figure 1.2: Stickiness phenomenon

constructed by this minimization procedure have relevant features in terms of differential geometry and geometric measure theory, since their energy functional can be considered as a nonlocal approximation of the classical perimeter functional and the nonlocal minimal surfaces as a fractional variant of the classical minimal surfaces, see [13, 39, 101, 24, 4, 25]. Critical points of the nonlocal perimeter energy functional satisfy an integral relation that can be seen as a vanishing nonlocal mean curvature prescription (see [22, 1, 41, 34]) and accordingly the study of volume prescribed minimizers leads to the analysis of surfaces with constant nonlocal mean curvature (see [40, 20, 21, 36]). Moreover, nonlocal minimal surfaces arise as the large-scale limit of long-range phase coexistence models (see [105]), as discrete iterations of fractional heat equations (see [23]) and as continuous approximations of interfaces of long-range Ising models (see [38]).

Given the importance of nonlocal minimal surfaces from all these perspectives, it is desirable to develop some intuition about their basic geometric features. For this, since it is very rare to have explicit solutions and precise formulas which entirely describe nonlocal minimal surfaces, it is often convenient to focus on some simplified cases in which the reference domain and the external data possess some special characteristics which lead to a deep understanding of at least some cardinal aspects of the object under investigation. Before our works on the shape of the nonlocal (s-fractional) minimal surfaces, S. Dipierro, O. Savin, and E. Valdinoci [50, 51, 52] have revealed several interesting properties; for instance, they discovered so-called "stickiness property" of the nonlocal (s-fractional) minimal surfaces. As shown in Figure 1, one may see the distinct properties from the classical minimal surfaces. For the left figure in Figure 1, the nonlocal perimeter gets minimized in the cylinder Ω with the given data colored in grey, which looks like a step. The the boundary of the minimizer tends to stick to the boundary of the domain, while the boundary of the minimizer of the classical perimeter does not. For the right figure in Figure 1, the nonlocal perimeter gets minimized in the cylinder Ω with the given data colored in grey, which is similar to the half space with small bumps.

Our topic in Chapter 3 follows precisely in this line of research, namely we consider a very simple domain, that is a vertical cylinder in $\mathbb{R}^N$, and a very special external data, that is the complement of a horizontal slab. In this setting, we detect how the minimizers of the nonlocal (s-fractional) perimeter change when the width of the slab varies. On the one hand, we obtain

Theorem (Theorem 3.1.1 in Chapter 3). *If the width of the slab is sufficiently small, then the minimizers become connected, more precisely, the minimizers coincide inside the cylinder with the cylinder itself.*

This change of topology is in agreement with the classical case, since minimizers of the classical perimeter constrained to two nearby parallel and co-axial circumferences are connected necks of catenoids. Nonetheless, the specific geometry exhibited in this case by nonlocal minimal surfaces is rather different from that of catenoids.

On the other hand, we obtain

Theorem (Theorem 3.1.2 in Chapter 3). *If the width of the slab is sufficiently large, then the minimizers in the domain are disconnected. Moreover, the minimizers contains a half ball of small radius inside the cylinder, and if $N = 2$, the nonlocal minimal surfaces sticks to the boundary of the domain (see Proposition 3.3.1 for the detail).*

The first part of the claim is the nonlocal counterpart of the fact that the classical perimeter gets minimized by far-away parallel and co-axial discs. However, the other part of the claim is totally distinct from the properties of the classical minimal surfaces.

Regularity for "nonlocal" denoising problem

The classical denoising model has been studied by many authors since the celebrated work by L. Rudin, S. Osher, and E. Fatemi [104], and plays an important role in image denoising and restoration (see for instance [31, 18]). Recently, a nonlocal version of the classical denoising model has attracted attentions to many authors in image processing. Given $s \in (0,\, 1)$ and $f \in L^2(\mathbb{R}^2)$, we define the functional $\mathcal{F}_{s,f}$ as

$$\mathcal{F}_{s,f}(u) := \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u(x)-u(y)|}{|x-y|^{N+s}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^2}(u-f)^2\,dx \tag{1.0.9}$$

for any $u \in W^{s,1}(\mathbb{R}^2)$, and in this thesis we study the minimization problem

$$\inf\left\{\mathcal{F}_{s,f}(u) \mid u \in W^{s,1} \cap L^2(\mathbb{R}^2)\right\}. \tag{1.0.10}$$

In Chapter 4, we discuss Problem (1.0.10) in detail, and basically focus on the regularity of the minimizer for Problem (1.0.10).

Our minimization problem is motivated by the classical minimization problem and the classical problem in the denoising model is given as

$$\inf\left\{\mathcal{F}_f(u) \mid u \in BV(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)\right\} \tag{1.0.11}$$

where $\mathcal{F}_f(u)$ is defined as

$$\mathcal{F}_f(u) := \int_{\mathbb{R}^N} |\nabla u| + \frac{1}{2}\int_{\mathbb{R}^N} |u-f|^2\,dx, \tag{1.0.12}$$

where $\int_{\mathbb{R}^N} |\nabla u|$ is the total variation of u in $\mathbb{R}^N$.

In image processing, the data f in the functional $\mathcal{F}_f$ indicates an observed image and, when the given image has poor quality, then the minimizers of $\mathcal{F}_f$ or solutions to the Euler-Lagrange equation associated with $\mathcal{F}_f$ correspond to regularized images. It is easy to show that the minimizer of (1.0.12) exists and is unique, as a result of strict convexity, lower semicontinuity and coercivity of the functional. Moreover, the minimizer turns out to be the solution, in a suitable sense, of the Euler-Lagrange equation

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + u - f = 0 \quad \text{in } \mathbb{R}^N. \tag{1.0.13}$$

The regularity of minimizers of $\mathcal{F}_f$ have been studied by several authors. In particular, the global and local regularity was investigated in a series of papers by V. Caselles, A. Chambolle and M. Novaga (see [29, 30, 31]), who proved that the solution of (1.0.13) inherits the local Hölder or Lipschitz regularity of the data f, when $N \leq 7$. In addition, if f is globally Hölder or Lipschitz in a convex domain $\Omega \subset \mathbb{R}^N$, the global regularity also holds for the solution of (1.0.13) with homogeneous Neumann boundary condition. In the recent papers [92, 102], some of these results were extended to general dimensions. G. Mercier [92] has proved that the continuity of f implies the continuity of a solution u and, in the case of convex domains, the modulus of continuity is also inherited globally by the solution. Eventually, A. Porretta [102] was able to remove the condition that the dimension of $\mathbb{R}^N$ is less than or equal to 7 considered in [30].

For the variational problems associated with the nonlocal total variation, G. Aubert and P. Kornprobst [7], and G. Gilboa and S. Osher [65] have proposed the methods for approximating the solutions to (1.0.11) with a sequence of nonlocal total variations associated with non-singular smooth kernels. Moreover, G. Gilboa and S. Osher [66] considered a similar nonlocal model to the functional (1.0.9) and did some numerical experiments. The authors showed some better functionality of the nonlocal model than that of the classical model. For instance, their nonlocal model in [66] can recover the original image from the inpainting image better than the classical model. However, as far as we know, there are no results on the regularity of minimizers of the functional $\mathcal{F}_{K,f}$.

In this thesis, we study the local Hölder regularity of minimizers for Problem (1.0.10) in two dimension as an analogy of the regularity results shown in [29, 30]. Precisely, we prove

Theorem (Theorem 4.1.1 in Chapter 4). *Let $N = 2$ and $K(x) = |x|^{-(2+s)}$. Assume that $f \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. If $f \in C^{0,\beta}_{loc}$ with $\beta \in (1-s,\, 1]$, then the minimizer of $\mathcal{F}_{K,f}$ is of class $C^{0,\beta}_{loc}$.*

Remark that we are not able to show the regularity result in higher dimensions because the singularities on the boundary of each superlevel sets of minimizers can appear and our method depends on the pointwise computations. Meanwhile, two-dimension case is of a particular interest for the application to image denoising.

On minimizers for "nonlocal" liquid drop model

The liquid drop model has been studied by many authors from a both physical and mathematical point of view. In Chapter 5, we investigate some minimization problems whose motivation comes from the classical liquid drop model by G. Gamow [63]. Let $m > 0$ be any number. We study the following minimization problem.

$$E_{K,g,\mu,\beta}[m] := \inf \{\mathcal{E}_{K,g,\mu,\beta}(E) \mid |E| = m\} \tag{1.0.14}$$

where we define the functional $\mathcal{E}_{K,g,\mu,\beta}$ as

$$\mathcal{E}_{K,g,\mu,\beta}(E) := P_K(E) + V_g(E) - R_{\mu,\beta}(E) \tag{1.0.15}$$

for any $E \subset \mathbb{R}^N$. We recall that P_K is the nonlocal perimeter associated with the kernel K and is given as

$$P_K(E) := \int_E \int_{E^c} K(x-y)\, dx\, dy \tag{1.0.16}$$

for any $E \subset \mathbb{R}^N$. Moreover, V_g is the Riesz potential associated with a general kernel g defined as

$$V_g(E) := \int_E \int_E g(x-y)\, dx\, dy, \tag{1.0.17}$$

and $R_{\mu,\beta}$ is the background potential with a parameter $\mu \in \mathbb{R}$ defined as

$$R_{\mu,\beta}(E) := \mu \int_E \frac{1}{|x|^\beta}\, dx$$

for any $E \subset \mathbb{R}^N$ and $\beta > 0$. Notice that, if $\mu = 0$, then $R_{\mu,\beta}$ is equal to zero no matter what β is, and thus we write $\mathcal{E}_{K,g}$ whenever $\mu = 0$. From a physical point of view, the background potential term $-R_{\mu,\beta}$ can behave as an attractive energy of electrons to a background nucleus as a point charge with the electrical charge μ along the potential function $|x|^{-\beta}$.

The study of Problem (1.0.14) can be seen as a nonlocal generalization of a series of the previous works [3, 12, 33, 58, 69, 71, 72, 74, 83, 84, 93, 98, 100] and the further references are therein. In their works, the authors treated the classical perimeter instead of the nonlocal perimeter. The classical minimization problem related to the liquid drop model is described as

$$\inf \{\mathcal{E}_{g,\mu,\beta}(E) \mid |E| = m\} \tag{1.0.18}$$

where we define the functional $\mathcal{E}_{g,\mu,\beta}$ as

$$\mathcal{E}_{g,\mu,\beta}(E) := P(E) + V_g(E) - R_{\mu,\beta}(E) \tag{1.0.19}$$

for any $E \subset \mathbb{R}^N$. Here P is the classical perimeter defined as

$$P(E) := \sup \left\{ \int_E \operatorname{div} g(x)\, dx \mid g \in C^1_c(\mathbb{R}^N; \mathbb{R}^N),\ |g| \leq 1 \right\}$$

for any set $E \subset \mathbb{R}^N$. In physics, it is important to consider the problem (1.0.18) when $N = 3$, $g(x) = |x|^{-1}$, and $\mu \equiv 0$. It is known as the liquid drop model, introduced by G. Gamow [63] to model the stability of atomic nuclei and explain nuclear fission. See [33] for the history of the Gamow's model. This model was developed by C.F. von Weizsäcker [108], N. Bohr [11], and so on. On the other hand, if $\mu \neq 0$, then Problem (1.0.18) is related to the ionization conjecture in quantum mechanics, which states that the number of electrons that can be bound to an atomic nucleus of charge $\mu > 0$ cannot exceed $\mu + 1$

Now we briefly review the previous works on the two problems; the classical problem (1.0.18) and the nonlocal problem (1.0.14). The following three topics are basically of much interest to us; the existence, the nonexistence, and the rigidity of minimizers. Here the rigidity means that a sequence of minimizers converges to the unit ball by rescaling properly as the volume converges to zero or diverges.

For the classical problem (1.0.18), H. Knüpfer and C.M. Muratov [71, 72] firstly proved the following results: if $N = 2$, $g(x) = |x|^{-\alpha}$ with $\alpha \in (0, 2)$, and $\mu = 0$, the ball is the only minimizer under the volume constraint $|E| = m$ for sufficiently small $m > 0$. In addition, for sufficiently large $m > 0$, there are no minimizers. Finally, in higher dimensions, if $3 \leq N \leq 7$, $\alpha \in (0, N-1)$, and $\mu = 0$, then the ball is the only minimizer for sufficiently small $m > 0$. Later, V. Julin [69] proved that, if $N \geq 3$, $g(x) = |x|^{-(N-2)}$, and $\mu = 0$, the ball is the unique minimizer of (1.0.19) whenever m is sufficiently small. Also, M. Bonacini and R. Cristoferi extended in [12] some of the results by H. Knüpfer and C.M. Muratov when $N \geq 2$, $g(x) = |x|^{-(N-\alpha)}$ with $\alpha \in (0, N-1)$, and $\mu = 0$. The authors showed that the ball is the unique minimizer for sufficiently small $m > 0$. Moreover, for small $\alpha > 0$, there exists a critical mass $m_1 > 0$ such that for $m \in (0, m_1]$, the ball is the unique minimizer under the constraint $|E| = m$, while for $m > m_1$ a solution to the minimization problem fails to exist. Regarding the nonexistence of minimizers of (1.0.19), not only H. Knüpfer and C.M. Muratov but also J. Lu and F. Otto [83] showed the following result: if $N = 3$, $g(x) = |x|^{-1)}$, $\mu \not\equiv 0$, and $\beta = 1$, then there exists a number $m_0 > 0$ such that for any $m \geq m_0$, Problem (1.0.18) has no solution. The authors in [83] were motivated by the ionization conjecture as we see in the above. Moreover, J. Lu and F. Otto [84] considered Thomas-Fermi-Dirac-von Weizsäcker model and showed the nonexistence of minimizers of the model. The model (1.0.19) can be regarded as a "sharp interface" version of Thomas-Fermi-Dirac-von Weizsäcker model. In a similar context to [83, 84], R.L. Frank, R. Killip, and P.T. Nam [58] showed the nonexistence of minimizers for large volumes in the case that $N = 3$, $g(x) = |x|^{-1}$, and $\mu = 0$. Later, R.L. Frank, P.T. Nam, and H. Van Den Bosch [61] studied the ionization conjecture in Thomas-Fermi-Dirac-von Weizsäcker theory and showed that a nucleus of charge $\mu > 0$ can bind at most $\mu + c$ for some universal constant. In contrast to these nonexistence results, in the case that $g(x) = |x|^{-\alpha}$ with $\alpha \in (0,\, N)$, $\mu \not\equiv 0$, and $\beta \in (0,\, \alpha)$, S. Alama, L. Bronsard, R. Choksi, and I. Topaloglu [3] proved that the functional (1.0.19) admits minimizers for any volumes, due to the effects from the background potential against the Riesz potential. The authors also considered the asymptotic behaviour of the minimizers when μ converges to zero. Even without the background potential, if the kernel g has a compact support, S. Rigot [103] proved the existence of minimizers for any volumes in the minimization problem (1.0.18) with $\mu = 0$. Very recently, M. Novaga and A. Pratelli [98] showed the existence of generalized minimizers for any volumes of the functional (1.0.19) with a general kernel g and $\mu = 0$. After this work, D. Carazzato, N. Fusco, and A. Pratelli in [28] showed that the ball is the unique minimizer for small volumes in any dimensions under general assumption on the kernel g. Concerning the behavior of the minimizers for large volumes, M. Pegon in [100] showed that, if the kernel g decays sufficiently fast at infinity and $\mu = 0$, then the minimizers of (1.0.19) exist and converge to a ball, up to rescaling, when the volume goes to infinity. Shortly after, B. Merlet and M. Pegon in [93] proved that, in dimension two, minimizers are actually balls for large enough volumes. Finally, we remark that F. Générau and E. Oudet [64] studied a similar problem to the problem (1.0.18) in the context of numerical analysis.

For the nonlocal problem (1.0.14), A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini [56] studied the isoperimetric problems in the case that $K(x) = |x|^{-(N+s)}$, $g(x) = |x|^{-(N-\alpha)}$ with $\alpha \in (0,\, N)$, and $\mu = 0$. The authors [56] showed that, if the volume m is sufficiently small, then the nonlocal minimization problem (1.0.14), up to multiplying a constant, admits the ball with the volume m as the unique minimizer, up to translations. Apart from the result in [56], there are almost no results on the problem (1.0.14).

Our works in this chapter closely follow this line of research on the nonlocal minimization problems. Intuitively, we can observe that the existence and nonexistence of minimizers for the nonlocal minimization problem (1.0.14) are valid if we use a dilation argument in the following way: if we consider each term in (1.0.15) separately and if $K(x) = |x|^{-(N+s)}$ with $s \in (0,\, 1)$ and $g(x) = |x|^{-(N-\alpha)}$ with $\alpha \in (0,\, N)$, then one can easily observe that, by the isoperimetric inequality of the s-fractional perimeter (see, for instance, [59]), a ball $B \subset \mathbb{R}^N$ is the only minimizer for P_K among the sets with the volume $|B|$ and, by the Riesz rearrangement inequality (see, for instance, [78, Theorem 3.4 and Theorem 3.7]), a ball $B \subset \mathbb{R}^N$ is the only maximizer for both V_α and $R_{\mu,\beta}$ among the sets with the volume $|B|$. Here V_α is defined as V_g in the case that $g(x) = |x|^{-(N-\alpha)}$ with $\alpha \in (0,\, N)$. Thus the non-trivial competition among P_K, V_α, and $R_{\mu,\beta}$ occurs. Letting $E \subset \mathbb{R}^N$ with $|E| = |B_1|$ and considering the dilated set $\lambda\, E$, we observe

$$\mathcal{E}_{s,\alpha,\mu,\beta}(\lambda\, E) = \lambda^{N-s}\, P_s(E) + \lambda^{N+\alpha}\, V_\alpha(E) - \lambda^{N-\beta}\, R_{\mu,\beta}(E)$$

As λ gets large, then the Riesz potential term V_α is the dominating term, while V_α does not admit minimizers. More precisely, we can see that it is more efficient for the minimizer to split into small pieces and the minimizer does not exist. In contrast, as λ gets small, then the dominating term is $\lambda^{N-s}\, P_K(E)$ or $-\lambda^{N-\beta}\, R_{\mu,\beta}(E)$. Recalling that both P_K and $-R_{\mu,\beta}$ admit a ball as the unique minimizer, we can expect that a ball is also the unique minimizer of our functional (1.0.15) if the volume is sufficiently small.

Following these lines of research, in this thesis we study a nonlocal generalization of the classical liquid drop model. We first obtain the nonexistence of minimizers for large volumes for Problem (1.0.14), namely, we obtain

Theorem (Theorem 5.1.2 in Chapter 5). *Let $\mu \not\equiv 0$ and $\beta = 1$. Assume that the kernel K behaves like $|x|^{-(N+s)}$ and the kernel g is given by $|x|^{-1}$. Then, for sufficiently large volume $m > 0$, Problem* (1.0.14) *admits no minimizers.*

See Theorem 5.1.2 in Chapter 5 for the precise assumptions on K. The idea of the proof is based on the arguments done in [58, 83, 84], and we can say that the essential point is to find the proper competitor against minimizers by doing the "cutting and pasting" procedure. Secondly, we obtain the existence of minimizers for any volumes under the fast decay of the kernel g of the Riesz potential. Precisely, we prove

Theorem (Theorem 5.2.3 in Chapter 5). *Let $\mu = 0$. Assume that the kernel $K(x)$ is given by $|x|^{-(N+s)}$ and the kernel g decays faster than the kernel K. Then, for any volume $m > 0$, Problem* (1.0.14) *admits minimizers.*

See in Chapter 5 for the precise assumptions on g. The idea of the proof is inspired by the "concentration compactness" lemma developed by P.L. Lions [79, 80]. Thirdly, we prove the existence of generalized minimizers of a generalized functional $\widetilde{\mathcal{E}}_{s,g}$, which we define later, under the assumption that the kernel g vanishes at infinity. For convenience, we here give the definitions of the generalized functional and generalized minimizers. For any $m > 0$, we define a *generalized functional* of $\mathcal{E}_{s,g}$ over the family of sequences of the sets $\{E^k\}_{k\in\mathbb{N}}$ with $\sum_{k=1}^{\infty} |E^k| = m$ as

$$\widetilde{\mathcal{E}}_{s,g}\left(\{E^k\}_{k\in\mathbb{N}}\right) := \sum_{k=1}^{\infty} \mathcal{E}_{s,g}(E^k). \tag{1.0.20}$$

Notice that in this functional the interaction between different components is excluded, which corresponds to the idea that the different components are placed "at infinity" from each other. Then we consider

$$\inf\left\{\widetilde{\mathcal{E}}_{s,g}\left(\{E^k\}_{k\in\mathbb{N}}\right) \mid E^k\text{: measurable for any } k,\ \sum_k |E^k| = m\right\} \tag{1.0.21}$$

and prove

Theorem (Theorem 5.2.4 in Chapter 5). *Let $\mu = 0$. Assume that the kernel $K(x)$ is given by $|x|^{-(N+s)}$ and the kernel g vanishes at infinity. Then, for any volume $m > 0$, Problem* (1.0.21) *admits minimizers.*

We call such a minimizer of the functional (1.0.20) the *generalized minimizer* for $\mathcal{E}_{s,g}$. The idea of the proof is also based on the "concentration compactness" lemma developed by P.L. Lions [79, 80] and a sort of reduction to Problem (1.0.14) with $K(x) = |x|^{-(N+s)}$ and $\mu = 0$.

Finally, we investigate the asymptotic behavior of minimizers as the volume goes to infinity, under the assumption that g decays faster at infinity than the kernel $|x|^{-(N+s)}$ of the s-fractional perimeter P_s. Here we require the assumption on g which is stronger than the one we assume in the existence result. To study the asymptotic behavior, we consider an equivalent minimization problem. More precisely, one can have two problems equivalent to $E_{s,g}[m]$ for $m > 0$ under some decay assumption on g. Indeed, since the kernel g is integrable over $\mathbb{R}^N$ under some proper assumptions, one can rewrite the Riesz potential as

$$\int_E \int_E g(x-y)\,dx\,dy = |E|\,\|g\|_{L^1(\mathbb{R}^N)} - \int_E \int_{E^c} g(x-y)\,dx\,dy$$

for any measurable set $E \subset \mathbb{R}^N$ with $|E| < \infty$. Hence, Problem (1.0.14) with $K(x) = |x|^{-(N+s)}$ and $\mu = 0$ becomes

$$\widehat{E}_{s,g}[m] \coloneqq \inf \left\{ P_s(E) - \int_E \int_{E^c} g(x-y)\, dx\, dy \mid |E| = m \right\} \tag{1.0.22}$$

for any $m > 0$. Moreover, by rescaling, one can further modify Problem (1.0.22) into the equivalent problem

$$\widehat{E}^{\lambda}_{s,g}[|B_1|] \coloneqq \inf \left\{ \widehat{\mathcal{E}}^{\lambda}_{s,g}(F) \mid |F| = |B_1| \right\} \tag{1.0.23}$$

for any $\lambda > 0$ where we define

$$\widehat{\mathcal{E}}^{\lambda}_{s,g}(F) \coloneqq P_s(F) - \int_F \int_{F^c} \lambda^{N+s} g(\lambda(x-y))\, dx\, dy.$$

Note that we will revisit the notations (1.0.22) and (1.0.23) more precisely in Section 5.2 of Chapter 5. With this notation, our last theorem is as follows.

Theorem (Theorem 5.2.6 in Chapter 5). *Suppose that $\{F_n\}_n$ is any sequence of the minimizers of $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$ such that $\lambda_n \to \infty$ and $|F_n| = |B_1|$ for any n. Then we have that the full sequence satisfies*

$$|F_n \Delta B_1| \xrightarrow[n\to\infty]{} 0$$

up to translations.

The idea of the proof is based on the two factors: the first one is the compactness by Lions that we mention in the above, and the second one is the Γ-convergence of the functional $\widehat{\mathcal{E}}^{\lambda}_{s,g}$ to the s-fractional perimeter P_s as $\lambda \uparrow \infty$.

Chapter 2

Nonlocal Perimeters

In this chapter, we give some definitions and show some of the properties of the nonlocal perimeter and nonlocal minimal sets that we need to prove our main results. The nonlocal perimeter is defined by the double integral of either a "singular" or "non-singular" kernel over a set and its complement. Here we mean by the "singular" kernel that the kernel of the nonlocal perimeter is not integrable near the origin, and the "non-singular" kernel is defined as a function which is integrable near the origin. In this thesis, we mainly focus on the nonlocal perimeter associated with the "singular" kernel. In particular, we study some minimization problems involving the so-called s-fractional perimeter with $s \in (0, 1)$, and this nonlocal perimeter is associated with the "singular" kernel $x \mapsto |x|^{-(N+s)}$. We remark that, for the fruitful topics on the "non-singular" kernel, J. M. Mazón, J. D. Rossi, and J. Toledo [89, 90] intensively studied this sort of the nonlocal perimeter, curvature, and minimal surfaces.

2.1 Nonlocal Perimeter

In this section, we give a rigorous notion of the nonlocal perimeter and its basic properties. Intuitively, as the classical perimeter does, a functional called perimeter should generally measure a sort of the boundary between a set and its complement. In the case of nonlocal perimeter, this perspective may be attained by considering some interaction between all the points in $\mathbb{R}^N$ via a measurable function $K : \mathbb{R}^N \to [0, \infty]$ which can be singular at the origin. Precisely, the *nonlocal perimeter* is defined in the following manner:

Definition 2.1.1 (Nonlocal Perimeter). The nonlocal perimeter $P_K(E)$ of a set $E \in \mathbb{R}^N$ associated with K is defined by

$$\int_E \int_{E^c} K(x-y)\, dx\, dy \tag{2.1.1}$$

where E^c is the complement of E given by $\mathbb{R}^N \setminus E$.

The reason we call (2.1.1) "perimeter" is that the interaction between points in E and E^c is measured with the function K that can be concentrated on the origin. Notice that from the change of variables, we have

$$\int_E \int_{(E+y)^c} K(x)\, dx\, dy. \tag{2.1.2}$$

Thus another interpretation on the nonlocal perimeter could be that, for each point y in a set E, we first consider the interaction by means of K in the complement of the translated set $E+y$ and then we integrate all the effects in E.

Next we give the notion of a "localized" nonlocal perimeter in a given domain. Let $\Omega \subset \mathbb{R}^N$ be an open set and $K : \mathbb{R}^N \to [0, \infty]$ be a measurable function. Before giving the definition, we define a map $L_K : \mathcal{E}_n \times \mathcal{E}_n \to [0, \infty]$ associated with K by

$$L_K(E, F) := \int_E \int_F K(x-y)\, dx\, dy$$

where we set $\mathcal{E}_n$ as the family of all $\mathcal{L}^N$-measurable sets. From the definition of L_K, one may easily observe the following properties: first, by definition, we can have that

$$L_K(E, E^c) = P_K(E)$$

for any $E \in \mathcal{E}_n$. Second, from Fubini-Tonelli's theorem and the non-negativity of K, one has that L_K is symmetric, i.e.,

$$L_K(E, F) = L_K(F, E) = \int\int_{E\times F} K(x-y)\, dx\, dy \tag{2.1.3}$$

for any $E,\, F \in \mathcal{E}_n$. One also observes that

$$\begin{aligned} L_K(E_1, F) &= L_K(E_2, F) \quad \text{if } \mathcal{L}^N(E_1 \Delta E_2) = 0, \\ L_K(E_1 \cup E_2, F) &= L_K(E_1, F) + L_K(E_2, F) \quad \text{if } \mathcal{L}^N(E_1 \cap E_2) = 0. \end{aligned}$$

for any $E_1,\, E_2,\, F \in \mathcal{E}_n$.

We now define the "localized" nonlocal perimeter as follows:

Definition 2.1.2 (Localised Nonlocal Perimeter). A *nonlocal perimeter in* Ω of a set E associated with K, denoted by $P_K(E;\Omega)$, is defined by

$$L_K(E\cap\Omega,\, E^c\cap\Omega) + L_K(E\cap\Omega,\, E^c\cap\Omega^c) + L_K(E\cap\Omega^c,\, E^c\cap\Omega). \tag{2.1.4}$$

As shown in the definition, the "localized" nonlocal perimeter consists of the three contributions, which are weighted with the kernel K, from the following regions: the first is the one between a set E and its complement that exist only in a reference set Ω. The second one is the one between E in Ω and E^c in Ω^c, and this contribution can measure the effect on $\partial\Omega$ of E coming from the inside of Ω. The last one is the one between E in Ω and E^c in Ω, and this contribution can measure the effect on $\partial\Omega$ of E coming from the outside of Ω.

Observe that the nonlocal perimeter of $E \subset \mathbb{R}^N$ in $\Omega \subset \mathbb{R}^N$ also has the following expression:

$$\begin{aligned} P_K(E;\Omega) = \frac{1}{2}\int_\Omega\int_\Omega K(x-y)\, |\chi_E(x) - \chi_E(y)|\, dx\, dy \\ + \int_\Omega\int_{\Omega^c} K(x-y)\, |\chi_E(x) - \chi_E(y)|\, dx\, dy. \end{aligned} \tag{2.1.5}$$

Now we state one of the most important properties of the nonlocal perimeter. This sort of statement is also valid and somehow more intuitive in the case of the classical perimeter.

Proposition 2.1.3. *Let $\Omega \subset \mathbb{R}^N$ be any open set. Assume that the kernel K is non-negative. Then, we have*

$$\begin{aligned} P_K(E\cup F;\Omega) = P_K(E;\Omega) + P_K(F;\Omega) - 2\int_{E\cap\Omega}\int_{F\cap\Omega} K(x-y)\, dx\, dy \\ - 2\int_{E\cap\Omega}\int_{F\cap\Omega^c} K(x-y)\, dx\, dy - 2\int_{E\cap\Omega^c}\int_{F\cap\Omega} K(x-y)\, dx\, dy \end{aligned}$$

for any $E,\, F \subset \mathbb{R}^N$ with $|E\cap F| = 0$.

Proof. The proof is done by a handful of computations as follows. For simplicity, we will not write the kernel K in the integral. First we have

$$\begin{aligned}
P_K(E\cup F;\Omega) &= \int_{E\cap\Omega\cup F\cap\Omega}\int_{E^c\cap F^c} + \int_{E\setminus\Omega\cup F\setminus\Omega}\int_{\Omega\cap E^c\cap F^c} \\
&= \int_{E\cap\Omega}\int_{E^c\cap F^c} + \int_{F\cap\Omega}\int_{E^c\cap F^c} + \int_{E\setminus\Omega}\int_{\Omega\cap E^c\cap F^c} + \int_{F\setminus\Omega}\int_{\Omega\cap E^c\cap F^c} \\
&= \int_{E\cap\Omega}\int_{E^c} - \int_{E\cap\Omega}\int_{E^c\cap F} + \int_{F\cap\Omega}\int_{F^c} - \int_{F\cap\Omega}\int_{E\cap F^c} \\
&\quad + \int_{E\setminus\Omega}\int_{\Omega\cap E^c} - \int_{E\setminus\Omega}\int_{\Omega\cap E^c\cap F} + \int_{F\setminus\Omega}\int_{\Omega\cap F^c} - \int_{F\setminus\Omega}\int_{\Omega\cap E\cap F^c}.
\end{aligned} \tag{2.1.6}$$

Notice that, since $|E\cap F| = 0$, we have that $E^c\cap F = F$ and $E\cap F^c = E$ in measure sense. Thus, from (2.1.6) and the definition of the nonlocal perimeter, we obtain

$$\begin{aligned}
P_K(E\cup F;\Omega) &= P_K(E;\Omega) + P_K(F;\Omega) - \int_{E\cap\Omega}\int_{F\cap\Omega} - \int_{E\cap\Omega}\int_{F\cap\Omega^c} \\
&\quad - \int_{F\cap\Omega}\int_{E\cap\Omega} - \int_{F\cap\Omega}\int_{E\cap\Omega^c} - \int_{E\setminus\Omega}\int_{\Omega\cap F} - \int_{F\setminus\Omega}\int_{\Omega\cap E} \\
&= P_K(E;\Omega) + P_K(F;\Omega) - 2\int_{E\cap\Omega}\int_{F\cap\Omega} \\
&\quad - 2\int_{E\cap\Omega}\int_{F\cap\Omega^c} - 2\int_{F\cap\Omega}\int_{E\cap\Omega^c}
\end{aligned}$$

as desired. □

Note that, if $\Omega = \mathbb{R}^N$, then we obtain another version of the equality

$$P_K(E\cup F) = P_K(E) + P_K(F) - 2\int_E\int_F K(x-y)\,dx\,dy.$$

Example 2.1.4. *The best known example of the nonlocal perimeter is given when $K(x) := |x|^{-(n+s)}$ with $s\in(0,1)$. This nonlocal perimeter was called "s-fractional perimeter", and introduced by L. Caffarelli, J.M. Roquejoffre, and O. Savin [22]. The authors studied the sets minimizing the s-fractional perimeter in some reference set, and their motivation for the study comes from phase field models where long range interactions occur.*

Remark 2.1.5. If the kernel K is given as $|x|^{-(N+s)}$ shown in Example 2.1.4, then the nonlocal perimeter $P_K(E)$ of a set $E\subset\mathbb{R}^N$ coincides with the s-fractional perimeter $P_s(E)$, and the s-fractional perimeter can be seen as the fractional Sobolev semi-norm defined as

$$[\chi_E]_{W^{s,1}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{N+s}}\,dx\,dy = 2P_s(E).$$

See for instance [46] for more details.

We remark that the nonlocal perimeter $P_K(E)$ of a set $E\subset\mathbb{R}^N$ may not be convergent, and the finiteness of $P_K(E)$ depends on the shape of the function K and the regularity of the set E. If the kernel K is constant almost everywhere in $\mathbb{R}^N\setminus\{0\}$ and a set E coincides wit the unit ball $B_1(0)\subset\mathbb{R}^N$, then the nonlocal perimeter $P_K(E)$ is obviously infinite.

We collect several properties of the nonlocal perimeter under suitable assumptions on the function K. The assumptions that we state here are natural as long as we discuss our results in this thesis, as we may weaken these assumptions. Remark that these properties are also valid in the case of the classical perimeter (see, for instance, [85, Chapter 12]).

Proposition 2.1.6. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Assume that K satisfies the following conditions:*

(NK1) *K is non-negative.*

(NK2) *$K(x) = K(-x)$ for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$.*

(NK3)

$$\int_{\mathbb{R}^N} K(x)\, \min\{1,\, |x|\}\, dx < \infty.$$

Then, we have the following properties:

1. *$P_K(U) < \infty$ for any bounded open set $U \subset \mathbb{R}^N$ with a Lipschitz boundary.*
2. *$P_K(\cdot;\Omega)$ is lower semicontinuous with respect to the L^1_{loc}-convergence, namely, if $\chi_{E_i} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^N)$ as $i \to \infty$, then*
$$P_K(E;\Omega) \leq \liminf_{i\to\infty} P_K(E_i;\Omega).$$
3. *$P_K(\cdot;\Omega)$ is sub-modular, namely, it holds that*
$$P_K(E \cup F;\Omega) + P_K(E \cap F;\Omega) \leq P_K(E;\Omega) + P_K(F;\Omega) \tag{2.1.7}$$
for any sets $E,\, F \subset \mathbb{R}^N$. Moreover, the equality holds if and only if either $|E\setminus F| = 0$, $|F \setminus E| = 0$, or $|E \Delta F \cap \Omega| = 0$ holds.

Proof. We can easily see that $P_K(U)$ is finite for any bounded open set U with a Lipschitz boundary. Indeed, from the assumptions (NK1), (NK2), and (NK3), we have

$$\begin{aligned} P_K(U) = L_K(U,\, U^c) &= \int_U \int_{U^c - y} K(-x)\, dx\, dy \\ &= \int_{\mathbb{R}^N} K(x) |U \cap (U-x)^c|\, dx \\ &\leq C \int_{\mathbb{R}^n} K(x)\, \min\{1,\, |x|\}\, dx < \infty \end{aligned}$$

for some constant $C = C(U) > 0$.

Secondly, the lower semicontinuity of $P_K(\cdot;\Omega)$ is proved by the expression (2.1.5) and Fatou's lemma because of (NK1). Note that this is true even though the assumptions (NK2) and (NK3) are not assumed.

Finally, we prove the sub-modularity of $P_K(\cdot;\Omega)$. For any sets $E,\, F \subset \mathbb{R}^N$, we compute the contributions $L_K(E \cap F \cap \Omega, (E^c \cup F^c) \cap \Omega)$ and $L_K((E \cup F) \cap \Omega, E^c \cap F^c \cap \Omega)$ as follows:

$$\begin{aligned} &L_K(E \cap F \cap \Omega, (E^c \cup F^c) \cap \Omega) \\ &= L_K(E \cap F \cap \Omega, E^c \cap F \cap \Omega) + L_K(E \cap F \cap \Omega, E^c \cap F^c \cap \Omega) \\ &\quad + L_K(E \cap F \cap \Omega, E^c \cap F^c \cap \Omega) \end{aligned}$$

and

$$\begin{aligned} &L_K((E \cup F) \cap \Omega, E^c \cap F^c \cap \Omega) \\ &= L_K(E \cap \Omega, E^c \cap \Omega) + L_K(F \cap \Omega, F^c \cap \Omega) \\ &\quad - L_K(E \cap F \cap \Omega, E^c \cap F^c \cap \Omega) - L_K(E \cap F \cap \Omega, E^c \cap F \cap \Omega) \\ &\quad - L_K(E \cap F^c \cap \Omega, E^c \cap F \cap \Omega) - L_K(E \cap F \cap \Omega, E \cap F^c \cap \Omega) \\ &\quad - L_K(E^c \cap F \cap \Omega, E \cap F^c \cap \Omega) - L_K(E^c \cap F \cap \Omega, E^c \cap F^c \cap \Omega). \end{aligned}$$

Thus, from the above two computations, we obtain

$$\begin{aligned} &L_K(E\cap F\cap\Omega,(E^c\cup F^c)\cap\Omega)+L_K((E\cup F)\cap\Omega,E^c\cap F^c\cap\Omega)\\ &\le L_K(E\cap\Omega,E^c\cap\Omega)+L_K(F\cap\Omega,F^c\cap\Omega). \end{aligned} \tag{2.1.8}$$

For the rest of the contributions in $P_K(\cdot;\Omega)$, we may repeat the same computations, and thus the inequality (2.1.7) is valid.

Now we show that the equality (2.1.7) holds if and only if either $|E\backslash F|=0$, $|F\backslash E|=0$, or $|E\Delta F\cap\Omega|=0$ holds. Indeed, by using Proposition 2.1.3 several times, one obtains the following identities:

$$\begin{aligned} P_K(E;\Omega)&=P_K(E\cap F;\Omega)+P_K(E\setminus F;\Omega)\\ &\quad-2\int_{E\cap F\cap\Omega}\int_{E\cap F^c\cap\Omega}-2\int_{E\cap F\cap\Omega}\int_{E\cap F^c\cap\Omega^c}-2\int_{E\cap F\cap\Omega^c}\int_{E\cap F^c\cap\Omega}\\ P_K(F;\Omega)&=P_K(F\cap E;\Omega)+P_K(F\setminus E;\Omega)\\ &\quad-2\int_{F\cap E\cap\Omega}\int_{F\cap E^c\cap\Omega}-2\int_{F\cap E\cap\Omega}\int_{F\cap E^c\cap\Omega^c}-2\int_{F\cap E\cap\Omega^c}\int_{F\cap E^c\cap\Omega}\\ P_K(E\cup F;\Omega)&=P_K(E\cap F;\Omega)+P_K(E\Delta F;\Omega)\\ &\quad-2\int_{E\cap F\cap\Omega}\int_{E\Delta F\cap\Omega}-2\int_{E\cap F\cap\Omega}\int_{E\Delta F\cap\Omega^c}-2\int_{E\cap F\cap\Omega^c}\int_{E\Delta F\cap\Omega}\\ P_K(E\Delta F;\Omega)&=P_K(E\setminus F;\Omega)+P_K(F\setminus E;\Omega)\\ &\quad-2\int_{E\cap F^c\cap\Omega}\int_{E^c\cap F\cap\Omega}-2\int_{E\cap F^c\cap\Omega}\int_{E^c\cap F\cap\Omega^c}-2\int_{E\cap F^c\cap\Omega^c}\int_{E^c\cap F\cap\Omega}. \end{aligned}$$

Note that we omit the kernel K in the integrals for simplicity. From all of the above computations, we have

$$\int_{E\cap F^c}\int_{E^c\cap F\cap\Omega}+\int_{E\cap F^c\cap\Omega}\int_{E^c\cap F\cap\Omega^c}=0. \tag{2.1.9}$$

Since $K\ge 0$ and K does not identically vanish, we can obtain that the equality of (2.1.7) holds if and only if either of the followings holds: $|E\cap F^c\cap\Omega|=0$ and $|E\cap F^c\cap\Omega^c|=0$ hold, $|E\cap F^c\cap\Omega|=0$ and $|E^c\cap F\cap\Omega|=0$ hold, or $|E^c\cap F\cap\Omega|=0$ and $|E^c\cap F\cap\Omega^c|=0$. This actually implies the claim as we desired. $\square$

Finally in this section, we show the compactness of sets of finite nonlocal perimeters, as an analogy of the compactness of sets of finite classical perimeters. To see this, we introduce the space of functions with finite nonlocal bounded variations. Let $\Omega\subset\mathbb{R}^N$ be an open set. Then we defined the space $BV_K(\Omega)$ as

$$BV_K(\Omega):=\left\{u\in L^1(\Omega)\mid [u]_K(\Omega)<\infty\right\} \tag{2.1.10}$$

where we set, for any measurable function u,

$$[u]_K(\Omega):=\int_\Omega\int_\Omega K(x-y)\,|u(x)-u(y)|\,dx\,dy. \tag{2.1.11}$$

Then, setting $\|\cdot\|_{K(\Omega)}$ as

$$\|u\|_{K(\Omega)}:=\|u\|_{L^1(\Omega)}+[u]_K(\Omega)$$

for any $u\in BV_K(\Omega)$, then we have that $\|\cdot\|_K$ is the norm of BV_K and BV_K is the Banach space with respect to this norm. We observe that the space $BV_K(\Omega)$ coincides with the fractional Sobolev space $W^{s,1}(\Omega)$ when the kernel K is given as $K(x) = |x|^{-(N+s)}$ with $s \in (0, 1)$ (see, for instance, [46]).

Recalling Definition 2.1.1 in Section 2.1, one readily sees that

$$[\chi_E]_K(\mathbb{R}^N) = 2P_K(E)$$

for any set $E \subset \mathbb{R}^N$. Moreover, given a domain $\Omega \subset \mathbb{R}^N$, we recall the definition of a localized version of the nonlocal perimeter P_K, and the localized nonlocal perimeter $P_K(E; \Omega)$ is given as

$$P_K(E; \Omega) = \frac{1}{2}\left([\chi_E]_K(\mathbb{R}^N) - [\chi_E]_K(\Omega^c)\right) \tag{2.1.12}$$

for any set $E \subset \mathbb{R}^N$.

Before proving the compactness of the nonlocal perimeter, we show the compactness of the space $BV_K(\Omega)$ as follows:

Theorem 2.1.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz boundary. Assume that the kernel K is given in such way that $BV_K(\Omega)$ is continuously embedded in $BV_K(\mathbb{R}^N)$ as shown in Lemma A.0.1 in Appendix A. Then, for any bounded subset $\mathcal{A} \subset BV_K(\Omega)$, $\mathcal{A}$ is relatively compact in $L^1(\Omega)$.*

Proof of Theorem 2.1.7. We show that $\mathcal{A} \subset BV_K(\Omega)$ is totally bounded in $L^1(\Omega)$, namely, for any $\varepsilon > 0$ there exist a finite number of functions $v_1, \cdots, v_M \in L^1(\Omega)$ such that for any $u \in \mathcal{A}$, there exists $j \in \{1, \cdots, M\}$ such that

$$\|u - v_j\|_{L^1(\Omega)} < \varepsilon. \tag{2.1.13}$$

From Lemma A.0.1, we can choose a function $\widetilde{u} \in BV_K(\mathbb{R}^N)$ such that $\|\widetilde{u}\|_{K(\mathbb{R}^N)} \leq C\,\|u\|_{K(\Omega)}$ for some constant $C > 0$. Thus, for any cube $Q \subset \mathbb{R}^N$ containing Ω, we have

$$\|\widetilde{u}\|_{K(Q)} \leq \|\widetilde{u}\|_{K(\mathbb{R}^N)} \leq C\,\|u\|_{K(\Omega)}.$$

Now, for any $\varepsilon \in (0, 1)$, we set

$$C_0 := \sup_{u \in \mathcal{A}}\left(\|\widetilde{u}\|_{L^1(\mathbb{R}^N)} + [\widetilde{u}]_{K(\mathbb{R}^N)}\right) \leq \sup_{u \in \mathcal{A}}\left(\|u\|_{L^1(\Omega)} + [u]_{K(\Omega)}\right) < \infty, \quad \rho := \varepsilon^{\frac{1}{t}}$$

where $t > 0$ is as in the assumption (A4) of Lemma A.0.1, and we can choose a collection of disjoint cubes $\{Q_i\}_{i=1}^{\tilde{M}}$ of side $\rho > 0$ such that

$$\Omega \subset \bigcup_{i=1}^{\tilde{M}} Q_i = Q.$$

For any $x \in \Omega$, we can choose the unique integer $i(x) \in \mathbb{N}$ such that $x \in Q_{i(x)}$ (#). Thus, we define the function $P : \mathcal{A} \to L^1(\Omega)$ as

$$P(u)(x) := \frac{1}{|Q_{i(x)}|}\int_{Q_{i(x)}} \widetilde{u}(y)\,dy$$

for any $u \in \mathcal{A}$ and $x \in \Omega$. Notice that P is additive and constant in any Q_i for $i \in \{1, \cdots, \tilde{M}\}$. We denote this constant by $q_i(u)$. Hence, we can define the vector function $R : \mathcal{A} \to \mathbb{R}^{\tilde{M}}$ as

$$R(u) := \rho^N(q_1(u), \cdots, q_{\tilde{M}}(u)) \in \mathbb{R}^{\tilde{M}}$$

for any $iu \in \mathcal{A}$, and it is easy to observe that the range $R(\mathcal{A}) \subset \mathbb{R}^{\tilde{M}}$ is totally bounded with respect to the ℓ^1-norm. Therefore there exist $L \in \mathbb{N}$ and $b_1, \cdots, b_L \in \mathbb{R}^{\tilde{M}}$ such that

$$R(\mathcal{A}) \subset \bigcup_{l=1}^{L} B_{\varepsilon/2}(b_l),$$

where the ball $B_{\varepsilon/2}$ are taken in the ℓ^1-norm of $\mathbb{R}^{\tilde{M}}$. Now, denoting the k-th coordinate of the vector $b_l \in \mathbb{R}^{\tilde{M}}$ by b_l^k for $k = 1, \cdots, \tilde{M}$ and $l = 1, \cdots, L$, we define functions $\{v_j\}_{j=1}^{M} \subset L^1(\mathbb{R}^N)$ as $v_j(x) \coloneqq \rho^{-N} b_j^{i(x)}$ for any $x \in \Omega$ where $i(x)$ is given in (#). By definition, we have that, if $x \in Q_i$, then

$$P(v_j)(x) = \frac{1}{|Q_{i(x)}|} \int_{Q_{i(x)}} \varepsilon^{-N} b_j^{i(x)}\, dy = \varepsilon^{-N} b_j^{i(x)} = v_j(x)$$

and we obtain $q_i(v_j) = \rho^{-N} b_j^i$, which implies $R(v_j) = b_j$.

Furthermore, recalling the assumptions (A1) and (A4) on K in Lemma A.0.1, we have that

$$\begin{aligned}
\|u - P(u)\|_{L^1(\Omega)} &= \sum_{i=1}^{\tilde{M}} \int_{Q_i \cap \Omega} |u(x) - P(u)(x)|\, dx \\
&\le \sum_{i=1}^{\tilde{M}} \frac{1}{\rho^N} \int_{Q_i \cap \Omega} \int_{Q_i} |u(x) - \widetilde{u}(y)|\, dy\, dx \\
&\le \sum_{i=1}^{\tilde{M}} \frac{1}{\rho^N\, k(\sqrt{2}\rho)} \int_{Q_i \cap \Omega} \int_{Q_i} K(|x-y|)\, |u(x) - \widetilde{u}(y)|\, dy\, dx \\
&\le \frac{1}{\rho^N\, k(\sqrt{2}\rho)} \int_{Q} \int_{Q} K(|x-y|)\, |\widetilde{u}(x) - \widetilde{u}(y)|\, dy\, dx \\
&\le C\, \rho^t\, [\widetilde{u}]_{K(Q)} \le C\, C_0\, \varepsilon
\end{aligned} \tag{2.1.14}$$

for any $u \in \mathcal{A}$ where $C > 0$ is a constant independent of u and ε.

As a consequence, from the definition of P and (2.1.14), we have

$$\begin{aligned}
\|u - v_j\|_{L^1(\Omega)} &\le \|u - P(u)\|_{L^1(\Omega)} + \|P(v_j) - v_j\|_{L^1(\Omega)} + \|P(u - v_j)\|_{L^1(\Omega)} \\
&\le C\, C_0\, \varepsilon + |R(u) - R(v_j)|_{\ell^1(\mathbb{R}^{\tilde{M}})}
\end{aligned}$$

for any $j \in \{1, \cdots, L\}$ where $|a|_{\ell^1(\mathbb{R}^{\tilde{M}})} \coloneqq \sum_{i=1}^{\tilde{M}} |a_i|$ for any $a \in \mathbb{R}^{\tilde{M}}$. Now given $u \in \mathcal{A}$, we can choose a number $j \in \{1, \cdots, L\}$ such that $|R(u) - b_j|_{\ell^1(\mathbb{R}^{\tilde{M}})} < \varepsilon$. Then, recalling that $R(v_j) = b_j$ for a number $j \in \{1, \cdots, L\}$, we obtain

$$\|u - v_j\|_{L^1(\Omega)} \le (C\, C_0 + 1)\varepsilon.$$

This proves the totally boundedness of $\mathcal{A}$, as desired. □

Now we prove the compactness of sets finite nonlocal bounded variations, using the compactness of the space BV_K.

