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Generalized Coarea Formula and Fractal Sets

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For any domain Ω of $\mathbf{R}^N (N \geq 1)$, the class $GC(\Omega)$ of functionals $\Lambda : L^1(\Omega) \to [0, +\infty]$ fulfilling the following *generalized coarea formula* is introduced:

(1) $\Lambda(u) = \int_{\mathbf{R}} \Lambda(H_s(u))ds \ (\leq +\infty) \quad \forall u \in L^1(\Omega)$,

where $H_s(\xi) = 0$ if $\xi < s$ and $H_s(\xi) = 1$ if $\xi \geq s$, for any $\xi, s \in \mathbf{R}$.
Examples are

(2) $V(u) := \int_\Omega |\nabla u| := \sup\limits_{\eta \in C_c^1(\Omega)^N, |\eta| \leq 1} \int_\Omega u \operatorname{div} \eta dx$,

(3) $\Lambda_r(u) := \iint_{\Omega^2} |u(x) - u(y)| \cdot |x - y|^{-(N+r)} dxdy \quad \forall r \in]0, 1[$,

(4) $\widetilde{\Lambda}_r(u) := \iint_{\Omega \times \mathbf{R}^+} (\operatorname*{ess\,sup}_{B_h(x) \cap \Omega} u - \operatorname*{ess\,inf}_{B_h(x) \cap \Omega} u) h^{-(1+r)} dxdh \quad \forall r \in]0, 1[$.

The main properties of this class of functionals are here investigated. Λ_r and $\widetilde{\Lambda}_r$ also allow us to construct two new definitions of *fractional dimension* for set boundaries. Applications to models of *surface tension* effects in two-phase systems are then outlined. In particular, Λ_r and $\widetilde{\Lambda}_r$ allow us to represent very irregular phase interfaces.

Key words: fractional dimension, pattern formation, two-phase systems, surface tension

1. Introduction

Surface tension effects occur in several multi-phase systems. For instance, they are responsible for the supercooling required for nucleating either a solid phase in a completely liquid system, or a liquid in a vapour, cf. [1; chap. 3], [5; chap. 9], [8]. These effects are also at the basis of morphologies like dendritic formations. It is clear that a mathematical description of these phenomena can be of interest, for both theoretical and applicative reasons.

Here we shall confine ourselves to two-phase systems. Three- and more- phase systems are also important, and there is no reason for expecting that they might be treated as simple extensions of the two-phase theory. However, one can guess that the study of two-phase systems already exhibits the main features of these types of problems.

Surface tension effects are usually modelled by means of a term proportional to the area of the interface separating different phases; this term is added to an appropriate potential, which is minimized by the system at equilibrium. Thus a problem of *calculus of variations* arises.

Following Caccioppoli, cf. [2,7], the interface area can be represented as the total variation (cf. the functional V below) of the characteristic function of one of the two phases. The proportionality factor σ (named *surface tension coefficient*) is usually very small, and this explains why in many models the surface term is neglected. As an effect of the smallness of σ, the area of the interface can be very large, although finite; this is consistent with complicated morphologies observed for instance in snowflakes, in dendritic formations, and the similar.

We propose to represent such interfaces as sets of *fractional dimension*, in a sense to be specified, having codimension strictly comprised between 0 and 1. Several (non-equivalent) definitions of *fractional dimension* can be given, as proved by much of the recent literature on *fractals sets.* Here we propose two different concepts, related to two classes of functionals, cf. Λ_r and $\widetilde{\Lambda}_r$ later on. As we shall see, these functionals are lower semi-continuous, and their domains have compact injection into the space L^1, as also occurs for the functional V of the standard model.

It is natural to compare the two definitions of *fractional dimension* given here with more classical ones. As we shall see, one of these concepts is essentially equivalent to the *Minkowski-Bouligand fractional dimension.* However, the present formulation in terms of functionals is especially adequate for problems in the calculus of variations, because of the aforementioned properties of compactness and lower semi-continuity. On the other hand, the theory we develope here can be applied only to set boundaries, and it does not seem easy to extend it to general sets.

There is another property which is shared by V, the Λ_r's and the $\widetilde{\Lambda}_r$'s; this will be named *generalized coarea formula*, as it extends the *classical coarea formula*, valid for the total variation functional V. This property is at the basis of several results concerning the analytical formulation of a class of problems in which the shape of a domain depends on a field; a typical example is the determination of a surface of prescribed mean curvature. In particular, the generalized coarea formula plays a key role in a model of set evolution presented in [13].

The aim of this paper is to offer examples of classes of functionals fulfilling the *generalized coarea formula*, and to investigate its implications.

Let Ω be any domain of $\mathbf{R}^N (N \geq 1)$, and consider the functional

$$V(u) := \int_\Omega |\nabla u| := \sup_{\eta \in C_c^1(\Omega)^N, |\eta| \leq 1} \int_\Omega u \, \mathrm{div}\eta \, dx \ (\leq +\infty) \quad \forall u \in L^1(\Omega). \tag{1.1}$$

Setting $H_s(\xi) = 0$ if $\xi < s$, $H_s(\xi) = 1$ if $\xi \geq s$ for any $\xi, s \in \mathbf{R}$, one has

$$V(u) = \int_{\mathbf{R}} V(H_s(u)) ds \qquad \forall u \in L^1(\Omega). \tag{1.2}$$

This formula was proven in its generality by Fleming and Rishel [6], after several other authors had obtained it for more restricted classes of functions. (1.2) is also known as the *coarea formula*, cf. [7; p. 20], because of its geometric interpretation. The same denomination is also used for more general formulae, which maintain the geometric meaning of (1.2) [4]; in the present paper we extend (1.2) in a completely different direction.

For any $r \in]0,1[$, also the functionals

$$(1.3) \qquad \Lambda_r(u) := \iint_{\Omega^2} |u(x) - u(y)| \cdot |x-y|^{-(N+r)} dxdy \ (\le +\infty), \qquad \forall u \in L^1(\Omega),$$

$$(1.4) \qquad \widetilde{\Lambda}_r(u) := \iint_{\Omega \times \mathbf{R}^+} (\operatorname*{ess\,sup}_{B_h(x)\cap\Omega} u - \operatorname*{ess\,inf}_{B_h(x)\cap\Omega} u) h^{-(1+r)} dxdh \ (\le +\infty), \qquad \forall u \in L^1(\Omega),$$

where $B_h(x) := \{y \in \mathbf{R}^N : |y - x| \le h\}$, fulfil a formula like (1.2).

As it was seen in [11], the functionals V, Λ_r's and $\widetilde{\Lambda}_r$'s have interesting applications in modelling *surface tension effects* in *multi-phase systems* and in more general *pattern formation* phenomena. In this respect the property (1.2) plays a key role. In order to study its implications, we introduce the class $GC(\Omega)$ of functionals $\Lambda : L^1(\Omega) \to [0, +\infty]$ which fulfil the following *generalized coarea formula*:

$$(1.5) \qquad \Lambda(u) = \int_{\mathbf{R}} \Lambda(H_s(u)) ds \quad (\le +\infty) \quad \forall u \in L^1(\Omega).$$

This is quite different from other extensions of (1.2), which are also called *coarea formulae.*

In Section 2, we present the basic properties of these functionals and give some further examples. In particular V, Λ_r and $\widetilde{\Lambda}_r$ $(0 < r < 1)$ are convex and lower semicontinuous in $L^1(\Omega)$; moreover, if Ω is bounded and "smooth", then the injections of their domains into $L^1(\Omega)$ are compact.

There is an obvious relationship between the functionals of $GC(\Omega)$ and the set applications $\mathcal{P}(\Omega) \to [0, +\infty]$. Let us denote by χ_A the characteristic function of any set $A \subset \Omega$. Any $\Lambda \in GC(\Omega)$ induces the set application

$$(1.6) \qquad \mathcal{F}_\Lambda(A) := \Lambda(\chi_A) \quad (\le +\infty) \quad \forall A \in \mathcal{P}(\Omega).$$

Conversely, any set application $F : \mathcal{P}(\Omega) \to [0, +\infty]$ determines the restriction of a functional $\mathcal{L}_F$ to the family of characteristic functions:

$$(1.7) \qquad \mathcal{L}_F(\chi_A) := F(A) \qquad \forall A \in \mathcal{P}(\Omega);$$

then $\mathcal{L}_F$ can be extended to all the functions $\Omega \to \mathbf{R}$ by means of the *generalized coarea formula* (1.5):

$$(1.8) \quad \mathcal{L}_F(u) := \int_{\mathbf{R}} \mathcal{L}_F(H_s(u)) ds \ (= \int_{\mathbf{R}} F(\{x \in \Omega : u(x) \ge s\}) ds) \quad \forall u : \Omega \to \mathbf{R}.$$

The properties of this construction are studied in Section 3.
Then we deal with sets with *boundary of fractional dimension.* For any $r \in]0,1[$, consider the functional J_r-corresponding to a set application F_r such that

$$(1.9) \qquad F_r(A) := \mathcal{H}_{N-r}(\partial_e A) \ (\le +\infty) \quad \forall A \subset \Omega, \quad A \text{ measurable};$$

here $\mathcal{H}_{N-r}$ denotes the $(N-r)$-dimensional *Hausdorff measure*, and $\partial_e A$ the *essential boundary* of A. It is not difficult to check that J_r is *not* lower semi-continuous with respect to the strong topology of $L^1(\Omega)$, and that the injection $L^1(\Omega) \cap \mathrm{Dom}(J_r) \to L^1(\Omega)$ is *not* compact, even if Ω is bounded and "smooth". Thus the standard tools of the calculus of variations cannot be applied to J_r.

On the other hand, for any $r \in]0,1[$, the functionals Λ_r and $\widetilde{\Lambda}_r$ fulfil properties of L^1-lower semi-continuity and compactness, as we said. Moreover for any measurable set A, $\Lambda_r(\chi_A)$ and $\widetilde{\Lambda}_r(\chi_A)$ mimic the properties of $(N-r)$-dimensional measures of $\partial_e A$. This leads to the introduction of the corresponding concepts of *fractional dimensions of* $\partial_e A$ *relative to the scales of functionals* $\{\Lambda_r\}_{0<r<1}$ and $\{\widetilde{\Lambda}_r\}_{0<r<1}$. We also show that in the case of $\{\widetilde{\Lambda}_r\}$ this definition is strictly related to the *Minkowski-Bouligand dimension*, defined in [9; p. 287], e.g.