Theorem 2.1.8 (Compactness for P_K). *Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of sets satisfying*

$$\sup_{n\in\mathbb{N}} P_K(E_n) < \infty$$

for every open bounded set $\Omega \subset \mathbb{R}^N$ with a smooth boundary. Then, there exist a subsequence $\{E_{n_i}\}_{i\in\mathbb{N}}$ and a set $E_\infty \subset \mathbb{R}^N$ such that

$$\chi_{E_{n_i}} \xrightarrow[i\to\infty]{} \chi_{E_\infty} \quad \text{in } L^1_{loc}, \quad P_K(E_\infty) < \infty. \tag{2.1.15}$$

Proof. We take any bounded open set $\Omega \subset \mathbb{R}^N$ with a smooth boundary. From the assumption of $\{E_n\}_n$ and the definition of $[\cdot]_K(\Omega)$, we obtain the uniform boundedness saying that

$$\sup_n \left([\chi_{E_n}]_K(\Omega) + \|\chi_{E_n}\|_{L^1(\Omega)}\right) \leq \sup_n \left(P_K(E_n) + |E_n \cap \Omega|\right) < \infty. \tag{2.1.16}$$

This implies that the sequence $\{\chi_{E_n}\}_n$ is uniformly bounded in $BV_K(\Omega)$. Thus, by Theorem 2.1.7, we can extract a subsequence of $\{E_n\}$ and choose a set E_∞ such that

$$\chi_{E_{n_i}} \xrightarrow[i\to\infty]{} \chi_{E_\infty} \quad \text{in } L^1(\Omega),$$

which proves the first part of the claim. From the lower semicontinuity of P_K, we further obtain

$$P_K(E_\infty) \leq \liminf_{i\to\infty} P_K(E_{n_i}) < \infty.$$

This completes the proof. □

2.2 s-Fractional Minimal Sets

In the sequel, we further investigate the nonlocal perimeter associated only with the kernel $K(x) = |x|^{-(N+s)}$ with $s \in (0, 1)$. We begin with the definition of s-fractional minimal sets. As an analogy of the classical theory of finite perimeter, one can also consider the sets that minimize the nonlocal perimeter P_K in the following way.

Definition 2.2.1 (**s-Fractional Minimal Sets).** Let $\Omega \subset \mathbb{R}^N$ be an open set. A measurable set $E \subset \mathbb{R}^N$ is called an *s-fractional minimal set* in Ω if

$$P_s(E;\Omega') \leq P_s(F;\Omega')$$

for any bounded open set $\Omega' \subset \Omega$ and any set $F \subset \mathbb{R}^N$ with $E \setminus \Omega' = F \setminus \Omega'$.

Taking into account the above definition, we can define the so-called *s-fractional minimal surface* as follows (see also [22]):

Definition 2.2.2 (**s-Fractional Minimal Surfaces).** Let $\Omega \subset \mathbb{R}^N$ be an open set. If a measurable set $E \subset \mathbb{R}^N$ is an s-fractional minimal set in Ω, then the boundary ∂E is called the s-fractional minimal surface in Ω.

When one considers the existence of minimizers of some functional, a sort of continuity of the functional with respect to a proper topology is necessary. The lower semicontinuity is one reasonable continuity to obtain the existence.

Proposition 2.2.3 ([22])**.** *If $\chi_{E_i} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^N)$ as $i \to \infty$, then*

$$P_s(E;\Omega) \leq \liminf_{i\to\infty} P_s(E_i;\Omega). \tag{2.2.1}$$

The proof of the proposition follows simply from the definition of (2.1.4) and Fatou's lemma.

We now review the existence of a s-fractional minimal set in a bounded set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary. This is shown by L. Caffarelli, J.M. Roquejoffre, and O. Savin in [22, Theorem 3.2] in the following.

Theorem 2.2.4 ([22])**.** *Let Ω be a bounded open set with Lipschitz boundary and $E_0 \subset \Omega^c$ be a given set. Then there exists an s-fractional minimal set E with $E \setminus \Omega = E_0$, namely, it holds*

$$P_s(E;\Omega) = \inf_{F\setminus\Omega=E_0} P_s(F;\Omega).$$

Proof. The infimum is bounded from below since $P_s(E_0;\Omega) < \infty$. Let $\{F_n\}_{n\in\mathbb{N}}$ be any minimizing sequence such that $P_s(F_n) \to \inf_{F\setminus\Omega=E_0} P_s(F;\Omega)$ as $n\to\infty$. Then one can show the boundedness of $\{\chi_{F_n\cap\Omega}\}_{n\in\mathbb{N}}$ with respect to the fractional Sobolev norm $\|\cdot\|_{W^{s/2,2}}$. From the compactness of $W^{s/2,2}$ (see [46, Theorem 7.1]), one can extract a subsequence of $\{\chi_{F_n\cap\Omega}\}_n$ converging to $\chi_{E\cap\Omega}$ for some $E\subset\mathbb{R}^N$. Therefore, the claim follows from the lower semicontinuity of s-fractional perimeter as in Proposition 2.2.3. □

2.3 Euler-Lagrange Equation

The authors in [22] also established the Euler-Lagrange equation in the viscosity sense associated to the s-fractional perimeter P_s in an open set $\Omega\subset\mathbb{R}^N$.

Theorem 2.3.1 ([22])**.** *Let $E\subset\mathbb{R}^N$ be a s-fractional minimal set in an open set $\Omega\subset\mathbb{R}^N$. If $x\in\partial E\cap\Omega$ and $E\cap\Omega$ has an interior tangential ball at x, then*

$$H^s_E(x) := P.V.\int_{\mathbb{R}^N}\frac{\chi_{E^c}(y)-\chi_E(y)}{|y-x|^{N+s}}\,dy \geq 0. \tag{2.3.1}$$

where "P.V." means the Cauchy principle value.

We remark that, in Theorem 2.3.1, if $E\cap\Omega$ has an exterior tangential ball at x, then we have

$$H^s_E(x)\leq 0.$$

Indeed, if $E\subset\mathbb{R}^N$ is a s-fractional minimal set in Ω, then E^c is also a s-fractional minimal set in Ω.

Taking into account the above, we obtain that, if $E\cap\Omega$ has both interior and exterior tangential balls at x, then the so-called *fractional minimal surface equation* $H^s_E(x) = 0$ holds.

Remark 2.3.2. From a geometrical point of view, the equation (2.3.1) is a nonlocal version of the classical mean curvature equation, which is given as $H_E = 0$ on ∂E of a set E where H_E is the mean curvature on ∂E. The reason we can call the quantity on the right-hand side in (2.3.1) is as follows; when we compute the first variation of the s-fractional perimeter of a smooth set E, we obtain

$$\frac{d}{dt}\lfloor_{t=0}P_s(\Phi_t(E)) = -\int_{\partial E}\left(\int_{\mathbb{R}^N}\frac{\chi_{E^c}(y)-\chi_E(y)}{|y-x|^{N+s}}\,dy\right)(X(x)\cdot\nu(x))\,d\mathcal{H}^{N-1}(x)$$

where $\{\Phi_t\}_{|t|\ll 1}$ is the flow along a smooth vector field X and ν is the unit normal vector on ∂E. Thus, the quantity H^s_E defined in (2.3.1) can be regarded as the "s-fractional mean curvature". We can say that, intuitively, the quantity defined in (2.3.1) measures the balance between the contributions of a set E and its complement E^c from its boundary ∂E with a weight function singular at each point of ∂E.

Moreover, it may be interesting to notice that the equation (2.3.1) can be interpreted as the s-fractional Laplace equation $(-\Delta)^{s/2}(\chi_E-\chi_{E^c}) = 0$ on ∂E.

Remark 2.3.3. If ∂E is of class $C^{1,\alpha}$ with $\alpha > s$, then the s-fractional mean curvature defined in (2.3.1) is well-defined at every point of ∂E (see [1, 37]). Recall that the classical mean curvature is well-defined at every point of a surface if the surface is of class C^2.

Remark 2.3.4. The definition of the s-fractional mean curvature may not look natural at a first glance. To make it clearer in another way, one can observe that the classical mean curvature of the boundary of a smooth set $E \subset \mathbb{R}^N$ at a point $x \in \partial E$ can be derived via the averaging procedure

$$\lim_{r \downarrow 0} \frac{1}{r^{N+1}} \int_{B_r(x)} (\chi_{E^c}(y) - \chi_E(y)) \, dy. \tag{2.3.2}$$

Indeed, we assume that x is the origin and the neighborhood of x can be described as the graph of some smooth function $u : \mathbb{R}^{N-1} \to \mathbb{R}$ with $u(0) = 0$ and $\nabla' u(0) = 0$. In this setting, E can be expressed as the super-graph of u in the neighborhood of x. Then we have that

$$\begin{aligned}
\lim_{r \downarrow 0} \frac{1}{r^{N+1}} \int_{B_r(x)} (\chi_{E^c}(y) - \chi_E(y)) \, dy &= \lim_{r \downarrow 0} \frac{1}{r^{N+1}} \int_{\{|y'|<r,\, |y_N|<r\}} (\chi_{E^c}(y) - \chi_E(y)) \, dy \\
&= \lim_{r \downarrow 0} \frac{2}{r^{N+1}} \int_{\{|y'|<r\}} u(y') \, dy' \\
&= \lim_{r \downarrow 0} \frac{1}{r^{N+1}} \int_{\{|y'|<r\}} \left(\nabla^2 u(0)\, y' \cdot y' + o(|y'|^2)\right) \, dy' \\
&= c \, \Delta u(0)
\end{aligned}$$

for some constant $c > 0$. This argument is also done in [54].

Remark 2.3.5. One may observe that a local minimizer $E \subset \mathbb{R}^N$ with a C^2-boundary in a domain $\Omega \subset \mathbb{R}^N$ possesses zero s-fractional mean curvature, in the sense that

$$\int_{\mathbb{R}^N} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y-q|^{N+s}} = 0 \quad \text{for any } q \in \partial E \cap \Omega. \tag{2.3.3}$$

As we see in the previous remark, the s-fractional mean curvature H^s_E given in (2.3.1) is defined in the Cauchy principle value, and the smoothness of E is exploited to employ cancellations for removing singularities. Without knowing any regularity of a set E, the Euler-Lagrange equation (2.3.3) is not understood in the pointwise sense, but must be in the viscosity sense. As is shown in Theorem 2.3.1, the notion of the viscosity sense is described by using the interior or exterior ball tangential to the boundary ∂E. This is an interesting geometric information. In particular, we notice that the simple fact that a ball is contained in E (respectively, in E^c) makes the quantity $\chi_{E^c} - \chi_E$ accordingly small (respectively, large), regardless of the minimality of the set. The useful information encoded in the above inequalities is that for local minimizers one is also provided with a partial knowledge with respect to "the opposite sign": specifically, in this case the fact that a ball is contained contained in E (respectively, in E^c) makes the quantity $\chi_{E^c} - \chi_E$ accordingly large (respectively, small) after averaging with respect to the singular kernel. We observe that for smooth sets one can obtain this Euler-Lagrange equation (or the corresponding Euler-Lagrange inequalities) simply by looking at energy perturbations under domain variations (see for instance [54]), but without assuming any smoothness on the set suitable cancellations need to be detected.

The proof of Theorem 2.3.1 is quite complicated, and we only explain some ideas of the proof in the following. Since one has an interior tangential ball at $x_0 \in \partial E \cap \Omega$, one may construct a proper "perturbation" set A outside E in a small neighborhood of $x_0 \in \partial E \cap \Omega$. From the minimality of E, one can compare the energies $P_s(E;\Omega)$ and $P_s(E \cup A;\Omega)$. Then, considering the cancellation of the contribution between the sets E and $E^c \cap A^c$, one obtains

$$\int_A \int_E \frac{1}{|y-x|^{N+s}} \, dy \, dx \le \int_A \int_{E^c \cap A^c} \frac{1}{|y-x|^{N+s}} \, dy \, dx,$$

and thus

$$\frac{1}{|A|}\int_A \left(\int_{\mathbb{R}^N} \frac{\chi_E(y)-\chi_{E^c}(y)}{|y-x|^{N+s}}\,dy\right) dx \le 0. \tag{2.3.4}$$

Choosing the "perturbation" set A in such a way that $|A| \downarrow 0$ and $x_0 \in A$, one may formally obtain, from (2.3.4), that $H^s_E(x_0) \le 0$.

The rigorous proof of Theorem 2.3.1 was conducted more carefully by L. Caffarelli, J.M. Roquejoffre, and O. Savin [22] because of the singularity of the s-fractional mean curvature H^s_E at $x_0 \in \partial E \cap \Omega$ and, in Theorem 5.1 of [22], the authors constructed a delicate perturbation set A by using the interior tangential ball.

2.4 Regularity of (Almost) s-Fractional Minimal Sets

In this section, we review several results on the regularity of both s-fractional minimal sets, and "almost" s-fractional minimal sets. Originally, the regularity of s-fractional minimal sets was obtained by L. Caffarelli, J.M. Roquejoffre, and O. Savin [22, Theorem 2.4]. Precisely they proved

Theorem 2.4.1 ([22])**.** *Let $s \in (0,\,1)$. If $E \subset \mathbb{R}^N$ is a minimizer of s-fractional perimeter P_s in a ball B_1, then $\partial E \cap B_{1/2}$ is , to the possible exception of a closed set of Hausdorff dimension $N-2$, a $C^{1,\alpha}$-hypersurface around each of its points for any $\alpha \in (0,\,s)$.*

Regarding the closed set of Hausdorff dimension $N-2$, O. Savin and E. Valdinoci [106] proved that the singular set of s-fractional minimal sets has Hausdorff dimension at most $N-3$.

Theorem 2.4.2 ([106])**.** *Assume that $E \subset \mathbb{R}^2$ is a s-fractional minimal cone, namely, E satisfies that $E = t\,E$ for any $t > 0$. Then E is a half-plane.*

In particular, by combining the blow-up and blow-down arguments in [22], one may obtain that s-fractional minimal surfaces in $\mathbb{R}^2$ are fully $C^{1,\alpha}$-regular for any $\alpha \in (0,\,s)$.

Corollary 2.4.3 ([106])**.** *If E is an s-fractional minimal set in $\Omega \subset \mathbb{R}^2$, then $\partial E \cap \Omega'$ is a $C^{1,\alpha}$-curve for any $\Omega' \Subset \Omega$.*

Finally in higher dimensions, by using the dimension reduction argument performed in [22], one may obtain the following corollary:

Corollary 2.4.4 ([106])**.** *Let ∂E be a s-fractional minimal surface in $\Omega \subset \mathbb{R}^N$ and let $\Sigma_E \subset \partial E \cap \Omega$ denote its singular set. Then $\mathcal{H}^d(\Sigma_E) = 0$ for any $d > N-3$.*

Now we study the regularity of "almost" s-fractional minimal sets. To see this, we recall the following two results: one is the C^1-regularity result of "almost" s-fractional minimal sets shown by M.C. Caputo and N. Guillen [26], and the other is the $C^{1,\alpha}$-regularity result shown by A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini [56]. Although there seems to be a conflict between the two results, the definition of "almost" s-fractional minimal sets given by M.C. Caputo and N. Guillen [26] is in a more general form than the one by A. Figalli, et al. [56]. To see the difference of the definitions, we will refer to the two statements (Theorem 2.4.8 and 2.4.9) in this section. The regularity of almost s-fractional minimal sets will be applied to show our main results in Chapter 4 and 5 of this thesis.

First of all, we give the definition of "almost" s-fractional minimal sets. After giving the definition, we state the results on the regularity of "almost" s-fractional minimal sets investigated in [26] and [56].

Definition 2.4.5 (Almost s-Fractional Minimal Sets). Let $s \in (0, 1)$ and $\Lambda > 0$. We say that a set $E \subset \mathbb{R}^N$ is an almost s-fractional minimal set with $\Lambda > 0$ if

$$P_s(E; B_R) \leq P_s(F; B_R) + \frac{\Lambda}{1-s} |E \Delta F|$$

for any set $F \subset \mathbb{R}^N$.

This definition is due to [56] and, in our thesis, we adopt Definition 2.4.5 in Chapter 4 and 5. we also review the definition and regularity of almost s-fractional minimal sets shown by M.C. Caputo and N. Guillen [26, Theorem 1.1]. The definition of almost s-fractional minimal sets given in [26] is as follows:

Definition 2.4.6 ([26])**.** Let $s \in (0, 1)$ and $\delta > 0$, and let $\Omega \subset \mathbb{R}^N$ be an bounded domain with Lipschitz boundary. Assume that $\rho : (0, \delta) \to \mathbb{R}$ is a non-decreasing and bounded function with some growth condition (for instance, $\rho(t) = t^\alpha$ with $\alpha \in (0, s]$). We say that a set $E \subset \mathbb{R}^N$ is a $(P_s, \rho, \delta,)$-minimal set in Ω if

$$P_s(E; B_R) \leq P_s(F; B_R) + \rho(r)\, r^{N-s},$$

for any $x_0 \in \partial E$, a set $F \subset \mathbb{R}^N$, and $0 < r < \min\{\delta, \operatorname{dist}(x_0, \partial\Omega)\}$ with $E \triangle F \subset B_r(x_0)$.

Remark 2.4.7. One may choose, for instance, a function $r \mapsto C\, r^\beta$ with $0 < \beta \leq s$ for some constant $C > 0$ as the function ρ in Theorem 2.4.8. Moreover, one can see that an almost s-fractional minimal set E in Definition 2.4.5 satisfies Definition 2.4.6 with $\rho(r) = \frac{\Lambda}{1-s} r^s$.

Now we state the C^1-regularity of almost s-fractional minimal sets in the sense of Definition 2.4.6 as follows:

Theorem 2.4.8 ([26])**.** *Let $s \in (0, 1)$, $\delta > 0$, and ρ be as in Definition 2.4.6. Suppose that a set $E \subset \mathbb{R}^N$ is a (P_s, ρ, δ)-minimal set in B_1 in the sense of Definition 2.4.6. Then ∂E is of class C^1 in $B_{\frac{1}{2}}$, except a closed set of Hausdorff dimension $N - 2$.*

Compared with Theorem 2.4.8, if a set is an almost s-fractional minimal set in the sense of Definition 2.4.5, then one may obtain the $C^{1,\alpha}$-regularity of the set. This result was due to [56, Theorem 3.4]. Precisely, we have

Theorem 2.4.9 ([56])**.** *Let $N \geq 2$, $\Lambda \geq 0$, and $s_0 \in (0, 1)$. Assume that E is an almost s-fractional minimal set in B_1 in the sense of Definition 2.4.5 for some $s \in (s_0, 1)$. Then $\partial E \cap B_{\frac{1}{2}}$ is of class $C^{1,\alpha}$ except a closed set of Hausdorff dimension $N - 2$.*

As we mentioned, the definition of almost s-fractional minimal sets given by M.C. Caputo and N. Guillen [26] is more general than the one given by A. Figalli, et al. [56].

Remark 2.4.10. From the regularity of the minimal cone by O. Savin and E. Valdinoci [106] in 2 dimension (see Theorem 2.4.2), one may obtain that the singular set of an almost s-fractional minimal set in the sense of either Definition 2.4.6 or 2.4.5 has Hausdorff dimension at most $N - 3$.

Finally, in this section, we briefly mention the density estimate for the almost s-fractional minimal sets. Thanks to [56, Lemma 3.1], we have the following claim:

Lemma 2.4.11 (Density Estimates of Almost s-Fractional Minimal Sets). *Let $s \in (0, 1)$, $\Lambda > 0$, and E be an almost s-fractional minimal sets with Λ in the sense of Definition 2.4.5. Then we have*

$$|B_1|(1 - c_0)\, r^N \geq |E \cap B_r(x_0)| \geq |B_1| c_0\, r^N$$

for any $r \in (0, r_0)$ and $x_0 \in \mathbb{R}^N$ such that $|E \cap B_r(x_0)| > 0$ and $|E^c \cap B_r(x_0)| > 0$ for any $r > 0$, where c_0 and r_0 are positive constants depending only on N, s, and Λ.

The proof is shown in [56] and we do not write it here.

Chapter 3

Topology of Nonlocal Minimal Sets

In this chapter, we investigate several qualitative properties of s-fractional minimal sets. Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^N$ be an open subset with a Lipschitz boundary. As is shown in Definition 2.1.2 of Chapter 2, a localized version of the s-fractional perimeter $P_s(E\,;\,\Omega)$ in Ω for a set $E \subset \mathbb{R}^N$ is defined as

$$P_s(E\,;\,\Omega) := \int_{E\cap\Omega}\int_{E^c} \frac{dx\,dy}{|x-y|^{N+s}} + \int_{E\cap\Omega^c}\int_{\Omega\cap E^c} \frac{dx\,dy}{|x-y|^{N+s}},$$

where we denote by E^c the complement of a set E. Then we give the notion of *nonlocal minimal surface* in unbounded open set $\Omega \subset \mathbb{R}^N$ in the following way:

Definition 3.0.1 (**s-Fractional Minimal Surfaces in Unbounded Domain).** Let $\Omega \subset \mathbb{R}^N$ be an unbounded open set and let $E \subset \mathbb{R}^N$. Then we say that the topological boundary of E is a nonlocal(s-fractional) minimal surface in Ω if E is an s-fractional minimal set in Ω, namely, it holds that

$$P_s(E;\Omega') \le P_s(F;\Omega')$$

for any bounded and open set Ω' contained in Ω and any set $F \subset \mathbb{R}^N$ with $F \setminus \Omega' = E \setminus \Omega'$.

We remark that the definition of the s-fractional minimal sets in Definition 2.2.1 of Chapter 2 is slightly different because of the existence of the minimizers are not trivial in unbounded domain (see [81]). Thus we slightly abuse the language of the notion of s-fractional minimal sets in this chapter. See [81, 82] for the details regarding the minimization procedure in bounded or unbounded domains.

3.1 Problem Setting and Main Results

We focus on the case that the reference domain Ω is given as the "cylindrical" domain of the form

$$\Omega_c := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid |x'| < 1\}. \tag{3.1.1}$$

We are interested in sets E whose exterior prescription outside Ω_c is the complement of a strip and which minimise the s-fractional perimeter in Ω_c. Namely, given $M > 0$, we define

$$E_0 := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid |x_N| > M\} \tag{3.1.2}$$

and we consider s-fractional minimal sets in Ω_c such that $E \setminus \Omega_c = E_0 \setminus \Omega_c$. See, for instance, [82, Theorem 0.2.5] for the existence results for this type of s-fractional minimal sets.

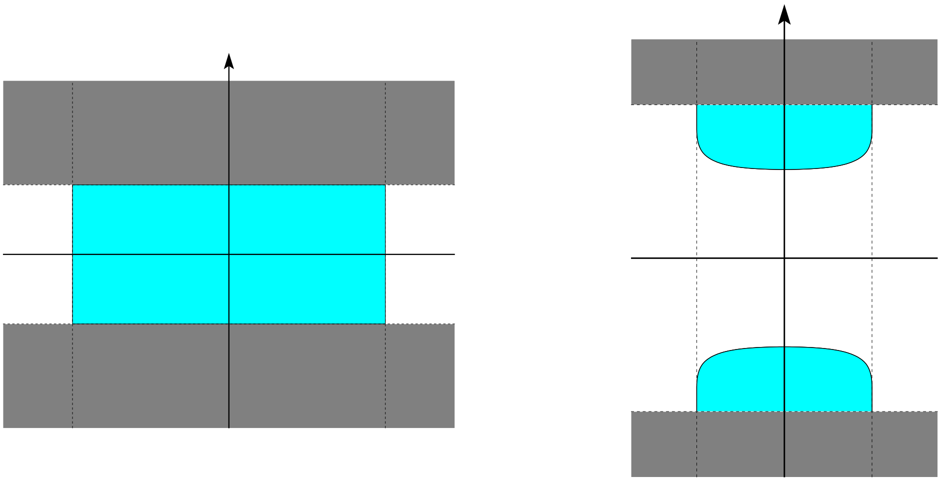

Figure 3.1: Minimizers in Theorem 3.1.1 (left) and Theorem 3.1.2 (right).

Our main concern in this chapter is how the variation of the parameter M affects the topological property of the s-fractional minimal surfaces and we will show that *for small values of M the s-minimizer is connected* while *for large values it is disconnected.*

Furthermore, we will show that *for small values of M the s-fractional minimal sets in Ω_c coincides with Ω_c itself,* and this is an interesting difference with respect to the case of classical minimal sets, which means the sets minimising the classical perimeter. Indeed, when $N \geq 3$, classical minimal sets in a cylinder do not coincide with the cylinder itself and, when connected, they develop a "neck" inside the cylinder, as exhibited by the classical example of the catenoid (as a matter of fact, when $N \geq 3$ the cylinder does not have vanishing mean curvature, hence it cannot be a minimizer for the classical perimeter functional).

Therefore, our construction of s-fractional minimal sets that coincide with the cylinder in their free domain heavily relies on the nonlocal character of the problem taken into consideration and can be seen as a new example of the *stickiness theory for nonlocal minimal surfaces* which was introduced in [51] and developed in [50, 19, 52, 53]. See also [54, 47] for surveys on s-fractional minimal surfaces (sets) discussing, among other topics, the stickiness phenomenon (and, for instance [68] to appreciate the structural differences with respect to the classical case).

In further detail, the precise result that we have concerning the connectedness of the s-minimizer and its stickiness properties for small values of M goes as follows:

Theorem 3.1.1. *Let Ω_c be as in* (3.1.1) *and let E_0 be defined by* (3.1.2)*. Then, there exists $M_0 \in (0, 1)$, depending only on n and s, such that, for any $M \in (0, M_0)$, the minimizer E_M in Ω_c of P_s coincides with Ω_c. In particular, E_M is connected.*

The minimizer described in Theorem 3.1.1 is depicted in Figure 3.1. As a counterpart of Theorem 3.1.1, the disconnectedness result for large values of M is the following:

Theorem 3.1.2. *Let Ω_c be as in* (3.1.1) *and let E_0 be defined by* (3.1.2)*. Then, there exists $M_0 > 1$, depending only on n and s, such that, for any $M > M_0$, the minimizer E_M in Ω_c of P_s is disconnected.*

To understand the situation intuitively, some sketch on how the minimizer in Theorem 3.1.2 could look like is given in Figure 3.1.

Interestingly, the situation described in Theorem 3.1.2 is similar, but structurally different from the one exhibited by classical minimal surfaces. Indeed, the analogy with the

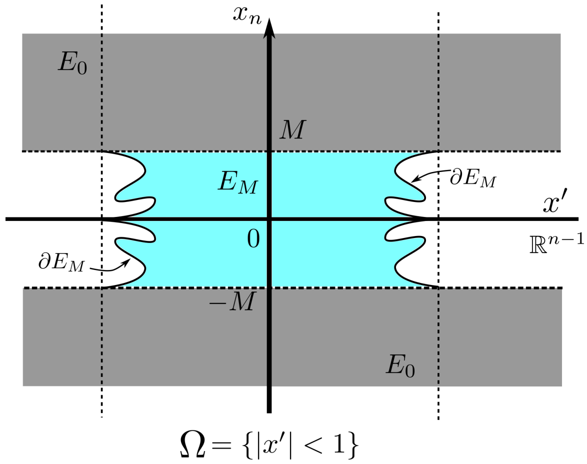

Figure 3.2: The situation in the proof of Theorem 3.1.1.

classical case is given by the disconnectedness of the minimizers. The difference in the pattern is that classical minimal surfaces in the framework of Theorem 3.1.2 are just flat disc, and this is not the case for their corresponding nonlocal counterpart (as we will make precise in Proposition 3.3.1).

The forthcoming Sections 3.2.1 and 3.2.2 contain the proofs of Theorems 3.1.1 and 3.1.2 respectively. In Section 3.3 we will present further similarities and differences with respect to the classical case in the framework of large M given by Theorem 3.1.2.

3.2 (Dis)connectedness of Nonlocal Minimal Surfaces

In this section, we give the proof of our main theorems (see Theorem 3.1.1 and Theorem 3.1.2 for the statements), which is related to the topology of s-fractional minimal sets. Precisely, we prove the following two things: if the width M of the slab is sufficiently large, then the the s-fractional minimal sets in $\Omega \subset \mathbb{R}^N$ are disconnected. On the other hand, if M is sufficiently small, then the s-fractional minimal sets in Ω are connected, and coincide with the reference set Ω itself.

3.2.1 Proof of Theorem 3.1.1

Let E_M be the minimizer selected in Theorem 3.1.1, see Figure 3.2 (at this stage of the proof, we do not really know how this minimizer looks like, so the one depicted in Figure 3.2 will not be the "real" minimizer after all).

By [22, Corollary 5.3], we know that

$$\{x_N > M\} \cup \{x_N < -M\} \subset E_M. \tag{3.2.1}$$

Given $t \in \mathbb{R}$ and $r \in (0,1)$, we consider the ball of radius r with a center te_N, where $e_N = (0,\dots,0,1)$. By (3.2.1), we have that $B_r(te_N) \subset E_M$ for every $t > M+1$. Hence, we can slide such a ball downwards till it touches ∂E_M inside Ω_c. The content of Theorem 3.1.1 is precisely that this touching does not occur, hence, by contradiction, we suppose instead that there exist $t_0 \in \mathbb{R}$ and $r_0 \in (0,1)$ such that

$$B_{r_0}(te_N) \subset E_M \quad \text{for all } t > t_0 \tag{3.2.2}$$

with

$$\partial B_{r_0}(t_0 e_N) \cap \partial E_M \neq \varnothing.$$

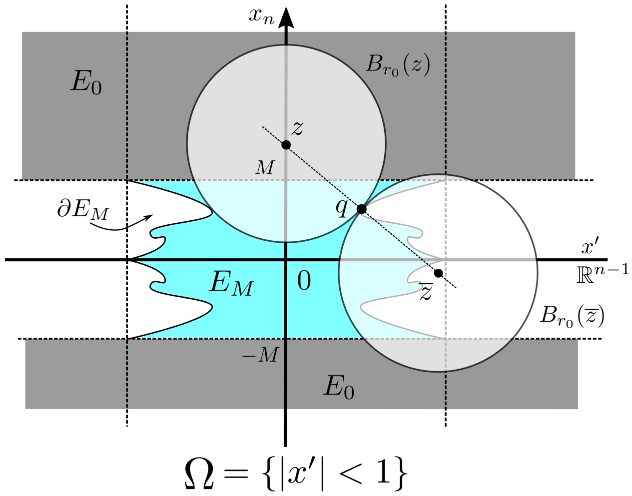

Figure 3.3: The touching between the ball $B_{r_0}(z)$ and the symmetric ball $B_{r_0}(\overline{z})$ at the point q.

Then, setting $z := t_0 e_N$, we can choose a point $q = (q', q_N) \in \partial B_{r_0}(z) \cap \partial E_M$.

Since E_M is a local minimizer of P_s in Ω_c, we obtain, by using the Euler-Lagrange equation in the viscosity sense shown in Theorem 2.3.1 of Chapter 2 (see also [22, Theorem 5.1] and [19, Theorem B.9]), that

$$\int_{\mathbb{R}^N} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\, dy \geq 0. \tag{3.2.3}$$

Our goal is now to produce a contradiction with (3.2.3) by showing that the left hand side is strictly negative. To this end, we let

$$S_M := \mathbb{R}^{N-1} \times [q_N - 2M, q_N + 2M].$$

We remark that

$$E_M^c \subset S_M \setminus B_{r_0}(z). \tag{3.2.4}$$

Indeed, by (3.2.1) we know that $q_N \in [-M, M]$ and $E_M^c \subset \{x_N \in [-M, M]\}$, whence $E_M^c \subset S_M$. This and (3.2.2) give (3.2.4).

We also observe that $S_M \supset \{|x_N| \leq M\}$, and therefore, in light of (3.2.1),

$$S_M^c \subset E_M. \tag{3.2.5}$$

Moreover, using the change of variable $y \mapsto y + q$,

$$\begin{aligned} \int_{S_M^c} \frac{dy}{|y-q|^{N+s}} &= \int_{\mathbb{R}^{N-1}\times((-\infty,-2M)\cup(2M,\infty))} \frac{dy}{|y|^{N+s}} \\ &\geq \int_{B_M(3Me_n)} \frac{dy}{|y|^{N+s}} \geq cM^{-s}, \end{aligned} \tag{3.2.6}$$

for a constant $c > 0$ depending only on N.

Now we set $\overline{z} := z + 2(q - z)$ and we consider the symmetric ball $B_{r_0}(\overline{z})$ with respect to q, see Figure 3.3. Moreover, we take a free parameter $\Lambda \geq 4$, to be chosen conveniently large in what follows and we observe that, by symmetry,

$$\int_{S_M \cap B_{\Lambda M}(q) \cap B_{r_0}(z)} \frac{dy}{|y-q|^{N+s}} = \int_{S_M \cap B_{\Lambda M}(q) \cap B_{r_0}(\overline{z})} \frac{dy}{|y-q|^{N+s}}.$$

Also, by (3.2.4),

$$\int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{N+s}}\,dy = -\int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{dy}{|y-q|^{N+s}},$$

and consequently

$$\begin{aligned}
&\int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{N+s}}\,dy + \int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \\
&\le -\int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{dy}{|y-q|^{N+s}} + \int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{dy}{|y-q|^{N+s}} = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{S_M\cap B_{\Lambda M}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \\
&= \int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(z)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{N+s}}\,dy + \int_{S_M\cap B_{\Lambda M}(q)\cap B_{r_0}(\overline{z})} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \\
&\quad + \int_{S_M\cap\big(B_{\Lambda M}(q)\setminus\big(B_{r_0}(z)\cup B_{r_0}(\overline{z})\big)\big)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \\
&\le \int_{S_M\cap\big(B_{\Lambda M}(q)\setminus\big(B_{r_0}(z)\cup B_{r_0}(\overline{z})\big)\big)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \\
&\le \int_{B_{\Lambda M}(q)\setminus\big(B_{r_0}(z)\cup B_{r_0}(\overline{z})\big)} \frac{dy}{|y-q|^{N+s}} \\
&\le C\Lambda^{1-s}M^{1-s},
\end{aligned} \tag{3.2.7}$$

for some $C>0$ depending only on N and s, where [50, Lemma 3.1] has been used in the last inequality (here with $R:=1$ and $\lambda:=\Lambda M$).

Furthermore,

$$\begin{aligned}
\int_{S_M\setminus B_{\Lambda M}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{N+s}}\,dy &\le \int_{S_M\setminus B_{\Lambda M}(q)} \frac{dy}{|y-q|^{N+s}} \\
&= \int_{(\mathbb{R}^{N-1}\times[-2M,2M])\setminus B_{\Lambda M}} \frac{dy}{|y|^{N+s}} \\
&\le \int_{(\mathbb{R}^{N-1}\times[-2M,2M])\setminus B_{\Lambda M}} \frac{dy}{|y'|^{N+s}} \\
&\le \int_{\{|y'|\ge\Lambda M/2,\ |y_N|\le 2M\}} \frac{dy}{|y'|^{N+s}} \\
&= \frac{C_0}{\Lambda^{1+s}M^s},
\end{aligned}$$

for some $C_0>0$ depending only on N and s.

Hence, combining this information with (3.2.7),

$$\int_{S_M} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \le C\Lambda^{1-s}M^{1-s} + \frac{C_0}{\Lambda^{1+s}M^s}.$$

This, (3.2.5) and (3.2.6) lead to

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \\
&= -\int_{S_M^c} \frac{dy}{|y-q|^{N+s}} + \int_{S_M} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \\
&\le -cM^{-s} + C\Lambda^{1-s}M^{1-s} + \frac{C_0}{\Lambda^{1+s}M^s} \\
&= -cM^{-s}\left(1 - \frac{C\Lambda^{1-s}M}{c} - \frac{C_0}{c\Lambda^{1+s}}\right).
\end{aligned}$$

Now we choose $\Lambda := \max\left\{4, \left(\frac{2C_0}{c}\right)^{\frac{1}{1+s}}\right\}$ and we thus obtain that

$$\int_{\mathbb{R}^N} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \le -cM^{-s}\left(\frac{1}{2} - \frac{C\Lambda^{1-s}M}{c}\right).$$

Taking now M conveniently small, we conclude that

$$\int_{\mathbb{R}^N} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \le -\frac{cM^{-s}}{4} < 0,$$

which produces the desired contradiction with (3.2.3).

3.2.2 Proof of Theorem 3.1.2

We let $M > 1$ to be chosen conveniently large. Given $t \in \mathbb{R}$, we consider the ball $B_{\sqrt{M}}(te_1)$, where $e_1 = (1, 0, \dots, 0)$, and we slide it from left to right till it touches ∂E_M. Notice indeed that $B_{\sqrt{M}}(te_1) \subset E_0^c$ when $t < -\sqrt{M}$ and, to prove Theorem 3.1.2, we suppose by contradiction that there exists $t_0 \in \mathbb{R}$ such that $B_{\sqrt{M}}(te_1) \subset E_M^c$ for all $t < t_0$ with $\partial B_{\sqrt{M}}(t_0e_1) \cap \partial E_M \neq \varnothing$.

We set $z := t_0e_1$ and we pick a point $q = (q', q_N) \in \partial B_{\sqrt{M}}(z) \cap \partial E_M$. By the Euler-Lagrange equation in the viscosity sense in Theorem 2.3.1 of Chapter 2 (see also [22, Theorem 5.1] and [19, Theorem B.9]), we know that

$$\int_{\mathbb{R}^N} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \le 0. \tag{3.2.8}$$

We consider the symmetric ball with respect to q, by defining $\overline{z} := z + 2(q - z)$ and taking into account the ball $B_{\sqrt{M}}(\overline{z})$, see Figure 3.4.

We define

$$S := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \text{ s.t. } |x' - q'| \le 3\}.$$

By symmetry,

$$\int_{S \cap B_{\sqrt{M}}(z)} \frac{dy}{|y-q|^{N+s}} = \int_{S \cap B_{\sqrt{M}}(\overline{z})} \frac{dy}{|y-q|^{N+s}}$$

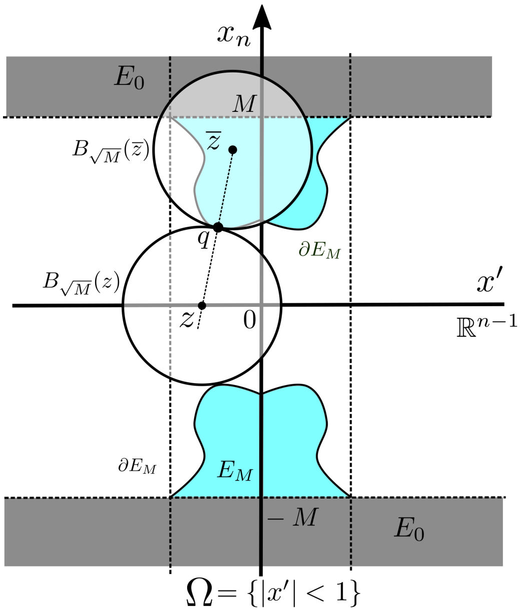

Figure 3.4: The touching between the ball $B_{\sqrt{M}}(z)$ and the symmetric ball $B_{\sqrt{M}}(\overline{z})$ at the point q.

and therefore

$$
\begin{aligned}
&\int_S \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \\
&= \int_{S\cap B_{\sqrt{M}}(z)} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\,dy + \int_{S\cap B_{\sqrt{M}}(\overline{z})} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \\
&\qquad + \int_{S\setminus(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z}))} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \\
&\geq \int_{S\cap B_{\sqrt{M}}(z)} \frac{dy}{|y-q|^{N+s}} - \int_{S\cap B_{\sqrt{M}}(\overline{z})} \frac{dy}{|y-q|^{N+s}} \\
&\qquad + \int_{S\setminus(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z}))} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\,dy \\
&\geq -\int_{S\setminus\left(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z})\right)} \frac{dy}{|y-q|^{N+s}}.
\end{aligned}
\tag{3.2.9}
$$

Now, in view of [50, Lemma 3.1], used here with $R := \sqrt{M}$ and $\lambda := 1/\sqrt[4]{M}$, we know that

$$
\int_{B_{\sqrt[4]{M}}(q)\setminus\left(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z})\right)} \frac{dy}{|y-q|^{N+s}} \leq CM^{-\frac{1+s}{4}},
$$

for some $C > 0$ depending only on N and s. As a result,

$$
\begin{aligned}
&\int_{S\setminus\left(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z})\right)} \frac{dy}{|y-q|^{N+s}} \\
&\leq \int_{B_{\sqrt[4]{M}}(q)\setminus\left(B_{\sqrt{M}}(z)\cup B_{\sqrt{M}}(\overline{z})\right)} \frac{dy}{|y-q|^{N+s}} + \int_{S\setminus B_{\sqrt[4]{M}}(q)} \frac{dy}{|y-q|^{N+s}} \\
&\leq CM^{-\frac{1+s}{4}} + \int_{\mathbb{R}^N\setminus B_{\sqrt[4]{M}}(q)} \frac{dy}{|y-q|^{N+s}}
\end{aligned}
$$

$$= CM^{-\frac{1+s}{4}} + C_1 M^{-\frac{s}{4}} \leq C_2 M^{-\frac{s}{4}}$$

for some $C_1 > 0$ depending only on N and s, with $C_2 := C + C_1$.

This and (3.2.9) lead to

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\, dy \\
= \quad & \int_S \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\, dy + \int_{S^c} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\, dy \\
\geq \quad & -C_2 M^{-\frac{s}{4}} + \int_{S^c} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\, dy \\
\geq \quad & -C_2 M^{-\frac{s}{4}} - \int_{S^c \cap \{|y_N| \geq M\}} \frac{dy}{|y-q|^{N+s}} + \int_{S^c \cap \{|y_N| < M\}} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\, dy \\
\geq \quad & -C_2 M^{-\frac{s}{4}} - \int_{\{|y-q| \geq M/2\}} \frac{dy}{|y-q|^{N+s}} + \int_{S^c \cap \{|y_N| < M\}} \frac{dy}{|y-q|^{N+s}} \\
= \quad & -C_2 M^{-\frac{s}{4}} - C_3 M^{-s} + \int_{S^c \cap \{|y_N| < M\}} \frac{dy}{|y-q|^{N+s}},
\end{aligned}$$

for some $C_3 > 0$ depending only on N and s.

Thus, since $S^c \cap \{|y_N| < M\} \supset B_1(q + 5e_1)$, letting $C_4 := C_2 + C_3$ we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{N+s}}\, dy &\geq -C_4 M^{-\frac{s}{4}} + \int_{B_1(q+5e_1)} \frac{dy}{|y-q|^{n+s}} \\
&= -C_4 M^{-\frac{s}{4}} + \int_{B_1(5e_1)} \frac{dy}{|y|^{N+s}} = -C_4 M^{-\frac{s}{4}} + c,
\end{aligned}$$

for some $c > 0$ depending only on N and s. In particular, if M is sufficiently large, we deduce that the left hand side of (3.2.8) is strictly positive, thus reaching a contradiction with (3.2.8).

3.3 Stickiness of Nonlocal Minimal Surfaces

Finally in this section, we would like to point out that, on the one hand, the result shown in Theorem 3.1.2 is related to the case of classical minimal surfaces, since both the classical and the nonlocal regimes exhibit disconnected minimizers for large width M of the slabs. On the other hand, there are some significant structural differences between the classical and s-fractional minimal surfaces (sets).

More precisely, differently from the classical minimal surfaces, the s-fractional minimal sets constructed in Theorem 3.1.2 exhibit the features listed below:

Proposition 3.3.1. *Let M and E_M be as in Theorem 3.1.2. Then,*

$$E_M \supsetneqq \{x_n > M\} \cup \{x_N < -M\}. \tag{3.3.1}$$

Moreover,

$$E_M \supset B_{cM^{-s}}(0, \dots, 0, -M) \cup B_{cM^{-s}}(0, \dots, 0, M), \tag{3.3.2}$$

for some $c > 0$ depending only on N and s.

In addition, if $N = 2$, given any $\epsilon_0 > 0$ there exists $c_\star > 0$, depending only on s and ϵ_0, such that

$$E_M \supset \big((-1,1) \times \big(-\infty, -M + c_\star M^{-\gamma}\big)\big) \cup \big((-1,1) \times \big(M - c_\star M^{-\gamma}, \infty\big)\big) \tag{3.3.3}$$

where $\gamma := \frac{(2+\epsilon_0)s}{1-s}$.

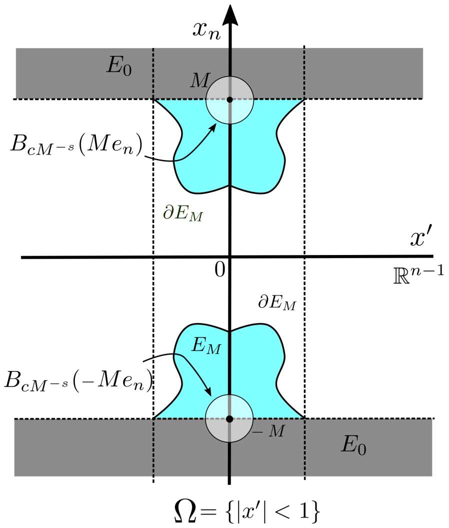

Figure 3.5: A sketch of an argument in Proposition 3.3.1.

Remark that (3.3.2) and (3.3.3) are quantitative versions of (3.3.1) and a sketch of an argument used in the proof of Proposition 3.3.1 is depicted in Figure 3.5. Though (3.3.2) and (3.3.3) provide a stronger result than (3.3.1), we give an independent proof of (3.3.1) based on a simple symmetry argument, while the proofs of (3.3.2) and (3.3.3) rely on finer quantitative arguments based on the result in [51, Corollary 7.2]. We also point out that (3.3.3) provides an explicit quantitative bound on the stickiness property in dimension 2.

Proof of Proposition 3.3.1. To prove (3.3.1), we need to show that the inclusion in (3.2.1) is strict. For this, we argue by contradiction and suppose that $E_M = \{x_n > M\} \cup \{x_n < -M\}$. Then we can use the Euler-Lagrange equation in the viscosity sense shown in Theorem 2.3.1 in Chapter 2 (see also [22, Theorem 5.1]) at the point $q := (0, \dots, 0, -M) \in \partial E_M$, thus finding that

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^N} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-q|^{n+s}}\, dy \\
&= \int_{\{|y_n|<M\}} \frac{dy}{|y-q|^{n+s}} - \int_{\{|y_n|\geq M\}} \frac{dy}{|y-q|^{n+s}} \\
&= \int_{\{z_n \in (0,2M)\}} \frac{dz}{|z|^{n+s}} - \int_{\{z_n \in (-\infty,0]\cup[2M,\infty)\}} \frac{dz}{|z|^{n+s}}.
\end{aligned} \tag{3.3.4}
$$

Also, by the transformation $(z', z_n) \mapsto (z', -z_n)$, we see that

$$
\int_{\{z_n \in (0,2M)\}} \frac{dz}{|z|^{n+s}} = \int_{\{z_n \in (-2M,0)\}} \frac{dz}{|z|^{n+s}},
$$

and therefore (3.3.4) gives that

$$
0 = -\int_{\{z_n \in (-\infty,-2M]\cup[2M,\infty)\}} \frac{dz}{|z|^{n+s}} < 0.
$$

This contradiction proves (3.3.1), and we now deal with the proof of (3.3.2). To this end, we let $\phi \in C_0^\infty(\mathbb{R}^{N-1}, [0,1])$ with $\phi(x') = 1$ if $|x'| \leq 1/2$ and $\phi(x') = 0$ if $|x'| \geq 3/4$. Given $\eta > 0$, we define

$$
F := \{x_N < \eta\phi(x')\}
$$

and we claim that, for every $p \in \partial F$,

$$\int_{\mathbb{R}^N} \frac{\chi_{F^c}(y) - \chi_F(y)}{|y-p|^{N+s}}\,dy \le C_0\eta, \tag{3.3.5}$$

for some $C_0 > 0$ depending only on N, s and ϕ. To prove this, we let

$$\Psi(x', x_n) := (x', x_N + \eta\phi(x')) \quad \text{and} \quad \Phi(x) := \Psi(x) - x = (0, \dots, 0, \eta\phi(x'))$$

Notice that $F = \Psi(\{x_N < 0\})$ and the Jacobian of Φ is bounded by $C\eta$, together with its derivatives, for some $C > 0$ depending only on N and η. Furthermore, the inverse of Ψ is given by

$$\Psi^{-1}(x) = (x', x_N - \eta\phi(x'))$$

and, setting $\Xi(x) := \Psi^{-1}(x) - x = -(0, \dots, 0, \eta\phi(x'))$, we find that also the Jacobian of Ξ is bounded by $C\eta$. Consequently, we are in the position of exploiting [37, Theorem 1.1] and deduce that

$$\int_{\mathbb{R}^N} \frac{\chi_{F^c}(y) - \chi_F(y)}{|y-p|^{N+s}}\,dy \le \int_{\mathbb{R}^N} \frac{\chi_{\{y_N>0\}}(y) - \chi_{\{y_N<0\}}(y)}{|y-\Psi^{-1}(p)|^{n+s}}\,dy + C_0\eta = C_0\eta,$$

for some $C_0 > 0$ depending only on N, s and ϕ, thus completing the proof of (3.3.5).

Now we define

$$G := F \cup \{x_N > 4M\},$$

we point out that this union is disjoint for large M and small η, and we claim that there exists $c > 0$, depending only on N, s and ϕ, such that if $\eta \in (0, cM^{-s}]$ then, for every $p \in \partial F$,

$$\int_{\mathbb{R}^N} \frac{\chi_{G^c}(y) - \chi_G(y)}{|y-p|^{N+s}}\,dy < 0. \tag{3.3.6}$$

Indeed, we have that $\chi_G = \chi_F + \chi_{\{x_N>4M\}}$, whence $\chi_{G^c} = 1 - \chi_G = 1 - \chi_F - \chi_{\{x_N>4M\}} = \chi_{F^c} - \chi_{\{x_N>4M\}}$. Accordingly, we have that $\chi_{G^c} - \chi_G = \chi_{F^c} - \chi_F - 2\chi_{\{x_N>4M\}}$ and therefore, using (3.3.5),

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{\chi_{G^c}(y) - \chi_G(y)}{|y-p|^{N+s}}\,dy &= \int_{\mathbb{R}^N} \frac{\chi_{F^c}(y) - \chi_F(y)}{|y-p|^{N+s}}\,dy - 2\int_{\{y_N>4M\}} \frac{dy}{|y-p|^{N+s}} \\
&\le C_0\eta - 2\int_{(-M,M)^{N-1}\times(4M,5M)} \frac{dy}{|y-p|^{N+s}} \\
&\le C_0\eta - c_0 M^{-s},
\end{aligned}$$

for some $c_0 > 0$ depending only on N and s, which plainly leads to (3.3.6).

By means of (3.3.6), we can thus use the set G as a sliding barrier from below with $\eta := cM^{-s}$ where $c > 0$ is such that $C_0 c M^{-s} - c_0 M^{-s} < 0$. (starting the sliding from a vertical translation of the set G equal to $-2M$) and find that $E_M \supset \{x_n < -M + cM^{-s}\phi(x')\}$. Indeed, we assume that there exits a touching point $p \in \partial F \cap \partial E_M$ such that $G \subset E_M$. Then, from the minimality of E_M, the following Euler-Lagrange equation holds:

$$0 \le \int_{\mathbb{R}^N} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y-p|^{N+s}}\,dy. \tag{3.3.7}$$

On the other hand, from the inclusion that $G \subset E_M$, we have

$$H^s_{E_M}(p) \le H^s_G(p). \tag{3.3.8}$$

Thus, from (3.3.6), (3.3.7), and (3.3.8), we obtain the contradiction. Therefore, we see that $E_M \supset \left[-\frac{1}{2}, \frac{1}{2}\right]^{N-1} \times (-\infty, -M + cM^{-s}] \supset B_{cM^{-s}}(0, \ldots, 0, -M)$.

Similarly, one proves that $E_M \supset B_{cM^{-s}}(0, \ldots, 0, M)$, thus completing the proof of (3.3.2).

Now we suppose that $N = 2$ and we establish (3.3.3). For this, we fix $\epsilon_0 > 0$, we consider a suitably small $\delta > 0$ and we exploit [51, Corollary 7.2] to construct a set $H \subset \mathbb{R}^2$ such that

$$\begin{aligned} & H \subset \{x_2 < \delta\}, \\ & H \cap \{x_1 < -1\} = (-\infty, -1) \times (-\infty, 0), \\ & H \cap \{x_1 > 1\} = (1, \infty) \times (-\infty, 0), \\ & H \supset (-1, 1) \times \left(-\infty, \delta^{\frac{2+\epsilon_0}{1-s}}\right), \\ \text{and} \quad & \int_{\mathbb{R}^2} \frac{\chi_{H^c}(y) - \chi_H(y)}{|y-p|^{2+s}}\, dy \leq \bar{C}\delta \end{aligned}$$

for every $p = (p_1, p_2) \in \partial H$ with $|p_1| < 1$, where $\bar{C} > 0$ depends only on s and ϵ_0. See Corollary 3.3.3 for the detail of this construction.

We define

$$L := H \cup \{x_2 > 4M\},$$

and we see that $\chi_{L^c} - \chi_L = \chi_{H^c} - \chi_H - 2\chi_{\{x_2 > 4M\}}$ and thus

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{\chi_{L^c}(y) - \chi_L(y)}{|y-p|^{2+s}}\, dy &\leq \int_{\mathbb{R}^2} \frac{\chi_{H^c}(y) - \chi_H(y)}{|y-p|^{2+s}}\, dy - 2\int_{\{y_2 > 4M\}} \frac{dy}{|y-p|^{2+s}} \\ &\leq \bar{C}\delta - 2\int_{(-M,M)\times(4M,5M)} \frac{dy}{|y-p|^{2+s}} \leq \bar{C}\delta - \bar{c}M^{-s} < 0 \end{aligned}$$

for every $p = (p_1, p_2) \in \partial H$ with $|p_1| < 1$, where $\bar{c} > 0$ depends only on s, and $\delta := \frac{\bar{c}M^{-s}}{2\bar{C}}$.

In this way, we can use L as sliding barrier from below (starting the sliding from a vertical translation of the set L equal to $-2M$) and deduce that

$$E_M \cap \{|x_1| < 1\} \supset (-1,\, 1) \times \left(-\infty, -M + \delta^{\frac{2+\epsilon_0}{1-s}}\right) = (-1,\, 1) \times \left(-\infty, -M + c_\star\, M^{-\frac{(2+\epsilon_0)s}{1-s}}\right)$$

for some $c_\star > 0$. Similarly, one finds that

$$E_M \cap \{|x_1| < 1\} \supset (-1,\, 1) \times \left(M - c_\star\, M^{-\frac{(2+\epsilon_0)s}{1-s}}, \infty\right).$$

The proof of (3.3.3) is thereby complete. □

As the last remark of this section, we recall the proposition and its corollary proved in [51], which we have used in the above proof to show the property (3.3.3).

Proposition 3.3.2 ([51])**.** *Let $\varepsilon_0 > 0$ be a small number depending only on $s \in (0, 1)$. For sufficiently small number $\delta > 0$, there exist constants $a_\delta > 0$, $L_\delta > A_\delta > d_\delta > 1$, $c_\delta \in \mathbb{R}$, $C_0 > 0$, and a set $F_\delta \subset \mathbb{R}^2$ such that $\partial F_\delta \cap \{x_2 > 0\}$ is of class $C^{1,1}$ and the following holds:*

$$\begin{aligned} & F_\delta \cap \{x_1 < 0\} = (-\infty,\, 0) \times (-\infty,\, 0), \\ & F_\delta \supset \mathbb{R} \times (-\infty,\, 0), \\ & F_\delta \supset (0,\, L_\delta + 1) \times (-\infty\, a_\delta], \\ & F_\delta \subset \{x_2 \leq C_0 \delta L_\delta^{\frac{1}{2}+s/2+\varepsilon_0}\}, \\ \textit{and} \quad & F_\delta \cap \{d_\delta < x_1 < L_\delta\} = \{x_2 < v_1(x_1),\, d_\delta < x_1 < L_\delta\} \end{aligned}$$

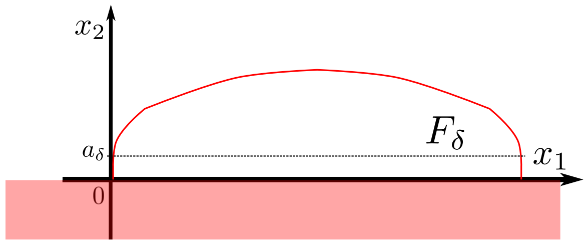

Figure 3.6: The barrier constructed in Proposition 3.3.2

where v is given as $v(x_1) = \gamma^{-1}\,\delta\,(x_1 + c_\delta)_+^\gamma$ with $\gamma = 1/2 + s/2 + \varepsilon_0$.

Moreover,

$$\int_{\mathbb{R}^2} \frac{\chi_{F_\delta^c}(y) - \chi_{F_\delta}(y)}{|y-p|^{2+s}}\, dy \le 0$$

for any $p \in \partial F_\delta \cap \{0 < x_1 < A_\delta\}$ and

$$\int_{\mathbb{R}^2} \frac{\chi_{F_\delta^c}(y) - \chi_{F_\delta}(y)}{|y-p|^{2+s}}\, dy \le \frac{C\delta^s}{L_\delta^{\frac{1}{2}+s/2+\varepsilon_0}}$$

for any $p \in \partial F_\delta \cap \{A_\delta \le x_1 \le L_\delta + 1\}$.

This proposition says that one can construct a "good" barrier for the s-fractional mean curvature equation $H_E^s(p) \le 0$ for $p \in \partial E \subset \mathbb{R}^2$, and this barrier is flat and horizontal outside a vertical slab, and whose geometric properties inside the slab are under control. See Figure 3.6 for a rough sketch of the barrier. The proof is quite technical and requires us to construct a piecewise linear barrier near the vertical slab and refine it in such a way that the barrier grows linearly with an almost horizontal slope. We here skip the proof of the theorem and we refer to [51, Proposition 7.1]. As a corollary of this proposition, the authors in [51] proved

Corollary 3.3.3 ([51])**.** *Fix $\varepsilon_0 > 0$ arbitrarily small. There exists an infinitesimal sequence of positive numbers $\{\delta_i\}_i$ and sets $\{H_i\}_i$ in $\mathbb{R}^2$ such that H_i is symmetric with respect to the axis $\{x_1 = 0\}$, $\partial H_i \cap \{x_2 > 0\}$ is of class $C^{1,1}$, and satisfy the following properties:*

$$\begin{aligned} & H_i \cap \{x_1 < -1\} = (-\infty,\, -1) \times (-\infty,\, 0), \\ & H_i \supset \mathbb{R} \times (-\infty,\, 0), \\ & H_i \supset (-1,\, 1) \times (-\infty\, \delta^{\frac{2+\varepsilon_0}{1-s}}], \\ \textit{and} \quad & H_i \subset \{x_2 \le \delta_i\}. \end{aligned}$$

Moreover,

$$\int_{\mathbb{R}^2} \frac{\chi_{H_i^c}(y) - \chi_{H_i}(y)}{|y-p|^{2+s}}\, dy \le \delta_i$$

for any $p \in \partial H_i \cap \{-1 + \frac{1}{100} \le x_1 \le 0\}$.

The proof follows directly from Proposition 3.3.2 and is based on the proper choice of $\delta_i := L_\delta^{-\frac{1}{2}+\frac{s}{2}+\varepsilon_0}$ where L_δ and ε_0 are as in Proposition 3.3.2 and scaling of the sets.

Chapter 4

Nonlocal Denoising Problem

4.1 Problem Setting and Main Results

Let $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be a given function and $f : \mathbb{R}^N \to \mathbb{R}$ be a given data. We study

$$\inf \left\{ \mathcal{F}_{K,f}(u) \mid u \in BV_K(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \right\} \tag{4.1.1}$$

where the functional $\mathcal{F}_{K,f}$ is defined as

$$\mathcal{F}_{K,f}(u) \coloneqq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x-y)\, |u(x) - u(y)| \, dx\, dy + \frac{1}{2} \int_{\mathbb{R}^N} (u(x) - f(x))^2 \, dx \tag{4.1.2}$$

for any function $u : \mathbb{R}^N \to \mathbb{R}$, and the space $BV_K(\mathbb{R}^N)$ is a set of all functions such that the first term of (4.1.2) is finite (see Section 4.2 in this chapter for the definition). Recall that the function K is called a "singular" kernel, as we briefly explain in Chapter 2.

In this chapter, we mainly focus on the typical kernel $K(x) = |x|^{-(N+s)}$ of the nonlocal perimeter, with $s \in (0,1)$, and we study the regularity of the minimizers of $\mathcal{F}_{K,f}$, under some suitable conditions on the data f.

As discussed in the case of the denoising problem in [29, 30, 31], our regularity result is based on the following observation: if u is a minimizer of the functional $\mathcal{F}_{K,f}$, then the superlevel set $\{u > t\}$ for each $t \in \mathbb{R}$ is also a minimizer of the functional associated with the prescribed nonlocal mean curvature problem

$$\inf \left\{ \mathcal{E}_{K,f,t}(E) \mid E \subset \mathbb{R}^N \right\}$$

where we define the functional $\mathcal{E}_{K,f,t}$ by

$$\mathcal{E}_{K,f,t}(E) \coloneqq P_K(E) + \int_E (t - f(x))\, dx$$

for any measurable set $E \subset \mathbb{R}^N$ and $t \in \mathbb{R}$. Recall that P_K is the nonlocal perimeter associated with the kernel K (see Section 2.1 in Chapter 2 for the precise definition). If K satisfies "good" conditions, E_t is a minimizer of $\mathcal{E}_{K,f,t}$ for each t, f is locally Lipschitz, and ∂E_t is smooth (C^2-regularity is sufficient), then we can obtain that the boundary ∂E_t satisfies the following *prescribed nonlocal mean curvature equation*

$$H^K_{E_t}(x) + t - f(x) = 0 \tag{4.1.3}$$

for any $x \in \partial E_t$. One may easily obtain this equation by computing the first variation of $\mathcal{E}_{K,f,t}$. Here $H^K_{E_t}$ is the so-called *nonlocal mean curvature* defined by

$$H^K_{E_t}(x) \coloneqq \mathrm{P.V.} \int_{\mathbb{R}^N} K(x-y)(\chi_{E_t}(x) - \chi_{E_t}(y))\, dy \tag{4.1.4}$$

for any $x \in \mathbb{R}^N$ where "P.V." means the Cauchy principal value (see also Section 2.3 of Chapter 2 for the notion of s-fractional mean curvature).

In the above setting, our main interest is on the regularity of the minimizer of $\mathcal{F}_{K,f}$ according to the one of the given data f. As we discussed in Chapter 1, it was proved that, in the classical denoising model, the minimizer is as regular as the data f if f is Hölder continuous. As an analogy of this result, we prove

Theorem 4.1.1. *Let $N = 2$, $s \in (0, 1)$, $K(x) = |x|^{-(N+s)}$ and $f \in L^2 \cap L^\infty(\mathbb{R}^2)$. If f is locally β-Hölder continuous with $\beta \in (1-s, 1]$, then the minimizer of the functional $\mathcal{F}_{K,f}$ is also locally β-Hölder continuous in $\mathbb{R}^2$.*

Our idea to show the local Hölder regularity of a minimizer is as follows: if we take any minimizer u of $\mathcal{F}_{K,f}$, then the distance between the boundaries of the two superlevel sets $\{u > t\}$ and $\{u > t'\}$ for $t,\, t' \in \mathbb{R}$ with $t \neq t'$ should not be too close to each other. To see this, we compare the two s-fractional mean curvatures of $\partial\{u > t\}$ and $\partial\{u > t'\}$ at the points where the smallest distance between the boundaries $\partial\{u > t\}$ and $\partial\{u > t'\}$ is attained. The comparison can be done thanks to the computations of the first variation of the nonlocal mean curvature shown in [41, 70]. Thus, using the Euler-Lagrange equation (4.1.3), we are able to derive some local estimate to assert the local Hölder regularity with the assumption of the local Hölder regularity of f.

4.2 Notations

In this section, we give several definitions and properties of the space of functions with finite nonlocal total variations. First of all, we recall the space $BV_K(\Omega)$ of functions with nonlocal bounded variations associated with the kernel K, which is defined in Chapter 2

Secondly, we give the definition of solutions to the so-called *nonlocal 1-Laplacian equation* associated with the kernel K.

Definition 4.2.1. Let $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a measurable function in $L^2(\mathbb{R}^N \times \mathbb{R})$. We say that $u \in BV_K \cap L^2(\mathbb{R}^N)$ is a solution to the nonlocal equation

$$- \Delta_1^K u(x) = F(x,\, u(x)) \quad \text{for a.e. } x \in \mathbb{R}^N \tag{4.2.1}$$

if there exists a function $z : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ with $|z| \le 1$ a.e. in $\mathbb{R}^N \times \mathbb{R}^N$ and $z(x,\, y) = -z(y,\, x)$ for a.e. $(x,\, y) \in \mathbb{R}^N \times \mathbb{R}^N$ such that

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\, z(x,\, y)(v(x) - v(y))\, dx\, dy = \int_{\mathbb{R}^N} F(x,\, u(x))\, v(x)\, dx \tag{4.2.2}$$

for every $v \in C_c^\infty(\mathbb{R}^N)$ with $[v]_K(\mathbb{R}^N) < \infty$ and

$$z(x,\, y) \in \operatorname{sgn}(u(y) - u(x)) \quad \text{for a.e. } (x,\, y) \in \mathbb{R}^N \times \mathbb{R}^N$$

where $\operatorname{sgn}(x)$ is a generalized sign function satisfying that

$$\operatorname{sgn}(x) \in [-1,\, 1], \quad \operatorname{sgn}(x)\, x = |x| \quad \text{for any } x \in \mathbb{R}.$$

In particular, the case that $F(x,\, u(x)) = u(x) - f(x)$ for a given data f is of our main interest in this chapter. The concept of the definition is motivated by the Euler-Lagrange equation of the functional

$$\mathcal{I}_K(u) := \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)|u(x) - u(y)|\, dx\, dy.$$

Indeed, when we assume that u is a minimizer of $\mathcal{I}_K$ and consider the first variation of the functional $\mathcal{I}_K$ in a formal way, namely, the quantity $\frac{d}{d\varepsilon}\lfloor_{\varepsilon=0}\mathcal{I}_K(u+\varepsilon\phi)$ for any suitable test function ϕ, we may obtain

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\frac{u(x)-u(y)}{|u(x)-u(y)|}\,(\phi(x)-\phi(y))\,dx\,dy = 0.$$

However, it is quite problematic for one to give a rigorous meaning to the ratio $\frac{u(x)-u(y)}{|u(x)-u(y)|}$. To overcome this difficulty, we adopt Definition 4.2.1 and this can be one of the proper treatments for this issue. Indeed, in Definition 4.2.1, we may consider the condition that $z(x,\,y)(u(y)-u(x)) = |u(y)-u(x)|$ for a.e. $(x,\,y)\in\mathbb{R}^N\times\mathbb{R}^N$ with $u(x)\neq u(y)$ as a natural requirement. Note that the framework of solutions in the sense of Definition 4.2.1 has been originally developed by, for instance, J.M. Mazón, J.D. Rossi, and J. Toledo in [88] and one may see this framework as a nonlocal counterpart of the framework given in [6] and [87].

Finally in this section, let us briefly mention the existence and uniqueness of the minimizer of the functional (4.1.2) with a general kernel K. These properties are true because of the general theory of functional analysis.

Existence of Minimizer in L^2

We assume that the kernel K is non-negative and satisfies that $P_K(B)<\infty$ for some ball $B\subset\mathbb{R}^N$ and moreover, the given data f is in $L^2(\mathbb{R}^N)$. We show the existence of the minimizer of the functional (4.1.2) in $L^2(\mathbb{R}^N)$. This is a simple consequence of the classical theory of functional analysis. Indeed, let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence in $L^2(\mathbb{R}^N)$, namely,

$$\lim_{n\to\infty}\mathcal{F}_{K,f}(u_n) = \inf\{\mathcal{F}_{K,f}(u)\mid u\in L^2(\mathbb{R}^N)\}. \tag{4.2.3}$$

Notice that, since $f\in L^2(\mathbb{R}^N)$ and $\chi_B\in L^2(\mathbb{R}^N)$ for some ball $B\subset\mathbb{R}^N$, we have

$$\inf_{u\in L^2(\mathbb{R}^N)}\mathcal{F}_{K,f}(u)\le\mathcal{F}_{K,f}(\chi_B) = P_K(B)+\int_{B^c}\frac{|f|^2}{2}+\int_B\frac{(f-1)^2}{2}<\infty.$$

From the minimality of each u_n and by choosing tu_n as the competitor with $t\in(0,\,1)$, we can observe

$$\sup_{n\in\mathbb{N}}\left([u_n]_K(\mathbb{R}^N)+\int_{\mathbb{R}^N}|u_n|^2\,dx\right)\le\frac{1}{1+t}\int_{\mathbb{R}^N}f^2\,dx<\infty. \tag{4.2.4}$$

From the weak compactness of L^2, we can extract a subsequence (denoted by the same indices) such that $u_n\to u$ weakly in $L^2(\mathbb{R}^N)$ as $n\to\infty$ for some $u\in L^2$. One can see that the functional $\mathcal{F}_{K,f}$ is lower semi-continuous in the strong topology of L^2 and convex. Hence, by Mazur's lemma, $\mathcal{F}_{K,f}$ is also lower semi-continuous in the weak topology of L^2. Therefore, we obtain that $u\in L^2$ is a minimizer of $\mathcal{F}_{K,f}$.

Uniqueness of Minimizer in L^2

Due to the L^2-fidelity term in (4.1.2), we can observe that the minimizer of the functional (4.1.2) is actually unique, up to multiple constants. Assume that u_1 and u_2 are the minimizers of (4.1.2), and K satisfies the conditions that the existence of the minimizer holds. Then, from the inequality

$$\left[\frac{u_1+u_2}{2}\right]_K(\mathbb{R}^N)\le\frac{1}{2}[u_1]_K(\mathbb{R}^N)+\frac{1}{2}[u_2]_K(\mathbb{R}^N)$$

and the convexity of the functional (4.1.2), we have

$$\mathcal{F}_{K,f}(\frac{u_1+u_2}{2})\le\frac{1}{2}\mathcal{F}_{K,f}(u_1)+\frac{1}{2}\mathcal{F}_{K,f}(u_2) \tag{4.2.5}$$

where the equality holds if and only if $u_1 = u_2$ a.e. in $\mathbb{R}^N$. On the other hand, from the minimality of u_1 and u_2, the equality in (4.2.5) holds, and thus we obtain the uniqueness.

4.3 Preliminary Results

In this section, we collect a number of properties of the minimizer of the functional (4.1.2) in order to prove the main theorem of this chapter.

4.3.1 Euler-Lagrange Equation for $\mathcal{F}_{K,f}$

In this subsection, we show the necessary and sufficient condition for the minimizers of the functional $\mathcal{F}_{K,f}$ in $\mathbb{R}^N$. Before stating the claim, we give some conditions on the kernel K which we assume in the sequel.

(K1) $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is a non-negative measurable function.

(K2) K is symmetric with respect to the origin, namely $K(-x) = K(x)$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

We observe that a typical example of the kernel K is given as $K(x) = |x|^{-(N+s)}$ with $s \in (0, 1)$ and this function satisfies all of the above assumptions.

In the following lemma, we show that the minimizer of $\mathcal{F}_{K,f}$ satisfies a prescribed nonlocal mean curvature equation. This equation can be regarded as the Euler-Lagrange equation. Moreover, we show that the converse statement is also valid.

Lemma 4.3.1. *Assume that the kernel K satisfies (K1) and (K2) and a given data f is $L^2(\mathbb{R}^2)$. If $u \in BV_K \cap L^2(\mathbb{R}^N)$ is a minimizer of the functional $\mathcal{F}_{K,f}$, then u satisfies the equation*

$$-\Delta_1^K u = u - f \quad \text{in } \mathbb{R}^N \tag{4.3.1}$$

in the sense of Definition 4.2.1. Conversely, if $u \in BV_K \cap L^2(\mathbb{R}^N)$ is a solution of the equation (4.3.1) *in the sense of Definition 4.2.1, then u is a minimizer of $\mathcal{F}_{K,f}$.*

Proof. First, we recall the definition of the functional $\mathcal{I}_K$ and the non-negativity of K and thus, find that $\mathcal{I}_K$ is convex, lower semi-continuous, and positive homogeneous of degree one. Then, by using the same argument as in [89, 90], we can show the characterization of the sub-differential of $\mathcal{I}_K(u)$ as follows:

$$\begin{aligned}&\partial\mathcal{I}_K(u)\\ &\quad = \left\{v \in L^2(\mathbb{R}^N) \mid -\Delta_1^K u = v \text{ in the sense of Definition 4.2.1}\right\}.\end{aligned} \tag{4.3.2}$$

Here we recall the definition of the sub-differential $\partial\mathcal{E}(u)$ for $u \in X$ of the functional $\mathcal{E} : X \to \mathbb{R} \cup \{\infty\}$ where X is the Hilbert space with the inner product $(\cdot, \cdot)_X$. We say that $v \in X$ belongs to $\partial\mathcal{E}(u)$ for each $u \in X$ if it holds that, for any $w \in X$,

$$\mathcal{E}(w) - \mathcal{E}(u) \geq (w, v)_X.$$

Note that $u \in X$ is a minimizer of $\mathcal{E}$ if and only if $0 \in \partial\mathcal{E}(u)$. Then, from the general theory on the sub-differential, we can also show the identity

$$\partial\mathcal{F}_{K,f}(u) = \partial\mathcal{I}_K(u) + u - f. \tag{4.3.3}$$

for any $u \in L^2$. Indeed, if $v \in \partial \mathcal{F}_{K,f}(u)$, then we can compute the functional of u as follows; for any $w \in L^2(\mathbb{R}^N)$,

$$\begin{aligned}
\mathcal{I}_K(w) - \mathcal{I}_K(u) &= \mathcal{F}_{K,f}(w) - \mathcal{F}_{K,f}(u) + \frac{1}{2}\int_{\mathbb{R}^N} (u-f)^2\,dx - \frac{1}{2}\int_{\mathbb{R}^N} (w-f)^2 \\
&\geq \int_{\mathbb{R}^N} v(w-u)\,dx - \frac{1}{2}\int_{\mathbb{R}^N} (w-u)(w+u-2f)\,dx \\
&= \int_{\mathbb{R}^N} (v-u+f)(w-u)\,dx + \int_{\mathbb{R}^N} (u-f)(w-u)\,dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^N} (w-u)(w+u-2f)\,dx \\
&= \int_{\mathbb{R}^N} (v-u+f)(w-u)\,dx + \frac{1}{2}\int_{\mathbb{R}^N} (w-u)^2\,dx \\
&\geq \int_{\mathbb{R}^N} (v-u+f)(w-u)\,dx. \qquad (4.3.4)
\end{aligned}$$

Therefore we obtain $v - u + f \in \partial \mathcal{I}_K(u)$. On the other hand, if $v \in \partial \mathcal{I}_K(u) + u - f$, then we can compute in the following manner; for any $w \in L^2(\mathbb{R}^N)$, we have

$$\begin{aligned}
\mathcal{F}_{K,f}(w) - \mathcal{F}_{K,f}(u) &= \mathcal{I}_K(w) - \mathcal{I}_K(u) + \frac{1}{2}\int_{\mathbb{R}^N} (w-f)^2\,dx - \frac{1}{2}\int_{\mathbb{R}^N} (u-f)^2 \qquad (4.3.5) \\
&\geq \int_{\mathbb{R}^N} (v-u+f)(w-u)\,dx + \frac{1}{2}\int_{\mathbb{R}^N} (w-u)(w+u-2f)\,dx \\
&= \int_{\mathbb{R}^N} v(w-u)\,dx + \frac{1}{2}\int_{\mathbb{R}^N} (w-u)^2\,dx \\
&\geq \int_{\mathbb{R}^N} v(w-u)\,dx, \qquad (4.3.6)
\end{aligned}$$

and thus we have that $v \in \partial \mathcal{F}_{K,f}(u)$. Therefore, from the computations (4.3.4) and (4.3.5), we conclude that the first part of the claim is valid. Then from (4.3.3), we can easily obtain the equity

$$\begin{aligned}
&\partial \mathcal{F}_{K,f}(u) \\
&\quad = \left\{ v + u - f \in L^2(\mathbb{R}^N) \mid -\Delta_1^K u = v \text{ in the sense of Definition 4.2.1} \right\}. \qquad (4.3.7)
\end{aligned}$$

We can readily see that $0 \in \partial \mathcal{F}_{K,f}(u)$ whenever u is a minimizer of $\mathcal{F}_{K,f}$ Therefore, we conclude that, if u is a minimizer of $\mathcal{F}_{K,f}$, then u is a solution of the equation (4.3.1).

Conversely, if u is a solution of the equation (4.3.1), then from (4.3.7) we have that 0 belongs to the set in the right-hand side of (4.3.7), and thus we obtain $0 \in \partial \mathcal{F}_{K,f}(u)$. □

4.3.2 Comparison between Minimizers

In this subsection, we prove a comparison principle for the minimizers of $\mathcal{F}_{K,f}$ by using the uniqueness of the minimizer in L^2. We assume that K satisfies the assumptions (K1) and (K2) shown in Subsection 4.3.1 and the data f_1 and f_2 satisfy that $f_1 \leq f_2$. Then we show that the minimizers u_1 and u_2 associated with f_1 and f_2, respectively, preserves the inequality. Precisely, we prove the following result:

Lemma 4.3.2. *Let f_i be in $L^2(\mathbb{R}^N)$ for each $i \in \{1, 2\}$ and $u_i \in BV_K \cap L^2(\mathbb{R}^N)$ be a minimizer of $\mathcal{F}_{K,f_i}$ for each $i \in \{1, 2\}$. Assume that the kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies (K1) and (K2). If $f_1 \leq f_2$ $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$, then $u_1 \leq u_2$ $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$.*

Proof. Let $u_1,\, u_2 \in BV_K(\mathbb{R}^N)$ be minimizers of $\mathcal{F}_{K,f}$ associated with given data $f_1,\, f_2 \in L^2(\mathbb{R}^N)$, respectively. First of all, we prove the following inequality:

$$[u_+]_K(\mathbb{R}^N) + [u_-]_K(\mathbb{R}^N) \leq [u_1]_K(\mathbb{R}^N) + [u_2]_K(\mathbb{R}^N). \tag{4.3.8}$$

Indeed, setting

$$u_+(x) \coloneqq \max\{u_1(x),\, u_2(x)\}, \quad u_-(x) \coloneqq \min\{u_1(x),\, u_2(x)\} \tag{4.3.9}$$

for any $x \in \mathbb{R}^N$ and by the co-area formula, we have that

$$\begin{aligned}
[u_i]_K(\mathbb{R}^N) &= \int_{-\infty}^{\infty} \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x-y)\, |\chi_{\{u_i > t\}}(x) - \chi_{\{u_i > t\}}(y)|\, dx\, dy\, dt \\
&= \int_{-\infty}^{\infty} P_K(\{u_i > t\})\, dt
\end{aligned} \tag{4.3.10}$$

for any $i \in \{1,\, 2,\, +,\, -\}$. From Proposition 2.1.6, we recall that the nonlocal perimeter P_K is sub-modular, namely, it holds that

$$P_K(E \cup F) + P_K(E \cap F) \leq P_K(E) + P_K(F) \tag{4.3.11}$$

for any $E,\, F \subset \mathbb{R}^N$. Therefore from (4.3.11) and the definitions of u_+ and u_-, we obtain the claim.

Now from the general theory of calculus of variations, the minimizer of $\mathcal{F}_{K,f}$ is unique in $L^2(\mathbb{R}^N)$ and thus, it is sufficient to prove that

$$\mathcal{F}_{K,f_2}(u_+) \leq \mathcal{F}_{K,f_2}(u_2)$$

where u_+ is defined in (4.3.9) to obtain the lemma. From a simple computation, we can easily see that the inequality

$$(u_- - f_1)^2 + (u_+ - f_2)^2 \leq (u_1 - f_1)^2 + (u_2 - f_2)^2 \tag{4.3.12}$$

in $\mathbb{R}^N$. From the minimality of u_i for $i \in \{1,\, 2\}$, we have

$$\mathcal{F}_{K,f_1}(u_1) + \mathcal{F}_{K,f_2}(u_2) \leq \mathcal{F}_{K,f_1}(u_-) + \mathcal{F}_{K,f_2}(u_+). \tag{4.3.13}$$

On the other hand, from (4.3.8) and (4.3.12), we have

$$\begin{aligned}
&\mathcal{F}_{K,f_1}(u_-) + \mathcal{F}_{K,f_2}(u_+) \\
&\leq [u_-]_K(\mathbb{R}^N) + \frac{1}{2} \int_{\mathbb{R}^N} (u_- - f_1)^2\, dx + [u_+]_K(\mathbb{R}^N) + \frac{1}{2} \int_{\mathbb{R}^N} (u_+ - f_2)^2\, dx \\
&= [u_1]_K(\mathbb{R}^N) + \frac{1}{2} \int_{\mathbb{R}^N} (u_1 - f_1)^2\, dx + [u_2]_K(\mathbb{R}^N) + \frac{1}{2} \int_{\mathbb{R}^N} (u_2 - f_2)^2\, dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} (u_- - f_1)^2\, dx - \frac{1}{2} \int_{\mathbb{R}^N} (u_1 - f_1)^2\, dx \\
&\quad\quad + \frac{1}{2} \int_{\mathbb{R}^N} (u_+ - f_2)^2\, dx - \frac{1}{2} \int_{\mathbb{R}^N} (u_2 - f_2)^2\, dx \\
&\leq \mathcal{F}_{K,f_1}(u_1) + \mathcal{F}_{K,f_2}(u_2).
\end{aligned} \tag{4.3.14}$$

Thus from (4.3.13) and (4.3.14), we obtain

$$\mathcal{F}_{K,f_1}(u_1) + \mathcal{F}_{K,f_2}(u_2) = \mathcal{F}_{K,f_1}(u_-) + \mathcal{F}_{K,f_2}(u_+) \tag{4.3.15}$$

Now suppose by contradiction that $\mathcal{F}_{K,f_2}(u_+) > \mathcal{F}_{K,f_2}(u_2)$. Then from (4.3.15) we have

$$\mathcal{F}_{K,f_1}(u_1) > \mathcal{F}_{K,f_1}(u_-)$$

which contradicts the minimality of u_1. Thus we obtain the inequality $\mathcal{F}_{K,f_2}(u_+) \leq \mathcal{F}_{K,f_2}(u_2)$. Therefore, by the uniqueness of the minimizer of $\mathcal{F}_{K,f}$, this implies that $u_+ = u_2$ a.e. in $\mathbb{R}^N$, which implies that $u_2 \geq u_1$ a.e. in $\mathbb{R}^N$. $\square$

Corollary 4.3.3. *Assume that the kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions (K1) and (K2) in Subsection 4.3.1. If a data $f \in L^2(\mathbb{R}^N)$ is non-negative a.e. in $\mathbb{R}^N$, then the minimizer $u \in BV_K \cap L^2(\mathbb{R}^N)$ is also non-negative a.e. in $\mathbb{R}^N$.*

Proof. Since it holds that

$$\mathcal{F}_{K,0}(0) = 0 \leq \mathcal{F}_{K,0}(v)$$

for every $v \in BV_K \cap L^2(\mathbb{R}^N)$, we have that the unique solution of the problem

$$\inf\{\mathcal{F}_{K,0}(v) \mid v \in BV_K \cap L^2\}$$

is $v = 0$. Hence, by applying Lemma 4.3.2 to the case that $f_1 = 0$ and $f_2 = f$, we obtain that $0 \leq u$ a.e. in $\mathbb{R}^N$. □

Finally, we show a sort of comparison property of minimizers under the assumption that a data f is bounded in $\mathbb{R}^N$. We do not derive the following proposition directly from Lemma 4.3.2 but from a simple computation.

Proposition 4.3.4. *Let $u \in BV_K \cap L^2(\mathbb{R}^N)$ be a minimizer of $\mathcal{F}_{K,f}$ with $f \in L^2(\mathbb{R}^N)$. Assume that the kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is non-negative measurable function. If there exists a constant $C > 0$ such that $|f(x)| \leq C$ for a.e. $x \in \mathbb{R}^N$, then $|u(x)| \leq C$ for a.e. $x \in \mathbb{R}^N$ with the same constant C.*

Proof. It is sufficient to show that, if $f \leq C$ a.e. in $\mathbb{R}^N$ with some constant $C > 0$, then $u \leq C$ a.e. in $\mathbb{R}^N$ with the same constant C because we only repeat the same argument as we show in this proof. We define $v(x) := \min\{u(x), C\}$ for $x \in \mathbb{R}^N$. It is sufficient to show that $u = v$ for a.e. in $\mathbb{R}^N$. From the definition, we can show the claim that $|v(x) - v(y)| \leq |u(x) - u(y)|$ for $x, y \in \mathbb{R}^N$. Indeed, if $u(x) \leq C$ and $u(y) \leq C$ or $u(x) > C$ and $u(y) > C$, then we can readily obtain the claim. If $u(x) \leq C$ and $u(y) > C$, then we have

$$\begin{aligned}|u(x) - u(y)|^2 - |v(x) - v(y)|^2 &= u^2(y) - C^2 - 2u(x)\, u(y) + 2u(x)\, C \\ &= (u(y) - C)(u(y) + C - 2u(x)) \geq 0.\end{aligned}$$

In the same way, we can prove the claim if $u(x) > C$ and $u(y) \leq C$. Moreover, we can show that $(v - f)^2 \leq (u - f)^2$ in $\mathbb{R}^N$. Therefore we compute the functional associated with v as follows:

$$\begin{aligned}\mathcal{F}_{K,f}(v) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\,|v(x) - v(y)|\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N}(v-f)^2\,dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\,|u(x) - u(y)|\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N}(u-f)^2\,dx \\ &= \mathcal{F}_{K,f}(u).\end{aligned}$$

Thus, from the uniqueness of the minimizer of the functional $\mathcal{F}_{K,f}$, we obtain $v = u$ a.e. in $\mathbb{R}^N$ and this concludes the proof. □

4.3.3 Characterization of Minimizers for $\mathcal{F}_{K,f}$

In this section, we show the following claim which gives a relation between the minimizers of $\mathcal{F}_{K,f}$ and $\mathcal{E}_{K,f,t}$ for $t \in \mathbb{R}$. Recall that $\mathcal{E}_{K,f,t}(E)$ as

$$\mathcal{E}_{K,f,t}(E) := P_K(E) + \int_E (t - f(x))\,dx \tag{4.3.16}$$

for every measurable set $E \subset \mathbb{R}^N$ where we assume that $f \in L^2(\mathbb{R}^N)$ is a given data and $t \in \mathbb{R}$ is any number.

Lemma 4.3.5. *Assume that the kernel* $K(x) = |x|^{-(N+s)}$ *for* $x \in \mathbb{R}^N \setminus \{0\}$ *with* $s \in (0, 1)$ *and a data* $f \in L^2 \cap L^\infty(\mathbb{R}^N)$. *If* $u \in BV_K \cap L^2(\mathbb{R}^N)$ *be a minimizer of* $\mathcal{F}_{K,f}$, *then the set* $\{x \in \mathbb{R}^N \mid u(x) > t\}$ *is also a minimizer of* $\mathcal{E}_{K,f,t}(E)$ *for every* $t \in \mathbb{R}$ *among measurable sets* $E \subset \mathbb{R}^N$.

Proof. Let $F \subset \mathbb{R}^N$ be any measurable set. We may assume that $P_K(F) < \infty$; otherwise this set cannot minimize the functional $\mathcal{E}_{K,f,t}$. Moreover, we may assume that $\|\chi_F\|_{L^1} = |F| < \infty$ because of the nonlocal isoperimeteric inequality. Then it suffices to show that the superlevel set $\{u > t\}$ for each $t \in \mathbb{R}$ satisfies the inequality

$$P_K(\{u > t\}) + \int_{\{u>t\}} (t - f(x))\,dx \leq P_K(F) + \int_F (t - f(x))\,dx. \tag{4.3.17}$$

From Lemma 4.3.1 and the assumption that u is a minimizer of the functional $\mathcal{F}_{K,f}$, we have that u is also a solution of the equation

$$-\Delta_1^K u = u - f \quad \text{in } \mathbb{R}^N \tag{4.3.18}$$

in the sense of Definition 4.2.1. Thus, by definition, there exists a function $z_u \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ with $|z_u| \leq 1$ and z_u being antisymmetric such that

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\, z_u(x, y)\,(w(x) - w(y))\,dx\,dy = \int_{\mathbb{R}^N} (u - f)\,w(x)\,dx \tag{4.3.19}$$

for any $w \in BV_K \cap L^2(\mathbb{R}^N)$ with a compact support and moreover

$$z_u(x, y)(u(y) - u(x)) = |u(y) - u(x)| \tag{4.3.20}$$

for a.e. $x, y \in \mathbb{R}^N$. From the co-area formula, we have the following two identities:

$$|u(x) - u(y)| = \int_{-\infty}^{\infty} |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|\,dt \tag{4.3.21}$$

and

$$(u(x) - u(y)) = \int_{-\infty}^{\infty} (\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y))\,dt \tag{4.3.22}$$

for any measurable $u : \mathbb{R}^N \to \mathbb{R}$ and a.e. $x, y \in \mathbb{R}^N$. Thus from (4.3.20), (4.3.21), and (4.3.22), we obtain

$$z_u(x, y)(\chi_{\{u>t\}}(y) - \chi_{\{u>t\}}(x)) = |\chi_{\{u>t\}}(y) - \chi_{\{u>t\}}(x)| \tag{4.3.23}$$

for a.e. $t \in \mathbb{R}$. Now we fix $t \in \mathbb{R}$ such that (4.3.23) holds. From the specific choice of $K(x) = |x|^{-(N+s)}$, the function space $BV_K(\mathbb{R}^N)$ coincides with the fractional Sobolev space $W^{s,1}(\mathbb{R}^N)$. Recall that the space $C_c^\infty(\mathbb{R}^N)$ of smooth functions with compact supports is dense in $W^{s,1}(\mathbb{R}^N)$ (see [2] for the detail). Hence, from the fact that $P_K(\{u > t\})$ and $P_K(F)$ are finite, we can choose sequences $\{\eta_l^u\}_{l\in\mathbb{N}}$ and $\{\eta_l^F\}_{l\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^N)$ such that

$$\eta_l^u \xrightarrow[l\to\infty]{} \chi_{\{u>t\}}, \quad \eta_l^F \xrightarrow[l\to\infty]{} \chi_F \quad \text{in } W^{s,1}(\mathbb{R}^N). \tag{4.3.24}$$

From the choice of the approximation, we notice that the difference function $\eta_l^u - \eta_l^F$ is also in $W^{s,1} \cap L^2(\mathbb{R}^N)$ and has a compact support for each $l \in \mathbb{N}$. Hence, from the definition of solutions to the equation (4.3.18), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} (u-f)\,(\eta_l^u - \eta_l^F)\,dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\,z_u(x,\,y)\,[(\eta_l^u - \eta_l^F)(y) - (\eta_l^u - \eta_l^F)(x)]\,dx\,dy \\
&= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\,z_u(x,\,y)\,(\eta_l^u(y) - \eta_l^u(x))\,dx\,dy \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\,z_u(x,\,y)\,(\eta_l^F(y) - \eta_l^F(x))\,dx\,dy. \qquad (4.3.25)
\end{aligned}$$

By applying Proposition 4.3.4 and from the assumption that $f \in L^\infty(\mathbb{R}^N)$, we have that the minimizer u is also in $L^\infty(\mathbb{R}^N)$ and thus

$$\left|\int_{\mathbb{R}^N} (u-f)(\eta_l^u - \eta_l^F)\,dx - \int_{\mathbb{R}^N} (u-f)(\chi_{\{u>t\}} - \chi_F)\,dx\right| \xrightarrow[l\to\infty]{} 0. \qquad (4.3.26)$$

Hence by applying the dominated convergence theorem and from (4.3.24), (4.3.25), and (4.3.26), we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (u-f)(\chi_{\{u>t\}} - \chi_F)\,dx \\
&= \lim_{l\to\infty}\int_{\mathbb{R}^N} (u-f)\,(\eta_l^u - \eta_l^F)\,dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\,z_u(x,\,y)\,(\chi_{\{u>t\}}(y) - \chi_{\{u>t\}}(x))\,dx\,dy \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\,z_u(x,\,y)\,(\chi_F(y) - \chi_F(x))\,dx\,dy. \qquad (4.3.27)
\end{aligned}$$

From the definition of z_u, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\,z_u(x,\,y)\,(\chi_F(x) - \chi_F(y))\,dx\,dy \\
&\le \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)|\chi_F(x) - \chi_F(y)|\,dx\,dy = P_K(F). \qquad (4.3.28)
\end{aligned}$$

Taking into account (4.3.23), (4.3.27), and (4.3.28), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} (u-f)\,(\chi_{\{u>t\}} - \chi_F)\,dx \\
&\le -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)|\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|\,dx\,dy + P_K(F). \qquad (4.3.29)
\end{aligned}$$

Regarding the left-hand side of (4.3.29), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} (u-f)\,(\chi_{\{u>t\}} - \chi_F)\,dx &= \int_{\mathbb{R}^N} (u-t+t-f)\,(\chi_{\{u>t\}} - \chi_F)\,dx \\
&\ge \int_{\{u>t\}\cap F^c} (t-f)\,dx - \int_{\{u\le t\}\cap F} (u-f)\,dx \\
&\ge \int_{\{u>t\}\cap F^c} (t-f)\,dx - \int_{\{u\le t\}\cap F} (t-f)\,dx \\
&= \int_{\mathbb{R}^N} (t-f)\,(\chi_{\{u>t\}} - \chi_F)\,dx \qquad (4.3.30)
\end{aligned}$$

for a.e. $t \in \mathbb{R}$. Hence, from (4.3.29) and (4.3.30), we have

$$
\begin{aligned}
&P_K(\{u > t\}) + \int_{\mathbb{R}^N} (t - f)\, \chi_{\{u>t\}}\, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x - y) |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|\, dx\, dy + \int_{\mathbb{R}^N} (t - f)\, \chi_{\{u>t\}}\, dx \\
&\leq P_K(F) + \int_{\mathbb{R}^N} (t - f)\, \chi_F\, dx
\end{aligned} \tag{4.3.31}
$$

for a.e. $t \in \mathbb{R}$. Therefore we conclude that the inequality (4.3.17) holds for a.e. $t \in \mathbb{R}$. Notice that, for any $t \in \mathbb{R}$ such that (4.3.23) does not hold, we can choose a sequence $\{t_j\}_{j \in \mathbb{N}}$ such that $t_j \to t$ as $j \to \infty$ and (4.3.23) holds for any t_j; otherwise we can choose a constant $\delta > 0$ such that $B_\delta(t) \subset \{t \in \mathbb{R} \mid \text{(4.3.23) is not true}\}$. Since the condition (4.3.23) holds true for a.e. $t \in \mathbb{R}$, we have that

$$
0 < 2\delta = |B_\delta(t)| \leq |\{t \in \mathbb{R} \mid \text{(4.3.23) is not true}\}| = 0,
$$

which is a contradiction. Thus from the lower semi-continuity of P_K and the continuity of the map $t \mapsto |\{u > t\}|$, we conclude that (4.3.17) holds for every $t \in \mathbb{R}$. ☐

4.3.4 Boundedness of Superlevel Sets of Minimizers

Let $u \in BV_K \cap L^2(\mathbb{R}^N)$ be a minimizer of $\mathcal{F}_{K,f}$ with a data $f \in L^p(\mathbb{R}^N)$ with $p \in (\frac{n}{s}, \infty]$. In this section, we show that the superlevel set $\{u > t\}$ for each $t \in \mathbb{R}$ is bounded up to negligible sets. Precisely, we prove

Lemma 4.3.6. *Assume that the kernel $K(x) = |x|^{-(N+s)}$ for $x \in \mathbb{R}^N \setminus \{0\}$ with $s \in (0, 1)$ and $f \in L^p(\mathbb{R}^N)$ with $p \in (\frac{n}{s}, \infty]$. If E_T is a minimizer of $\mathcal{E}_{K,f,T}$ among sets with finite volumes for any $T \in \mathbb{R}$, then there exists a constant $R_T > 0$ such that $|E_T \setminus B_{R_T}| = 0$.*

Proof. We basically follow the proof shown in [32, Proposition 3.2]. Suppose by contradiction that $|E_T \setminus B_r| > 0$ for any $r > 0$. By setting $\phi_T(r) := |E_T \setminus B_r|$ for any $r > 0$, we have

$$
(\phi_T)'(r) = -\mathcal{H}^{N-1}(E_T \cap \partial B_r)
$$

for a.e. $r > 0$. We fix any $R > 1$. From the minimality of E_T, we have

$$
\mathcal{E}_{K,f,T}(E_T) \leq \mathcal{E}_{K,f,T}(E_T \cap B_r). \tag{4.3.32}
$$

From Proposition 2.1.3 with $\Omega = \mathbb{R}^N$, we have

$$
P_K(E_T \setminus B_r) \leq 2 \int_{E_T \cap B_r} \int_{E_T \setminus B_r} K(x - y)\, dx\, dy - \int_{E_T \setminus B_r} (T - f(x))\, dx. \tag{4.3.33}
$$

From the isoperimetric inequality of the nonlocal perimeter, we can have the following lower bound of the term of the left-hand side in (4.3.33) (see for instance [56]):

$$
P_K(E_T \setminus B_r) \geq \frac{P_K(B_1)}{|B_1|^{\frac{n-s}{n}}} |E_T \setminus B_r|^{\frac{n-s}{n}} = C(n, s)\, \phi_T^{\frac{n-s}{n}}(r) \tag{4.3.34}
$$

for $r \geq R$, where we set $C(n, s) := |B_1|^{-\frac{n-s}{n}} P_K(B_1)$. Secondly, from Fubini-Tonelli's theorem and the co-area formula, we can compute the first term of the right-hand side in

(4.3.33) as follows:

$$
\begin{aligned}
\int_{E_T\cap B_r}\int_{E_T\setminus B_r} K(x-y)\,dx\,dy &\le \int_{E_T\setminus B_r}\int_{B_{|y|-r}(y)} \frac{1}{|x-y|^{N+s}}\,dx\,dy \\
&= \int_{E_T\setminus B_r} |\mathbb{S}^{N-1}| \int_{|y|-r}^{\infty} \frac{1}{r^{1+s}}\,dr\,dy \\
&\le \frac{|\mathbb{S}^{N-1}|}{s}\int_{E_T\setminus B_r} (|y|-r)^{-s}\,dy \\
&= \frac{|\mathbb{S}^{N-1}|}{s}\int_r^{\infty} \frac{\mathcal{H}^{N-1}(E_T\cap\partial B_\sigma)}{(\sigma-r)^s}\,d\sigma \\
&= -\frac{|\mathbb{S}^{N-1}|}{s}\int_r^{\infty}\frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,d\sigma
\end{aligned}
\tag{4.3.35}
$$

for any $r \ge R$. Finally, regarding the second term of the right-hand side in (4.3.33), from the assumption of f and Cauchy-Schwartz inequality (if $p \neq \infty$), we have

$$
\begin{aligned}
\int_{E_T\setminus B_r} (-T+f(x))\,dx &\le T\,|E_T\setminus B_r| + \|f\|_{L^p(\mathbb{R}^N)}\,|E_T\setminus B_r|^{\frac{1}{q}} \\
&= T\,\phi_T(r) + \|f\|_{L^p(\mathbb{R}^N)}\,\phi_T^{\frac{1}{q}}(r) < \infty
\end{aligned}
\tag{4.3.36}
$$

for any $r \ge R > 1$ where $q \ge 1$ satisfies $p^{-1}+q^{-1}=1$. By combining all the computations (4.3.34), (4.3.35), and (4.3.36) with (4.3.33), we obtain

$$
C(n,s)\,\phi_T^{\frac{n-s}{n}}(r) \le -C_1\int_r^{\infty}\frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,d\sigma + T\,\phi_T(r) + \|f\|_{L^p(\mathbb{R}^N)}\,\phi_T^{\frac{1}{q}}(r) \tag{4.3.37}
$$

for any $r \ge R$ where we set $C_1 := \frac{|\mathbb{S}^{N-1}|}{s}$. Since $\phi_T(r)$ vanishes as $r \to \infty$ and $\frac{1}{q} > \frac{n-s}{n}$, we can have that

$$
2T\,\phi_T(r) + 2\|f\|_{L^p(\mathbb{R}^N)}\phi_T^{\frac{1}{q}}(r) \le C(n,s)\,\phi_T^{\frac{n-s}{n}}(r)
$$

for sufficiently large $r \ge R$. Hence, by integrating the both sides of (4.3.37) over $r \in (R,\infty)$, we obtain

$$
\frac{C(n,s)}{2}\int_R^{\infty}\phi_T^{\frac{n-s}{n}}(r)\,dr \le -C_1\int_R^{\infty}\int_r^{\infty}\frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,d\sigma\,dr. \tag{4.3.38}
$$

By exchanging the order of the integration, we have

$$
\int_R^{\infty}\int_r^{\infty}\frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,d\sigma\,dr = \int_R^{\infty}\int_R^{\sigma}\frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,dr\,d\sigma. \tag{4.3.39}
$$

Then by employing the similar computation shown in [32], we obtain

$$
\int_R^{\infty}\int_R^{\sigma}\frac{(\phi_T)'(\sigma)}{(\sigma-r)^s}\,dr\,d\sigma \ge -\frac{\phi_T(R)}{1-s} - \int_{R+1}^{\infty}\frac{\phi_T(r)}{(\sigma-R)^s}\,d\sigma.
$$

Therefore, from (4.3.38), we have

$$
\begin{aligned}
\frac{C(n,s)}{2}\int_R^{\infty}\phi_T^{\frac{n-s}{n}}(r)\,dr &\le C_1\frac{\phi_T(R)}{1-s} + C_1\int_{R+1}^{\infty}\frac{\phi_T(\sigma)}{(\sigma-R)^s}\,d\sigma \\
&\le C_1\frac{\phi_T(R)}{1-s} + C_1\int_{R+1}^{\infty}\phi_T(\sigma)\,d\sigma.
\end{aligned}
$$

Again, by choosing R sufficiently large so that the inequality

$$C_1 \int_{R+1}^{\infty} \phi_T(r)\, dr \le \frac{C(n,s)}{4} \int_R^{\infty} \phi_T^{\frac{n-s}{n}}(r)\, dr$$

holds, we have

$$\int_R^{\infty} \phi_T^{\frac{n-s}{n}}(r)\, dr \le \frac{4C_1}{C(n,s)(1-s)}\, \phi_T(R).$$

Then by applying the method shown in, for instance, [45, 32], we obtain the contradiction to the assumption that $\phi_T(r) > 0$ for any $r > 0$. Therefore, we conclude the essential boundedness of the set E_T. □

We assume that $u \in BV_K \cap L^2(\mathbb{R}^N)$ is a minimizer of the functional $\mathcal{F}_{K,f}$ and u is bounded from below with the constant $c \in \mathbb{R}$. Then, since the superlevel set $\{u > c\}$ is also a minimizer of $\mathcal{E}_{K,f,c}$, we may obtain from Lemma 4.3.6 that there exists a constant $R_c > 1$ such that $|\{u > c\} \setminus B_{R_c}| = 0$. In addition to this, we have the inclusion of the superlevel sets that $\{u > t'\} \subset \{u > t\}$ for any $t' > t$. Thus, we conclude that the following corollary holds.