In Section 4, we outline how the functionals of $GC(\Omega)$ can be used to construct models of *surface tension* effects in *two-phase systems*. There we mainly review the developments of [11; Sect. 5], which are based on a result here recalled in the Appendix.

According to the classical theory of Caccioppoli and De Giorgi, cf. [2,7], for any measurable set A, $V(\chi_A)$ represents the (generalized) perimeter of A, namely the $(N-1)$-dimensional area of $\partial_e A$. Hence the functional V can be used to represent the surface tension contribution to the free enthalpy of a two-phase system (solid-liquid, e.g.). This also accounts for the classical *Gibbs-Thomson law*, which prescribes that at any interface between two different phases the temperature is proportional to the mean curvature of the interface itself. More generally, the functionals of $GC(\Omega)$ fulfilling the properties of convexity, L^1-lower semi-continuity and L^1-compactness mentioned above, hence in particular Λ_r and $\widetilde{\Lambda}_r$ for any $0 < r < 1$, can represent *generalized surface tension* contributions to the free enthalpy. This yields very irregular phase boundaries, which are reminiscent of *dendritic* interfaces, and of the shape of *snowflakes*.

Variational models of surface tension effects in stationary multi-phase systems were already proposed by the present author in [10,11]; those results were announced in [12]. The functionals of $GC(\Omega)$ are also used in the formulation of a variational model of set evolution proposed in [13]. The results of the present paper were announced in [14,15].

§2. Generalized Coarea Formula

Let Ω be any non-empty set; we shall denote by $\mathbf{R}^\Omega$ the family of all functions $\Omega \to \mathbf{R}$, and set for any $s, y \in \mathbf{R}$,

$$H_s^0(y) := \begin{cases} 0 & \text{if} \quad y \le s, \\ 1 & \text{if} \quad y > s, \end{cases}$$

$$H_s^1(y) := \begin{cases} 0 & \text{if} \quad y < s, \\ 1 & \text{if} \quad y \ge s. \end{cases}$$

DEFINITION. For $i = 0$ or 1, we denote by $GC^i(\Omega)$ the family of the functionals

$\Lambda : \mathbf{R}^\Omega \to [0,+\infty]$ which are proper (i.e. $\Lambda(u) \neq +\infty$ for some $u \in \mathbf{R}^\Omega$), and which fulfill the following *generalized coarea formula*

$$\Lambda(u) = \int_{\mathbf{R}} \Lambda(H^i_s(u))ds \quad (\leq +\infty) \quad \forall u \in \mathbf{R}^\Omega, \tag{2.1}$$

with the convention that the integral is set equal to $+\infty$ if the function $\mathbf{R} \to [0,+\infty] : s \mapsto \Lambda(H^i_s(u))$ is not measurable. We also set

$$GC^\cup(\Omega) := GC^0(\Omega) \cup GC^1(\Omega), \quad GC^\cap(\Omega) := GC^0(\Omega) \cap GC^1(\Omega), \tag{2.2}$$

$$\mathrm{Dom}(\Lambda) := \{u \in \mathbf{R}^\Omega : \Lambda(u) \neq +\infty\}, \tag{2.3}$$

$$\widehat{\Lambda}(u) := \Lambda(-u) \quad \forall \Lambda \in GC^\cup(\Omega),\ \forall u \in \mathbf{R}^\Omega. \tag{2.4}$$

REMARK. The classes of functionals $GC^i(\Omega)$ are here introduced having in mind applications to the calculus of variations, cf. Section 4; so it looks convenient to exclude from their domains any function which might lead to pathologies due to non-measurability. This is the main reason why the integrals of non-measurable functions are set equal to $+\infty$.

An essentially equivalent definition would be obtained by assuming the convention that, whenever the function $s \mapsto \Lambda(H^i_s(u))$ is not measurable, in (2.1) the integral is replaced by the *superior integral*. Actually the following developments hold also if the latter convention is assumed, just with very minor modifications. □

We list some properties of $GC^\cup(\Omega)$, whose justification is straightforward:

PROPOSITION 1. *For any $\Lambda \in GC^\cup(\Omega)$,*

$$\Lambda(u+c) = \Lambda(u) \qquad \forall u \in \mathbf{R}^\Omega, \quad \forall c \in \mathbf{R}, \tag{2.5}$$

$$\Lambda(cu) = c\Lambda(u) \qquad \forall u \in \mathbf{R}^\Omega, \quad \forall c > 0, \tag{2.6}$$

$$\Lambda(c) = 0 \qquad \forall c \in \mathbf{R}, \tag{2.7}$$

$$\Lambda(u) = \Lambda(u \wedge c) + \Lambda(u \vee c) \quad \forall u \in \mathbf{R}^\Omega, \quad \forall c \in \mathbf{R} \tag{2.8}$$

$$(\textit{here} \quad (u \wedge c)(x) := \min(u(x), c),\ (u \vee c)(x) := \max(u(x), c)),$$

$$\lim_{\mathbf{R} \ni c \to +\infty} \Lambda(u \wedge c) = \lim_{\mathbf{R} \ni c \to -\infty} \Lambda(u \vee c) = \Lambda(u), \tag{2.9}$$

$$\text{(2.10)} \qquad \begin{cases} \textit{if } \Lambda \in GC^i(\Omega) \ (i = 0,1), \ \textit{then } \forall u \in \mathbf{R}^\Omega \\ \Lambda(u) = \displaystyle\int_{c_1}^{c_2} \Lambda(H_s^i(u))ds \quad \forall c_1 \le \inf_\Omega u, \quad \forall c_2 \ge \sup_\Omega u, \end{cases}$$

$$\text{(2.11)} \qquad \textit{if } \Lambda \textit{ is convex, then } \mathrm{Dom}(\Lambda) \textit{ is a convex cone,}$$

$$\text{(2.12)} \qquad \begin{cases} \textit{if } \Lambda \textit{ is convex, then } \mathrm{Dom}(\Lambda + \widehat{\Lambda}) \textit{ is a linear space} \\ \textit{and } \Lambda + \widehat{\Lambda} \textit{ is a seminorm on it,} \end{cases}$$

$$\text{(2.13)} \qquad GC^0(\Omega),\ GC^1(\Omega) \textit{ and } GC^\cap(\Omega) \textit{ are convex cones,}$$

$$\text{(2.14)} \qquad \forall \Lambda \in GC^i(\Omega),\ \widehat{\Lambda} \in GC^{1-i}(\Omega) \quad (i = 0,1). \qquad \square$$

We shall denote by $\langle\, , \rangle$ the pairing between $L^1(\Omega)$ and $L^\infty(\Omega)$.

PROPOSITION 2. *For any $\Lambda \in GC^\cup(\Omega)$ and for any $u \in L^1(\Omega)$ such that $\partial\Lambda(u) \neq \emptyset$, one has*

$$\text{(2.15)} \qquad \partial\Lambda(cu) = \partial\Lambda(u) \qquad \forall c > 0,$$

$$\text{(2.16)} \qquad \Lambda(u) = \langle \xi, u\rangle \qquad \forall \xi \in \partial\Lambda(u).$$

Proof. For any $c > 0$, any $\eta \in \partial\Lambda(cu)$ and any $v \in L^1(\Omega)$, we have

$$\langle \eta, cu - cv\rangle \ge \Lambda(cu) - \Lambda(cv),$$

that is by (2.6)

$$c\langle \eta, u - v\rangle \ge c[\Lambda(u) - \Lambda(v)];$$

hence dividing by c we get that $\eta \in \partial\Lambda(u)$. Thus $\partial\Lambda(cu) \subset \partial\Lambda(u)$. The opposite inclusion can be shown similarly. Thus (2.15) holds.

Now we note that for any $\phi \in L^1(\Omega)$, any $\xi \in \partial\Lambda(u)$ and any $\eta \in \partial\Lambda(u + \phi)$ (assumed $\neq \emptyset$), one has

$$\langle \xi, \phi\rangle \le \Lambda(u + \phi) - \Lambda(u) \le \langle \eta, \phi\rangle;$$

hence taking $\phi = u$ and $\eta = \xi$, as possible by (2.15), one gets

$$\langle \xi, u\rangle \le \Lambda(2u) - \Lambda(u) \le \langle \xi, u\rangle,$$

which yields (2.16), by (2.6). □

Remark. The previous proposition holds for any functional Λ fulfilling the homogeneity property (2.6), as it is clear by the proof. □

Examples of functionals of $GC^{\cup}(\Omega)$.

(1) Trivial cases: $\Lambda^0(u) := 0$; $\Lambda_c(u) := 0$ if u=constant in Ω, $\Lambda_c(u) := +\infty$ otherwise.

(2) For any family $\mathcal{A} \subset \mathcal{P}(\Omega)$ (set of the parts of Ω), let us set

$$\Lambda^0_{\mathcal{A}}(u) := \begin{cases} 0 & \text{if } u^{-1}(]s,+\infty[) \in \mathcal{A}, \text{ for a.e. } s \in \mathbf{R}, \\ +\infty & \text{otherwise;} \end{cases} \tag{2.17}$$

$$\Lambda^1_{\mathcal{A}}(u) := \begin{cases} 0 & \text{if } u^{-1}([s,+\infty[) \in \mathcal{A}, \text{ for a.e. } s \in \mathbf{R}, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.18}$$

Then $\Lambda^i_{\mathcal{A}} \in GC^i(\Omega)$ $(i = 0,1)$. In particular, if $\mathcal{A}$ is a topology over Ω, then $\mathrm{Dom}(\Lambda^0_{\mathcal{A}})$ (Dom $(\widehat{\Lambda^0_{\mathcal{A}}})$, respectively) is the family of lower (upper, respectively) semi-continuous functions $\Omega \to \mathbf{R}$. Note that $\mathrm{Dom}[\Lambda^0_{\mathcal{A}} + \widehat{\Lambda^0_{\mathcal{A}}}] = C^0(\Omega)$, but $\Lambda^0_{\mathcal{A}} + \widehat{\Lambda^0_{\mathcal{A}}} \notin GC^{\cup}(\Omega)$.