Corollary 4.3.7. *Assume that the kernel $K(x) = |x|^{-(N+s)}$ for $x \in \mathbb{R}^N \setminus \{0\}$ with $s \in (0,1)$. Let $u \in BV_K \cap L^2(\mathbb{R}^N)$ be a minimizer of $\mathcal{F}_{K,f}$. If f is in $L^\infty(\mathbb{R}^N)$ and $u \ge c$ a.e. in $\mathbb{R}^N$ for some $c \in \mathbb{R}$, then the superlevel set $\{u > t\}$ is uniformly bounded with respect to $t \ge c$. Namely, there exists $R_c > 0$, independent of t, such that $\{u > t\} \subset B_{R_c}$ for any $t \ge c$.*

4.4 Hölder Regularity of Minimizers

First of all, we prove that, if the boundary of $\{u > t\}$ is regular, then u is continuous. This claim will not be applied here to prove our main theorem; however the proof itself contains some ideas for the proof of our main theorem.

Proposition 4.4.1. *Assume that $K(x) = |x|^{-(N+s)}$ for any $x \in \mathbb{R}^N \setminus \{0\}$ with $s \in (0,1)$ and the data f is locally Lipschitz and in $L^\infty(\mathbb{R}^N)$. Let $u \in BV_K(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ be a minimizer of $\mathcal{F}_{K,f}$, and we assume that $\partial\{u > t\}$ is of class $C^{1,\alpha}$ with $\alpha \in (s,1]$ for each $t \in \mathbb{R}$. Then u is continuous in $\mathbb{R}^N$.*

Proof. From Lemma 4.3.5, we have that the set $E_t := \{u > t\}$ is a minimizer of $\mathcal{E}_{K,f,t}$ for each $t \in \mathbb{R}$. Suppose by contradiction that u is not continuous in $\mathbb{R}^N$. Then there exist a point $x_0 \in \mathbb{R}^N$ and $-\infty < t' < t < \infty$ such that $x_0 \in \partial E_t \cap \partial E_{t'}$. Indeed, if u is not continuous at x_0, then it holds that $t_+ := \limsup_{x \to x_0} u(x) > \liminf_{x \to x_0} u(x) =: t_-$. Note that $t_+ \ge u(x_0) \ge t_-$ by definition. Setting $\delta := t_+ - t_- > 0$ and the definition of t_+, we can choose a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_k \to x_0$ and $u(x_k) > t_+ - \frac{\delta}{2^k}$ for any $k \in \mathbb{N}$ with $k \ge 1$. If $u(x_0) = t_+$, then we have that $x_k \in \{u > u(x_0) - \frac{\delta}{2}\}$ for large $k \in \mathbb{N}$ and thus, we obtain that $x_0 \in \overline{\{u > u(x_0) - \frac{\delta}{2}\}}$. However, from the definition of δ, x_0 cannot be a interior point of $\{u > u(x_0) - \frac{\delta}{2}\}$; otherwise we can choose a sequence $\{y_k\}_{k \in \mathbb{N}}$ such that

$$u(x_0) - \frac{\delta}{2} < u(y_k) < t_- + \frac{\delta}{2^k} \tag{4.4.1}$$

for any large k. From the definition of δ and the fact that $u(x_0) = t_+$, we obtain a contradiction. Therefore, we have $x_0 \in \partial\{u > u(x_0) - \frac{\delta}{2}\}$. In the same way, we also have $x_0 \in \partial\{u > u(x_0) - \frac{\delta}{4}\}$. This implies the validity of the claim. Now we assume that $u(x_0) < t_+$ and set $\tilde{\delta} := t_+ - u(x_0) > 0$. Then, since $u(x_k) > t_+ - \frac{\delta}{2^k}$ for any

$k \in \mathbb{N}$, we have that $u(x_k) > u(x_0) + \frac{1}{2}\tilde{\delta}$ for any $k \in \mathbb{N}$ with $k \geq (2\delta)^{-1}\tilde{\delta}$ and that $x_k \in \{u > u(x_0) + \frac{1}{2}\tilde{\delta}\}$ for large $k \in \mathbb{N}$. Hence, recalling that $x_k \to x_0$ as $k \to \infty$, we obtain that $x_0 \in \partial\{u > u(x_0) + \frac{1}{2}\tilde{\delta}\}$. In the same way, we can show that $x_0 \in \partial\{u > u(x_0) + \frac{3}{4}\tilde{\delta}\}$. Therefore, we conclude that, if u is not continuous at x_0, we can find distinct constants t, $t' \in \mathbb{R}$ such that $x_0 \in \partial\{u > t\} \cap \partial\{u > t'\}$.

Since E_t and $E_{t'}$ are the minimizers of $\mathcal{E}_{K,f,t}$ and $\mathcal{E}_{K,f,t'}$, respectively, and we assume that the boundaries of E_t and $E_{t'}$ have the $C^{1,\alpha}$-regularity with $\alpha > s$, we obtain the Euler-Lagrange equations

$$H^s_{E_t}(x) + t - f(x) = 0 \tag{4.4.2}$$

and

$$H^s_{E_{t'}}(x) + t' - f(x) = 0 \tag{4.4.3}$$

for each $x \in \partial E_t \cap \partial E_{t'}$ where H^s_E is the s-fractional mean curvature of E (see Section 2.3 of Chapter 2 for the definition). Note that the s-fractional mean curvature is well-defined on each point of the boundary if the boundary is at least of class $C^{1,\alpha}$ with $\alpha > s$ (see, for instance, [37, Corollary 3.5]). Moreover, the Euler-Lagrange equations can be derived by directly computing the first variation of $\mathcal{E}_{K,f,t}$ associated with the one-parameter family of diffeomorphisms.

Now we can readily see that, if two sets E, F satisfy that $E \subset F$ and $\partial E \cap \partial F \neq \emptyset$, then it holds that $H^s_E \geq H^s_F$ on $\partial E \cap \partial F$. Indeed, by definition, we have

$$\begin{aligned} H^s_E(x) - H^s_F(x) &= \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_E(y)}{|x-y|^{N+s}}\, dy \\ &\quad - \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{\chi_F(x) - \chi_F(y)}{|x-y|^{N+s}}\, dy \\ &= \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_F(x) - \chi_E(y) + \chi_F(y)}{|x-y|^{N+s}}\, dy \end{aligned} \tag{4.4.4}$$

for any $x \in \partial E \cap \partial F$. Since $E \subset F$, it holds $\chi_E \leq \chi_F$ in $\mathbb{R}^N$ and $\chi_E(x) = \chi_F(x)$ for any $x \in \partial E \cap \partial F$. Thus from (4.4.4) and the non-negativity of K, we obtain the claim.

Therefore, from (4.4.2), (4.4.3), and the fact that $H^s_{E_{t'}} \geq H^s_{E_t}$, we obtain

$$t' - f(x_0) \geq t - f(x_0)$$

and it turns out that $t' \geq t$. This contradicts the fact that $t' < t$. □

4.4.1 Regularity of Boundaries of Superlevel Sets of Minimizers

Now we show some regularity results of the boundary of the set $\{u > t\}$ for each t under suitable assumptions on the data f, where u is a minimizer of the functional $\mathcal{F}_{K,f}$ with $K(x) = |x|^{-(N+s)}$. From Proposition 4.3.4, we have that $u \in L^\infty(\mathbb{R}^N)$ whenever $f \in L^\infty(\mathbb{R}^N)$. Since $\{u > t\} = \mathbb{R}^N$ if $t < -\|u\|_{L^\infty}$ and $\{u > t\} = \emptyset$ if $t \geq \|u\|_{L^\infty}$, we focus on the set $\{u > t\}$ only for $t \in [-\|u\|_{L^\infty}, \|u\|_{L^\infty})$ in the sequel. Note that, from Corollary 4.3.7, the superlevel set $\{u > t\}$ is bounded uniformly in $t[-\|u\|_{L^\infty}, \|u\|_{L^\infty})$ if $f \in L^2 \cap L^\infty(\mathbb{R}^N)$.

To obtain our main result on the regularity of minimizers, we exploit the regularity results proved by A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini [56], O. Savin and E. Valdinoci [106], and B. Barrios, A. Figalli, and E. Valdinoci [9] (see also [26]). All of the results by these authors are stated in Section 2.4 of Chapter 2.

First, we recall the following two results: one is on the regularity of almost s-fractional minimal sets, which was shown by A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini

[56], and the other is on the regularity of s-fractional minimal cones in 2 dimension, which was shown by O. Savin and E. Valdinoci [106] (see Theorem 2.4.9 and Remark 2.4.10 in Section 2.4 of Chapter 2). From these results, we can obtain the following result:

Lemma 4.4.2 ($C^{1,\alpha}$-regularity of Boundary of Superlevel Sets of Minimizers)**.** *Let $f \in L^2 \cap L^\infty(\mathbb{R}^N)$. Assume that $K(x) = |x|^{-(N+s)}$ for $x \in \mathbb{R}^N \setminus \{0\}$ with $s \in (0, 1)$ and $u \in BV_K \cap L^2(\mathbb{R}^N)$ is a minimizer of the functional $\mathcal{F}_{K,f}$. Then, for each $t \in [0, \infty)$, the boundary of the superlevel set $\{u > t\}$ is of class $C^{1,\alpha}$ with some $0 < \alpha < 1$, except a closed set of Hausdorff dimension $N - 3$.*

Proof. We fix $t \in \mathbb{R}$. Let $x_0 \in \partial\{u > t\}$ and $r > 0$ be any number. First, from the assumption on f, Proposition 4.3.4, and Corollary 4.3.7, there exists a constant $R_0 > 0$ such that $E_t := \{u > t\} \subset B_{\frac{R_0}{2}}$ for any $t \geq 0$. In order to apply the regularity result to our case, it is sufficient to show that the set E_t is an almost s-fractional minimal sets in B_{R_0} in the sense of Definition 2.4.6 for some constant $\Lambda > 0$. Note that this also indicates that E_0 is a minimizer in the sense of Theorem 2.4.8. From Lemma 4.3.5, we know that $\{u > t\}$ is a solution to the problem

$$\inf\{\mathcal{E}_{K,f,t}(E) \mid |E| < \infty\}$$

for each $t \in \mathbb{R}$. Hence, from the minimality and boundedness of E_t, we have that

$$\mathcal{E}_{K,f,t}(E_t) \leq \mathcal{E}_{K,f,t}(F) \tag{4.4.5}$$

for any $F \subset \mathbb{R}^N$ and $E_t \triangle F \subset B_r(x_0)$. From the definition of $P_s(\cdot\,; B_{R_0})$ and the fact that $E_t \subset B_{\frac{R_0}{2}}$, we have the identity that $P_K(E_t; B_{R_0}) = P_K(E_t)$. Hence, from (4.4.5), we can compute as follows: for any set F and $r > 0$ with $E_t \triangle F \subset B_r(x_0) \subset B_{\frac{R_0}{2}}$, we have

$$\begin{aligned} P_K(E_t; B_{R_t}) - P_K(F; B_{R_t}) &= \mathcal{E}_{K,f,t}(E_t) - \int_{E_t} (t - f(x))\, dx \\ &\quad - \mathcal{E}_{K,f,t}(F) + \int_F (t - f(x)\, dx \\ &\leq \int_{\mathbb{R}^N} |\chi_{E_t} - \chi_F|\, |t - f(x)|\, dx \\ &\leq \int_{B_r(x_0)} |t - f(x)|\, dx. \end{aligned} \tag{4.4.6}$$

Since we assume that $f \in L^\infty(\mathbb{R}^N)$, we have

$$\int_{B_r(x_0)} |t - f(x)|\, dx \leq (t + \|f\|_{L^\infty(\mathbb{R}^N)})\, |E_t \Delta F|. \tag{4.4.7}$$

Hence, from (4.4.6) and (4.4.7), we have

$$P_K(E_t; B_{R_0}) \leq P_K(F; B_{R_0}) + (t + \|f\|_{L^\infty(\mathbb{R}^N)})\, |E_t \Delta F|$$

or equivalently, since $E_t \cup F \subset B_{\frac{R_0}{2}}$, we have

$$P_K(E_t) \leq P_K(F) + (t + \|f\|_{L^\infty(\mathbb{R}^N)})\, |E_t \Delta F|$$

for any $F \subset \mathbb{R}^N$ with $E_t \triangle F \subset B_r(x_0)$. Therefore, we apply the regularity of the s-fractional minimal sets and s-fractional minimal cones in Theorem 2.4.9 and Remark 2.4.10 in Section 2.4 of Chapter 2 to conclude that the claim is valid. ☐

In the previous proof, we have shown that, if $u \in BV_K \cap L^2(\mathbb{R}^N)$ is a minimizer of $\mathcal{F}_{K,f}$, then each superlevel sets $\{u > t\}$ is not only a minimizer of $\mathcal{E}_{K,f,t}$ but also an almost s-fractional minimal set in a ball. Hence, recalling Lemma 2.4.11 in Section 2.4 of Chapter 2, one can obtain the density estimate of each superlevel set $\{u > t\}$ as follows:

Lemma 4.4.3 (Density Estimates of Each Superlevel Set)**.** *Let $t > 0$ and $f \in L^\infty(\mathbb{R}^N)$. Assume that the kernel $K(x) = |x|^{-(N+s)}$ for $x \in \mathbb{R}^N \setminus \{0\}$ with $s \in (0, 1)$ and $u \in BV_K(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is a minimizer of $\mathcal{F}_{K,f}$. Then $E_t := \{u > t\}$ satisfies the following condition: there exist constants $c_0 \in (0, 1)$ and $r_0 \in (0, 1)$, independent of t, such that we have*

$$|B_1|(1 - c_0)\, r^N \geq |E_t \cap B_r(x)| \geq c_0 |B_1|\, r^N$$

for any $r \in (0, r_0)$ and $x \in \partial E_t$ such that $|E_t \cap B_r(x)| > 0$ and $|(E_t)^c \cap B_r(x)| > 0$ for any $r > 0$.

The reason why the constants c_0 and r_0 are independent of t is that we have the finiteness of $\|u\|_{L^\infty}$ from the assumption that $f \in L^\infty(\mathbb{R}^N)$.

We now exploit another result of the regularity of solutions to integro-differential equations via the bootstrap argument. This result is obtained by B. Barrios, A. Figalli, and E. Valdinoci [9]. The authors proved the following regularity theorem on the solutions to a certain integro-differential equation. For simplicity, we do not describe the whole statement. See [9, Theorem 1.6] for the full statement.

Theorem 4.4.4 ([9])**.** *Let $v \in L^\infty(\mathbb{R}^{N-1})$ be a solution (in the viscosity sense) to the integro-differential equation*

$$\int_{\mathbb{R}^{N-1}} A_r(x', y') \left(v(x' + y') + v(x' - y') - 2v(x') \right) dy' = F(x')$$

for any $x' \in B'_r(0) \subset \mathbb{R}^{N-1}$ where A_r satisfies the following assumptions:

(A1) There exist constants a_0, $r_0 > 0$ and $\eta \in (0, \frac{a_0}{4})$ such that

$$\frac{(1-s)(a_0 - \eta)}{|y'|^{N+s}} \leq A_r(x', y') \leq \frac{(1-s)(a_0 + \eta)}{|y'|^{N+s}}$$

for any $x' \in B'_r(0)$ and $y' \in B'_{r_0}(0) \setminus \{0\}$.

(A2) There exists a constant $C_0 > 0$ such that

$$\|A_r(\cdot, y')\|_{C^{0,\beta}(B'_1)} \leq \frac{C_0}{|y'|^{N+s}}$$

for any $y' \in B'_{r_0}(0) \setminus \{0\}$.

and $F \in C^{0,\beta}(B'_r(0))$ with $\beta \in (0, 1]$. Then, we have that $v \in C^{1,s+\alpha}(B'_{\frac{r}{2}}(0))$ for any $\alpha < \beta$.

Taking into account all the above arguments, we can obtain that the boundary of the superlevel set of the minimizers of $\mathcal{F}_{K,f}$ has the $C^{2,s+\beta-1}$-regularity under the β-Hölder regularity of a given data f with $\beta \in (1-s, 1]$. Precisely, we prove

Lemma 4.4.5. *Let $f \in L^2 \cap L^\infty(\mathbb{R}^N)$. Assume that $K(x) = |x|^{-(N+s)}$ for $x \in \mathbb{R}^N \setminus \{0\}$ with $s \in (0, 1)$ and $u \in BV_K \cap L^2(\mathbb{R}^N)$ is a minimizer of the functional $\mathcal{F}_{K,f}$. If a data f is in $C^{0,\beta}_{loc}(\mathbb{R}^N)$ with $\beta \in (1-s, 1]$, then, for each $t \in [-\|u\|_{L^\infty}, \|u\|_{L^\infty})$, the boundary of the superlevel set $\{u > t\}$ is of class $C^{2,s+\delta-1}$ with some $1 - s < \delta < \beta$ except a closed set of Hausdorff dimension $N - 3$.*

Proof. Recalling Proposition 4.3.4, we first observe that the assumption that $f \in L^\infty(\mathbb{R}^N)$ implies that $u \in L^\infty(\mathbb{R}^N)$. Since minimizers $\{E_t = \{u > t\}\}_t$ are bounded uniformly in $t \geq -\|u\|_{L^\infty}$ due to Corollary 4.3.7, we also observe that each E_t is also an almost s-fractional minimizer of P_s in B_{R_c} where $R_c > 0$ is the constant in Corollary 4.3.7.

Now by closely following the argument in [22, Theorem 5.1] and [26, Theorem 5.3] (see Appendix B for the precise argument), we can obtain the Euler-Lagrange equation

$$H^s_{E_t} + t - f = 0 \quad \text{on } \partial E_t \tag{4.4.8}$$

in the viscosity sense. We note that the equation (4.4.8) holds in the viscosity sense if (4.4.8) is valid at any $x \in \partial E_t$ where ∂E_t has both interior and exterior tangential balls. See also Chapter 2.

From Lemma 4.4.2 and the assumption that $f \in L^\infty(\mathbb{R}^N)$, the boundary of the super-level set $\{u > t\}$ of the minimizer u has the $C^{1,\gamma}$-regularity with some $\gamma \in (0,\, 1)$ except a closed set Σ of Hausdorff dimension $N-3$, and thus we can represent the boundary $\{u > t\} \setminus \Sigma$ locally as the graph of a $C^{1,\gamma}$-function (denoted by v_t).

Now, in the similar way to [9, Section 3], we show that the graph function v_t of ∂E_t satisfies the following integro-differential equation in the viscosity sense:

$$\int_{\mathbb{R}^{N-1}} A_r(x',\, y') \frac{v(x'+y') + v(x'-y') - 2v(x')}{|y'-x'|^{(n-1)+(1+s)}}\, dy' = F(x',\, v_t(x')) + t - f(x',\, v_t(x'))$$

locally on ∂E_t, where F and A_r are "suitable" functions. To see this, we first take any boundary point $p = (p',\, p_N) \in \partial E_t$, and choose an open cylinder $C_r(p) := B'_r(p') \times (p_N - r,\, p_N + r) \subset \mathbb{R}^{N-1} \times \mathbb{R}$ with some $r \in (0,\, 1)$ in such a way that, up to a coordinate,

$$\partial E_t \cap C_r(p) = \{(y',\, y_N) \mid y_N = v_t(y'),\, y' \in B'_r(x')\}, \quad \nabla' v_t(p') = 0$$

where we mean by ∇' the gradient in $\mathbb{R}^{N-1}$. We may also assume that $\|\nabla' v_t\|_{L^\infty(B'_r(p'))} \leq 1$. In terms of the graph function v_t, we can rewrite the nonlocal mean curvature $H^s_{E_t}$ in the following manner: first we choose a smooth function $\xi_r : [0,\, \infty) \to [0,\, 1]$ such that

$$\xi_r \equiv 1 \quad \text{in } \left[0,\, \frac{r}{8}\right), \quad \xi_r \equiv 0 \quad \text{in } \left[\frac{r}{4},\, \infty\right).$$

Then, for any $x \in \partial E_t \cap C_r(p)$, we have the following decomposition of the nonlocal mean curvature.

$$\begin{aligned} H^s_{E_t}(x) &= \int_{\mathbb{R}^N} \xi_r(|y'-x'|)\, \xi_r(|y_N - x_N|)\, \frac{\chi_{E^c_t}(y) - \chi_{E_t}(y)}{|y-x|^{N+s}}\, dy \\ &\quad + \int_{\mathbb{R}^N} (1 - \xi_r(|y'-x'|)\, \xi_r(|y_N - x_N|))\, \frac{\chi_{E^c_t}(y) - \chi_{E_t}(y)}{|y-x|^{N+s}}\, dy \\ &=: S_r(x) + \Psi^{v_t}_r(x') \end{aligned} \tag{4.4.9}$$

Notice that, due to the choice of the function ξ, we may easily observe that $\Psi^{v_t}_r \in C^\infty(B'_r(p'))$. Moreover, we have that

$$S_r(x) = 2 \int_{\mathbb{R}^{N-1}} G\left(\frac{v_t(x') - v_t(x'-y')}{|y'|}\right) \frac{\xi_r(|y'|)}{|y'|^{N-1+s}}\, dy' \tag{4.4.10}$$

for any $x \in \partial E_t \cap C_r(p)$ where we set

$$G(t) := \int_0^t \frac{1}{(1+\tau^2)^{\frac{N+s}{2}}}\, d\tau.$$

Indeed, we can prove this identity by doing the similar computation conducted in [9]. By the change of variables, we have

$$S_r(x) = 2\int_{\mathbb{R}^{N-1}} G\left(\frac{v_t(x') - v_t(x'+y')}{|y'|}\right) \frac{\xi_r(|y'|)}{|y'|^{N-1+s}}\,dy'. \tag{4.4.11}$$

By combining (4.4.9), (4.4.10), and (4.4.11) with (4.4.8), we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}} \left(G\left(\frac{v_t(x') - v_t(x'-y')}{|y'|}\right) + G\left(\frac{v_t(x') - v_t(x'+y')}{|y'|}\right)\right) \frac{\xi_r(|y'|)}{|y'|^{N-1+s}}\,dy' \\ &= f(x',\, v_t(x')) - t - \Psi_r^{v_t}(x',\, v_t(x'))\end{aligned} \tag{4.4.12}$$

for any $x \in \partial E_t \cap C_r(p)$. From the fundamental theorem of calculus and the fact that $G(-t) = -G(t)$ for $t > 0$, we have that

$$\begin{aligned}&G\left(\frac{v_t(x') - v_t(x'-y')}{|y'|}\right) - G\left(-\frac{v_t(x') - v_t(x'+y')}{|y'|}\right) \\ &= \int_0^1 G'\left(\lambda\,\frac{v_t(x') - v_t(x'-y')}{|y'|} + (\lambda-1)\,\frac{v_t(x') - v_t(x'+y')}{|y'|}\right) d\lambda \\ &\quad \times \frac{2v_t(x') - v_t(x'-y') - v_t(x'+y')}{|y'|}.\end{aligned} \tag{4.4.13}$$

Thus, from (4.4.12) and (4.4.13) and by setting $\mathcal{A}_r$ as

$$\mathcal{A}_r(x',\, y') := \frac{\xi_r(|y'|)}{|y'|^{N+s}} \int_0^1 G'\left(\lambda\,\frac{v_t(x') - v_t(x'-y')}{|y'|} + (\lambda-1)\,\frac{v_t(x') - v_t(x'+y')}{|y'|}\right) d\lambda$$

for any $x',\, y' \in B'_r(p')$, we finally obtain the equation

$$\int_{\mathbb{R}^{N-1}} \mathcal{A}_r(x',\, y')\,(2v_t(x') - v_t(x'-y') - v_t(x'+y')), dy' = f(x',\, v_t(x')) - t - \Psi_r^{v_t}(x') \tag{4.4.14}$$

in the viscosity sense.

We now claim that $|y'|^{N+s}\mathcal{A}_r(\cdot,\, y') \in C^{0,\gamma}(B'_r(p'))$ for any $y' \in B'_r(p')$ where γ is the same Hölder exponent of the graph function v_t. Indeed, by setting $p_{v_t}(\lambda, x', y')$ as

$$p_{v_t}(\lambda, x', y') := \lambda\,\frac{v_t(x') - v_t(x'-y')}{|y'|} + (\lambda-1)\,\frac{v_t(x') - v_t(x'+y')}{|y'|},$$

we have that

$$\mathcal{A}_r(x',\, y') = \frac{\xi_r(|y'|)}{|y'|^{N+s}} \int_0^1 \frac{1}{(1 + p_{v_t}(\lambda, x', y')^2)^{\frac{N+s}{2}}}\,d\lambda. \tag{4.4.15}$$

By using the assumption that $v_t \in C^{1,\gamma}(B'_r(p'))$, we can derive the two estimates that

$$|p_{v_t}(\lambda, x', y')| \leq 2\|\nabla' v_t\|_{L^\infty(B'_r(p'))} < \infty,$$

and

$$\begin{aligned}&\left|\left(1 + p_{v_t}(\lambda, x', y')^2\right)^{\frac{N+s}{2}} - \left(1 + p_{v_t}(\lambda, z', y')^2\right)^{\frac{N+s}{2}}\right| \\ &\leq 2(N+s)(1 + 4\|\nabla' v_t\|^2_{L^\infty(B'_r(p'))})^{\frac{N-1+s}{2}} \|\nabla' v_t\|_{L^\infty(B'_r(p'))}\,|x' - z'|^\gamma\end{aligned} \tag{4.4.16}$$

for any $x',\, y',\, z' \in B'_r(p')$ and $\lambda \in [0,\, 1]$. From the definition of $\mathcal{A}_r$, the assumption that $\|\nabla' v_t\|_{L^\infty(B'_r(p'))} \leq 1$, (4.4.15), and (4.4.16), we have that

$$|\mathcal{A}_r(x',\, y') - \mathcal{A}_r(z',\, y')| \leq C\,|x' - z'|^\gamma\,\frac{1}{|y'|^{N+s}} \tag{4.4.17}$$

for any x', y', $z' \in B'_r(p')$ and some constant $C > 0$ depending only on N, s, and t. Therefore, we obtain that $\mathcal{A}_r$ satisfies the condition $(A2)$ with $\beta := \gamma$ in Theorem 4.4.4.

Now we confirm that $\mathcal{A}_r$ also satisfies the condition $(A1)$ in Theorem 4.4.4. Indeed, from (4.4.15), we have that

$$\frac{1}{|y'|^{N+s}} \geq \mathcal{A}_r(x', y') \geq \frac{1}{(1 + 4\|\nabla' v_t\|^2_{L^\infty(B'_r)})^{\frac{N+s}{2}}} \frac{1}{|y'|^{N+s}} \tag{4.4.18}$$

for any $x' \in B'_{r/2}(0)$ and $y' \in B'_{r/8}(0)$, and, by choosing $r > 0$ small if necessary, we conclude that $\mathcal{A}_r$ satisfies $(A1)$.

If $\gamma \geq \beta(> 1 - s)$, then we have that $|y'|^{N+s}\mathcal{A}_r(\cdot, y') \in C^{0,\beta}(B'_r(p'))$ for $y' \in B'_r(p')$ and thus, we apply the bootstrap argument in [9, Theorem 1.6] to obtain that v_t is of class $C^{1+s+\alpha}$ for any $\alpha < \beta$. Sine the Hölder exponent β is greater than $1 - s$, we further obtain that v_t is of class $C^{2,s+\alpha-1}$ with $1 - s < \alpha < \beta$.

If $\gamma < \beta$, then, from the fact that $f \in C^{0,\beta}_{loc} \subset C^{0,\gamma}_{loc}$, we can apply the bootstrap argument in [9, Theorem 1.6] to obtain that v_t is of class $C^{1,s+\alpha}$ for any $\alpha < \gamma$. If $s+\gamma \geq \beta(> 1-s)$, then we further apply the bootstrap argument to obtain the $C^{2,s+\alpha-1}$-regularity of the graph function v_t with some $1-s < \alpha < \beta$; otherwise, we again apply the bootstrap argument for $f \in C^{0,s+\alpha}$ and $|y'|^{N+s}\mathcal{A}_r(\cdot, y') \in C^{0,s+\alpha}$ with any $\alpha < \gamma$ to obtain that v_t is of class $C^{1+s+\alpha'}$ with $\alpha' < s+\gamma$. If $2s+\gamma \geq \beta(1-s)$, then we are done; otherwise, we repeat the above argument finite times until we obtain the $C^{2,s+\delta-1}$-regularity for some $1 - s < \delta < \beta$.

Therefore, we conclude the proof. □

4.4.2 Proof of Theorem 4.1.1

By using Lemma 4.4.5, we are now ready to prove the main result of this chapter.

Before proceeding with the proof, we briefly explain the strategy of the proof of Theorem 3.1.1. Let t_1, $t_2 \in [-\|u\|_{L^\infty}, \|u\|_{L^\infty})$ with $t_1 < t_2$ and we set $E_1 := \{u > t_1\}$ and $E_2 := \{u > t_2\}$. We show that the boundaries of E_1 and E_2 are not too close. Precisely, using the regularity of $f \in C^{0,\beta}$, we show the inequality that

$$t_2 - t_1 \lesssim (\mathrm{dist}\,(\partial E_1, \partial E_2))^\beta\,. \tag{4.4.19}$$

To see this, we compare the nonlocal mean curvatures on the boundaries ∂E_1 and ∂E_2. Notice that one cannot directly compare the curvatures unless the boundaries share some point with each other. Thus, we slide ∂E_1 (denoted by ∂E_1^ν) along the outer unit normal ν of ∂E_1 until ∂E_1^ν touches ∂E_2. At the touching point, we can now compare the curvatures between ∂E_1^ν and ∂E_2. Moreover, by employing the computation by J. Dávila, M. del Pino, and J. Wei [41], we can also compare the curvatures between ∂E_1 and ∂E_1^ν. Finally, using (4.4.19), we seek for the estimate of $|u(y_1) - u(y_2)|$ for $y_i \in \partial E_i$ with $i \in \{1, 2\}$ to conclude the proof of our main theorem.

Proof of Theorem 4.1.1. Let $d_t := d_{E_t}$ for $t \in [0, \infty)$ be a signed distance function from $\partial\{u > t\}$, which is negative inside $\{u > t\}$. We set $E_t := \{x \mid u(x) > t\}$ for any t. Since $N = 2$, from Lemma 4.4.5 it follows that the surface ∂E_t is $C^{2,\delta}$-regular for each $t \in \mathbb{R}$ with some $\delta > 0$. Hence, the signed distance function d_t is of class $C^{2,\delta}$ in a neighborhood of ∂E_t with some $\delta > 0$ (see, for instance, [107, 42, 43, 10] for the relation between the distance function and regularity of surfaces).

We first recall that, from the assumption on f and Proposition 4.3.4, we have that $\|u\|_{L^\infty} \leq \|f\|_{L^\infty} < \infty$. We now take any $t_1 \in [-\|u\|_{L^\infty}, \|u\|_{L^\infty})$ and set $E_1 := E_{t_1}$. Then we can choose a neighborhood $U_1 \subset \mathbb{R}^2$ of the boundary ∂E_1 such that $d_1 :=$

$d_{t_1} \in C^{2,s+\alpha-1}(U_1)$. Moreover, we take any $t_2 \in (-\|u\|_{L^\infty}, \|u\|_{L^\infty})$ with $t_2 > t_1$ and set $E_2 := E_{t_2}$. Then, from Corollary 4.3.7, we obtain that there exists a constant $R_c > 0$ independent of t_1 and t_2 such that $E_2 \subset E_1 \subset B_{R_c}$. We can choose points $x_1 \in \partial E_1$ and $x_2 \in \partial E_2$ such that

$$\tilde{\delta} := \operatorname{dist}(\partial E_1, \partial E_2) = |x_1 - x_2|.$$

Note that, since we study the local Hölder regularity of u, it is sufficient to consider the case that $x_2 \in U_1$.

Step 1. We first show that the following inequality holds:

$$t_2 - t_1 \leq ([f]_\beta + C\,\tilde{\delta}^{1-\beta})\,\tilde{\delta}^\beta \tag{4.4.20}$$

where $C > 0$ is a constant depending only on s and d_1.

Without loss of generality, we may assume that $\widetilde{\delta} > 0$. Indeed, if $\widetilde{\delta} = 0$, then, from the definition of $\widetilde{\delta}$, we can easily see that $t_2 = t_1$. This implies that the inequality (4.4.20) is valid. Thus, in the sequel, we always assume that $\tilde{\delta} > 0$.

Now we define E_1^δ as

$$E_1^\delta := \{x \in E_1 \mid \operatorname{dist}(x, \partial E_1) \leq \delta\}$$

for any $\delta \in (0, \tilde{\delta}]$. Then, from the choice of t_2 and the definition of $\widetilde{\delta}$, the boundary of E_1^δ can be described as $\partial E_1^\delta = \{x - \delta \nabla d_1(x) \mid x \in \partial E_1\}$ for any $\delta \in (0, \widetilde{\delta}]$ where ∇d_1 is the outer unit normal vector of ∂E_1. Note that we can readily see that $E_2 \subset E_1^\delta$ and $x_2 \in \partial E_2 \cap \partial E_1^{\tilde{\delta}}$. From the definition of the nonlocal mean curvature, we can easily see that the following comparison inequality holds:

$$H^K_{E_1^{\tilde{\delta}}}(x_2) \leq H^K_{E_2}(x_2). \tag{4.4.21}$$

From the choice of x_1 and x_2, we have $x_2 = x_1 - \delta\,\nabla d_1(x_1)$. Now we compare the two nonlocal curvatures $H^K_{E_1^\delta}(x_2)$ and $H^K_{E_1}(x_1)$. To do this, we employ the computation shown by Dávila, del Pino, and Wei in [41] (see also [37, 70]). This computation is on the variation of the s-fractional mean curvature. Precisely, we have that, for any set $E \subset \mathbb{R}^2$ with a smooth boundary (at least $C^{1,\alpha}$ with $\alpha > \frac{1+s}{2}$), it holds that

$$\begin{aligned}
&- \frac{d}{d\delta}\bigg|_{\delta=0} H^K_{E_{\delta h}}(x - \delta h(x) \nabla d_E(x)) \\
&= 2\int_{\partial E} \frac{h(y) - h(x)}{|y - x|^{2+s}}\, d\mathcal{H}^{N-1}(y) \\
&\quad + 2\int_{\partial E} \frac{(\nabla d_E(y) - \nabla d_E(x)) \cdot \nabla d_E(x)}{|y - x|^{2+s}}\, d\mathcal{H}^{N-1}(y)
\end{aligned} \tag{4.4.22}$$

for $x \in \partial E$ where $h \in L^\infty(\partial E)$ and h is as smooth as ∂E. Here we define $E_{\delta h}$ in such a way that its boundary is given by $\partial E_{\delta h} := \{x - \delta\, h(x)\, \nabla d_E(x) \mid x \in \partial E\}$ for any $\delta > 0$. Then from (4.4.22) and by some computation, we have the estimate of the variation of the nonlocal mean curvature $H^s_{E_1^\delta}$ for small $\delta > 0$. Precisely we can obtain that there exist constants $C > 0$ and $\delta_0 > 0$, which depends on the dimension of $\mathbb{R}^N$ $N = 2$, s, and the L^∞-norm of $\nabla^2 d_1$ (equivalently the second fundamental form of ∂E_1), such that

$$- \frac{d}{d\delta} H^K_{E_1^\delta}(\Psi_\delta(x_1)) \leq C \int_{\partial E_1} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{N-1}(y) \tag{4.4.23}$$

for any $\delta \in (0, \delta_0)$ where we set $\Psi_\delta(x_1) \coloneqq x - \delta \nabla d_1(x)$. Indeed, choosing any smooth cut-off function η_ε such that $\operatorname{spt}\eta_\varepsilon \subset B^c_\varepsilon(0)$, $\eta_\varepsilon \equiv 1$ in $B^c_{2\varepsilon}(0)$, and $0 \leq \eta_\varepsilon \leq 1$, we can write the nonlocal curvature as follows:

$$\begin{aligned} &- H^s_{E_1^\delta}(\Psi_\delta(x_1)) \\ &= \int_{\mathbb{R}^2} \frac{\chi_{E_1^\delta}(y) - \chi_{(E_1^\delta)^c}(y)}{|y - \Psi_\delta(x_1)|^{2+s}} \eta_\varepsilon(y - \Psi_\delta(x_1))\, dy \\ &\quad + \int_{\mathbb{R}^2} \frac{\chi_{E_1^\delta}(y) - \chi_{(E_1^\delta)^c}(y)}{|y - \Psi_\delta(x_1)|^{2+s}} (1 - \eta_\varepsilon(y - \Psi_\delta(x_1)))\, dy \\ &=: A_\varepsilon(\delta) + B_\varepsilon(\delta). \end{aligned} \tag{4.4.24}$$

Then we can compute the derivative of $A_\varepsilon(\delta)$ in (4.4.24) for small $\delta > 0$ in the following manner: setting $\widetilde{y}_\delta \coloneqq y - \Psi_\delta(x_1)$ for simplicity, we have

$$\begin{aligned} &\frac{d}{d\delta} \int_{\mathbb{R}^2} \frac{\chi_{E_1^\delta}(y) - \chi_{(E_1^\delta)^c}(y)}{|\widetilde{y}_\delta|^{2+s}} \eta_\varepsilon(\widetilde{y}_\delta)\, dy \\ &= \int_{\partial E_1^\delta} \frac{\eta_\varepsilon(\widetilde{y}_\delta)}{|\widetilde{y}_\delta|^{2+s}}\, d\mathcal{H}^{N-1}(y) + \int_{\partial (E_1^\delta)^c} \frac{\eta_\varepsilon(\widetilde{y}_\delta)}{|\widetilde{y}_\delta|^{2+s}}\, d\mathcal{H}^{N-1}(y) \\ &\quad - (2+s) \int_{\mathbb{R}^2} \frac{\chi_{E_1^\delta}(y) - \chi_{(E_1^\delta)^c}(y)}{|\widetilde{y}_\delta|^{4+s}} (y - x_1 + \delta \nabla d_1(x_1)) \cdot \nabla d_1(x_1)\, \eta_\varepsilon(\widetilde{y}_\delta)\, dy \\ &\qquad + \int_{\mathbb{R}^2} \frac{\chi_{E_1^\delta}(y) - \chi_{(E_1^\delta)^c}(y)}{|\widetilde{y}_\delta|^{2+s}} \nabla \eta_\varepsilon(\widetilde{y}_\delta) \cdot \nabla d_1(x_1)\, dy \end{aligned} \tag{4.4.25}$$

for any $\delta \in (0, 1)$ with $\Psi_\delta(x_1) \in U_1$. Then by using the Gauss-Green theorem, we have

$$\begin{aligned} &- (2+s) \int_{\mathbb{R}^2} \frac{\chi_{E_1^\delta}(y) - \chi_{(E_1^\delta)^c}(y)}{|\widetilde{y}_\delta|^{4+s}} (y - x_1 + \delta \nabla d_1(x_1)) \cdot \nabla d_1(x_1)\, \eta_\varepsilon(\widetilde{y}_\delta)\, dy \\ &= \int_{\mathbb{R}^2} (\chi_{E_1^\delta}(y) - \chi_{(E_1^\delta)^c}(y)) \nabla_y \left(\frac{1}{|\widetilde{y}_\delta|^{2+s}} \right) \cdot \nabla d_1(x_1)\, \eta_\varepsilon(\widetilde{y}_\delta)\, dy \\ &= \int_{\partial E_1^\delta} \frac{\nabla d_1(x_1) \cdot \nabla d_{E_1^\delta}(y)}{|\widetilde{y}_\delta|^{2+s}} \eta_\varepsilon(\widetilde{y}_\delta)\, d\mathcal{H}^{N-1} \\ &\quad - \int_{\partial (E_1^\delta)^c} \frac{\nabla d_1(x_1) \cdot (-\nabla d_{E_1^\delta}(y))}{|\widetilde{y}_\delta|^{2+s}} \eta_\varepsilon(\widetilde{y}_\delta)\, d\mathcal{H}^{N-1} \\ &\quad - \int_{\mathbb{R}^2} \frac{\chi_{E_1^\delta}(y) - \chi_{(E_1^\delta)^c}(y)}{|\widetilde{y}_\delta|^{2+s}} \nabla \eta_\varepsilon(\widetilde{y}_\delta) \cdot \nabla d_1(x_1)\, dy. \end{aligned} \tag{4.4.26}$$

Thus from (4.4.25) and (4.4.26), we obtain

$$\begin{aligned} \frac{d}{d\delta} A_\varepsilon(\delta) &= \int_{\partial E_1^\delta} \frac{2 - 2(\nabla d_1(x_1) \cdot \nabla d_{E_1^\delta}(y))}{|\widetilde{y}_\delta|^{2+s}} \eta_\varepsilon(\widetilde{y}_\delta)\, d\mathcal{H}^{N-1}(y) \\ &= \int_{\partial E_1^\delta} \frac{|\nabla d_1(x_1) - \nabla d_{E_1^\delta}(y)|^2}{|\widetilde{y}_\delta|^{2+s}} \eta_\varepsilon(\widetilde{y}_\delta)\, d\mathcal{H}^{N-1}(y) \end{aligned}$$

for any small $\delta > 0$ with $\Psi_\delta(x_1) \in U_1$. Hence from the change of variables, we obtain

$$\frac{d}{d\delta} A_\varepsilon(\delta) = \int_{\partial E_1} \frac{|\nabla d_1(x_1) - \nabla d_1(y)|^2}{|\Psi_\delta(y) - \Psi_\delta(x_1)|^{2+s}} \eta_\varepsilon(\Psi_\delta(y) - \Psi_\delta(x_1))\, J_{\partial E_1} \Psi_\delta(y)\, d\mathcal{H}^{N-1}(y)$$

where $J_{\partial E_1}\Psi_\delta(y)$ is the tangential Jacobian of ∂E_1 at y. As is shown in [41], we can have that there exist constants $c' > 0$ and $\delta' > 0$, depending on the dimension of $\mathbb{R}^N$ with $N = 2$ and s but independent of $\varepsilon > 0$, such that $|\frac{d}{d\delta}B_\varepsilon(\delta)| \le c'\varepsilon^{1-s}$ for any $\delta \in (0,\, \delta')$ and $\varepsilon \in (0,\, 1)$. Therefore, we conclude that

$$\begin{aligned}
-\frac{d}{d\delta}H^s_{E_1^\delta}(\Psi_\delta(x_1)) &= \lim_{\varepsilon\downarrow 0}\left(\frac{d}{d\delta}A_\varepsilon(\delta) + \frac{d}{d\delta}B_\varepsilon(\delta)\right) \\
&= \int_{\partial E_1} \frac{|\nabla d_1(x_1) - \nabla d_1(y)|^2}{|\Psi_\delta(y) - \Psi_\delta(x_1)|^{2+s}} J_{\partial E_1}\Psi_\delta(y)\, d\mathcal{H}^{N-1}(y)
\end{aligned}$$

for any $\delta \in (0,\, \delta_0')$ where $\delta_0' > 0$ is a constant depending on the dimension of $\mathbb{R}^N$ with $N = 2$, s, and the L^∞-norm of $\nabla^2 d_1$. From the definition of Ψ_δ, we have that there exists a constant $C_0 > 0$ depending on the dimension of $\mathbb{R}^N$ with $N = 2$, s, and the L^∞-norm of $\nabla^2 d_1$, such that

$$\frac{J_{\partial E_1}\Psi_\delta(y)}{|\Psi_\delta(y) - \Psi_\delta(x_1)|^{2+s}} \le \frac{C_0}{|y - x_1|^{2+s}}$$

for any $y \in \partial E_1$ and $\delta \in (0,\, \delta_0')$. Therefore we obtain that there exist constants $C > 0$ and $\delta_0 > 0$, depending on the dimension of $\mathbb{R}^N$ $N = 2$, s, and the second derivative of d_1 but independent of δ, such that the inequality (4.4.23) with the constant C holds for any $\delta \in (0,\, \delta_0)$. Thus, from the fundamental theorem of calculus and (4.4.23), we obtain that

$$\begin{aligned}
-H^K_{E_1^\delta}(x - \delta\,\nabla d_1(x)) &= -H^K_{E_1}(x_1) - \delta \int_0^1 \frac{d}{d\delta}H^K_{E_1^\delta}(x - \lambda\delta\,\nabla d_1(x))\, d\lambda \\
&\le -H^K_{E_1}(x_1) + C\,\delta \int_{\partial E_1} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{N-1}(y) \qquad (4.4.27)
\end{aligned}$$

for any $\delta \in (0,\, \delta_0)$. Now we show that the integral

$$\int_{\partial E_1} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{N-1}(y)$$

is uniformly bounded for any $x_1 \in V$ and any open set $V \subsetneq U_1$. Indeed, we define the set $U_1^r := \{x \in U_1 \mid \mathrm{dist}\,(x,\, \partial U_1) > r\}$ for any $r > 0$ satisfying that $B_{2r}(x) \subset U_1$ for any $x \in U_1$. Then we can compute the integral as follows: for any $x_1 \in U_1^r$, it holds that

$$\begin{aligned}
&\int_{\partial E_1} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{N-1}(y) \\
&= \int_{\partial E_1 \cap B_r(x_1)} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{N-1}(y) \\
&\quad + \int_{\partial E_1 \cap B_r^c(x_1)} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}}\, d\mathcal{H}^{N-1}(y) \\
&\le \int_{\partial E_1 \cap B_r(x_1)} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^2} \frac{1}{|y - x_1|^{n-2+s}}\, d\mathcal{H}^{N-1}(y) \\
&\quad + \int_{\partial E_1 \cap B_r^c(x_1)} \frac{4}{|y - x_1|^{2+s}}\, d\mathcal{H}^{N-1}(y). \qquad (4.4.28)
\end{aligned}$$

From the fundamental theorem of calculus and the fact that $B_r(x_1) \subset U_1$ for any $x_1 \in U_1^r$, we have that

$$\frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^2} \le \|\nabla^2 d_1\|^2_{L^\infty(B_r(x_1))} \qquad (4.4.29)$$

for any $y \in B_r(x_1)$. Thus from (4.4.28) and (4.4.29) and noticing that $x_1 \in U_1^r$ and $E_t \subset B_{R_c}$ holds uniformly in $t \geq c$ where $c := -\|u\|_{L^\infty} > -\infty$, we obtain

$$\int_{\partial E_1} \frac{|\nabla d_1(y) - \nabla d_1(x_1)|^2}{|y - x_1|^{2+s}} \, d\mathcal{H}^{N-1}(y) \leq c_1 \, \|\nabla^2 d_1\|^3_{L^\infty(B_r(x_1))} \, r^{1-s} + \frac{c_2 \, \|\nabla^2 d_1\|_{L^\infty(U_1)}}{r^s} \tag{4.4.30}$$

where $c_1 > 0$ and $c_2 > 0$ are constants depending on the dimension of $\mathbb{R}^N$ with $N = 2$ and s. Since we choose any r in such a way that $B_r(x_1) \subset U_1$, we conclude the claim is valid. Thus, from (4.4.27) and (4.4.30), we finally obtain the inequality

$$- H^K_{E_1^\delta}(x_1 - \delta \, \nabla d_1(x)) \leq -H^K_{E_1}(x_1) + C(n, s, R_c) \, \delta \tag{4.4.31}$$

for any $\delta \in (0, \, \delta_0)$ where $C(n, s, R_c) > 0$ ($N = 2$ is the dimension of $\mathbb{R}^N$) and $\delta_0 > 0$ are some constants, which also depend on the L^∞-norm of $\nabla^2 d_1$. Note that the constant δ_0 can be bounded by the inverse of the L^∞-norm of $\nabla^2 d_1$. Thus from (4.4.21) and (4.4.31), we have that, for any $\delta \in (0, \, \delta_0)$,

$$- H^K_{E_2}(x_2) \leq -H^K_{E_1}(x_1) + C(n, s, R_c) \, \delta. \tag{4.4.32}$$

Now we consider the following two cases:

Case 1: $0 < \tilde{\delta} < \delta_0$. In this case, we simply substitute $\delta = \tilde{\delta}$ with (4.4.32) and obtain

$$- H^K_{E_2}(x_2) \leq -H^K_{E_1}(x_1) + C(n, s, R_c) \, \tilde{\delta}$$

where $\tilde{\delta} = \operatorname{dist}(\partial E_1, \, \partial E_2)$.