If $\mathcal{A}$ is a σ-algebra over Ω, then $\Lambda^0_{\mathcal{A}} = \Lambda^1_{\mathcal{A}} \in GC^{\cap}(\Omega)$ and $\mathrm{Dom}(\Lambda^i_{\mathcal{A}})$ $(i = 0,1)$ coincides with the family of $\mathcal{A}$-measurable functions $\Omega \to \mathbf{R}$.

(3) Henceforth we shall consider a measure space $(\Omega, \mathcal{B}, \nu)$, with ν positive and finite, and deal with functionals defined over measurable functions. That is, for any functional Λ we set $\Lambda(u) = +\infty$ $\forall u \notin M(\Omega) :=$ family of (equivalence classes of) ν-measurable functions. All of these functionals fulfill the natural invariance condition

$$\Lambda(u) = \Lambda(v) \quad \forall u, v : \Omega \to \mathbf{R},\ u = v \text{ a.e. in } \Omega.$$

As a first example, we consider

$$\Lambda_{\sup}(u) := \int_\Omega [\operatorname*{ess\,sup}_\Omega\ u - u(x)]\, d\nu(x) \quad (\le +\infty). \tag{2.19}$$

We notice that for any function $v : \Omega \to \mathbf{R}$, defined everywhere, the following identity holds

$$\sup_\Omega v - v(x) = \int_{\mathbf{R}} [\sup_\Omega H^i_s(v) - H^i_s(v(x))]\, ds \quad \forall x \in \Omega,\ i = 0,1; \tag{2.20}$$

moreover this formula is valid a.e. in Ω, if v is a measurable function and *sup* is replaced by *ess sup*. Thus by *Fubini's theorem*, one has

$$\begin{aligned} \Lambda_{\sup}(u) &= \int_\Omega d\nu(x) \int_{\mathbf{R}} [\operatorname*{ess\,sup}_\Omega H^i_s(u) - H^i_s(u(x))]\, ds \\ &= \int_{\mathbf{R}} ds \int_\Omega [\operatorname*{ess\,sup}_\Omega H^i_s(u) - H^i_s(u(x))]\, d\nu(x) = \int_{\mathbf{R}} \Lambda_{\sup}(H^i_s(u))\, ds,\ i = 0,1; \end{aligned}$$

so $\Lambda_{\sup} \in GC^{\cap}(\Omega)$. We also set

$$\text{(2.21)} \qquad \Lambda_{\inf}(u) := \Lambda_{\sup}(-u) = \int_{\Omega} [u(x) - \operatorname*{ess\,inf}_{\Omega} u] d\nu(x) \quad (\leq +\infty);$$

hence

$$\text{(2.22)} \qquad (\Lambda_{\sup} + \Lambda_{\inf})(u) = \nu(\Omega)[\operatorname*{ess\,sup}_{\Omega}\ u - \operatorname*{ess\,inf}_{\Omega}\ u] \quad (\leq +\infty).$$

REMARK. By (2.20), the above functionals can be modified by replacing *ess sup* and *ess inf* with *sup* and *inf*, respectively. In this case the functionals are defined on functions defined everywhere in Ω. □

(4) Let $(\Omega, \mathcal{B}, \nu)$ still be a measure space, with ν positive and σ-finite. We fix a $\nu \otimes \nu$-measurable function $g: \ \Omega^2 \to \mathbf{R}^+$ and set

$$\text{(2.23)} \quad \Lambda_g^{(+)}(u) := \begin{cases} \displaystyle\iint_{\Omega^2} [u(x) - u(y)]^+ g(x,y) d\nu(x) d\nu(y) \quad (\leq +\infty) & \forall u \in M(\Omega), \\ +\infty & \forall u \notin M(\Omega), \end{cases}$$

where $\xi^+ := \max(\xi, 0)$ for any $\xi \in \mathbf{R}$. By the identity

$$\text{(2.24)} \qquad [\xi_1 - \xi_2]^+ = \int_{\mathbf{R}} [H_s^i(\xi_1) - H_s^i(\xi_2)]^+ ds \quad \forall \xi_1, \xi_2 \in \mathbf{R}, \quad i = 0, 1,$$

and by *Fubini's theorem*, we have

$$\begin{aligned} \text{(2.25)} \quad \Lambda_g^{(+)}(u) &= \iint_{\Omega^2} g(x,y) d\nu(x) d\nu(y) \int_{\mathbf{R}} [H_s^i(u(x)) - H_s^i(u(y))]^+ ds \\ &= \int_{\mathbf{R}} ds \iint_{\Omega^2} [H_s^i(u(x)) - H_s^i(u(y))]^+ g(x,y) d\nu(x) d\nu(y) \\ &= \int_{\mathbf{R}} \Lambda_g^{(+)}(H_s^i(u)) ds \qquad \forall u \in M(\Omega), \quad i = 0, 1. \end{aligned}$$

This identity trivially holds also if $u \notin M(u)$; thus $\Lambda_g^{(+)} \in GC^{\cap}(\Omega)$. $\Lambda_g^{(+)}$ is obviously convex; by *Fatou's lemma*, it is also lower-semicontinuous in $L^1(\Omega, \mathcal{B}, \nu)$. We set

$$\text{(2.26)} \qquad \Lambda_g^{(-)}(u) := \Lambda_g^{(+)}(-u) = \iint_{\Omega^2} [u(x) - u(y)]^- g(x,y) d\nu(x) d\nu(y) \qquad (\leq +\infty) \quad \forall u \in \mathbf{R}^{\Omega},$$

where $\xi^- := \max(-\xi, 0)$ for any $\xi \in \mathbf{R}$, and

$$\begin{aligned} \text{(2.27)} \ \Lambda_g(u) :&= \Lambda_g^{(+)}(u) + \Lambda_g^{(-)}(u) \\ &= \iint_{\Omega^2} |u(x) - u(y)| g(x,y) d\nu(x) d\nu(y) \quad (\leq +\infty) \quad \forall u \in \mathbf{R}^{\Omega}. \end{aligned}$$

Note that if $g(x,y) = g(y,x)$ for almost any $x, y \in \Omega$, then $\Lambda_g^{(+)} = \Lambda_g^{(-)}$.

The subdifferential of Λ_g can be easily characterized. For any $u \in \mathrm{Dom}(\Lambda_g)$, $\xi \in \partial\Lambda_g(u)$ if and only if there exists a measurable function $\sigma : \Omega^2 \to \mathbf{R}$ such that a.e. in Ω^2

$$\sigma(x,y) \begin{cases} = -1 & \text{if } u(x) < u(y), \\ \in [-1,1] & \text{if } u(x) = u(y), \\ = 1 & \text{if } u(x) > u(y), \end{cases}$$

and

$$(2.28) \qquad {}_{L^\infty(\Omega)}\langle \xi, v \rangle_{L^1(\Omega)} = \iint_{\Omega^2} \sigma(x,y)[v(x) - v(y)]g(x,y)d\nu(x)d\nu(y).$$

(5) Henceforth we shall assume that Ω is a domain of $\mathbf{R}^N (N \geq 1)$ endowed with the ordinary Lebesgue measure. For any $r \in]0,1[$, we set

$$(2.29) \qquad g_r(x,y) := |x-y|^{-(N+r)} \qquad \forall x, y \in \Omega \quad (x \neq y),$$

and denote by Λ_r the corresponding functional Λ_{g_r} defined in (2.27). Note that for any $r \in]0,1[$

$$(2.30) \qquad \begin{aligned} &L^1(\Omega) \cap \mathrm{Dom}(\Lambda_r) = W^{r,1}(\Omega) : \text{ fractional Sobolev space,} \\ &\|v\|_{L^1(\Omega)} + \Lambda_r(v) = \|v\|_{W^{r,1}(\Omega)}. \end{aligned}$$

Moreover if

$$(2.31) \qquad \begin{cases} \Omega \text{ fulfils the regularity assumptions required by the} \\ \text{classical } \textit{Rellich compactness theorem}, \end{cases}$$

(for instance, Ω is bounded and of Lipschitz class), then

$$(2.32) \qquad \text{the injection } W^{r,1}(\Omega) \to L^1(\Omega) \text{ is compact.}$$

We also have

$$(2.33) \qquad W^{r_1,1}(\Omega) \subset W^{r_2,1}(\Omega) \qquad \forall r_1, r_2 \in]0,1[, \quad r_1 > r_2.$$

Remarks. (i) The functionals Λ_g can represent an anisotropic and nonhomogeneous material; this is not possible for the Λ_r's. However the latter can be generalized by replacing g_r with a weight of the form $\gamma(x,y)|x-y|^{-(N+r)}$, with γ measurable and larger than some positive constant.

(ii) If u is a characteristic function, namely $u(x) = 0$ or 1 a.e. in Ω, then for any $m \in]0,r[$, setting $p = \frac{r}{m}$, we have

$$(2.34) \qquad \Lambda_r(u) = \iint_\Omega |u(x) - u(y)|^p \cdot |x-y|^{-(N+mp)} dx dy = |||u|||^p_{W^{m,p}(Q)}.$$

(iii) The case $r = 1$ is not interesting: $\Lambda_1(u) = 0$ for any u=constant, $\Lambda_1(u) = +\infty$ otherwise. $\square$

(6) Still with Ω domain of $\mathbf{R}^N$, we fix a measurable function $f : \Omega\times\mathbf{R}^+ \to \mathbf{R}^+$. For any $x \in \mathbf{R}^N$ and any $h \in \mathbf{R}^+$, we set $B_h(x) := \{y \in \mathbf{R}^N : |x-y| \le h\}$. For any $u \in M(\Omega)$, then we define

$$\widetilde{\Lambda}_f^{(+)}(u) := \iint_{\Omega\times\mathbf{R}^+} [\operatorname*{ess\,sup}_{B_h(x)\cap\Omega} u - u(x)] f(x,h)dxdh \quad (\le +\infty), \tag{2.35}$$

$$\widetilde{\Lambda}_f^{(-)}(u) := \Lambda_f^{(+)}(-u) = \iint_{\Omega\times\mathbf{R}^+} [u(x) - \operatorname*{ess\,inf}_{B_h(x)\cap\Omega} u] f(x,h)dxdh \quad (\le +\infty), \tag{2.36}$$

$$\begin{aligned} \widetilde{\Lambda}_f(u) :&= \widetilde{\Lambda}_f^{(+)}(u) + \widetilde{\Lambda}_f^{(-)}(u) \\ &= \iint_{\Omega\times\mathbf{R}^+} (\operatorname*{ess\,sup}_{B_h(x)\cap\Omega} u - \operatorname*{ess\,inf}_{B_h(x)\cap\Omega} u) f(x,h)dxdh \quad (\le +\infty); \end{aligned} \tag{2.37}$$

as above we extend these functionals with value $+\infty$ for any $u \notin M(\Omega)$.

As we said, the identity (2.20) holds also a.e. in Ω if *sup* is replaced by *ess sup*. Hence, applying *Fubini's theorem*, we have

$$\begin{aligned} \widetilde{\Lambda}_f^{(+)}(u) &= \iint_{\Omega\times\mathbf{R}^+} dxdh\ f(x,h) \int_{\mathbf{R}} [\operatorname*{ess\,sup}_{B_h(x)\cap\Omega} H_s^i(u) - H_s^i(u(x))]ds \\ &= \int_{\mathbf{R}} ds \iint_{\Omega\times\mathbf{R}^+} [\operatorname*{ess\,sup}_{B_h(x)\cap\Omega} H_s^i(u) - H_s^i(u(x))] f(x,h)dxdh \\ &= \int_{\mathbf{R}} \widetilde{\Lambda}_f^{(+)}(H_s^i(u))ds \qquad \forall u \in M(\Omega), \quad i = 0,1. \end{aligned} \tag{2.38}$$

This trivially holds also if $u \notin M(\Omega)$. It is easy to check that $\widetilde{\Lambda}_f^{(+)}$ is convex. As the *ess sup* is lower semi-continuous with respect to the L^1-topology, applying *Fatou's lemma* we get that $\widetilde{\Lambda}_f^{(+)}$ is lower semi-continuous in $L^1(\Omega, \mathcal{A}, \mu)$. Obviously these properties hold also for $\widetilde{\Lambda}_f^{(-)}$ and $\widetilde{\Lambda}_f$.

(7) For any $x \in \mathbf{R}^N$ and any $h > 0$, let us set $S_h(x) := \{y \in \mathbf{R}^N : |x-y| = h\}$ and denote by ω_{N-1} the area of the $(N-1)$-dimensional sphere of radius 1. For almost any $(x,h) \in \Omega \times \mathbf{R}^+$, we have

$$\int_{S_h(x)\cap\Omega} |u(x) - u(y)|dy \le \omega_{N-1} h^{N-1} (\operatorname*{ess\,sup}_{B_h(x)\cap\Omega} u - \operatorname*{ess\,inf}_{B_h(x)\cap\Omega} u);$$

hence, setting

$$g(x,y) := f(x, |x-y|) \cdot |x-y|^{1-N} \qquad \forall x,y \in \Omega, \tag{2.39}$$

we have

$$(2.40)\quad \widetilde{\Lambda}_f(u) \geq \omega_{N-1}^{-1} \int_\Omega dx \int_{\mathbf{R}^+} dh\; h^{1-N} \int_{S_h(x)\cap\Omega} |u(x)-u(y)|dy\; f(x,h)$$

$$= \omega_{N-1}^{-1} \int_\Omega dx \int_\Omega dy |u(x)-u(y)| f(x,|x-y|)|x-y|^{1-N}$$

$$= \omega_{N-1}^{-1} \Lambda_g(u).$$

For any $r \in]0,1[$, by the transformation formula (2.39), g_r defined in (2.29) corresponds to

$$(2.41)\qquad f_r(x,h) := h^{-(1+r)} \qquad \forall x \in \Omega,\ \forall h > 0.$$

Let us set $\widetilde{\Lambda}_r := \widetilde{\Lambda}_{f_r}$ for any $r \in]0,1[$. By (2.12), also $L^1(\Omega)\cap \mathrm{Dom}(\widetilde{\Lambda}_r)$ is a Banach space, which here we shall denote by $\widetilde{W}^{r,1}(\Omega)$. By (2.40) we have

$$(2.42)\qquad \widetilde{W}^{r,1}(\Omega) \subset W^{r,1}(\Omega) \qquad \forall r \in]0,1[;$$

hence by (2.32) we conclude that, if (2.31) holds, then

$$(2.43)\qquad \text{the injection } \widetilde{W}^{r,1}(\Omega) \to L^1(\Omega) \text{ is compact.}$$

Moreover, if Ω is bounded, then the integrand of (2.37) is constant for h large enough; hence

$$(2.44)\qquad \widetilde{W}^{r_1,1}(\Omega) \subset \widetilde{W}^{r_2,1}(\Omega) \qquad \forall r_1, r_2 \in]0,1[,\ r_1 > r_2.$$

REMARKS. (i) The functionals $\widetilde{\Lambda}_f$ can represent non-homogeneous but isotropic materials. However they admit a natural generalization to the anisotropic case, consisting in replacing the balls $B_h(x)$ with a different basis of (measurable) neighbourhoods of x.

(ii) Also here the case $r = 1$ yields a trivial functional. □

(8) Still for Ω domain of $\mathbf{R}^N$, let us set

$$(2.45)\quad V(u) := \int_\Omega |\nabla u| := \sup_{\eta \in C_c^1(\Omega)^N, |\eta|\leq 1} \int_\Omega u\; div\; \eta\; dx \quad (\leq +\infty) \quad \forall u \in L^1_{\mathrm{loc}}(\Omega);$$

here $C_c^1(\Omega)^N$ denotes the subspace of C^1-functions $\Omega \to \mathbf{R}^N$ with compact support in Ω. Thus

$$\begin{cases} L^1(\Omega) \cap \mathrm{Dom}(V) = BV(\Omega): \text{ Banach space of} \\ \qquad \text{functions with bounded total variation,} \end{cases}$$

$$\|v\|_{L^1(\Omega)} + V(v) = \|v\|_{BV(\Omega)}.$$

Here (2.1) (with $i = 0$ or 1) coincides with the standard Fleming-Rishel *coarea formula* [6,7; p. 20]; thus $V \in GC^{\cap}(\Omega)$. Note that V is convex and lower semicontinuous in $L^1(\Omega)$ [7; p. 9]. Moreover, if (2.31) holds, then [7; p. 17]

$$\text{the injection } BV(\Omega) \to L^1(\Omega) \text{ is compact.} \tag{2.46}$$

Finally, we recall that

$$BV(\Omega) \subset W^{r,1}(\Omega) \quad \forall r \in]0,1[. \tag{2.47}$$

(9) Still for Ω domain of $\mathbf{R}^N$, let us set

$$\overline{V}(u) := \int_{\overline{\Omega}} |\nabla u| := \sup_{\eta \in C^1(\mathbf{R}^N)^N, |\eta| \leq 1} \int_\Omega u \; div \; \eta \; dx \qquad (\leq +\infty) \quad \forall u \in L^1_{\mathrm{loc}}(\Omega). \tag{2.48}$$

Also $\overline{V} \in GC^{\cap}(\Omega)$; $\overline{V}$ has the same properties pointed out for V.

We summarize the properties of the previous functionals:

PROPOSITION 3. *Let either $\Lambda = \Lambda_r$ $(0 < r < 1)$, or $\Lambda = \widetilde{\Lambda}_r$ $(0 < r < 1)$, or $\Lambda = V$, or $\Lambda = \overline{V}$. Then $\Lambda \in GC^{\cup}(\Omega)$, is convex and lower semi-continuous in $L^1(\Omega)$. Moreover, if* (2.31) *holds, then the injection $L^1(\Omega) \cap \mathrm{Dom}(\Lambda) \to L^1(\Omega)$ is compact. Finally the inclusions* (2.42) *and* (2.47) *hold.* □

The compactness property can be improved as follows:

PROPOSITION 4. *Let $0 < r < 1$ and $\Omega \subset \mathbf{R}^N$ be a bounded domain such that*

$$\int_{\mathbf{R}^N \setminus \Omega} dy \int_\Omega dx |x - y|^{-(N+r)} < +\infty. \tag{2.49}$$

Then any family $\mathcal{F}$ of functions $\Omega \to \mathbf{R}$ such that

$$\Lambda_r(v) + \|v\|_{L^\infty(\Omega)} \leq \textit{Constant independent of } v \in \mathcal{F}, \tag{2.50}$$

is precompact with respect to the strong topology of $L^1(\Omega)$.

Proof. Let us fix a ball B of $\mathbf{R}^N$ such that $\Omega \subset B$. Here we shall denote by $\Lambda_{r,\Omega}$ the functional Λ_r defined in (2.27), (2.29), and by $\Lambda_{r,B}$ the same functional with Ω replaced by B. Moreover, for any $v : \Omega \to \mathbf{R}$ we set

$$\widehat{v}(x) := \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in B \setminus \Omega. \end{cases} \tag{2.51}$$

Then we have

$$\begin{aligned}
\Lambda_{r,B}(\widehat{v}) &= \iint_{B^2} |\widehat{v}(x) - \widehat{v}(y)| \cdot |x-y|^{-(N+r)} dx dy \\
&= \iint_{\Omega^2} |v(x) - v(y)| \cdot |x-y|^{-(N+r)} dx dy \\
&\qquad + 2 \int_{B\setminus\Omega} dy \int_{\Omega} dx |v(x)| |x-y|^{-(N+r)} \\
&\leq \Lambda_{r,\Omega}(v) + 2\|v\|_{L^\infty(\Omega)} \int_{B\setminus\Omega} dy \int_{\Omega} dx |x-y|^{-(N+r)} \\
&\leq \text{Constant independent of v} \in \mathcal{F}.
\end{aligned}$$

Hence applying *Proposition 3* we get the thesis. □

PROPOSITION 5. *Let $0 < r < 1$ and Ω be a bounded domain of $\mathbf{R}^N$ such that for a suitable ball B of $\mathbf{R}^N$, $\Omega \subset B$ and*

$$\int_{\mathbf{R}^+} \mu(\{x \in B : d(x, \Omega) \leq h\}) h^{-1-r} dh < +\infty. \tag{2.52}$$

Then any family F of functions $\Omega \to \mathbf{R}$ such that

$$\widetilde{\Lambda}_r(v) + \|v\|_{L^\infty(\Omega)} \leq \textit{ Constant independent of } v \in F, \tag{2.53}$$

is precompact with respect to the strong topology of $L^1(\Omega)$.