Case 2: $\tilde{\delta} \geq \delta_0$. In this case, there exists a number $N \in \mathbb{N}$ such that $\frac{\tilde{\delta}}{N} < \|\nabla^2 d_1\|^{-1}_{L^\infty(U_1)}$. Then setting $\tilde{\delta}_k := \frac{k}{N} \tilde{\delta}$ for each $k \in \{1, \cdots, N\}$ and taking into account all the above arguments, we obtain the inequality that

$$- H^K_{E_1^{\tilde{\delta}_k}}(x_1^{\tilde{\delta}_k}) \leq -H^K_{E_1^{\tilde{\delta}_{k-1}}}(x_1^{\tilde{\delta}_{k-1}}) + C(n, s, R_c) \, \frac{\tilde{\delta}}{N} \tag{4.4.33}$$

for each $k \in \{1, \cdots, N\}$ where we understand the notation $x_1^{\tilde{\delta}_0} = x_1$ and $E_1^{\tilde{\delta}_0} = E_1$. Thus by summing the inequality (4.4.33) for all $i \in \{1, \cdots, N\}$, we obtain

$$\begin{aligned} -H^K_{E_1^{\tilde{\delta}}}(x_2) &= -H^K_{E_1^{\tilde{\delta}_N}}(x_1^{\tilde{\delta}_N}) \\ &\leq -H^K_{E_1^{\tilde{\delta}_0}}(x_1^{\tilde{\delta}_0}) + N \, C(n, s, R_c) \, \frac{\tilde{\delta}}{N} = -H^K_{E_1}(x_1) + C(n, s, R_c) \, \tilde{\delta} \end{aligned}$$

where $\tilde{\delta} = \operatorname{dist}(\partial E_1, \, \partial E_2)$. In both cases, we finally obtain the inequality

$$- H^K_{E_2}(x_2) \leq -H^K_{E_1}(x_1) + C(n, s, R_c) \, \tilde{\delta}. \tag{4.4.34}$$

Now we recall that, thanks to Lemma 4.4.5, the Euler-Lagrange equation

$$H^s_{E_t}(x) + t - f(x) = 0$$

holds for any $x \in \partial E_t$ and $t \in [-\|u\|_{L^\infty} \, \|u\|_{L^\infty})$. Then, since E_i is the minimizer of $\mathcal{E}_{K,f,t_i}$ for $i \in \{1, \, 2\}$ and from (4.4.34), we obtain

$$t_2 - t_1 \leq f(x_2) - f(x_1) + C(n, s, R_c) \, \tilde{\delta}.$$

Recalling the definition of x_2, the Hölder continuity of f, and the fact that $E_t \subset B_{R_c}$ for any $t \geq c$, we conclude that

$$t_2 - t_1 \leq ([f]_\beta(B_{R_c}) + C(n, s, R_c)\,\tilde{\delta}^{1-\beta})\,\tilde{\delta}^\beta \tag{4.4.35}$$

where $[f]_\beta(B_{R_c})$ is the Hölder constant of f in B_{R_c} given as

$$[f]_\beta(B_{R_c}) \coloneqq \sup_{x,\,y \in B_{R_c},\, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\beta}$$

and the constant $\tilde{\delta}$ is defined as $\tilde{\delta} \coloneqq \mathrm{dist}\,(\partial E_1,\, \partial E_2)$. Note that the constant $C(n, s, R_c) > 0$ also depends on the L^∞-norm of $\nabla^2 d_1$.

Step 2. We are now ready to prove the local Hölder continuity of u.

Let $B_{r_0}(x_0) \subset \mathbb{R}^2$ be any open ball of radius r_0 with $x_0 \in \{u = t_0\}$ for a number $t_0 \geq c \coloneqq -\|u\|_{L^\infty}$. We take any points $x,\, y \in B_{r_0}(x_0)$ with $x \neq y$ and set $t_1,\, t_2 \in \mathbb{R}$ as $t_1 \coloneqq u(x)$ and $t_2 \coloneqq u(y)$. We may assume that $t_1 > t_2 \geq c$ because we only repeat the same argument in the case of $t_1 < t_2$. In addition to this, we also assume that $t_1 > t_0 > t_2$. Indeed, in the case of $t_1 > t_2 \geq t_0$ or $t_0 \geq t_1 > t_2$, it is sufficient to take another point $x_0' \in B_{r_0}(x_0)$ and $t_0' \in \mathbb{R}$ such that $x_0' \in \{u = t_0'\}$ and $t_1 > t_0' > t_2$, and do the argument that we will show below. Moreover, since we only observe the local regularity of u, it is sufficient to consider the case that $B_{r_0}(x_0) \subset U_0$ where U_0 is a neighborhood of $\partial\{u > t_0\}$ such that the signed distance function from $\partial\{u > t_0\}$ is of class $C^{1,s+\alpha}(U_0)$. Indeed, if $x \in B_{r_0}(x_0) \setminus U_0$ and $y \in B_{r_0}(x_0)$, then, from the continuity of u, we can choose a point z_0 in $B_{r_0}(x_0)$ and close to x such that the estimate $|u(x) - u(z_0)| \leq |x-y|^\beta$ holds and $t_1 = u(x) > u(z_0) \geq u(y) = t_2$. In the case of $z_0 \in U_0$, we just apply the argument that we will show below with (4.4.35) for z_0, x_0, and y; otherwise we can repeat the above argument until we have the point belonging to U_0.

Now we choose sufficiently small $\varepsilon > 0$ such that $t_1 - \varepsilon > t_0$ and $t_0 - \varepsilon > t_2$ and then we have that $x \in \{u > t_1 - \varepsilon\}$, $y \in \{u > t_2 - \varepsilon\}$, and $x_0 \in \{u > t_0 - \varepsilon\}$. Hence, from (4.4.35) and the fact that $x,\, y \in B_{r_0}(x_0)$, we obtain the two inequalities

$$\begin{aligned} u(x) - u(x_0) = t_1 - \varepsilon - (t_0 - \varepsilon) &\leq ([f]_\beta(B_{R_c}) + C(n, s, R_c)\,\tilde{\delta}_1^{1-\beta})\,\tilde{\delta}_1^\beta \\ &\leq ([f]_\beta(B_{R_c}) + C(n, s, R_c)\,r_0^{1-\beta})\,\tilde{\delta}_1^\beta. \end{aligned} \tag{4.4.36}$$

and

$$\begin{aligned} u(x_0) - u(y) = t_0 - \varepsilon - (t_2 - \varepsilon) &\leq ([f]_\beta(B_{R_c}) + C(n, s, R_c)\,\tilde{\delta}_2^{1-\beta})\,\tilde{\delta}_2^\beta \\ &\leq ([f]_\beta(B_{R_c}) + C(n, s, R_c)\,r_0^{1-\beta})\,\tilde{\delta}_2^\beta \end{aligned} \tag{4.4.37}$$

where we set $\tilde{\delta}_1 \coloneqq \mathrm{dist}\,(\partial E_{t_0}, \partial E_{t_1})$ and $\tilde{\delta}_2 \coloneqq \mathrm{dist}\,(\partial E_{t_0}, \partial E_{t_2})$. Note that the constant $C(n, s, R_c) > 0$ also depends on the L^∞-norm of $\nabla^2 d_{t_0}$, which can be uniformly bounded in $B_{r_0}(x_0)$. Notice that the inequality

$$\tilde{\delta}_1 + \tilde{\delta}_2 = \mathrm{dist}\,(\partial E_{t_0},\, \partial E_{t_1}) + \mathrm{dist}\,(\partial E_{t_0},\, \partial E_{t_2}) \leq \mathrm{dist}\,(\partial E_{t_1},\, \partial E_{t_2}) \leq |x-y|$$

holds because of the fact that $E_{t_1} \subset E_{t_0} \subset E_{t_2}$. Therefore from (4.4.36) and (4.4.37), we obtain that there exists a constant $C = C(n, s, f, R_c, r_0, x_0) > 0$ (we have assumed that the dimension of $\mathbb{R}^N$ is two) such that

$$\begin{aligned} |u(x) - u(y)| &= |u(x) - u(x_0) + u(x_0) - u(y)| \\ &\leq C\,(\tilde{\delta}_1^\beta + \tilde{\delta}_2^\beta) \leq C\,2^{1-\beta}(\tilde{\delta}_1 + \tilde{\delta}_2)^\beta \leq 2^{1-\beta} C\,|x-y|^\beta. \end{aligned}$$

Here, in the second inequality, we have used the fact that $2^{1-\beta}(x+1)^\beta \geq x^\beta + 1$ for any $x \geq 1$ and $\beta \in (0,\,1)$ and applied this fact with $x = \widetilde{\delta}_1\,\widetilde{\delta}_2^{-1}$ if $\widetilde{\delta}_1 \geq \widetilde{\delta}_2$ or $x = \widetilde{\delta}_2\,\widetilde{\delta}_1^{-1}$ if $\widetilde{\delta}_1 < \widetilde{\delta}_1$. $\square$

Chapter 5

Nonlocal Liquid Drop Model

In this chapter, we investigate a nonlocal extension of the classical liquid drop model, which was originally introduced by G. Gamow [63] to explain the behavior of atoms and predict a nuclear fission. The liquid drop model consists of the following two energies: one is an attractive term (classical or nonlocal perimeter) and the other is a repulsive term (Riesz potential). Then, in this model, the non-trivial competition between these energies occurs and the model has been studied by many authors.

In this thesis, we focus on a nonlocal version of the classical liquid drop model. To discuss our problem, we mainly divide this chapter into two sections; one is on the nonexistence of minimizers for the model and the other is on the existence and asymptotic behavior of minimizers for the model. Strictly speaking, in the first section, we study a nonlocal version of the classical liquid drop model in a more general framework, and show the nonexistence of minimizers for large volumes. Our model in the first section is also closely related to the ionization conjecture in quantum mechanics. In the second section, we consider a nonlocal version of the classical liquid drop model, and show the existence of minimizers for any volume under suitable assumptions on the repulsive term.

5.1 Nonexistence of Minimizers

In this section, we first consider the nonexistence of minimizers of the functional $\mathcal{E}_{K,\alpha,\mu,\beta}$, which is defined in (1.0.15), for large volumes. We obtain the following two results; the first one is that, if we assume that $\mu \geq 0$ and $m > 0$ and K satisfies several conditions (see the first subsection in Section 5.1.1 for the detail), then every minimizer of $\mathcal{E}_{K,\alpha,\mu,\beta}$ is bounded. The second one is as follows: we assume that K satisfies several conditions which are weaker than the ones in the first result. Then there exists a constant $m_0 = m_0(N,\, s,\, \varepsilon,\, \mu) > 0$, which can be determined explicitly, such that $\inf\{\mathcal{E}_{K,\alpha,\mu,\beta}(E) \mid |E| = m\}$ has no bounded solutions for any $m > m_0$. According to these two results, we can say that, if K satisfies the assumptions as in the first result, it holds that $\inf\{\mathcal{E}_{K,\alpha,\mu,\beta}(E) \mid |E| = m\}$ has no solutions. We emphasize that our result in this section is a partial extension of the results shown in [56]. Indeed, if we set $K(x) = |x|^{-(N+s)}$ with $s \in (0,\, 1)$, $\alpha = N-1$, and $\mu = 0$, then we prove the nonexistence of minimizers of the functional $P_s(E)+V_\alpha(E)$ for large volume $|E| = m$. Meanwhile, the authors [56] showed the existence of minimizers for the functional $P_s(E)+V_\alpha(E)$ with $\alpha \in (0,\, N)$ for small volume $|E| = m$.

Our idea for proving the boundedness of minimizers is based on [32]. First, we will show that the functional (1.0.15) is continuous under some sufficiently small and smooth perturbation near a point in the measure-theoretic boundary of a minimizer (see Lemma 5.1.4). This continuity is what we call "Almgren's lemma". Secondly, if we suppose that a minimizer is not bounded, this continuity and the minimality give us some inequality for the volume of a minimizer outside of some ball, which leads to a contradiction. On the other hand, the strategy for proving the main theorem, namely, the nonexistence of minimizers is based on [58, 69, 71, 72, 73, 84]. Precisely, we take the following strategy: first, we separate $\mathbb{R}^N$ into two parts by a hyperplane which is parametrized by a directional parameter $\nu \in \mathbb{S}^{N-1}$ and a translating parameter $l \in \mathbb{R}$. Then, taking any minimizer of (1.0.15) and considering the intersection of the minimizer by each separated part (either of them can be an empty set), we compare the sum of the functional for each intersection with that of the original set. Integrating the resulting inequality with respect to l and then ν, we are able to obtain the inequality for the volume of a minimizer to find that it actually shows the upper bound of that volume.

5.1.1 Statement of Main Results

In this subsection, we first state some assumptions on the kernel K and then show the main results of Section 5.1. Note that we apply the following assumptions on K only to this section. We suppose that K satisfies

(K1) K is non-negative and symmetry with respect to the origin, namely, $K(-x) = K(x)$ for any $x \in \mathbb{R}^N$.

(K2) $K \in L^1(B_1^c(0))$.

(K3) There exists a constant $\gamma_0 > 0$ such that

$$K(w) \leq \left(1 + \frac{|z-w|}{|w|}\right)^{\gamma_0} K(z)$$

for any z, $w \in \mathbb{R}^N$ with $|z-w| \leq \frac{1}{2}|w|$.

(K4) There exists constants $0 < s < 1$, $\varepsilon > 0$, and $\lambda > 1$ such that

$$\begin{aligned} |x|^{-(N+s)} \leq K(x) &\leq \lambda |x|^{-(N+s)} \quad \text{for any } 0 < |x| < 1+\varepsilon, \\ |x|^{-(N+s-1)} \leq |x|K(x) &\leq (1+\varepsilon)^{-(N+s-1)} \quad \text{for any } |x| \geq 1+\varepsilon. \end{aligned}$$

We can say that (K3) would correspond to the control of the gradient of the kernel K in the neighborhood of some point. This assumption can allow us to control the difference $|P_K(\Phi_t(E)) - P_K(E)|$ of a set $E \subset \mathbb{R}^N$ where $\{\Phi_t\}_{|t|<1}$ is a one-parameter diffeomorphism (see Lemma 5.1.4 in Subsection 5.1.2 for the detail). We use this control of the perturbation for P_K in order to obtain the boundedness of minimizers of (1.0.15), and one may weaken (K3) if one could find another method to prove the boundedness of the minimizers.

If we consider the nonexistence of bounded minimizers of the functional (1.0.15), then we impose (K5), which is weaker than (K4), in the following way:

(K5) There exists constants $0 < s < 1$, $\varepsilon > 0$, and $\lambda > 1$ such that

$$\begin{aligned} K(x) &\leq \lambda\, |x|^{-(N+s)} \quad \text{for any } 0 < |x| < 1+\varepsilon, \\ |x|K(x) &\leq (1+\varepsilon)^{-(N+s-1)} \quad \text{for any } |x| \geq 1+\varepsilon. \end{aligned}$$

In the following figures (Figure 5.1 and 5.2), we illustrate a rough shape of the kernel K with the above assumptions.

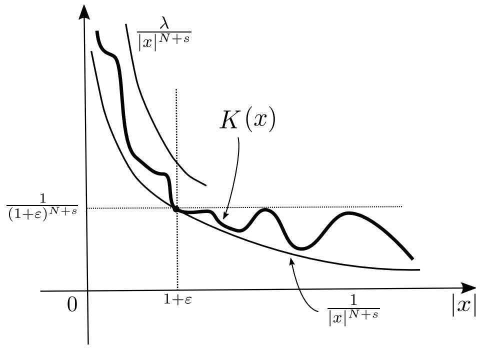

Figure 5.1: Graph of K with (K4)

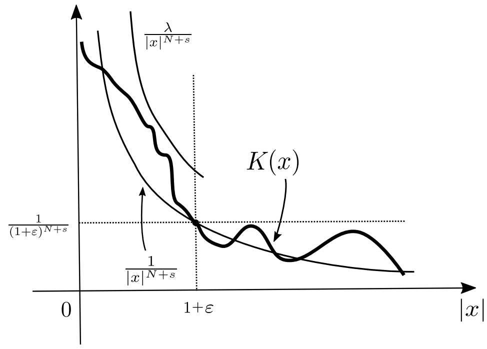

Figure 5.2: Graph of K with (K5)

Remark that the function $|x|^{-(N+s)}$, denoted by K_1, is a typical example of the kernel K satisfying all the above assumptions. Indeed, one easily see that K_1 satisfies (K1), (K2), (K4), and (K5). Moreover, we can observe that K_1 satisfies (K3) in the following manner: taking any $z,\, w \in \mathbb{R}^N$, we have from the triangle inequality

$$\frac{|z|^{N+s}}{|w|^{N+s}} \leq \left(\frac{|z-w|+|w|}{|w|}\right)^{N+s} \leq \left(1+\frac{|z-w|}{|w|}\right)^{N+s}.$$

Thus we obtain that, for any $z,\, w \in \mathbb{R}^N$,

$$K_1(w) = \frac{1}{|w|^{N+s}} \leq \left(1+\frac{|z-w|}{|w|}\right)^{N+s} \frac{1}{|z|^{N+s}} = \left(1+\frac{|z-w|}{|w|}\right)^{N+s} K_1(z)$$

and we can conclude that K_1 satisfies (K3) with $\gamma_0 = N+s$.

Another non-trivial example of the kernel K is given by

$$K_2(x) := \begin{cases} -|x|^{-(N+s-1)} \log|x| & \text{if} \quad |x| \leq \dfrac{1}{e} \\ \dfrac{1}{e}|x|^{-(N+s)} & \text{if} \quad |x| > \dfrac{1}{e} \end{cases}$$

where e is the Napier's number. We can observe that this K_2 satisfies (K1), (K2), (K3), and (K5). Note that this function does not satisfy (K4) and is not homogeneous while the function $|x|^{-(N+s)}$ is. Indeed, it is easy to see that K_2 satisfies (K1) and (K2) by definition. From the fact that th function $-r\log r$ is increasing in $(0,\, e^{-1}]$, one can see that, for any $|x| \leq \frac{1}{e}(< 1+\varepsilon)$,

$$K_2(x) = |x|^{-(N+s)}(-|x|\log|x|) \leq \frac{1}{e}\,|x|^{-(N+s)}.$$

On the other hand, we can easily observe that $K_2(x) \leq |x|^{-(N+s)}$ or $K_2(x) \leq (1+\varepsilon)^{-(N+s-1)}\,|x|^{-1}$ holds for any $|x| \geq \frac{1}{e}$. Therefore K_2 satisfies (K5). Finally, we show that K_2 satisfies (K3). Actually, if $|z| \leq |w|$, then K_2 is a decreasing function and thus we can conclude that

$$K_2(w) \leq K_2(z) \leq \left(1+\frac{|z-w|}{|w|}\right)^{\gamma_0} K_2(z)$$

holds for any $\gamma_0 > 0$. If $|z| > |w|$, then we consider the following three cases; $\frac{1}{e} > |z| > |w|$, $|z| \geq \frac{1}{e} > |w|$, and $|z| > |w| \geq \frac{1}{e}$. First, in the case of $\frac{1}{e} > |z| > |w|$, from the fact that the function $r \mapsto -r\log r$ is increasing in $(0,\, \frac{1}{e})$, we have that $-|z|\log|z| > -|w|\log|w|$. Thus we obtain

$$\frac{K_2(w)}{K_2(z)} = \frac{|z|^{N+s}}{|w|^{N+s}} \frac{-|w|\log|w|}{-|z|\log|z|} < \left(1+\frac{|z-w|}{|w|}\right)^{N+s}$$

and this indicates (K3). Secondly, in the case of $|z| \geq \frac{1}{e} > |w|$, from the definition of K_2 and by the same reason in the first case, we have that

$$\frac{K_2(w)}{K_2(z)} = \frac{e\,|z|^{N+s}}{|w|^{N+s}} - |w| \log |w| < \left(1 + \frac{|z-w|}{|w|}\right)^{N+s}.$$

Lastly, in the case of $|z| > |w| \geq \frac{1}{e}$, from the definition of K_2, we can easily see that K_2 satisfies (K3). Therefore we may conclude that K_2 satisfies (K3).

Assuming that the kernel $g(x) = |x|^{-1}$ and $\beta = 1$, we now give our main results in this section. The first one is the boundedness of minimizers, the second one is the nonexistence of bounded minimizers, and the last one is the corollary of the second result.

Proposition 5.1.1. *Let $N \geq 2$, $\mu \geq 0$, and $m \in (0, \infty)$. Assume that K satisfies (K1), (K2), (K3), and (K4). Then every minimizer E of* (1.0.15) *is essentially bounded, namely there exists $R > 0$ such that $|E \setminus B_R(0)| = 0$.*

Theorem 5.1.2. *Let $N \geq 2$ and $\mu \geq 0$. Assume that K satisfies (K1), (K2), and (K5). Then, there exists a constant $m_c > 0$ given by*

$$m_c := \left(1 - \frac{1}{(1+\varepsilon)^{N+s-1}}\right)^{-1} \left(\frac{\omega_{N-1}\,\lambda}{1-s}\,(1+\varepsilon)^{1-s} + \mu\right) \tag{5.1.1}$$

such that for any $m > m_c$,

$$\inf\{\mathcal{E}_{K,\alpha,\mu,\beta}(E) \mid E \subset \mathbb{R}^N,\, |E| = m\} \tag{5.1.2}$$

has no bounded solutions.

Since (K5) is weaker than (K4), then from Proposition 5.1.1 and Theorem 5.1.2 we may obtain

Corollary 5.1.3. *Assume that K satisfies (K1), (K2), (K3), and (K4) and let $\mu \geq 0$. Then for any $m > m_c$ where m_c is as in Theorem 5.1.2,* (5.1.2) *has no solutions.*

Remark that we cannot say that the mass m_c given in (5.1.1) is optimal since we just show the necessary condition for the existence of the minimizers.

5.1.2 Boundedness of Minimizers

In this subsection, we prove Proposition (5.1.1), namely, the boundedness of minimizers for the functional (1.0.15) under some conditions on a kernel K. First of all, we start to show a generalized version of so-called Almgren's lemma (see [94] for the classical results, [57] in the case of an anisotropic perimeter, or [32] in the case of a nonlocal s-perimeter).

Lemma 5.1.4. *Suppose that K satisfies (K1), (K2), and (K3). Let $E \subset \mathbb{R}^N$ be a measurable set with $P_K(E) + V_{N-1}(E) + R_\mu(E) < \infty$. Let $x_0 \in \mathbb{R}^N$ and $r_0 > 0$ be such that*

$$|B_{r_0}(x_0) \cap E| > 0, \quad |B_{r_0}(x_0) \cap E^c| > 0.$$

Then, there exist k_0, $C > 0$ such that, for any $k \in (-k_0, k_0)$, there exists a measurable set $F_k \subset \mathbb{R}^N$ such that $P_K(F_k) + V_{N-1}(F_k) + R_\mu(F_k) < \infty$ and the following properties hold:

1. *$E \Delta F_k \subset\subset B_{r_0}(x_0)$*
2. *$|F_k| - |E| = k$*
3. *$|\mathcal{E}_{K,\alpha,\mu,\beta}(F_k) - \mathcal{E}_{K,\alpha,\mu,\beta}(E)| < C\,|k|$.*

Proof. From the density assumption, we have that there exists a function $T \in C^1_c(B_{r_0}(x_0); \mathbb{R}^N)$ such that

$$M := \int_E \operatorname{div} T(x)\, dx > 0.$$

If this fails, then we have that, for any $T \in C^1_c(B_{r_0}(x_0); \mathbb{R}^N)$, $\int_E \operatorname{div} T(x)\, dx = 0$. Then, by the definition of the classical perimeter (see, for instance, [55] for the definition), it holds that $P(E, B_{r_0}(x_0)) = 0$. However, by the classical isoperimetric inequality (see also [55]), we have $|E \cap B_{r_0}(x_0)| = 0$, which contradicts the assumption of E.

For any $t \in (-1, 1)$, we define the maps $\Psi_t(x) := x + t\, T(x)$ for all $x \in \mathbb{R}^N$. Then, we may easily see that there exists $\delta_0 \in (0, 1)$ such that the maps Ψ_t are diffeomorphisms from $B_{r_0}(x_0)$ onto itself for any $t \in (-\delta_0, \delta_0)$. Moreover, we have that $\det(\nabla\Psi_t(x)) = 1 + t\operatorname{div} T(x) + o(t)$ for any $t \in (-\delta_0, \delta_0)$. By the definition of Ψ_t, we also have $E \Delta \Psi_t(E) \subset\subset \mathbb{R}^N$ and thus, applying the change of variables,

$$\begin{aligned} |\Psi_t(E)| &= \int_E |\det \nabla\Psi_t(x)|\, dx \\ &= \int_E (1 + t\operatorname{div} T(x) + o(t))\, dx = |E| + tM + o(t), \end{aligned} \tag{5.1.3}$$

for sufficiently small $t \in (-\delta_0, \delta_0)$. Therefore, there exists a constant $k_0 > 0$ such that, if we set $F_k := \Psi_{t(k)}(E)$, where $t(k) := k/M + o(k)$, for any $k \in (-k_0, k_0)$, F_k satisfies the first and second properties in Lemma 5.1.4.

Now we derive the upper bound of the difference between $P_K(\Psi_t(E))$ and $P_K(E)$. Take any $x, y \in \mathbb{R}^N$ with $x \neq y$. If we set $z = \Psi_t(x) - \Psi_t(y)$ and $w = x - y$, then we have that

$$\begin{aligned} |z - w| &= |t|\ |T(x) - T(y)| \\ &= |t|\,|x - y| \frac{|T(x) - T(y)|}{|x - y|} \\ &\le |w|\,|t|\,\|T\|_{C^1} \le \frac{1}{2}\,|w| \end{aligned} \tag{5.1.4}$$

for any $|t| < \frac{1}{2}\|T\|_{C^1}^{-1}$. Thus, from the assumption (K3), there exists $\gamma_0 > 0$ such that

$$\begin{aligned} K(x - y) = K(w) &\le \left(1 + \frac{|t|\,|T(x) - T(y)|}{|x - y|}\right)^{\gamma_0} K(z) \\ &\le (1 + \|T\|_{C^1}\,|t|)^{\gamma_0}\, K(\Psi_t(x) - \Psi_t(y)). \end{aligned} \tag{5.1.5}$$

By taking t sufficiently small, we have

$$\begin{aligned} K(\Psi_t(x) - \Psi_t(y)) &\ge (1 + \|T\|_{C^1}\,|t|)^{-\gamma_0}\ K(x - y) \\ &\ge (1 - 2\gamma_0\,\|T\|_{C^1}\,|t|)\, K(x - y). \end{aligned} \tag{5.1.6}$$

where we have used the Taylor expansion of $(1 + x)^{-\gamma_0}$ for small x at the last inequality in (5.1.6). On the other hand, if we set $z = x - y$ and $w = \Psi_t(x) - \Psi_t(y)$, then we have that, if $T(x) \neq T(y)$,

$$\begin{aligned} \frac{1}{2}|w| &= \frac{1}{2}|\Psi_t(x) - \Psi_t(y)| \\ &\ge \frac{1}{2}|t||T(x) - T(y)| \left(\frac{|x - y|}{|t|\,|T(x) - T(y)|} - 1\right) \\ &\ge |z - w|\,\frac{1}{2}\left(\frac{1}{|t|\,\|T\|_{C^1}} - 1\right) \\ &\ge |z - w| \end{aligned} \tag{5.1.7}$$

for any $0 < |t| < \frac{1}{3}\|\nabla T\|_{C^1}^{-1}$. If $T(x) = T(y)$ or $t = 0$, then we have $|z-w| = 0$ by definition and thus (5.1.7) also holds. Hence, from (K3) we obtain that

$$\begin{aligned} K(\Psi_t(x) - \Psi_t(y)) &= K(w) \\ &\le \left(1 + \frac{|t|\,|T(x) - T(y)|}{|\Psi_t(x) - \Psi_t(y)|}\right)^{\gamma_0} K(z) \\ &\le \left(1 + \frac{|t|\,|T(x) - T(y)|}{|x-y|\left(1 - |t|\frac{|T(x)-T(y)|}{|x-y|}\right)}\right)^{\gamma_0} K(x-y) \\ &\le \left(1 + \frac{\|T\|_{C^1}\,|t|}{1 - \|T\|_{C^1}\,|t|}\right)^{\gamma_0} K(x-y) \\ &= (1 - \|T\|_{C^1}\,|t|)^{-\gamma_0} K(x-y) \\ &\le (1 + 2\gamma_0\,\|T\|_{C^1}\,|t|)\,K(x-y) \end{aligned} \tag{5.1.8}$$

for sufficiently small t. Here we also have used the Taylor expansion of $(1-x)^{-\gamma_0}$ for small x at the last inequality in (5.1.8). Therefore from (5.1.6) and (5.1.8) we obtain

$$|K(\Psi_t(x) - \Psi_t(y)) - K(x-y)| \le 2\gamma_0\,\|T\|_{C^1}\,|t|\,K(x-y) \tag{5.1.9}$$

for sufficiently small t and $x,\, y \in \mathbb{R}$ with $x \neq y$. Since we can easily see that $x = y$ if and only if $\Psi_t(x) = \Psi_t(y)$ for any $0 < |t| \le \frac{1}{2\|\nabla T\|_{L^\infty}}$, then the estimate (5.1.9) is also valid when $x = y$. Then, we may compute $|P_K(\Psi_t(E)) - P_K(E)|$ as follows: first, we can see from (5.1.9) that

$$\begin{aligned} &\Big|K(\Psi_t(x) - \Psi_t(y))\,|\det\nabla\Psi_t(x)|\,|\det\nabla\Psi_t(y)| - K(x-y)\Big| \\ &= \Big|K(\Psi_t(x) - \Psi_t(y))\,(1 + t\,\mathrm{div}\,(T(x) + T(y)) + o(|t|)) - K(x-y)\Big| \\ &\le C_0(T,\, d_0)\,|t|\,K(x-y) \end{aligned} \tag{5.1.10}$$

for any $x,\, y \in \mathbb{R}^N$ with $x \neq y$ and sufficiently small $|t|$ where

$$C_0(T,\, \gamma_0) := 2\|T\|_{C^1}(\gamma_0 + 1 + 2\gamma_0\,\|T\|_{C^1}). \tag{5.1.11}$$

Hence setting $J\Psi_t := |\det\nabla\Psi_t|$, we conclude from (5.1.10) that

$$\begin{aligned} &|P_K(\Psi_t(E)) - P_K(E)| \\ &\le \int_E\int_{E^c} |K(\Psi_t(x) - \Psi_t(y))\,J\Psi_t(x)\,J\Psi_t(y) - K(x-y)|\,dx\,dy \\ &\le |t|\,C_0(T,\, d_0)\int_E\int_{E^c} K(x-y)\,dx\,dy = |t|\,C_0(T,\, d_0)\,P_K(E) < \infty. \end{aligned} \tag{5.1.12}$$

Secondly, we also have that

$$\begin{aligned} &|V_{N-1}(\Psi_t(E)) - V_{N-1}(E)| \\ &\quad\le \int_E\int_E \left|\frac{|\det\nabla\Psi_t(x)|\,|\det\nabla\Psi_t(y)|}{|\Psi_t(x) - \Psi_t(y)|} - \frac{1}{|x-y|}\right| dx\,dy \\ &\quad= \int_E\int_E \left|\frac{1 + t(\mathrm{div}\,T(x) + \mathrm{div}\,T(y)) + o(t)}{|x - y + t(T(x) - T(y))|} - \frac{1}{|x-y|}\right| dx\,dy \\ &\quad\le \int_E\int_E \left(\frac{C_1(T)|t|}{|x-y|} + \frac{o(t)}{|x-y|}\right) dx\,dy \\ &\quad\le |t|\,C_2(T)\,V_{N-1}(E) < \infty, \end{aligned} \tag{5.1.13}$$

where $C_1(T)$, $C_2(T)$ are positive constants independent of t. Finally, we estimate the difference of the potentials $R_\mu(\Psi_t(E))$ and $R_\mu(E)$. Since t is sufficiently small, there exists a constant $c_0(T) > 0$ such that $|x|(1 - c_0(T)\,|t|) \le |x + t\,T(x)| \le |x|(1 + c_0(T)\,|t|)$. Hence, we obtain

$$
\begin{aligned}
|R_\mu(\Psi_t(E)) - R_\mu(E)| &\le \mu \int_E \left| \frac{1 + t \operatorname{div} T(x) + o(t)}{|x + t\,T(x)|} - \frac{1}{|x|} \right| dx \\
&\le \mu \int_E \left(\frac{C_3(T)|t|}{|x|} + \frac{o(t)}{|x|} \right) dx \\
&\le |t|\, C_4(T)\, R_\mu(E) < \infty \qquad (5.1.14)
\end{aligned}
$$

where $C_3(T)$, $C_4(T)$ are positive constants independent of t. Therefore, (5.1.12), (5.1.13), and (5.1.14) imply that the set F_k satisfies the third property in Lemma 5.1.4 and this completes the proof. □

Before starting to prove Proposition 5.1.1, we need the following isoperimetric inequality of the nonlocal perimeter P_K for sets with small volumes if K satisfies the conditions (K1) and (K4).

Lemma 5.1.5. *Suppose that K satisfies (K1), (K2), and (K4) and let $\varepsilon > 0$ and $\lambda \ge 1$ be the constants given in (K4). Then, there exists a constant $C = C(N,\, s,\, \lambda) > 0$ such that, for any measurable set $F \subset \mathbb{R}^N$ with $0 < |F| \le \omega_N (1+\varepsilon)^N$, we have*

$$
C\,|F|^{\frac{N-s}{N}} \le P_K(F).
$$

Proof. If $P_K(F) = \infty$, then the lemma is proved. Thus, we can assume that $P_K(F)$ is finite. Setting $\rho := \omega_N^{-1/N} |F|^{1/N} \in (0,\, 1+\varepsilon]$, we may compute the nonlocal perimeter $P_K(F)$ as follows:

$$
\begin{aligned}
\int_F \int_{F^c} K(x-y)\,dx\,dy &= \int_F \int_{F^c \cap B_\rho(y)} K(x-y)\,dx\,dy \\
&\quad + \int_F \int_{F^c \cap B^c_\rho(y)} K(x-y)\,dx\,dy \\
&\ge \frac{1}{\rho^{N+s}} \int_F |F^c \cap B_\rho(y)|\,dy \\
&\quad + \int_F \int_{F^c \cap B^c_\rho(y)} K(x-y)\,dx\,dy. \qquad (5.1.15)
\end{aligned}
$$

Here, by the assumption of F, we have $|F| = \omega_N \rho^N = |B_\rho(y)|$ for any y. Thus, it holds that

$$
\begin{aligned}
|F^c \cap B_\rho(y)| &= |B_\rho(y)| - |F \cap B_\rho(y)| \\
&= |F| - |F \cap B_\rho(y)| = |F \cap B^c_\rho(y)| \qquad (5.1.16)
\end{aligned}
$$

for any $y \in F$. Thus, from (5.1.15) and (5.1.16) and recalling the assumption of K and the fact that $\lambda > 1$ and $\rho < 1 + \varepsilon$, we obtain the following inequality:

$$\begin{aligned}
&\int_F \int_{F^c} K(x-y)\,dx\,dy \\
&\geq \frac{1}{\rho^{N+s}} \int_F |F \cap B^c_\rho(y)|\,dy + \int_F \int_{F^c \cap B^c_\rho(y)} K(x-y)\,dx\,dy \\
&\geq \frac{1}{\lambda} \int_F \int_{F \cap B^c_\rho(y)} K(x-y)\,dx\,dy + \int_F \int_{F^c \cap B^c_\rho(y)} K(x-y)\,dx\,dy \\
&\geq \frac{1}{\lambda} \int_F \int_{B^c_\rho(y)} K(x-y)\,dx\,dy \\
&\geq \frac{1}{\lambda} \int_F \int_{B^c_\rho(y)} \frac{1}{|x-y|^{N+s}}\,dx\,dy.
\end{aligned} \tag{5.1.17}$$

Moreover, by applying the change of variables, we can further calculate the last term in (5.1.17) in the following manner:

$$\begin{aligned}
\int_F \int_{B^c_\rho(y)} \frac{1}{|x-y|^{N+s}}\,dx\,dy &= |F|\,\omega_{N-1} \int_\rho^\infty \frac{1}{r^{1+s}}\,dr \\
&= |F| \frac{\omega_{N-1}}{s} \rho^{-s} = \frac{\omega_{N-1}\omega_N^{\frac{s}{N}}}{s} |F|^{\frac{N-s}{N}}.
\end{aligned} \tag{5.1.18}$$

Therefore, from (5.1.17) and (5.1.18)we obtain

$$P_K(F) \geq \frac{\omega_{N-1}\omega_N^{\frac{s}{N}}}{\lambda\, s} |F|^{\frac{N-s}{N}}$$

which is a required inequality. □

Now we are ready to prove Proposition 5.1.1 as we stated in Subsection 5.1.1 by applying the above two claims. The proof is based on the strategy shown in [32] for instance.

Proof of Proposition 5.1.1. Suppose that K satisfies (K1), (K2), (K3), and (K4) and let $E \subset \mathbb{R}^N$ be a minimizer of $\inf_{|E|=m} \mathcal{E}_{K,\alpha,\mu,\beta}(E)$. In the following, we use the notation B_r as the open ball $B_r(0)$ with radius r centred at 0. For any $r > 0$, we define $f(r) := |E \setminus B_r|$. Then, by the continuity of measure and $|E| = m$, f is a non-increasing function and converges to zero as $r \to \infty$. Moreover, the coarea formula implies

$$f'(r) = -\mathcal{H}^{N-1}(E \cap \partial B_r).$$

We now show that there exists $R > 0$ such that $f(r) = 0$ for all $r > R$. This implies the boundedness of minimizers because $\mathcal{E}_{K,\alpha,\mu,\beta}(E)$ coincides with $\mathcal{E}_{K,\alpha,\mu,\beta}(E')$ if $E \setminus E'$ is a set of Lebesgue measure zero and thus we can identify E with E'. Suppose by contradiction that $f(r) > 0$ for any $r > 0$. Without loss of generality, we can assume that $|E \cap B_1(0)| > 0$ and $|\mathcal{C}E \cap B_1(0)| > 0$. Choosing k_0 as in Lemma 5.1.4, we may fix $R_0 > 0$ such that $f(r) < k_0$ for any $r \geq R_0$. Then, by Lemma 5.1.4, for any $r \geq R_0$, there exists $F_r \subset \mathbb{R}^N$ such that the followings are true:

1. $E \Delta F_r \subset\subset B_1(0) \subset B_r$.
2. $|F_r| = |E| + f(r)$.
3. $|\mathcal{E}_{K,\alpha,\mu,\beta}(E) - \mathcal{E}_{K,\alpha,\mu,\beta}(F_r)| < C\, f(r)$ for a constant $C > 0$ independent of r.

Now, letting $G_r := F_r \cap B_r$ and recalling the first and second properties of Lemma 5.1.4, we have that $|G_r| = |E|$. Here we recall the following identity:

$$V_{N-1}(U \cup W) = V_{N-1}(U) + V_{N-1}(W) + 2 \int_U \int_W \frac{1}{|x-y|} \, dx \, dy. \tag{5.1.19}$$

for any measurable U, $W \subset \mathbb{R}^N$ such that $|U \cap W| = 0$. Therefore, by the minimality of E and using Proposition 2.1.3 with $\Omega = \mathbb{R}^N$ and (5.1.19), we obtain

$$\begin{aligned}
\mathcal{E}_{K,\alpha,\mu,\beta}(E) \le \mathcal{E}_{K,\alpha,\mu,\beta}(G_r) &= P_K(G_r) + V_{N-1}(G_r) - R_\mu(G_r) \\
&\le P_K(F_r) - P_K(F_r \setminus B_r) \\
&\quad + V_{N-1}(F_r) - V_{N-1}(F_r \setminus B_r) \\
&\quad\quad - R_\mu(F_r) + R_\mu(F_r \setminus B_r) \\
&\quad\quad\quad + 2 \int_{F_r \setminus B_r} \int_{F_r \cap B_r} K(x-y) \, dx \, dy \\
&\le \mathcal{E}_{K,\alpha,\mu,\beta}(E) + C\, f(r) \\
&\quad - P_K(F_r \setminus B_r) + \mu \int_{F_r \setminus B_r} \frac{1}{|x|} \, dx \\
&\quad\quad + 2 \int_{F_r \setminus B_r} \int_{F_r \cap B_r} K(x-y) \, dx \, dy.
\end{aligned} \tag{5.1.20}$$

Note that we also used the third property of Lemma 5.1.4 in the last inequality of (5.1.20). Since $E \setminus B_r = F_r \setminus B_r$, it holds that

$$\int_{F_r \setminus B_r} \frac{1}{|x|} \, dx = \int_{E \setminus B_r} \frac{1}{|x|} \, dx \le \frac{1}{r} \, |E \setminus B_r| = \frac{f(r)}{r}.$$

From the assumption of K, we have that

$$\begin{aligned}
\int_{F_r \setminus B_r} \int_{F_r \cap B_r} K(x-y) \, dx \, dy &= \int_{E \setminus B_r} \int_{F_r \cap B_r \cap B_{1+\varepsilon}(y)} K(x-y) \, dx \, dy \\
&\quad + \int_{E \setminus B_r} \int_{F_r \cap B_r \cap B^c_{1+\varepsilon}(y)} K(x-y) \, dx \, dy \\
&\le \int_{E \setminus B_r} \int_{B_r \cap B_{1+\varepsilon}(y)} K(x-y) \, dx \, dy \\
&\quad + \int_{E \setminus B_r} \int_{F_r \cap B^c_{1+\varepsilon}(y)} K(x-y) \, dx \, dy.
\end{aligned} \tag{5.1.21}$$

From the assumption on K, we have that $K(x-y) \le \lambda \, |x-y|^{-N-s}$ for $|x-y| < 1+\varepsilon$. Moreover, for any $y \in E \setminus B_r$, it holds that $B_r \subset B^c_{|y|-r}(y)$. By using the coarea formula, we can continue the calculations in (5.1.21) as follows:

$$\begin{aligned}
\int_{F_r \setminus B_r} \int_{F_r \cap B_r} K(x-y) \, dx \, dy &\le \lambda \int_{E \setminus B_r} \int_{B^c_{|y|-r}(y)} \frac{1}{|x-y|^{N+s}} \, dx \, dy \\
&\quad + \frac{1}{(1+\varepsilon)^{N+s}} \int_{E \setminus B_r} |F_r| \, dy \\
&\le \frac{\lambda \omega_{N-1}}{s} \int_{E \setminus B_r} \frac{1}{(|y|-r)^s} \, dy + \frac{f(r)\,|E|}{(1+\varepsilon)^{N+s}} \\
&\le \frac{\lambda \omega_{N-1}}{s} \int_r^\infty \frac{\mathcal{H}^{N-1}(E \cap \partial B_t)}{(t-r)^s} \, dt + \frac{m\, f(r)}{(1+\varepsilon)^{N+s}} \\
&= -\frac{\lambda \omega_{N-1}}{s} \int_r^\infty \frac{f'(t)}{(t-r)^s} \, dt + \frac{m\, f(r)}{(1+\varepsilon)^{N+s}}.
\end{aligned} \tag{5.1.22}$$

Substituting (5.1.22) in (5.1.20), we obtain

$$P_K(E \setminus B_r) \leq C_5\, f(r) - \frac{\lambda\omega_{N-1}}{s} \int_r^{\infty} \frac{1}{(t-r)^s} f'(t)\, dt \tag{5.1.23}$$

for some positive constant C_5 independent of $r > 0$. For sufficiently large $r > 0$, we can assume that $|E \setminus B_r| < \omega_N (1+\varepsilon)^N$ and thus, by the isoperimetric inequality stated in Lemma 5.1.5, it holds that

$$C_6\, |E \setminus B_r|^{\frac{N-s}{N}} \leq P_K(E \setminus B_r), \tag{5.1.24}$$

where C_6 is a positive constant independent of r. Thus, combining (5.1.23) with (5.1.24), we have

$$C_6\, f(r)^{\frac{N-s}{N}} \leq C_5\, f(r) - \frac{\lambda\omega_{N-1}}{s} \int_r^{\infty} \frac{1}{(t-r)^s} f'(t)\, dt.$$

Recalling the fact that $f(r) \to 0$ as $r \to \infty$, we can choose $R_1 > R_0$ such that

$$C_5\, f(r) \leq \frac{C_6}{2} f(r)^{\frac{N-s}{N}}$$

for any $r \geq R_1$. Therefore, for all $r \geq R_1$, we obtain

$$\frac{s\, C_6}{\lambda\omega_{N-1}} f(r)^{\frac{N-s}{N}} \leq - \int_r^{\infty} \frac{1}{(t-r)^s} f'(t)\, dt.$$

We integrate over (R, ∞) where $R \geq R_1$ and then we exchange the order of integration to obtain

$$\begin{aligned}
\frac{s\, C_6}{\lambda_1\omega_{N-1}} \int_R^{\infty} f(r)^{\frac{N-s}{N}}\, dr &\leq - \int_R^{\infty} \int_r^{\infty} \frac{1}{(t-r)^s} f'(t)\, dt\, dr \\
&= - \int_R^{\infty} \int_R^{t} \frac{1}{(t-r)^s} f'(t)\, dr\, dt \\
&= - \frac{1}{1-s} \int_R^{\infty} f'(t)(t-R)^{1-s}\, dt.
\end{aligned} \tag{5.1.25}$$

Moreover, recalling $f(r) \to 0$ as $r \to \infty$, we have that

$$\begin{aligned}
- \int_R^{\infty} f'(t)(t-R)^{1-s}\, dt &= - \int_R^{R+1} f'(t)(t-R)^{1-s}\, dt \\
&\quad - \int_{R+1}^{\infty} f'(t)(t-R)^{1-s}\, dt \\
&\leq f(R) - f(R+1) - \int_{R+1}^{\infty} f'(t)(t-R)^{1-s}\, dt \\
&\leq f(R) + \int_{R+1}^{\infty} f'(t)(1 - (t-R)^{1-s})\, dt \\
&\leq f(R) + (1-s) \int_{R+1}^{\infty} f(t)(t-R)^{-s}\, dt \\
&\leq f(R) + \int_R^{\infty} f(t)\, dt.
\end{aligned} \tag{5.1.26}$$

Thus, substituting (5.1.26) with (5.1.25), we obtain

$$\frac{s\, C_6}{\lambda\omega_{N-1}} \int_R^{\infty} f(r)^{\frac{N-s}{N}}\, dr \leq f(R) + \int_R^{\infty} f(r)\, dr.$$

Since $f(r)$ is small for sufficiently large $R > R_1$, we may assume

$$2\int_R^\infty f(r)\,dr \le \frac{s\,C_6}{\lambda\omega_{N-1}}\int_R^\infty f(r)^{\frac{N-s}{N}}\,dr$$

for any $R > R_2$ and some $R_2 > R_1$. Therefore, we conclude that, for any $R > R_2$,

$$C_7\int_R^\infty f(r)^{\frac{N-s}{N}}\,dr \le f(R), \tag{5.1.27}$$

where $C_7 = C_7(N,\, s,\, \lambda) := \frac{s\,C_6}{2\lambda\omega_{N-1}} > 0$.