Proof. Also here we shall denote by $\widetilde{\Lambda}_{r,\Omega}$ the functional $\widetilde{\Lambda}_r$ defined in (2.37), (2.41), and by $\widetilde{\Lambda}_{r,B}$ the same functional with Ω replaced by B. We shall still use the notation (2.51), set ess osc:= ess sup–ess inf, and denote by χ_Ω the characteristic function of Ω (i.e., $\chi_\Omega = 1$ in Ω, $\chi_\Omega = 0$ in $\mathbf{R}^N \setminus \Omega$). Then we have

$$\begin{aligned}
\widetilde{\Lambda}_{r,B}(\widehat{v}) &= \int_{\mathbf{R}^+} dh\ h^{-1-r} \int_B \operatorname*{ess\,osc}_{B_h(x)\cap B} \widehat{v}\ dx \\
&\leq \int_{\mathbf{R}^+} dh\ h^{-1-r} \left\{ \|v\|_{L^\infty(\Omega)} \int_B \operatorname*{ess\,osc}_{B_h(x)\cap B} \chi_\Omega dx + \int_\Omega \operatorname*{ess\,osc}_{B_h(x)\cap\Omega} v\ dx \right\} \\
&= \|v\|_{L^\infty(\Omega)} \int_{\mathbf{R}^+} dh\ h^{-1-r} \mu(\{x \in B : d(x,\Omega) \leq h\}) + \widetilde{\Lambda}_{r,\Omega}(v) \\
&\leq \text{Constant independent of } v \in F,
\end{aligned}$$

hence by *Proposition 3* we get the thesis. □

REMARK. The assumptions (2.49) and (2.52) can be rewritten in the form

$$\Lambda_{r,\mathbf{R}^N}(\chi_\Omega) < +\infty; \quad \widetilde{\Lambda}_{r,B}(\chi_\Omega) < +\infty. \quad \square \tag{2.54}$$

Added in proofs. The classes $GC^i(\Omega)$ $(i = 0,1)$ are closed with respect to pointwise convergence in $L^1(\Omega)$, in the following sense:

THEOREM. *Let $\{\Lambda_j\}_{j\in\mathbf{N}}$ be a sequence of functionals of $GC^i(\Omega)$, with either $i = 0$ or $i = 1$, and the functional $\Lambda : L^1(\Omega) \to [0,+\infty]$ be convex, lower semicontinuous and homogeneous of degree 1. If $\Lambda_j(u) \to \Lambda(u)$ for any $u \in L^1(\Omega)$, then $\Lambda \in GC^i(\Omega)$.*

This result allows to construct several other examples of functionals fulfilling the generalized coarea formula.

§3. Boundaries of Fractional Dimension

3.1 Functionals and set applications.

For any set $A \subset \Omega$ we shall denote by χ_A its characteristic function in Ω, namely

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \Omega\backslash A. \end{cases}$$

For any property Q about the points of Ω, we shall set

$$\{Q\} := \{x \in \Omega : Q(x) \text{ holds}\};$$

for instance, for any function $u : \Omega \to \mathbf{R}$ we set

$$\{u \geq 0\} := \{x \in \Omega : u(x) \geq 0\}.$$

Any functional $\Lambda : \mathbf{R}^\Omega \to [0,+\infty]$ determines the non-negative set application

$$\mathcal{F}_\Lambda(A) := \Lambda(\chi_A) \qquad \forall A \in \mathcal{P}(\Omega). \tag{3.1}$$

On the other hand any set application $F : \mathcal{P}(\Omega) \to [0,+\infty]$ determines the functionals

$$\mathcal{L}_F^0(u) := \int_{\mathbf{R}} F(\{u > s\})ds \qquad \forall u \in \mathbf{R}^\Omega, \tag{3.2}$$

$$\mathcal{L}_F^1(u) := \int_{\mathbf{R}} F(\{u \geq s\})ds \qquad \forall u \in \mathbf{R}^\Omega; \tag{3.3}$$

also here the integrals of non-measurable functions are assumed to be equal to $+\infty$; however the remark following formula (2.4) could be repeated here.

PROPOSITION 6. *For any $F : \mathcal{P}(\Omega) \to [0,+\infty]$, if*

$$F(\emptyset) = F(\Omega) = 0, \tag{3.4}$$

then

$$\mathcal{L}^i_F \in GC^i(\Omega) \quad (i=0,1). \tag{3.5}$$

Proof. Let us take $i=0$; for $i=1$ the argument would be quite similar. We note that for any s, $\xi \in \mathbf{R}$

$$\{H^0_s(u) > \xi\} = \begin{cases} \Omega & \text{if } \xi < 0, \\ \{u > s\} & \text{if } 0 \le \xi < 1, \\ \phi & \text{if } \xi \ge 1; \end{cases}$$

hence, by (3.2) and (3.4), we have

$$\mathcal{L}^0_F(H^0_s(u)) = \int_{\mathbf{R}} F(\{H^0_s(u) > \xi\})d\xi = F(\{u > s\});$$

thus (3.2) can be rewritten in the form

$$\mathcal{L}^0_F(u) = \int_{\mathbf{R}} \mathcal{L}^0_F(H^0_s(u))ds \qquad \forall u \in \mathbf{R}^\Omega, \tag{3.6}$$

that is $\mathcal{L}^0_F \in GC^0(\Omega)$. $\square$

REMARK. It would have been equivalent to set

$$\mathcal{L}^0_F(\chi_A) := F(A) \qquad \forall A \in \mathcal{P}(\Omega), \tag{3.7}$$

and then to extend $\mathcal{L}^0_F$ by means of the *generalized coarea formula* (3.6). The previous argument shows that this extension is consistent with (3.7). $\square$

We introduce the family of *non-negative set applications* such that (3.4) holds:

$$SA(\Omega) := \{F : \mathcal{P}(\Omega) \to [0,+\infty],\ F(\emptyset) = F(\Omega) = 0\}.$$

PROPOSITION 7. (i) *For $i = 0,1$, the transformations*

$$\mathcal{F} : GC^i(\Omega) \to SA(\Omega) : \Lambda \mapsto \mathcal{F}_\Lambda, \tag{3.9}$$

$$\mathcal{L}^i : SA(\Omega) \to GC^i(\Omega) : F \mapsto \mathcal{L}^i_F, \tag{3.10}$$

are each one the inverse of the other one.

(ii) *If $(\Omega, \mathcal{B}, \nu)$ is a measure space, then for any $\Lambda \in GC(\Omega)$*

$$\Lambda(u) = \Lambda(v) \qquad \forall u,v \in \Omega^{\mathbf{R}}\ \nu\text{-measurable},\ u=v \qquad \nu - a.e.\ in\ \Omega, \tag{3.11}$$

if and only if, setting $A\Delta B := (A \cup B)\backslash(A \cap B)$,

$$\mathcal{F}_\Lambda(A) = \mathcal{F}_\Lambda(B) \qquad \forall A,B \in \mathcal{B},\ \nu(A\Delta B) = 0. \tag{3.12}$$

(iii) *Setting*

$$\widehat{F}(A) := F(\Omega\backslash A) \qquad \forall A \in \mathcal{P}(\Omega), \quad \forall F \in SA(\Omega), \tag{3.13}$$

and using the notation (2.4), *for any* $\Lambda \in GC^{\cup}(\Omega)$,

$$\mathcal{F}_{\widehat{\Lambda}} = \widehat{\mathcal{F}}_{\Lambda}. \tag{3.14}$$

(iv) *For any* $F \in SA(\Omega)$,

$$\mathcal{L}^{i}_{\widehat{F}} = \widehat{\mathcal{L}^{i-1}_{F}} \qquad (i = 0, 1). \tag{3.15}$$

Proof. It is easy to check the first two statements. By (2.5) we have

$$\begin{aligned}\mathcal{F}_{\widehat{\Lambda}}(A) &= \widehat{\Lambda}(\chi_A) = \Lambda(-\chi_A) = \Lambda(1 - \chi_A) = \Lambda(\chi_{\Omega\backslash A}) \\ &= \mathcal{F}_{\Lambda}(\Omega\backslash A) = \widehat{\mathcal{F}}_{\Lambda}(A) \qquad \forall A \in \mathcal{P}(\Omega),\end{aligned}$$

so (3.14) holds. Finally, we check (3.15) for $i = 0$, e.g. By (3.2) we have

$$\begin{aligned}\mathcal{L}^{0}_{\widehat{F}}(u) &= \int_{\mathbf{R}} \widehat{F}(\{u > s\})ds = \int_{\mathbf{R}} F(\{u \le s\})ds = \int_{\mathbf{R}} F(\{-u \ge s\})ds \\ &= \mathcal{L}^{1}_{F}(-u) = \widehat{\mathcal{L}}^{1}_{F}(u) \qquad \forall u \in \mathbf{R}^{\Omega}. \qquad \square\end{aligned}$$

Above, in formulae (2.1), (3.2) and (3.3), the integrals of non-measurable functions were set equal to $+\infty$. However, it was also remarked that essentially equivalent definitions would be obtained, if for non-measurable integrands the integral were replaced by the *superior integral.* Actually the previous developments hold also under the latter convention, with very minor modifications. Now we present a result which holds only if the convention prescribing the use of the superior integral is assumed, as it will be clear from the proof. Here we shall denote the superior integral over $\mathbf{R}$ by $\int^{*}_{\mathbf{R}}$, and the corresponding functionals defined as in (3.2) and (3.3) by $\overline{\mathcal{L}}^{i}_{F}(u)$ $(i = 0, 1)$.

PROPOSITION 8. *Let* $(\Omega, \mathcal{B}, \nu)$ *be a measure space, with* ν *positive and finite, and let* $F \in SA(\Omega)$. *Then*

(3.16) $\forall$ *sequence* $\{A_n \in \mathcal{A}\}_{n\in\mathbf{N}}$, $\nu(A_n \Delta A) \to 0$ *entails* $\liminf_{n\to\infty} F(A_n) \ \ge F(A)$,

if and only if

(3.17) $\overline{\mathcal{L}}^{i}_{F}$ *is lower semi-continuous with respect to the strong topology of* $L^1(\Omega, \mathcal{B}, \nu)$.

Proof. (3.16) is equivalent to the statement that

$$(3.18)\quad \begin{cases} \text{the restriction of } \overrightarrow{L}^i_F \text{ to the family of characteristic functions is lower} \\ \text{semi-continuous with respect to the strong topology of } L^1(\Omega, \mathcal{B}, \nu). \end{cases}$$

Thus we are left with showing that (3.18) entails (3.17).
Let us fix any sequence $\{u_n\}_{n\in\mathbf{N}}$ such that

$$u_n \to u \quad \text{strongly in} \quad L^1(\Omega, \mathcal{B}, \nu),$$

that is

$$\int_{\mathbf{R}} ds \int_\Omega |H_s(u_n) - H_s(u)| d\nu = \int_\Omega |u_n - u| d\nu \to 0;$$

then, possibly extracting a subsequence, for almost any $s \in \mathbf{R}$

$$\int_\Omega |H_s(u_n) - H_s(u)| d\nu \to 0,$$

namely

$$H_s(u_n) \to H_s(u) \quad \text{strongly in} \quad L^1(\Omega, \mathcal{B}, \nu).$$

Then by (3.18), setting $\Lambda := \overrightarrow{\mathcal{L}}^i_F$, still for almost any $s \in \mathbf{R}$ we have

$$\liminf_{n\to\infty} \Lambda(H^i_s(u_n)) \geq \Lambda(H^i_s(u));$$

finally, as $\Lambda \in GC^i(\Omega)$ and by *Fatou's lemma*, we get

$$\begin{aligned} \liminf_{n\to\infty} \Lambda(u_n) &= \liminf_{n\to\infty} \int_{\mathbf{R}} \Lambda(H^i_s(u_n)) ds \\ &\geq \int_{\mathbf{R}} \liminf_{n\to\infty} \Lambda(H^i_s(u_n)) ds \geq \int^*_{\mathbf{R}} \Lambda(H^i_s(u)) ds = \Lambda(u). \end{aligned}$$

Note that the function $s \mapsto \Lambda(H^i_s(u))$ may be non-measurable, whence the need of using the superior integral. As the previous conclusion holds for any extracted subsequence, then it holds for the whole sequence, too. □

3.2 Fractal boundaries.

Henceforth Ω will be a domain of $\mathbf{R}^N$ $(N \geq 1)$, endowed with the ordinary Lebesgue measure μ; the corresponding family of measurable subsets of Ω will be denoted by $\mathcal{A}$. We shall use the standard equivalence relation:

$$(3.19)\qquad \forall A, B \in \mathcal{A}, \quad A \sim B \quad \text{if and only if} \quad \mu(A \Delta B) = 0.$$

For any set $A \in \mathcal{A}$, we shall denote by ∂A its *topological* boundary in Ω, and by $\partial_e A$ its *essential* boundary in Ω,

$$(3.20)\qquad \partial_e A := \{x \in \Omega : \forall h > 0,\ \mu(A \cap B_h(x)) > 0,\ \mu((\Omega \backslash A) \cap B_h(x)) > 0\}.$$

We have

$$\partial_e A \subset \partial B \qquad \forall A, B \in \mathcal{A}, \quad A \sim B; \tag{3.21}$$

moreover, setting

$$D_1(A) := \{x \in \Omega : \lim_{h \to 0^+} \frac{\mu(A \cap B_h(x))}{\mu(B_h(x))} = 1\} \quad \forall A \in \mathcal{A},$$

it is not difficult to check that

$$D_1(A) \sim A, \qquad \partial D_1(A) = \partial_e A, \qquad \forall A \in \mathcal{A}. \tag{3.22}$$

By (3.21) and (3.22) we conclude that

$$\partial_e A = \bigcap_{B \sim A} \partial B \qquad \forall A \in \mathcal{A}, \tag{3.23}$$

$$\partial_e A = \partial_e B \qquad \forall A, B \in \mathcal{A}, \quad A \sim B. \tag{3.24}$$

Now, for any application $\psi : \{\partial_e A : A \in \mathcal{A}\} \to [0, +\infty]$ such that $\psi(\emptyset) = 0$, we set

$$F_\psi(A) := \begin{cases} \psi(\partial_e A) & \text{if } A \in \mathcal{A}, \\ +\infty & \text{if } A \in \mathcal{P}(\Omega) \backslash \mathcal{A}. \end{cases} \tag{3.25}$$

For $i = 0, 1$, setting

$$\mathcal{L}^i_\psi := \mathcal{L}^i_{F_\psi} \qquad (i = 0, 1), \tag{3.26}$$

by *Proposition* 7 (ii) and by (3.24), one has

$$\mathcal{L}^i_\psi(u) = \mathcal{L}^i_\psi(v) \qquad \forall u, v \in \mathbf{R}^\Omega, \ u = v \text{ a.e. in } \Omega. \tag{3.27}$$

For instance, one can take ψ equal to an outer measure; in that case the formulae (3.2) and (3.3) have the same geometric interpretation as the classical *coarea formula* (1.2).

Let us recall the definition of d-dimensional (spherical) *Hausdorff outer measure* $\mathcal{H}_d$ for any $d \in [0, N]$. For any $C \subset \mathbf{R}^N$,

$$\mathcal{H}_d(C) := \lim_{h \to 0^+} \mathcal{H}^h_d(C), \tag{3.28}$$

where, setting $\omega_d := [\Gamma(\frac{1}{2})]^d / \Gamma(1 + \frac{d}{2})$ (Γ denoting the classical Euler function)

$$\mathcal{H}^h_d(C) := \omega_d \cdot \inf\{\sum_{i \in I} \rho_i^d : \bigcup_{i \in I} B_{\rho_i}(x_i) \subset C \ \ |\rho_i| \leq h \quad \forall i \in I\} \tag{3.29}$$

(i.e., the inf is taken over all the coverings of C with balls of radius not larger than h). The *Hausdorff dimension* is then defined by

$$\text{(3.30)}\qquad \begin{aligned}\mathrm{Dim}_H(C) &:= \sup\{d \in [0,N] : \mathcal{H}_d(C) = +\infty\}\\ &(= \inf\{d \in [0,N] : \mathcal{H}_d(C) = 0\}).\end{aligned}$$

We recall that, cf. [7, p. 5],

$$\text{(3.31)}\qquad V(\chi_A) = \mathcal{H}_{N-1}(\partial_e A) \qquad \forall A \in \mathcal{A},\ A \text{ of class } C^2;$$

it is then natural to consider the functionals of $GC(\Omega)$ associated to $\mathcal{H}_{N-r}$, with $0 < r < 1$. We anticipate a lemma:

LEMMA 1. *Let Ω be a bounded domain of $\mathbf{R}^N (N \geq 1)$. Then for any closed set $A \subset \Omega$, there exists a sequence $\{A_n\}_{n\in\mathbf{N}}$ of sets of class C^∞ such that*

$$\text{(3.32)}\qquad \mu(A_n \Delta A) \to 0.$$

Proof. As the application $d(\cdot, A) : \Omega \to \mathbf{R}^+$ is Lipschitz-continuous, for any $n \in \mathbf{N}$ the set $B_n := \{x \in \Omega : d(x, A) \leq \frac{1}{n}\}$ is of Lipschitz class; moreover $A \subset B_n$ and $B_n \to A$; hence $\mu(B_n \backslash A) \to 0$. Note that, for any $n > d(A, \mathbf{R}^N \backslash \Omega)^{-1}$

$$d(x,y) \geq \frac{1}{2n} \qquad \forall x \in \partial B_n, \quad \forall y \in \partial B_{2n};$$

then, by means of a convolution with a suitable mollifier, it is easy to construct a set A_n of class C^∞ such that $B_{2n} \subset A_n \subset B_n$. Hence (3.32) holds. □

For the sets A_n of *Lemma* 1 we have

$$\text{(3.33)}\qquad \mathcal{H}_{N-r}(\partial_e A_n) = 0 \qquad \forall r \in]0,1[;$$

then, as one can take $A \in \mathcal{A}$ such that $\mathcal{H}_{N-r}(\partial_e A) = +\infty$, we get the following result:

PROPOSITION 9. *For any $r \in]0,1[$, if*

$$\text{(3.34)}\qquad F(A) = \mathcal{H}_{N-r}(\partial_e A) \qquad \forall A \in \mathcal{A},$$

then $\mathcal{L}^i_F$ $(i = 0, 1)$ is not lower semi-continuous with respect to the strong topology of $L^1(\Omega)$. Moreover, even if Ω fulfils (2.31), *the injection $L^1(\Omega) \cap \mathrm{Dom}(\mathcal{L}^i_F) \to L^1(\Omega)$ is not compact.* □

The same result holds if $\mathcal{H}_{N-r}$ is replaced by the *Minkowski-Bouligand contents*, whose definitions we shall recall later on in this section.

3.3 New classes of fractal boundaries.

Because of the negative result of *Proposition 9*, the most standard tools of the calculus of variations cannot be applied to the functionals constructed by means of the $(N-r)$*-dimensional Hausdorff measures.* On the other hand, the functionals Λ_r and $\widetilde{\Lambda}_r$ have the useful properties of lower semi-continuity and compactness pointed out in *Proposition 3.* Now we shall show that by means of the latter functionals it is possible to mimic the concept of *set of fractional dimension.* Our construction is different from the standard one of [3,9].