Let $R > R_2$ be fixed such that $w_0 = |E \setminus B_R(0)| > 0$ is sufficiently small. For any $k \in \mathbb{Z}$ with $k \ge 0$, we set $\alpha := \frac{N-s}{N}$, $R_k := R + 1 - 2^{-k}$, and $w_k := f(R_k)$. Then, from (5.1.27), we have that $R_k \to R_\infty := R+1$ as $k \to \infty$ and

$$C_{10}\,2^{-(k+1)}w_{k+1}^{\alpha} \le w_k$$

for any k. Then, by iterating this estimate and recalling that w_0 can be chosen sufficiently small, we obtain that $w_k \to 0$ as $k \to \infty$. However, by the assumption, we also have that $\lim_{k\to\infty} w_k = f(R+1) = |E \setminus B_{R+1}(0)| > 0$, which is a contradiction. □

5.1.3 Proof of Theorem 5.1.2

Now we are ready to show the proof of Theorem 5.1.2. First of all, given $\alpha \in (0,\, N)$, $\mu \ge 0$, and $\beta > 0$, we define the quantity

$$E_{K,\alpha,\mu,\beta}[m] := \inf\{\mathcal{E}_{K,\alpha,\mu,\beta}(E) \mid |E| = m,\, E : \text{bounded}\} \tag{5.1.28}$$

for any $m \in (0,\, \infty)$. Because of the last term of the functional (1.0.15), we cannot expect the subadditivity of the functional $\mathcal{E}_{K,\alpha,\mu,\beta}$. Indeed, if we decompose a set $E \subset \mathbb{R}^N$ into two parts E_1 and E_2, then the Riesz potential $V_{N-1}(E)$ could be larger than the sum of $V_{N-1}(E_1)$ and $V_{N-1}(E_2)$, while the opposite happens in the case of the nonlocal perimeter P_K. This may imply that the subadditivity of (1.0.15) is not necessarily true. However, since we can move the two bounded sets far away from each other to decrease the extra potential energy arising from the decomposition, we may have a weak version of the subadditivity in terms of the quantity (5.1.28) as follows:

Lemma 5.1.6. *Let $\alpha \in (0,\, N)$, $\mu \ge 0$, and $\beta > 0$. Assume that K satisfies (K1) and (K2). Let $m_1,\, m_2 > 0$. Then, it holds that*

$$E_{K,\alpha,\mu,\beta}[m_1 + m_2] \le E_{K,\alpha,\mu,\beta}[m_1] + E_{K,\alpha,0}[m_2].$$

Proof of Lemma 5.1.6. The proof can be done in a similar way with in [84], and thus, we basically follow their strategy. Let $\eta > 0$ be arbitrary. Then, by the definition of (5.1.28), there exist bounded subsets $E_1,\, E_2 \subset \mathbb{R}^N$ with the volume constraints $|E_1| = m_1$ and $|E_2| = m_2$ such that

$$\mathcal{E}_{K,\alpha,\mu,\beta}(E_1) + \mathcal{E}_{K,\alpha,0}(E_2) \le E_{K,\alpha,\mu,\beta}[m_1] + E_{K,\alpha,0}[m_2] + \eta.$$

Since $E_1,\, E_2$ are bounded, we can find a sufficiently large number $d = d(\eta) > 0$ such that $\operatorname{dist}(E_1,\, (E_2 + d\,e_1)) \ge d/2$. Then, from Proposition 2.1.3 with $\Omega = \mathbb{R}^N$ and (5.1.19), we have

$$\begin{aligned}\mathcal{E}_{K,\alpha,\mu,\beta}(E_1 \cup (E_2 + d\,e_1)) &= P_K(E_1 \cup (E_2 + d\,e_1)) + V_{N-1}(E_1 \cup (E_2 + d\,e_1)) \\ &\quad - R_\mu(E_1 \cup (E_2 + d\,e_1) \\ &\le P_K(E_1) + P_K(E_2 + d\,e_1) \\ &\quad + V_{N-1}(E_1) + V_{N-1}(E_2 + d\,e_1) \\ &\quad + 2\int_{E_1}\int_{E_2 + d\,e_1} \frac{1}{|x-y|}\,dx\,dy - R_\mu(E_1) \\ &\le \mathcal{E}_{K,\alpha,\mu,\beta}(E_1) + \mathcal{E}_{K,\alpha,0}(E_2) + \frac{2m_1\,m_2}{d}. \end{aligned} \tag{5.1.29}$$

Note that P_K and V_{N-1} is invariant under translations and $|x-y| \ge d/2$ for any $x \in E_1$ and $y \in E_2 + d\,e_1$. Hence, by the definition of (5.1.28), we obtain

$$E_{K,\alpha,\mu,\beta}[m_1 + m_2] \le E_{K,\alpha,\mu,\beta}[m_1] + E_{K,\alpha,0}[m_2] + \eta + \frac{2m_1\,m_2}{d}.$$

Letting $d \to \infty$, and then $\eta \to 0$, we conclude that the lemma holds. □

Proof of Theorem 5.1.2 and Corollary 5.1.3. First of all, we prove Theorem 5.1.2. To do this, we assume that K satisfies (K1), (K2), and (K5) and suppose that there exists a bounded minimizer $E \subset \mathbb{R}^N$ with $|E| = m$ of (5.1.28) for given m. Then we will show that m actually satisfies the opposite inequality to (5.1.1). As we stated in Introduction, our strategy is to divide a minimizer into two parts and obtain the differential inequality which implies the upper bound of the volume of the minimizer. In order to divide a minimizer, we define the hyperplane $H_{\nu,l}$ by $H_{\nu,l} := \{x \in \mathbb{R}^N \mid x \cdot \nu = l\}$ for any parameters $\nu \in \mathbb{S}^{N-1}$ and $l \in \mathbb{R}$. Moreover, we set

$$H^+_{\nu,\,l} := \{x \in \mathbb{R} \mid x\cdot\nu \ge l\}, \quad H^-_{\nu,\,l} := \mathbb{R}^N \setminus H^+_{\nu,\,l}.$$

and

$$E^+_{\nu,\,l} := E \cap H^+_{\nu,\,l}, \quad E^-_{\nu,\,l} := E \cap H^-_{\nu,\,l}$$

for any set $E \subset \mathbb{R}^N$ for any $\nu \in \mathbb{S}^{N-1}$ and $l \in \mathbb{R}$. Next, we want to compare the sum of the functionals of $E^+_{\nu,\,l}$ and $E^-_{\nu,\,l}$ with the functional for E. To do this, we apply the Lemma 5.1.6 and use the minimality of E and then we have

$$\begin{aligned}\mathcal{E}_{K,\alpha,\mu,\beta}(E) = E_{K,\alpha,\mu,\beta}[m] &\le E_{K,\alpha,\mu,\beta}[|E^+_{\nu,\,l}|] + E_{K,\alpha,0}[|E^-_{\nu,\,l}|] \\ &\le \mathcal{E}_{K,\alpha,\mu,\beta}(E^+_{\nu,\,l}) + \mathcal{F}_{(K,\,0)}(E^-_{\nu,\,l}),\end{aligned} \tag{5.1.30}$$

Thus, it can be rewritten as

$$\begin{aligned}&P_K(E) + V_{N-1}(E) - R_\mu(E) \\ &\le P_K(E^+_{\nu,\,l}) + V_{N-1}(E^+_{\nu,\,l}) - R_\mu(E^+_{\nu,\,l}) + P_K(E^-_{\nu,\,l}) + V_{N-1}(E^-_{\nu,\,l}).\end{aligned} \tag{5.1.31}$$

Therefore, from Proposition 2.1.3 with $\Omega = \mathbb{R}^N$ and (5.1.19), we obtain

$$2\int_{E^+_{\nu,\,l}}\int_{E^-_{\nu,\,l}} \frac{1}{|x-y|}\,dx\,dy \le 2\int_{E^+_{\nu,\,l}}\int_{E^-_{\nu,\,l}} K(x-y)\,dx\,dy + \mu\int_{E^-_{\nu,\,l}} \frac{1}{|x|}\,dx. \tag{5.1.32}$$

Integrating the inequality (5.1.32) with respect to l from $-\infty$ to 0 and substituting (5.1.36) for (5.1.32), we have

$$\begin{aligned}&2\int_{-\infty}^0\int_{E^+_{\nu,\,l}}\int_{E^-_{\nu,\,l}} \frac{1}{|x-y|}\,dx\,dy\,dl \\ &\le 2\int_{-\infty}^0\int_{E^+_{\nu,\,l}}\int_{E^-_{\nu,\,l}} K(x-y)\,dx\,dy\,dl + \mu\int_{-\infty}^0\int_{E^-_{\nu,\,0}} \frac{1}{|x|}\,dx\,dl.\end{aligned} \tag{5.1.33}$$

By interchanging the role of $E^+_{\nu,\,l}$ and $E^-_{\nu,\,l}$ in the above calculations, we obtain

$$\begin{aligned}
&2\int_0^\infty \int_{E^+_{\nu,\,l}} \int_{E^-_{\nu,\,l}} \frac{1}{|x-y|}\, dx\, dy\, dl \\
&\leq 2\int_0^\infty \int_{E^+_{\nu,\,l}} \int_{E^-_{\nu,\,l}} K(x-y)\, dx\, dy\, dl + \mu \int_0^\infty \int_{E^+_{\nu,\,0}} \frac{1}{|x|}\, dx\, dl.
\end{aligned} \tag{5.1.34}$$

Thus, summing up (5.1.33) and (5.1.34), we have that

$$\begin{aligned}
&2\int_{-\infty}^\infty \int_{E^+_{\nu,\,l}} \int_{E^-_{\nu,\,l}} \frac{1}{|x-y|}\, dx\, dy\, dl \\
&\leq 2\int_{-\infty}^\infty \int_{E^+_{\nu,\,l}} \int_{E^-_{\nu,\,l}} K(x-y)\, dx\, dy\, dl \\
&\quad + \mu \int_0^\infty \int_{E^+_{\nu,\,l}} \frac{1}{|x|}\, dx\, dl + \mu \int_{-\infty}^0 \int_{E^-_{\nu,\,l}} \frac{1}{|x|}\, dx\, dl
\end{aligned} \tag{5.1.35}$$

for any $\nu \in \mathbb{S}^{N-1}$ and $l \in \mathbb{R}$. Considering the third term of the right-hand side in (5.1.35), by the layer cake formula and Fubini's theorem (see, for instance, [78] for the statements), we may obtain the following result:

$$\begin{aligned}
\int_{-\infty}^0 \int_{E^-_{\nu,\,l}} \frac{1}{|x|}\, dx\, dl &= \int_{-\infty}^0 \int_{E^-_{\nu,\,0}} \frac{\chi_{\{x\cdot\nu<l\}}(x)}{|x|}\, dx\, dl \\
&= \int_{E^-_{\nu,\,0}} \int_{-\infty}^0 \frac{\chi_{(x\cdot\nu,\,0)}(l)}{|x|}\, dl\, dx = \int_{E^-_{\nu,\,0}} \frac{-x\cdot\nu}{|x|}\, dx
\end{aligned} \tag{5.1.36}$$

where χ_A is the characteristic function on a set $A \subset \mathbb{R}^N$. The similar result to (5.1.36) can be obtained if we consider the second term of the right-hand side in (5.1.35). Therefore, from (5.1.35) we obtain

$$\begin{aligned}
&2\int_{-\infty}^\infty \int_{E^+_{\nu,\,l}} \int_{E^-_{\nu,\,l}} \frac{1}{|x-y|}\, dx\, dy\, dl \\
&\leq 2\int_{-\infty}^\infty \int_{E^+_{\nu,\,l}} \int_{E^-_{\nu,\,l}} K(x-y)\, dx\, dy\, dl \\
&\quad + \mu \int_0^\infty \int_{E^+_{\nu,\,l}} \frac{1}{|x|}\, dx\, dl + \mu \int_{-\infty}^0 \int_{E^-_{\nu,\,l}} \frac{1}{|x|}\, dx\, dl
\end{aligned} \tag{5.1.37}$$

Now, considering the left-hand side in (5.1.37) and using the layer cake formula and Fubini's theorem again, we have that

$$\begin{aligned}
&\int_{-\infty}^\infty \int_{E^+_{\nu,\,l}} \int_{E^-_{\nu,\,l}} \frac{1}{|x-y|}\, dx\, dy\, dl \\
&= \int_E \int_E \int_{-\infty}^\infty \frac{\chi_{\{x\cdot\nu<l\}}(x)\, \chi_{\{y\cdot\nu\geq l\}}(y)}{|x-y|}\, dl\, dx\, dy \\
&= \int_E \int_E \int_{-\infty}^\infty \frac{\chi_{\{x\cdot\nu<l<y\cdot\nu\}}(l)}{|x-y|}\, dl\, dx\, dy \\
&= \int_E \int_E \frac{((y-x)\cdot\nu)_+}{|x-y|}\, dx\, dy.
\end{aligned} \tag{5.1.38}$$

Similarly, regarding to the first term of the right-hand side in (5.1.37), we also have that

$$\int_{-\infty}^{\infty}\int_{E_{\nu,l}^{+}}\int_{E_{\nu,l}^{-}} K(x-y)\,dx\,dy\,dl = \int_{E}\int_{E}((y-x)\cdot\nu)_{+}\,K(x-y)\,dx\,dy. \tag{5.1.39}$$

For any fixed $x \in \mathbb{R}^N$, by the spherical polar coordinates with x located on the x_N-axis, we obtain

$$\int_{\mathbb{S}^{N-1}} (x\cdot\nu)_{+}\,d\mathcal{H}^{N-1}(\nu) = \int_{\mathbb{S}^{N-2}}\int_{0}^{\frac{\pi}{2}} |x|\,\cos\theta\,d\theta\,d\mathcal{H}^{N-2} = \omega_{N-2}\,|x|. \tag{5.1.40}$$

Since $\nu \in \mathbb{S}^{N-1}$ is any element, we integrate the left-hand side of (5.1.38) over $\nu \in \mathbb{S}^{N-1}$ and, by (5.1.40) and by using Fubini's theorem again we have

$$\begin{aligned}
&\int_{\mathbb{S}^{N-1}}\int_{E}\int_{E}\frac{((y-x)\cdot\nu)_{+}}{|x-y|}\,dx\,dy\,d\mathcal{H}^{N-1}(\nu)\\
&= \int_{E}\int_{E}\int_{\mathbb{S}^{N-1}}\frac{((y-x)\cdot\nu)_{+}}{|x-y|}\,d\mathcal{H}^{N-1}(\nu)\,dx\,dy\\
&= \omega_{N-2}\,|E|^2.
\end{aligned} \tag{5.1.41}$$

Moreover, by Fubini's theorem and (5.1.40), we also obtain

$$\int_{\mathbb{S}^{N-1}}\int_{E}\frac{|x\cdot\nu|}{|x|}\,dx\,d\mathcal{H}^{N-1}(\nu) = 2\omega_{N-2}\,|E|. \tag{5.1.42}$$

With respect to the first term in (5.1.37), by (K5), (5.1.39), (5.1.38), and by using Fubini's theorem, we may compute as follows:

$$\begin{aligned}
&\int_{\mathbb{S}^{N-1}}\int_{-\infty}^{\infty}\int_{E_{\nu,l}^{+}}\int_{E_{\nu,l}^{-}} K(x-y)\,dx\,dy\,dl\,d\mathcal{H}^{N-1}(\nu)\\
&\quad= \int_{E}\int_{E}\int_{\mathbb{S}^{N-1}}((y-x)\cdot\nu)_{+}\,K(x-y)\,d\mathcal{H}^{N-1}(\nu)dx\,dy\\
&\quad= \omega_{N-2}\int_{E}\int_{E}|x-y|\,K(x-y)\,dx\,dy.\\
&\quad\le \omega_{N-2}\int_{E}\int_{B_{1+\varepsilon}(y)}|x-y|\,K(x-y)\,dx\,dy\\
&\qquad\quad+\omega_{N-2}\int_{E}\int_{E\cap B^{c}_{1+\varepsilon}(y)}|x-y|\,K(x-y)\,dx\,dy\\
&\quad\le \omega_{N-2}\,\lambda\int_{E}\int_{B_{1+\varepsilon}(y)}\frac{dx\,dy}{|x-y|^{N+s-1}}+\omega_{N-2}\int_{E}\int_{E}\frac{1}{(1+\varepsilon)^{N+s-1}}\,dx\,dy\\
&\quad= \omega_{N-2}\,\lambda\int_{\mathbb{S}^{N-1}}\int_{0}^{1+\varepsilon}\frac{1}{r^s}\,dr\,d\mathcal{H}^{N-1}+\frac{\omega_{N-2}}{(1+\varepsilon)^{N+s-1}}|E|^2\\
&\quad= \frac{\omega_{N-2}\,\omega_{N-1}\,\lambda}{1-s}(1+\varepsilon)^{1-s}|E|+\frac{\omega_{N-2}}{(1+\varepsilon)^{N+s-1}}|E|^2.
\end{aligned} \tag{5.1.43}$$

Therefore, integrating both sides in (5.1.37) with respect to ν in $\mathbb{S}^{N-1}$ and substituting (5.1.41), (5.1.42), and (5.1.43), we obtain

$$2\omega_{N-2}\,|E|^2 \le \frac{2\omega_{N-2}\,\omega_{N-1}\,\lambda}{1-s}(1+\varepsilon)^{1-s}|E|+\frac{2\omega_{N-2}}{(1+\varepsilon)^{N+s-1}}|E|^2+2\mu\omega_{N-2}\,|E|.$$

Hence, recalling that $\varepsilon > 0$ and $|E| = m$, we obtain

$$\left(1-\frac{1}{(1+\varepsilon)^{N+s-1}}\right)m \le \frac{\omega_{N-1}\,\lambda\,(1+\varepsilon)^{1-s}}{1-s}+\mu.$$

This completes the proof of Theorem 5.1.2.

Finally, Corollary 5.1.3 can be readily obtained in the following manner: If we assume that K satisfies (K1), (K2), (K3), and (K4), then every minimizer of the functional $\mathcal{E}_{K,\alpha,\mu,\beta}$ under the volume constraint is bounded, thanks to Proposition 5.1.1. Therefore, applying the same argument shown in the proof of Theorem 5.1.2, we may conclude that the claim of Corollary 5.1.3 is valid and this completes the proof. □

5.2 Existence and Asymptotic Behaviour of Minimizers

In the previous section, we studied a generalization of the liquid drop model in which the background potential term $-R_{\mu,\beta}$ of (1.0.15) exists. On the other hand, in this section, we focus on Problem (1.0.14) without any background potential, namely, the case that $\mu = 0$. Moreover, we restrict ourselves to consider the case that $K(x) \coloneqq |x|^{-(N+s)}$, while in general we assume that g decays fast at infinity. In this case, Problem (1.0.14) can be regarded as a nonlocal extension of the classical liquid drop model of Gamow type.

5.2.1 Problem Setting and Main Results

Let $s \in (0, 1)$ and $m > 0$. We study Problem (1.0.14) in the case that $K(x) = |x|^{-(N+s)}$ and $\mu = 0$, namely, the problem

$$E_{s,g}[m] \coloneqq \inf \left\{ \mathcal{E}_{s,g}(E) \mid E \subset \mathbb{R}^N : \text{measurable}, \, |E| = m \right\} \tag{5.2.1}$$

for any $m > 0$, where we define $\mathcal{E}_{s,g}$ as

$$\mathcal{E}_{s,g}(E) \coloneqq P_s(E) + V_g(E) \tag{5.2.2}$$

for any measurable set $E \subset \mathbb{R}^N$. Note that the first term P_s of (5.2.2) is the *s-fractional perimeter* with $s \in (0, 1)$ (see also Chapter 2), and the second term V_g of (5.2.2) is the Riesz potential defined in (1.0.17), where $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is a non-negative and measurable function.

Now we give several assumptions on the kernel g of the Riesz potential in the functional $\mathcal{E}_{s,g}$. Throughout this section, we assume that $s \in (0, 1)$ and $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is in $L^1_{loc}(\mathbb{R}^N)$ and not identically equal to zero. We basically assume that g satisfies

(g1) g is non-negative and radially non-increasing, namely,

$$g(\lambda\, x) \leq g(x) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\} \text{ and } \lambda \geq 1.$$

(g2) g is symmetric with respect to the origin, namely, $g(-x) = g(x)$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

When we prove the existence of minimizers of $\mathcal{E}_{s,g}$ in Section 5.2.6, we further assume the following condition on g:

(g3) There exist constants $R_0 > 1$ and $\beta \in (0, 1)$ such that

$$g(x) \leq \frac{\beta}{|x|^{N+s}} \quad \text{for any } |x| \geq R_0.$$

(g decays faster than the kernel of P_s far away from the origin.)

On the other hand, when we prove the existence of generalized minimizers of $\widetilde{\mathcal{E}}_{s,g}$ in Subsection 5.2.5, we assume the following condition, weaker than (g3):

(g4) g vanishes at infinity, namely, $g(x) \to 0$ as $|x| \to \infty$.

Note that (g4) is weaker than (g3). Moreover, when we study the asymptotic behavior of rescaled minimizers with large volumes in Section 5.2.6, we further impose the following assumption on g:

(g5) There exists a constant $\gamma \in (0,\,1)$ such that

$$g(x) \le \frac{\gamma}{|x|^{N+s}} \quad \text{for any } x \in \mathbb{R}^N \setminus \{0\}, \quad g(x) = o\left(\frac{1}{|x|^{N+s}}\right) \quad \text{as } |x| \to \infty.$$

Note that (g5) is stronger than (g3). Thus, we have the following implication of the conditions on g:

$$\text{(g5)} \Rightarrow \text{(g3)} \Rightarrow \text{(g4)}$$

where $p \Rightarrow q$ means that p implies q.

Remark 5.2.1. From the assumption that $g \in L^1_{loc}(\mathbb{R}^N)$, we can easily show that $V_g(B) < \infty$ for any ball $B \subset \mathbb{R}^N$. Indeed, one may compute

$$V_g(B) \le \int_B \int_{2B(y)} g(x-y)\,dx\,dy = |B| \int_{2B(0)} g(x)\,dx < \infty.$$

Moreover, if g satisfies (g3), we have that g is integrable in $\mathbb{R}^N$. Indeed, if $g \in L^1_{loc}(\mathbb{R}^N)$, we have that $\|g\|_{L^1(B_{R_0})} < \infty$. On the other hand, from (g3) and the integrability of $|x|^{-(N+s)}$ in $B^c_{R_0}$, we also have that $\|g\|_{L^1(B^c_{R_0})} < \infty$. Hence, the claim holds true.

Remark 5.2.2. A condition ensuring assumption (g5) is the existence of constants $R_0 > 1$, $\gamma \in (0,\,1)$, and $t > s$ such that

$$g(x) \le \begin{cases} \dfrac{\gamma}{|x|^{N+s}} & \text{if } 0 < |x| < R_0 \\ \dfrac{1}{|x|^{N+t}} & \text{if } |x| \ge R_0. \end{cases} \tag{5.2.3}$$

One can readily observe that (5.2.3) is stronger than (g3).

Now we state our main results of this chapter. For the first result, we show the existence of minimizers of $\mathcal{E}_{s,g}$ for any volume under the assumption that g decays faster than the kernel of P_s at infinity (precisely (g3)).

Theorem 5.2.3. *Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions* (g1), (g2), *and* (g3). *Then, there exists a minimizer of $\mathcal{E}_{s,g}$ with the volume m for any $m > 0$.*

Moreover, the boundary of every minimizer has the regularity of class $C^{1,\alpha}$ with $\alpha \in (0,\,1)$ except a closed set of Hausdorff dimension $N-3$.

The strategy of the proof is inspired by the "concentration-compactness" lemma by P.L. Lions [79, 80] and has been adapted by many authors (see, for instance, [67, 45, 32] for topics closely related to ours). We will give some intuitive explanation of the strategy before proving the claim in Section 5.2.6.

For the second theorem, we show the existence of generalized minimizer of $\widetilde{\mathcal{E}}_{s,g}$ for any volume, under the assumption that g vanishes at infinity (precisely (g4)).

Theorem 5.2.4. *Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions* (g1), (g2), *and* (g4). *Then, there exists a generalized minimizer of $\widetilde{\mathcal{E}}_{s,g}$ for any $m > 0$, namely, there exist a number $M \in \mathbb{N}$ and a sequence of sets $\{E^k\}_{k\in\mathbb{N}}$ such that*

$$\sum_{k=1}^{M} \mathcal{E}_{s,g}(E^k) = \inf\left\{\widetilde{\mathcal{E}}_{s,g}(\{E^k\}_k) \mid \sum_{k=1}^{M} |E^k| = m\right\},$$

and E^k is also a minimizer of $\mathcal{E}_{s,g}$ among sets of volume $|E^k|$ for every $k \in \mathbb{N}$.

The idea of the proof is to show the identity

$$\inf\{\mathcal{E}_{s,g}(E) \mid |E| = m\} = \inf\left\{\widetilde{\mathcal{E}}_{s,g}(\{E^k\}_k) \mid \sum_{k=1}^{\infty}|E^k| = m\right\}$$

for any $m > 0$ and to apply the "concentration-compactness" method that we use in the proof of our first result, namely Theorem 5.2.3.

Finally, we study the asymptotic behavior of minimizers of $\mathcal{E}_{s,g}$ when the volume goes to infinity, under the assumption that g decays much faster than the kernel $|x|^{-(N+s)}$ of P_s at infinity.

In order to study the behavior of the minimizers of Problem $E_{s,g}[m]$ for any $m > 0$, it is convenient to lift the volume constraint onto the functional itself and work with fixed volume $|B_1|$. To see this, we first define the rescaled kernel g_λ by

$$g_\lambda(x) \coloneqq \lambda^{N+s}\, g(\lambda\, x) \tag{5.2.4}$$

for any $x \neq 0$ and $\lambda > 0$. Then we show the equivalence of the rescaled problem in the following proposition.

Proposition 5.2.5 (Equivalent Problem)**.** *Let $m > 0$. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is in $L^1_{loc}(\mathbb{R}^N)$. Then, setting $\lambda^N \coloneqq m\,|B_1|^{-1}$, we have that the problem $E_{s,g}[m]$ is equivalent to*

$$E^{\lambda}_{s,g}(B_1) \coloneqq \inf\left\{P_s(F) + V_{g_\lambda}(F) \mid F \subset \mathbb{R}^N : \textit{measurable, } |F| = |B_1|\right\}$$

where g_λ is given in (5.2.4)*.*

Moreover, if g is integrable in $\mathbb{R}^N$, Problem $E_{s,g}[m]$ is also equivalent to Problem (1.0.23)*.*

Proof. Given any E with $|E| = m$ and setting $F \coloneqq \lambda^{-1}\, E$ where $\lambda^N = m\,|B_1|^{-1}$, we have that $|F| = |B_1|$ and

$$\begin{aligned}
\mathcal{E}_{s,g}(E) &= \lambda^{N-s}\, P_s(F) + \lambda^{2N} \int_F\int_F g(\lambda(x-y))\,dx\,dy \\
&= \lambda^{N-s}\left(P_s(F) + \int_F\int_F \lambda^{N+s} g(\lambda(x-y))\,dx\,dy\right) \\
&= \lambda^{N-s}\left(P_s(F) + V_{g_\lambda}(F)\right)
\end{aligned} \tag{5.2.5}$$

where $g_\lambda(x) \coloneqq \lambda^{N+s}\, g(\lambda\, x)$ as in (5.2.4). For the latter part of the claim, we first recall the equivalent minimization problem

$$\widehat{E}_{s,g}[m] \coloneqq \inf\left\{P_s(E) - \int_E\int_{E^c} g(x-y)\,dx\,dy\right\},$$

which is equivalent to the problem $E_{s,g}[m]$ for any $m > 0$. Thus, from (5.2.5), we obtain that

$$\mathcal{E}_{s,g}(E) = \lambda^{N-s}\left(P_s(F) - \int_F\int_{F^c} g_\lambda(x-y)\,dx\,dy + m\,\|g\|_{L^1(\mathbb{R}^N)}\right).$$

Hence, we conclude that the claim is valid. □

Now we are ready to state the last theorem of this chapter.

Theorem 5.2.6. *Let $s \in (0, 1)$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset (1, \infty)$ with $\lambda_n \to \infty$ as $n \to \infty$. Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of minimizers for $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$ with $|F_n| = |B_1|$ for each $n \in \mathbb{N}$. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions* (g1)*,* (g2)*, and* (g5)*. Then, the sequence $\{F_n\}_{n\in\mathbb{N}}$ converges to the unit ball B_1, up to translations, in the sense of L^1-topology, namely,*

$$|F_n \Delta B_1| \xrightarrow[n\to\infty]{} 0.$$

The idea of the proof is based on the same argument of Theorem 5.2.3, and we will give the precise description of the proof in Section 5.2.6. Moreover, we show the Γ-convergence of the functional $\widehat{\mathcal{E}}^{\lambda}_{s,g}$ as $\lambda \to \infty$. For the detail, we refer to Proposition 5.2.16 in Section 5.2.6.

Remark 5.2.7. A remarkable feature of our main results is that, if the kernel g of the Riesz potential decays faster than that of the s-fractional perimeter (and is not necessarily compactly supported), then there always exists a minimizer of $\mathcal{E}_{s,g}$ in Problem (5.2.1) for any volumes. This phenomena is not well-understood in the classical case. Indeed, S. Rigot [103] proved the existence of minimizers of $\mathcal{E}_g$ for any volumes if the kernel g has a compact support. For the case that the kernel g does not have a compact support, M. Pegon [100] recently showed the existence of minimizers of $\mathcal{E}_g$ only for sufficiently large volumes whenever g decays sufficiently fast. In contrast, we reveal that, even if the kernel g does not necessarily have a compact support, there exists a minimizer of the energy $\mathcal{E}_{s,g}$ for any volumes whenever g decays sufficiently fast. Thus, we can say that, unlike the classical cases studied in [103] and [100], the s-fractional perimeter, which can be understood as an interpolation between the classical perimeter and volume measure, would play an important role of ensuring the existence of minimizers for any volumes.

5.2.2 Preliminary Results on Minimizers of $\mathcal{E}_{s,g}$

In this subsection, we collect several properties for minimizers of $\mathcal{E}_{s,g}$ under suitable assumptions on g described in Subsection 5.2.1.

First of all, we recall one important property on the s-fractional perimeter P_s with $0 < s < 1$.

Proposition 5.2.8. *For any $s \in (0, 1)$ and measurable set $E \subset \mathbb{R}^N$ with $|E| < \infty$, it follows that $P_s(E \cap K) \le P_s(E)$ for every convex set $K \subset \mathbb{R}^N$.*

The proof can be found in [56, Lemma B.1] and we do not give a proof of this proposition here. We also refer to [22, Corollary 5.3] and [4] for related properties to Proposition 5.2.8.

The assumption that g is radially non-increasing enables us to show the scaling property of $\mathcal{E}_{s,g}$ by simple computations.

Lemma 5.2.9 (Scaling lemma)**.** *Let $E \subset \mathbb{R}^N$ be a measurable set with $|E| < \infty$. Assume that the kernel $g : \mathbb{R}^N\setminus \to \mathbb{R}$ satisfies* (g1) *and* (g2)*. Then, for any $\lambda \ge 1$, it follows that*

$$\mathcal{E}_{s,g}(\lambda E) \le \lambda^{2N} \mathcal{E}_{s,g}(E).$$

Proof. From the change of variables and the choice of $\lambda > 1$, we have

$$P_s(\lambda E) = \int_{\lambda E}\int_{\lambda E^c} \frac{dx\,dy}{|x-y|^{N+s}} = \lambda^{N-s}\int_E\int_{E^c} \frac{dx\,dy}{|x-y|^{N+s}} = \lambda^{N-s} P_s(E) \le \lambda^{2N} P_s(E) \tag{5.2.6}$$

for any $E \subset \mathbb{R}^N$ and $\lambda > 1$. From the assumptions on g and the change of variables again, we can compute the Riesz potential as follows:

$$V_g(\lambda E) = \lambda^{2N} \int_E \int_E g(\lambda(x-y))\, dx\, dy \le \lambda^{2N} \int_E \int_E g(x-y)\, dx\, dy = \lambda^{2N} V_g(E) \quad (5.2.7)$$

for any $E \subset \mathbb{R}^N$ with $|E| < \infty$ and $\lambda > 1$. Therefore, from (5.2.6) and (5.2.7), we obtain

$$\mathcal{E}_{s,g}(\lambda E) \le \lambda^{2N}(P_s(E) + V_g(E)) \le \lambda^{2N} \mathcal{E}_{s,g}(E)$$

and this completes the proof. □

We next prove the boundedness of minimizers of $\mathcal{E}_{s,g}$ among sets of volume m.

Lemma 5.2.10 (Boundedness of minimizers)**.** *Let $m > 0$. Assume that the kernel g satisfies* (g1) *and* (g2)*. If $E \subset \mathbb{R}^N$ is a minimizer of $\mathcal{E}_{s,g}$ with $|E| = m$, then E is essentially bounded, namely, there exists a constant $\hat{R} > 0$ such that $|E \setminus B_{\hat{R}}(0)| = 0$.*

Proof. Let E be a minimizer of $\mathcal{E}_{s,g}$ with $|E| = m$. By setting $\phi(r) := |E \setminus B_r(0)|$ for any $r > 0$, we have that $\phi'(r) = -\mathcal{H}^{N-1}(E \cap \partial B_r(0))$ for a.e. $r > 0$. In order to prove the claim, we suppose by contradiction that $\phi(r) > 0$ for any $r > 0$. Setting $E_r := E \cap B_r(0)$ for any $r > 0$ and $\lambda_r := \frac{m}{m-\phi(r)}$ for any $r > 0$, then we choose $\lambda_r E_r$ as the competitor of E if $\phi(r) < m$ and thus, we have that

$$\begin{aligned} \mathcal{E}_{s,g}(E) &\le \mathcal{E}_{s,g}(\lambda_r E_r) \le (\lambda_r)^{N-s} P_s(E_r) + (\lambda_r)^{2N} V_{g_{\lambda_r}}(E_r) \\ &\le \mathcal{E}_{s,g}(E_r) + \left((\lambda_r)^{N-s} - 1\right) P_s(E_r) + \left((\lambda_r)^{2N} - 1\right) V_g(E_r). \quad (5.2.8) \end{aligned}$$

Since $\phi(r) \to 0$ as $r \to \infty$, we can choose a constant $R_0 > 0$ such that $\phi(r) \le m/2$ for any $r \ge R_0$ and thus, we may assume that

$$(\lambda_r)^{N-s} - 1 \le c_0\, \phi(r), \quad (\lambda_r)^{2N} - 1 \le c_0'\, \phi(r) \quad (5.2.9)$$

for any $r \ge R_0$ where c_0 and c_0' are some positive constants depending only on N, s, and m. Then, by using the decomposition property of P_s and V_g and combining (5.2.9) with (5.2.8), we have that

$$\begin{aligned} P_s(E \setminus B_r(0)) &\le P_s(E \setminus B_r(0)) + V_g(E \setminus B_r(0)) \\ &\le 2 \int_{E \cap B_r(0)} \int_{E \setminus B_r(0)} \frac{dx\, dy}{|x-y|^{N+s}} + c_0\, \phi(r)\, P_s(E_r) + c_0'\, \phi(r)\, V_g(E_r) \end{aligned} \quad (5.2.10)$$

for any $r \ge R_0$. From Proposition 5.2.8 and the definition of V_g, we have that, for any $r > 0$,

$$P_s(E_r) + V_g(E_r) \le P_s(E) + V_g(E) = E_{s,g}[m].$$

Thus, from (5.2.10), we obtain that

$$P_s(E \setminus B_r(0)) \le 2 \int_{E \cap B_r(0)} \int_{E \setminus B_r(0)} \frac{dx\, dy}{|x-y|^{N+s}} + (c_0 + c_0') E_{s,g}[m]\, \phi(r)$$

for any $r \geq R_0$. Now using the isoperimetric inequality of P_s and the fact that $E \cap B_r(0) \subset B^c_{|y|-r}(y)$ for any $y \in E \setminus B_r(0)$, we obtain

$$\begin{aligned}
\frac{P_s(B_1)}{|B_1|^{\frac{N-s}{N}}}\, \phi(r)^{\frac{N-s}{N}} &\leq 2 \int_{E \setminus B_r(0)} \int_{B^c_{r-|y|}(y)} \frac{dx\, dy}{|x-y|^{N+s}} + (c_0 + c_0')E_{s,g}[m]\, \phi(r) \\
&= \frac{2|\partial B_1|}{s} \int_{E \setminus B_r(0)} \int_{|y|-r}^{\infty} \frac{1}{t^{1+s}}\, dt\, dy + (c_0 + c_0')E_{s,g}[m]\, \phi(r) \\
&= \frac{2|\partial B_1|}{s} \int_{E \setminus B_r(0)} \frac{1}{(|y|-r)^s}\, dy + (c_0 + c_0')E_{s,g}[m]\, \phi(r) \\
&= \frac{2|\partial B_1|}{s} \int_r^{\infty} \frac{-\phi'(\sigma)}{(\sigma - r)^s}\, d\sigma + (c_0 + c_0')E_{s,g}[m]\, \phi(r) \qquad (5.2.11)
\end{aligned}$$

for any $r > 0$. Here we have used the co-area formula in the last equality. Since ϕ is non-increasing, there exists a constant $R_0' = R_0'(N, s, m) > 0$ such that

$$(c_0 + c_0')E_{s,g}[m]\, \phi(r) \leq \frac{P_s(B_1)}{2|B_1|^{\frac{N-s}{N}}}\, \phi(r)^{\frac{N-s}{N}} \qquad (5.2.12)$$

for any $r \geq \max\{R_0,\, R_0'\}$. From (5.2.11) and (5.2.12), we obtain

$$c_1\, \phi(r)^{\frac{N-s}{N}} \leq c_2 \int_r^{\infty} \frac{-\phi'(\sigma)}{(\sigma - r)^s}\, d\sigma \qquad (5.2.13)$$

for any $r \geq \max\{R_0,\, R_0'\}$ where we set $c_1 := (2|B_1|^{\frac{N-s}{N}})^{-1}\, P_s(B_1)$ and $c_2 := 2s^{-1}|\partial B_1|$. By integrating the both sides in (5.2.13) over $r \in [R, \infty)$ for any fixed constant $R \geq \max\{R_0,\, R_0'\}$ and changing the order of the integration, we obtain

$$\begin{aligned}
c_1 \int_R^{\infty} \phi(r)^{\frac{N-s}{N}}\, dr \leq c_2 \int_R^{\infty} \int_r^{\infty} \frac{-\phi'(\sigma)}{(\sigma - r)^s}\, d\sigma\, dr &= c_2 \int_R^{\infty} \int_R^{\sigma} \frac{-\phi'(\sigma)}{(\sigma - r)^s}\, dr\, d\sigma \\
&= -\frac{c_2}{1-s} \int_R^{\infty} \phi'(\sigma)\, (\sigma - R)^{1-s}\, d\sigma.
\end{aligned} \qquad (5.2.14)$$

Hence, by employing the same argument shown in [45, Lemma 4.1] and [32, Proposition 3.2] together with (5.2.14), we obtain that $\phi(R) = 0$, which contradicts the assumption that $\phi(r) > 0$ for any $r > 0$. Therefore, we conclude the existence of the constant $\hat{R} > 0$ such that $|E \setminus B_{\hat{R}}| = 0$. □

Next, by using assumption (g4), we show the sub-additivity result of the function $m \mapsto E_{s,g}[m]$. We recall that $E_{s,g}[m]$ is defined by

$$\inf \left\{ \mathcal{E}_{s,g}(E) \mid E \subset \mathbb{R}^N \text{: measurable, } |E| = m \right\}.$$

for any $m > 0$.

Lemma 5.2.11 (Sub-additivity of $E_{s,g}$)**.** *Let $m > 0$ be any number. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies* (g1), (g2), *and* (g4). *Then, for any $m_1 \in (0, m]$, it holds*

$$E_{s,g}[m] \leq E_{s,g}[m_1] + E_{s,g}[m - m_1].$$

Proof. The idea is in the same spirit as the one shown in [84, Lemma 3] (see also [99]).

Let $m > 0$ be any constant and we take any $m_1 \in (0, m)$. By definition, for any $\eta > 0$, there exist measurable sets E_1, $E_2 \subset \mathbb{R}^N$ with the volume constraints $|E_1| = m_1$ and $|E_2| = m - m_1$ such that

$$\mathcal{E}_{s,g}(E_1) + \mathcal{E}_{s,g}(E_2) \leq E_{s,g}[m_1] + E_{s,g}[m_2] + \eta. \tag{5.2.15}$$

Now we may assume that E_1 and E_2 are bounded. Indeed, we can observe that the minimum of $\mathcal{E}_{s,g}$ among unbounded sets of volume m is not smaller than the minimum of $\mathcal{E}_{s,g}$ among bounded sets of volume m. To see this, for any unbounded set E with $|E| = m$, we can choose sufficiently large $R > 1$ in such a way that $|E \setminus B_R(0)|$ is as small as possible. Then, setting $\widehat{E} := \lambda(R)\,(E \cap B_R(0))$ where $\lambda(R)^N := \frac{m}{m-|E\setminus B_R(0)|} \geq 1$, we obtain, from Lemma 5.2.9, that

$$|\widehat{E}| = \lambda(R)^N\,(m - |E \setminus B_R(0)|) = m$$

and

$$\begin{aligned}\mathcal{E}_{s,g}(\widehat{E}) &\leq \lambda(R)^{2N}\mathcal{E}_{s,g}(E \cap B_R(0))\\ &\leq \lambda(R)^{2N}\mathcal{E}_{s,g}(E) - P_s(E \setminus B_R(0)) + 2\int_{E\cap B_R(0)}\int_{E\setminus B_R(0)} \frac{dx\,dy}{|x-y|^{N+s}}.\end{aligned} \tag{5.2.16}$$

Here we have used the following identity of the s-fractional perimeter:

$$P_s(E \cup F) = P_s(E) + P_s(F) - 2\int_E\int_F \frac{1}{|x-y|^{N+s}}\,dx\,dy$$

for any disjoint sets E, $F \subset \mathbb{R}^N$. From the isoperimetric inequality and the computation in (5.2.11) in Lemma 5.2.10, we have that

$$\mathcal{E}_{s,g}(\widehat{E}) \leq \lambda(R)^{2N}\mathcal{E}_{s,g}(E) - C_1\,|E \setminus B_R(0)|^{\frac{N-s}{N}} + C_2\int_R^\infty \frac{\mathcal{H}^{N-1}(E \cap \partial B_\sigma(0))}{(\sigma - R)^s}\,d\sigma \tag{5.2.17}$$

where we set $C_1 := P_s(B_1)\,|B_1|^{-\frac{N-s}{N}}$ and $C_2 := 2s^{-1}|\partial B_1|$. Since E is unbounded, we have that the function $R \mapsto |E \setminus B_R(0)|$ is non-increasing and not equal to zero for any $R > 0$. Thus, by applying the same argument in Lemma 5.2.10, we can find that there exists a sequence $\{R_i\}_{i\in\mathbb{N}}$ such that $R_i \to \infty$ as $i \to \infty$ and

$$-C_1\,|E \setminus B_{R_i}(0)|^{\frac{N-s}{N}} + C_2\int_{R_i}^\infty \frac{\mathcal{H}^{N-1}(E \cap \partial B_\sigma(0))}{(\sigma - R_i)^s}\,d\sigma < 0 \tag{5.2.18}$$

for any $i \in \mathbb{N}$. Hence, from (5.2.17) and (5.2.18), it follows that

$$\inf\{\mathcal{E}_{s,g}(E) \mid E\colon \text{bounded}, |E| = m\} \leq \mathcal{E}_{s,g}(\widehat{E}) < \lambda(R_i)^{2N}\mathcal{E}_{s,g}(E)$$

for any $i \in \mathbb{N}$. From the fact that $\lambda(R_i) \to 1$ as $i \to \infty$, the arbitrariness of E and by letting $i \to \infty$, we finally obtain that

$$\inf\{\mathcal{E}_{s,g}(E) \mid E\colon \text{bounded}, |E| = m\} \leq \inf\{\mathcal{E}_{s,g}(E) \mid E\colon \text{unbounded}, |E| = m\},$$

as we desired.

Now we focus on the case that both E_1 and E_2 are bounded. Since E_1, E_2 are bounded, we can find a vector $e \in \mathbb{S}^{N-1}$ such that it follows that

$$\operatorname{dist}(E_1, (E_2 + d\,e)) \xrightarrow[d\to\infty]{} \infty.$$

Then, computing $\mathcal{E}_{s,g}(E_1 \cup (E_2 + de))$, we have the following:

$$\begin{aligned}\mathcal{E}_{s,g}(E_1 \cup (E_2 + d\,e)) &= P_s(E_1 \cup (E_2 + d\,e)) + V_g(E_1 \cup (E_2 + d\,e))\\ &\leq P_s(E_1) + P_s(E_2 + d\,e)\\ &\qquad + V_g(E_1) + V_g(E_2 + d\,e) + 2\int_{E_1}\int_{E_2+d\,e} g(x-y)\,dx\,dy\\ &\leq \mathcal{E}_{s,g}(E_1) + \mathcal{E}_{s,g}(E_2) + 2\int_{E_1}\int_{E_2+d\,e} g(x-y)\,dx\,dy.\end{aligned}$$

Here we have used the translation invariance of P_s and V_g. From assumption (g4), which says that g vanishes at infinity, we can show that

$$\int_{E_1}\int_{E_2+d\,e} g(x-y)\,dx\,dy \xrightarrow[d\to\infty]{} 0.$$

Since $|E_1 \cup (E_2 + d\,e)| = |E_1| + |E_2| = m$ for sufficiently large $d > 0$ and from (5.2.15), we obtain

$$E_{s,g}[m_1 + m - m_1] \leq E_{s,g}[m_1] + E_{s,g}[m - m_1] + \eta + o(1).$$

Letting $d \to \infty$ and then $\eta \to 0$, we conclude that the lemma is valid. □

Finally in this subsection, we prove the density estimate of minimizers of $\mathcal{E}_{s,g}$ for any $m \geq 1$.

Lemma 5.2.12 (Density Estimates for Minimizers of $\mathcal{E}_{s,g}$). *Let $m \geq 1$. We assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is integrable in $\mathbb{R}^N$ and satisfies the assumptions* (g1) *and* (g2)*. Then there exist constants $c_0 > 0$, $c_1 > 0$, and $r_0 > 0$, depending only on N, s, and g, such that, if E is a minimizer of $\mathcal{E}_{s,g}$ with $|E| = m$, then it holds that*

$$|E \cap B_r(x_0)| \geq c_0\, r^N, \quad |E^c \cap B_r(x_0)| \geq c_1\, r^N$$

for any $0 < r < \min\{r_0, (\frac{m}{|B_1|})^{1/N}\}$ and $x \in \mathbb{R}^N$ with $|E \cap B_r(x_0)| > 0$ and $|E^c \cap B_r(x_0)| > 0$ for any $r > 0$.

Proof. Let E be a minimizer of $\mathcal{E}_{s,g}$ with $|E| = m$ and $x_0 \in E$ be any point such that $|E \cap B_r(x_0)| > 0$ for any $r > 0$. We set $\lambda_r^N := \frac{m}{m - |E\cap B_r(x_0)|} \geq 1$ for any $r > 0$. We may assume that $|E \cap B_r(x_0)| < m$ for any $0 < r < (\frac{m}{|B_1|})^{1/N}$. Then, from the minimality of E, we have the inequality

$$\mathcal{E}_{s,g}(E) \leq \mathcal{E}_{s,g}\left(\lambda_r(E \setminus B_r(x_0))\right)$$

for any $r > 0$. Then, from Proposition 2.1.3 with $\Omega = \mathbb{R}^N$ and $K(x) = |x|^{-(N+s)}$, Lemma 5.2.9, (5.2.17), and the fact that $V_g(E \setminus B_r(x_0)) \leq V_g(E)$, we have that

$$\begin{aligned}&\mathcal{E}_{s,g}(E)\\ &\leq \mathcal{E}_{s,g}(E \setminus B_r(x_0)) + (\lambda_r^{2N} - 1)\,\mathcal{E}_{s,g}(E \setminus B_r(x_0))\\ &\leq \mathcal{E}_{s,g}(E \setminus B_r(x_0))\\ &\qquad + (\lambda_r^{2N} - 1)\left(\mathcal{E}_{s,g}(E) - P_s(E \cap B_r(x_0)) + \int_{E\cap B_r(x_0)}\int_{E\setminus B_r(x_0)} \frac{2\,dx\,dy}{|x-y|^{N+s}}\right) \end{aligned} \tag{5.2.19}$$

for $0 < r < (\frac{m}{|B_1|})^{1/N}$. Similarly to (5.2.17), we also have the following identity on the Riesz potential:

$$V_g(E \cup F) = V_g(E) + V_g(F) + 2\int_E\int_F g(x-y)\,dx\,dy \tag{5.2.20}$$

for any measurable sets E, $F \subset \mathbb{R}^N$ with $E \cap F = \emptyset$. Thus, from (5.2.17), (5.2.19), and (5.2.20), we further have that

$$\lambda_r^{2N} P_s(E \cap B_r(x_0)) \leq \lambda_r^{2N} \int_{E\cap B_r(x_0)} \int_{E\setminus B_r(x_0)} \frac{2\,dx\,dy}{|x-y|^{N+s}} + (\lambda_r^{2N} - 1)\, E_{s,g}[m] \tag{5.2.21}$$

for any $0 < r < (\frac{m}{|B_1|})^{1/N}$. Recalling the definition of λ_r, we have that

$$\lambda_r^{2N} = \frac{m^2}{(m - |E\cap B_r|)^2}, \quad \lambda_r^{2N} - 1 = \frac{|E\cap B_r|}{m - |E\cap B_r|}\left(2 + \frac{|E\cap B_r|}{m - |E\cap B_r|}\right) \tag{5.2.22}$$

for any $0 < r < (\frac{m}{|B_1|})^{1/N}$. From (5.2.21) and (5.2.22), we finally obtain

$$P_s(E\cap B_r(x_0)) \leq \int_{E\cap B_r(x_0)} \int_{E\setminus B_r(x_0)} \frac{2\,dx\,dy}{|x-y|^{N+s}} + \frac{2E_{s,g}[m]}{m}|E\cap B_r(x_0)|$$

for any $0 < r < (\frac{m}{|B_1|})^{1/N}$. Hence, from the nonlocal isoperimetric inequality, we have that

$$\begin{aligned}
&\frac{P_s(B_1)}{|B_1|^{\frac{N-s}{N}}}|E\cap B_r(x_0)|^{\frac{N-s}{N}} \\
&\leq P_s(E\cap B_r(x_0)) + V_g(E\cap B_r(x_0)) + 2\int_{E\cap B_r(x_0)}\int_{E\setminus B_r(x_0)} g(x-y)\,dx\,dy \\
&\leq 2\int_{E\cap B_r(x_0)}\int_{E\setminus B_r(x_0)} \frac{1}{|x-y|^{N+s}}\,dx\,dy + \frac{2\,E_{s,g}[m]}{m}|E\cap B_r(x_0)|
\end{aligned} \tag{5.2.23}$$

for any small $r > 0$. Noticing that $E \setminus B_r(x_0) \subset B^c_{r-|y-x_0|}(y)$ for any $y \in E \cap B_r(x_0)$ and from the co-area formula, we have the following estimate:

$$\begin{aligned}
\int_{E\cap B_r(x_0)}\int_{E\setminus B_r(x_0)} \frac{1}{|x-y|^{N+s}}\,dx\,dy &\leq \int_{E\cap B_r(x_0)}\int_{B^c_{r-|y-x_0|}(y)} \frac{1}{|x-y|^{N+s}}\,dx\,dy \\
&\leq \frac{|\partial B_1|}{s}\int_{E\cap B_r(x_0)} \frac{1}{(r-|y-x_0|)^s}\,dy \\
&= \frac{|\partial B_1|}{s}\int_0^r \frac{\mathcal{H}^{N-1}(E\cap\partial B_\sigma)}{(r-\sigma)^s}\,d\sigma.
\end{aligned} \tag{5.2.24}$$

Now we set a function $\phi(r) := |E\cap B_r(x_0)|$ for any $r > 0$ and we have that, for a.e. $r > 0$, $\phi'(r) = \mathcal{H}^{N-1}(E\cap\partial B_r)$. Thus we obtain from (5.2.23) and (5.2.24), that

$$C(N,s)\,\phi(r)^{\frac{N-s}{N}} \leq \frac{2|\partial B_1|}{s}\int_0^r \frac{\phi'(\sigma)}{(r-\sigma)^s}\,d\sigma + \frac{2\,E_{s,g}[m]}{m}\,\phi(r) \tag{5.2.25}$$

for any small $r > 0$ where $C(N,s) := |B_1|^{-\frac{N-s}{N}}\,P_s(B_1)$. Now we show that $m^{-1}E_{s,g}[m]$ is bounded by the constant independent of $m \geq 1$. Indeed, from the definition of $E_{s,g}[m]$ and by changing the variable $x \mapsto r_m\,x$, we first have that

$$\begin{aligned}
E_{s,g}[m] &\leq \mathcal{E}_{s,g}(B_{r_m}) \\
&= P_s(B_{r_m}) + V_g(B_{r_m}) \\
&\leq \left(\frac{m}{|B_1|}\right)^{\frac{N-s}{N}} P_s(B_1) + \left(\frac{m}{|B_1|}\right)^2 2\int_{B_1}\int_{B_1} g(r_m(x-y))\,dx\,dy,
\end{aligned} \tag{5.2.26}$$

where $r_m > 0$ is the constant with $|B_{r_m}| = m$. Moreover, from the assumptions on g and by changing the variable again, we have that

$$\begin{aligned}\int_{B_1}\int_{B_1} g(r_m(x-y))\,dx\,dy &\leq \int_{B_1(0)} \sup_{y\in\mathbb{R}^N}\int_{B_1(0)} g(r_m\,(x-y))\,dx\,dy\\ &= |B_1|\,r_m^{-N}\int_{B_{r_m}(0)} g(x)\,dx \leq |B_1|^2\|g\|_{L^1(\mathbb{R}^N)}\,m^{-1}. \qquad (5.2.27)\end{aligned}$$

Thus, from (5.2.26), (5.2.27) and the assumption that $m \geq 1$, we obtain

$$m^{-1}E_{s,g}[m] \leq \frac{P_s(B_1)}{|B_1|^{\frac{N-s}{N}}} m^{-\frac{s}{N}} + 2\|g\|_{L^1(\mathbb{R}^N)} \leq \frac{P_s(B_1)}{|B_1|^{\frac{N-s}{N}}} + 2\|g\|_{L^1(\mathbb{R}^N)} =: \tilde{C}(N,s,g)$$

and this completes the proof of the claim. Since ϕ is non-decreasing and $\phi(r) \leq |B_1|\,r^N$ for any $r > 0$, we have that

$$4\,\tilde{C}(N,s,g)\,\phi(r) \leq C(N,s)\,\phi(r)^{\frac{N-s}{N}}$$

for any $r \in (0,\,r_0]$ where r_0 is defined by

$$r_0 := \left(\frac{P_s(B_1)}{4\tilde{C}(N,s,g)|B_1|}\right)^{\frac{1}{s}}.$$

Then integrating the both side of (5.2.25) over $r \in [0,\,r']$ for any $r' \in (0,\,r_1)$ where we set $r_1 := \min\{r_0, (\frac{m}{|B_1|})^{1/N}\}$, we obtain that

$$\frac{C(N,s)}{2}\int_0^{r'} \phi(r)^{\frac{N-s}{N}}\,dr \leq \frac{2|\partial B_1|}{s}\int_0^{r'}\int_0^{r} \frac{\phi'(\sigma)}{(r-\sigma)^s}\,d\sigma\,dr.$$

By changing the order of the integral, we have that

$$\begin{aligned}C'(N,s)\int_0^{r'} \phi(r)^{\frac{N-s}{N}}\,dr &\leq \int_0^{r'}\int_\sigma^{r'} \frac{\phi'(\sigma)}{(r-\sigma)^s}\,dr\,d\sigma\\ &= \frac{1}{1-s}\int_0^{r'} \phi'(r)\,(r'-r)^{1-s}\,dr\\ &\leq \frac{(r')^{1-s}}{1-s}\phi(r')\end{aligned}$$

for any $r' \in (0,\,r_1)$ where we set $C'(N,s) := 4^{-1}\,|\partial B_1|^{-1}\,s\,C(N,s)$. Now in order to prove the uniform density estimate, we suppose by contradiction that there exists a constant $r_2 \in (0,\,r_1)$ such that

$$|E \cap B_{r_2}(x_0)| \leq c_0^{-\frac{N}{s}}\,r_2^N, \quad c_0 := \frac{s(1-s)\,P_s(B_1)}{16|\partial B_1|\,|B_1|^{\frac{N-s}{N}}}.$$

Then by applying the same argument in [56, Lemma 3.1], we can obtain $|E\cap B_{\frac{r_2}{2}}(x_0)| = 0$, which is a contradiction to the choice of x_0. Notice that the constants c_0 and r_0 are independent of E, x_0, and r. □

5.2.3 Existence of Minimizers for $\mathcal{E}_{s,g}$

In this subsection, we prove Theorem 5.2.3, namely, the existence of minimizers of the functional $\mathcal{E}_{s,g}$ for any volume $m > 0$ under the assumption that the kernel g of the Riesz potential decays faster than the kernel $|x|^{-(N+s)}$ of the s-fractional perimeter P_s.