For any measurable function $u : \Omega \to \mathbf{R}$ and any $h > 0$, we set

$$\lambda(u,h) := h^{1-N} \int_\Omega dx \int_{S_h(x)\cap\Omega} |u(x) - u(y)|dy, \tag{3.35}$$

$$\widetilde{\lambda}(u,h) := \int_\Omega \big(\operatorname*{ess\,sup}_{B_h(x)\cap\Omega} u - \operatorname*{ess\,inf}_{B_h(x)\cap\Omega} u\big)dx; \tag{3.36}$$

hence for any $r \in]0,1[$ we have

$$\Lambda_r(u) = \int_{\mathbf{R}^+} \lambda(u,h)h^{-(1+r)}dh, \tag{3.37}$$

$$\widetilde{\Lambda}_r(u) = \int_{\mathbf{R}^+} \widetilde{\lambda}(u,h)h^{-(1+r)}dh. \tag{3.38}$$

Note that for any $A \in \mathcal{A}$, $\widetilde{\lambda}(\chi_A,h)$ is the *N-dimensional Lebesgue measure* of the neighbourhood of thickness h of the essential boundary of A in Ω; namely

$$\widetilde{\lambda}(\chi_A,h) = \mu(\{x \in \Omega : d(x,\partial_e A) \le h\}). \tag{3.39}$$

PROPOSITION 10. *For any measurable function $u : \Omega \to \mathbf{R}$, there exists one and only one $R(u) \in [0,1]$ such that*

$$\begin{cases} \forall r \in]0,R(u)[, & \Lambda_r(u) < +\infty, \\ \forall r \in]R(u),1[, & \Lambda_r(u) = +\infty; \end{cases} \tag{3.40}$$

that is

$$\begin{aligned} R(u) &= \sup\{r \in]0,1[: \Lambda_r(u) < +\infty\} \\ &= \inf\{r \in]0,1[: \Lambda_r(u) = +\infty\}, \end{aligned} \tag{3.41}$$

with the convention that $\sup\emptyset = 1$, $\inf\emptyset = 0$, *as it is natural here.*

(ii) *If Ω is bounded, then the same result holds with Λ_r replaced by $\widetilde{\Lambda}_r$. The corresponding limit value is then denoted by $\widetilde{R}(u)$.*

Proof. These statements stem from the inclusions (2.33) and (2.44). □

DEFINITIONS. For any $A \in \mathcal{A}$ such that $\partial_e A \neq \emptyset$, we set

$$\mathrm{Dim}_{\{\Lambda_r\}}(\partial_e A) := N - R(\chi_A); \tag{3.42}$$

this number is called the *dimension of the essential boundary* $\partial_e A$ of A in Ω, *relative to the functionals* $\{\Lambda_r\}_{0<r<1}$.

Similarly, under the condition that the set Ω be bounded, we introduce the *dimension of* $\partial_e A$ *relative to the functionals* $\{\widetilde{\Lambda}_r\}_{0<r<1}$, which will be denoted by $\mathrm{Dim}_{\{\widetilde{\Lambda}_r\}}(\partial_e A)$. □

PROPOSITION 11. *Let Ω be bounded. Then for any $A \in \mathcal{A}$ such that $\partial_e A \neq \emptyset$,*

$$\mathrm{Dim}_{\{\Lambda_r\}}(\partial_e A) \leq \mathrm{Dim}_{\{\widetilde{\Lambda}_r\}}(\partial_e A). \tag{3.43}$$

Proof. By (2.42) we have

$$\widetilde{R}(\chi_A) \leq R(\chi_A). \qquad \square \tag{3.44}$$

REMARK. These concepts can be easily extended to anisotropic and heterogeneous cases, by modifying the functionals Λ_r and $\widetilde{\Lambda}_r$ as indicated in Section 2. □

We recall the definitions of the *Minkowski-Bouligand contents* (relative to an ambient set $\Omega \subset \mathbf{R}^N$), cf. [9; p. 287], e.g.. For any set $\Gamma \subset \Omega$ and any $h > 0$, first we set

$$I_h(r) := \{x \in \Omega : d(x, \Gamma) \leq h\}; \tag{3.45}$$

then the numbers

$$\liminf_{h \to 0^+} \mu(I_h(\Gamma))h^{-r}, \qquad \limsup_{h \to 0^+} \mu(I_h(\Gamma))h^{-r} \tag{3.46}$$

are named *inferior* and *superior* $(N-r)$*-dimensional Minkowski-Bouligand contents*, respectively. Moreover, cf. still [9; p. 287], if there exists an $R \in [0, N]$ such that

$$\lim_{h \to 0^+} \mu(I_h(\Gamma))h^{-r} = 0 \qquad \forall r < R, \tag{3.47}$$

$$\lim_{h \to 0^+} \mu(I_h(\Gamma))h^{-r} = +\infty \qquad \forall r > R, \tag{3.48}$$

then $N - r$ is called the *Minkowski-Bouligand dimension* of Γ, (relative to the set Ω), and is here denoted by $\mathrm{Dim}_{MB}(\Gamma)$.

It is then natural to introduce also the concepts of *inferior* and *superior* *Minkowski-Bouligand dimension* of Γ, (relative to Ω), respectively defined as follows:

$$\underline{\mathrm{Dim}}_{MB}(\Gamma) := N\text{-}\sup\{r \in [0,N] : \liminf_{h\to 0^+} \mu(I_h(\Gamma))h^{-r} = 0\} \tag{3.49}$$

$$\overline{\mathrm{Dim}}_{MB}(\Gamma) := N\text{-}\inf\{r \in [0,N] : \limsup_{h\to 0^+} \mu(I_h(\Gamma))h^{-r} = +\infty\}; \tag{3.50}$$

(here we assume the natural convention that $\sup\emptyset = N$, $\inf\emptyset = 0$). Note that both of these numbers always exist, whereas $\mathrm{Dim}_{MB}(\Gamma)$ may fail to exist.

It is easy to check the following statement:

PROPOSITION 12. *For any set $\Gamma \subset \mathbf{R}^N$,*

$$\underline{\mathrm{Dim}}_{MB}(\Gamma) \le \overline{\mathrm{Dim}}_{MB}(\Gamma). \tag{3.51}$$

Moreover $\mathrm{Dim}_{MB}(\Gamma)$ *exists if and only if*

$$\underline{\mathrm{Dim}}_{MB}(\Gamma) = \overline{\mathrm{Dim}}_{MB}(\Gamma). \tag{3.52}$$

Finally, if (3.52) *holds then*

$$\mathrm{Dim}_{MB}(\Gamma) = \underline{\mathrm{Dim}}_{MB}(\Gamma) = \overline{\mathrm{Dim}}_{MB}(\Gamma). \qquad \square \tag{3.53}$$

Now we return to set boundaries.

PROPOSITION 13. *Assume that Ω is a bounded domain of $\mathbf{R}^N (N \ge 1)$. Then, for any $A \in \mathcal{A}$ such that $\partial_e A \ne \emptyset$,*

$$\underline{\mathrm{Dim}}_{MB}(\partial_e A) \le \mathrm{Dim}_{\{\widetilde{\Lambda}_r\}}(\partial_e A) \le \overline{\mathrm{Dim}}_{MB}(\partial_e A). \tag{3.54}$$

Proof. Note that, cf. (3.39),

$$\mu(I_h^{\Omega}(\partial_e A)) = \widetilde{\lambda}(\chi_A, h). \tag{3.55}$$

By elementary properties of improper integrals, one has

$$\text{if } \liminf_{h\to 0^+} \mu(I_h^{\Omega}(\partial_e A))h^{-r} > 0, \tag{3.56}$$

$$\text{then } \forall \widehat{r} \ge r \int_{\mathbf{R}^+} \widetilde{\lambda}(\chi_A, h)h^{-(1+\widehat{r})}dh = +\infty;$$

$$\text{if } \int_{\mathbf{R}^+} \widetilde{\lambda}(\chi_A, h)h^{-(1+r)}dh = +\infty, \tag{3.57}$$

$$\text{then } \forall\, \widehat{r} > r \ \limsup_{h\to 0^+} \mu(I_h^{\Omega}(\partial_e A))h^{-\widehat{r}} = +\infty.$$

These two statements yield the thesis. □

COROLLARY 1. *Assume that Ω is a bounded domain. Then for any $A \in \mathcal{A}$ such that $\partial_e A \neq \emptyset$, if* $\mathrm{Dim}_{MB}(\partial_e A)$ *exists then*

$$\mathrm{Dim}_{MB}(\partial_e A) = \mathrm{Dim}_{\{\widetilde{\Lambda}_r\}}(\partial_e A). \tag{3.58}$$

Proof. Direct consequence of *Propositions 12* and *13.* □

REMARKS. (i) On the model of the relation between the functionals $\{\widetilde{\Lambda}_r\}_{0<r<1}$ and the *Minkowski-Bouligand dimension*, it is possible to construct a class of functionals of $GC^{\cup}(\Omega)$ strictly related to the *Hausdorff dimension.* In fact, for any set $\Gamma \subset \Omega$.

$$\begin{aligned} \mathrm{Dim}_H(\Gamma) : &= \sup\{d \in [0,N] : \lim_{h\to 0^+} \mathcal{H}_d^h(\Gamma) = +\infty\} \\ &= \sup\{d \in [0,N] : \psi_d(\Gamma) := \int_{\mathbf{R}^+} \mathcal{H}_d^h(\Gamma)\frac{dh}{h} = +\infty\}; \end{aligned} \tag{3.59}$$

hence for any $A \in \mathcal{A}$ the *Hausdorff dimension* of $\partial_e A$ coincides with the dimension of $\partial_e A$ relative to the functionals $\mathcal{L}^0_{\mathcal{H}_d}$, cf. (3.25), (3.26).