The idea to prove the existence is based on the argument by Di Castro, M. Novaga, B. Ruffini, and E. Valdinoci [45] (see also [67, 32]). As we mentioned in the introduction, the idea was originally inspired by the so-called "concentration-compactness" principle introduced by P.L. Lions in [79, 80]. When one studies the variational problems in unbounded domain, the possible loss of compactness may occur from the vanishing or splitting into many pieces of minimizing sequences, so that one cannot naively apply the direct method of calculus of variations to these problems.

Although the proof of this method may be technical, we briefly explain the strategy of it in the following; when we obtain the existence of minimizers of the minimization problems of isoperimetric type, we usually apply the direct method in the calculus of variations. More precisely, we first take any minimizing sequence; then we try to construct another sequence from the minimizing sequence in such a way that the new elements are uniformly bounded and the functional of the new sequence is smaller than that of the original sequence (one may often refer to this procedure as "truncation"); thus, by some compactness, we can extract a convergent subsequence in proper topology; finally, we may conclude that, by lower semi-continuity, the limit of the subsequence should be a minimizer as desired.

Unfortunately, in our problem, we might not be able to easily construct another sequence, which satisfies "good" properties we want, from the original minimizing sequence. One possible reason is as follows; as is well-known, the s-fractional perimeter P_s behaves like an attracting term, while the Riesz potential associated with the kernel g could disaggregate minimizers into many different components. Moreover, in general, as the volume of a minimizer gets larger, the effect that separates minimizers into pieces from the Riesz potential may get stronger. However it is not obvious whether or not the nonlocal perimeter term can overcome such an effect from the Riesz potential because we cannot easily capture the precise behavior of a general kernel g. Therefore, we select the following strategy to handle the problems: first, taking any minimizing sequence $\{E_n\}_n$ of $\mathcal{E}_{s,g}$ with $|E_n| = m > 0$, we decompose each element E_n into many pieces with the cubes $\{Q^i_n\}_i$ in such a way that each piece has non-negligible volumes. Then we "properly" collect all the components $\{E_n \cap Q^i_n\}_i$ of E_n such that $\mathrm{dist}\,(Q^i_n, Q^j_n) \to c^{ij} < \infty$ as $n \to \infty$ for $i \neq j$ (the case that $c^{ij} = \infty$ for $i \neq j$ is called the "dichotomy" in the sense of Lions'). Thanks to the uniformly boundedness of $\{P_s(E_n)\}_n$ and the isoperimetric inequality of P_s, we can obtain a sequence of the limit sets $\{G^i\}_i$ of the components of E_n that we have "properly" collected such that $\{G^i\}_i$ is the collection with $c^{ij} = \infty$ for any $i \neq j$. Now we need to show that the amount of the volume of $\{G^i\}_i$ is equal to m (this means that we exclude the "vanishing phenomena" in the sense of Lions'). Once we have shown that $\sum_i |G^i| = m$, the faster decay of the kernel g in the Riesz potential enables us to prove that the only one element in $\{G_i\}_i$ must be the true minimizer of $\mathcal{E}_{s,g}$ among sets of volume m.

Proof of Theorem 5.2.3. Let $m > 0$ be any number and let $\{E_n\}_{n\in\mathbb{N}}$ be any minimizing sequence for the energy $\mathcal{E}_{s,g}$ with $|E_n| = m$.

Step1. We first show that, under (g1) and (g2), there exist sets $\{G^j\}_j$ such that

$$\sum_j \mathcal{E}_{s,g}(G^j) \leq \liminf_{n\to\infty} \mathcal{E}_{s,g}(E_n), \quad \sum_j |G^j| = m.$$

We first decompose $\mathbb{R}^N$ into infinitely many disjoint cubes of side 1 (denoted by $\{Q^i\}_{i=1}^{\infty}$). Thanks to Lemma 5.2.10, we may assume that each E_n is bounded. Then, we can choose a number $I_n \in \mathbb{N}$ in such a way that $|E_n \cap Q^i| > 0$ for any $i \in \{1, \cdots, I_n\}$ and $|E_n \cap Q^i| = 0$ for any $i > I_n$. We set $x_n^i := |E_n \cap Q^i|$ and we have that

$$\sum_{i=1}^{I_n} x_n^i = |E_n| = m \tag{5.2.28}$$

for any $n \in \mathbb{N}$. Since E_n is a minimizer with $|E_n| = m$ for any n, we can choose a ball with the volume m as a competitor and then, from the local integrability of g, have

$$\sup_{n\in\mathbb{N}} P_s(E_n) \leq P_s(B_m) + V_g(B_m) \leq \left(\frac{m}{|B_1|}\right)^{\frac{N-s}{N}} P_s(B_1) + m\,\|g\|_{L^1(2B_m)} < \infty \tag{5.2.29}$$

where B_m is the open ball with the volume m for each $m > 0$. From (5.2.29) and the isoperimetric inequality shown in [45, Lemma 2.5], we obtain

$$\sum_{i=1}^{\infty} (x_n^i)^{\frac{N-s}{N}} \leq C \sum_{i=1}^{\infty} P_s(E_n; Q^i) \leq 2C P_s(E_n) \leq C_1 < \infty \tag{5.2.30}$$

for any $n \in \mathbb{N}$, where $C > 0$ and $C_1 > 0$ are the constants independent of n. Here we recall that $P_s(E;Q)$ is the localized s-fractional perimeter of E in Q defined by

$$P_s(E;Q) := \int_{E\cap Q}\int_{E^c} \frac{dx\,dy}{|x-y|^{N+s}} + \int_{E\cap Q^c}\int_{E^c\cap Q} \frac{dx\,dy}{|x-y|^{N+s}}.$$

Up to reordering the cubes $\{Q^i\}_i$, we may assume that $\{x_n^i\}_i$ is a non-increasing sequence for any $n \in \mathbb{N}$. Thus, applying the technical result shown in [67, Lemma 4.2] or [45, Lemma 7.4] with (5.2.28) and (5.2.30), we obtain that

$$\sum_{i=k+1}^{\infty} x_n^i \leq \frac{C_1\, m^{\frac{s}{N}}}{(k+1)^{\frac{s}{N}}} \tag{5.2.31}$$

for any $k \in \mathbb{N}$. Indeed, since we may assume that $\{x_n^i\}_i$ is a non-increasing sequence, we have that, for $K \in \mathbb{N}$,

$$\sum_{i=K+1}^{\infty} x_n^i = \sum_{i=K+1}^{\infty} (x_n^i)^{\frac{s}{N}} (x_n^i)^{\frac{N-s}{N}} \leq (x_{K+1}^n)^{\frac{s}{N}} \sum_{i=K+1}^{\infty} (x_n^i)^{\frac{N-s}{N}} \leq (x_{K+1}^n)^{\frac{s}{N}} C_1 < \infty. \tag{5.2.32}$$

On the other hand, we also have that, for $K \in \mathbb{N}$,

$$m \geq \sum_{i=1}^{K+1} x_n^i \geq (K+1)\, x_{K+1}^n. \tag{5.2.33}$$

From (5.2.32) and (5.2.33), we finally obtain

$$\sum_{i=K+1}^{\infty} x_n^i \leq (x_{K+1}^n)^{\frac{s}{N}} C_1 \leq \frac{C_1\, m^{\frac{s}{N}}}{(k+1)^{\frac{s}{N}}}.$$

Hence, by using the diagonal argument, we have that, up to extracting a subsequence, $x_n^i \to \alpha^i \in [0, m]$ as $n \to \infty$ for every $i \in \mathbb{N}$. From (5.2.28) and (5.2.31), we obtain that

$$\sum_{i=1}^{M} \alpha^i = \lim_{n\to\infty} \sum_{i=1}^{M} x_n^i = \lim_{n\to\infty} \left(\sum_{i=1}^{\infty} x_n^i - \sum_{i=M+1}^{\infty} x_n^i \right) = m - \lim_{n\to\infty} \sum_{i=M+1}^{\infty} x_n^i$$

for any $M \in \mathbb{N}$ and thus, letting $M \to \infty$, we have

$$\sum_{i=1}^{\infty} \alpha^i = m. \tag{5.2.34}$$

Now we choose a point $z_n^i \in E_n \cap Q^i$ for each i and n. Up to extracting a further subsequence, we may assume that $|z_n^i - z_n^j| \to c^{ij} \in [0, \infty]$ as $n \to \infty$ for each $i, j \in \mathbb{N}$ and, since we have, from (5.2.29), the uniform bound of the sequence $\{P_s(E_n - z_n^i)\}_{n\in\mathbb{N}}$ and its upper-bound is independent of i, there exists a measurable set $G^i \subset \mathbb{R}^N$ such that, up to a subsequence,

$$\chi_{E_n - z_n^i} \xrightarrow[n\to\infty]{} \chi_{G^i} \quad \text{in } L^1_{loc}\text{-topology.}$$

We define the relation $i \sim j$ for every $i, j \in \mathbb{N}$ as $c^{ij} < \infty$ and we denote by $[i]$ the equivalent class of i. Moreover, we define the set of the equivalent class by $\mathcal{I}$. Then, in the following, we show a sort of lower semi-continuity, More precisely,

$$\sum_{[i]\in\mathcal{I}} P_s(G^{[i]}) \leq \liminf_{n\to\infty} P_s(E_n), \quad \sum_{[i]\in\mathcal{I}} V_g(G^{[i]}) \leq \liminf_{n\to\infty} V_g(E_n). \tag{5.2.35}$$

Indeed, we first fix $M \in \mathbb{N}$ and $R > 0$ and we take the equivalent classes $i_1, \cdots, i_M$. Notice that, if $p \neq q$, then $|z_n^{i_p} - z_n^{i_q}| \to \infty$ as $n \to \infty$ and thus we have that $\{z_n^{i_p} + Q_R\}_p$ are disjoint sets for large n and

$$\int_{z_n^{i_p}+Q_R} \int_{z_n^{i_q}+Q_R} \frac{1}{|x-y|^{N+s}}\, dx\, dy \xrightarrow[n\to\infty]{} 0$$

where Q_R is the cube of side R. We recall the inequality of the s-fractional perimeter as follows:

$$P_s(E; A) + P_s(E; B) \leq P_s(E; A \cup B) + 2\int_A \int_B \frac{dx\, dy}{|x-y|^{N+s}} \tag{5.2.36}$$

for any measurable disjoint sets $A, B \subset \mathbb{R}^N$. Here $P_s(E; A)$ is as in As a consequence, from the lower semi-continuity of P_s, we obtain

$$\begin{aligned}
\sum_{p=1}^{M} P_s(G^{i_p}; Q_R) &\leq \liminf_{n\to\infty} \sum_{p=1}^{M} P_s(E_n - z_n^{i_p}; Q_R) \\
&= \liminf_{n\to\infty} \sum_{p=1}^{M} P_s(E_n; z_n^{i_p} + Q_R) \\
&\leq \liminf_{n\to\infty} P_s\left(E_n; \bigcup_{p=1} \left(z_n^{i_p} + Q_R\right)\right) \\
&\qquad + \liminf_{n\to\infty} 2\sum_{p\neq q} \int_{z_n^{i_p}+Q_R} \int_{z_n^{i_q}+Q_R} \frac{dx\, dy}{|x-y|^{N+s}} \\
&\leq \liminf_{n\to\infty} P_s(E_n).
\end{aligned}$$

Letting $R \to \infty$ and then $M \to \infty$, we obtain the first claim of (5.2.35). For the second claim, we again take any $M \in \mathbb{N}$ and $R > 0$. We recall the identity

$$V_g(A) + V_g(B) = V_g(A \cup B) - 2\int_A \int_B g(x-y)\, dx\, dy$$

for any measurable disjoint set A, $B \subset \mathbb{R}^N$. Then, in the same way as we have observed in the first claim, we have, from Fatou's lemma and the non-negativity of g, that

$$\begin{aligned}\sum_{p=1}^{M} V_g(G^{i_p} \cap Q_R) &\leq \liminf_{n\to\infty} \sum_{p=1}^{M} V_g\left(\left(E_n - z_n^{i_p}\right) \cap Q_R\right) \\ &= \liminf_{n\to\infty} \sum_{p=1}^{M} V_g\left(E_n \cap \left(z_n^{i_p} + Q_R\right)\right) \\ &\leq \liminf_{n\to\infty} V_g\left(E_n \cap \bigcup_{p=1}^{M} \left(z_n^{i_p} + Q_R\right)\right) \\ &\leq \liminf_{n\to\infty} V_g(E_n).\end{aligned}$$

Here we have used the fact that the sets $\{z_n^{i_p} + Q_R\}_{p=1}^M$ are disjoint if n is sufficiently large from the choice of the points $\{z_n^{i_p}\}_{p=1}^M$. Thus, letting $R \to \infty$ and then $M \to \infty$, we obtain the second claim.

Now we show that

$$\sum_{[i]\in\mathcal{I}} |G^{[i]}| = m.$$

Indeed, from the L^1_{loc}-convergence of $\{\chi_{E_n - z_n^i}\}_{n\in\mathbb{N}}$ for any i, we have that, for any $R > 0$ sufficiently large,

$$|G^i| \geq |G^i \cap Q_R| = \lim_{n\to\infty} |(E_n - z_n^i) \cap Q_R|. \tag{5.2.37}$$

If $j \in \mathbb{N}$ is such that $j \sim i$ and $c^{ij} < \frac{R}{100}$, then we have that $Q^j - z_n^i \subset Q_R$ for large $R > 0$ and all n. Thus, from (5.2.37), it follows

$$\begin{aligned}|(E_n - z_n^i) \cap Q_R| &= \sum_{j\in[i]} |(E_n - z_n^i) \cap Q_R \cap \left(Q^j - z_n^i\right)| \\ &\geq \sum_{j:\, c^{ij} < \frac{R}{100}} |(E_n - z_n^i) \cap Q_R \cap \left(Q^j - z_n^i\right)| \\ &= \sum_{j:\, c^{ij} < \frac{R}{100}} |E_n \cap Q^j|\end{aligned} \tag{5.2.38}$$

for all n and large $R > 0$. Therefore, combining (5.2.38) with (5.2.37), we obtain

$$|G^i| \geq \sum_{j:\, c^{ij} < \frac{R}{100}} \alpha^j$$

and, letting $R \to \infty$, we have

$$|G^i| \geq \sum_{j:\, c^{ij} < \infty} \alpha^j = \sum_{j\in[i]} \alpha^j.$$

Hence, recalling (5.2.34), we have

$$\sum_{[i]\in\mathcal{I}} |G^{[i]}| \geq \sum_{[i]\in\mathcal{I}} \sum_{j\in[i]} \alpha^j = m. \tag{5.2.39}$$

For the other inequality, we can easily obtain from the choice of $\{G^i\}_i$ in the following manner; for any $M \in \mathbb{N}$ and $R > 0$, we take the equivalent classes $i_1, \cdots, i_M$ and then have that

$$\sum_{p=1}^{M} |G^{i_p} \cap Q_R| = \lim_{n\to\infty} \sum_{p=1}^{M} \left| \left(E_n - z_n^{i_p} \right) \cap Q_R \right|$$
$$= \lim_{n\to\infty} \sum_{p=1}^{M} \left| E_n \cap \left(z_n^{i_p} + Q_R \right) \right|. \tag{5.2.40}$$

Recalling the condition that $|z_n^{i_p} - z_n^{i_q}| \to \infty$ as $n \to \infty$ if $p \neq q$, we have that, for sufficiently large $n \in \mathbb{N}$, $\left(z_n^{i_p} + Q_R \right) \cap \left(z_n^{i_q} + Q_R \right) = \emptyset$ for any $p \neq q$. From (5.2.40), we have that

$$\sum_{p=1}^{M} |G^{i_p} \cap Q_R| = \lim_{n\to\infty} \left| E_n \cap \bigcup_{p=1}^{M} \left(z_n^{i_p} + Q_R \right) \right| \leq m$$

and thus, letting $R \to \infty$ and then $M \to \infty$, we obtain that

$$\sum_{[i]\in\mathcal{I}} |G^{[i]}| = \sum_{p=1}^{\infty} |G^{i_p}| \leq m.$$

This completes the proof of the claim. Taking into account all the above arguments, we obtain the existence of sets $\{G^{[i]}\}_{[i]\in\mathcal{I}}$ satisfying the properties that

$$\sum_{[i]\in\mathcal{I}} \mathcal{E}_{s,g}(G^{[i]}) \leq \liminf_{n\to\infty} \mathcal{E}_{s,g}(E_n), \qquad \sum_{[i]\in\mathcal{I}} |G^{[i]}| = m. \tag{5.2.41}$$

Step 2. We now claim that, under (g1), (g2), and (g3), each particle G^i for $[i] \in \mathcal{I}$ is a minimizer of $\mathcal{E}_{s,g}$ among sets with volume $|G^i|$. Moreover, we show that G^i is bounded for each $[i] \in \mathcal{I}$.

Indeed, we first recall the definition of $E_{s,g}$, which says that

$$E_{s,g}[m] \coloneqq \inf \{\mathcal{E}_{s,g}(E) \mid |E| = m\}$$

for any $m > 0$, and the sub-additivity result of the function $m \mapsto E_{s,g}[m]$ as is shown in Lemma 5.2.11. Notice that, in this theorem, we impose assumption (g3) as we show in Subsection 5.2.2, which is stronger than (g4). Thus, we can apply Lemma 5.2.11 to the case in the present proof. Then, from (5.2.41), we have that

$$\begin{aligned}
\sum_{p=1}^{M} \left(\mathcal{E}_{s,g}(G^{i_p}) - E_{s,g}[|G^{i_p}|] \right) &\leq E_{s,g}[m] - \sum_{p=1}^{M} E_{s,g}[|G^{i_p}|] \\
&\leq E_{s,g}\left[\sum_{p=M+1}^{\infty} |G^{i_p}| \right] + E_{s,g}\left[\sum_{p=1}^{M} |G^{i_p}| \right] - \sum_{p=1}^{M} E_{s,g}[|G^{i_p}|] \\
&\leq E_{s,g}\left[\sum_{p=M+1}^{\infty} |G^{i_p}| \right]
\end{aligned} \tag{5.2.42}$$

for any $M \in \mathbb{N}$. We can observe that $E_{s,g}[m] \to E_{s,g}[0] = 0$ as $m \to 0$ because $E_{s,g}[m]$ can be bounded by the quantity $C_1\, m^{\frac{N-s}{N}} + C_2\, m$ for small $m > 0$, where C_1 and C_2 are the constants depending only on N, s, and g. Hence, letting $M \to \infty$ in (5.2.42), we obtain that

$$\sum_{[i]\in\mathcal{I}} \left(\mathcal{E}_{s,g}(G^{i_p}) - E_{s,g}[|G^{i_p}|] \right) = \sum_{p=1}^{\infty} \left(\mathcal{E}_{s,g}(G^{i_p}) - E_{s,g}[|G^{i_p}|] \right) \leq 0$$

and, from the fact that each term of the series is non-negative, we conclude that each term of the series is equal to zero. This implies that, for every $[i] \in \mathcal{I}$, G^i is a minimizer of $\mathcal{E}_{s,g}$ among sets with the volume $|G^i|$. To see the boundedness of $\{G^i\}_{[i]\in\mathcal{I}}$, it is sufficient to apply Lemma 5.2.10 to G^i for each $[i] \in \mathcal{I}$. This completes the proof of Step 2.

Step 3. We now show that, under (g1) and (g2), there exist a number $H \in \mathbb{N}$ and a family of bounded sets $\{\widetilde{G}^p\}_{p=1}^H$ such that

$$\sum_{p=1}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) \le \sum_{[i]\in\mathcal{I}} \mathcal{E}_{s,g}(G^{[i]}) \le \liminf_{n\to\infty} \mathcal{E}_{s,g}(E_n), \quad \sum_{p=1}^{H} |\widetilde{G}^p| = m. \tag{5.2.43}$$

Indeed, letting $\{G^{i_p}\}_{p=1}^\infty$ be as in Step 2, we first set $m^p := |G^{i_p}|$ for any $p \in \mathbb{N}$ and, since $\sum_{p=1}^\infty m^p = m$, we can observe that $m^p \to 0$ as $p \to \infty$ and, moreover, $\mu_\ell := \sum_{p=\ell+1}^\infty m^p \to 0$ as $\ell \to \infty$. Then, we can choose $\widetilde{p} \in \mathbb{N}$ such that $m^{\widetilde{p}} \ge \frac{m}{2^{\widetilde{p}+1}}$. Now using the sets $\{G^{i_p}\}_{p=1}^\infty$, we construct a new family of sets $\{\widetilde{G}^p\}_{p=1}^H$ for some $H \in \mathbb{N}$, depending only on N, s, and m, in the following manner; we choose $H \in \mathbb{N}$ so large that $H \ge \widetilde{p}$ and set $\widetilde{G}^p := G^{i_p}$ for any $p \in \{1, \cdots, H\}$ with $p \neq \widetilde{p}$ and $\widetilde{G}^{\widetilde{p}} := \lambda\, G^{i_{\widetilde{p}}}$ where $\lambda^N := \frac{m^{\widetilde{p}}+\mu_H}{m^{\widetilde{p}}}$. Then, we have the volume identity that

$$\sum_{p=1}^{H} \left|\widetilde{G}^p\right| = \sum_{p=1,\, p\neq\widetilde{p}}^{H} \left|G^{i_p}\right| + \lambda^N \left|G^{i_{\widetilde{p}}}\right| = \sum_{p=1,\, p\neq\widetilde{p}}^{H} m^p + m^{\widetilde{p}} + \mu_H = m. \tag{5.2.44}$$

Now we compute the functional for $\{\widetilde{G}^p\}_{p=1}^H$ as follows to show that the total energy of each elements of $\{\widetilde{G}^p\}_{p=1}^H$ is more efficient than that of $\{G^{i_p}\}_{p=1}^\infty$; from the definition of $\lambda \ge 1$ and Lemma 5.2.9, we have that

$$\begin{aligned}
\sum_{p=1}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) &\le \sum_{p=1,\, p\neq\widetilde{p}}^{H} \mathcal{E}_{s,g}(G^{i_p}) + \lambda^{2N}\, \mathcal{E}_{s,g}(G^{i_{\widetilde{p}}}) \\
&= \sum_{p=1}^{\infty} \mathcal{E}_{s,g}(G^{i_p}) + \left(\lambda^{2N} - 1\right) \mathcal{E}_{s,g}(G^{i_{\widetilde{p}}}) - \sum_{p=H+1}^{\infty} \mathcal{E}_{s,g}(G^{i_p}) \\
&\le \sum_{[i]\in\mathcal{I}} \mathcal{E}_{s,g}(G^{[i]}) + \frac{2^{\widetilde{p}+1}\, E_{s,g}[m]}{m}\, \mu_H - \sum_{p=H+1}^{\infty} P_s(G^{i_p}).
\end{aligned} \tag{5.2.45}$$

Here, in the last inequality, we have also used (5.2.41). From the isoperimetric inequality of P_s and (5.2.45), we further obtain that

$$\begin{aligned}
\sum_{p=1}^{H} \mathcal{E}_{s,g}(\widetilde{G}^p) &\le \sum_{[i]\in\mathcal{I}} \mathcal{E}_{s,g}(G^{[i]}) + \frac{2^{\widetilde{p}+1}\, E_{s,g}[m]}{m}\, \mu_H - C \sum_{p=H+1}^{\infty} (m^p)^{\frac{N-s}{N}} \\
&\le \sum_{[i]\in\mathcal{I}} \mathcal{E}_{s,g}(G^{[i]}) + \frac{2^{\widetilde{p}+1}\, E_{s,g}[m]}{m}\, \mu_H - C \left(\sum_{p=H+1}^{\infty} m^p \right)^{\frac{N-s}{N}} \\
&= \sum_{[i]\in\mathcal{I}} \mathcal{E}_{s,g}(G^{[i]}) + \frac{2^{\widetilde{p}+1}\, E_{s,g}[m]}{m}\, \mu_H - C\, (\mu_H)^{\frac{N-s}{N}}.
\end{aligned}$$

Taking the number H so large that $H \ge \widetilde{p}$ and

$$\frac{2^{\widetilde{p}+1}\, E_{s,g}[m]}{m}\, \mu_H - C\, (\mu_H)^{\frac{N-s}{N}} \le 0,$$

then we finally obtain (5.2.43) and this completes the proof of Step 3.

Step 4. We finally show that, under (g1), (g2), and (g3), there exists $p' \in \{1, 2, \cdots, H\}$ such that

$$\mathcal{E}_{s,g}(\widetilde{G}^{p'}) \leq \liminf_{n\to\infty} \mathcal{E}_{s,g}(E_n) = E_{s,g}[m], \quad |\widetilde{G}^{p'}| = m$$

where $H \in \mathbb{N}$ and $\{\widetilde{G}^p\}_{p=1}^H$ are given in the previous step.

From (5.2.39), there exists at least $p' \in \{1, 2, \cdots, H\}$ such that $|G^{i_{p'}}| > 0$. Moreover, since the sets $\{\widetilde{G}^p\}_{p=1}^H$ are bounded, we can choose the points $\{z^p\}_{p=1,\, p\neq p'}^H$ such that each set $\widetilde{G}^p + R\, z^p$ is disjoint with another for large $R > 1$. We can thus compute the functional as follows; from the translation invariance of $\mathcal{E}_{s,g}$, it holds that

$$\begin{aligned}
\sum_{p=1}^H \mathcal{E}_{s,g}(\widetilde{G}^p) &= \sum_{p=1\, p\neq p',q}^H \mathcal{E}_{s,g}(\widetilde{G}^p) + \mathcal{E}_{s,g}(\widetilde{G}^{p'}) + \mathcal{E}_{s,g}(\widetilde{G}^q) \\
&= \sum_{p=1\, p\neq p',q}^H \mathcal{E}_{s,g}(\widetilde{G}^p) + \mathcal{E}_{s,g}(\widetilde{G}^{p'}) + \mathcal{E}_{s,g}(\widetilde{G}^q + R\, z^{i_q}) \\
&= \sum_{p=1\, p\neq p',q}^H \mathcal{E}_{s,g}(\widetilde{G}^p) + \mathcal{E}_{s,g}(\widetilde{G}^{p'} \cup (\widetilde{G}^q + R\, z^{i_q})) \\
&\qquad + 2\int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^q + R\, z^{i_q}} \frac{dx\, dy}{|x-y|^{N+s}} - 2\int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^q + R\, z^{i_q}} g(x-y)\, dx\, dy
\end{aligned}$$

for any $q \in \{1, \cdots, H\}$ with $q \neq p'$ and sufficiently large $R > 1$. Recalling the assumption (g3) that $g(x) \leq \beta |x|^{-(N+s)}$ for any $|x| \geq R_0$ and some $\beta \in (0, 1)$, and choosing $R > 1$ in such a way that the set $\widetilde{G}^q + R\, z^{i_q}$ has the distance of more than R_0 from $\widetilde{G}^{p'}$, we obtain that

$$\begin{aligned}
\sum_{p=1}^H \mathcal{E}_{s,g}(\widetilde{G}^p) \geq &\sum_{p=1,\, p\neq p',\, q}^H \mathcal{E}_{s,g}(\widetilde{G}^p) + \mathcal{E}_{s,g}(\widetilde{G}^{p'} \cup (\widetilde{G}^q + R\, z^{i_q})) \\
&+ 2(1-\beta) \int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^q + R\, z^{i_q}} \frac{dx\, dy}{|x-y|^{N+s}}. \qquad (5.2.46)
\end{aligned}$$

By repeating the same argument finite times for the rest of the sets $\{\widetilde{G}^p\}_{p=1,\, p\neq p',q}^H$ with sufficiently large $R > 1$, we obtain the similar inequalities to (5.2.46) and, finally, we can derive the inequality that

$$\begin{aligned}
\sum_{p=1}^H \mathcal{E}_{s,g}(\widetilde{G}^p) \geq &\ \mathcal{E}_{s,g}\left(\widetilde{G}^{p'} \cup \bigcup_{p=1,\, p\neq p'}^H \left(\widetilde{G}^p + R\, z^p \right) \right) \\
&+ 2(1-\beta) \sum_{p=1,\, p\neq p'}^H \int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^p + R\, z^p} \frac{dx\, dy}{|x-y|^{N+s}}. \qquad (5.2.47)
\end{aligned}$$

Since $\widetilde{G}^{p'} \cup \bigcup_{p=1,\, p\neq p'}^H \left(\widetilde{G}^p + R\, z^p \right)$ are the union of disjoint sets, we have, from (5.2.44), that

$$\left| \widetilde{G}^{p'} \cup \bigcup_{p=1,\, p\neq p'}^H \left(\widetilde{G}^p + R\, z^p \right) \right| = \sum_{p=1}^H |\widetilde{G}^{i_p}| = m.$$

Thus, from (5.2.47), we obtain

$$
\begin{aligned}
2(1-\beta) &\sum_{p=1,\, p\neq p'}^{H} \int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^{p}+R\,z^p} \frac{dx\,dy}{|x-y|^{N+s}} + E_{s,g}[m] \\
&\leq \sum_{p=1,\, p\neq p'}^{H} \int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^{p}+R\,z^p} \frac{dx\,dy}{|x-y|^{N+s}} + \mathcal{E}_{s,g}\left(\widetilde{G}^{p'} \cup \bigcup_{p=1,\, p\neq p'}^{H} \left(\widetilde{G}^{p} + R\,z^p\right)\right) \\
&\leq \sum_{p=1}^{H} \mathcal{E}_{s,g}(\widetilde{G}^{p}) \leq E_{s,g}[m]
\end{aligned}
$$

and it follows that

$$
2(1-\beta) \sum_{p=1,\, p\neq p'}^{H} \int_{\widetilde{G}^{p'}} \int_{\widetilde{G}^{p}+R\,z^p} \frac{dx\,dy}{|x-y|^{N+s}} \leq 0
$$

for large $R > 1$. Since each term of the sum is non-negative, $\beta < 1$, and $|\widetilde{G}^{p'}| > 0$, we conclude that $|\widetilde{G}^{p}| = 0$ for all $p \neq p'$. Therefore, the final claim is valid and this completes the proof of Theorem 5.2.3. □

5.2.4 Regularity of Boundaries of Minimizers

In this subsection, we study the regularity of the boundary of minimizers of $\mathcal{E}_{s,g}$ under suitable assumptions on the kernel g. To see this, we employ the regularity results of the so-called *almost s-fractional minimal sets* shown in Section 2.4 of Chapter 2.

As a consequence of these regularity results, we obtain the regularity of minimizers of $\mathcal{E}_{s,g}$. To do this, we reduce Problem $E_{s,g}[m]$ for any $m > 0$ to another minimization problem. More precisely, we show that any solutions of Problem $E_{s,g}[m]$ are also the solutions of the unconstrained minimization problem

$$
\inf \left\{ \mathcal{E}_{s,g,\mu_0}(E) \mid E \subset \mathbb{R}^N \text{: measurable} \right\}
$$

for some constant $\mu_0 > 0$ and any $m > 0$, where we define $\mathcal{E}_{s,g,\mu_0}$ as

$$
\mathcal{E}_{s,g,\mu}(F) \coloneqq \mathcal{E}_{s,g}(F) + \mu \, ||F| - m|
$$

for any $F \subset \mathbb{R}^N$ and $\mu > 0$.

Proposition 5.2.13 (Reduction to Unconstrained Problem). *Let $m > 0$. Assume that the kernel g satisfies the conditions* (g1) *and* (g2). *Then there exists a constant $\mu_0 = \mu_0(N, s, g, m) > 0$ such that, if E is a minimizer of $\mathcal{E}_{s,g}$ with $|E| = m$, then E is also a minimizer of $\mathcal{E}_{s,g,\mu}$ among sets in $\mathbb{R}^N$ for any $\mu \geq \mu_0$.*

Proof. Suppose by contradiction that, for any $n \in \mathbb{N}$, there exist a minimizer E_n of $\mathcal{E}_{s,g}$ with $|E_n| = m$ and a constant $\mu(n) \geq n$ such that E_n is not a minimizer of $\mathcal{E}_{s,g,\mu(n)}$. Then, by assumption, we can choose a sequence $\{F_n\}_{n\in\mathbb{N}}$ such that

$$
\mathcal{E}_{s,g,\mu(n)}(F_n) < \mathcal{E}_{s,g,\mu(n)}(E_n) \tag{5.2.48}
$$

for any $n \in \mathbb{N}$. First of all, we show that $|F_n| \xrightarrow[n\to\infty]{} m$. Indeed, we set B_m as a open ball in $\mathbb{R}^N$ whose volume is equal to m. Then from (5.2.48) and the minimality of E_n with $|E_n| = m$ for any $n \in \mathbb{N}$, we have that

$$
\mathcal{E}_{s,g,\mu(n)}(F_n) < \mathcal{E}_{s,g,\mu(n)}(E_n) = \mathcal{E}_{s,g}(E_n) = E_{s,g}[m]. \tag{5.2.49}
$$

Thus, denoting r_m by the radius of the ball B_m and using the change of variables, we obtain

$$\mu(n)\,||F_n| - m| < E_{s,g}[m] < \infty \tag{5.2.50}$$

for any $n \in \mathbb{N}$. From the definition of r_m, the right-hand side in (5.2.50) is finite and independent of n. Hence, letting $n \to \infty$ in (5.2.50), we obtain the claim that $|F_n| \to m$ as $n \to \infty$. Finally, we derive a contradiction in the following manner. We first define $\widetilde{F}_n$ as $\widetilde{F}_n := \lambda_n F_n$ where $\lambda_n^N := m\,|F_n|^{-1}$ and, by definition, we can observe that $|\widetilde{F}_n| = m$. In the sequel, we may assume that, up to extracting a subsequence, $|F_n| \leq m$ for $n \in \mathbb{N}$. Indeed, we suppose by contradiction that, for any subsequence $\{F_{n_k}\}_{k\in\mathbb{N}}$ of $\{F_n\}_{n\in\mathbb{N}}$, we always have that $|F_{n_k}| > m$ for any $k \in \mathbb{N}$. From the continuity of the Lebesgue measure, for each $k \in \mathbb{N}$, there exists a constant $R_k > 0$ such that $|F_{n_k} \cap B_{R_{n_k}}(0)| = m$ for every $k \in \mathbb{N}$. Thus, from the minimality of E_n for any $n \in \mathbb{N}$ and Proposition 5.2.8, we have the estimate that

$$\mathcal{E}_{s,g,\mu(n)}(E_{n_k}) = \mathcal{E}_{s,g}(E_{n_k}) \leq \mathcal{E}_{s,g}(F_{n_k} \cap B_{R_{n_k}}(0)) \leq P_s(F_{n_k}) + V_g(F_{n_k}) = \mathcal{E}_{s,g}(F_{n_k})$$

for any $k \in \mathbb{N}$, which contradicts the estimate (5.2.48) since $\mathcal{E}_{s,g}(F_{n_k}) \leq \mathcal{E}_{s,g,\mu(n)}(F_{n_k})$ for any $k \in \mathbb{N}$. Hence, from (5.2.48), the minimality of E_n, the assumption that $\lambda_n \geq 1$ for any $n \in \mathbb{N}$, and Lemma 5.2.9, we have

$$\mathcal{E}_{s,g,\mu(n)}(F_n) < \mathcal{E}_{s,g}(E_n) \leq \mathcal{E}_{s,g}(\widetilde{F}_n) \leq \lambda_n^{2N}\mathcal{E}_{s,g}(F_n). \tag{5.2.51}$$

From the definition, we notice that $||F_n| - m| = |\lambda_n^{-N}\,m - m| = m\,\lambda_n^{-N}\,|\lambda_n^N - 1|$ for any n. Hence, from (5.2.51) and dividing the both side of (5.2.51) by $||F_n| - m|$, we obtain

$$\mu(n) \leq m^{-1}\,\lambda_n^N \frac{|\lambda_n^{2N} - 1|}{|\lambda_n^N - 1|} P_s(F_n) + m^{-1}\,\lambda_n^N \frac{|\lambda_n^{2N} - 1|}{|\lambda_n^N - 1|} V_g(F_n) \tag{5.2.52}$$

for any $n \in \mathbb{N}$. Recalling (5.2.48) and (5.2.49), we have that $P_s(F_n) + V_g(F_n) < E_{s,g}[m] < \infty$. Moreover, we observe that $\frac{|\lambda_n^{2N}-1|}{|\lambda_n^N-1|} \leq 2$ for sufficiently large $n \in \mathbb{N}$. Therefore, from (5.2.52), we obtain

$$\mu(n) \leq 6m^{-1}\,E_{s,g}[m] \tag{5.2.53}$$

for sufficiently large $n \in \mathbb{N}$ and thus obtain a contradiction. $\square$

Now we are ready to show the regularity of minimizers for $\mathcal{E}_{s,g}$

Lemma 5.2.14 (Regularity of Minimizers of $\mathcal{E}_{s,g}$). *Let $s \in (0, 1)$ and let $m > 0$. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies* (g1), (g2), *and* (g3). *If $E \subset \mathbb{R}^N$ is a minimizer of $\mathcal{E}_{s,g}$ among sets of volume m, then ∂E is of class $C^{1,\alpha}$ with some $0 < \alpha < 1$, except a closed set of Hausdorff dimension $N - 3$.*

Proof. First of all, from Lemma 5.2.10, we have the essential boundedness of the minimizer $E \subset \mathbb{R}^N$, namely, $E \subset B_{R_1}(0)$ up to negligible sets for some $R_1 > 0$. Without loss of generality, we may assume that $R_1 \geq R_0$ where R_0 is given in assumption (g3) in Subsection 5.2.2. In order to apply the regularity result of Theorem 2.4.9 to our case, it is sufficient to show that the set E is almost s-fractional minimal set in the sense of Definition 2.4.6 for some constant $\Lambda > 0$ independent of E. From Proposition 5.2.13, we know that E with $|E| = m$ is also a solution to

$$\inf\{\mathcal{E}_{s,g,\mu_0}(E) \mid E \subset \mathbb{R}^N\}.$$

where $\mu_0 > 0$ is as in Proposition 5.2.13 and is independent of E. Hence, from the minimality of E, we have that

$$\mathcal{E}_{s,g,\mu_0}(E) \leq \mathcal{E}_{s,g,\mu_0}(F) \tag{5.2.54}$$

for any bounded measurable set $F \subset \mathbb{R}^N$. We may assume that F is finite with respect to the s-fractional perimeter; otherwise the inequality (5.2.54) is obviously valid. Then from the fact that $|E| = m$, we have

$$\begin{aligned} P_s(E) &\leq P_s(F) + V_g(F) - V_g(E) + \mu_0 \,||F| - |E|| \\ &\leq P_s(F) + V_g(F) - V_g(E) + \mu_0 \,|F \Delta E|. \end{aligned} \tag{5.2.55}$$

Regarding the Riesz potential, we can compute the difference $V_g(F) - V_g(E)$ as follows:

$$\begin{aligned} |V_g(F) - V_g(E)| &\leq \left| \int_F \int_{F\cup E} g(x-y)\,dx\,dy - \int_E \int_{F\cup E} g(x-y)\,dx\,dy \right| \\ &\leq 2 \int_{F\Delta E} \int_{F\cup E} g(x-y)\,dx\,dy \\ &\leq 2|F\Delta E| \int_{\mathbb{R}^N} g(x)\,dx. \end{aligned} \tag{5.2.56}$$

Note that, from the local integrability of g and assumption (g3), the kernel g is integrable in $\mathbb{R}^N$ and thus, the right-hand side in (5.2.56) is finite. Hence, by combining (5.2.56) with (5.2.55), we obtain that

$$P_s(E) \leq P_s(F) + \left(2\|g\|_{L^1(\mathbb{R}^N)} + \mu_0\right) |F\Delta E|$$

for any measurable set $F \subset \mathbb{R}^N$. Therefore, by employing Theorem 2.4.9 in Section 2.4 of Chapter 2, we can conclude that the claim is valid. □

5.2.5 Existence of Generalized Minimizers for $\widetilde{\mathcal{E}}_{s,g}$

In this subsection, we prove Theorem 5.2.4, namely, the existence of generalized minimizers for the generalized functional $\widetilde{\mathcal{E}}_{s,g}$ given as (1.0.20) for any volumes. To see this, we impose slightly more general assumptions on g than we do to prove the existence of minimizers of $\mathcal{E}_{s,g}$ for any volumes in Section 5.2.6. More precisely, we assume that the kernel $g \in L^1_{loc}(\mathbb{R}^N)$ satisfies the assumptions (g1), (g2), and (g4) in Subsection 5.2.2.

Before proving the main theorem, we show one lemma, saying that one can modify a "generalized" minimizing sequence for the generalized functional $\widetilde{\mathcal{E}}_{s,g}$ into a "usual" minimizing sequence for the functional $\mathcal{E}_{s,g}$. More precisely, we prove

Lemma 5.2.15. *Let $s \in (0,1)$. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions* (g1)*,* (g2)*, and* (g4)*. Then, for any $m > 0$, it follows that*

$$\inf \{\mathcal{E}_{s,g}(E) \mid |E| = m\} = \inf \left\{ \widetilde{\mathcal{E}}_{s,g}(\{E^k\}_k) \mid \sum_{k=1}^{\infty} |E^k| = m \right\}.$$

Proof. The proof of this lemma proceeds in a similar manner to the method in the proof of Theorem 5.2.3; however, it seems a little technical and thus we do not omit the detail.

First of all, we can readily observe that the inequality

$$\inf \{\mathcal{E}_{s,g}(E) \mid |E| = m\} \geq \inf \left\{ \widetilde{\mathcal{E}}_{s,g}(\{E^k\}_k) \mid \sum_{k=1}^{\infty} |E^k| = m \right\}$$

holds true. Hence, it remains for us to prove the other inequality. To see this, we take any minimizing sequence $\{\{E^k_n\}_k\}_n$ for the generalized functional $\widetilde{\mathcal{E}}_{s,g}$. Then it follows that, for any $\varepsilon > 0$, there exists a number $n_0 \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} \mathcal{E}_{s,g}(E^k_n) = \widetilde{\mathcal{E}}_{s,g}(\{E^k_n\}_k) \leq \widetilde{E}_{s,g}[m] + \varepsilon, \quad \sum_{k=1}^{\infty} |E^k_n| = m \tag{5.2.57}$$

for any $n \geq n_0$. Since the minimum is attained with a minimizing sequence of which each element is bounded, we may assume that E_n^k is bounded for each k, $n \in \mathbb{N}$. In the sequel, we fix one $n \in \mathbb{N}$ with $n \geq n_0$ until we give another remark.

Step 1. We first show that, under (g1) and (g2), there exist a number $K_n \in \mathbb{N}$ and a sequence $\{\widetilde{E}_n^k\}_{k=1}^{K_n}$, constructed from $\{E_n^k\}_k$, such that

$$\sum_{k=1}^{K_n} \mathcal{E}_{s,g}(\widetilde{E}_n^k) \leq \sum_{k=1}^{\infty} \mathcal{E}_{s,g}(E_n^k), \quad \sum_{k=1}^{K_n} |\widetilde{E}_n^k| = m. \tag{5.2.58}$$

The proof of this claim is the same as Step 3 in the proof of Theorem 5.2.3 of Section 5.2.3, since we assume that the kernel g only satisfies (g1) and (g2). Thus we do not repeat the proof here.

Step 2. We now prove Lemma 5.2.15 under (g4), which says that g vanishes at infinity.

Let $\{\widetilde{E}_n^k\}_{k=1}^{K_n}$ be as in the previous step. Since we have that $\sum_{k=1}^{K_n} |E_n^k| = m$, we can choose one $k' \in \mathbb{N}$ with $|E_n^{k'}| > 0$. Since we have assumed that the sets $\{E_n^k\}_{k=1}^{K_n}$ are bounded, we can choose the points $\{z_n^k\}_{k=1,\, k\neq k'}^{K_n}$ such that each set $E_n^k + R\, z_n^k$ is far away from the others for sufficiently large $R > 1$. We can thus compute the functional as follows; from the translation invariance of $\mathcal{E}_{s,g}$, it holds that

$$\begin{aligned}
\sum_{k=1}^{K_n} \mathcal{E}_{s,g}(E_n^k) &= \sum_{k=1,\, k\neq k',\ell}^{K_n} \mathcal{E}_{s,g}(E_n^k) + \mathcal{E}_{s,g}(E_n^{k'}) + \mathcal{E}_{s,g}(E_n^{\ell}) \\
&= \sum_{k=1,\, k\neq k',\ell}^{K_n} \mathcal{E}_{s,g}(E_n^k) + \mathcal{E}_{s,g}(E_n^{k'}) + \mathcal{E}_{s,g}(E_n^{\ell} + R\, z_n^{\ell}) \\
&= \sum_{k=1,\, k\neq k',\ell}^{K_n} \mathcal{E}_{s,g}(E_n^k) + \mathcal{E}_{s,g}(E_n^{k'} \cup (E_n^{\ell} + R\, z_n^{\ell})) \\
&\qquad + 2\int_{E_n^{k'}} \int_{E_n^{\ell} + R\, z_n^{\ell}} \frac{dx\, dy}{|x-y|^{N+s}} - 2\int_{E_n^{k'}} \int_{E_n^{\ell} + R\, z_n^{\ell}} g(x-y)\, dx\, dy
\end{aligned}$$

for any $\ell \in \{1, \cdots, K_n\}$ with $\ell \neq k'$ and sufficiently large $R > 1$. Thus, we obtain that

$$\begin{aligned}
&\sum_{k=1,\, k\neq k',\ell}^{K_n} \mathcal{E}_{s,g}(E_n^k) + \mathcal{E}_{s,g}(E_n^{k'} \cup (E_n^{\ell} + R\, z_n^{\ell})) \\
&\qquad \leq \sum_{k=1}^{K_n} \mathcal{E}_{s,g}(E_n^k) + 2\int_{E_n^{k'}} \int_{E_n^{\ell} + R\, z_n^{\ell}} g(x-y)\, dx\, dy
\end{aligned} \tag{5.2.59}$$

for any $\ell \in \{1, \cdots, K_n\}$ with $\ell \neq k'$ and sufficiently large $R > 1$. By repeating the same argument finite times for the rest of the sets $\{E_n^k\}_{k=1,\, k\neq k',\ell}^{K_n}$ with sufficiently large $R > 1$ and from the translation invariance of $\mathcal{E}_{s,g}$, we can derive the inequality

$$\begin{aligned}
&\mathcal{E}_{s,g}\left(E_n^{k'} \cup \bigcup_{k=1,\, k\neq k'}^{K_n} \left(E_n^k + R\, z_n^k \right) \right) \\
&\qquad \leq \sum_{k=1}^{K_n} \mathcal{E}_{s,g}(E_n^k) + 2\sum_{k=1}^{K_n - 1} \sum_{\ell=k+1}^{K_n} \int_{F_n^k(R)} \int_{F_n^{\ell}(R)} g(x-y)\, dx\, dy
\end{aligned} \tag{5.2.60}$$

where we define the sets $\{F_n^k(R)\}_{k=1}^{K_n}$ in such a way that $F_n^k(R) := E_n^k + R\, z_n^k$ if $k \neq k'$ and $F_n^{k'}(R) := E_n^{k'}$. Note that the sets $\{F_n^k(R)\}_{k=1}^{K_n}$ satisfy

$$\mathrm{dist}\,(F_n^k(R),\, F_n^{\ell}(R)) \xrightarrow[R\to\infty]{} \infty \tag{5.2.61}$$

for any $k,\, \ell \in \{1, \cdots, K_n\}$ with $k \neq \ell$. Since $\sum_{k=1}^{K_n} |E_n^k| = m$ and $E_n^{k'} \cup \bigcup_{k=1,\, k\neq k'}^{K_n} \left(E_n^k + R\, z_n^p\right)$ are the union of disjoint sets, we have that

$$\left| E_n^{k'} \cup \bigcup_{k=1,\, k\neq k'}^{K_n} \left(E_n^k + R\, z_n^p\right) \right| = \sum_{k=1}^{K_n} |E_n^k| = m.$$

Thus, from (5.2.57) and (5.2.60), we obtain

$$\begin{aligned} E_{s,g}[m] &\leq \mathcal{E}_{s,g}\left(E_n^{k'} \cup \bigcup_{k=1,\, k\neq k'}^{K_n} \left(E_n^k + R\, z_n^k\right) \right) \\ &\leq \sum_{k=1}^{K_n} \mathcal{E}_{s,g}(E_n^k) \\ &\qquad + 2 \sum_{1\leq k\neq \ell \leq K_n} \int_{F_n^k(R)} \int_{F_n^\ell(R)} g(x-y)\, dx\, dy \\ &\leq \widetilde{E}_{s,g}[m] + \varepsilon \\ &\qquad + 2 \sum_{1\leq k\neq \ell \leq K_n} \int_{F_n^k(R)} \int_{F_n^\ell(R)} g(x-y)\, dx\, dy \end{aligned} \tag{5.2.62}$$

Hence, if we show that the last term of the right-hand side in (5.2.62) converges to zero as $R \to \infty$ for each $n \geq n_0$, then we conclude that the inequality

$$\inf\left\{\mathcal{E}_{s,g}(E) \mid |E| = m\right\} = E_{s,g}[m] \leq \widetilde{E}_{s,g}[m] = \inf\left\{ \widetilde{\mathcal{E}}_{s,g}(\{E^k\}_k) \mid \sum_{k=1}^{\infty} |E^k| = m \right\}$$

holds and this completes the proof of the lemma. To conclude the proof of the lemma, it is sufficient to show that, under assumption (g4), the convergence

$$\sum_{1\leq k\neq \ell \leq K_n} \int_{F_n^k(R)} \int_{F_n^\ell(R)} g(x-y)\, dx\, dy \xrightarrow[R\to\infty]{} 0$$

holds for each $n \geq n_0$. We fix $n \geq n_0$. From assumption (g4), we have that, for any $\varepsilon > 0$, there exists a constant $R(\varepsilon) > 0$ such that $g(z) < \varepsilon$ for any $|z| \geq R(\varepsilon)$. On the other hand, from (5.2.61), we can also choose a constant $R'(\varepsilon) > 0$ such that $|x-y| \geq R(\varepsilon)$ for any $R > R'(\varepsilon)$, $x \in F_n^k(R)$, $y \in F_n^\ell(R)$, and $k,\, \ell \in \{1, \cdots, K_n\}$ with $k \neq \ell$. Thus, taking these into account, we obtain that, for any $R > R'(\varepsilon)$,

$$\sum_{1\leq k\neq \ell \leq K_n} \int_{F_n^k(R)} \int_{F_n^\ell(R)} g(x-y)\, dx\, dy < \varepsilon \sum_{k=1}^{K_n} |F_n^k(R)| \sum_{\ell=1}^{K_n} |F_n^\ell(R)|.$$

Recalling the definition of the sets $\{F_n^k(R)\}_k$, we have that $\sum_{k=1}^{K_n} |F_n^k(R)| \leq m$. Therefore, we obtain that

$$\sum_{1\leq k\neq \ell \leq K_n} \int_{F_n^k(R)} \int_{F_n^\ell(R)} g(x-y)\, dx\, dy < m^2\, \varepsilon$$

for any $R > R'(\varepsilon)$ and this completes the proof of Lemma 5.2.15.