(ii) So far, we have considered functionals corresponding to set applications F_ψ defined as in (3.25); there

$$F_\psi(A) = F_\psi(\Omega\backslash A) \qquad \forall A \in \mathcal{P}(\Omega),$$

as $\partial_e A = \partial_e(\Omega\backslash A)$. One can also deal with oriented boundaries. For instance, let us consider $\widetilde{F}_f^{(+)} := \mathcal{F}_{\widetilde{\Lambda}_f^{(+)}}$, namely the set application corresponding to $\widetilde{\Lambda}_f^{(+)}$, cf. (2.35) and (3.1); one has

$$\widetilde{F}_f^{(+)}(A) = \widetilde{\Lambda}_f^{(+)}(\chi_A) \neq \widetilde{\Lambda}_f^{(+)}(1-\chi_A) = \widetilde{F}_f^{(+)}(\Omega\backslash A), \quad \forall A \in \mathcal{P}(\Omega). \tag{3.60}$$

Also here one can introduce the concept of *dimension of an (oriented) boundary with respect to a scale of functionals*, and so on. □

§4. Surface Tension in Two-Phase Systems

The study of functionals fulfilling the *generalized coarea formula* (2.1) is justified by applications to models of *surface tension effects* in *multi-phase systems.*

Let us consider a domain Ω of $\mathbf{R}^3$ occupied by a two-phase system (e.g. liquid and solid), composed of a homogeneous substance and subject to a prescribed relative temperature field (proportional to) $\theta \in L^1(\Omega)^{(*)}$. If $c(x)$ denotes the local

(*) In this discussion several physical constants will be assumed equal to 1.

concentration of liquid, we set $u(x) := 2c(x) - 1$. Thus $u = -1$ ($u = 1$, respectively) in the interior of the solid (liquid, respectively) phase, and $-1 < u < 1$ in the so-called *mushy region*, corresponding to a very fine mixture of solid and liquid.

At equilibrium the phase distribution is either a relative or absolute minimum of a functional representing the free enthalpy of the system maintained at constant pressure, or the free energy of the system if it is at constant volume.

Here we shall exclude the occurrence of any *mushy region*, i.e., we shall deal with the case $|u| = 1$ a.e. in Ω; a more general model is outlined in [11; Sect. 5]. We shall distinguish the contribution of the free enthalpy relative to the interior of the solid and liquid phases, and that relative to the interface between the phases. The contribution of the solid and liquid phases are respectively (proportional to)

$$\int_{\{u=-1\}} \theta(x)dx, \quad -\int_{\{u=1\}} \theta(x)dx.$$

Now we fix a positive constant α, whose meaning will be explained later on, and introduce the lower semi-continuous non-convex function

$$\varphi(y) := \begin{cases} \frac{\alpha}{2}(1 - y^2) & \text{if } \quad |y| \leq 1, \\ +\infty & \text{if } \quad |y| > 1; \end{cases} \tag{4.1}$$

note that $\varphi(\pm 1) = 0$. Then, assuming that $|u| = 1$ a.e. in Ω, the bulk contribution to the free enthalpy can be represented in the form

$$\Phi_\theta(u) := \int_\Omega [\varphi(u(x)) - \theta(x)u(x)]dx. \tag{4.2}$$

Still assuming that there is no *mushy region*, the (generalized) area of the solid-liquid interface S is equal to $\frac{1}{2}V(u)$, cf. (2.45). Hence the interface contribution to the free enthalpy of the system is equal to $\frac{\sigma}{2}V(u)$, where σ is the surface tension coefficient.

So we can conclude that, if $|u| = 1$ a.e. in Ω, then the total free enthalpy of the two-phase system, subject to a prescribed temperature field θ, is equal to

$$\overline{\Psi}_\theta(u) := \Phi_\theta(u) + \frac{\sigma}{2}V(u), \tag{4.3}$$

plus boundary terms which will be neglected here.

The results we shall present hold for a more general class of functionals. So, for any functional $\Lambda \in GC^{\cup}(\Omega)$ and for any fixed $\theta \in L^1(\Omega)$, we set

$$\Psi_\theta(u) := \Phi_\theta(u) + \Lambda(u) \quad (\leq +\infty) \quad \forall u \in L^1(\Omega). \tag{4.4}$$

By means of the direct method of the calculus of variations, it is easy to verify the following statement:

PROPOSITION 14. *If the injection $L^1(\Omega) \cap \mathrm{Dom}(\Lambda) \to L^1(\Omega)$ is compact, and if Λ is lower semi-continuous with respect to the strong topology of $L^1(\Omega)$, then Ψ_θ has an absolute minimum.* □

As Ψ_θ is non-convex, it can also have one or more relative (non-absolute) minima with respect to the strong topology of $L^1(\Omega)$.

If Ψ_θ represents the free enthalpy of the system, then its absolute minima correspond to the states of *stable equilibrium*; moreover, if the thresholds of liquid and solid nucleation are both equal to α (a restriction which could be easily avoided by modifying the function φ), then the relative minima of Ψ_θ correspond to the states of *metastable equilibrium*, as discussed in [11; Sect. 5]. This model also accounts for *supercooling* and *superheating* effects.

PROPOSITION 15. *If $\Lambda \in GC^{\cup}(\Omega)$, then for any absolute or relative minimum u of Ψ_θ in $L^1(\Omega)$,*

$$|u| = 1 \quad a.e.\ in\ \Omega. \qquad \square \tag{4.5}$$

This statement is a particular case of a result proved in [11; Sect. 3] and here recalled in the *Appendix*.

Note that, after *Proposition 3*, if (2.31) holds then the functionals V, $\widehat{V}$, Λ_r, $\widetilde{\Lambda}_r$ fulfill the assumptions of *Propositions 14* and *15*. Applying *Proposition 15* with $\Lambda = \frac{\sigma}{2}V$, we find that for any absolute or relative minimum u of $\overline{\Psi}_\theta$, cf. (4.3), no *mushy region* may occur; hence $\overline{\Psi}_\theta(u)$ represents the free enthalpy of the system. Moreover the absolute and the relative minima of $\overline{\Psi}_\theta$ fulfill the classical *Gibbs-Thomson law* [10]. Namely, denoting by $\mathcal{S}$ the corresponding liquid-solid interface and by κ its local mean curvature (assumed positive for an ice ball), if θ is continuous (on $\mathcal{S}$) one has

$$\theta = -2\sigma\kappa \qquad \text{on } \mathcal{S}. \tag{4.6}$$

Terms proportional to Λ_r or to $\widetilde{\Lambda}_r$, with $0 < r < 1$, can be regarded as *generalized surface tension* contributions to the free enthalpy. They can be used to represent very irregular interfaces $\mathcal{S}$, of fractional dimension in the sense introduced in Section 3. So they are models of *dendritic interfaces*, for instance.

A generalized surface tension contribution to the free enthalpy (proportional to) $\Lambda_g(u)$, cf. (2.27), can be interpreted as follows; any couple of *particles* staying at the points x, y and belonging to different phases gives a contribution $2[g(x,y) + g(y,x)]$ to the free enthalpy. This was already discussed in [11; Sect. 5].

In Section 3 we saw the relation between the functionals $\widetilde{\Lambda}_r$ and the *Minkowski-Bouligand dimension*. Also the more general functional $\widetilde{\Lambda}_f$, cf. (2.37), has a natural geometrical interpretation. This is especially clear in the homogeneous case, in which f is independent of x: for any measurable set $A \subset \Omega$,

$$\widetilde{\Lambda}_f(\chi_A) = \int_{\mathbf{R}^+} \mu(\{x \in \Omega : d(x, \partial_e A) \leq h\}) f(h) dh. \tag{4.7}$$

The physical interpretation of $\widetilde{\Lambda}_f(u)$ in terms of phase interaction is less natural: any particle P would give a contribution $2f(h)$ to the free enthalpy, for any $h > 0$ such that the set of particles of the other phase staying at a distance not larger than h from P had a strictly positive measure; thus this contribution would be independent of that measure.

Finally we note that the functionals $\Lambda_g^{(+)}$, $\Lambda_g^{(-)}$ and $\widetilde{\Lambda}_f^{(+)}$, $\widetilde{\Lambda}_f^{(-)}$, defined in (2.23), (2.26), (2.35), (2.36), can be used to construct models in which the two phases give asymmetric contributions to the interface enthalpy. In fact $\Lambda_g^{(+)}$ and $\widetilde{\Lambda}_f^{(+)}$ ($\Lambda_g^{(-)}$ and $\widetilde{\Lambda}_f^{(-)}$, respectively) penalize a protrusion of the liquid phase through the solid one (of the solid phase through the liquid one, respectively).

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Appendix. On a Class of Non-Convex Functionals

We recall the main result of [11], whose proof is based on the *generalized coarea formula* (2.1).

Let still Ω be a domain of $\mathbf{R}^N (N \geq 1)$ and

$$\text{(A1)} \qquad \varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\} \text{be lower semi-continuous, } \varphi \not\equiv +\infty,$$

$$\text{(A2)} \qquad \varphi(y) \geq -C_1|y| - C_2 \ (C_1, C_2 \in \mathbf{R}^+; C_2 = 0 \text{ if } \mu(\Omega) = +\infty);$$

set

$$\text{(A3)} \qquad (-\infty <) \ \Phi(u) := \int_\Omega \varphi(u(x))dx \ (\leq +\infty) \quad \forall u \in L^1(\Omega).$$

THEOREM [11]. *Let $\Lambda \in GC(\Omega)$ and $\Lambda = \Lambda^{**}$. Assume that* (A1), (A2) *hold and that*

(A4) *any connected component of* $\{y \in \mathbf{R} : \varphi^{**}(y) < \varphi(y)\}$ *is bounded.*

Then for any $u \in L^1(\Omega)$

$$\partial(\Phi + \Lambda)(u) = \partial\Phi(u) + \partial\Lambda(u) \quad \text{in } L^\infty(\Omega), \tag{A5}$$

$$(\Phi + \Lambda)^{**}(u) = \Phi^{**}(u) + \Lambda(u). \qquad \square \tag{A6}$$

COROLLARY [11]. *Under the previous assumptions,*

$$\text{if } \partial(\Phi + \Lambda)(u) \neq \emptyset \text{ then } \partial\varphi(u(x)) \neq \emptyset \text{ a.e. in } \Omega. \qquad \square \tag{A7}$$

Otherwise stated: for any $\xi \in L^\infty(\Omega)$, if $u \in L^1(\Omega)$ is an *absolute* minimum of $v \mapsto (\Phi + \Lambda)(v) - \int_\Omega \xi(x)v(x)dx$, then $\partial\varphi(u(x)) \neq \emptyset$ a.e. in Ω. A similar statement holds for *relative* minima in the strong topology of $L^1(\Omega)$. In both cases, if φ is non-convex then certain values are a priori excluded from the range of u, for any $\xi \in L^\infty(\Omega)$. Proposition 15 of Section 4 is an application of this result.