□

Now we prove Theorem 5.2.4, namely, the existence of generalized minimizers of $\widetilde{\mathcal{E}}_{s,g}$ under the assumptions (g1), (g2), and (g4) in Subsection 5.2.2.

Proof of Theorem 5.2.4. Let $m > 0$. Thanks to Lemma 5.2.15, it is sufficient to take any sequence $\{E_n\}_{n\in\mathbb{N}}$ such that $|E_n| = m$ for any $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \mathcal{E}_{s,g}(E_n) = \widetilde{E}_{s,g}[m] \tag{5.2.63}$$

instead of taking a minimizing sequence for $\widetilde{E}_{s,g}[m]$.

We now apply the same argument as we conducted in Step 1, 2, and 3 in the proof of Theorem 5.2.3 because we only need the assumptions (g1), (g2), and (g4) in order for the arguments in Step 1, 2, and 3 to work. Thus, we can choose a finite number of measurable sets $\{G^i\}_{i=1}^H$ with $H \in \mathbb{N}$ such that

$$\sum_{i=1}^{H} \mathcal{E}_{s,g}(G^i) \leq \liminf_{n\to\infty} \mathcal{E}_{s,g}(E_n), \quad \sum_{i=1}^{H} |G^i| = m. \tag{5.2.64}$$

Moreover, each G^i is a minimizer of $\mathcal{E}_{s,g}$ among sets with the volume $|G^i|$. Therefore, from (5.2.63) and (5.2.64), we conclude that the sequence $\{G^i\}_{i=1}^H$ is a generalized minimizer of $\widetilde{\mathcal{E}}_{s,g}$ with $\sum_{i=1}^H |G^i| = m$ as we desired. □

5.2.6 Asymptotic Behavior of Minimizers for Large Volumes

In this subsection, we study the asymptotic behavior of minimizers of $\mathcal{E}_{s,g}$ for large volumes under the assumption that the kernel g decays sufficiently fast. We first show the Γ-convergence result in L^1-topology of the functional associated with Problem (1.0.23) to the s-fractional perimeter P_s as the volume m goes to infinity.

Γ-convergence of $\widehat{\mathcal{E}}^{\lambda}_{s,g}$ to s-Fractional Perimeter as $\lambda \to \infty$

We recall the definition of the s-fractional Sobolev semi-norm $[f]_{W^{s,1}}(\mathbb{R}^N)$ as follows:

$$[f]_{W^{s,1}}(\mathbb{R}^N) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x) - f(y)|}{|x-y|^{N+s}}\,dx\,dy$$

for $f \in L^1$. Note that $[\chi_E]_{W^{s,1}}(\mathbb{R}^N) = P_s(E)$ for any measurable set $E \subset \mathbb{R}^N$. As is shown in [15, Proposition 4.2 and Corollary 4.4], it follows that any integrable function of bounded variation is also finite with respect to the fractional semi-norm $[\cdot]_{W^{s,1}}$. Secondly, in order to study the Γ-convergence of the sequence $\{\widehat{\mathcal{E}}^{\lambda}_{s,g}\}_{\lambda>1}$ given in Proposition 5.2.5, we define the functional $\widehat{\mathcal{F}}^{\lambda_n}_{s,g}$ as

$$\widehat{\mathcal{F}}^{\lambda}_{s,g}(f) := \begin{cases} [f]_{W^{s,1}}(\mathbb{R}^N) - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x) - f(y)|\, g_\lambda(x-y)\,dx\,dy & \\ \qquad\qquad \text{if } f = \chi_F \text{ for some bounded set } F \subset \mathbb{R}^N \text{ with } P_s(F) < \infty, & \\ \infty \qquad\qquad \text{otherwise.} & \end{cases} \tag{5.2.65}$$

Note that the functional $\widehat{\mathcal{F}}^{\lambda}_{s,g}(f)$ for any $\lambda > 0$ is well-defined. Moreover, if $f = \chi_E$ for some bounded set E with $P_s(E) < \infty$, then we can easily see that $\widehat{\mathcal{F}}^{\lambda}_{s,g}(f) = \widehat{\mathcal{E}}^{\lambda}_{s,g}(E)$.

Now we prove the Γ-convergence of the functional $\widehat{\mathcal{F}}^{\lambda_n}_{s,g}$ to $\widehat{\mathcal{F}}^{\infty}_{s}$ (we give the definition of $\widehat{\mathcal{F}}^{\infty}_{s}$ in the following proposition) as $n \to \infty$ in the L^1-topology.

Proposition 5.2.16 (**Γ-convergence to s-Fractional Semi-norm).** *Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be any sequence of positive real numbers going to infinity as $n \to \infty$. Assume that the kernel $g : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies the assumptions* (g1), (g2), *and* (g5) *in Subsection 5.2.1. Then the sequence $\{\widehat{\mathcal{F}}^{\lambda_n}_{s,g}\}_{n\in\mathbb{N}}$ Γ-converges, with respect to L^1-topology, to the functional $\widehat{\mathcal{F}}^{\infty}_s$ defined by*

$$\widehat{\mathcal{F}}^{\infty}_s(f) := \begin{cases} [f]_{W^{s,1}}(\mathbb{R}^N) & \text{if } f = \chi_F \text{ for some bounded } F \subset \mathbb{R}^N \text{ with } P_s(F) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. We recall the definition of the Γ-convergence. We say that $\{\widehat{\mathcal{F}}^{\lambda_n}_{s,g}\}_{n\in\mathbb{N}}$ Γ-converges to $\widehat{\mathcal{F}}^{\infty}_s$ with respect to L^1-topology if the two estimates hold

$$\Gamma_{L^1}\text{–}\limsup_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f) \le \widehat{\mathcal{F}}^{\infty}_s(f), \quad \widehat{\mathcal{F}}^{\infty}_s(f) \le \Gamma_{L^1}\text{–}\liminf_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f)$$

for any $f \in L^1(\mathbb{R}^N)$, where we set

$$\Gamma_{L^1}\text{–}\limsup_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f) := \min\left\{\limsup_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) \mid f_n \xrightarrow[n\to\infty]{} f \text{ in } L^1(\mathbb{R}^N)\right\} \tag{5.2.66}$$

and

$$\Gamma_{L^1}\text{–}\liminf_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f) := \min\left\{\liminf_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) \mid f_n \xrightarrow[n\to\infty]{} f \text{ in } L^1(\mathbb{R}^N)\right\}. \tag{5.2.67}$$

Note that the minimum in (5.2.66) and (5.2.67) is attained by the diagonal argument.

First of all, we prove that $\Gamma_{L^1}\text{–}\limsup_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f) \le \widehat{\mathcal{F}}^{\infty}_s(f)$ for any $f \in L^1(\mathbb{R}^N)$. In the case that f is not a characteristic function of some bounded set with a finite nonlocal perimeter, we obviously have that $\widehat{\mathcal{F}}^{\infty}_s(f) = \infty$ and the inequality holds. Thus, we may assume that $f = \chi_F$ for a bounded set $F \subset \mathbb{R}^N$ with $P_s(F) < \infty$. Setting a sequence $\{f_n\}_{n\in\mathbb{N}}$ as $f_n = f = \chi_F$ for any $n \in \mathbb{N}$, we obtain, from the non-negativity of g, that

$$\widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) \le \widehat{\mathcal{F}}^{\infty}_s(f)$$

for any $n \in \mathbb{N}$ and thus, it follows that $\Gamma_{L^1}\text{–}\limsup_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f) \le \widehat{\mathcal{F}}^{\infty}_s(f)$.

Next we prove that $\widehat{\mathcal{F}}^{\infty}_s(f) \le \Gamma_{L^1}\text{–}\liminf_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f)$ for any $f \in L^1(\mathbb{R}^N)$. We take any sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^1(\mathbb{R}^N)$ such that $f_n \to f$ in L^1 as $n \to \infty$. In the case that f is not a characteristic function of some bounded set with a finite nonlocal perimeter, we claim that there exists a number $n_0 \in \mathbb{N}$ such that f_n is also not a characteristic function of a measurable set for any $n \ge n_0$. Indeed, we suppose by contradiction that there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} = \chi_{F_{n_k}}$ for some measurable set $F_{n_k} \subset \mathbb{R}^N$ for any $k \in \mathbb{N}$. Since $f_{n_k} \to f$ in L^1 as $k \to \infty$ and $f_{n_k} \in \{0, 1\}$ for any $k \in \mathbb{N}$, we obtain that $f \in \{0, 1\}$ a.e. in $\mathbb{R}^N$ and f can be written as $f = \chi_F$ for some measurable $F \subset \mathbb{R}^N$. This contradicts the assumption that f is not a characteristic function. Hence, we conclude that, for large $n \in \mathbb{N}$, $\widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) = \infty$ and the claim holds true. Thus, in the sequel, we may assume that $f = \chi_F$ for some bounded set $F \subset \mathbb{R}^N$. Moreover, we may assume that $P_s(F) < \infty$ due to the lower semi-continuity of the fractional Sobolev semi-norm $[\cdot]_{W^{s,1}}$ and (g5). Indeed, from (g5), we have that

$$\widehat{\mathcal{F}}^{\lambda_n}_s(f_n) \ge \frac{1-\gamma}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f_n(x) - f_n(y)|}{|x-y|^{N+s}}\,dx\,dy = (1-\gamma)[f_n]_{W^{s,1}}$$

for any $n \in \mathbb{N}$ where $\gamma \in (0, 1)$ is given in (g5). Then, if $P_s(F) = \infty$, from the convergence $f_n \to f$ in L^1 as $n \to \infty$ and the lower semi-continuity of $[\cdot]_{W^{s,1}}$, we obtain

$$\liminf_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_s(f_n) \ge (1-\gamma) \liminf_{n\to\infty} [f_n]_{W^{s,1}} \ge (1-\gamma)[f]_{W^{s,1}} = (1-\gamma)P_s(F) = \infty.$$

Under the above assumption, we first compute the second term of the functional $\widehat{\mathcal{F}}^{\lambda_n}_{s,g}$ in (5.2.65). Let $\varepsilon \in (0, 1)$. From (g5), we can choose a constant $R_\varepsilon > 1$ such that $g(x) \leq \frac{\varepsilon}{|x|^{N+s}}$ for $|x| \geq R_\varepsilon$. Then, from the definition of g_{λ_n} for any $n \in \mathbb{N}$, we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f_n(x) - f_n(y)|\, g_{\lambda_n}(x-y)\,dx\,dy \\
&= \iint_{\{(x,y)|\,\lambda_n|x-y|<R_\varepsilon\}} |f_n(x) - f_n(y)|\, g_{\lambda_n}(x-y)\,dx\,dy \\
&\quad + \iint_{\{(x,y)|\,\lambda_n|x-y|\geq R_\varepsilon\}} |f_n(x) - f_n(y)|\, g_{\lambda_n}(x-y)\,dx\,dy \\
&\leq \iint_{\{(x,y)|\,\lambda_n|x-y|<R_\varepsilon\}} \frac{|f_n(x) - f_n(y)|}{|x-y|^{N+s}}\,dx\,dy \\
&\quad + \varepsilon \iint_{\{(x,y)|\,\lambda_n|x-y|\geq R_\varepsilon\}} \frac{|f_n(x) - f_n(y)|}{|x-y|^{N+s}}\,dx\,dy
\end{aligned}
\tag{5.2.68}
$$

for any $n \in \mathbb{N}$. Thus, from the definition of $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$ and (5.2.68), we obtain

$$
\begin{aligned}
\widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) &\geq [f_n]_{W^{s,1}}(\mathbb{R}^N) - \frac{1}{2}\iint_{\{(x,y)|\,\lambda_n|x-y|<R_\varepsilon\}} \frac{|f_n(x) - f_n(y)|}{|x-y|^{N+s}}\,dx\,dy \\
&\quad - \frac{\varepsilon}{2}\iint_{\{(x,y)|\,\lambda_n|x-y|\geq R_\varepsilon\}} \frac{|f_n(x) - f_n(y)|}{|x-y|^{N+s}}\,dx\,dy \\
&\geq \frac{1-\varepsilon}{2}\iint_{\{(x,y)|\,\lambda_n|x-y|\geq R_\varepsilon\}} \frac{|f_n(x) - f_n(y)|}{|x-y|^{N+s}}\,dx\,dy
\end{aligned}
\tag{5.2.69}
$$

for any $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$, where $\gamma \in (0, 1)$ is given in (g5). Thus, letting first $n \to \infty$ and then $\varepsilon \to 0$ with Fatou's lemma and the monotone convergence theorem, we finally obtain

$$
\begin{aligned}
\liminf_{n\to\infty} \widehat{\mathcal{F}}^{\lambda_n}_{s,g}(f_n) &\geq \limsup_{\varepsilon\to 0} \frac{1-\varepsilon}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|f(x) - f(y)|}{|x-y|^{N+s}}\,dx\,dy \\
&= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x) - f(y)|}{|x-y|^{N+s}}\,dx\,dy = [f]_{W^{s,1}}(\mathbb{R}^N).
\end{aligned}
$$

Therefore, from the above arguments, we complete the proof. □

Convergence of Minimizers of $\widehat{\mathcal{E}}^{\lambda}_{s,g}$ to Ball as $\lambda \to \infty$

Now we prove Theorem 5.2.6, which is the last main theorem in this chapter. In this theorem, the control of the kernel g by the one of the s-fractional perimeter, namely the assumption (g5), is crucial because . To see the convergence, we consider the modified minimization problem (Problem 1.0.23) and finally we take the limit $\lambda \to \infty$ instead of Problem 5.2.1 with the limit $m \to \infty$.

Our idea for the proof is to apply the "concentration-compactness" lemma by P.L. Lions as we did in the proof of the existence of minimizers. Precisely, we proceed in the following way; we first take any sequence $\{F_n\}_n$ of the minimizers for $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$ with $|F_n| = |B_1|$. Then we apply so-called "concentration-compactness" lemma that we use to show the existence of minimizers in Subsection 5.2.3 and 5.2.5. As a consequence of the lemma, we can choose a sequence of sets $\{G^i\}_i$ and points $\{z^i_n\}_{i,n}$ such that, up to extracting a subsequence,

$$
\sum_i P_s(G^i) \leq \liminf_{n\to\infty} \widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n), \quad F_n - z^i_n \xrightarrow[n\to\infty]{} G^i \quad \text{in } L^1_{loc}, \quad \sum_i |G^i| = |B_1| \tag{5.2.70}
$$

thanks to the assumptions on g. Then, from the isoperimetric inequality of P_s and the minimality of F_n, we can actually obtain that each G^i coincides with the Euclidean ball, up to translations and negligible sets, whenever $|G^i| > 0$. Finally, from (5.2.70), we can show that the only one element in $\{G^i\}_i$ has a positive volume and its volume is equal to $|B_1|$. From Brezis-Lieb lemma, the convergence in (5.2.70) is improved to the L^1-convergence. Combining the Γ-convergence result, we conclude the proof.

Proof of Theorem 5.2.6. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be any sequence going to infinity as $n \in \mathbb{N}$ and we take any sequence $\{F_n\}_{n\in\mathbb{N}}$ of the minimizers for $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$ with $|F_n| = |B_1|$ for any $n \in \mathbb{N}$. From the assumption (g5), we can choose a constant $\gamma \in (0,\,1)$ such that $g_{\lambda_n}(x) \le \gamma|x|^{-(N+s)}$ for any $|x| \neq 0$. From the minimality of F_n for each $n \in \mathbb{N}$, we have that

$$P_s(F_n) \le P_s(B_1) + P_{g_{\lambda_n}}(F_n) = P_s(B_1) + \gamma\, P_s(F_n)$$

for any $n \in \mathbb{N}$ and thus, we obtain that $\{P_s(F_n)\}_n$ is uniformly bounded with respect to n, namely, $\sup_{n\in\mathbb{N}} P_s(F_n) \le (1-\gamma)^{-1}P_s(B_1) < \infty$. As a consequence of the uniform bound of $\{P_s(F_n)\}_n$, we can now apply the same method as in the proof of Theorem 5.2.3 (see also [45]) to the sequence $\{F_n\}_n$. Although we discuss in the proof of Theorem 5.2.3, we rewrite the argument in the sequel for convenience.

Step 1. We first show that, under (g1), (g2), and (g5), there exist sets $\{G^j\}_j$ such that

$$\sum_j P_s(G^j) \le \liminf_{n\to\infty} \widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n), \quad \sum_j |G^j| = |B_1|. \tag{5.2.71}$$

Indeed, we decompose $\mathbb{R}^N$ into the unit cubes and denote by $\{Q^i\}_{i=1}^{\infty}$. We set $x^i_n \coloneqq |F_n \cap Q^i|$ and have that

$$\sum_{i=1}^{\infty} x^i_n = |F_n| = |B_1| \tag{5.2.72}$$

for any $n \in \mathbb{N}$. Moreover, from the isoperimetric inequality shown in [45, Lemma 2.5], we obtain

$$\sum_{i=1}^{\infty} (x^i_n)^{\frac{N-s}{N}} \le C \sum_{i=1}^{\infty} P_s(F_n; Q^i) \le 2C P_s(F_n) \le C_1 < \infty \tag{5.2.73}$$

for any $n \in \mathbb{N}$, where C and C_1 are the positive constants independent of n. Up to reordering the cubes $\{Q^i\}_i$, we may assume that $\{x^i_n\}_i$ is a non-increasing sequence for any $n \in \mathbb{N}$. Thus, applying the technical result shown in [67, Lemma 4.2] or [45, Lemma 7.4] with (5.2.72) and (5.2.73), we obtain that

$$\sum_{i=k+1}^{\infty} x^i_n \le \frac{C_2}{k^{\frac{s}{N}}} \tag{5.2.74}$$

for any $k \in \mathbb{N}$ where C_2 is the positive constant independent of n and k. Hence, by using the diagonal argument, we have that, up to extracting a subsequence, $x^i_n \to \alpha^i \in [0,\,|B_1|]$ as $n \to \infty$ for every $i \in \mathbb{N}$ and, from (5.2.72) and (5.2.74),

$$\sum_{i=1}^{\infty} \alpha^i = |B_1|. \tag{5.2.75}$$

Now we fix the centre of the cube $z^i_n \in Q^i$ for each i and n. Up to extracting a further subsequence, we may assume that $|z^i_n - z^i_j| \to c^{ij} \in [0,\,\infty]$ as $n \to \infty$ for each $i,\, j \in \mathbb{N}$. As already seen in the above, we have the uniform bound of the sequence $\{P_s(F_n - z^i_n)\}_{n\in\mathbb{N}}$

and its upper-bound is independent of i and thus, from the compactness, there exists a measurable set $G^i \subset \mathbb{R}^N$ such that, up to extracting a further subsequence,

$$\chi_{F_n - z_n^i} \xrightarrow[n\to\infty]{} \chi_{G^i} \quad \text{in } L^1_{loc}\text{-topology}.$$

We define the relation $i \sim j$ for every $i, j \in \mathbb{N}$ as $c^{ij} < \infty$ and we denote by $[i]$ the equivalent class of i. Moreover, we define the set of the equivalent class by $\mathcal{I}$. Then, by applying the same argument as in the proof of Theorem 5.2.3, it is easy to show that

$$\sum_{[i]\in\mathcal{I}} |G^{[i]}| = |B_1|.$$

The last thing we need to show is the following inequality;

$$\sum_{[i]\in\mathcal{I}} P_s(G^{[i]}) \le \liminf_{n\to\infty} \widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n) = \liminf_{n\to\infty} \left(P_s(F_n) - P_{g_{\lambda_n}}(F_n)\right). \tag{5.2.76}$$

Indeed, we first fix $M \in \mathbb{N}$ and $R > 0$ and we take the equivalent classes $i_1, \cdots, i_M$. Notice that, if $p \neq q$, then $|z_n^{i_p} - z_n^{i_q}| \to \infty$ as $n \to \infty$ and thus we have that $\{z_n^{i_p} + Q_R\}_p$ are disjoint sets for large n and

$$\int_{z_n^{i_p}+Q_R} \int_{z_n^{i_q}+Q_R} \frac{1}{|x-y|^{N+s}}\, dx\, dy \xrightarrow[n\to\infty]{} 0 \tag{5.2.77}$$

where Q_R is the cube of side R. Then, by using the similar argument to the one shown in the proof of the Γ-liminf inequality in Proposition 5.2.16 with (5.2.77), we have the following computation: let $\varepsilon \in (0, 1)$ and, from (g5), we can choose a constant $R_\varepsilon > 1$ such that $g(x) \le \frac{\varepsilon}{|x|^{N+s}}$ for any $|x| \ge R_\varepsilon$. Then it holds that

$$\begin{aligned}
&\liminf_{n\to\infty} \left(P_s(F_n) - P_{g_{\lambda_n}}(F_n)\right) \\
&\ge (1-\varepsilon) \liminf_{n\to\infty} \left(\int_{F_n \cap A_n^{M,R}} \int_{F_n^c} \frac{\chi_{\{|x-y|\ge r_n^\varepsilon\}}(x, y)}{|x-y|^{N+s}}\, dx\, dy \right) \\
&\quad + (1-\varepsilon) \liminf_{n\to\infty} \left(\int_{F_n \setminus A_n^{M,R}} \int_{A_n^{M,R}\setminus F_n} \frac{\chi_{\{|x-y|\ge r_n^\varepsilon\}}(x, y)}{|x-y|^{N+s}}\, dx\, dy \right) \\
&\quad + (1-\varepsilon) \liminf_{n\to\infty} 2 \sum_{p\neq q} \int_{z_n^{i_p}+Q_R} \int_{z_n^{i_q}+Q_R} \frac{\chi_{\{|x-y|\ge r_n^\varepsilon\}}(x, y)}{|x-y|^{N+s}}\, dx\, dy
\end{aligned} \tag{5.2.78}$$

for any $\varepsilon \in (0, 1)$ where we set $r_n^\varepsilon := \lambda_n^{-1} R_\varepsilon$ for each n and $A_n^{M,R} := \cup_{p=1}^M (z_n^{i_p} + Q_R)$. Hence, from (5.2.78), (5.2.36), and the lower semi-continuity of P_s in L^1_{loc}-topology with Fatou's lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \left(P_s(F_n) - P_{g_{\lambda_n}}(F_n)\right) \\
&\ge (1-\varepsilon) \liminf_{n\to\infty} \sum_{p=1}^M \left(\int_{F_n \cap (z_n^{i_p}+Q_R)} \int_{F_n^c} \frac{\chi_{\{|x-y|\ge r_n^\varepsilon\}}(x, y)}{|x-y|^{N+s}}\, dx\, dy \right. \\
&\quad \left. + \int_{F_n \setminus (z_n^{i_p}+Q_R)} \int_{(z_n^{i_p}+Q_R)\setminus F_n} \frac{\chi_{\{|x-y|\ge r_n^\varepsilon\}}(x, y)}{|x-y|^{N+s}}\, dx\, dy \right) \\
&\ge (1-\varepsilon) \sum_{p=1}^M P_s(G^{i_p}; Q_R)
\end{aligned}$$

for any $\varepsilon \in (0, 1)$. Letting $R \to \infty$, $M \to \infty$, and $\varepsilon \to 0$, we finally conclude that the inequality (5.2.76) holds true. Taking into account all of the above arguments, we obtain the existence of sets $\{G^{[i]}\}_{[i]\in\mathcal{I}}$ satisfying (5.2.71).

Step 2 We next show that each G^i actually coincides, up to translations and negligible sets, with the Euclidean ball with volume $|G^i|$, whenever $|G^i| > 0$.

Indeed, we first set $B_{[i]}$ as the ball of radius $r_{[i]} := |B_1|^{-1/N}\,|G^{[i]}|^{1/N}$ for each $[i] \in \mathcal{I}$. Then, from (5.2.71) and the minimality of F_n, we have that

$$\begin{aligned}
\sum_{[i]\in\mathcal{I}} \left(P_s(G^{[i]}) - P_s(B_{[i]})\right) &\le \liminf_{n\to\infty} \widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n) - \sum_{[i]\in\mathcal{I}} P_s(B_{[i]}) \\
&\le P_s(B_1) - \sum_{[i]\in\mathcal{I}} \left(\frac{|G^{[i]}|}{|B_1|}\right)^{\frac{N-s}{N}} P_s(B_1) \\
&\le P_s(B_1) - P_s(B_1) \left(\sum_{[i]\in\mathcal{I}} \frac{|G^{[i]}|}{|B_1|}\right)^{\frac{N-s}{N}} = 0. \qquad (5.2.79)
\end{aligned}$$

From the isoperimetric inequality of P_s, we know that $P_s(B_{[i]}) \le P_s(G^{[i]})$ for any $[i] \in \mathcal{I}$ and the equality holds if and only if $G^{[i]} = B_{[i]}$ up to translation and negligible sets. Hence, from (5.2.79), we conclude that the claim holds true.

Step 3. We finally show that, under (g1), (g2), and (g5), there exist a number i_0 such that $|G^{[i]}| = 0$ for any $[i] \in \mathcal{I}$ with $i \neq i_0$.

Indeed, from the isoperimetric inequality of the fractional perimeter P_s and (5.2.71), we obtain the following:

$$\begin{aligned}
\liminf_{n\to\infty} \widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n) &\ge \sum_{[i]\in\mathcal{I}} P_s(G^{[i]}) \\
&\ge \sum_{[i]\in\mathcal{I}} \frac{P_s(B_1)}{|B_1|^{\frac{N-s}{N}}} |G^{[i]}|^{\frac{N-s}{N}} \\
&\ge \frac{P_s(B_1)}{|B_1|^{\frac{N-s}{N}}} \left(\sum_{[i]\in\mathcal{I}} |G^{[i]}|\right)^{\frac{N-s}{N}} = P_s(B_1) \qquad (5.2.80)
\end{aligned}$$

where, in the last inequality of (5.2.80), we have used the inequality that

$$\sum_i a_i^\alpha \ge \left(\sum_i a_i\right)^\alpha \qquad (5.2.81)$$

for any $i \in \mathbb{N}$, $a_i \in [0, 1]$, and $\alpha \in (0, 1)$ with $\sum_i a_i < \infty$. Note that the equality in (5.2.81) holds if and only if $a_1 = \sum_i a_i$ and $a_i = 0$ for any $i > 1$ up to reordering. In addition, from the definition of $\widehat{\mathcal{E}}^{\lambda_n}_{s,g}$ and the minimality of each F_n, we have that

$$\widehat{\mathcal{E}}^{\lambda_n}_{s,g}(F_n) \le P_s(B_1) \qquad (5.2.82)$$

for any $n \in \mathbb{N}$. Thus, from (5.2.80) and (5.2.82), we obtain

$$|B_1|^{\frac{N-s}{N}} = \sum_{[i]\in\mathcal{I}} |G^{[i]}|^{\frac{N-s}{N}} = \left(\sum_{[i]\in\mathcal{I}} |G^{[i]}|\right)^{\frac{N-s}{N}}, \quad \sum_{[i]\in\mathcal{I}} |G^{[i]}| = |B_1|. \qquad (5.2.83)$$

Then, from (5.2.81), (5.2.83), and Step 2, we obtain that there exists i_0 such that $|G^{[i]}| = 0$ for any $[i] \in \mathcal{I}$ with $i \neq i_0$ and $G^{i_0} = B_1$ up to translations and negligible sets. This completes Step 4.

Finally, taking into account all of the above arguments, we may conclude that there exist points $\{z'_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that, up to extracting a subsequence, we have

$$\chi_{F_n - z'_n} \xrightarrow[n\to\infty]{} \chi_{B_1} \quad \text{in } L^1_{loc}.$$

From Brezis-Lieb lemma in [16] and the fact that $|F_n - z'_n| = |B_1|$ for any $n \in \mathbb{N}$, we obtain that the convergence

$$\chi_{F_n - z'_n} \xrightarrow[n\to\infty]{} \chi_{G'} \quad \text{in } L^1_{loc}$$

holds in L^1 sense. Finally, we may repeat the above argument for any subsequence of $\{F_n\}_{n\in\mathbb{N}}$ and therefore, we conclude that Theorem 5.2.6 is valid. $\square$

Appendix A

Extension Result of Functions with Nonlocal Bounded Variations

In this section, we show the extension of the function space $BV_K(\Omega)$ into $BV_K(\mathbb{R}^{\mathbb{N}})$ for each bounded open set $\Omega \subset \mathbb{R}^N$ with a smooth boundary (in other words, we show the embedding $BV_K(\Omega) \hookrightarrow BV_K(\mathbb{R}^N)$). This extension result is required when we consider the compact embedding of BV_K into L^1 with a general kernel K as is shown in Section 2.1 of Chapter 2, where the definition of BV_K is also given. Note that one may have the compact embedding of the fractional Sobolev space $W^{s,1}$ into L^1 (with the kernel K given as $|x|^{-(N+s)}$ for $s \in (0, 1)$) thanks to the work by, for instance, [46, Theorem 7.1]. The extension property of $BV_K(\Omega)$ into $BV_K(\mathbb{R}^N)$ as follows:

Lemma A.0.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set with a bounded Lipschitz boundary. Assume that the kernel K satisfies the following conditions:*

(A1) K is non-negative and radially symmetric, namely, $K(x) = k(|x|)$ for some measurable function $k : (0, \infty) \to [0, \infty)$.

(A2) K is radially non-increasing, namely, $K(x) \leq K(y)$ for any $|x| \geq |y| > 0$.

(A3) K satisfies

$$\int_{\mathbb{R}^N} K(x) \min\{1, |x|\} \, dx < \infty$$

.

(A4) There exists a constant $t > 0$ such that $|x|^N K(x) = O(|x|^{-t})$ as $|x| \to 0$.

Then $BV_K(\Omega)$ is continuously embedded in $BV_K(\mathbb{R}^N)$, namely for any $u \in BV_K(\Omega)$ there exists $\widetilde{u} \in BV_K(\mathbb{R}^N)$ such that

$$\widetilde{u}\lfloor_{\Omega} \equiv u, \quad \|\widetilde{u}\|_{K(\mathbb{R}^N)} \leq C\|u\|_{K(\Omega)} \tag{A.0.1}$$

for some constant $C = C(N, K, \Omega) > 0$

Proof of Lemma A.0.1. The proof is conducted in the same way as in the proof of [46, Theorem 5.4] because of the assumptions on K.

Since $\partial\Omega$ is closed and bounded, we can find a finite number of balls $\{B_j\}_{j=1}^M$ such that $\partial\Omega \subset \cup_{j=1}^M B_j$, and thus we write $\mathbb{R}^N = \cup_{j=1}^M B_j \cup (\mathbb{R}^N \setminus \partial\Omega)$. With this covering, we can further find a finite number of smooth functions $\{\psi_j\}_{j=1}^{M+1}$ such that $0 \leq \psi_j \leq 1$

for any $j \in \{1, \cdots, M\}$, $\operatorname{spt}\psi_j \subset B_j$ for any $j \in \{1, \cdots, M\}$, $\operatorname{spt}\psi_{M+1} \subset \mathbb{R}^N \setminus \partial\Omega$, and $\sum_{j=1}^{M+1} \psi_j \equiv 1$. Then, one can easily see that

$$\sum_{j=1}^{M+1} \psi_j \, u = u \quad \text{in } \mathbb{R}^N.$$

By simple computations, we can show that $\psi_{M+1}u$ belongs to $BV_K(\Omega)$ (see also [46, Lemma 5.3]). Since $\psi_{M+1}u \equiv 0$ in a neighborhood of $\partial\Omega$, we can extend the domain of $\psi_{M+1}u$ to the whole of $\mathbb{R}^N$ by setting

$$\widetilde{\psi_{M+1}u}(x) := \begin{cases} \psi_{M+1}(x)\, u(x) & \text{for } x \in \Omega, \\ 0 & \text{in } x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

and we have $\widetilde{\psi_{M+1}u} \in BV_K(\mathbb{R}^N)$ with the estimate

$$\|\widetilde{\psi_{M+1}u}\|_{K(\mathbb{R}^N)} \leq \|\psi_{M+1}u\|_{K(\Omega)} \leq C\, \|u\|_{K(\Omega)}$$

where $C = C(N, K, \Omega) > 0$ is a constant.

Since $\partial\Omega$ is of class $C^{0,1}$, we can construct a finite number of the bi-Lipschitz isomorphisms $\{T_j : Q \to B_j\}_{j=1}^M$ such that

$$\|T_j\|_{C^{0,1}} + \|T_j^{-1}\|_{C^{0,1}} \leq C_0, \quad T_j(Q_+) = B_j \cap \Omega, \quad T_j(Q_0) = B_j \cap \partial\Omega,$$

where $C_0 > 0$ is a constant independent of j, and we set

$$\begin{aligned} & Q := \{(x', x_N) \mid |x'| < 1,\ |x_N| < 1\}, \\ & Q_+ := \{(x', x_N) \mid |x'| < 1,\ 0 < x_N < 1\}, \\ \text{and} \quad & Q_0 := \{(x', x_N) \mid x_N = 0\}. \end{aligned}$$

Now for any $j \in \{1, \cdots, M\}$, we consider the restricted function $u\lfloor_{B_j \cap \Omega}$ and set

$$w_j(y) := u\left(T_j(y)\right) \quad \text{for } y \in Q_+.$$

Then we prove that $w_j \in BV_K(Q_+)$. Indeed, by using the change of variables, we have

$$\begin{aligned} & \int_{Q_+}\int_{Q_+} K(p-q)\, |w_j(p) - w_j(q)|\, dp\, dq \\ & = \int_{Q_+}\int_{Q_+} K(p-q)\, |u(T_j(p)) - u(T_j(q))|\, dp\, dq \\ & = \int_{B_j\cap\Omega}\int_{B_j\cap\Omega} K\left(T_j^{-1}(x) - T_j^{-1}(y)\right)\, |u(x) - u(y)|\, JT_j^{-1}(x)\, JT_j^{-1}(y)\, dx\, dy \end{aligned}$$

where $JT_j^{-1}(x)$ is the Jacobian of the bi-Lipschitz function T_j^{-1} at $x \in B_j \cap \Omega$ for $j \in \{1, \cdots, M\}$. Recalling the bi-Lipschitz regularity of T_j for all $j \in \{1, \cdots, M\}$ and using the assumption (A2), we obtain

$$\begin{aligned} \int_{Q_+}\int_{Q_+} K(p-q)\, |w_j(p) - w_j(q)|\, dp\, dq & \lesssim \int_{B_j\cap\Omega}\int_{B_j\cap\Omega} K\left(x-y\right)\, |u(x) - u(y)|\, dx\, dy \\ & = [u]_K(B_j \cap \Omega) < \infty. \end{aligned} \tag{A.0.2}$$

By using the reflection argument, we can extend the domain of w_j to all Q so that the extension $\widetilde{w_j}$ belongs to $BV_K(Q)$ and

$$\|\widetilde{w_j}\|_{K(Q)} \leq 4\|w_j\|_{K(Q_+)}. \tag{A.0.3}$$

We now set $z_j(x) := \widetilde{w_j}(T_j^{-1}(x))$ for any $x \in B_j$ and $j \in \{1, \cdots, M\}$. From the bi-Lipschitz continuity of T_j, we have $z_j \in BV_K(B_j)$. Notice that $z_j(x) \equiv u(x)$ for any $x \in B_j \cap \Omega$, and by definition, $\psi_j\, z_j$ has a compact support in B_j for all $j \in \{1, \cdots, M\}$. As we see the extension of $\psi_{M+1}\, u$, we can consider the extension $\widetilde{\psi_j\, z_j}$ in such a way that $\widetilde{\psi_j\, z_j} \in BV_K(\mathbb{R}^N)$. Thus, from (A.0.2) and (A.0.3), we may observe that

$$\begin{aligned} \|\widetilde{\psi_j\, z_j}\|_{K(\mathbb{R}^N)} &\lesssim \|\psi_j\, z_j\|_{K(B_j)} \\ &\lesssim \|z_j\|_{K(B_j)} \\ &\lesssim \|\widetilde{w_j}\|_{K(Q)} \\ &\leq 4\|w_j\|_{K(Q_+)} \lesssim \|u\|_{K(B_j \cap \Omega)}. \end{aligned} \tag{A.0.4}$$

Finally, defining the function $\widetilde{u}$ in $\mathbb{R}^N$ as

$$\widetilde{u} := \sum_{j=1}^{M} \widetilde{\psi_j\, z_j} + \widetilde{\psi_{M+1}\, u}, \tag{A.0.5}$$

we can see, by definition, that $\widetilde{u}\lfloor_{\Omega} \equiv u$ and, from (A.0.4) and (A.0.5), we obtain

$$\begin{aligned} \|\widetilde{u}\|_{K(\mathbb{R}^N)} &\leq \sum_{j=1}^{M} \|\widetilde{\psi_j\, z_j}\|_{K(\mathbb{R}^N)} + \|\widetilde{\psi_{M+1}\, u}\|_{K(\mathbb{R}^N)} \\ &\leq \sum_{j=1}^{M} \|u\|_{K(B_j \cap \Omega)} + C\|u\|_{K(\Omega)} \\ &\leq (1 + C)\|u\|_{K(\Omega)} \end{aligned} \tag{A.0.6}$$

for some constant $C > 0$ independent of u. $\square$

Appendix B

Euler-Lagrange Equation in the Viscosity Sense

In this appendix, we will show that each superlevel set $\{u > t\}$ of the minimizer u of the functional $\mathcal{F}_{K,f}$ satisfies the Euler-Lagrange equation in the viscosity sense. Recall that each set $\{u > t\}$ is also a minimizer of the functional $\mathcal{E}_{K,f,t}$.

First of all, by following the same line of the argument by L. Caffarelli, J.M. Roquejoffre, and O. Savin [22] and M.C. Caputo and N. Guillen [26], we can obtain the following theorem:

Theorem B.0.1 (Euler-Lagrange Inequalities). *Let E be a set satisfying the condition that*

$$\int_A \int_E \frac{1}{|x-y|^{N+s}}\,dx\,dy - \int_A \int_{E^c \cap A^c} \frac{1}{|x-y|^{N+s}}\,dx\,dy \le \int_A (t - f(x))\,dx \tag{B.0.1}$$

for any $A \subset E^c \cap B_r(x_0)$ and any $x_0 \in \partial E$ with some $r > 0$ where $s \in (0, 1)$, $f \in C^0(\mathbb{R}^N)$, and $t \in \mathbb{R}$. Suppose that $0 \in \partial E$ and that $E \cap \Omega$ contains the ball $B_{2R}(-2Re_N)$, $R \ge 1$. Then there exist constants $C_0 = C_0(N, s, t, f) > 0$ and $r_0 = r_0(N, s, t, f) > 0$ such that the following holds: for any $0 < \varepsilon \ll \delta < r_0$, one can choose $\varepsilon/2 < \varepsilon^* < \varepsilon$ such that

$$\begin{aligned} &\int_{A_{\varepsilon^*}} \int_{E \setminus B_\delta(x_0)} \frac{1}{|x-y|^{N+s}}\,dx\,dy - \int_{A_{\varepsilon^*}} \int_{E^c \setminus B_\delta(x_0)} \frac{1}{|x-y|^{N+s}}\,dx\,dy \\ &\le C_0 \left(\int_{A_{\varepsilon^*}} (t - f(x))\,dx + R^{-1} \delta^{\frac{1-s}{2}} \, |A^-_{\varepsilon^*}| \right), \end{aligned} \tag{B.0.2}$$

where A_{ε^*} and $A^-_{\varepsilon^*}$ are sets contained in E^c, which are defined with the perturbation constructed in [26] (see also [22]).

Since the essential point of the proof is the same as in [22, Theorem 5.1] and [26, Theorem 5.3], we can prove Theorem B.0.1 in the same manner, and thus we omit the proof here (see also [15, Lemma 6.6] for a similar argument). Roughly speaking, the idea of the proof is to construct some proper perturbation sets $A^-_{\varepsilon^*}$ and A_{ε^*} outside E around some boundary point $x_0 \in \partial E$. These perturbation sets can be constructed by some reflection with respect to a slightly deviated ball from the ball tangential to ∂E at x_0. Then, we test the condition (B.0.1) of E against the set of the union of E and $A_{\varepsilon^*} \subset E^c$. Note that the last term of the right-hand side in (B.0.2) can be obtained by deriving the upper bound of the double integrals

$$-\int_{A_{\varepsilon^*}} \int_{E \cap B_\delta(x_0)} \frac{dx\,dy}{|x-y|^{N+s}} + \int_{A_{\varepsilon^*}} \int_{E^c \cap A^c \cap B_\delta(x_0)} \frac{dx\,dy}{|x-y|^{N+s}}.$$

The reason we decompose the integral into two parts by using the ball $B_\delta(x_0)$ is that the s-fractional mean curvature H^s_E at $x_0 \in \partial E$ is defined by the singular integral at x_0. Thus we must avoid computing the integral inside $B_\delta(x_0)$.

We remark that the term $\int_{A_\varepsilon}(t - f(x))\,dx$ does not appear in the original claim in [26]; however, this extra term may not affect the essential point of the proof in [26] because the set E is not involved in the term.

Notice that, if E is a minimizer of $\mathcal{E}_{s,f,t}$ defined in (4.3.16), then one can show that E satisfies the condition (B.0.1). Indeed, let $A \subset \mathbb{R}^N$ be in $E^c \cap B$ for some ball B centered at a point on ∂E. Then, if we choose a set $E \cup A$ as a competitor against the minimizer E, then we have

$$\begin{aligned} P_s(E) &\leq P_s(E \cup A) + \int_{E\cup A}(t - f(x))\,dx - \int_E (t - f(x))\,dx \\ &\leq P_s(E) + P_s(A) + \int_A (t - f(x))\,dx - 2\int_E\int_A \frac{1}{|x-y|^{N+s}}\,dx\,dy. \end{aligned} \tag{B.0.3}$$

Here we have used the fact that $A \subset E^c$ and Proposition 2.1.3 with $\Omega = \mathbb{R}^N$ and $K(x) = |x|^{-(N+s)}$ in Chapter 2. Then, by a handful of computations and from (B.0.3), we finally obtain

$$\int_A\int_E \frac{1}{|x-y|^{N+s}}\,dx\,dy - \int_A\int_{E^c\cap A^c} \frac{1}{|x-y|^{N+s}}\,dx\,dy \leq \int_A (t - f(x))\,dx.$$

Now we are ready to state the main claim in this appendix, and this can be obtained as a corollary of Theorem B.0.1 (see also [15, Theorem 6.7] for a similar statement).

Corollary B.0.2 (Euler-Lagrange Equations). *Let E be a minimizer of the functional $\mathcal{E}_{s,f,t}$ with $s \in (0,\,1)$, $f \in C^{0,\beta}_{loc}(\mathbb{R}^N)$, and $t \in \mathbb{R}$ as is defined in (4.3.16). Then, whenever E has both interior and exterior tangential balls at a point x_0, we have*

$$H^s_E(x_0) + t - f(x_0) = \int_{\mathbb{R}^N} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y - x_0|^{N+s}}\,dy + t - f(x_0) = 0. \tag{B.0.4}$$

Proof. Without loss of generality, we may assume that $x_0 = 0$. Let ε^* and δ be given as in Theorem B.0.1, and we observe that

$$\begin{aligned} &\left| \frac{1}{|A_{\varepsilon^*}|}\left(\int_{A_{\varepsilon^*}}\int_{E\setminus B_\delta} \frac{1}{|x-y|^{N+s}}\,dx\,dy - \int_{A_{\varepsilon^*}}\int_{E^c\setminus B_\delta} \frac{1}{|x-y|^{N+s}}\,dx\,dy\right) \right. \\ &\qquad \left. - \int_{B^c_\delta} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y|^{N+s}}\,dy \right| \\ &\leq \int_{B^c_\delta(0)} \frac{C\,\varepsilon^*}{|y|^{N+s}}\,dy \leq C\varepsilon^*\delta^{-s} \end{aligned} \tag{B.0.5}$$

for some constant $C > 0$. By testing the minimality of E against the set $E \cup A$ where $A \subset E^c \cap B_r(x)$ for any $x \in \partial E$ and some $r > 0$, we obtain that the condition (B.0.1) holds. Thus, from Theorem B.0.1 and (B.0.5) and by following the argument in [26, Corollary 5.4], we obtain

$$\int_{B^c_\delta} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y|^{N+s}}\,dy \leq C\varepsilon^*\delta^{-s} + C_0\left(\frac{1}{|A_{\varepsilon^*}|}\int_{A_{\varepsilon^*}}(t - f(x))\,dx + R^{-1}\delta^{\frac{1-s}{2}}\right) \tag{B.0.6}$$

where $x_0 = 0 \in \partial E$ is the point where ∂E has an interior tangential ball $B_{2R}(-2Re_N)$.

Now we claim the following: let $x \in \mathbb{R}^N$ and $\{C_\varepsilon\}_{\varepsilon>0}$ be a sequence of measurable sets such that $x \in C_\varepsilon$, $|C_\varepsilon| > 0$, and $C_\varepsilon \subset B_{r(\varepsilon)}$ for any $\varepsilon > 0$ and some increasing function $r : (0, \infty) \to (0, \infty)$. Then, for any $F \in C^0(\mathbb{R}^N)$, we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{|C_\varepsilon|} \int_{C_\varepsilon} F(y)\, dy = F(x). \tag{B.0.7}$$

Indeed, let $\eta > 0$ and, from the continuity of F, we choose a constant $\delta > 0$ such that $|F(y) - F(x)| < \eta$ for any $y \in \mathbb{R}^N$ with $|y - x| < \delta$. In addition, from the assumptions that $x \in C_\varepsilon$ and $C_\varepsilon \subset B_{r(\varepsilon)}$, we can choose a constant $\varepsilon_0 > 0$ such that $|z - x| < r(\varepsilon_0) < \delta$ for any $z \in A^{\varepsilon_0}$. Thus, setting $\tilde{\varepsilon}_0 \coloneqq \frac{1}{2} \min\{\varepsilon_0,\, \delta,\, r(\varepsilon_0)\}$, we obtain

$$y \in C_\varepsilon \Rightarrow |F(y) - F(x)| < \eta$$

for any $\varepsilon \in (0,\, \tilde{\varepsilon}_0)$. Thus, we have that

$$\left| \frac{1}{|C_\varepsilon|} \int_{C_\varepsilon} F(y)\, dy - F(x) \right| \leq \frac{1}{|C_\varepsilon|} \int_{C_\varepsilon} |F(y) - F(x)|\, dy < \eta,$$

and this concludes the proof of (B.0.7).

Now, from the construction of A_ε shown in [26], the sets $\{A_\varepsilon\}_\varepsilon$ satisfy the conditions that

$$x_0 \in A_\varepsilon, \quad |A_\varepsilon| > 0, \quad A_\varepsilon \subset B_\varepsilon$$

for any small $\varepsilon > 0$. Therefore, from (B.0.6), the assumption of f, and the above claim and by letting $\varepsilon \downarrow 0$, we have that

$$\int_{B_\delta^c} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y|^{N+s}}\, dy \leq t - f(x_0) + R^{-1} \delta^{\frac{1-s}{2}},$$

and thus, by letting $\delta \downarrow 0$, we conclude that

$$-\, H_E^s(x_0) \leq t - f(x_0), \tag{B.0.8}$$

for any $x_0 \in \partial E$ where ∂E has an interior tangential ball.

On the other hand, if ∂E has an exterior tangential ball, then, by testing the minimality of E against the set $E \setminus A$ where $A \subset E \cap B_r(x)$ for any $x \in \partial E$ and some $r > 0$, we obtain

$$\int_A \int_{E^c} \frac{1}{|x - y|^{N+s}}\, dx\, dy - \int_A \int_{E \cap A^c} \frac{1}{|x - y|^{N+s}}\, dx\, dy \leq \int_A -(t - f(x))\, dx.$$

This inequality means that E^c also satisfies the condition (B.0.1) by replacing $t - f(x)$ with $-(t - f(x))$ in the integral of the right-hand side in (B.0.1), and by hypothesis E^c contains a tangential ball at x_0. Then, by applying again Theorem B.0.1 to the set E^c, we obtain

$$\begin{aligned} &\int_{A_{\varepsilon^*}} \int_{E^c \setminus B_\delta} \frac{1}{|x - y|^{N+s}}\, dx\, dy - \int_{A_{\varepsilon^*}} \int_{E \setminus B_\delta} \frac{1}{|x - y|^{N+s}}\, dx\, dy \\ &\leq C_0 \left(- \int_{A_{\varepsilon^*}} (t - f(x))\, dx + R^{-1} \delta^{\frac{1-s}{2}}\, |A_{\varepsilon^*}^-| \right) \end{aligned}$$

for small $\varepsilon^* > 0$ where A_{ε^*} and $A_{\varepsilon^*}^-$ are as in Theorem B.0.1. Therefore, by following the same argument that we showed in the above, we conclude that

$$H_E^s(x_0) \leq -(t - f(x_0)) \tag{B.0.9}$$

for any $x_0 \in \partial E$ where ∂E has an exterior tangential ball.

From (B.0.8) and (B.0.9), we finally obtain the equality

$$H_E^s(x_0) + t - f(x_0) = 0$$

for any $x_0 \in \partial E$ where ∂E has both interior and exterior tangential balls. $\square$

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